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A VARIATIONAL PRINCIPLE FOR OBTAINING MAGNETOHYDRODYNAMIC EQUATIONS IN MIXED EULERIAN–LAGRANGIAN VARIABLES*

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A VARIATIONAL principle is formulated for the dynamic equations of adiabatic magnetohydrodynamics in mixed Eulerian–Lagrangian variables. The variational formulation of two-dimensional equations in the cases of plane and axial symmetry is considered in detail.

Introduction

One of the most general and fundamental approaches to the investigation of problems of theoretical and mathematical physics is based on the use of variational principles. Classical examples are the law of the minimality of energy for steady mechanical systems and the law of least action for dynamic systems with a finite number of degrees of freedom. It is possible to obtain from the corresponding variational principle the equations of electrodynamics, variational formulations are known for dissipative processes and variational principles exist describing the hydrodynamic motion of a continuous medium [1].

Variational principles have been long and successfully used to obtain efficient computational algorithms for analyzing a wide range of applied problems. Thus, in [2, 3] a variational principle similar to Hamilton's principle of least action was used as the basis for constructing discrete models of a continuous medium.

The variational approach to the construction of discrete models is based on the approximation of the action functional and of a number of conditions having the nature of constraints [3], by difference expressions involving the spatial variables. The latter use of the variational formalism leads to differential–difference equations (differential in time and difference is space), approximating the dynamic equations in Lagrangian variables. For sufficiently general requirements on the method of approximating the action functional the resulting differential–difference equations possess the property of conservativism. Moreover, the approximations of the difference equations, forming a complete system of equations of the mechanics of a continuous medium, are in a certain sense self-adjoint [2], [3]. Self-adjoint discrete models reproduce well the fine features of the flows modelled even on coarse computing meshes.

However, since the computational algorithms obtained on the basis of the variational approach of [1] are Lagrangian, they have a limited sphere of application. Thus, it is difficult to discuss flows with strong deformations on this basis.

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Flows with strong deformations are usually computed by using Eulerian [4], [5] or mixed Eulerian–Lagrangian [6] variables. As a rule discrete models in these variables are constructed on the basis of the integro-interpolational approach [7]. However, it is unfortunate that in this approach it is not always possible to preserve a number of important properties possessed by the variational models in Lagrangian variables, for example, the property of complete conservativeness.

Attempts to obtain discrete models of a continuous medium in mixed Eulerian–Lagrangian variables on the basis of the variational approach have not led to the desired result, mainly because of the absence of a satisfactory form of the corresponding variational principle [1].

Such a form of the variational principle if presented in this paper.

1. A variational formulation of the two-dimensional magnetohydrodynamic equations in mixed Eulerian–Lagrangian variables

1. Let x, y be Eulerian variables, α, β Lagrangian variables, u and v be the components of the velocity vector of the medium, and t the time. We will denote by ξ, μ the Eulerian–Lagrangian, or "base", coordinates. Since the variables x, y and α, β are connected by a one-to-one correspondence $x = x(\alpha, \beta, t)$, $y = y(\alpha, \beta, t)$, which below will be assumed to be sufficiently smooth, we can write

$$
\begin{aligned}
dx &= \frac{\partial x}{\partial \alpha} d\alpha + \frac{\partial x}{\partial \beta} d\beta + \frac{\partial x}{\partial t} dt, \\
dy &= \frac{\partial y}{\partial \alpha} d\alpha + \frac{\partial y}{\partial \beta} d\beta + \frac{\partial y}{\partial t} dt.
\end{aligned}
\tag{1.1}
$$

We also assume that (x, y) and (α, β) are one-to-one functions of the base variables (ξ, μ), so that the following relations hold:

$$
\begin{aligned}
dx &= \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \mu} d\mu + \left(\frac{\partial x}{\partial t}\right)_{\xi,\mu} dt, \\
dy &= \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \mu} d\mu + \left(\frac{\partial y}{\partial t}\right)_{\xi,\mu} dt, \\
d\alpha &= \frac{\partial \alpha}{\partial \xi} d\xi + \frac{\partial \alpha}{\partial \mu} d\mu + \left(\frac{\partial \alpha}{\partial t}\right)_{\xi,\mu} dt, \\
d\beta &= \frac{\partial \beta}{\partial \xi} d\xi + \frac{\partial \beta}{\partial \mu} d\mu + \left(\frac{\partial \beta}{\partial t}\right)_{\xi,\mu} dt.
\end{aligned}
\tag{1.2}
$$

If we substitute (1.2) into (1.1) and require that $d\xi = d\mu = 0$, we obtain

$$
\begin{aligned}
\dot{x} &= \frac{\partial x}{\partial \alpha} \dot{\alpha} + \frac{\partial x}{\partial \beta} \dot{\beta} + u, \\
\dot{y} &= \frac{\partial y}{\partial \alpha} \dot{\alpha} + \frac{\partial y}{\partial \beta} \dot{\beta} + v,
\end{aligned}
\tag{1.3}
$$

where $\dot{x} = (\partial x/\partial t)_{\xi,\mu}$, $\dot{y} = (\partial y/\partial t)_{\xi,\mu}$, $\dot{\alpha} = (\partial \alpha/\partial t)_{\xi,\mu}$, $\dot{\beta} = (\partial \beta/\partial t)_{\xi,\mu}$ are the velocities of displacement of the base coordinate system (ξ, μ) relative to the coordinates x, y, α, β respectively.

We rewrite (1.3) in a form more symmetric in the variables (x, y) and (α, β):

$$\frac{\partial(\alpha,\beta)}{\partial(\xi,\mu)}(\dot{x}-u)=\frac{\partial(x,\beta)}{\partial(\xi,\mu)}\dot{\alpha}+\frac{\partial(\alpha,x)}{\partial(\xi,\mu)}\dot{\beta},$$

$$\frac{\partial(\alpha,\beta)}{\partial(\xi,\mu)}(\dot{y}-v)=\frac{\partial(y,\beta)}{\partial(\xi,\mu)}\dot{\alpha}+\frac{\partial(\alpha,y)}{\partial(\xi,\mu)}\dot{\beta}. \tag{1.3'}$$

Solving (1.3′) for $\dot{\alpha}$ and $\dot{\beta}$, we arrive at another form of notation, equivalent to (1.3), for the kinematic connections between the variables (x, y), (α, β) and (ξ, μ):

$$\frac{\partial(x,y)}{\partial(\xi,\mu)}\dot{\alpha}=\frac{\partial(\alpha,y)}{\partial(\xi,\mu)}(\dot{x}-u)+\frac{\partial(x,\alpha)}{\partial(\xi,\mu)}(\dot{y}-v),$$

$$\frac{\partial(x,y)}{\partial(\xi,\mu)}\dot{\beta}=\frac{\partial(\beta,y)}{\partial(\xi,\mu)}(\dot{x}-u)+\frac{\partial(x,\beta)}{\partial(\xi,\mu)}(\dot{y}-v). \tag{1.3''}$$

The following notation is used below: ρ is the density, E is the specific internal energy, P is the pressure, S is the specific entropy, $\eta = 1/\rho$ is the specific volume, T is the absolute temperature of the medium, and $\tilde{\rho} = \tilde{\rho}(\alpha)$ is the density of the medium in Lagrangian variables.

The time derivatives for fixed ξ and μ will be denoted by the symbol D/Dt.

2. In Lagrangian variables the law of conservation of mass is written in the form [1]

$$\rho x^{l-1}\frac{\partial(x,y)}{\partial(\alpha,\beta)}=\tilde{\rho}(\alpha,\beta). \tag{1.4}$$

Here l is an integral parameter equal to 1 or 2.

