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A NON-LINEAR ELLIPTIC PROBLEM WITH A COMPLEX SPECTRUM OF SOLUTIONS*

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Radially-symmetric positive solutions of a non-linear elliptic equation in R^N, which arises when examining the unbounded selfsimilar solutions of a quasilinear parabolic equation with a source, are considered. It is shown that the elliptic problem has four different families of solutions, three discrete (denumerable) solutions and one continual solution. In the one-dimensional case the solutions are constructed numerically, and the branching pattern is given.

1. Introduction.

1. In this paper we examine a class of unbounded selfsimilar solutions of a quasilinear parabolic equation

$$u_t=\nabla(|\nabla u|^\sigma\nabla u)+u^\beta,\quad t>0,\quad x\in R^N,\tag{1}$$

where $\sigma>0$, $\beta>\sigma+1$ are fixed constants, and $\nabla(\cdot)=\mathrm{grad}_x(\cdot)$. Under defined conditions, we can consider (1) as the equation of heat diffusion in a continuous non-linear medium with thermal conductivity $k=|\nabla u|^\sigma\geqslant0$, which depends on the temperature gradient $u=u(t, x)$. At the same time there is a volume energy release in the medium, and the strength of the heat source $Q=u^\beta\geqslant0$ at each point of the space is determined by the value of the temperature. Many properties of the solutions of Cauchy's problem for (1) have been examined in detail at the present time. It is well-known, for example, that due to the intense energy release $(\beta>1)$ the combustion process can occur in a mode with peaking. In other words, the Cauchy problem for (1) cannot have a (time) global solution, and at some instant of time $t=T_0<+\infty$ (the peaking moment) the amplitude of the solution becomes infinitely large:

$$\sup_{x\in R^N} u(t,x)\to+\infty,\quad t\to T_0^-\tag{2}$$

(see the review in /1, 2/).

The conditions of global solvability and non-solvability as a whole of the Cauchy problem for (1) are obtained in /3/ (see also the bibliography therein), where it is shown that for any $1<\beta<\sigma+1+(\sigma+2)/N$ all the non-trivial $(u\not\equiv0)$ solutions are unbounded in the sense (2). If $\beta>\sigma+1+(\sigma+2)/N$, then as a function of the value of the initial perturbation $u_0(x)=u(0, x)\geqslant 0$ in R^N, both global solutions - determined for all $t\geqslant0$ - and unbounded solutions are possible. In addition, the conditions of the localization of unbounded solutions of Eq.(1) are obtained in /3/. The unbounded solution is termed localized /2/ if it increases to infinity as $t\to T_0^-$ in a domain of bounded dimensions. And conversely: if $u(t, x)\to\infty$ as $t\to T_0^-$ everywhere in R^N, there is no localization. It is established in /3/ that for (1) the latter occurs for all $\beta\in(1, \sigma+1)$. Using numerical calculations, and also some qualitative estimates, it was also shown in /3/ that in when $\beta\geqslant\sigma+1$ the solutions are localized. In particular, an example was given of a localized solution, represented in explicit form, when $\beta=\sigma+1$.

The range of parameters β and σ, for which the localization effect exists in Cauchy's problem, in the same for (1) as in the case of the equation $u_t=\nabla(u^\sigma\nabla u)+u^\beta$. The proof of the localization of the finite unbounded solutions of the latter for $\beta\geqslant\sigma+1$, carried out in /4/ for $N=1$, is obviously transferred to the case of Eq.(1).

2. Below we examine the "thin structure" of unbounded localized selfsimilar solutions of Cauchy's problem for (1) when $\beta>\sigma+1$ of the following form:

$$u(t,x)=(T_0-t)^{-1/(\beta-1)}\theta(\eta),\qquad \eta=\frac{x}{(T_0-t)^{[\beta-(\sigma+1)]/(\sigma+2)(\beta-1)}}.\tag{3}$$

Substitution of Eq.(3) into (1) gives the following quasilinear elliptic equation for the function $\theta(\eta)\geqslant0$:

$$\nabla_\eta(|\nabla_\eta\theta|^\sigma\nabla_\eta\theta)-\frac{\beta-(\sigma+1)}{(\sigma+2)(\beta-1)}\nabla_\eta\theta\cdot\eta-\frac{1}{\beta-1}\theta+\theta^\beta=0,\quad \eta\in R^N.\tag{4}$$

In accordance with the formulation of the initial problem we will concern ourselves with the non-trivial $(\theta\not\equiv0)$ non-negative solutions of Eq.(4), which satisfy, at infinitely distant points, the condition

$$\theta(\eta)\to0,\qquad \|\eta\|\to+\infty.\tag{5}$$

The spatial-temporal structure of the selfsimilar solution (3) gives a represenation of the character of the evolution of the thermal perturbation in the above non-linear problem. In particular, it is obvious from (3) that when $\beta>\sigma+1$ the combustion process is localized, and the effective width of the thermal structure $x_{ef}(t)$ generated is reduced in time:

