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ADDITIVE DIFFERENCE SCHEMES AND ITERATION METHODS FOR PROBLEMS OF MATHEMATICAL PHYSICS

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We construct additive difference schemes for first-order differential–operator equations. The exposition is based on the general theory of stability for operator–difference schemes in lattice Hilbert spaces. The main focus is on the case of additive decomposition with an arbitrary number of mutually noncommuting operator terms. Additive schemes for second-order evolution equations are considered in the same way. Bibliography: 9 titles.

1. Introduction

In finding an approximate solution of initial-condition boundary-value problems, a great deal of attention is paid to constructing additive schemes [1, 2]. A transition to a chain of simpler problems allows one to construct more economical difference schemes, separated with respect to the spatial variable. In some cases, it is useful to separate subproblems that have a different nature, according to the physical nature of the process. Recently (see, for example, [3, 4]), there has been an active discussion about regional–additive schemes (the region-decomposition schemes) that are aimed at the construction of computational algorithms for parallel computers.

Here we construct additive difference schemes for first-order differential–operator equations. The exposition is based on the general theory of stability for operator–difference schemes in lattice Hilbert spaces [1]. Our main focus is on the case of additive decomposition with an arbitrary number of mutually noncommuting operator terms. Additive schemes for second-order evolution equations are treated in the same way. Of special interest is the general principle of constructing schemes of a given quality, the principle of difference-scheme regularization.

The theory of iteration methods for finding an approximate solution of problems of mathematical physics is interpreted as a part of the general theory of stability of operator–difference schemes [1]. Using additive schemes, one constructs new classes of iteration methods aimed at applications in parallel computers. In [5], a sufficiently general class of cluster aggregating iteration methods is considered. This class includes, in particular, multiplicative (scalar, synchronous) and additive (parallel, asynchronous) versions of the Schwartz iteration method. We show that there is a close connection between the considered class of cluster aggregating iteration methods and componentwise decomposition schemes.

It became possible to consider new versions of multicomponent decomposition iteration methods with a sequential and parallel organization of the calculation process. For a multicomponent decomposition without the condition that the operators pairwise commute, it is possible to get a speed increase along the adjoint gradients.

2. Additive schemes

We consider the Cauchy problem for a first-order evolution equation in a lattice real Hilbert space H obtained after the discretization of the space, when finding a solution of the initial-condition boundary-value problem for a parabolic equation. We look for a function $y(t) \in H$ satisfying the equation

$$\frac{dy}{dt} + \Lambda y = f(t), \qquad 0 < t \le T, \tag{1}$$

and the initial condition

$$y(0) = u_0. \tag{2}$$

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We assume that, for the operator $\Lambda > 0$, there exists the following additive representation:

$$\Lambda = \sum_{\alpha-1}^{p} \Lambda_\alpha, \qquad \Lambda_\alpha \geq 0, \qquad \alpha = 1(1)p. \tag{3}$$

Additive difference schemes are constructed on the basis of representation (3). Here, the transition from one time layer t^n to another one, $t^{n+1} = t^n + \tau$, is related to solving the problems for separate operators Λ_α, $\alpha = 1(1)p$, in the additive decomposition (3), that is, the problem is decomposed into p subproblems.

Examples of additive schemes are given by classical componentwise decomposition schemes (locally one-dimensional schemes) [1, 2]. Keeping in mind applications to modern parallel computers, special consideration should be paid to additive-averaging componentwise decompositions [2, 6]. Such schemes were constructed not only for the first-order evolution equation but for second-order equations, too [7]. Vector additive difference schemes are used [8, 9] for a broad class of nonstationary problems. One can also note that there are new possibilities for obtaining unconditionally stable factored schemes.

3. Integral approximation schemes

Additive difference schemes for problems with a decomposition into three or more mutually noncommuting operators are traditionally constructed using the notion of integral approximation, a componentwise decomposition scheme (locally one-dimensional schemes) [1, 2]. For problem (1)–(3), we have

$$\frac{y^{n+\alpha/p} - y^{n+(\alpha-1)/p}}{\tau} + \Lambda_\alpha \left(\sigma y^{n+\alpha/p} + (1-\sigma) y^{n+(\alpha-1)/p}\right) = f_\alpha^n, \tag{4}$$

$$\alpha = 1(1)p, \qquad n = 0, 1, \ldots,$$

where

$$f^n = \sum_{\alpha=1}^{n} f_\alpha^n.$$

It is well known that the componentwise decomposition scheme (4) is unconditionally stable for $\sigma \geq \frac{1}{2}$.