In the first case Eq. (1.4) holds for plane-parallel flows, in the second case it holds for axisymmetric flows, x for $l = 2$ corresponding to the radius r, and y to the z-axis in cylindrical coordinates. In the base variables ξ, μ introduced above, relation (1.4) assumes the form

$$\rho x^{l-1}\frac{\partial(x,y)}{\partial(\xi,\mu)}=\tilde{\rho}\frac{\partial(\alpha,\beta)}{\partial(\xi,\mu)}. \tag{1.4'}$$

Regarding ξ, μ as independent of time, we differentiate (1.4′) with respect to time:

$$\frac{D\rho}{Dt}\Delta+\rho\frac{D\Delta}{Dt}=\frac{D\tilde{\rho}}{Dt}\tilde{\Delta}+\tilde{\rho}\frac{D\tilde{\Delta}}{Dt}, \tag{1.4''}$$

where

$$\Delta=x^{l-1}\frac{\partial(x,y)}{\partial(\xi,\mu)},\qquad \tilde{\Delta}=\frac{\partial(\alpha,\beta)}{\partial(\xi,\mu)}. \tag{1.5}$$

We transform the right side of Eq. (1.4″):

$$\frac{D\bar\rho}{Dt}\Delta+\bar\rho\frac{D\Delta}{Dt}=\left[\frac{\partial(\bar\rho,\beta)}{\partial(\alpha,\beta)}\dot\alpha+\frac{\partial(\alpha,\bar\rho)}{\partial(\alpha,\beta)}\dot\beta\right]\frac{\partial(\alpha,\beta)}{\partial(\xi,\mu)}$$
$$+\bar\rho\left[\frac{\partial(\dot\alpha,\beta)}{\partial(\xi,\mu)}+\frac{\partial(\alpha,\dot\beta)}{\partial(\xi,\mu)}\right]=\frac{\partial(\bar\rho\dot\alpha,\beta)}{\partial(\xi,\mu)}+\frac{\partial(\alpha,\bar\rho\dot\beta)}{\partial(\xi,\mu)} \tag{1.6}$$
$$=\Delta\left(\frac{\partial\bar\rho\dot\alpha}{\partial\alpha}+\frac{\partial\bar\rho\dot\beta}{\partial\beta}\right).$$

Taking into account (1.3″) and (1.4), we write

$$\bar\rho\dot\alpha=\rho x^{l-1}\left[\frac{\partial y}{\partial\beta}(\dot x-u)-\frac{\partial x}{\partial\beta}(\dot y-v)\right],$$
$$\bar\rho\dot\beta=\rho x^{l-1}\left[\frac{\partial x}{\partial\alpha}(\dot y-v)-\frac{\partial y}{\partial\alpha}(\dot x-u)\right], \tag{1.7}$$

from which we find

$$\frac{\partial\bar\rho\dot\alpha}{\partial\alpha}+\frac{\partial\bar\rho\dot\beta}{\partial\beta}=\frac{\partial(\rho x^{l-1}(\dot x-u),y)}{\partial(\alpha,\beta)}+\frac{\partial(x,\rho x^{l-1}(\dot y-v))}{\partial(\alpha,\beta)}. \tag{1.8}$$

Substituting (1.8) in (1.6), and (1.6) in (1.4″), we arrive at the continuity equation in mixed Eulerian–Lagrangian variables:

$$x^{l-1}\frac{\partial(x,y)}{\partial(\xi,\mu)}\frac{D\rho}{Dt}+\rho\left[\frac{\partial(x^{l-1}\dot x,y)}{\partial(\xi,\mu)}+\frac{1}{l}\frac{\partial(x^{l},\dot y)}{\partial(\xi,\mu)}\right]$$
$$=\frac{\partial(\rho x^{l-1}(\dot x-u),y)}{\partial(\xi,\mu)}+\frac{\partial(x,\rho x^{l-1}(\dot y-v))}{\partial(\xi,\mu)}. \tag{1.5′}$$

Multiplying (1.5′) by $\Delta^{-1}.x^{l-1}$, we obtain finally

$$x^{l-1}\frac{D\rho}{Dt}+\rho\left(\frac{\partial x^{l-1}\dot x}{\partial x}+\frac{\partial x^{l-1}\dot y}{\partial y}\right)=\frac{\partial}{\partial x}\rho x^{l-1}(\dot x-u)+\frac{\partial}{\partial y}\rho x^{l-1}(\dot y-v). \tag{1.5″}$$

We write in mixed Eulerian–Lagrangian variables the law of variation of the specific internal energy.

Since dissipative processes are regarded as absent, we can write $S = S(\alpha, \beta)$.

Differentiating the internal energy $E+E(\eta, S)$ with respect to time and taking into account the thermodynamic identity $(\partial E/\partial\eta)_S=-P$, $(\partial E/\partial S)_\eta=T$, we obtain

$$\frac{DE}{Dt}=\frac{\partial E}{\partial\eta}\frac{\partial\eta}{\partial\rho}\frac{D\rho}{Dt}+\frac{\partial E}{\partial S}\frac{DS}{Dt}=\frac{P}{\rho^2}\frac{D\rho}{Dt}+T\left(\frac{\partial S}{\partial\alpha}\dot\alpha+\frac{\partial S}{\partial\beta}\dot\beta\right). \tag{1.9}$$

We substitute in (1.9) $D\rho/Dt$ from (1.5″) and $\dot\alpha$ and $\dot\beta$ from (1.3″), we find

$$\frac{DE}{Dt}=-\frac{P}{\rho}\frac{1}{x^{l-1}}\left[\frac{\partial}{\partial x}(x^{l-1}\dot x)+\frac{\partial}{\partial y}(x^{l-1}\dot y)\right]$$
$$+\frac{P}{\rho^2}\frac{1}{x^{l-1}}\left[\frac{\partial}{\partial x}\rho x^{l-1}(\dot x-u)+\frac{\partial}{\partial y}\rho x^{l-1}(\dot y-v)\right]$$
$$+T\left[\frac{\partial S}{\partial x}(\dot x-u)+\frac{\partial S}{\partial y}(\dot y-v)\right].$$

3. Let the magnetic field be represented by the components H_x and H_y. The condition of freezing in of the magnetic field is written in the form [3]

$$x^{l-1}\left[H_x\frac{\partial y}{\partial\beta}-H_y\frac{\partial x}{\partial\beta}\right]=H_\alpha,\qquad x^{l-1}\left[H_y\frac{\partial x}{\partial\alpha}-H_x\frac{\partial y}{\partial\alpha}\right]=H_\beta, \tag{1.10}$$

where $H_\alpha = H_\alpha(\alpha,\beta)$ and $H_\beta = H_\beta(\alpha,\beta)$ are the components of the magnetic field vector in Lagrangian variables.

Solving (1.10) for H_x and H_y and changing to the base variables ξ, μ, we obtain

$$\begin{aligned}x^{l-1}\frac{\partial(x,y)}{\partial(\xi,\mu)}H_x&=H_\alpha\frac{\partial(x,\beta)}{\partial(\xi,\mu)}+H_\beta\frac{\partial(\alpha,x)}{\partial(\xi,\mu)},\\ x^{l-1}\frac{\partial(x,y)}{\partial(\xi,\mu)}H_y&=H_\alpha\frac{\partial(y,\beta)}{\partial(\xi,\mu)}+H_\beta\frac{\partial(\alpha,y)}{\partial(\xi,\mu)}.\end{aligned} \tag{1.10'}$$