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$\|x_{ef}(t)\| \sim (T_0-t)^{[\beta-(\sigma+1)]/(\sigma+2)(\beta-1)} \to 0$ as $t \to T_0^-$. Thus, the elliptic problem (4), (5) obtained enables us to separate the manifold of thermal structures peculiar to this medium, and describe in detail their spatial geometry ("architecture", in the terminology of /5/). However an analysis of the whole set of solutions of this non-linear elliptic problem in R^N in the general formulation is hardly possible, in spite of the recent successes in the theory of elliptic problems. Note, in this connection, that the basic advances in the theory are connected with variational methods (see, for example, /6–8/ and their bibliography); as far as problem (4), (5) is concerned, it does not assume an equivalent variational formulation.

We will therefore confine ourselves to an investigation of the special class of radially-symmetric solutions of problem (4), (5), which depend on one coordinate $\xi=\|\eta\|\geqslant 0$. All of them satisfy the boundary value problem for the ordinary differential equation:

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}|\theta'|^\sigma\theta')' - \frac{\beta-(\sigma+1)}{(\sigma+2)(\beta-1)}\theta'\xi - \frac{1}{\beta-1}\theta+\theta^\beta=0, \qquad \xi>0, \tag{6a}$$

$$\theta'(0)=0, \qquad \theta(+\infty)=0. \tag{6b}$$

It turns out that even in the one-dimensional case ($N=1$) the set of solutions of problem (6) is extremely complex. Roughly speaking, the spectrum of generalized solutions counsists of four families of non-monotonic non-negative functions $\theta=\theta(\xi)$, see Sect.3. At the same time three of them are discrete, whilst at least two consist of an infinite (denumerable) set of solutions. The fourth family is a "continuous" (continual) set. In Sect.3 for the case $\sigma=1$, $\beta=3$, $N=1$ a curious branching pattern is presented of the solutions of problem (6), arranged along some parameter. In addition, the functions θ of three of the families can be combined in pairs, and we obtain as a result new solutions which are radially-asymmetric when $N=1$. The corresponding examples are presented in Sect.3. The functions $\theta=\theta(\xi)$ are constructed using numerical methods.

Some preliminary investigations are made in Sect.2, enabling us to clarify the character of the oscillations of all the possible solutions of problems (6) with respect to the spatially homogeneous solution $\theta=(\beta-1)^{-1/(\beta-1)}$. On the basis of the approaches developed in /2, 5, 9, 10/, this "local" analysis gives important (even exhaustive) information on the principles of constructing solutions of problem (6), which is used in numerical calculations, and also determines the main content of theorems of solvability and the "number" of solutions.

3. It will be no exaggeration to say that the theory of modes with peaking in non-linear media (see, for example, /2–5, 9, 10/ and the review in /1, 2/) formulated a number of new non-linear elliptic problems with unique properties. This refers both to problem (4), (5), and to an equation of another type /2, 5, 9, 10/:

$$\nabla_\eta(\theta^\sigma\nabla_\eta\theta) - \frac{\beta-(\sigma+1)}{2(\beta-1)}\nabla_\eta\theta\cdot\eta - \frac{1}{\beta-1}\theta+\theta^\beta=0, \qquad \eta\in R^N, \tag{7}$$

which arises when constructing unbounded selfsimilar solutions of a parabolic equation with a form of non-linearity that is different from that in (1):

$$u_t=\nabla(u^\sigma\nabla u)+u^\beta, \qquad t>0, \quad x\in R^N. \tag{8}$$

The selfsimilar solution of Eq.(8) is sought in the form

$$u(t,x)=(T_0-t)^{-1/(\beta-1)}\theta(\eta), \qquad \eta=x(T_0-t)^{-[\beta-(\sigma+1)]/[2(\beta-1)]};$$

then $\theta(\eta)\geqslant 0$ satisfies elliptic Eq.(7). Problem (7), (5) when $\sigma>0, \beta>\sigma+1$ has a discrete and obviously finite set of positive solutions, which possess an extremely diverse spatial structure. In particular, in the one-dimensional case $M\sim(\beta-1)/[\beta-(\sigma+1)]$ different non-monotonic solutions exist /2, 9, 10/. When $N>1$ these solutions were constructed numerically /5/.

