Difference Schemes with Operator Factors

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Difference Schemes with Operator Factors

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Contents

Chapter 1

INTRODUCTION

Two- and three-level difference schemes for discretisation in time, in conjunction with finite difference or finite element approximations with respect to the space variables, are often used to solve numerically non-stationary problems of mathematical physics. In the theoretical analysis of difference schemes our basic attention is paid to the problem of stability of a difference solution (or well posedness of a difference scheme) with respect to small perturbations of the initial conditions and the right hand side.

The theory of stability of difference schemes develops in various directions. The most important results on this subject can be found in the book by A.A. Samarskii and A.V. Goolin [Samarskii and Goolin, 1973]. The survey papers of V. Thomee [Thomée, 1969, Thomée, 1990], A.V. Goolin and A.A. Samarskii [Goolin and Samarskii, 1976], E. Tadmore [Tadmor, 1987] should also be mentioned here. The stability theory is a basis for the analysis of the convergence of an approximative solution to the exact solution, provided that the mesh width tends to zero. In this case the required estimate for the truncation error follows from consideration of the corresponding problem for it and from *a priori* estimates of stability with respect to the initial data and the right hand side. Putting it briefly, this means the known result that consistency and stability imply convergence.

At present the most valuable results on the stability of difference schemes have been obtained in Hilbert spaces of mesh functions. General stability conditions for two- and three-level schemes are determined by A.A. Samarskii [Samarskii, 1967a, Samarskii, 1967b, Samarskii, 1968, Samarskii, 1970]. Fundamentally these results are exact in the sense

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that necessary and sufficient conditions for stability coincide. The general theory of stability of operator-difference schemes is based on the following new methodological principles:

- a difference scheme is considered as an independent target for research, which does not refer to the original differential problem;
- a unified (canonical) form of representation of difference schemes in Hilbert mesh spaces is introduced;
- stability conditions are stated in the form of operator inequalities.

Easily verifiable necessary and sufficient conditions for stability exhibit the constructibility of the theory.

The general stability theory gives a powerful tool for theoretical investigations of specific difference schemes. Instead of work on obtaining *a priori* estimates for each particular difference scheme, it suffices to reduce it into the canonical form and apply the known results. A few useful examples on the application of general results from the theory of the stability of operator-difference schemes to typical problems of mathematical physics can be found in the books [Samarskii, 1971, Samarskii, 1989, Samarskii and Goolin, 1973, Samarskii and Vabishchevich, 1995a].

A.A. Samarskii in [Samarskii, 1967b] suggested and completely illustrated with examples the regularization principle as a generic approach to the construction of difference schemes of necessary quality. This principle plays a key role in obtaining stable difference schemes starting from certain simplest difference schemes. The properties of a difference scheme can be improved by applying perturbations to operators of the original (generating) difference scheme with regard for general stability conditions. Standard criteria for stability with respect to the initial data (for a homogeneous right hand side) are usually combined with the lack of growth of the norm of a difference solution. In some problems of mathematical physics (such as reaction–diffusion problems) the norm of the exact solution may have a complex behaviour, for example, a growth according to some rule. This leads us to the concept of ρ-stability of difference schemes in which the parameter ρ defines the rate of a possible change of the norm as we pass from one time level to another level. For iterative methods the quantity ρ takes a value of relative variation of the error norm and therefore it suffices for convergence results to confine consideration to the case $\rho < 1$.

Nowadays the inverse problems for ill posed (in the classical sense) equations of mathematical physics have a significant value in applications. Amongst ill posed problems for evolutionary equations we distinguish the inverse time problem for parabolic equations (the so called retrospective inverse problem of heat transfer) and the Cauchy problem for elliptic equations. These problems belong to the class of conditionally well posed problems. The continuous dependence of a solution with respect to the initial data can be observed by contraction of the set of admissible solutions (or by selection of correctness classes). The general theory of stability of difference schemes is used to analyze the stability of finite difference methods for these problems [Samarskii and Vabishchevich, 1990, Samarskii and Vabishchevich, 1992, Samarskii and Vabishchevich, 1995a]. In this case $\rho > 1$ and thus the norm of the numerical solution can grow.

It is natural to try to design a similar stability theory for finite element schemes as well, and which is expected to be very closely connected to the theory of the stability of finite difference schemes. It should be admitted that there are two possible ways for the development of investigations in this direction. The first of them is related to the consideration of the corresponding difference schemes for coefficients of a series expansion of the approximate solution in the finite element basis. Here we can confine ourselves to an operator representation of the scheme in a Hilbert linear vector space with the ordinary Euclidean norm. In this case it suffices to apply directly the results on the stability of difference schemes.

Another way of constructing the general stability theory takes into account more precisely the specific character of finite element schemes, as far as they are considered in the projective form. The main results [Vabishchevich and Samarskii, 1995, Samarskii and Vabishchevich, 1996b] can be successfully obtained from the properties of bilinear forms only, without explicit consideration of matrix problems for coefficients of the finite element expansion. Sufficient conditions for stability with respect to the initial data are obtained for schemes written in canonical form. *A priori* estimates of the stability with respect to the right hand side are also given. Coincident necessary and sufficient conditions are determined for schemes with symmetric bilinear forms.

Stability conditions for difference schemes with self-adjoint operators can be formulated most simply. Some important classes of two- and three-level difference schemes having non-self-adjoint operators are considered in [Samarskii and Goolin, 1973, Goolin, 1979]. In particular, special attention should be paid to difference schemes with a subordinate skew-symmetric part.

Difference schemes with weights have a special meaning for computational practice. In some cases weighted factors can be varying in time and space. As an example, we mention the "red–white" (in other words, "chess board") scheme of V.K. Saul'ev [Saul'ev, 1960], which combines implicit and explicit schemes for adjacent nodes. Bearing in mind such class of difference schemes, A.A. Samarskii and A.V. Goolin [Samarskii and Goolin, 1993, Goolin and Samarskii, 1993] have stated the problem of stability and then obtained fundamental results on necessary and sufficient conditions for the stability of symmetrizable difference schemes. These schemes have non-self-adjoint original operators, but they can be transformed into schemes with symmetric operators, particularly, by a suitable choice of special norms. It is shown that such a class of schemes includes also difference schemes with variable weighted factors. In [Goolin and Degtyariov, 1996, Goolin and Yukhno, 1996, Goolin and Yukhno, 1998, Degtyariov, 1994] A.V. Goolin and other authors dealt with finding necessary and sufficient conditions of stability of two-level difference schemes with variable weighted factors for parabolic equations.

Various classes of difference schemes with operator factors, including schemes with variable weighted factors, are studied in [Vabishchevich and Matus, 1993b, Vabishchevich et al., 1994a, Vabishchevich et al., 1994b]. In this case non-self-adjoint operators of a difference scheme contain a term which is the product of two difference operators. Different ways of symmetrization of a difference scheme in accordance with its operator factor are discussed. Similar schemes with operator factors were considered earlier by A.V. Goolin (see, for example, [Goolin, 1979]) as very important classes of schemes with non-self-adjoint operators.

The results mentioned above provided the basis for our subsequent investigations. It turned out that a lot of both known and new difference schemes can be written in the form of difference schemes with operator factors. In this connection we consider schemes with local grid refinement with respect to the time and space variables and domain decomposition schemes for implementation on parallel computers. This book was intended as an attempt to present in details the general results on the stability of difference schemes with operator factors and to gain insights into their application to the study of new perspective difference schemes. The reader can judge to what extent we have succeeded in achieving our intentions. Let us consider the contents of the book.

The main results from the general stability theory [Samarskii, 1989, Samarskii and Goolin, 1973] for two-level operator-difference schemes are given in Chapter 2. We introduce the concepts of stability of two-level operator-difference schemes in finite-dimensional Hilbert spaces and give the canonical form for difference schemes under consideration. Basic criteria for stability are written in the form of corresponding operator inequalities. We also study separately the stability with respect to the initial data and the right hand side as well as conditions for ρ-stability of two-level difference schemes. Peculiarities of stability in different norms for difference schemes with variable (nonconstant in time) operators are marked. Two-level difference schemes with weights are examined in details. A difference scheme is called strongly stable [Samarskii, 1989] provided that the scheme is stable with respect to the initial data, the right hand side and coefficients. Conditions of coefficient stability for two-level operator-difference schemes are formulated according to [Samarskii et al., 1997d].

The third chapter is devoted to two-level operator-difference schemes with operator factors. We study difference schemes with non-self-adjoint operators, which adjoin closely schemes with variable weighted factors. Depending on how the operator factor enters the scheme, we distinguish three basic classes of schemes. Concerning schemes with variable weighted factors, for these classes of schemes one can weigh the solution itself, either the fluxes or the whole part of the equation associated with the space discretization. Necessary and sufficient conditions for stability with respect to the initial data are formulated, and *a priori* estimates of stability with respect to the right hand side are given. These theoretical investigations are then used in subsequent chapters of this book where particular difference schemes are studied.

The next two chapters of this book are devoted to analogous questions for three-level operator-difference schemes. The common criteria for stability and ρ-stability of three-level schemes are formulated in Chapter 4. These schemes are studied by writing the scheme in the form of a vector two-level operator-difference scheme on the direct sum of two Hilbert spaces. Keeping in mind the use of three-level difference schemes for first-order and second-order evolutionary equations, we consider two canonical forms for the representation of three-level difference schemes.

Three-level operator-difference schemes with operator factors are considered in Chapter 5. Starting from two-parameter three-level operator-difference schemes, we study schemes with two operator factors. Again, as in Chapter 3, three classes of schemes with different entering of operator factors are the basis for our consideration. The considerable attention is paid to deriving *a priori* estimates of stability with respect to the initial data and the right hand side in various norms.

Chapters 6–8 examine how the general results obtained in the stability theory for operator-difference schemes can be applied to the study of particular difference scheme. Among the most important questions is the problem of accuracy of a difference solution. The general theory of stability of operator–difference schemes and the corresponding *a priori* estimates of stability with respect to the initial data and the right hand side constitute a mathematical tool for the analysis of convergence of the approximate solution to the exact solution in different classes of smoothness.

Some non-stationary problems of mathematical physics are considered in Chapter 6, which begins with the classical results [Samarskii, 1989] concerning accuracy of two-level difference schemes for parabolic equations. It turns out that *a priori* estimates in norms integral with respect to time are often more suitable for the case of boundary value problems with generalized solutions. Here, in particular, we obtain (see also [Samarskii et al., 1997c]) the stability criteria for two-level operator-difference schemes in the integral with respect to time norms, which complement the stability results in norms that are uniform with respect to time.

Then, as a basic equation of continuum mechanics, one model non-stationary convection–diffusion equation is investigated. We consider the problems of approximating the non-self-adjoint part (i.e., convective terms) of a second-order parabolic equation. Two-level difference schemes are constructed and studied. We discuss also the questions of constructing difference schemes for a model nonlinear equation, that is, the Korteweg–de Vries equation. Here two- and three-level schemes are obtained which belong to the class of conservative schemes in either sense.

The questions concerning the construction and accuracy of difference schemes for second-order hyperbolic equations are considered with an example of a one-dimensional oscillation equation. A special attention is given here to the important case of problems with piecewise smooth solutions. In the final section of Chapter 6 we study difference schemes applied to the numerical solution of a model boundary value problem for equations of mixed (hyperbolo–parabolic) type. Specifically we consider the initial boundary value problem in which we have a second-order hyperbolic equation in one part of the domain of definition, and a parabolic equation in another part. In fact, for such a problem one can use a homogeneous three-level difference scheme with discontinuous weighted factors.

The stability theory for operator–difference schemes with operator factors are most widely applied in Chapter 7, where schemes of adaptive type for basic problems of mathematical physics are considered. In many problems, in order to handle local singularities of the solution it is necessary to refine the computational grid with respect to time. These schemes can be constructed in two different ways, the first of which is connected with a special organization of computations outside the adaptation domain. The second way is connected with linear interpolation algorithms. We consider difference schemes on locally condensing grids for parabolic and hyperbolic equations of the second order. This research is based on the representation of difference schemes on adaptive grids in the form of schemes with variable weighted factors. Approaches with local refinement of the space grid in a sub-domain which varies in time (dynamic adaptation) are frequently used in computational practice. The main technical issues here are related to the construction of a difference scheme at the new nodes on the upper time level. We analyze the convergence of difference schemes for a model one-dimensional problem, in which the case of variable coefficients are separately examined.

Traditionally non-uniform grids are widely used for improving the accuracy of the approximate solution. However, the disadvantageous property of approximating second-order elliptic operators on non-uniform grids is that, generally speaking, the second order of local approximation on standard minimal stencils is lost (see, e.g., [Samarskii, 1989]). In this book we observe the possibility of obtaining an increased order of the approximations on non-uniform grids not at the nodes but at some special intermediate points. This technique is illustrated by applying it to boundary value problems for basic stationary and non-stationary equations of mathematical physics.

Nowadays parallel algorithms for solving partial differential equations are built mainly by means of domain decomposition methods, which consist in partitioning the solution domain into overlapping or non-overlapping sub-domains. The effective implementation of a numerical method on contemporary supercomputing machines is attained by solving smaller problems separately in sub-domains. In this way, for solving non-stationary problems numerically one can design iteration-free splitting computational algorithms (so called regionally additive difference schemes). The problems arising in the construction and study of domain decomposition schemes are considered in Chapter 8. Note that these schemes can be constructed by applying different splitting algorithms. We consider separately the case of two-component ('two-colored') splitting. Decomposition schemes for second-order hyperbolic and parabolic equations are proposed. The main attention here is given to setting exchange boundary conditions at the interfaces between sub-domains as well as to issues of how the accuracy of the numerical approximations depends on the partition used in this method and on the choice of a difference domain decomposition scheme. The research undertaken is based on the stability theory of additive difference schemes (splitting schemes), in which the regionally additive schemes themselves are interpreted as schemes with operator factors.

This book has been written with the attentive and helpful participation of many of our colleagues from the Institute for Mathematical Modelling, the Russian Academy of Sciences, the Institute of Mathematics, the National Academy of Sciences of Belarus and the Chair of Computational Methods of the Lomonosov Moscow State University. For helpful and fruitful discussions, the authors thank especially A.V. Goolin whose works on the stability theory for difference schemes define this direction of investigations. We are also grateful to I.V. Gaishun for his active interest and collaboration in the current work and constant encouragement throughout this project. Direct help in preparing the manuscript was given to us by young colleague S.V. Lemeshevsky. The authors wish to express their gratitude to I. Tselishcheva and G.R. Malyavskaya who edited this monograph.

Chapter 2

TWO-LEVEL DIFFERENCE SCHEMES

1. Introduction

After having replaced a non–stationary problem of mathematical physics by its finite difference or finite element discretization in space, we obtain the Cauchy problem for a system of ordinary differential equations which is considered in a Hilbert grid space. The discretisation in time yields an operator–difference scheme. Let us mention some most important directions in the development of the theory of operator–difference schemes for nonstationary problems of mathematical physics.