We will obtain the magnetic induction equations in mixed Eulerian–Lagrangian variables. For this we differentiate (1.10′) with respect to time, regarding ξ, μ as independent of time. Differentiating the first equation of (1.10′), we find

$$\begin{aligned}\Delta\frac{DH_x}{Dt}+H_x\frac{D\Delta}{Dt}&=\left[H_\alpha\frac{\partial(\dot x,\beta)}{\partial(\xi,\mu)}+H_\beta\frac{\partial(\alpha,\dot x)}{\partial(\xi,\mu)}\right]\\ &+\left[H_\alpha\frac{\partial(x,\dot\beta)}{\partial(\xi,\mu)}+H_\beta\frac{\partial(\dot\alpha,x)}{\partial(\xi,\mu)}+\frac{DH_\alpha}{Dt}\frac{\partial(x,\beta)}{\partial(\xi,\mu)}+\frac{DH_\beta}{Dt}\frac{\partial(\alpha,x)}{\partial(\xi,\mu)}\right].\end{aligned} \tag{1.11}$$

Using (1.10) we transform the first expression in square brackets on the right side of (1.11):

$$H_\alpha\frac{\partial(\dot x,\beta)}{\partial(\xi,\mu)}+H_\beta\frac{\partial(\alpha,\dot x)}{\partial(\xi,\mu)}=x^{l-1}\left[H_x\frac{\partial(\dot x,y)}{\partial(\xi,\mu)}+H_y\frac{\partial(x,\dot x)}{\partial(\xi,\mu)}\right]. \tag{1.12}$$

Substituting the expressions DH_α/Dt and DH_β/Dt in the second square brackets of (1.11), we obtain

$$\begin{aligned}&H_\alpha\frac{\partial(x,\dot\beta)}{\partial(\xi,\mu)}+H_\beta\frac{\partial(\dot\alpha,x)}{\partial(\xi,\mu)}+\left(\frac{\partial H_\alpha}{\partial\alpha}\dot\alpha+\frac{\partial H_\alpha}{\partial\beta}\dot\beta\right)\frac{\partial(x,\beta)}{\partial(\xi,\mu)}\\ &+\left(\frac{\partial H_\beta}{\partial\alpha}\dot\alpha+\frac{\partial H_\beta}{\partial\beta}\dot\beta\right)\frac{\partial(\alpha,x)}{\partial(\xi,\mu)}=\frac{\partial(x,H_\alpha\dot\beta)}{\partial(\xi,\mu)}+\frac{\partial(\dot\alpha H_\beta,x)}{\partial(\xi,\mu)}.\end{aligned} \tag{1.13}$$

Substituting (1.13) and (1.12) into (1.11), we find

$$\frac{DH_x}{Dt}\Delta+H_x\frac{D\Delta}{Dt}=x^{l-1}\left[H_x\frac{\partial(\dot{x},y)}{\partial(\xi,\mu)}+H_v\frac{\partial(x,\dot{x})}{\partial(\xi,\mu)}\right] \tag{1.14}$$
$$+\left\{\frac{\partial(x,H_\alpha\beta)}{\partial(\xi,\mu)}+\frac{\partial(\dot{\alpha}H_\beta,y)}{\partial(\xi,\mu)}\right\}.$$

In the same way the second equation of (1.10′) is transformed to the form

$$\frac{DH_v}{Dt}\Delta+H_v\frac{D\Delta}{Dt}=x^{l-1}\left[H_x\frac{\partial(\dot{y},y)}{\partial(\xi,\mu)}+H_v\frac{\partial(x,\dot{y})}{\partial(\xi,\mu)}\right] \tag{1.15}$$
$$+\left\{\frac{\partial(y,H_\alpha\beta)}{\partial(\xi,\mu)}+\frac{\partial(\dot{\alpha}H_\beta,y)}{\partial(\xi,\mu)}\right\}.$$

In (1.14) we transform the terms contained in the curly brackets:

$$\frac{\partial(x,H_\alpha\dot{\beta})}{\partial(\xi,\mu)}+\frac{\partial(\dot{\alpha}H_\beta,x)}{\partial(\xi,\mu)}=\frac{\partial(x,y)}{\partial(\xi,\mu)}\frac{\partial}{\partial y}(H_\alpha\dot{\beta}-\dot{\alpha}H_\beta).$$

Using (1.10) and (1.3″), we find

$$H_\alpha\dot{\beta}-\dot{\alpha}H_\beta=x^{l-1}[H_x(\dot{y}-v)-H_v(\dot{x}-u)].$$

After substituting the latter two expressions into (1.14) we obtain

$$x^{l-1}\frac{DH_x}{Dt}+H_x\left[\frac{\partial}{x}(x^{l-1}\dot{x})+\frac{\partial}{\partial y}(x^{l-1}\dot{y})\right]$$
$$=x^{l-1}\left(H_x\frac{\partial\dot{x}}{\partial y}+H_v\frac{\partial\dot{x}}{\partial y}\right)+\frac{\partial}{\partial y}x^{l-1}[H_x(\dot{y}-v)-H_v(\dot{x}-u)].$$

Similarly from (1.15) we obtain the second induction equation also in mixed Eulerian–Lagrangian variables:

$$x^{l-1}\frac{DH_v}{Dt}+H_v\left[\frac{\partial}{\partial x}(x^{l-1}\dot{x})+\frac{\partial}{\partial y}(x^{l-1}\dot{y})\right]$$
$$=x^{l-1}\left[H_x\frac{\partial\dot{y}}{\partial x}-H_v\frac{\partial\dot{y}}{\partial y}\right]+\frac{\partial}{\partial x}x^{l-1}[H_v(\dot{x}-u)-H_x(\dot{y}-v)].$$

4. We formulate the variational principle for the two-dimensional dynamic equations of magnetohydrodynamics in mixed Eulerian–Lagrangian variables.

Variational principle. The dynamic equations of magnetohydrodynamics follow from the condition that the first variation of the functional

$$\Phi=\int_{t_0}^{t_1}\left[\iint_{\Omega(\xi,\mu)}\rho x^{l-1}\frac{\partial(x,y)}{\partial(\xi,\mu)}\left(\frac{u^2+v^2}{2}-E-\frac{H_x^2+H_v^2}{8\pi\rho}\right)d\xi\,d\mu\right]dt, \tag{1.16}$$

vanishes, it being assumed that the base variables ξ, μ are not varied, and the variations of all the other quantities are interconnected by the conditions of conservation of mass (1.4′), conservation of magnetic flux (1.10′), by the kinematic relations (1.3′) and by the first law of thermodynamics:

$$dE = -P d\eta + T dS. \tag{1.16'}$$

5. We illustrate the technique for obtaining the dynamic equations of magnetohydrodynamics from the variational principle formulated above.

We write down the first variation of the functional (1.16):

$$\begin{aligned}\delta\Phi = \int_{t_0}^{t_1}\Big\{\iint_{\Omega}\Big[\,&\delta(\rho\Delta)\left(\frac{u^2+v^2}{2}-E\right)+\rho\Delta(u\delta u+v\delta v-\delta E)\\ &-\delta\Delta\frac{H_x^2+H_y^2}{8\pi}-\frac{\Delta}{4\pi}(H_x\delta H_x+H_y\delta H_y)\Big]\,d\xi\,d\mu\Big\}\,dt.\end{aligned} \tag{1.17}$$

From (1.4′) we express the first variation of the density:

$$\delta\rho = -\rho\frac{\delta\Delta}{\Delta}+\frac{\bar\Delta}{\Delta}\left(\frac{\partial\bar\rho\delta\alpha}{\partial\alpha}+\frac{\partial\bar\rho\delta\beta}{\partial\beta}\right), \tag{1.18}$$

from which we find

$$\delta(\rho\Delta) = \bar\Delta\left(\frac{\partial\bar\rho\delta\alpha}{\partial\alpha}+\frac{\partial\bar\rho\delta\beta}{\partial\beta}\right).$$