We emphasize that the presence of a "discrete" spectrum of solutions of Eqs.(7) or (4) is connected with the fact that they occur when investigating the selfsimilar solutions that develop in a mode with peaking. Here it is pertinent to make a comparison with the similar elliptic problems that appear with a selfsimilar description of the usual, non-peaking thermal processes. For example, an equation with a runoff

$$u_t=\nabla(u^\sigma\nabla u)-u^\beta, \qquad t>0, \quad x\in R^N, \quad \sigma\geqslant 0, \quad \beta>1,$$

which differs from (8) in the sign in front of the lowest term, also allows of the selfsimilar solution

$$u(t,x)=(T+t)^{-1/(\beta-1)}\theta(\eta), \qquad \eta=x(T+t)^{-[\beta-(\sigma+1)]/[2(\beta-1)]}, \tag{9}$$
$$T=\text{const}\geqslant 0.$$

It is defined for all t ($u(t,x)\to 0$ as $t\to+\infty$); the function $\theta(\eta)\geqslant 0$ is such that

$$\nabla_\eta(\theta^\sigma\nabla_\eta\theta) + \frac{\beta-(\sigma+1)}{2(\beta-1)}\nabla_\eta\theta\cdot\eta + \frac{1}{\beta-1}\theta-\theta^\beta=0, \qquad \eta\in R^N. \tag{10}$$

This equation differs from (7) in the signs for the three last terms, which sharply alters the set of solutions. A recent paper /11/ shows that the elliptic problem (10), (5) when $\sigma=0$, $\beta>1+2/N$, $N\geqslant 2$ always has an infinite-dimensional set of solutions (when $N=1$ it is two-dimensional). The radially-symmetric solutions of Eq.(10) when $\sigma=0$ and for arbitrary $\beta>1$ form one continuous branch of the solutions /12/ (a continual set). In addition, all the solutions of the form (9) are asymptotically stable /11, 12/.

Thus, "discretization" of the spectrum of solutions (or, in other words, of the eigenfunctions of the non-linear medium /2/) occurs precisely for the development of modes with peaking which confirms the unusual reinforcement of the principles of evolutionary selection of stable solutions at a highly intensive stage of the thermal process.

2. The "linearized" non-linear equation.

1. It is easy to see that (6a) (like, however, the initial elliptic equation (4)) has an obvious spatially homogeneous solution $\theta=\theta_H=(\beta-1)^{-1/(\beta-1)}$. For convenience we will transform (6a), such that is converts to $\theta=1$. This is achieved by the substitution

$$\theta\to(\beta-1)^{-1/(\beta-1)}\theta,\quad \xi\to(\beta-1)^{[\beta-(\sigma+1)]/(\sigma+2)(\beta-1)}\xi.$$

As a result we obtain the problem

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}|\theta'|^\sigma\theta')'-\frac{\beta-(\sigma+1)}{\sigma+2}\theta'\xi-\theta+\theta^\beta=0,\qquad \xi>0, \tag{11a}$$

$$\theta'(0)=0,\qquad \theta(+\infty)=0. \tag{11b}$$

As already noted in the Introduction, the set (number and form of non-monotony) of solutions of problem (11) depends on the number and spatial behaviour of the different families of solutions of the equation, obtained as a result of linearizing the initial solution with respect to the homogeneous solution $\theta=1$. Assuming $\theta(\xi)=1+z(\xi)$ and carrying out the linearization procedure in (11a) (formally assuming that $|z(\xi)|\ll 1$), we obtain another (but all the same non-linear!) equation in the function $z(\xi)$:

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}|z'|^\sigma z')'-\frac{\beta-(\sigma+1)}{\sigma+2}z'\xi+(\beta-1)z=0,\qquad \xi>0. \tag{12}$$

Naturally, $z(\xi)$ need only satisfy one boundary condition of symmetry when $\xi=0$:

$$z'(0)=0. \tag{13}$$

Problem (12), (13) in fact describes the possible oscillations of the soltuion $\theta(\xi)$ of the initial problem (11) with respect to its homogeneous solution $\theta=1$.

2. We shall now formulate the main result of this section.

Proposition. Suppose $\sigma>0,\beta>\sigma+1$ and, in addition,

$$a_*=\frac{\beta(\sigma-1)+1}{\sigma}\geqslant 0. \tag{14}$$

Then problem (12), (13) for any $N\geqslant 1$ has four different families of solutions, which are infinitely oscillating in any neighbourhood of the point $\xi=+\infty$.

($\mathscr{P}$) is a single-parametric family of solutions $\mathscr{P}_\mu=\{z=z(\xi,\mu),\mu\neq 0\}$, which satisfy the boundary conditions $z'(0,\mu)=0, z(0,\mu)=\mu$;

(Q) is a single-parametric family of solutions $Q_a=\{z=z(\xi,a),a>0\}$, such that $z(\xi,a)=0$ when $\xi\in[0,a)$, $z(\xi,a)\neq 0$ in some right-hand neighbourhood of the point $\xi=a$ and $z(a,a)=z'(a,a)=0$.

($\mathscr{R}$) is a double-parametric family of non-trivial solutions $\mathscr{R}_\nu=\{z=z(\xi,\nu),\nu=(\nu_1,\nu_2)\}$, which satisfy the conditions $z(0,\nu)=z'(0,\nu)=0$ and are of constant sign in the fairly small right-hand neighbourhood of the point $\xi=0$;

(S) is a single-parametric family of solutions $S_\lambda=\{z=z(\xi,\lambda)\}$, $z(0,\lambda)=z'(0,\lambda)=0$ and, unlike the functions from $\mathscr{R}_\nu$, are infintely oscillating in some unlimited neighbourhood $\xi=0$.