The general theory of stability (well posedness) of operator–difference schemes is widely used to study finite difference schemes for non–stationary problems. At present exact (coinciding necessary and sufficient) conditions are obtained for a broad class of two- and three-level difference schemes in finite-dimensional Hilbert spaces.

It is necessary to emphasize the constructibility of the general stability theory for operator–difference schemes in which the criteria for stability are stated in the form of easily verifiable inequalities for operators. Amongst the most important generalizations, we can note making use of the general stability theory for ill posed evolution problems and also for the analysis of projective–difference schemes (finite elements schemes). Also new *a priori* estimates of stability are obtained in norms that are integral with respect to time. Based on these estimates one can study, in particular, convergence properties of difference schemes for problems with generalized solutions.

A question of principle in the case of initial value problems for non-stationary equations of mathematical physics is not only the stability of a solution with respect to the initial data and the right hand side, but

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also the continuous dependence of the solution on a small perturbation of the operators of the problem, that is, strong stability. In this part of the book we obtain estimates of stability under perturbations of the operator in the Cauchy problem, of the right hand side, and of the initial condition for first-order evolutionary equations considered in Hilbert spaces. This chapter also contains *a priori* estimates of strong stability for two-level operator-difference schemes which are coordinated with the corresponding estimates for a differential-operator equation.

Let H be a finite-dimensional Hilbert space, B, A be linear operators in H. A homogeneous (with zero on the right hand side) two-level operator-difference scheme can be written in a common (canonical) form

$$B\frac{y^{n+1}-y^n}{\tau} + Ay^n = 0, \qquad n = 0, 1, \dots, \tag{2.1}$$

for given y_0. We consider that in (2.1) the operators A and B are constant (independent of n), the operator A is self-adjoint and positive ($A = A^* > 0$).

The difference scheme (2.1) is *stable* with respect to the initial data in H_D if the following estimates are satisfied:

$$\|y^{n+1}\|_D \le \|y^n\|_D, \quad n = 0, 1, \dots.$$

The main result (Theorem 2.1) about the stability of two-level difference schemes can be formulated in such a way. The necessary and sufficient condition for the stability of difference scheme (2.1) with the operator $A = A^* > 0$ in H_A is the inequality

$$B \ge \frac{\tau}{2}A. \tag{2.2}$$

Under more general conditions it is necessary to apply the criteria of ρ-stability with respect to the initial data, where a difference solution satisfies the *a priori* layerwise bound

$$\|y^{n+1}\|_D \le \rho\|y^n\|_D, \quad n = 0, 1, \dots,$$

and also $\rho^n \le M$. The constant ρ is usually chosen to be one of the following quantities

$$\begin{aligned} \rho &= 1, \\ \rho &= 1 + c\tau, \quad c > 0, \\ \rho &= e^{c\tau}, \end{aligned}$$

where c a constant independent of the mesh parameters τ, n.

For difference scheme (2.1) with the operators $A = A^*,\ B = B^* > 0$, the conditions

$$\frac{1-\rho}{\tau} B \leq A \leq \frac{1+\rho}{\tau} B \tag{2.3}$$

are necessary and sufficient (see Theorem 2.5) for stability in H_B.

The convergence of difference schemes is analyzed in different norms, which must be conforming to a class of smoothness of solutions to a differential problem. By virtue of this there is a need to have a few estimates for a difference solution. In particular, if we consider difference schemes for non-stationary boundary value problems with generalized solutions, the emphasis in this case is on taking into consideration estimates of a difference solution in integral (with respect to time) norms.

In the study of well posedness of initial value problems for non-stationary equations of mathematical physics, principal attention should be paid to questions concerning the stability of the solution with respect to the initial data and the right hand side. Based on *a priori* estimates for the solution of a difference problem with an inhomogenous right hand side, we consider the relevant problem for the error to study the rate of convergence of the approximate solution to the exact one. The estimates of stability with respect to the right hand side can be often obtained appropriately from the estimates of stability with respect to the initial data. In this case it is said that the stability of a difference scheme with respect to the initial data implies its stability with respect to the right hand side as well. Of course, this statement is valid only for the corresponding (consistent) choice of norms for the difference solution, the initial condition and the right hand side. The estimates of such kind are given in this part of our book.

In a more general situation it is necessary to demand the continuous dependence of a solution to be also on perturbations of operators in the problem under consideration, for example, on coefficients of the equation. *A priori* estimates reflecting the continuity of the problem's solution with respect to the right hand side and the operator have been previously obtained under different conditions for stationary problems (for operator equations of the first kind). We give some new *a priori* estimates of strong stability for two-level operator-difference schemes, which are coordinated with the corresponding estimates for a differential-operator equation. The theoretical material stated in the chapter is illustrated by consideration of two-level difference schemes with weights, which are applied to the Cauchy problem for first-order evolution equations.

2. Stability of Difference Schemes

In this section we give a general definition of stability of a two-level operator-difference scheme written in canonical form. A more general concept of ρ–stability of a difference scheme is introduced for problems in which the rate of change of the solution norm is essential.

2.1 Canonical Form

Let us introduce some fundamental notations and definitions from the theory of finite difference schemes. Let H be a real finite-dimensional Hilbert space and

$$\bar{\omega}_\tau = \{t_n = n\tau,\ n = 0, 1, ..., N_0;\ \tau N_0 = T\} = \omega_\tau \cup \{T\} \tag{2.4}$$

be a uniform grid in time. We denote by $A, B\colon H \to H$ linear operators in H which depend, generally speaking, on τ and t_n.

Consider *Cauchy problem for the operator-difference equation*

$$B(t_n)\frac{y_{n+1} - y_n}{\tau} + A(t_n)y_n = \varphi_n, \quad t_n \in \omega_\tau, \tag{2.5}$$

$$y_0 = u_0, \tag{2.6}$$

where φ_n and $u_0 \in H$ are given functions, and $y_n = y(t_n) \in H$ is the unknown function that we seek for.

We use the notation

$$y = y_n, \quad \hat{y} = y_{n+1}, \quad \check{y}=y_{n-1}, \quad y_{\bar{t}} = \frac{y - \check{y}}{\tau}, \quad y_t = \frac{\hat{y} - y}{\tau}.$$

Then equation (2.5) can be written in the form

$$By_t + Ay = \varphi, \qquad t \in \omega_\tau. \tag{2.7}$$

The set of Cauchy problems (2.5), (2.6), which depend on the parameter τ, is called a *two-level scheme*. We call the notation (2.5) and (2.6) the *canonical form of two-level schemes*.

For the solvability of the Cauchy problem at a new time level, we assume that B^{-1} exists. Then equation (2.7) can be written down as follows:

$$y_{n+1} = Sy_n + \tau\widetilde{\varphi}_n, \quad S = E - \tau B^{-1}A, \quad \widetilde{\varphi}_n = B^{-1}\varphi_n, \tag{2.8}$$

where E is the identity operator. The operator S is called a *transition operator*.

2.2 General Concept of Stability

Let us consider the set of the solutions $y(t)$ of the Cauchy problem (2.5), (2.6) depending on the problem's *input* data, namely, on the right hand side $\varphi(t)$ and the initial condition u_0.

A two-level difference scheme is called *well-posed* if the following statements are valid for sufficiently small $\tau < \tau_0$:

1) a solution of this scheme exists and is unique for any initial data $u_0 \in H$ and right hand sides $\varphi \in H$, for all $t \in \bar{\omega}_\tau$;

2) there exist positive constants M_1 and M_2, both independent of τ and the choice of u_0, φ, such that for any $u_0 \in H$, $\varphi \in H$ and $t \in \bar{\omega}_\tau$ the solution of the Cauchy problem (2.5), (2.6) satisfies the estimate

$$\|y_{n+1}\| \leq M_1 \|u_0\| + M_2 \max_{0 \leq t' \leq t_n} \|\varphi(t')\|_*. \tag{2.9}$$

Here $\|\cdot\|$ and $\|\cdot\|_*$ are some norms in the space H.

The inequality (2.9) for linear problems under consideration reflects such a property as the continuous dependence of the solution of the Cauchy problem (2.5), (2.6) on the input data (or *stability*). The problem satisfying this property is said to be *stable.*

We usually use the following concepts of stability with respect to the initial data and the right hand side.

The difference scheme

$$B\frac{y_{n+1} - y_n}{\tau} + Ay_n = 0, \qquad t_n \in \omega_\tau, \tag{2.10}$$

$$y_0 = u_0 \tag{2.11}$$

is called *stable with respect to the initial data* if the solution of problem (2.10), (2.11) satisfies the bound

$$\|y_{n+1}\| \leq M_1 \|u_0\|, \qquad n = 0, 1, \ldots, N_0 - 1. \tag{2.12}$$

The two-level scheme

$$B\frac{y_{n+1} - y_n}{\tau} + Ay_n = \varphi_n, \qquad t_n \in \omega_\tau, \tag{2.13}$$

$$y_0 = 0 \tag{2.14}$$

is called *stable with respect to the right hand side* if the solution of problem (2.13) with the homogeneous initial condition (2.14) satisfies the inequality

$$\|y_{n+1}\| \leq M_2 \max_{0 \leq t' \leq t_n} \|\varphi(t')\|_*, \qquad n = 0, 1, \ldots, N_0 - 1. \tag{2.15}$$

The solution of problem (2.7), (2.6) can be represented in the form $y = y^{(1)} + y^{(2)}$, where $y^{(1)}$ is the solution of problem (2.10), (2.11), and $y^{(2)}$ is the solution of problem (2.13), (2.14). By virtue of the triangle inequality

$$\|y_{n+1}\| \leq \|y_{n+1}^{(1)}\| + \|y_{n+1}^{(2)}\|$$

and then inequalities (2.12) and (2.15) imply estimate (2.9).

2.3 ρ–Stability of Difference Schemes

Stability estimates can be obtained by using *a priori* estimates of a difference solution on different successive time levels.

The difference scheme (2.10), (2.11) is called *ρ-stable* (*uniformly stable*) with respect to the initial data if there are constants $\rho > 0$ and M_1, independent of τ and n, such that for any n and for all $y_n \in H$ the solution y_{n+1} of the difference equation (2.10) satisfies the estimate

$$\|y_{n+1}\| \leq \rho \|y_n\| \tag{2.16}$$

with $\rho^n \leq M_1$.

In the theory of difference schemes the constant ρ usually takes one of the following values:

$$\begin{aligned} \rho &= 1, \\ \rho &= 1 + c\tau, \qquad c > 0, \\ \rho &= e^{c\tau}, \end{aligned}$$

where the constant c does not depend on τ and n. If, for example, $\rho = 1 + c\tau,\ c > 0$ then

$$\rho^n \leq e^{cn\tau} = e^{ct_n} \leq e^{cT} = M_1.$$

Taking account of (2.8) we rewrite equation (2.10) in the form

$$y_{n+1} = S y_n. \tag{2.17}$$

The criterion of ρ-stability is equivalent to the fulfilment of the inequality

$$\|S\| \leq \rho, \tag{2.18}$$

which can be written in such a form

$$-\rho E \leq S \leq \rho E \tag{2.19}$$

if the operator S is self-adjoint. For an arbitrary operator we have

$$S^* S \leq \rho^2 E. \tag{2.20}$$

The *a priori* estimate of the difference solution by levels

$$\|y_{n+1}\| \le \rho\|y_n\| + \tau\|\varphi_n\|_* \tag{2.21}$$

implies the inequality

$$\|y_{n+1}\| \le \rho^{n+1}\|y_0\| + \sum_{k=0}^{n} \tau\rho^{n-k}\|\varphi_k\|_* . \tag{2.22}$$

This is *a discrete analog of the Gronwall lemma.*

Let $(\cdot,\cdot)$ be an inner (or scalar) product in H and D be a self-adjoint positive operator defined in H, i.e., for any $y, v \in H$

$$(Dy, v) = (y, Dv), \qquad (Dy, y) > 0, \quad \text{if} \quad y \neq 0.$$

H_D denotes a Hilbert space consisting of elements of the space H provided with the following inner product and norm

$$(y, v)_D = (Dy, v), \qquad \|y\|_D = \sqrt{(Dy, y)}$$

respectively.

The difference scheme (2.10), (2.11) is called ρ-stable in H_D, if

$$\|y_{n+1}\|_D \le \rho\|y_n\|_D. \tag{2.23}$$

The analogues of operator inequalities (2.19) and (2.20) (for the transition operator S) are

$$-\rho D \le DS \le \rho D, \qquad DS = S^*D, \tag{2.24}$$

$$S^*DS \le \rho^2 D. \tag{2.25}$$

These inequalities ensure the ρ-stability of the operator difference scheme in H_D.

3. Conditions of Stability with Respect to the Initial Data

In this section the constructive conditions for stability of two-level operator-difference schemes in Hilbert spaces are formulated in the form of operator inequalities. Necessary and sufficient conditions for stability of two-level schemes have been obtained in [Samarskii, 1968]. More recent results are presented in [Goolin and Samarskii, 1976, Samarskii and Goolin, 1973].

3.1 Stability in the Space H_A

In order to study the stability of scheme (2.5), (2.6) with respect to the initial data, we will estimate the solution of the problem (2.10), (2.11).

THEOREM 2.1 *Let the operator A in equation (2.10) be a self-adjoint positive operator independent of n. The condition*

$$B \geq 0.5\tau A, \qquad t \in \omega_\tau, \tag{2.26}$$

is necessary and sufficient for the stability of scheme (2.10), (2.11) in H_A with respect to the initial data with the constant $M_1 = 1$, i.e., in order that the following estimate is satisfied:

$$\|y_n\|_A \leq \|u_0\|_A, \qquad n = 0, 1, ..., N_0. \tag{2.27}$$

Proof. *Sufficiency.* Assume that the condition (2.26) is satisfied. Taking the inner product of equation (2.10) and y_t in H, we obtain the identity

$$(By_t, y_t) + (Ay, y_t) = 0. \tag{2.28}$$

Using the formula

$$y = 0.5(y + \hat{y}) - 0.5\tau y_t \tag{2.29}$$

we rewrite (2.28) in the form

$$((B - 0.5\tau A)y_t, y_t) + 0.5\tau^{-1}(A(\hat{y} + y), \hat{y} - y) = 0. \tag{2.30}$$

Since the operator A is self-adjoint we have $(Ay, \hat{y}) = (y, A\hat{y}) = (A\hat{y}, y)$ and therefore

$$\begin{aligned}(A(\hat{y} + y), \hat{y} - y) &= (A\hat{y}, \hat{y}) + (Ay, \hat{y}) - (A\hat{y}, y) - (Ay, y) \\ &= (A\hat{y}, \hat{y}) - (Ay, y) = \|y_{n+1}\|_A^2 - \|y_n\|_A^2.\end{aligned} \tag{2.31}$$

Substituting the last identity (2.31) into (2.30) and using the inequality (2.26) gives us the recursion relation

$$\|y_{n+1}\|_A \leq \|y_n\|_A. \tag{2.32}$$

Since n is an arbitrary number, the statement of the theorem follows from the last estimate (2.32).