Relations (1.3′) enable the variations δu and δv to be expressed in terms of the variations δx, δy, $\delta\alpha$, $\delta\beta$:

$$\begin{aligned}\delta u &= \delta\dot x-\frac{\partial\delta x}{\partial\alpha}\dot\alpha-\frac{\partial\delta x}{\partial\beta}\dot\beta+\left(\frac{\partial\delta\alpha}{\partial\alpha}+\frac{\partial\delta\beta}{\partial\beta}\right)(\dot x-u)-\frac{\partial x}{\partial\alpha}\delta\dot\alpha\\ &\quad-\frac{\partial x}{\partial\beta}\delta\dot\beta-\frac{\partial(x,\delta\beta)}{\partial(\alpha,\beta)}\dot\alpha-\frac{\partial(\delta\alpha,x)}{\partial(\alpha,\beta)}\dot\beta,\\ \delta v &= \delta\dot y-\frac{\partial\delta y}{\partial\alpha}\dot\alpha-\frac{\partial\delta y}{\partial\beta}\dot\beta+\left(\frac{\partial\delta\alpha}{\partial\alpha}+\frac{\partial\delta\beta}{\partial\beta}\right)(\dot y-v)-\frac{\partial y}{\partial\alpha}\delta\dot\alpha\\ &\quad-\frac{\partial y}{\partial\beta}\delta\dot\beta-\frac{\partial(y,\delta\beta)}{\partial(\alpha,\beta)}\dot\alpha-\frac{\partial(\delta\alpha,y)}{\partial(\alpha,\beta)}\dot\beta.\end{aligned} \tag{1.19}$$

and the quantity δE to be determined allowing for (1.16′):

$$\delta E = \frac{\partial E}{\partial\eta}\frac{\partial\eta}{\partial\rho}\delta\rho+\frac{\partial E}{\partial S}\delta S = \frac{P}{\rho^2}\delta\rho+T\left(\frac{\partial S}{\partial\alpha}\delta\alpha+\frac{\partial S}{\partial\beta}\delta\beta\right). \tag{1.20}$$

Varying (1.10′), we find δH_x and δH_y:

$$\begin{aligned}\delta H_x &= -H_x\frac{\delta\Delta}{\Delta}+\frac{1}{\Delta}\left[H_\alpha\frac{\partial(\delta x,\beta)}{\partial(\xi,\mu)}+H_\beta\frac{\partial(\alpha,\delta x)}{\partial(\xi,\mu)}\right]\\ &\quad+\frac{1}{\Delta}\left[\frac{\partial(x,H_\alpha\delta\beta)}{\partial(\xi,\mu)}+\frac{\partial(\delta\alpha H_\beta,x)}{\partial(\xi,\mu)}\right],\end{aligned} \tag{1.21}$$

(1.21 Cont'd.)

$$\delta H_y = -H_y \frac{\delta\Delta}{\Delta} - \frac{1}{\Delta}\left[H_\alpha \frac{\partial(\delta y, \beta)}{\partial(\xi, \mu)} + H_\beta \frac{\partial(\alpha, \delta y)}{\partial(\xi, \mu)}\right] + \frac{1}{\Delta}\left[\frac{\partial(y, H_\alpha \delta\beta)}{\partial(\xi, \mu)} + \frac{\partial(H_\beta \delta\alpha, y)}{\partial(\xi, \mu)}\right].$$

Substituting (1.18)–(1.21) into (1.17), we obtain

$$\delta\Phi = \int_{t_0}^{t_1}\Bigg\{ \iint_\Omega \Bigg\{ \Bigg\{ \rho\Delta\Big[u\Big(\delta\dot{x} - \frac{\partial\delta x}{\partial\alpha}\dot{\alpha} - \frac{\partial\delta x}{\partial\beta}\dot{\beta}\Big)$$
$$+v\Big(\delta\dot{y} - \frac{\partial\delta y}{\partial\alpha}\dot{\alpha} - \frac{\partial\delta y}{\partial\beta}\dot{\beta}\Big)\Big] + \Big(P + \frac{H_x^2 + H_y^2}{8\pi}\Big)\delta\Delta$$
$$-\frac{1}{4\pi}H_x\Big[H_\alpha \frac{\partial(\delta x, \beta)}{\partial(\xi, \mu)} + H^\beta \frac{\partial(\alpha, \delta x)}{\partial(\xi, \mu)}\Big]$$
$$-\frac{1}{4\pi}H_y\Big[H_\alpha \frac{\partial(\delta y, \beta)}{\partial(\xi, \mu)} + H_\beta \frac{\partial(\alpha, \delta y)}{\partial(\xi, \mu)}\Big]\Bigg\}$$
$$+\Bigg\{\Big(\frac{u^2+v^2}{2} - E - \frac{P}{\rho}\Big)\bar{\Delta}\Big(\frac{\partial\bar{\rho}\delta\alpha}{\partial\alpha} + \frac{\partial\bar{\rho}\delta\beta}{\partial\beta}\Big) + \rho\Delta u\Big[\Big(\frac{\partial\delta\alpha}{\partial\alpha} + \frac{\partial\rho\beta}{\partial\beta}\Big)$$
$$\times(\dot{x} - u) - \frac{\partial x}{\partial\alpha}\delta\dot{\alpha} - \frac{\partial x}{\partial\beta}\delta\dot{\beta} - \frac{\partial(x, \delta\beta)}{\partial(\alpha, \beta)}\dot{\alpha} - \frac{\partial(\delta\alpha, x)}{\partial(\alpha, \beta)}\dot{\beta}\Big]$$
$$+\rho\Delta v\Big[\Big(\frac{\partial\delta\alpha}{\partial\alpha} + \frac{\partial\delta\beta}{\partial\beta}\Big)(\dot{y} - v) - \frac{\partial y}{\partial\alpha}\delta\dot{\alpha} - \frac{\partial y}{\partial\beta}\delta\dot{\beta}$$
$$-\frac{\partial(y, \delta\beta)}{\partial(\alpha, \beta)}\dot{\alpha} - \frac{\partial(\delta\alpha, y)}{\partial(\alpha, \beta)}\dot{\beta}\Big] - \rho\Delta T\Big(\frac{\partial S}{\partial\alpha}\delta\alpha + \frac{\partial S}{\partial\beta}\delta\beta\Big)$$
$$-\frac{1}{4\pi}H_x\Big[\frac{\partial(x, H_\alpha\delta\beta)}{\partial(\xi, \mu)} + \frac{\partial(\delta\alpha H_\beta, x)}{\partial(\xi, \mu)}\Big]$$
$$-\frac{1}{4\pi}H_y\Big[\frac{\partial(y, H_\alpha\delta\beta)}{\partial(\xi, \mu)} + \frac{\partial(\delta\alpha H_\beta, y)}{\partial(\xi, \mu)}\Big]\Bigg\}\Bigg\} d\xi\, d\mu\Bigg\} dt.$$

Integrating by parts in the last expression and putting $\delta x = \delta y = \delta\alpha = \delta\beta = 0$ on the boundary of the domain of integration $\Omega \times t$, we obtain

$$\delta\Phi = \int_{t_0}\Bigg\{\iint_\Omega\Bigg\{\Big[-\frac{D}{Dt}(\rho\Delta u) + \frac{\partial(\bar{\rho}u\dot{\alpha}, \beta)}{\partial(\xi, \mu)} + \frac{\partial(\alpha, \bar{\rho}u\dot{\beta})}{\partial(\xi, \mu)}$$
$$-x^{l-1}\frac{\partial(P, y)}{\partial(\xi, \mu)} - x^{l-1}\frac{1}{8\pi}\frac{\partial(H_x^2 + H_y^2, y)}{\partial(\xi, \mu)} + \frac{1}{4\pi}\frac{\partial(H_x H_\alpha, \beta)}{\partial(\xi, \mu)}$$
$$+\frac{1}{4\pi}\frac{\partial(\alpha, H_x H_\beta)}{\partial(\xi, \mu)}\Big]\delta x + \Big[-\frac{D}{Dt}(\rho\Delta v) + \frac{\partial(\bar{\rho}v\dot{\alpha}, \beta)}{\partial(\xi, \mu)} + \frac{\partial(\alpha, \bar{\rho}v\dot{\beta})}{\partial(\xi, \mu)}$$
$$-\frac{\partial(x, Px^{l-1})}{\partial(\xi, \mu)} - \frac{1}{8\pi}\frac{\partial(x, x^{l-1}(H_x^2 + H_y^2))}{\partial(\xi, \mu)} + \frac{1}{4\pi}\frac{\partial(H_\alpha H_y, \beta)}{\partial(\xi, \mu)}$$
$$+\frac{1}{4\pi}\frac{\partial(\alpha, H_y H_\beta)}{\partial(\xi, \mu)}\Big]\delta y + \Big\{\frac{D}{Dt}\rho\Delta\Big(u\frac{\partial x}{\partial\alpha} + v\frac{\partial y}{\partial\alpha}\Big)$$