Note. These families of solutions also exist in the case $a_*<0$, when (14) does not hold. However, at the same time the proof that the solution can oscillate in the neighbourhood of infinity (the fundamentally important result) requires a fairly cumbersome analysis of the phase plane of a first-order equation, equivalent to (12).

This proposition is proved in fairly general form in /13/. We shall briefly describe the main stages of the proof. The local existence of the solutions of $\mathscr{P}_\mu$ and Q_a is proved in /13/ using theorems on the fixed points of continuous transformations. The global properties (in particular, oscillability) of the solutions of $\mathscr{P}_\mu$, Q_a, $\mathscr{R}_\nu$ and S_λ are established by reducing Eq.(12) using the substitution

$$z(\xi)=\xi^{(\sigma+2)/\sigma}\varphi(\eta),\quad \eta=\ln\xi,\qquad P=d\varphi/d\eta \tag{15}$$

to the first-order equation

$$\frac{dP}{d\varphi}=-\frac{1}{(\sigma+1)P}\left[\frac{a_*\varphi-mP}{|\varphi(\sigma+2)/\sigma+P|^\sigma}+b\varphi+cP\right], \tag{16}$$

where the following notation is introduced:

$$a_*=\frac{\beta(\sigma-1)+1}{\sigma},\qquad m=\frac{\beta-(\sigma+1)}{\sigma+2},$$

$$b=\left[N-1+\frac{2(\sigma+1)}{\sigma}\right]\frac{\sigma+2}{\sigma}>0,\qquad c=N-1+\frac{(\sigma+1)(\sigma+4)}{\sigma}>0.$$

The main result of the analysis of Eq.(16) is as follows: for any $\beta>\sigma+1$ its phase portrait contains a limiting cycle encompassing the point $(0,0)$. This conclusion means, by virtue of (15), that all the solutions of the initial problem (12), (13) are infinitely oscillating in the neighbourhood $\xi=+\infty$. The pattern of phase trajectories of Eq.(16) looks particularly simple when $a_*\geqslant 0$ (see Fig.1, where $N=1$, $\sigma=1$, $\beta=3$). It is obtained during the numerical solution of different Cauchy problems for Eq.(12) with a simultaneous treatment in accordance with the transformation (15). Fig.1 illustrates the interior of the limiting point S.

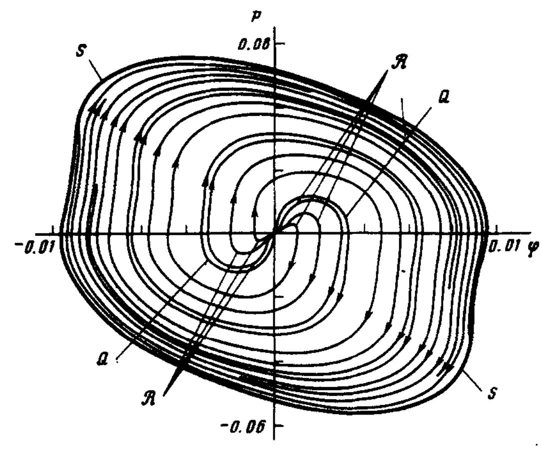

Fig.1

The letters Q, $\mathscr{R}$ and S denote the trajectories which, as a result of the inverse transformation to (15), generate the families of solutions Q_a, $\mathscr{R}_\nu$, S_λ. The trajectories Q and S are unique; therefore they generate single-parametric families of solutions. At the same time the trajectory S (the limiting cycle) determines the solutions $z(\xi)$ from S_λ, which are infinitely oscillating in both the neighbourhood of $\xi=+\infty$ and in the neighbourhood of the point $\xi=0$. The trajectories $\mathscr{R}$ form a bunch, and therefore the transformation, inverse to (15), determines the double-parametric family of solutions $\mathscr{R}_\nu$, $\nu=(\nu_1, \nu_2)$. The trajectory $\mathscr{P}$, which determines the family $\mathscr{P}_\mu$ is "wound" on to the limiting cycle S from the outside, and is not shown in Fig.1.