Necessity. Assume that the scheme is stable and let the inequality (2.27) be satisfied. We will prove that this assumption implies the operator inequality (2.26), i.e

$$(Bv, v) \geq 0.5\tau(Av, v) \qquad \text{for any} \quad v \in H. \tag{2.33}$$

Starting from the identity (2.30) at the first level (for $n = 0$), we have

$$((B - 0.5\tau A)y_{t0}, y_{t0}) + 0.5\tau^{-1}(Ay_1, y_1) = 0.5\tau^{-1}(Ay_0, y_0).$$

By virtue of (2.27) this identity can be valid only provided that

$$((B - 0.5\tau A)y_{t0}, y_{t0}) = 0.5\tau^{-1}\left((Au_0, u_0) - (Ay_1, y_1)\right) \geq 0,$$

that is, if

$$((B - 0.5\tau A)y_{t0}, y_{t0}) \geq 0.$$

Since $y_0 = u_0 \in H$ is an arbitrary element it is easily seen that $v = y_{t0} = -B^{-1}Au_0 \in H$ is also arbitrary. In fact, given an arbitrary element $v = y_{t0} \in H$ and taking into account the fact that A^{-1} exists, we have $u_0 = -A^{-1}Bv \in H$. Thus, the last inequality holds for any $v = y_{t0} \in H$, i.e., the operator inequality (2.26) is valid.

REMARK 2.1 Under the condition

$$B(t) \geq 0.5\tau A, \quad A = A^*, \tag{2.34}$$

where A is a constant operator of not fixed sign, the solution of equation (2.10), (2.11) satisfies the estimate

$$(Ay_{n+1}, y_{n+1}) \leq (Ay_0, y_0). \tag{2.35}$$

The last inequality (2.35) is simply a consequence of (2.30) and (2.31); it implies the stability estimate with respect to the initial data in the energy space H_A when $A > 0$.

Up to this point in the book we have assumed that the operator A is constant, that is, independent of t. But if the operator A depends on t and $A(t) = A^*(t) > 0$, then we shall require that the following condition of*Lipschitz continuity* of $A(t)$ in t holds:

$$|((A(t) - A(t-\tau))v, v)| \leq \tau c_0 (A(t-\tau)v, v) \tag{2.36}$$

for all $v \in H$, $0 < t \leq T$, where c_0 is a positive constant independent of τ.

THEOREM 2.2 *Let the operators $A(t)$, $B(t)$ for $t \in \omega_\tau$ comply with the conditions*

$$A(t) = A^*(t) > 0, \tag{2.37}$$

$$B(t) \geq 0.5\tau A(t), \tag{2.38}$$

respectively. Then the difference scheme (2.10), (2.11) is ρ-stable with respect to the initial data for $\rho = e^{0.5c_0\tau}$, $M_1 = \rho^{N_0} = e^{0.5c_0T}$, and the following estimates hold:

$$\|y_{n+1}\|_{A_n} \leq \rho \|y_n\|_{A_{n-1}}, \qquad n = 1, 2, ..., N_0 - 1, \tag{2.39}$$

$$\|y_{n+1}\|_{A_n} \le M_1 \|u_0\|_{A_0}, \qquad n = 0, 1, ..., N_0 - 1. \tag{2.40}$$

Proof. Taking the inner product of (2.10) and $2\tau y_t$, and using (2.30) and (2.31) we obtain the energy identity

$$2\tau((B - 0.5\tau A)y_t, y_t) + (A\hat{y}, \hat{y}) = (Ay, y). \tag{2.41}$$

In order to obtain the recurrent inequality we transform the expression

$$(Ay, y) = (A_{n-1}y_n, y_n) + ((A_n - A_{n-1})y_n, y_n)$$

and then estimate the second term in the right hand side with the help of condition (2.36):

$$(A_n y_n, y_n) \le (1 + \tau c_0)\|y_n\|^2_{A_{n-1}} \le \rho^2 \|y_n\|^2_{A_{n-1}}. \tag{2.42}$$

Substituting (2.42) into (2.41) we obtain

$$2\tau((B - 0.5\tau A)y_t, y_t) + \|y_{n+1}\|^2_{A_n} \le \rho^2 \|y_n\|^2_{A_{n-1}}. \tag{2.43}$$

Using the condition (2.38) leads to the estimate

$$\|y_{n+1}\|_{A_n} \le \rho \|y_n\|_{A_{n-1}}, \tag{2.44}$$

which conveys the uniform stability of the scheme with respect to the initial data.

In particular, it follows from here that

$$\|y_{n+1}\|_{A_n} \le \rho^n \|y_1\|_{A_0}. \tag{2.45}$$

On the other hand, from the identity (2.41) for $n = 0$ by virtue of (2.38) we deduce the inequality

$$\|y_1\|_{A_0} \le \|u_0\|_{A_0}. \tag{2.46}$$

Substituting (2.46) into (2.45) and bearing in mind that $\rho^n \le M_1$, we come to the formula

$$\|y_{n+1}\|_{A_n} \le M_1 \|u_0\|_{A_0}, \tag{2.47}$$

which expresses the stability of the scheme with respect to the initial data.

3.2 Stability in the Space H_B

In many cases it is useful to have the stability estimates not only in the space H_A, but also in other norms. The following statement is typical.

THEOREM 2.3 *Let the operators A and B in the Cauchy problem (2.10), (2.11) be independent of t and satisfy the conditions*

$$B = B^* > 0, \qquad A = A^* > 0. \tag{2.48}$$

Then the condition (2.26) is sufficient for the stability of the scheme (2.10), (2.11) with respect to the initial data in H_B with constant $M_1 = 1$.

Proof. Consider the inner product of equation (2.10) and $2\tau\hat{y}$:

$$2\tau(By_t, \hat{y}) + 2\tau(Ay, \hat{y}) = 0. \tag{2.49}$$

Taking into account the formulas

$$\hat{y} = 0.5(\hat{y} + y) + 0.5\tau y_t, \qquad y = 0.5(\hat{y} + y) - 0.5\tau y_t$$

and using the assumption (2.48) we find

$$\begin{aligned} 2\tau(By_t, \hat{y}) &= (B(\hat{y} - y), \hat{y} + y) + \tau^2(By_t, y_t) \\ &= \|\hat{y}\|_B^2 - \|y\|_B^2 + \tau^2\|y_t\|_B^2, \\ 2\tau(Ay, \hat{y}) &= 0.5\tau(A(\hat{y} + y - \tau y_t), \hat{y} + y + \tau y_t) \\ &= 0.5\tau\|\hat{y} + y\|_A^2 - 0.5\tau^3\|y_t\|_A^2. \end{aligned}$$

Substituting these expressions into equation (2.49) we come to the equality

$$\|\hat{y}\|_B^2 + \tau^2(\|y_t\|_B^2 - 0.5\tau\|y_t\|_A^2) + 0.5\tau\|\hat{y} + y\|_A^2 = \|y\|_B^2. \tag{2.50}$$

Hence as we know $B \geq 0.5\tau A$ we obtain required inequality

$$\|\hat{y}\|_B^2 \leq \|y\|_B^2, \tag{2.51}$$

that is,

$$\|y_{n+1}\|_B \leq \|u_0\|_B. \tag{2.52}$$

REMARK 2.2 Further (see Theorem 2.5) by using the method of operator inequalities we will show that (2.26) is also a necessary condition for the stability of scheme (2.10), (2.11) with respect to the initial data in the space H_B.

REMARK 2.3 If the operators A and B are commutative then the condition (2.26) is necessary and sufficient for the stability of scheme (2.10), (2.11) in the space H_D:

$$\|y_n\|_D \leq \|u_0\|_D, \tag{2.53}$$

where $D = D^* > 0$ is any operator which is commutative with A and B, for example, $D = E$, $D = A^2$ or $D = B^2$ when $B = B^*$, so that

$$\|y_n\| \le \|u_0\|, \qquad \|Ay_n\| \le \|Au_0\|, \qquad \|By_n\| \le \|Bu_0\|.$$

We now analyze the stability of the scheme with respect to the initial data in the energy space H_B in the case of variable operators A and B.

THEOREM 2.4 *Assume that the operators $A(t)$ and $B(t)$ satisfy conditions (2.37), (2.38) and, in addition, $B(t)$ is a self-adjoint Lipschitz continuous operator*

$$|((B(t) - B(t-\tau))v, v)| \le \tau c_1 (B(t-\tau)v, v), \qquad t \in \omega_\tau. \tag{2.54}$$

Then the difference scheme (2.10), (2.11) is ρ-stable with respect to the initial data in the space H_B with

$$\rho = e^{0.5 c_1 \tau}, \qquad M_1 = \rho^{N_0} = e^{0.5 c_1 T}, \tag{2.55}$$

and the inequalities

$$\|y_{n+1}\|_{B_n} \le \rho \|y_n\|_{B_{n-1}}, \tag{2.56}$$

$$\|y_{n+1}\|_{B_n} \le M_1 \|u_0\|_{B_0} \tag{2.57}$$

are valid.

Proof. Considering the inner product of (2.10) and $2\tau y_t$, similarly to (2.50) we obtain the energy identity

$$\|\hat{y}\|^2_{B(t)} + \tau^2((B - 0.5\tau A)y_t, y_t) + 0.5\tau\|\hat{y} + y\|^2_{A(t)} = \|y\|^2_{B(t)}. \tag{2.58}$$

Hence by virtue of condition (2.38) we have

$$\|y_{n+1}\|^2_{B_n} \le \|y_n\|^2_{B_n}. \tag{2.59}$$

Having used the condition of Lipschitz continuity of the operator $B(t)$ and the inequality (2.42), we deduce from (2.59) that

$$\|y_{n+1}\|^2_{B_n} \le \rho^2 \|y_n\|^2_{B_{n-1}}, \qquad \rho = e^{0.5 c_1 \tau}, \tag{2.60}$$

or

$$\|y_{n+1}\|_{B_n} \le \rho \|y_n\|_{B_{n-1}}.$$

The inequality (2.57) is derived by proceeding in a similar way (see also the proof of Theorem 2.2).

3.3 Condition for ρ-Stability

The difference scheme (2.10), (2.11) with constant operators A and B can be transformed to the explicit scheme

$$\frac{v_{n+1} - v_n}{\tau} + Cv_n = 0 \tag{2.61}$$

or

$$v_{n+1} = (E - \tau C)v_n$$

with the initial data

$$v_0 = \bar{u}_0, \tag{2.62}$$

if we make the substitution as follows:

1) $v_n = B^{1/2}y_n$ for $B = B^* > 0,\ \ C = C_1 = B^{-1/2}AB^{-1/2}$,

2) $v_n = A^{1/2}y_n$ for $A = A^* > 0,\ \ C = C_2 = A^{1/2}B^{-1}A^{1/2}$.

Hence we have

$$\|v_n\| = \|y_n\|_B \quad \text{when } C = C_1, \quad v_n = B^{1/2}y_n,$$

$$\|v_n\| = \|y_n\|_A \quad \text{when } C = C_2, \quad v_n = A^{1/2}y_n.$$

We thus claim that the condition for ρ-stability of the implicit scheme (2.10), (2.11) in the space H_D for $D = B$ and $D = A$ is equivalent to the condition for ρ-stability of the explicit scheme (2.61), (2.62) in the space H:

$$\|v_n\| \le \rho\|v_{n-1}\|.$$

LEMMA 2.1 *The condition for ρ-stability of scheme (2.61), (2.62) with constant operator C is equivalent to the condition that the norm of the transition operator S is bounded:*

$$\|S\| = \|E - \tau C\| \le \rho.$$

LEMMA 2.2 *If the conditions $A = A^* > 0,\ \ B = B^* > 0$ are satisfied, then the inequalities*

$$\gamma_1 B \le A \le \gamma_2 B \qquad \text{and} \qquad \gamma_1 E \le C \le \gamma_2 E \tag{2.63}$$

are equivalent for $C = B^{-1/2}AB^{-1/2}$ and $C = A^{1/2}B^{-1}A^{1/2}$.

Proof. Assume that $C = B^{-1/2}AB^{-1/2}$ and γ is an integer. Consider the difference

$$(Cv, v) - \gamma(v, v) = (B^{-1/2}AB^{-1/2}v, v) - \gamma(v, v) = (Ay, y) - \gamma(By, y)$$

for $y = B^{-1/2}v$. Hence it is seen that the signs of the operators $C - \gamma E$ and $A - \gamma B$ coincide. Note that it does not require positivity of the operator A.

We assume now that $C = A^{1/2}B^{-1}A^{1/2}$. We first prove that the inequalities $C \geq \gamma E$ ($C \leq \gamma E$) and $E \geq \gamma C^{-1}$ ($E \leq \gamma C^{-1}$) are equivalent. Making the replacement $y = C^{1/2}v$ we obtain

$$(Cv, v) - \gamma(v, v) = (C^{1/2}v, C^{1/2}v) - \gamma(v, v) = (y, y) - \gamma(C^{-1}y, y).$$

Hence it follows that the operators $C - \gamma E$ and $E - \gamma C^{-1}$ have the same signs. Now, substituting $C^{-1} = A^{-1/2}BA^{-1/2}$ into the above identity and denoting $x = A^{-1/2}y$ we come to

$$(Cv, v) - \gamma(v, v) = (y, y) - \gamma(A^{-1/2}BA^{-1/2}y, y) = (Ax, x) - \gamma(Bx, x),$$

that is, the operators $C - \gamma E$ and $A - \gamma B$ have the same signs. The equivalence of the inequalities from (2.63) follows immediately if we take $\gamma = \gamma_1$ and $\gamma = \gamma_2$.

LEMMA 2.3 *If the operator $C = C^* > 0$, $\tau > 0$, then the conditions*

$$\|S\| = \|E - \tau C\| \leq \rho \tag{2.64}$$

and

$$\frac{1-\rho}{\tau}E \leq C \leq \frac{1+\rho}{\tau}E \tag{2.65}$$

are equivalent.

Proof. As the operator $S = E - \tau C = S^*$ is self-adjoint it follows that

$$\|S\| = \sup_{\|v\|=1} |(Sv, v)| = \sup_{\|v\|=1} |((E - \tau C)v, v)|.$$

Hence $-\|S\|E \leq S \leq \|S\|E$ or $-\rho E \leq S \leq \rho E$ and

$$-\rho E \leq E - \tau C \leq \rho E,$$

i.e., the condition

$$\frac{1-\rho}{\tau}E \leq C \leq \frac{1+\rho}{\tau}E$$

holds. In this way (2.65) follows from the inequality (2.64). The reasoning in the reverse direction is obvious.