$$-\bar{\rho}\frac{\partial([u^2+v^2)/2-E-P/\rho],\beta)}{\partial(\xi,\mu)}-\frac{\partial(\bar{\rho}u(\dot{x}-u),\beta)}{\partial(\xi,\mu)}$$
$$-\frac{\partial(\bar{\rho}v(\dot{y}-v),\beta)}{\partial(\xi,\mu)}+\frac{\partial(\bar{\rho}u\dot{\beta},x)}{\partial(\xi,\mu)}+\frac{\partial(\bar{\rho}v\dot{\beta},y)}{\partial(\xi,\mu)}-\rho\Delta T\frac{\partial S}{\partial\alpha}$$

$$+\frac{1}{4\pi}H_\beta\left[\frac{\partial(H_x,x)}{\partial(\xi,\mu)}+\frac{\partial(H_y,y)}{\partial(\xi,\mu)}\right]\Big\}\delta\alpha+\Big\{\frac{D}{Dt}\rho\Delta\left(u\frac{\partial x}{\partial\beta}+v\frac{\partial y}{\partial\beta}\right)$$
$$-\bar{\rho}\frac{\partial(\alpha,[(u^2+v^2)/2-E-P/\rho])}{\partial(\xi,\mu)}-\frac{\partial(\alpha,\bar{\rho}u(\dot{x}-u))}{\partial(\xi,\mu)}$$
$$-\frac{\partial(\alpha,\bar{\rho}v(\dot{y}-v))}{\partial(\xi,\mu)}+\frac{\partial(x,\bar{\rho}u\dot{\alpha})}{\partial(\xi,\mu)}+\frac{\partial(y,\bar{\rho}v\dot{\alpha})}{\partial(\xi,\mu)}-\rho\Delta T\frac{\partial S}{\partial\beta}$$
$$+\frac{1}{4\pi}H_\alpha\left[\frac{\partial(x,H_x)}{\partial(\xi,\mu)}+\frac{\partial(y,H_y)}{\partial(\xi,\mu)}\right]\Big\}\delta\beta\Big\}d\xi\,d\mu\Big\}\,dt.$$

The requirement that the first variation of $\delta\Phi$ vanishes, with the condition that the variations δx, δy, $\delta\alpha$, $\delta\beta$ are independent leads to the equations

$$\frac{D}{Dt}(\rho\Delta u)-\frac{\partial(\bar{\rho}u\dot{\alpha},\beta)}{\partial(\xi,\mu)}-\frac{\partial(\alpha,\bar{\rho}u\dot{\beta})}{\partial(\xi,\mu)}+x^{l-1}\frac{\partial(P,y)}{\partial(\xi,\mu)}$$
$$+x^{l-1}\frac{1}{8\pi}\frac{\partial(H_x^2+H_y^2,y)}{\partial(\xi,\mu)}-\frac{1}{4\pi}\frac{\partial(H_xH_\alpha,\beta)}{\partial(\xi,\mu)}-\frac{1}{4\pi}\frac{\partial(\alpha,H_xH_\beta)}{\partial(\xi,\mu)}=0, \tag{1.22}$$

$$\frac{D}{Dt}(\rho\Delta v)-\frac{\partial(\bar{\rho}v\dot{\alpha},\beta)}{\partial(\xi,\mu)}-\frac{\partial(\alpha,\bar{\rho}v\dot{\beta})}{\partial(\xi,\mu)}+\frac{\partial(x,Px^{l-1})}{\partial(\xi,\mu)}$$
$$+\frac{1}{8\pi}\frac{\partial(x,x^{l-1}(H_x^2+H_y^2))}{\partial(\xi,\mu)}-\frac{1}{4\pi}\frac{\partial(H_\alpha H_y,\beta)}{\partial(\xi,\mu)}-\frac{1}{4\mu}\frac{\partial(\alpha,H_yH_\beta)}{\partial(\xi,\mu)}=0, \tag{1.23}$$

$$\frac{D}{Dt}\rho\Delta\left(u\frac{\partial x}{\partial\alpha}+v\frac{\partial y}{\partial\alpha}\right)-\frac{\partial(\bar{\rho}[u(\dot{x}-u)+v(\dot{y}-v)],\beta)}{\partial(\xi,\mu)}$$
$$+\frac{\partial(\bar{\rho}u\dot{\beta},x)}{\partial(\xi,\mu)}+\frac{\partial(\bar{\rho}v\dot{\beta},y)}{\partial(\xi,\mu)}-\bar{\rho}\frac{\partial([(u^2+v^2)/2-E-P/\rho],\beta)}{\partial(\xi,\mu)}. \tag{1.24}$$
$$-\rho\Delta T\frac{\partial S}{\partial\alpha}+\frac{1}{4\pi}H_\beta\left[\frac{\partial(H_x,x)}{\partial(\xi,\mu)}+\frac{\partial(H_y,y)}{\partial(\xi,\mu)}\right]=0,$$

$$\frac{D}{Dt}\rho\Delta\left(u\frac{\partial x}{\partial\beta}+v\frac{\partial y}{\partial\beta}\right)-\frac{\partial(\alpha,\bar{\rho}[u(\dot{x}-u)+v(\dot{y}-v)])}{\partial(\xi,\mu)}$$
$$+\frac{\partial(x,\bar{\rho}u\dot{\alpha})}{\partial(\xi,\mu)}+\frac{\partial(y,\bar{\rho}v\dot{\alpha})}{\partial(\xi,\mu)}-\bar{\rho}\frac{\partial(\alpha,[(u^2+v^2)/2-E-P/\rho])}{\partial(\xi,\mu)} \tag{1.25}$$
$$-\rho\Delta T\frac{\partial S}{\partial\beta}+\frac{1}{4\pi}H_\alpha\left[\frac{\partial(x,H_x)}{\partial(\xi,\mu)}+\frac{\partial(y,H_y)}{\partial(\xi,\mu)}\right]=0.$$

6. We will show that any two equations from (1.22)–(1.25) are sufficient to describe the dynamics of the medium. We reduce Eqs. (1.22), (1.23) to a more usual form. We consider the second and third terms of (1.22). Substituting $\dot{\alpha}$ and from $\dot{\beta}$ from (1.3″), we find:

$$\bar{\rho}u\dot{\alpha}=A\frac{\partial y}{\partial\beta}-B\frac{\partial x}{\partial\beta},\qquad \bar{\rho}u\dot{\beta}=B\frac{\partial x}{\partial\alpha}-A\frac{\partial y}{\partial\alpha}, \tag{1.26}$$

where $A=x^{l-1}\rho u(\dot{x}-u)$, $B=x^{l-1}\rho u(\dot{y}-v)$. It is easy to show that

$$\frac{\partial(\bar{\rho}u\dot{\alpha},\beta)}{\partial(\xi,\mu)}+\frac{\partial(\alpha,\bar{\rho}u\dot{\beta})}{\partial(\xi,\mu)}=\bar{\Delta}\left(\frac{\partial\bar{\rho}u\dot{\alpha}}{\partial\alpha}+\frac{\partial\bar{\rho}u\dot{\beta}}{\partial\beta}\right)=\frac{\partial(A,y)}{\partial(\xi,\mu)}+\frac{\partial(x,B)}{\partial(\xi,\mu)}. \tag{1.27}$$