We shall now present the asymptotic form of the solutions $z(\xi)$ from $\mathscr{P}_\mu$, Q_a, $\mathscr{R}_\nu$ which determine the "local" dependence of the function $z(\xi)$ on the corresponding parameters:

$$(\mathscr{P}) \qquad z(\xi)=\mu\left(1-\frac{(\sigma+1)}{(\sigma+2)}\left(\frac{\beta-1}{N}\right)^{1/(\sigma+1)}\xi^{(\sigma+2)/(\sigma+1)}|\mu|^{-\sigma/(\sigma+1)}\right)+ o(\xi^{(\sigma+2)/(\sigma+1)}), \quad \xi\to 0,$$

$$(Q) \qquad z(\xi)=m^{1/\sigma}\left(\frac{\sigma+1}{\sigma}\right)^{-(\sigma+1)/\sigma}a^{1/\sigma}(\xi-a)^{(\sigma+1)/\sigma}+o((\xi-a)^{(\sigma+1)/\sigma}), \quad \xi\to a^+.$$

As far as the solutions from $\mathscr{R}_\nu$ are concerned, when $a^*>0$ (see /13/)

$$(\mathscr{R}) \qquad z(\xi)=\xi^{(\sigma+2)/\sigma}\left\{[\nu_1\xi^{a_*/m}+\ldots]+\left[\nu_2\xi^{\alpha}\exp\left(-\frac{\alpha_1^2}{\sigma}|\nu_1|^{-\sigma}\xi^{-\alpha}\right)+\ldots\right]\right\},$$

where $\alpha_1^2=m^2\{a_*(\sigma+1)[(\sigma+2)/\sigma+a_*/m]^\sigma\}^{-1}$, $\alpha=a_*\sigma/m$. Here the rational part of the asymptotic form (the first square brackets) and the essentially non-analytical part (the second square brackets) are naturally separated. It is impossible to write out the asymptotic form of the solutions from S_λ, each of which can be said to be an "essentially non-linear" solution (we recall that they are all produced by one trajectory of Eq.(16), which forms the limiting cycle). However, they indeed form a single-parametric family since problem (12), (13) is invariant with respect to the transformation

$$\xi\to\alpha\xi, \qquad z\to\pm\alpha^{(\sigma+2)/\sigma}z(\xi), \qquad \alpha=\mathrm{const}>0$$

(strictly, this fact determines the form of the transformation (15) which reduces the order of Eq.(12)).

3. Thus, two principal facts have been established:

1) the families of solutions $\mathscr{P}_\mu$, Q_a, $\mathscr{R}_\nu$ and S_λ of the "linearized" problem are continuous with respect to the corresponding parameters;

2) all the solutions infinitely oscillate in the neighbourhood of $\xi=+\infty$.

Then the representations developed in /2, 5, 9, 10/ enable us to draw the following conclusions. Each of the single-parametric families $\mathscr{P}_\mu$, Q_a, S_λ must, in principle, produce a discrete (denumerable) set of solutions of the initial problem (11) with the same character of non-monotony in the neighbourhood of $\theta=1$. These sets of solutions can be conveniently denoted by $\mathscr{P}=\{\mathscr{P}_i\}$, $Q=\{Q_i\}$ and $S=\{S_i\}$ respectively. As far as the family $\mathscr{R}_\nu$ is concerned, here owing to the appearance of a new parameter ($\mathscr{R}_\nu$, unlike the previous ones, is double-parametric) the set of corresponding solutions $\theta(\xi)$ must be not discrete, but continual. Henceforth it is denoted by $\mathscr{R}=\{\mathscr{R}_i^\nu\}$. Here the index $\nu=\{\nu_i\}$ underlines the continuousness of the spectrum of these solutions. The need for a discrete lower index i is explained in

Sect.3. The degree of authenticity of these qualitative conclusions is verified numerically in Sect.3.

3. Non-linear eigenfunctions (numerical results).

For Eq.(11a) we can obtain, using a formal expansion, the following representation of the solutions in the neighbourhood $\xi=+\infty$:

$$\theta(\xi)=C\xi^{-(\sigma+2)/[\beta-(\sigma+1)]}+o(\xi^{-(\sigma+2)/[\beta-(\sigma+1)]}),\quad \xi\to\infty, \tag{17}$$

where $C>0$ is an arbitrary constant. At the same time in the case $\beta>\sigma+1$ we have $\theta(\xi)\to 0$ as $\xi\to\infty$, such that condition (11b) holds.

Below the solutions of problem (11) with the asymptotic behaviour (17) in the one-dimensional case ($N=1$) when $\sigma=1$, $\beta=3$ are constructed numerically.

The qualitative conclusions of Sect.2 are basically confirmed. Indeed there exist four families of solutions of problem (11), which have topologically diverse local behaviour in the neighbourhood of the homothermal solution $\theta=1$. Each solution from these families has its now constant C from (17), and henceforth we call these solutions eigenfunctions. The parameter C from (17) (it enables us to arrange all the sets of solutions), which corresponds to the functions from the families $\{\mathcal{P}_i\}$, $\{Q_i\}$ and $\{\mathcal{R}_i^\nu\}$, will be denoted by $C_{\mathcal{P}}^i$, C_Q^i and $C_{\mathcal{R}}^{i,\nu}$ respectively. We will discuss the family S below.