THEOREM 2.5 *Let A and B be constant operators and assume that $A = A^*$, $B = B^* > 0$. Then the condition*

$$\frac{1-\rho}{\tau}B \leq A \leq \frac{1+\rho}{\tau}B \tag{2.66}$$

is necessary and sufficient for the ρ-stability of scheme (2.10), (2.11) in the space H_B, i.e., for the following estimate to be satisfied:

$$\|y_{n+1}\|_B \leq \rho\|y_n\|_B.$$

Furthermore, if A is a positive operator then the condition (2.66) is necessary and sufficient also for the ρ-stability in the space H_A and, respectively, for the validity of the estimate

$$\|y_{n+1}\|_A \leq \rho\|y_n\|_A \ .$$

Proof. To verify the veracity of this assertion we reduce the implicit scheme (2.10), (2.11) to the explicit one (2.61), (2.62) with operator $C = B^{-1/2}AB^{-1/2}$ (or $C = A^{1/2}B^{-1}A^{1/2}$ when $A > 0$) and then we use Lemmas 2.1 and 2.3.

REMARK 2.4 If the inequality (2.66) is satisfied then by Lemma 2.3 the estimate (2.64) for the norm of the transition operator holds.

3.4 Stability of Schemes with Weights

Let us denote

$$v_n{}^{(\sigma)} = \sigma v_{n+1} + (1-\sigma)v_n, \tag{2.67}$$

where $\sigma \geq 0$ is an arbitrary real parameter. Consider the following class of difference schemes with constant weights:

$$y_t + Ay_n{}^{(\sigma)} = 0, \quad t_n \in \omega_\tau, \tag{2.68}$$

$$y_0 = u_0. \tag{2.69}$$

The scheme (2.68) can be written down in the canonical form (2.10) with $B = E + \sigma\tau A$. We now investigate separately the case of a constant and self-adjoint operator A.

THEOREM 2.6 *Let $A = A^* > 0$ be a constant operator. Then the condition*

$$E + (\sigma - 0.5)\tau A \geq 0 \tag{2.70}$$

is necessary and sufficient for the stability of scheme (2.68), (2.69) with respect to the initial data in the space H_A.

Proof. To prove this theorem it suffices to consider the identity

$$B - 0.5\tau A = E + (\sigma - 0.5)\tau A$$

and to apply Theorem 2.1.

The condition (2.70), is in fact, a restrictive condition imposed on the parameter σ. In particular, the inequality (2.70) is true when $\sigma \geq 0.5$. Since A is a positive operator this inequality for $\sigma < 0.5$ can be written in the form

$$A \leq \frac{1}{\tau(0.5-\sigma)} E \,.$$

Since A is a self-adjoint positive operator the latter inequality is equivalent to the estimate

$$\|A\| \leq \frac{1}{\tau(0.5-\sigma)} \,,$$

which can be rewritten as

$$\sigma \geq \frac{1}{2} - \frac{1}{\tau\|A\|} \,. \tag{2.71}$$

So, we have seen that the necessary and sufficient condition for the stability of scheme (2.68), (2.69) in the space H_A for the case of a self-adjoint positive operator A can be written in the form (2.71).

Let us consider a more general class of two-level operator-difference schemes with constant weights

$$Dy_t + Ay_n{}^{(\sigma)} = 0, \quad t_n \in \omega_\tau, \tag{2.72}$$

$$y_0 = u_0. \tag{2.73}$$

In this case the scheme (2.72) can be reduced to the canonical form with $B = D + \sigma\tau A$.

Note that when $v = D^{1/2}y$, where D is a self-adjoint positive operator, the problem (2.72), (2.73) can be transformed into the form

$$v_t + \widetilde{A}v_n{}^{(\sigma)} = 0, \quad t_n \in \omega_\tau, \tag{2.74}$$

$$v_0 = \widetilde{u}_0. \tag{2.75}$$

Here $\widetilde{A} = D^{-1/2}AD^{-1/2}$, $\widetilde{u}_0 = D^{1/2}u_0$. Thus by substituting $v = D^{1/2}y$ the scheme (2.72), (2.73) with respect to v can be reduced to the above case (2.68), (2.69). On the basis of Theorem 2.6 we conclude that the condition

$$E + (\sigma - 0.5)\tau D^{-1/2}AD^{-1/2} \geq 0 \tag{2.76}$$

or

$$\sigma \geq \frac{1}{2} - \frac{1}{\tau\|D^{-1/2}AD^{-1/2}\|} \tag{2.77}$$

is a necessary and sufficient condition for the stability of scheme (2.74), (2.75) in the space $H_{\widetilde{A}}$. As $\|v\|_{\widetilde{A}} = \|y\|_A$, the next theorem follows from the above considerations.

THEOREM 2.7 *Let the operators $A = A^* > 0$, $D = D^* > 0$ in the difference scheme (2.72), (2.73) be constant self-adjoint and positive. Then the operator inequality (2.76) is a necessary and sufficient condition for the stability of the scheme in the space H_A, i.e., for the satisfaction of the estimate*

$$\|y_n\|_A \leq \|u_0\|_A, \quad n = 0, 1, ..., N_0. \tag{2.78}$$

We now turn to obtaining *a priori* estimates of stability in the case when the operator A is not self-adjoint.

THEOREM 2.8 *Assume that $A(t) \geq 0$ is a non-negative operator and $D = D^* > 0$ is a constant operator. Then the scheme (2.72), (2.73) is stable with respect to the initial data in the space H_D with constant $M_1 = 1$ provided that*

$$\sigma \geq 0.5. \tag{2.79}$$

Proof. Taking the inner product of equation (2.72) and $2\tau y^{(\sigma)}$ we obtain the identity

$$2\tau \left(Dy_t, y^{(\sigma)}\right) + 2\tau \left(Ay^{(\sigma)}, y^{(\sigma)}\right) = 0. \tag{2.80}$$

Hence using the formula

$$y^{(\sigma)} = y^{(0.5)} + \tau(\sigma - 0.5)y_t \tag{2.81}$$

we come to

$$(D(\hat{y}-y), \hat{y}+y) + 2\tau^2(\sigma - 0.5)\|y_t\|_D^2 + 2\tau\left(Ay^{(\sigma)}, y^{(\sigma)}\right) = 0. \tag{2.82}$$

As the operator D is positive and self-adjoint, and the operator A is non-negative, then from formula (2.82), provided $\sigma \geq 0.5$ we obtain, taking account of the identity (2.31), that

$$\|y_{n+1}\|_D \leq \|y_n\|_D \leq ... \leq \|y_0\|_D. \tag{2.83}$$

We present below more several results concerning stability of schemes with weights.

THEOREM 2.9 *Let A and D be constant operators which satisfy the conditions*

$$D = D^* > 0, \quad A > 0. \tag{2.84}$$

Then the following operator inequality

$$DA^{-1}D + (\sigma - 0.5)\tau D \geq 0 \tag{2.85}$$

is necessary and sufficient for the stability of scheme (2.72), (2.73) in the space H_D with constant $\rho = 1$, i.e., for the estimate

$$\|y_{n+1}\|_D \leq \|y_n\|_D, \quad n = 0, 1, ..., N_0 - 1, \tag{2.86}$$

to be satisfied.

Proof. We rewrite the difference equation (2.72) in the form

$$\widetilde{B} y_t + \widetilde{A} y = 0, \tag{2.87}$$

where $\widetilde{A} = D$, $\widetilde{B} = DA^{-1}(D + \sigma\tau A) = DA^{-1}D + \sigma\tau D$. In this case the necessary and sufficient condition for the stability of scheme (2.87) in $H_{\widetilde{A}}$ formulated as the operator inequality $\widetilde{B} \geq 0.5\tau\widetilde{A}$ takes the form

$$DA^{-1}D + \sigma\tau D \geq 0.5\tau D,$$

which coincides with the required condition (2.85).

THEOREM 2.10 *Let the inequalities (2.84) hold. Then the condition*

$$\widetilde{A}^* + (\sigma - 0.5)\tau\widetilde{A}^*\widetilde{A} \geq 0, \tag{2.88}$$

where $\widetilde{A} = D^{-1/2}AD^{-1/2}$ is necessary and sufficient for the stability of scheme (2.72), (2.73) in the space $H_{\widetilde{A}^*\widetilde{A}}$, i.e., for the following estimate to be valid:

$$\|Ay_{n+1}\|_{D^{-1}} \leq \|Ay_n\|_{D^{-1}}, \quad n = 0, 1, ..., N_0 - 1. \tag{2.89}$$

Proof. Multiplying (2.74) by the positive operator $\widetilde{A}^* > 0$ we come to the equation

$$\bar{B}v_t + \bar{A}v_n = 0, \quad t_n \in \omega_\tau, \tag{2.90}$$

where $\bar{A} = \widetilde{A}^*\widetilde{A} = \bar{A}^* > 0$, $\bar{B} = \widetilde{A}^* + \sigma\tau\bar{A} > 0$. In this case the necessary and sufficient condition for the stability of scheme (2.90), (2.75) in the space $H_{\bar{A}}$ $(\bar{B} \geq 0.5\tau\bar{A})$ becomes of the following kind $\widetilde{A}^* + (\sigma - 0.5)\tau\widetilde{A}^*\widetilde{A} \geq 0$. This inequality coincides with condition (2.88) of the theorem.

Let us give the corresponding estimates in the case of variable operators $D(t)$ and $A(t)$.

THEOREM 2.11 *Let $A(t) = A^*(t) > 0$ be a variable positive and self-adjoint operator for all $t \in \omega_\tau$. Then provided that $\sigma \geq 0.5$ and the operator $A(t)$ satisfies the Lipschitz continuity condition (2.36), the difference scheme (2.68), (2.69) is ρ-stable with respect to the initial data with $\rho = e^{0.5c_0\tau}$ $\left(M_1 = e^{0.5c_0T}\right)$ and the following estimates hold:*

$$\|y_{n+1}\|_{A_n} \leq \rho\|y_n\|_{A_{n-1}}, \quad n = 1, 2, \ldots, N_0 - 1, \tag{2.91}$$

$$\|y_{n+1}\|_{A_n} \leq M_1 \|u_0\|_{A_0}, \quad n = 0, 1, \dots, N_0 - 1. \tag{2.92}$$

Proof. Notice than for $\sigma \geq 0.5$ the inequality $B(t) \geq 0.5\tau A(t)$ holds because of (2.70). Hence by Theorem 2.2 the *a priori* estimates (2.91), (2.92) are valid.

THEOREM 2.12 *Let the operators $A(t)$, $D \neq D(t)$ be self-adjoint positive, and also the operator $\widetilde{A}(t) = D^{-1/2}AD^{-1/2}$ be Lipschitz continuous with constant c_1. Then provided $\sigma \geq 0.5$ the difference scheme (2.72), (2.73) is ρ-stable with respect to the initial data in the space H_A with $\rho = e^{0.5c_1\tau}$, and the estimate (2.91) holds.*

Proof. Since the conditions of Theorem 2.2 hold for scheme (2.74), then on the strength of *a priori* estimates (2.39), (2.40) we draw a conclusion that

$$\|v_{n+1}\|_{\widetilde{A}_n} \leq \rho \|v_n\|_{\widetilde{A}_{n-1}}. \tag{2.93}$$

Because of the accepted notations $v_n = D^{1/2}y_n$, $\widetilde{A}_n = D^{-1/2}A_nD^{-1/2}$, the required inequality (2.91) follows at once from (2.93).

THEOREM 2.13 *Let $A(t) \geq 0$ be a non-negative operator and $D(t) = D^*(t) > 0$ be a Lipschitz continuous operator with constant c_2. If also $\sigma \geq 0.5$, then the scheme (2.72), (2.73) is ρ-stable with respect to the initial data in the space H_D with $\rho = e^{0.5c_2\tau}$ and hence the estimate*

$$\|y_{n+1}\|_{D_n} \leq \rho \|y_n\|_{D_{n-1}}, \quad n = 1, 2, ..., N_0 - 1, \tag{2.94}$$

is valid.

Proof. Similarly to estimate (2.83) we deduce the inequality

$$\|y_{n+1}\|_{D_n} \leq \|y_n\|_{D_n}. \tag{2.95}$$

Taking into account inequalities (2.42) we estimate the expression $\|y_n\|_{D_n}$:

$$(D_n y_n, y_n) \leq (1 + \tau c_2) \|y_n\|^2_{D_{n-1}} \leq \rho^2 \|y_n\|^2_{D_{n-1}}.$$

Hence

$$\|y_n\|_{D_n} \leq \rho \|y_n\|_{D_{n-1}}. \tag{2.96}$$

Substituting the above estimate into (2.95) we come to the required statement of the theorem.

4. Stability with Respect to the Right Hand Side

The analysis of accuracy of difference schemes is based on the corresponding estimates for stability of operator-difference schemes with respect to right hand sides. In this subsection we give few *a priori* estimates of stability with respect to the right hand side in different norms for two-level operator-difference schemes.

4.1 Elementary Estimates in H_A, H_B

Let us show that the stability with respect to the initial data in H_D infers the stability with respect to the right hand side when using the norm $\|\varphi\|_* = \|B^{-1}\varphi\|_D$.

Consider the two-level difference scheme in the canonical form

$$B\frac{y_{n+1}-y_n}{\tau} + Ay_n = \varphi_n, \quad t_n \in \omega_\tau, \tag{2.97}$$

$$y_0 = u_0. \tag{2.98}$$

THEOREM 2.14 *Let the homogeneous scheme (2.97), (2.98) with $\varphi_n = 0$ be ρ-stable in H_D, i.e., the estimates*

$$\begin{gathered} \|y_{n+1}\|_D \le \rho\|y_n\|_D, \quad n = 0, 1, ..., N_0 - 1. \\ \|y_{n+1}\|_D \le M_1\|y_0\|_D, \quad M_1 = \rho^{N_0} = \text{const} \end{gathered} \tag{2.99}$$

are valid. Then the inhomogeneous difference scheme is stable with respect to the right hand side, and the solution of problem (2.97), (2.98) satisfies the a priori estimate

$$\|y_{n+1}\|_D \le \rho^{n+1}\|y_0\|_D + \sum_{k=0}^{n} \tau\rho^{n-k}\|B^{-1}\varphi_k\|_D. \tag{2.100}$$

Proof. Since the operator B^{-1} exists, then equation (2.97) can be written in the form

$$y_{n+1} = Sy_n + \tau\widetilde{\varphi}_n, \quad S = E - \tau B^{-1}A, \quad \widetilde{\varphi}_n = B^{-1}\varphi_n. \tag{2.101}$$

By assumption, $D = D^* > 0$ is a self-adjoint positive operator. Applying the triangle inequality, from equation (2.101) we deduce the inequality

$$\|y_{n+1}\|_D \le \|Sy_n\|_D + \tau\|B^{-1}\varphi_n\|_D. \tag{2.102}$$

In Subsection 2.2.3 we observed that the condition of ρ-stability with respect to the initial data is equivalent to the boundedness of the transition operator norm, i.e.,

$$\|Sy_n\|_D \leq \rho\|y_n\|_D, \quad \rho^{N_0} = M_1.$$

Then from the expression (2.102), by taking account of the last inequality, we find that

$$\|y_{n+1}\|_D \leq \rho\|y_n\|_D + \tau\|B^{-1}\varphi_n\|_D.$$

Hence applying the discrete Gronwall lemma, according to (2.22), we come to the required estimate (2.100). This gives us the stability of the scheme with respect to both initial data and right hand side. In particular, if $D = A$ or $D = B$ (for $A = A^* > 0$ or $B = B^* > 0$) then from the inequality (2.100) we would obtain the simplest estimates of stability in the energy space H_A or H_B.