We transform the last two terms of Eq. (1.22). Taking into account (1.10), we find

$$H_xH_\alpha=x^{l-1}\left(A'\frac{\partial y}{\partial\beta}-B'\frac{\partial x}{\partial\beta}\right),\qquad H_xH_\beta=x^{l-1}\left(B'\frac{\partial x}{\partial\alpha}-A'\frac{\partial y}{\partial\alpha}\right). \tag{1.28}$$

Here $A'=H_x{}^2$, $B'=H_xH_y$. We then have

$$\begin{aligned}\frac{1}{4\pi}\left[\frac{\partial(H_\alpha H_x,\beta)}{\partial(\xi,\mu)}+\frac{\partial(\alpha,H_xH_\beta)}{\partial(\xi,\mu)}\right]=\frac{1}{4\pi}\bar{\Delta}\left[\frac{\partial}{\partial\alpha}(H_\alpha H_x)\right.\\ \left.+\frac{\partial}{\partial\beta}(H_xH_\beta)\right]=\frac{1}{4\pi}\left[\frac{\partial(x^{l-1}A',y)}{\partial(\xi,\mu)}+\frac{\partial(x,x^{l-1}B')}{\partial(\xi,\mu)}\right].\end{aligned} \tag{1.29}$$

Substituting (1.29) and (1.27) into (1.22), we obtain

$$\begin{aligned}&\frac{D}{Dt}(\rho\Delta u)-\frac{\partial(x^{l-1}\rho u(\dot{x}-u),y)}{\partial(\xi,\mu)}-\frac{\partial(x,x^{l-1}\rho u(\dot{y}-v))}{\partial(\xi,\mu)}\\&+x^{l-1}\frac{\partial(P,y)}{\partial(\xi,\mu)}+x^{l-1}\frac{1}{8\pi}\frac{\partial(H_x^2+H_y^2,y)}{\partial(\xi,\mu)}\\&-\frac{1}{4\pi}\left[\frac{\partial(x^{l-1}H_x^2,y)}{\partial(\xi,\mu)}+\frac{\partial(x,x^{l-1}H_xH_y)}{\partial(\xi,\mu)}\right]=0.\end{aligned} \tag{1.22'}$$

Performing similar operations with (1.23), we reduce it to the form

$$\begin{aligned}&\frac{D}{Dt}(\rho\Delta v)-\frac{\partial(x^{l-1}\rho v(\dot{x}-u),y)}{\partial(\xi,\mu)}-\frac{\partial(x,x^{l-1}\rho v(\dot{y}-v))}{\partial(\xi,\mu)}\\&+\frac{\partial(x,Px^{l-1})}{\partial(\xi,\mu)}+\frac{1}{8\pi}\frac{\partial(x,(H_x^2+H_y^2)x^{l-1})}{\partial(\xi,\mu)}\\&-\frac{1}{4\pi}\left[\frac{\partial(x^{l-1}H_xH_y,y)}{\partial(\xi,\mu)}+\frac{\partial(x,x^{l-1}H_y^2)}{\partial(\xi,\mu)}\right]=0.\end{aligned} \tag{1.23'}$$

Integrating (1.22'), (1.23') over some domain Ω' in the plane of the variables (ξ,μ), it is easy to verify that these equations represent the law of conservation of momentum.

Let us transform Eqs. (1.24) and (1.25). We simplify the first term in (1.25):

$$\begin{aligned}\frac{D}{Dt}\rho\Delta\left(u\frac{\partial x}{\partial\beta}+v\frac{\partial y}{\partial\beta}\right)=\frac{\partial(x,y)}{\partial(\alpha,\beta)}\left[\frac{\partial\alpha}{\partial x}\frac{D}{Dt}(\rho\Delta v)\right.\\ \left.-\frac{\partial\alpha}{\partial y}\frac{D}{Dt}(\rho\Delta u)\right]+\rho\Delta\left[u\frac{D}{Dt}\left(\frac{\partial x}{\partial\beta}\right)+v\frac{D}{Dt}\left(\frac{\partial y}{\partial\beta}\right)\right].\end{aligned} \tag{1.30}$$

Let us write down the time derivatives of $\partial x/\partial\beta$ and $\partial y/\partial\beta$:

$$\begin{aligned}\frac{D}{Dt}\left(\frac{\partial x}{\partial\beta}\right)&=\frac{\partial(\dot{\alpha},x)}{\partial(\alpha,\beta)}+\frac{\partial\dot{x}}{\partial\beta}-\frac{\partial x}{\partial\beta}\left(\frac{\partial\dot{\alpha}}{\partial\alpha}+\frac{\partial\dot{\beta}}{\partial\beta}\right),\\ \frac{D}{Dt}\left(\frac{\partial y}{\partial\beta}\right)&=\frac{\partial(\dot{\alpha},y)}{\partial(\alpha,\beta)}+\frac{\partial\dot{y}}{\partial\beta}-\frac{\partial y}{\partial\beta}\left(\frac{\partial\dot{\alpha}}{\partial\alpha}+\frac{\partial\dot{\beta}}{\partial\beta}\right)\end{aligned} \tag{1.31}$$

Taking into account (2.3″) it is easy to show that

$$\begin{aligned}\frac{\partial\dot{\alpha}}{\partial\alpha}+\frac{\partial\dot{\beta}}{\partial\beta}&=\frac{\partial(\dot{x}-u)}{\partial x}+\frac{\partial(\dot{y}-v)}{\partial y}+(\dot{x}-u)\left(\frac{\partial}{\partial\alpha}\frac{\partial\alpha}{\partial x}+\frac{\partial}{\partial\beta}\frac{\partial\beta}{\partial x}\right)\\ &+(\dot{y}-v)\left(\frac{\partial}{\partial\alpha}\frac{\partial\alpha}{\partial y}+\frac{\partial}{\partial\beta}\frac{\partial\beta}{\partial y}\right).\end{aligned} \tag{1.32}$$

Substituting (1.32) into (1.31), and (1.31) into the second term on the right side of (1.30), we obtain

$$\begin{aligned}\rho\Delta\left[u\frac{D}{Dt}\left(\frac{\partial x}{\partial\beta}\right)+v\frac{D}{Dt}\left(\frac{\partial y}{\partial\beta}\right)\right]&=\bar{\rho}\frac{\partial(x,y)}{\partial(\xi,\mu)}\left\{\left(\frac{\partial\alpha}{\partial x}v-\frac{\partial\alpha}{\partial y}u\right)\frac{\partial(\alpha,\beta)}{\partial(x,y)}\right.\\ &+\left[(\dot{x}-u)\frac{\partial}{\partial x}\frac{\partial(x,y)}{\partial(\alpha,\beta)}+(\dot{y}-v)\frac{\partial}{\partial y}\frac{\partial(x,y)}{\partial(\alpha,\beta)}\right]\\ &+\frac{\partial\alpha}{\partial x}\left(u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}\right)\\ &+\frac{\partial\alpha}{\partial y}\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right)\\ &+(\dot{x}-u)\left[v\frac{\partial^2\alpha}{\partial x^2}-u\frac{\partial^2\alpha}{\partial x\partial y}\right]\\ &\left.-(\dot{y}-v)\left[u\frac{\partial^2\alpha}{\partial y^2}-v\frac{\partial^2\alpha}{\partial x\partial y}\right]\right\}.\end{aligned} \tag{1.33}$$