1. Eigenfunctions from the family $\mathcal{P}$. The eigenfunctions $\{\mathcal{P}_i\}$ have the values $\theta(0)=\theta_0^{\mathcal{P}_i}\neq 1$. Generally these solutions have a non-monotonic profile. The i index here and henceforth equals the number of extrema of the eigenfunctions. We are able to construct numerically several of the first eigenfunctions from $\mathcal{P}$ (see Fig.2). The parameter $C_{\mathcal{P}}^i$ increases as i increases: $C_{\mathcal{P}}^{i+1}>C_{\mathcal{P}}^i, C_{\mathcal{P}}^i$ is a finite non-vanishing quantity. It is obvious that, in the eigenfunctions which have extrema, a "buildup" of oscillations occurs alongside the homothermal solution $\theta=1$, as in the linearized problem, and the inequality $|\theta_0^{\mathcal{P}_{i+1}}-1|<|\theta_0^{\mathcal{P}_i}-1|$ holds. Bunching of the oscillations occurs in the leading eigenfunctions (for large values of i) as $\xi\to 0$.

2. Eigenfunctions from the family Q. The solutions $\{Q_i\}$ of this family satisfy the condition $\theta(\xi)=1$ when $\xi\in[0,\xi_i]$ and $\theta(\xi)\neq 1$ when $\xi>\xi_i$. The parameters ξ_i and $\theta_0^{\mathcal{P}_i}$ from the sets Q and $\mathcal{P}$ respectively can be called the eigenvalues of problem (11). The pairs (ξ_i, C_Q^i) are arranged in the following way: $\xi_{i+1}<\xi_i, C_Q^{i+1}<C_Q^i$ (ξ_i and C_Q^i are finite quantities). The form of the non-monotony of the behaviour of the eigenfunctions Q_i is the same as for $\mathcal{P}_i$.

3. Eigenfunctions from the family $\mathcal{R}$. The solutions from $\mathcal{R}=\{\mathcal{R}_i^\nu\}$ satisfy the condition $\theta(0)=1$ and in the fairly small neighbourhood $\xi=0$ the solutions $\theta(\xi)$ are monotonic. The non-monotonic behaviour of the solutions is the same as for the eigenfunctions $\mathcal{P}_i$, Q_i.

However, unlike the first two families of spectra the eigenfunction $\{\mathcal{R}_i^\nu\}$ has a continuous form. If they are expressed more exactly, then $\mathcal{R}=\{\mathcal{R}_i^\nu\}$ consists of a denumerable (infinite) set of continuous sets of different solutions $\theta(\xi)$. As the calculations show, when $i\geq 0$ any $C\in(C_Q^{i+1},C_Q^i)$ produces an eigenfunction from $\mathcal{R}_{i+1}^\nu$ (here we formally assume $C_Q^0=\infty$, such that the set $\mathcal{R}_0^\nu$ consists of one spatially-homogeneous solution $\theta=1$). The behaviour of these eigenfunctions in the domain of the monotonic approach to the homothermal solution as $\xi\to 0$ is characterized by the constant ν_i from the expansion ($\mathcal{R}$) (see Sect.2). The pair of constants $(\nu_i^i, C_{\mathcal{R}}^{i,\nu})$ corresponds to any eigenfunction from $\mathcal{R}_i^\nu$. The graph of the relation between ν_1^{i+1} and $C_{\mathcal{R}}^{i+1,\nu}\in(C_Q^{i+1},C_Q^i)$ has zero values ν_1^{i+1} at the boundary of the domain of variation of $C_{\mathcal{R}}^{i+1,\nu}$ and one finite extremum. Hence it is clear that the eigenfunctions Q_i are "limit" functions, in a specific sense, from the continuous family $\mathcal{R}_i^\nu$, i.e. the constants $\nu_i^i=0$ correspond to them.

Figs.3 and 4 illustrate the comparative behaviour of the functions Q_i, $\mathcal{R}_i^\nu$ respectively when $i=1,2$ in the domain of values which are "close" to the homothermal solution. For large ξ all these solutions monotonically approach zero as $\xi\to+\infty$ (in the same way as in Fig.2).

4. Branching pattern of the solutions; the eigenfunction is from the family S. Fig.5 illustrates the relation between the number of extrema M of the eigenfunctions and the values of the constant C from (17). It is obvious that as $i\to\infty$ the constants $C_{\mathcal{P}}^i$ converge from below, and $C_Q^i, C_{\mathcal{R}}^{i,\nu}$ converge from above to some value C_s. Bearing in mind that $\theta_0^{\mathcal{P}_i}\to 1$, $\xi_i\to 0$, $i\to\infty$, and also the results of Sect.2, we can conclude that C_s corresponds to some eigenfunction of the S family. This eigenfunction satisfies the condition $\theta(0)=1$ and oscillates infinitely in any small neighbourhood $\xi=0$ i.e. $M=+\infty$ corresponds to it. The S-type solution is apparently natural.