REMARK 2.5 If $D = D^* > 0$ is a constant operator, $M_1 = 1$, then the estimate (2.100) can be rewritten as

$$\|y_{n+1}\|_D \leq \|y_0\|_D + \sum_{k=0}^{n} \tau\|B^{-1}\varphi_k\|_D.$$

THEOREM 2.15 *Let the constant operators B, A satisfy the conditions:*

$$B = B^*, \quad A = A^* > 0. \tag{2.103}$$

Then provided that

$$B \geq 0.5\tau A \tag{2.104}$$

the solution of problem (2.97), (2.98) satisfies the following a priori estimate:

$$((B - 0.5\tau A)\, y_{n+1}, y_{n+1}) \leq ((B - 0.5\tau A)\, y_0, y_0) + \frac{1}{2}\sum_{k=0}^{n} \tau\|\varphi_k\|^2_{A^{-1}}. \tag{2.105}$$

Proof. Rewrite the equation (2.97) in the form

$$(B - 0.5\tau A)\, y_t + Ay^{(0.5)} = \varphi. \tag{2.106}$$

Considering the inner product of equation (2.106) and $2\tau y^{(0.5)}$, and recalling that the operator $R = B - 0.5\tau A$ is self-adjoint, we obtain the energy identity

$$(Ry_{n+1}, y_{n+1}) + 2\tau\|y^{(0.5)}\|^2_A = (Ry_n, y_n) + 2\tau\left(\varphi, y^{(0.5)}\right). \tag{2.107}$$

We estimate the right hand side of the last expression by using the generalized Cauchy–Bunyakovskii–Schwarz inequality and the ε-inequality as follows

$$2\tau\left(\varphi, y^{(0.5)}\right) \leq 2\tau\|\varphi\|_{A^{-1}}\|y^{(0.5)}\|_A \leq 2\tau\varepsilon_1\|y^{(0.5)}\|_A^2 + \frac{\tau}{2\varepsilon_1}\|\varphi\|_{A^{-1}}^2. \tag{2.108}$$

Substituting estimate (2.108) for $\varepsilon_1 = 1$ into the above identity (2.107), we come to the required inequality

$$(Ry_{n+1}, y_{n+1}) \leq (Ry_n, y_n) + \frac{\tau}{2}\|\varphi_n\|_{A^{-1}}^2 \leq \dots$$
$$\leq (Ry_0, y_0) + \frac{1}{2}\sum_{k=0}^{n}\tau\|\varphi_k\|_{A^{-1}}^2.$$

REMARK 2.6 If the strict inequality

$$B > 0.5\tau A \tag{2.109}$$

is valid instead of (2.104), then the estimate (2.105) means that the solution of the two-level difference scheme (2.97), (2.98) is stable with respect to both initial data and right hand side in the norm $H_{B-0.5\tau A}$.

4.2 Splitting off the Stationary Non-Homogeneity

The disadvantage of estimate (2.100) consists in difficulties of finding the right hand side norm. To obtain estimates in simpler norms, the *method of splitting off stationary non-homogeneities* [Samarskii and Goolin, 1973] is used.

THEOREM 2.16 *Let the following condition be satisfied:*

$$A = A^* > 0 \ \text{ is a constant operator.} \tag{2.110}$$

Then under the condition

$$B(t) \geq 0.5\tau A \tag{2.111}$$

the difference scheme (2.97), (2.98) is stable with respect to the right hand side, and the solution of the problem satisfies the following a priori estimate

$$\|y_{n+1}\|_A \leq \|y_0\|_A + \|\varphi_0\|_{A^{-1}} + \|\varphi_n\|_{A^{-1}} + \sum_{k=1}^{n}\tau\|\varphi_{\bar{t},k}\|_{A^{-1}}. \tag{2.112}$$

P r o o f. We represent the solution of problem (2.97), (2.98) as a sum

$$y_n = v_n + \omega_n, \tag{2.113}$$

where $\omega_n = \omega(t_n)$ is a solution of the stationary equation

$$A\omega_{n+1} = \varphi_n, \quad n = 0, 1, ..., \quad \omega(0) = \omega(\tau). \tag{2.114}$$

Substituting the expressions (2.113) and (2.114) into equation (2.97), we come to the following problem for $v(t)$:

$$B(t)v_t + Av = \widetilde{\varphi}_n, \quad v_0 = y_0 - \omega_0, \tag{2.115}$$

where $\widetilde{\varphi}_n = -(B - \tau A)\omega_{n,t}$, $\widetilde{\varphi}_0 = 0$.

For the solution of this auxiliary problem (2.115), in accordance with Remark 2.5 we have the estimate

$$\|v_{n+1}\|_A \le \|v_0\|_A + \sum_{k=0}^{n} \tau \|B^{-1}\widetilde{\varphi}_k\|_A. \tag{2.116}$$

Let us estimate now the term $\|B^{-1}\widetilde{\varphi}_k\|_A = \|A^{1/2}B^{-1}\widetilde{\varphi}_k\|$ in the inequality (2.116) taking into account that

$$\omega_t = A^{-1}\varphi_{\bar{t}} \quad \text{and} \quad \|A^{1/2}B^{-1}\widetilde{\varphi}\| = \|\,(E - \tau C)\,A^{-1/2}\varphi_{\bar{t}}\|,$$

where $C = A^{1/2}B^{-1}A^{1/2}$. Because of the assumption (2.111), the condition of Theorem 2.5 (see (2.66)) with $\rho = 1$ holds. Consequently (see Remark 2.4), the estimate

$$\|E - \tau C\| \le \rho \tag{2.117}$$

with $\rho = 1$ is valid. From (2.117) we find that

$$\|A^{1/2}B^{-1}\widetilde{\varphi}\| \le \|A^{-1/2}\varphi_{\bar{t}}\| = \|\varphi_{\bar{t}}\|_{A^{-1}},$$

where $\|\varphi_{\bar{t}}\|_{A^{-1}} = \sqrt{(A^{-1}\varphi_{\bar{t}}, \varphi_{\bar{t}})}$. Therefore the solution of problem (2.115) satisfies the estimate

$$\|v_{n+1}\|_A \le \|v_0\|_A + \sum_{k=1}^{n} \tau \|\varphi_{\bar{t},k}\|_{A^{-1}},$$

from which, taking into account the inequality

$$\|v_0\|_A \le \|y_0\|_A + \|\omega_0\|_A = \|y_0\|_A + \|\varphi_0\|_{A^{-1}},$$

we deduce in turn

$$\|v_{n+1}\|_A \leq \|y_0\|_A + \|\varphi_0\|_{A^{-1}} + \sum_{k=1}^{n} \tau \|\varphi_{\bar{t},k}\|_{A^{-1}}.$$

Finally, using the triangle inequality we obtain from (2.113) the estimate that was to be proved $\left(\sum\limits_{k=1}^{n} a_k = 0, \text{ where } n < k \right)$.

REMARK 2.7 Comparing the *a priori* estimates (2.105) and (2.112), which show stability with respect to the right hand side in the negative norm $\|\cdot\|_{A^{-1}}$, we see that the last case requires stronger conditions on the right hand side. Therefore an immediate consequence of (2.112) is the stability result in the much stronger norm H_A.

The estimate similar to (2.112) holds also in the case of a variable operator $A(t)$.

THEOREM 2.17 *Let the operator $A(t)$ in scheme (2.97), (2.98) be self-adjoint, positive and satisfy the Lipschitz condition (2.36). Then if the stability condition (2.111) is satisfied the solution of the problem satisfies the estimate*

$$\|y_{n+1}\|_{A_n} \leq M\left(\|y_0\|_{A_0} + \max_{0 \leq k \leq n} \left(\|\varphi_k\|_{A_k^{-1}} + \|\varphi_{\bar{t},k}\|_{A_k^{-1}} \right) \right), \tag{2.118}$$

where M is a positive constant, $\varphi_{-1} = \varphi_0$.

Proof. At first sight, to prove this theorem one could again use the method of splitting off stationary non-homogeneities. However, in our opinion, the proof using directly the method of energy inequalities is easier. For this purpose we rewrite equation (2.97) in the form

$$(B_n - 0.5\tau A_n)\, y_{t,n} + A_n y_n^{(0.5)} = \varphi_n. \tag{2.119}$$

Multiplying equation (2.119) by $2\tau y_t$ (in the sense of taking the inner product) and taking into account the stability condition (2.111) from the hypotheses of the theorem, by the equality (2.31) we obtain the energy expression

$$\|y_{n+1}\|_{A_n}^2 \leq \|y_n\|_{A_n}^2 + 2\tau\,(y_t, \varphi)\,. \tag{2.120}$$

Hence by virtue of the Lipschitz continuity condition and the inequality (2.42), we have

$$\|y_{n+1}\|_{A_n}^2 \leq (1 + \tau c_0)\, \|y_n\|_{A_{n-1}}^2 + 2\tau\,(y_t, \varphi)\,. \tag{2.121}$$

As the operator A is self-adjoint, then by introducing the notations

$$v_n = A_{n-1}^{1/2} y_n, \quad \widetilde{\varphi}_n = A_n^{-1/2}\varphi_n, \quad \widetilde{\varphi}_{\bar{t},n} = A_{n-1}^{-1/2}\varphi_{\bar{t},n}$$

and taking into account the obvious identities

$$2\tau\left(y_t, \varphi\right) = 2\left(\hat{v}, \widetilde{\varphi}\right) - 2\left(v, \check{\widetilde{\varphi}}\right) - 2\tau\left(v, \widetilde{\varphi}_{\bar{t}}\right),$$

$$\|\widetilde{\varphi}\|^2 - \|\check{\widetilde{\varphi}}\|^2 = 2\tau\left(\check{\widetilde{\varphi}}^{(0.5)}, \widetilde{\varphi}_{\bar{t}}\right),$$

we can rewrite the inequality (2.121) in the form

$$\|z_{n+1}\|^2 \le \|z_n\|^2 + \tau c_0\|v_n\|^2 + 2\tau\left(\widetilde{\varphi}_{n-1}^{(0.5)} - v_n, \widetilde{\varphi}_{\bar{t},n}\right), \qquad (2.122)$$

where $z_n = v_n - \widetilde{\varphi}_{n-1}$. By virtue of the Cauchy inequality with $\varepsilon = 1$

$$\begin{aligned}
\tau c_0\|v\|^2 &\le 2\tau c_0\|z\|^2 + 2\tau c_0\|\check{\widetilde{\varphi}}\|^2, \\
-2\tau\left(v, \widetilde{\varphi}_{\bar{t}}\right) &\le \tau\|v\|^2 + \tau\|\widetilde{\varphi}_{\bar{t}}\|^2 \\
&\le 2\tau\|z\|^2 + 2\tau\|\check{\widetilde{\varphi}}\|^2 + \tau\|\widetilde{\varphi}_{\bar{t}}\|^2, \\
2\tau\left(\check{\widetilde{\varphi}}^{(0.5)}, \widetilde{\varphi}_{\bar{t}}\right) &\le \tau\|\check{\widetilde{\varphi}}^{(0.5)}\|^2 + \tau\|\widetilde{\varphi}_{\bar{t}}\|^2,
\end{aligned}$$

we have

$$\|z_{n+1}\|^2 \le \left(1 + 2\tau c_1\right)\|z_n\|^2 + \tau\|\varphi_n\|_*^2, \qquad (2.123)$$

where $c_1 = c_0 + 1$ and

$$\|\varphi_n\|_*^2 = 2c_1\|\varphi_{n-1}\|_{A_{n-1}^{-1}}^2 + \|\varphi_{n-1}^{(0.5)}\|_{A_{n-1}^{-1}}^2 + 2\|\varphi_{\bar{t},n-1}\|_{A_{n-1}^{-1}}^2. \qquad (2.124)$$

Taking into account that $1 + 2\tau c_1 \le \rho^2$, $\rho = e^{\tau c_1}$, we reduce the inequality (2.123) to the form

$$\|z_{n+1}\|^2 \le \rho^2\|z_n\|^2 + \tau\|\varphi_n\|_*^2.$$

Hence we find the estimate

$$\|z_{n+1}\|^2 \le \rho^{2n}\|z_1\|^2 + \sum_{k=1}^{n}\tau\rho^{2(n-k)}\|\varphi_k\|_*^2$$

or

$$\|z_{n+1}\| \le \rho^n\left(\|z_1\| + \sqrt{t_n}\max_{1\le k\le n}\|\varphi_k\|_*\right). \qquad (2.125)$$

To obtain the estimate for the required grid function y_n we make use of the following inequalities

$$\begin{aligned}
\|z_{n+1}\| &\ge \|v_{n+1}\| - \|\widetilde{\varphi}_n\| = \|y_{n+1}\|_{A_n} - \|\varphi_n\|_{A_n^{-1}}, \\
\|z_1\| &\le \|y_1\|_{A_0} + \|\varphi_0\|_{A_0^{-1}}.
\end{aligned}$$

To estimate the term $\|y_1\|_{A_0}$ we use inequality (2.120) for $n = 0$. As a result we obtain

$$\|y_1\|_{A_0}^2 \leq \|y_0\|_{A_0}^2 + 2\,(y_1 - y_0, \varphi_0)\,. \tag{2.126}$$

Using the Cauchy–Bunyakovskii–Schwarz inequality with ε, we estimate the inner product in (2.126) as

$$2\,(y_1 - y_0, \varphi_0) \leq \frac{1}{4}\|y_1 - y_0\|_{A_0}^2 + 4\|\varphi_0\|_{A_0^{-1}}^2\,. \tag{2.127}$$

Taking this estimate (2.127) into account, from inequality (2.126) we deduce

$$\|y_1\|_{A_0} \leq \sqrt{3}\|y_0\|_{A_0} + 2\sqrt{2}\|\varphi_0\|_{A_0^{-1}}.$$

Substituting the estimates obtained above into (2.125), we come to the inequality

$$\begin{aligned}\|y_{n+1}\|_{A_n} \leq \|\varphi_n\|_{A_n^{-1}} + \rho^n(\|\varphi_0\|_{A_0^{-1}} &+ \sqrt{3}\|y_0\|_{A_0} \\ &+ 2\sqrt{2}\|\varphi_0\|_{A_0^{-1}} + \sqrt{t_n}\max_{1\leq k\leq n}\|\varphi_k\|_*).\end{aligned} \tag{2.128}$$

Hence because of the representation (2.124) the required estimate follows.