We transform the second term of (1.25) to the form

$$\begin{aligned}&-\frac{\partial(x,y)}{\partial(\xi,\mu)}\left\{\frac{\partial\alpha}{\partial x}\frac{\partial}{\partial y}\bar{\rho}[u(\dot{x}-u)+v(\dot{y}-v)]-\frac{\partial\alpha}{\partial y}\frac{\partial}{\partial x}\bar{\rho}\right.\\ &\left.\times[u(\dot{x}-u)+v(\dot{y}-v)]\right\}.\end{aligned} \tag{1.34}$$

Let us consider the third and fourth term of (1.25):

$$\begin{aligned}&\frac{\partial(x,\bar{\rho}u\dot{\alpha})}{\partial(\xi,\mu)}+\frac{\partial(y,\bar{\rho}v\dot{\alpha})}{\partial(\xi,\mu)}\\ &=\frac{\partial(x,y)}{\partial(\xi,\mu)}\left\{\frac{\partial\alpha}{\partial x}\left[\frac{\partial}{\partial y}\bar{\rho}u(\dot{x}-u)-\frac{\partial}{\partial x}\bar{\rho}v(\dot{x}-u)\right]\right.\\ &+\frac{\partial\alpha}{\partial y}\left[\frac{\partial}{\partial y}\bar{\rho}u(\dot{y}-v)-\frac{\partial}{\partial x}\bar{\rho}v(\dot{y}-v)\right]\\ &+\bar{\rho}\left[\frac{\partial^2\alpha}{\partial x\,\partial y}u(\dot{x}-u)+\frac{\partial^2\alpha}{\partial y^2}u(\dot{y}-v)\right.\\ &\left.\left.-\frac{\partial^2\alpha}{\partial x^2}v(\dot{x}-u)-\frac{\partial^2\alpha}{\partial x\,\partial y}v(\dot{y}-v)\right]\right\}.\end{aligned} \tag{1.35}$$

Taking into account (1.16′), it can be shown that

$$-\delta\frac{\partial(\alpha,\,(u^2+v^2)/2-E-P/\rho)}{\partial(\xi,\mu)}-\rho\Delta T\frac{\partial S}{\partial\beta}$$
$$=-\rho\Delta\left(u\frac{\partial u}{\partial\beta}+v\frac{\partial v}{\partial\beta}\right)+\Delta\frac{\partial(x,y)}{\partial(\alpha,\beta)}\frac{\partial(\alpha,P)}{\partial(x,y)}. \tag{1.36}$$

Substituting H_α from (1.10) in the last term of (1.25), we find

$$\frac{1}{4\pi}H_\alpha\left[\frac{\partial(x,H_x)}{\partial(\xi,\mu)}+\frac{\partial(y,H_y)}{\partial(\xi,\mu)}\right]$$
$$=\frac{1}{4\pi}\Delta\frac{\partial(x,y)}{\partial(\alpha,\beta)}\left(H_x\frac{\partial\alpha}{\partial x}+H_y\frac{\partial\alpha}{\partial y}\right)\left(\frac{\partial H_x}{\partial y}-\frac{\partial H_y}{\partial x}\right). \tag{1.37}$$

Adding (1.30), taking into account (1.33), to expressions (1.34)–(1.37), after a number of simple operations we obtain

$$\frac{\partial\alpha}{\partial x}\left\{\frac{D}{Dt}(\rho\Delta v)-\frac{D(x,y)}{D(\xi,\mu)}\left[\frac{\partial\rho x^{l-1}(\dot{x}-u)}{\partial x}\right.\right.$$
$$+\left.\frac{\partial\rho x^{l-1}v(\dot{y}-v)}{\partial y}\right]+\Delta\frac{\partial P}{\partial y}+\frac{1}{8\pi}\Delta\frac{\partial(H_x^2+H_y^2)}{\partial y}$$
$$\left.-\frac{1}{4\pi}\frac{\partial(x,y)}{\partial(\xi,\mu)}\left[\frac{\partial x^{l-1}H_xH_y}{\partial x}+\frac{\partial x^{l-1}H_y^2}{\partial y}\right]\right\}$$
$$-\frac{\partial\alpha}{\partial y}\left\{\frac{D}{Dt}(\rho\Delta u)-\frac{\partial(x,y)}{\partial(\xi,\mu)}\left[\frac{\partial\rho x^{l-1}u(\dot{x}-u)}{\partial x}\right.\right.$$
$$+\left.\frac{\partial}{\partial y}\rho x^{l-1}u(\dot{y}-v)\right]+\Delta\frac{\partial P}{\partial x}+\Delta\frac{1}{8\pi}\frac{\partial(H_x^2+H_y^2)}{\partial x}$$
$$\left.-\frac{1}{4\pi}\frac{\partial(x,y)}{\partial(\xi,\mu)}\left[\frac{\partial x^{l-1}H_x^2}{\partial x}+\frac{\partial x^{l-1}H_xH_y}{\partial y}\right]\right\}=0. \tag{1.25′}$$

By similar calculations Eq. (1.24) can be reduced to the form

$$\frac{\partial\beta}{\partial y}\left\{\frac{D}{Dt}(\rho\Delta u)-\frac{\partial(x,y)}{\partial(\xi,\mu)}\left[\frac{\partial}{\partial x}\rho x^{l-1}u(x-u)\right.\right.$$
$$+\left.\frac{\partial}{\partial y}\rho x^{l-1}u(\dot{y}-v)\right]+\Delta\frac{\partial P}{\partial x}+\Delta\frac{1}{8\pi}\frac{\partial(H_x^2+H_y^2)}{\partial x}$$
$$\left.-\frac{1}{4\pi}\frac{\partial(x,y)}{\partial(\xi,\mu)}\left[\frac{\partial x^{l-1}H_x^2}{\partial x}+\frac{\partial x^{l-1}H_xH_y}{\partial y}\right]\right\}$$
$$-\frac{D\beta}{\partial x}\left\{\frac{D}{Dt}(\rho\Delta v)-\frac{\partial(x,y)}{\partial(\xi,\mu)}\left[\frac{\partial\rho x^{l-1}v(\dot{x}-u)}{\partial x}\right.\right.$$
$$+\left.\frac{\partial\rho x^{l-1}v(\dot{y}-v)}{\partial y}\right]+\Delta\frac{\partial P}{\partial y}+\frac{1}{8\pi}\Delta\frac{\partial(H_x^2+H_y^2)}{\partial y}$$
$$\left.-\frac{1}{4\pi}\frac{\partial(x,y)}{\partial(\xi,\mu)}\left[\frac{\partial x^{l-1}H_xH_y}{\partial x}+\frac{\partial x^{l-1}H_y^2}{\partial y}\right]\right\}=0. \tag{1.24′}$$

It is easy to see that Eqs. (1.24′) and (1.25′) are equivalent to the system of equations (1.22′), (1.23′) and that any two equations of the system (1.22′)–(1.25′) are sufficient to describe the motion of the medium.

Equations (1.22′), (1.23′), (1.5″), (1.9′), (1.14′), (1.15′) together with the equation of state $P = P(\rho, E)$ for given $\dot{x} = \dot{x}(\xi, \mu, t), \dot{y} = \dot{y}(\xi, \mu, t)$ from a complete system of equations of two-dimensional magnetohydrodynamics in mixed Eulerian–Lagrangian variables.

2. General formulation of the variational principle in adiabatic magnetohydrodynamics

1. Let x^i be Eulerian variables, α^i Lagrangian variables, ξ^i the base coordinates, and H^i the magnetic field components in a Cartesian coordinate system. Here and below all the indices run through the values from one to three. We denote by q^i some coordinates with a metric defined by the tensor q_{ij}; H'^i are the magnetic field components in the coordinates q^i.