5. Non-symmetric eigenfunctions. It is obvious that when $N=1$ we can connect the eigenfunctions Q_i, $\mathcal{R}_i^\nu$, S smoothly together when $\xi=0$ and obtain, at the same time, non-symmetric eigenfunctions which are defined on the whole axis $\xi\in(-\infty,+\infty)$. Examples of these solutions are given in Fig.6. They are all generalized solutions of Eq.(11a) in R^1. From the form of the asymptotic behaviour $\theta(\xi)$ as $\xi\to 0$ it is easy to derive the conditions for which $\theta(\xi)$ in Fig.6 are classical solutions, $\theta\in C^2(R^1)$.

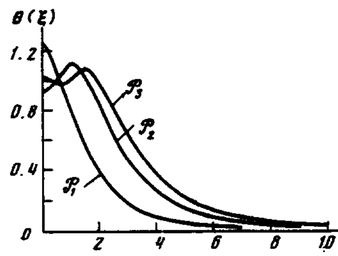

Fig.2 $C_{\mathcal{P}}^{1}=5.77$, $C_{\mathcal{P}}^{2}=18.82$, $C_{\mathcal{P}}^{3}=28.65$

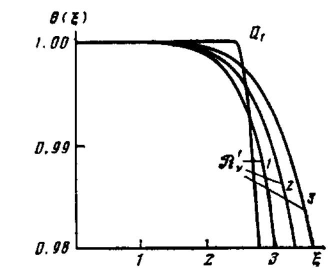

Fig.3 $C_{Q}^{1}=73.56$, $C_1=100$, $C_2=150$, $C_3=200$

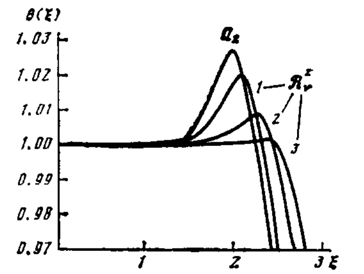

Fig.4 $C_{Q}^{2}=45.8$, $C_1=50$, $C_2=60$, $C_3=70$

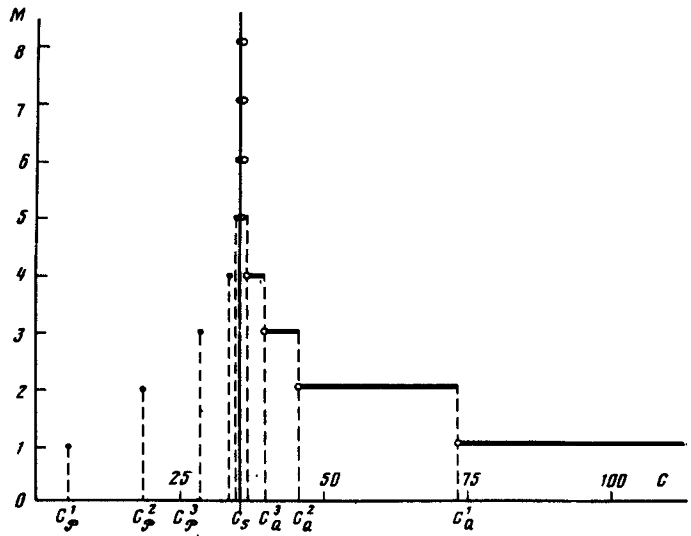

Fig.5

$C_S=35.6$, • are the solutions $\mathcal{P}_i$, ∘ are the solutions Q_i, — are the solutions $\mathcal{R}_i^{v}$, $i=1, 2, \ldots$