4.3 *A Priori* Estimates for Stability under More Severe Restrictions

Up to now we have used the stability condition of the kind

$$B \geq 0.5\tau A. \tag{2.129}$$

A few of the new *a priori* estimates for the non-homogeneous equation (2.97) can be obtained by the method of energy inequalities under stronger assumptions than (2.129).

THEOREM 2.18 *Let the linear constant operators A, B satisfy the conditions*

$$B \geq \frac{1+\varepsilon}{2}\tau A, \quad B = B^*, \quad A = A^* > 0. \tag{2.130}$$

Then for the difference scheme (2.97), (2.98) the following a priori *estimate is valid:*

$$\|y_{n+1}\|_A^2 \leq \|y_0\|_A^2 + \frac{1+\varepsilon}{2\varepsilon}\sum_{k=0}^{n}\tau\|\varphi\|_{B^{-1}}^2, \tag{2.131}$$

where ε is a positive constant which is independent of τ.

Proof. Multiplying equation (2.119) by $2\tau y_t$, similarly to the inequality (2.120) we obtain the energy identity

$$2\tau\left((B-0.5\tau A)\,y_t, y_t\right) + (A\hat{y},\hat{y}) = (Ay,y) + 2\tau\,(\varphi, y_t)\,. \qquad (2.132)$$

Using the generalized Cauchy–Bunyakovskii–Schwarz inequality and the ε-inequality we estimate the last term in the right hand side of the previous expression as

$$2\tau\,(\varphi, y_t) \le 2\tau\|\varphi\|_{B^{-1}}\|y_t\|_B \le 2\tau\varepsilon_1\|y_t\|_B^2 + \frac{\tau}{2\varepsilon_1}\|\varphi\|_{B^{-1}}^2.$$

Substituting this estimate into the identity (2.132), we come to

$$2\tau\left(\left((1-\varepsilon_1)\,B - \frac{\tau}{2}A\right)y_t, y_t\right) + (A\hat{y},\hat{y}) \le (Ay,y) + \frac{\tau}{2\varepsilon_1}\|\varphi\|_{B^{-1}}^2.$$

If conditions (2.130) are satisfied, then one can choose ε_1 such that $1/(1-\varepsilon_1) = 1+\varepsilon$, that is, $\varepsilon_1 = \varepsilon/\,(1+\varepsilon)$. Then

$$(1-\varepsilon_1)\,B - \frac{\tau}{2}A = (1-\varepsilon_1)\left(B - \frac{1+\varepsilon}{2}\tau A\right) \ge 0$$

and

$$(Ay_{k+1}, y_{k+1}) \le (Ay_k, y_k) + \frac{1+\varepsilon}{2\varepsilon}\tau\|\varphi_k\|_{B^{-1}}^2\,.$$

Summing up over all $k = 0, 1, ..., n$ yields the estimate (2.131).

When we study convergence of a difference scheme, the initial data in the problem for the error of a method are homogeneous. In this connection the following theorem is very useful.

THEOREM 2.19 *Let the linear constant operators A, B satisfy conditions (2.130). Then for the solution of difference scheme (2.13), (2.14) with the homogeneous initial data*

$$B\frac{y_{n+1}-y_n}{\tau} + Ay_n = \varphi_n, \quad t_n \in \omega_\tau, \qquad (2.133)$$

$$y_0 = 0 \qquad (2.134)$$

the following estimates are valid:

$$\|y_{n+1}\|_B \le \sqrt{\frac{1+\varepsilon}{2\varepsilon}}\left(\sum_{k=0}^{n}\tau\|\varphi_k\|_{A^{-1}}^2\right)^{1/2}, \qquad (2.135)$$

$$\|y_{n+1}\|_B^2 + \varepsilon\sum_{k=1}^{n}\tau\|y_k\|_A^2 \le \frac{2}{\varepsilon(2-\varepsilon)}\sum_{k=0}^{n}\tau\|\varphi_k\|_{A^{-1}}^2, \quad \varepsilon < 2. \qquad (2.136)$$

Proof. Rewrite equation (2.133) in the form

$$(B - \tau A)\, y_t + A\hat{y} = \varphi$$

and consider its inner product with $2\tau\hat{y}$. Since B and A are self-adjoint constant operators the following equality is true:

$$2\tau\left((B - \tau A)\, y_t, \hat{y}\right) = \tau\left((B - \tau A)\, y, y\right)_t + \tau^2\left((B - \tau A)\, y_t, y_t\right),$$

and we obtain the identity

$$\begin{aligned}(B\hat{y}, \hat{y}) + \tau\,(A\hat{y}, \hat{y}) + \tau\,(Ay, y) + \tau^2\left((B - \tau A)\, y_t, y_t\right)&\\ = 2\tau\,(\varphi, \hat{y}) + (By, y)\,.&\end{aligned} \tag{2.137}$$

Hence taking into account the equality

$$(A\hat{y}, \hat{y}) + (Ay, y) = \frac{1}{2}\|\hat{y} + y\|_A^2 + \frac{\tau^2}{2}\|y_t\|_A^2,$$

we find

$$\begin{aligned}\|\hat{y}\|_B^2 + \tau^2\left((B - 0.5\tau A)\, y_t, y_t\right) + \frac{\tau}{2}\|y + \hat{y}\|_A^2&\\ = \|y\|_B^2 + 2\tau\,(\varphi, \hat{y})\,.&\end{aligned} \tag{2.138}$$

Let us estimate the term $2\tau\,(\varphi, \hat{y})$. As

$$\hat{y} = 0.5(y + \hat{y}) + 0.5\tau y_t,$$

then

$$\begin{aligned}2\tau(\varphi, \hat{y}) &= \tau(\varphi, y + \hat{y}) + \tau^2(\varphi, y_t)\\ &\le \frac{\tau\varepsilon_1}{2}\|y + \hat{y}\|_A^2 + + \frac{\tau}{2\varepsilon_1}\|\varphi\|_{A^{-1}}^2 + \frac{\tau^3\varepsilon_2}{2}\|y_t\|_A^2 + \frac{\tau}{2\varepsilon_2}\|\varphi\|_{A^{-1}}^2.\end{aligned}$$

Therefore from the identity (2.138) we obtain

$$\begin{aligned}\|y_{k+1}\|_B^2 + \tau^2\left(\left(B - \frac{(1+\varepsilon_2)\tau}{2}A\right) y_{t,k}, y_{t,k}\right)&\\ + \frac{\tau}{2}(1 - \varepsilon_1)\|y_k + y_{k+1}\|_A^2&\\ \le \|y_k\|_B^2 + \frac{\tau}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\|\varphi_k\|_{A^{-1}}^2.&\end{aligned} \tag{2.139}$$

Consider two ways for choosing the constants ε_1 and ε_2. If $\varepsilon_1 = 1$, $\varepsilon_2 = \varepsilon > 0$ and $B \ge \dfrac{1+\varepsilon}{2}\tau A$, then from (2.139) for the problem (2.133), (2.134) under consideration we obtain

$$\|y_{k+1}\|_B^2 \le \|y_k\|_B^2 + \frac{1+\varepsilon}{2\varepsilon}\tau\|\varphi_k\|_{A^{-1}}^2.$$

Summing up this inequality over all $k = 0, 1, ..., n$, we verify the validity of the first assertion (2.135).

If $\varepsilon_1 = 1 - \frac{\varepsilon}{2}$, $\varepsilon_2 = \frac{\varepsilon}{2} > 0$ and $B \geq \frac{1+\varepsilon}{2}\tau A$, then on the basis of the inequality

$$\tau^2\left(\left(B - \frac{1+\varepsilon_2}{2}\tau A\right)y_t, y_t\right) + \frac{1-\varepsilon_1}{2}\tau\|y+\hat{y}\|_A^2$$
$$\geq \frac{\tau^3\varepsilon}{4}\|y_t\|_A^2 + \frac{\tau\varepsilon}{4}\|y+\hat{y}\|_A^2$$
$$= \frac{\tau\varepsilon}{4}\left(\|y+\hat{y}\|_A^2 + \tau^2\|y_t\|_A^2\right)$$
$$= \frac{\tau\varepsilon}{2}\left(\|y\|_A^2 + \|\hat{y}\|_A^2\right)$$

we deduce from the estimate (2.139) that

$$\|y_{k+1}\|_B^2 + \frac{\tau\varepsilon}{2}\left(\|y_k\|_A^2 + \|y_{k+1}\|_A^2\right) \leq \|y_k\|_B^2 + \frac{2\tau}{\varepsilon(2-\varepsilon)}\|\varphi_k\|_{A^{-1}}^2.$$

Hence we come to the required inequality (2.136).

The estimates (2.135), (2.136) exhibit the stability of the scheme with respect to the right hand side in the space H_B.

COROLLARY 2.1 *From (2.136) it follows that*

$$\sum_{k=0}^{n}\tau\|y_k\|_A^2 \leq \frac{2}{\varepsilon(2-\varepsilon)}\sum_{k=0}^{n}\tau\|\varphi_k\|_{A^{-1}}^2, \quad 0 < \varepsilon < 2. \tag{2.140}$$

By analogy with the space H we introduce the Hilbert space H^{n+1} consisting of the elements

$$Y = \{y_0, y_1, ..., y_n\}, \quad V = \{v_0, v_1, ..., v_n\};$$
$$y_k, v_k \in H, \quad k = 0, 1, ..., n.$$

For the elements of H^{n+1} we define an inner product as

$$(Y, V) = \sum_{k=0}^{n}\tau(y_k, v_k),$$

where (y_k, v_k) is the inner product in H, and a norm as

$$\|Y\| = (Y,Y)^{1/2} = \sqrt{\sum_{k=0}^{n}\tau\|y_k\|^2},$$

where $\|y_k\|$ is the norm in H.

By H_D^{n+1}, where $D = D^* > 0$ is a self-adjoint positive operator, we denote a Hilbert space consisting of the elements of the space H^{n+1} provided with the inner product

$$(Y, V)_D = \sum_{k=0}^{n} \tau (y_k, v_k)_D,$$

where $(y_k, v_k)_D$ is the inner product in H_D, and the norm

$$\|Y\|_D = (Y, Y)_D^{1/2} = \sqrt{\sum_{k=0}^{n} \tau \|y_k\|_D^2},$$

where $\|y_k\|_D$ is the norm in H_D.

Assume that $\Phi = \{\varphi_0, \varphi_1, \ldots, \varphi_n\} \in H_{A^{-1}}^{n+1}$. Then the inequality (2.140) for $\varepsilon = 1$ can be rewritten in the form

$$\|Y\|_A \leq \sqrt{2}\|\Phi\|_{A^{-1}}. \tag{2.141}$$

The estimate (2.141) guarantees the stability of scheme (2.133), (2.134) with respect to the right hand side in the energy norm in H_A^{n+1}.

The following theorem gives conditions for stability with respect to the right hand side in the norm $\|\varphi\|_* = \|\varphi\|$.

THEOREM 2.20 *Let the linear constant operators A and B satisfy the conditions*

$$B \geq \varepsilon E + 0.5\tau A, \quad A = A^* > 0, \tag{2.142}$$

where ε is an arbitrary positive number. Then the following a priori *estimate is valid for problem (2.97), (2.98):*

$$\|y_{n+1}\|_A^2 \leq \|y_0\|_A^2 + \frac{1}{2\varepsilon}\sum_{k=0}^{n} \tau \|\varphi\|^2. \tag{2.143}$$

Proof. Let us turn to the identity (2.132). The Cauchy–Bunyakovskii–Schwarz inequality and the ε-inequality give

$$2\tau\,(\varphi, y_t) \leq 2\tau\|\varphi\|\,\|y_t\| \leq 2\tau\varepsilon\|y_t\|^2 + \frac{\tau}{2\varepsilon}\|\varphi\|^2. \tag{2.144}$$

Substituting this estimate into the energy identity (2.132) and using condition (2.142), we obtain

$$\|y_{n+1}\|_A^2 \leq \|y_n\|_A^2 + \frac{\tau}{2\varepsilon}\|\varphi_n\|^2 \leq \ldots \leq \|y_0\|_A^2 + \frac{1}{2\varepsilon}\sum_{k=0}^{n} \tau \|\varphi_k\|^2.$$

REMARK 2.8 Theorem 2.20 remains valid also in the case of a variable operator $B = B(t)$. This fact is obvious from its proof.

REMARK 2.9 If $A = A^*$, $B = B^* > 0$ are constant operators and conditions (2.66) are satisfied, then the scheme (2.97), (2.98) is ρ-stable with respect to the initial data in H_B, i.e., the solution of problem (2.10), (2.11) satisfies the inequality

$$\|y_n\|_B \le \rho \|y_0\|_B.$$

Hence by Theorem 2.14 the estimate

$$\|y_{n+1}\|_B \le \rho^{n+1}\|y_0\|_B + \sum_{k=0}^{n} \tau \rho^{n-k} \|\varphi_k\|_{B^{-1}}$$

follows. Note that this does not require positivity of the operator A. For example, assume that the operator A satisfies the condition

$$A \ge -c_* E, \quad c_* > 0.$$

Then the relation

$$A \ge \frac{1-\rho}{\tau} B \quad \text{or} \quad A + \frac{\rho - 1}{\tau} B \ge 0$$

can be realized only if $\rho = e^{c_0 \tau} > 1$, i.e., if $c_0 \ge 0$.

Assume that

$$B \ge \varepsilon E, \quad \varepsilon > 0.$$

Then the inequality

$$A + \frac{\rho - 1}{\tau} B \ge A + c_0 B \ge (-c_* + c_0 \varepsilon)\, E$$

is valid, and consequently $A \ge \dfrac{1-\rho}{\tau} B$ if we take $c_0 \ge \dfrac{c_*}{\varepsilon}$. In particular, one can choose B as an operator of the kind

$$B = E + \tau R, \quad R = 0.5 A', \quad A' = A + c_* E \ge 0.$$

Then $\varepsilon = 1$, $c_0 = c_*$, $\rho = e^{c_* \tau}$.