Keeping in mind a realization on modern parallel computers, we should mention the additive-averaging schemes [2, 6, 7]:

$$\frac{y_\alpha^{n+1} - y^n}{\tau} + \Lambda_\alpha(\sigma y_\alpha^{n+1} + (1-\sigma)y^n) = f_\alpha^n, \tag{5}$$

$$\alpha = 1(1)p, \qquad n = 0, 1, \ldots,$$

$$y^{n+1} = \frac{1}{p}\sum_{\alpha=1}^{p} y_\alpha^{n+1}.$$

Stability conditions for such schemes are the same as in the standard componentwise decomposition schemes (4). A fundamental advantage of the additive-averaging schemes (5) is connected with the fact that the grid functions y_α^{n+1}, $\alpha = 1(1)p$, admit a parallel organization of calculation.

4. Factored schemes

For a two-component decomposition ($p = 2$ in representation (3)), different versions of factored schemes (variable direction schemes, alternately–triangular schemes) are used [1]. Let us briefly discuss the possibility of constructing multicomponent factored schemes.

The difference scheme is written in a canonical form,

$$B\frac{y^{n+1} - y^n}{\tau} + Ay^n = f^n, \tag{6}$$

with $A = \Lambda$. For a multicomponent decomposition, the standard form of the factored scheme becomes

$$B = \prod_{\alpha=1}^{p} (E + \sigma\tau\Lambda_\alpha).$$

For a general case of mutually noncommuting nonnegative operators Λ_α, $\alpha = 1(1)p$, we cannot guarantee that the operator B is self-adjoint and positive.

One can use the following double multiplicative representation of the operator B:

$$B = \prod_{\alpha=1}^{p} (E + \sigma\tau\Lambda_\alpha) \prod_{\beta=p}^{1} (E + \sigma\tau\Lambda_\beta^*), \qquad B = B^* > 0. \tag{7}$$

Using results of the general theory of stability, one easily obtains that the factored scheme (6), (7) is unconditionally stable if $\sigma \geq \frac{1}{4}$.

5. Vector additive schemes

According to [8, 9], define a vector $Y = (y_1, y_2, \ldots, y_p)$ such that each of its components is found by solving the system

$$\frac{dy_\alpha}{dt} + \sum_{\beta=1}^{p} \Lambda_\beta y_\beta = f(t), \tag{8}$$

$$y_\alpha(0) = u_0, \qquad \alpha = 1(1)p.$$

An arbitrary component of the vector $Y(t)$ can be chosen as a solution of the initial problem (1), (2).

Then one can construct various schemes for system (8). Let us give a representative example of a vector additive scheme. The scheme of complete approximation,

$$(E + \sigma\tau\Lambda_\alpha)\frac{y_\alpha^{n+1} - y_\alpha^n}{\tau} + \sum_{\beta=1}^{p} \Lambda_\beta y_\beta^n = f^n,$$

$\alpha = 1(1)p$, $n = 0, 1, \ldots$, is unconditionally stable if $\sigma \geq p/2$. To approximate system (8), one constructs schemes of higher accuracy, additive schemes for second-order evolution equations, etc.

6. Regularized additive schemes

To construct additive schemes, one can use a general constructive principle for regularizing difference schemes [1]. We give a brief illustration using the example of problem (1)–(3). As a primary scheme, we take the simplest explicit scheme:

$$\frac{y^{n+1} - y^n}{\tau} + \sum_{\alpha=1}^{p} \Lambda_\alpha y^n = f^n.$$

It can be written in canonical form (6) with the operators

$$B = E, \qquad A = \sum_{\alpha=1}^{p} \Lambda_\alpha.$$

The additive schemes are constructed by perturbing every operator term in the additive representation (3):

$$B = E, \qquad A = \sum_{\alpha=1}^{p} (E + \sigma\tau\Lambda_\alpha)^{-1}\Lambda_\alpha. \tag{9}$$

Scheme (6),(9) is unconditionally stable for $\sigma \geq p/2$.

One can take the parallel realization of scheme (6), (9):

$$\frac{y_\alpha^{n+1} - y^n}{\tau} + (E + \sigma\tau\Lambda_\alpha)^{-1}\Lambda_\alpha y^n = f_\alpha^n,$$

$$\alpha = 1(1)p, \qquad n = 0, 1, \dots,$$

$$y^{n+1} = \frac{1}{p}\sum_{\alpha=1}^{p} y_\alpha^{n+1}.$$

This gets us back to an additive-averaging scheme that now is constructed without a use of integral approximation.

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