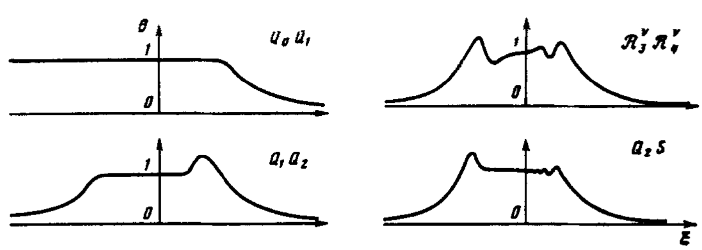

Fig.6

REFERENCES

1. GALAKTIONOV V.A., KURDYUMOV S.P. and SAMARSKII A.A., A parabolic set of quasilinear equations. I. Differents. ur-niya, 19, 12, 2123-2140, 1983.
2. KURDYUMOV S.P., Eigenfunctions of the combustion of a non-linear medium and the constructive laws for constructing its organization. In: Sovrem. probl. matem. fiz. i vychisl. matem. Moscow: Nauka, 217-243, 1982.
3. GALAKTIONOV V.A., Conditions for non-existence as a whole and localization of the solutions of Cauchy's problem for a class of non-linear parabolic equations. Zh. vychisl. mat. mat. fiz., 23, 6, 1341-1354, 1983.
4. GALAKTIONOV V.A., Proof of the localization of unbounded solutions of the non-linear parabolic equation $u_t=(u^{\sigma}u_x)_x+u^{\beta}$. Differents. ur-niya, 21, 1, 15-23, 1985.
5. KURDYUMOV S.P. et al., The architecture of multidimensional thermal structures. Dokl. AN SSSR, 274, 5, 1071-1075, 1984.
6. POKHOZHAEV S.I., An approach to non-linear equations. Dokl. AN SSSR, 247, 6, 1327-1331, 1979.
7. LIONS P.L., On the existence of positive solutions of semilinear elliptic equations. SIAM Rev., 24, 4, 441-467, 1982.
8. BERESTYCKI H. and LIONS P.-L., Non-linear scalar field equations. Arch. Ration. Mech. and Analys. 82, 4, 313-375, 1983

9. SAMARSKII A.A., et al., The combustion of a non-linear medium in the form of complex structures. Dokl. AN SSSR, 237, 6, 1330-1333, 1977.
10. AD'YUTOV M.M., KLOKOV YU.A. and MIKHAILOV A.P., Selfsimilar thermal structures with a reducing halfwidth. Differents. ur-niya, 19, 7, 1107-1114, 1983.
11. KAMIN S. and PELETIER L.A., Large time behaviour of solutions of the heat equation with absorption. Preprint Math. Inst. Univ. Leiden, 25, 1983.
12. GALAKTIONOV V.A., KURDYUMOV S.P. and SAMARSKII A.A., The asymptotic "eigenfunctions" of Cauchy's problem for a non-linear parabolic equation. Matem sb., 126, 4, 435-472, 1985.
13. GALAKTIONOV V.A. and POSASHKOV S.A., A "linearized" non-linear elliptic problem which arises in the theory of modes with peaking. Preprint IPMatem. AN SSSR. Moscow, 10, 1985.

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ITERATIONAL METHODS OF SOLVING EQUATIONS OF GAS DYNAMICS*

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Iterational methods of solving non-linear sets of difference equations, which are written in a divergent and non-divergent form and approximate the equations of gas dynamics for a heat-conducting gas, are described. For divergent equations, bearing in mind the physical viscosity, all the methods are applicable in the case of plane geometry, and for non-divergent equations some of them can be used after obvious modification in the case of any uniform geometry. The effectiveness of these methods is illustrated by the example of the solution of model problems.

1. Introduction.

When numerically solving the equations of gas dynamics implicit methods, as a rule, possess absolute stability and enable us to use a large step, with respect to the time variable, in the calculations. In implicit schemes the time step is determined according to the accuracy of the solution obtained and the condition of convergence of the iteration processes used to solve non-linear sets of difference equations. Greater effectiveness of implicit by comparison with explicit methods is achieved, on the one hand, by using conservative or completely conservative schemes /1/ enabling us to obtain the required solution with the required accuracy in coarse nets which must inevitably be used in practical calculations and, on the other, by using economical algorithms and iteration methods. This paper covers the latter problem. At the same time great attention is paid to constructing iteration methods of solving non-linear sets of difference equations which approximate equations of gas dynamics in divergent form (conservative schemes).

We will write out this system for general one-dimensional motion in Lagrange's mass variables:

$$\frac{\partial u}{\partial t}+R^{\nu}\frac{\partial}{\partial q}(p+\Pi+\omega)+1.5\nu uR^{\nu-1}\frac{\partial\mu}{\partial q}=0, \tag{1.1a}$$

$$\frac{\partial R}{\partial t}=u, \tag{1.1b}$$

$$\frac{\partial V}{\partial t}=\frac{\partial}{\partial q}(R^{\nu}u), \tag{1.1c}$$

$$\frac{\partial}{\partial t}\left(\varepsilon+\frac{u^2}{2}\right)+\frac{\partial}{\partial q}[R^{\nu}u(p+\Pi+\omega)]-\frac{\partial}{\partial q}\left(R^{2\nu}\varkappa\frac{\partial T}{\partial q}\right)+ \tag{1.1d}$$

$$1.5\nu\frac{\partial}{\partial q}(\mu u^2R^{\nu-1})=0,$$

$$\Pi=-\frac{\mu}{V}\frac{\partial}{\partial q}(R^{\nu}u),\qquad \mu=\mu_0T^{2.5}.$$

Here we use the following notation: u is the mass velocity, R is the Euler radius of the Lagrangian coordinate, V is the specific volume, T is the temperature, q is the Lagrangian mass variable; $\nu=0,1,2$ for plane, cylindrical and spherical geometry, respectively; ω is the Neumann - Richtmeyer mathematical viscosity, and μ_0 is a constant. The equations of state and the thermal conductivity are taken in the form

$$\varepsilon=\varepsilon(T,V),\qquad p=p(T,V),\qquad \varkappa=\varkappa(T,V). \tag{1.2}$$

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