Let us formulate the counterpart of Theorem 2.20 for the case of variable operators $B(t)$, $A(t)$ which satisfy the conditions

$$B(t) > 0, \quad A(t) = A^*(t) > 0, \quad t \in \omega_\tau, \tag{2.145}$$

$$\begin{gathered} |\left((A(t) - A(t-\tau))\, v, v\right)| \le \tau c_0 \left(A(t-\tau) v, v\right), \\ v \in H, \quad t \in \omega_\tau, \end{gathered} \tag{2.146}$$

$$B(t) \geq \varepsilon E + 0.5\tau A, \quad t \in \omega_\tau, \quad 0 < \varepsilon \leq 1. \tag{2.147}$$

THEOREM 2.21 *Let $A = A(t)$ and $B = B(t)$ depend on t and conditions (2.145)–(2.147) be satisfied. Then the following estimate is valid for scheme (2.97), (2.98):*

$$\|y_{n+1}\|^2_{A_n} \leq M_1^2 \left\{ \|y_0\|^2_{A_0} + \frac{1}{2\varepsilon} \sum_{k=0}^{n} \tau \|\varphi_k\|^2 \right\}, \tag{2.148}$$

where $M_1 = e^{0.5c_0 T}$.

P r o o f. The energy identity (2.132) with $A = A(t)$ serves as a starting point of our analysis. Substituting estimate (2.42) in this identity we obtain the inequality

$$2\tau \left(\left(B(t) - 0.5\tau A(t) \right) y_t, y_t \right) + Q_{n+1} \leq e^{c_0 \tau} Q_n + 2\tau (\varphi_n, y_t), \tag{2.149}$$

where we use the notation

$$Q_{n+1} = (A_n y_{n+1}, y_{n+1}) = \|y_{n+1}\|^2_{A_n}.$$

The energy identity for $t = 0$ can be rewritten in the form

$$\begin{aligned} 2\tau((B(0) - 0.5\tau A(0)) y_t, y_t) + Q_1 \\ = \|y(0)\|^2_{A(0)} + 2\tau \left(\varphi(0), y_t(0) \right). \end{aligned} \tag{2.150}$$

Substituting the estimate for the right hand side of (2.144) into inequality (2.149), we come to the relation

$$2\tau \left(\left(B(t) - (\varepsilon E + 0.5\tau A(t)) \right) y_t, y_t \right) + Q_{n+1} \leq e^{c_0 \tau} Q_n + \frac{\tau}{2\varepsilon} \|\varphi_n\|^2.$$

Hence by the assumption (2.147) the required estimate (2.148) follows.

From above arguments we observe that the case of variable operators differs only fundamentally from the case of constant ones.

4.4 Schemes with Weights

Let us show how the theorems proved above can be applied, for instance, to the weighted scheme

$$y_t + A y^{(\sigma)} = \varphi, \quad y_0 = u_0. \tag{2.151}$$

Using the identity (2.67), scheme (2.151) can be reduced to the canonical form (2.10) as

$$(E + \sigma\tau A)\, y_t + Ay = \varphi, \quad y_0 = u_0 \tag{2.152}$$

with $B = E + \sigma\tau A$. Let the operator A^{-1} exist. Applying A^{-1} to equation (2.152) we obtain the second canonical form of the weighted scheme

$$\widetilde{B}y_t + \widetilde{A}y = \widetilde{\varphi}, \quad y(0) = y_0, \quad \widetilde{B} = A^{-1} + \sigma\tau E,$$
$$\widetilde{A} = E, \quad \widetilde{\varphi} = A^{-1}\varphi. \tag{2.153}$$

We shall use problem (2.152) in the case of a self-adjoint operator A. For a non-self-adjoint positive operator $A = A(t)$ we shall take into consideration problem (2.153).

First, consider the case when A is a constant, self-adjoint and positive operator, i.e.,

$$A = A^* > 0.$$

In Subsection 2.2.4 we have shown that in this case the necessary and sufficient condition of stability with respect to the initial data for the weighted scheme (2.151) takes the form

$$\sigma \geq \sigma_0, \quad \sigma_0 = \frac{1}{2} - \frac{1}{\tau\|A\|}.$$

Under this condition the estimate (2.27) is valid for problem (2.152). In particular, for an explicit scheme (with $\sigma = 0$) the condition $\sigma \geq \sigma_0$ implies $\tau \leq 2/\|A\|$, i.e., the explicit scheme is stable in H_A provided that $\tau \leq 2/\|A\|$. The scheme with $\sigma \geq 0.5$ is unconditionally (i.e., for any τ) stable.

THEOREM 2.22 *Let A be a self-adjoint positive operator which is independent of $t = n\tau$, i.e., $A = A^* > 0$. Then the following estimates hold for the weigthed scheme (2.152): namely, estimate (2.112) for $\sigma \geq \sigma_0$, estimate (2.131) for*

$$\sigma \geq \frac{1+\varepsilon}{2} - \frac{1}{\tau\|A\|}, \quad \varepsilon > 0,$$

and estimate (2.143) for

$$\sigma \geq \sigma_\varepsilon, \quad \sigma_\varepsilon = 0.5 - (1-\varepsilon)/\left(\tau\|A\|\right), \quad 0 < \varepsilon \leq 1.$$

Here ε is a positive constant independent of τ.

Proof. It is sufficient to verify by using the corresponding inequality whether the conditions of Theorems 2.16, 2.18, 2.20 are satisfied. For instance, we have for Theorem 2.16

$$B - 0.5\tau A = E + (\sigma - 0.5)\tau A \geq \left(1/\|A\| + (\sigma - 0.5)\tau\right) A.$$

Further considerations are obvious.

THEOREM 2.23 *Let $A = A^* > 0$ be a constant operator. Then, if*

$$\sigma \geq \frac{1+\varepsilon}{2} - \frac{1}{\tau\|A\|}, \quad \varepsilon = \text{const} > 0, \tag{2.154}$$

the following a priori *estimate is valid for the weighted scheme (2.151):*

$$\|y_{n+1}\|^2_{\widetilde{A}} \leq \|y_0\|^2_{\widetilde{A}} + \frac{1+\varepsilon}{2\varepsilon} \sum_{k=0}^{n} \tau\|\varphi_k\|^2, \tag{2.155}$$

where

$$\widetilde{A} = (E + \sigma\tau A)\,A, \quad \|y\|^2_{\widetilde{A}} = \|y\|^2_A + \tau\sigma\|Ay\|^2.$$

Proof. Multiplying equation (2.151) by the operator $B = E + \sigma\tau A$ we reduce it to the canonical form

$$\widetilde{B}y_t + \widetilde{A}y = \widetilde{\varphi}, \quad y(0) = y_0, \quad \widetilde{B} = B^2, \quad \widetilde{\varphi} = B\varphi. \tag{2.156}$$

Using Theorem 2.18 we conclude that for

$$\widetilde{B} \geq \frac{1+\varepsilon}{2}\tau\widetilde{A} \tag{2.157}$$

the following *a priori* estimate in $H_{\widetilde{A}}$ is valid:

$$\|y_{n+1}\|^2_{\widetilde{A}} \leq \|y_0\|^2_{\widetilde{A}} + \frac{1+\varepsilon}{2\varepsilon} \sum_{k=0}^{n} \tau\|\widetilde{\varphi}_k\|^2_{\widetilde{B}^{-1}}. \tag{2.158}$$

Since $\|\widetilde{\varphi}_k\|^2_{\widetilde{B}^{-1}} = \|\varphi_k\|^2$ the inequality (2.158) implies the required estimate (2.155). We only need to show that by the choice of σ from the inequality (2.154) the condition (2.157) is satisfied. Indeed, since A is a self-adjoint positive operator it follows that

$$\begin{aligned}
\widetilde{B} - 0.5\tau\widetilde{A} &= (E + \sigma\tau A)\left(E + \left(\sigma - \frac{1+\varepsilon}{2}\right)\tau A\right) \\
&\geq (E + \sigma\tau A)\left(\frac{1}{\|A\|} + \left(\sigma - \frac{1+\varepsilon}{2}\right)\tau\right)A \geq 0.
\end{aligned}$$

THEOREM 2.24 *Let $A = A^* > 0$ be a constant operator. Then if*

$$\sigma \geq \frac{1+\varepsilon}{2} - \frac{1}{\tau\|A\|}, \quad \varepsilon = \text{const} > 0, \tag{2.159}$$

the difference scheme (2.151) with weighted factors is stable in the norm $H_{A^{-1}+\tau\sigma E}$, *and the* a priori *estimate*

$$\|y_{n+1}\|^2_{A^{-1}} + \sigma\tau\|y_{n+1}\|^2 \le \|y_0\|^2_{A^{-1}} + \sigma\tau\|y_0\|^2 + \frac{1+\varepsilon}{2\varepsilon}\sum_{k=0}^{n}\tau\|A^{-1}\varphi\|^2 \tag{2.160}$$

is valid.

Proof. Acting by the operator A^{-1} onto the equation

$$(E+\tau\sigma A)y_t + Ay = \varphi$$

we obtain another canonical form of the two-level operator difference scheme (2.153) with operators

$$\widetilde{B} = A^{-1} + \sigma\tau E, \quad \widetilde{A} = E, \quad \widetilde{\varphi} = A^{-1}\varphi.$$

Then by multiplying equation (2.153) by the operator $\widetilde{B} = \widetilde{B}^* > 0$ we find one more canonical form

$$\approx{B}\, y_t + \approx{A}\, y = \approx{\varphi}, \quad y_0 = u_0, \tag{2.161}$$

where now

$$\approx{B} = \widetilde{B}^2 = (A^{-1}+\sigma\tau E)^2, \quad \approx{A} = A^{-1} + \sigma\tau E, \quad \approx{\varphi} = \widetilde{B}\widetilde{\varphi}.$$

It is obvious that $\approx{B} = \approx{B}*$, $\approx{A} = \approx{A}* > 0$. The task is now to verify the validity of the sufficient condition of stability in $H_{\approx{A}}$

$$\approx{B} \ge \frac{1+\varepsilon}{2}\tau\approx{A}. \tag{2.162}$$

Since for the self-adjoint operator $A^{-1} \ge \dfrac{1}{\|A\|}E$ and

$$\approx{B} - \frac{1+\varepsilon}{2}\tau\approx{A} = (A^{-1}+\sigma\tau E)\left(A^{-1} + \left(\sigma - \frac{1+\varepsilon}{2}\right)\tau E\right)$$
$$\ge (A^{-1}+\sigma\tau E)\left(\frac{1}{\|A\|} + \left(\sigma - \frac{1+\varepsilon}{2}\right)\tau\right)E,$$

then with regard to condition (2.159) the inequality (2.162) is satisfied. Hence on the basis of Theorem 2.18 we conclude that

$$\|y_{n+1}\|^2_{\approx{A}} \le \|y_0\|^2_{\approx{A}} + \frac{1+\varepsilon}{2\varepsilon}\sum_{k=0}^{n}\tau\|\approx{\varphi}_k\|^2_{\approx{B}^{-1}}. \tag{2.163}$$

Owing to our notation we have

$$\|y\|^2_{\widetilde{\widetilde{A}}} = \|y\|^2_{A^{-1}} + \sigma\tau\|y\|^2,$$

$$\|\widetilde{\widetilde{\varphi}}\|^2_{\widetilde{\widetilde{B}}^{-1}} = \|\widetilde{B}\widetilde{\varphi}\|^2_{\widetilde{B}^{-2}} = \|\widetilde{\varphi}\|^2 = \|A^{-1}\varphi\|^2,$$

and the stability estimate (2.160) being proved follows from the inequality (2.163).

Consider now the weighted scheme of the kind (2.151) with homogeneous initial conditions

$$By_t + Ay = \varphi, \quad y(0) = 0, \quad B = E + \tau\sigma A. \tag{2.164}$$

The following theorem gives the stability estimates with respect to the right hand side.

THEOREM 2.25 *Let the conditions of Theorem 2.24 be satisfied. Then the solution of the difference scheme (2.164) is stable with respect to the right hand side, and the estimates*

$$\|y_{n+1}\|^2_R \le \frac{1+\varepsilon}{2\varepsilon} \sum_{k=0}^{n} \tau\|A^{-1}\varphi_k\|^2_D, \tag{2.165}$$

$$\sum_{k=0}^{n} \tau\|y_k\|^2_D \le \frac{2}{\varepsilon^2(2-\varepsilon)} \sum_{k=0}^{n} \tau\|A^{-1}\varphi_k\|^2_D, \quad \varepsilon < 2, \tag{2.166}$$

are valid, where

$$R = E + \sigma\tau A, \quad D = A, \tag{2.167}$$

$$R = A^{-1} + \sigma\tau E, \quad D = E, \tag{2.168}$$

$$R = (A^{-1} + \sigma\tau E)^2, \quad D = A^{-1} + \sigma\tau E \tag{2.169}$$

correspondingly.

Proof. Under our assumptions, all of the conditions of Theorem 2.19 are satisfied, and then from estimate (2.135), (2.136) we deduce the inequalities

$$\|y_{n+1}\|^2_B \le \frac{1+\varepsilon}{2\varepsilon} \sum_{k=0}^{n} \tau\|A^{-1}\varphi_k\|^2_A, \tag{2.170}$$

$$\sum_{k=0}^{n} \tau\|y_k\|^2_A \le \frac{2}{\varepsilon^2(2-\varepsilon)} \sum_{k=0}^{n} \tau\|A^{-1}\varphi_k\|^2_A. \tag{2.171}$$

The statement of the theorem is thus proved for the operators $R = B$ and D defined by (2.167). To prove these estimates (2.165), (2.166) for operators R and D of the form (2.168) and (2.169), it is necessary to use the canonical form (2.153) or (2.161) of the weighted scheme and again to apply Theorem 2.19.

We now assume that $A = A(t) > 0$ and $A^* \neq A$, i.e., $A = A(t)$ is a variable positive and non-self-adjoint operator.

THEOREM 2.26 *Let $A = A(t)$ be a positive non-self-adjoint operator. If the condition $\sigma \geq 0.5$ is satisfied, then the following estimate is valid for scheme (2.151):*

$$\begin{aligned}\|y_{n+1}\| \leq \|y_0\| + \| \left(A^{-1}\varphi\right)_0 \| + \| \left(A^{-1}\varphi\right)_n \| \\ + \sum_{k=1}^{n} \tau \| \left(A^{-1}\varphi\right)_{\bar{t},k} \| .\end{aligned} \tag{2.172}$$

P r o o f. Consider scheme (2.153) with the right hand side $\widetilde{\varphi} = A^{-1}\varphi$. If $\sigma \geq 0.5$ then for this scheme the conditions of Theorem 2.16 are satisfied. Therefore by using inequality (2.112) and taking into account that

$$\|y\|_{\widetilde{A}} = \|y\|, \quad \|\widetilde{\varphi}_{\bar{t},k}\|_{\widetilde{A}^{-1}} = \|\widetilde{\varphi}_{\bar{t},k}\| = \left\|\left(A^{-1}\varphi\right)_{\bar{t},k}\right\|,$$

we obtain the required relation (2.172).