The connection between the displacement velocities of the base coordinate system ξ^i with respect to the coordinates q^i and α^i and the velocity vector u^i of the medium in the coordinates q^i is determined in exactly the same way as in the two-dimensional case:

$$(\dot{q}^i - u^i) = \frac{\partial q^i}{\partial \alpha^j} \dot{\alpha}^j. \tag{2.1}$$

Expression (2.1) can also be rewritten in the form more symmetric in the variables $\dot{\alpha}^i$ and $\dot{q}^i$:

$$\begin{gathered} \frac{\partial(\alpha^1, \alpha^2, \alpha^3)}{\partial(\xi^1, \xi^2, \xi^3)}(\dot{q}^i - u^i) = \frac{\partial(q^i, \alpha^2, \alpha^3)}{\partial(\xi^1, \xi^2, \xi^3)} \dot{\alpha}^1 \\ + \frac{\partial(\alpha^1, q^i, \alpha^3)}{\partial(\xi^1, \xi^2, \xi^3)} \dot{\alpha}^2 + \frac{\partial(\alpha^1, \alpha^2, q^i)}{\partial(\xi^1, \xi^2, \xi^3)} \dot{\alpha}^3. \end{gathered} \tag{2.1'}$$

In Eulerian–Lagrangian variables the continuity equation has the form

$$\rho |g_{ij}|^{1/2} \frac{\partial(q^1, q^2, q^3)}{\partial(\xi^1, \xi^2, \xi^3)} = \bar{\rho} \frac{\partial(\alpha^1, \alpha^2, \alpha^3)}{\partial(\xi^1, \xi^2, \xi^3)}. \tag{2.2}$$

We obtain the freezing-in condition of the magnetic field. Let w and v be the internal coordinates of some fluid surface. The flux of the magnetic field through the area $dwdv$ is expressed by the formula

$$\begin{aligned} dF &= H^i e_{ijk} \frac{\partial x^j}{\partial w} \frac{\partial x^k}{\partial v} dw\, dv \\ &= \left[H^1 \frac{\partial(x^2, x^3)}{\partial(w, v)} + H^2 \frac{\partial(x^3, x^1)}{\partial(w, v)} + H^3 \frac{\partial(x^1, x^2)}{\partial(w, v)} \right] dwdv. \end{aligned}$$

We pass from the coordinate x^i to the coordinate q^i:

$$\begin{aligned} &\left[H^1 \frac{\partial(x^2, x^3)}{\partial(w, v)} + H^2 \frac{\partial(x^3, x^1)}{\partial(w, v)} + H^3 \frac{\partial(x^1, x^2)}{\partial(w, v)} \right] dwdv \\ &= \frac{\partial(x^1, x^2\ x^3)}{\partial(q^1, q^2, q^3)} \left[\frac{\partial(q^1, q^2, q^3)}{\partial(w, v, x^1)} H^1 + \frac{\partial(q^1, q^2, q^3)}{\partial(w, v, x^2)} H^2 \right. \\ &\left. + \frac{\partial(q^1, q^2, q^3)}{\partial(w, v, x^3)} H^3 \right] dwdv = |q_{ij}|^{1/2} \left[\frac{\partial(q^2, q^3)}{\partial(w, v)} H'^1 \right. \\ &\left. + \frac{\partial(q^3, q^1)}{\partial(w, v)} H'^2 + \frac{\partial(q^1, q^2)}{\partial(w, v)} H'^3 \right] dwdv. \end{aligned}$$

If we now take as (w, v) the Lagrangian variables (α^1, α^2), (α^2, α^3), (α^3, α^1) and require that the magnetic flux $dF^1 = \bar{H}^1 d\alpha^2 d\alpha^3$, $dF^2 = \bar{H}^2 d\alpha^1 d\alpha^3$, $dF^3 = \bar{H}^3 d\alpha^1 d\alpha^2$ does not change with time, then

$$
\begin{gathered}
|q_{ij}|^{1/2}\left[\frac{\partial(q^1, q^2, \alpha^i)}{\partial(\xi^1, \xi^2, \xi^3)} H'^3 + \frac{\partial(q^3, q^1, \alpha^i)}{\partial(\xi^1, \xi^2, \xi^3)} H'^2\right. \\
\left. + \frac{\partial(q^2, q^3, \alpha^i)}{\partial(\xi^1, \xi^2, \xi^3)} H'^1\right] = \bar{H}^i(\alpha^1, \alpha^2, \alpha^3)\frac{\partial(\alpha^1, \alpha^2, \alpha^3)}{\partial(\xi^1, \xi^2, \xi^3)}.
\end{gathered}
\tag{2.3}
$$

2. *The general form of the variational principle.* The dynamic equations of magnetohydrodynamics in mixed Eulerian–Lagrangian variables follow from the requirement that the first variation of the functional

$$
\begin{gathered}
\Phi = \int_{t_1}^{t_2}\left\{\int_{\Omega}\int_{(q^i)}\int \rho |q_{ij}|^{1/2}\frac{\partial(q^1, q^2, q^3)}{\partial(\xi^1, \xi^2, \xi^3)}\left[\frac{1}{2} q_{ij}u^i u^j\right.\right. \\
\left.\left. - E - \frac{1}{8\pi\rho} q_{ij}H'^i H'^j\right] d\xi^1\, d\xi^2\, d\xi^3\right\} dt
\end{gathered}
$$

vanishes, on the assumption that the base variables are not varied, and that the variations of all the other quantities are interconnected by the law of conservation of mass (2.2), by the condition of freezing in of the magnetic flux (2.3), by the kinematic relations (2.1′) and by the first law of thermodynamics. The variational principle formulated leads to six dynamic equations, any three of which can be represented as a linear combination of the others.

To obtain a system of three independent dynamic equations in the variational vector $\delta\Psi = (\delta q^1, \delta q^2, \delta q^3, \delta\alpha^1, \delta\alpha^2, \delta\alpha^3)$ it is necessary to impose the additional constaints $\delta\Psi^T = A\delta\mu^T$, where $A = A(\xi^1, \xi^2, \xi^3)$ is some matrix of dimension 6 X 3 with rank equal to 3, and $\delta\mu = (\delta\mu^1, \delta\mu^2, \delta\mu^3)$ is the vector of the generalized independent variations.

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REFERENCES

1. SELENGER, R. L. and WHITHAM, G. B. Variational principles in continuum mechanics. *Proc. Roy. Soc.*, **A305**, 1–25, 1968.

2. GOLOVIZNIN, V. M., SAMARSKII, A. A. and FAVORSKII, A. P. The variational approach to the construction of finite-difference mathematical models in hydrodynamics. *Dokl. Akad. Nauk SSSR*, **235**, 6, 1285–1288, 1977.

3. GOLOVIZNIN, V. M., SAMARSKII, A. A. and FAVORSKII, A. P. On the use of the principle of least action to construct discrete mathematical models in magnetohydrodynamics. *Dokl. Akad. Nauk SSSR*, **246**, 5, 1083–1088, 1979.

4. BELOTSERKOVSKII, O. M. and DAVYDOV, Yu. M. A non-stationary "coarse particle" method for gas-dynamic computations. *Zh. vȳchisl. Mat. mat. Fiz.*, **11**, 1, 182–207, 1971.

5. YANENKO, N. N. *The method of fractional steps for solving multidimensional problems of mathematical physics* (Metod drobnykh shagov dlya resheniya mnogomernykh zadach matematicheskoi fiziki), Nauka, Novosibirsk, 1967.

6. HIRT, C. W., AMSDEN, A. A. and COOK, J. L. An arbitrary Lagrangian–Eulerian computing method for all flow speeds. *J. Comput. Phys.*, **14**, 227–253, 1974.

7. SAMARSKII, A. A. *Theory of difference schemes* (Teoriya raznostnykh skhem). Nauka, Moscow, 1978.

8. SAMARSKII, A. A. and POPOV, Yu. P. *The difference schemes of gas dynamics* (Raznostnye skhemy gazovoi dinamiki). Nauka, Moscow, 1975.