REMARK 2.10 If $A^*(t) = A(t) > 0$ is a self-adjoint operator, then estimate (2.172) is valid for

$$\sigma \geq \sigma_0, \quad \sigma_0 = \frac{1}{2} - \frac{1}{\tau\|A\|}.$$

THEOREM 2.27 *Let $A(t) = A^*(t) > 0$ and $\sigma \geq \sigma_\varepsilon$. Then the solution of problem (2.151) satisfies the estimate*

$$\|y_{n+1}\|^2 \leq \|y_0\|^2 + \frac{1}{2\varepsilon} \sum_{k=0}^{n} \tau \|\varphi_k\|^2_{A_k^{-1}} . \tag{2.173}$$

P r o o f. The energy identity (2.132) for the difference scheme (2.153) takes the form

$$\begin{aligned}2\tau((A^{-1} + (\sigma - 0.5)\,\tau E)y_t, y_t) + \|\hat{y}\|^2 \\ = \|y\|^2 + 2\tau \left(A^{-1}\varphi, y_t\right) .\end{aligned} \tag{2.174}$$

From the condition $\sigma \geq \sigma_\varepsilon$ it follows that

$$\widetilde{B} - \frac{\tau}{2}\widetilde{A} \geq \varepsilon A^{-1}. \tag{2.175}$$

In fact,

$$\widetilde{B} - \frac{\tau}{2}\widetilde{A} = A^{-1} + (\sigma - 0.5)\,\tau E = \varepsilon A^{-1} + (1-\varepsilon)\,A^{-1}$$
$$+ (\sigma - 0.5)\,\tau E \geq \varepsilon A^{-1} + \frac{1-\varepsilon}{\|A\|}E + (\sigma - 0.5)\,\tau E$$
$$= \varepsilon A^{-1} + (\sigma - \sigma_\varepsilon)\,\tau E \geq \varepsilon A^{-1}$$

for $\sigma \geq \sigma_\varepsilon$. Here we have taken into account that

$$A^{-1} \geq \frac{1}{\|A\|}E.$$

Substituting the relation (2.175) into the identity (2.174), we obtain

$$2\tau\varepsilon\|y_t\|^2_{A^{-1}} + \|\hat{y}\|^2 \leq \|y\|^2 + 2\tau\left(A^{-1}\varphi, y_t\right). \qquad (2.176)$$

Applying the generalized Cauchy–Bunyakovskii–Schwarz inequality and the ε-inequality to the last term, we have

$$2\tau\left(A^{-1}\varphi, y_t\right) \leq 2\tau\|\varphi\|_{A^{-1}}\|y_t\|_{A^{-1}} \leq 2\tau\varepsilon\|y_t\|^2_{A^{-1}} + \frac{\tau}{2\varepsilon}\|\varphi\|^2_{A^{-1}}. \qquad (2.177)$$

Taking account of the inequality (2.177) we deduce from the relation (2.176) that

$$\|\hat{y}\|^2 \leq \|y\|^2 + \frac{\tau}{2\varepsilon}\|\varphi\|^2_{A^{-1}} \quad \text{or} \quad \|y_{k+1}\|^2 \leq \|y_k\|^2 + \frac{\tau}{2\varepsilon}\|\varphi_k\|^2_{A_k^{-1}}.$$

Summing the last inequality over $k = 0, 1, \ldots$ leads to the estimate (2.173).

5. Coefficient Stability

Stability with respect to the initial data and right hand side is a point of special attention when investigating well posedness of boundary initial problems for non-stationary equations of mathematical physics. In a more general situation one has necessarily to require the continuous dependence of a solution also on perturbations of operators, for instance, on coefficients of the equations. In this case it is said [Samarskii, 1989] about *strong stability*. *A priori* estimates expressing the continuous dependence on perturbations of the right hand side and the operator have been obtained (see, e.g., [Mikhlin, 1988, Samarskii, 1989]) under different conditions for stationary problems (operator equations of the first kind). For non-stationary problems we refer the reader to [Streit, 1986].

Following [Samarskii et al., 1997d] we establish estimates of stability under perturbations of the Cauchy problem operator, of the right hand side and of the initial condition for evolutionary equations considered in Hilbert spaces. We derive an *a priori* estimate for the error under natural assumptions about the perturbation of the problem operator. Using discretization with respect to time we obtain a operator-difference equation and then prove basic *a priori* estimates of strong stability for two-level operator-difference schemes, which are consistent with the corresponding estimates for the operator-differential equation under consideration. Main results are illustrated wit an example of an initial boundary value problem for a one-dimensional parabolic equation.

5.1 Strong Stability of Operator-Differential Schemes

Let H be a finite-dimensional Hilbert space, in which $(\cdot,\cdot)$, $\|\cdot\|$ denote an inner product and a norm respectively. Let A be a constant self-adjoint and positive linear operator in the space H, i.e.,

$$A \neq A(t) = A^* \geq \delta E, \quad \delta = \text{const} > 0.$$

By H_D, where $D = D^* > 0$, we denote, as usual, the space with the inner product and the norm

$$(y, v)_D = (Dy, v), \quad \|y\|_D = \sqrt{(Dy, y)},$$

respectively.

Consider the Cauchy problem

$$\frac{du}{dt} + Au = f(t), \quad 0 < t < T, \tag{2.178}$$

$$u(0) = u_0, \tag{2.179}$$

where $f(t)$, u_0 are given, and $u(t)$ is the unknown function with values in H. We denote by $\widetilde{u}(t)$ the solution of the following problem with perturbed right hand side, initial data, and operator:

$$\frac{d\widetilde{u}}{dt} + \widetilde{A}\widetilde{u} = \widetilde{f}(t), \quad 0 < t < T, \tag{2.180}$$

$$\widetilde{u}(0) = \widetilde{u}_0. \tag{2.181}$$

We set the problem of estimating the magnitude of the perturbation of the solution

$$z(t) = \widetilde{u}(t) - u(t)$$

through the magnitudes of the perturbations of f, u_0 and A.

The perturbed operator $\widetilde{A}$ is assumed to satisfy the same conditions as for the unperturbed operator A

$$\widetilde{A} \neq \widetilde{A}(t) = \widetilde{A}^* \geq \delta E, \quad \delta = \text{const} > 0.$$

The positive constant α in the inequality

$$\|(\widetilde{A} - A)v\| \leq \alpha \|\widetilde{A}v\|. \tag{2.182}$$

will be a measure of the operator perturbation. Much weaker restrictions are related, see [Samarskii, 1989], to the estimate for the energy of the operator under the additional assumption that the operator $\widetilde{A} - A$ is non-negative:

$$0 \leq ((\widetilde{A} - A)v, v) \leq \alpha(Av, v). \tag{2.183}$$

For the perturbation of the solution, from (2.178), (2.179) and (2.180), (2.181), we obtain the problem

$$\frac{dz}{dt} + Az = (\widetilde{f}(t) - f(t)) - (\widetilde{A} - A)\widetilde{u}, \quad 0 < t < T, \tag{2.184}$$

$$z(0) = \widetilde{u}_0 - u_0. \tag{2.185}$$

THEOREM 2.28 *If condition (2.182), imposed on the perturbation of the operator A, is satisfied then the following a priori estimate is valid for the perturbation of the solution:*

$$\begin{aligned} \|z(t)\|_A^2 \leq \|\widetilde{u}_0 - u_0\|_A^2 + \int_0^t \|\widetilde{f}(\theta) - f(\theta)\|^2 d\theta \\ + \alpha^2 \left(\|\widetilde{u}_0\|_{\widetilde{A}}^2 + \int_0^t \|\widetilde{f}(\theta)\|^2 d\theta \right). \end{aligned} \tag{2.186}$$

A similar estimate of strong stability under condition (2.183) has the form

$$\begin{aligned} \|z(t)\|^2 \leq \|\widetilde{u}_0 - u_0\|^2 + \int_0^t \|\widetilde{f}(\theta) - f(\theta)\|_{A^{-1}}^2 d\theta \\ + \alpha^2 \left(\|\widetilde{u}_0\|^2 + \int_0^t \|\widetilde{f}(\theta)\|_{\widetilde{A}^{-1}}^2 d\theta \right). \end{aligned} \tag{2.187}$$

Proof. Consider the inner product of the equation (2.184) and $2dz/dt$ as

$$2\left\|\frac{dz}{dt}\right\|^2+\frac{d}{dt}\|z\|_A^2=2\left((\widetilde{f}(t)-f(t)),\frac{dz}{dt}\right)-2\left((\widetilde{A}-A)\widetilde{u},\frac{dz}{dt}\right). \quad (2.188)$$

Let us estimate the right hand side of the above identity

$$2\left((\widetilde{f}(t)-f(t)),\frac{dz}{dt}\right)-2\left((\widetilde{A}-A)\widetilde{u},\frac{dz}{dt}\right)$$
$$\leq 2\left\|\frac{dz}{dt}\right\|^2+\|\widetilde{f}(t)-f(t)\|^2+\|(\widetilde{A}-A)\widetilde{u}\|^2.$$

Then from the identity (2.188) we deduce the estimate

$$\|z(t)\|_A^2 \leq \|z(0)\|_A^2+\int_0^t \|\widetilde{f}(\theta)-f(\theta)\|^2 d\theta + \int_0^t \|(\widetilde{A}-A)\widetilde{u}(\theta)\|^2 d\theta. \quad (2.189)$$

We now turn to the perturbed problem (2.180), (2.181). Taking the inner product of equation (2.180) and $2\widetilde{A}\widetilde{u}$, and then estimating the right hand side we obtain the inequality

$$\frac{d}{dt}\|\widetilde{u}\|_{\widetilde{A}}^2+2\|\widetilde{A}\widetilde{u}\|^2=2(\widetilde{f},\widetilde{A}\widetilde{u})\leq\|\widetilde{A}\widetilde{u}\|^2+\|\widetilde{f}(t)\|^2.$$

Integrating this from 0 to t yields

$$\int_0^t \|\widetilde{A}\widetilde{u}(\theta)\|^2 d\theta \leq \|\widetilde{u}(0)\|_{\widetilde{A}}^2+\int_0^t \|\widetilde{f}(\theta)\|^2 d\theta. \quad (2.190)$$

Taking into account the inequalities (2.182) and (2.190) we obtain the required estimate (2.186) from (2.189).

The proof of estimate (2.187) is analogous. In a similar manner, consider first the inner product of the perturbed equation (2.184) and $2z$:

$$\frac{d}{dt}\|z\|^2+2\|z\|_A^2=2((\widetilde{f}(t)-f(t)),z)-2((\widetilde{A}-A)\widetilde{u},z). \quad (2.191)$$

In virtue of (2.183), we have

$$2((\widetilde{f}(t) - f(t)), z) \leq \|\widetilde{f}(t) - f(t)\|^2_{A^{-1}} + \|z\|^2_A,$$

$$2((\widetilde{A} - A)\widetilde{u}, z) \leq \alpha((\widetilde{A} - A)\widetilde{u}, \widetilde{u}) + \frac{1}{\alpha}((\widetilde{A} - A)z, z)$$

$$\leq \alpha^2(A\widetilde{u}, \widetilde{u}) + (Az, z) \leq \alpha^2\|\widetilde{u}\|^2_{\widetilde{A}} + \|z\|^2_A.$$

Substituting the above estimates into the identity (2.191) and integrating this from 0 to t, we come to the inequality

$$\|z(t)\|^2 \leq \|z(0)\|^2 + \int_0^t \|\widetilde{f}(\theta) - f(\theta)\|^2_{A^{-1}} d\theta + \alpha^2 \int_0^t \|\widetilde{u}(0)\|^2_{\widetilde{A}} d\theta. \quad (2.192)$$

If we now consider the inner product of the perturbed equation (2.180) and $2\widetilde{u}$, estimating its right hand side we obtain

$$\frac{d}{dt}\|\widetilde{u}\|^2 + 2\|\widetilde{u}\|^2_{\widetilde{A}} = 2(\widetilde{f}, \widetilde{u}) \leq \|\widetilde{u}\|^2_{\widetilde{A}} + \|\widetilde{f}(t)\|^2_{\widetilde{A}^{-1}}.$$

Hence we find that

$$\int_0^t \|\widetilde{u}\|^2_{\widetilde{A}} d\theta \leq \|\widetilde{u}(0)\|^2 + \int_0^t \|\widetilde{f}(\theta)\|^2_{\widetilde{A}^{-1}} d\theta.$$

Substituting the last estimates into the relation (2.192) we come to the inequality (2.187).

5.2 Strong Stability of the Operator-Difference Schemes

Let $\tau > 0$ be a mesh width with respect to time and $y_n = y(t_n)$, $t_n = n\tau$. For the Cauchy problem (2.178), (2.179) in the operator-differential case, we consider the corresponding scheme with weights

$$\frac{y_{n+1} - y_n}{\tau} + A(\sigma y_{n+1} + (1 - \sigma)y_n) = f_n, \quad n = 0, 1, \ldots, \quad (2.193)$$

$$y_0 = u_0. \quad (2.194)$$

Let us write also the weighted difference scheme for the perturbed problem (2.180), (2.181),

$$\frac{\widetilde{y}_{n+1} - \widetilde{y}_n}{\tau} + \widetilde{A}(\sigma\widetilde{y}_{n+1} + (1 - \sigma)\widetilde{y}_n) = \widetilde{f}_n, \quad n = 0, 1, \ldots, \quad (2.195)$$

$$\widetilde{y}_0 = \widetilde{u}_0. \tag{2.196}$$

Analogously to Theorem 2.28 we now formulate the strong stability result for the difference scheme under consideration.

THEOREM 2.29 *For the perturbation of the solution $z_n = \widetilde{y}_n - y_n$ of the difference schemes (2.193), (2.194) and (2.195), (2.196) when $\sigma \geq 0.5$, and if condition (2.182) is satisfied, the following a priori estimate is valid:*

$$\begin{aligned} \|z_{n+1}\|_A^2 \leq \|\widetilde{y}_0 - y_0\|_A^2 + \sum_{k=0}^{n} \tau \|\widetilde{f}_k - f_k\|^2 \\ + \alpha^2 \left(\|\widetilde{y}_0\|_{\widetilde{A}}^2 + \sum_{k=0}^{n} \tau \|\widetilde{f}_k\|^2 \right). \end{aligned} \tag{2.197}$$

If only conditions (2.183) are satisfied then

$$\begin{aligned} \|z_{n+1}\|^2 \leq \|\widetilde{y}_0 - y_0\|^2 + \sum_{k=0}^{n} \tau \|\widetilde{f}_k - f_k\|_{A^{-1}}^2 \\ + \alpha^2 \left(\|\widetilde{y}_0\|^2 + \sum_{k=0}^{n} \tau \|\widetilde{f}_k\|_{\widetilde{A}^{-1}}^2 \right). \end{aligned} \tag{2.198}$$

P r o o f. Note that estimates (2.197) and (2.198) for the perturbation of the difference solution are the full analogues of the corresponding estimates (2.186) and (2.187) for the solution of the Cauchy problem in the operator-differential case.

Let us prove, for example, estimate (2.197). For the perturbation of the solution we have the problem

$$\begin{aligned} \frac{z_{n+1} - z_n}{\tau} + A(\sigma z_{n+1} + (1-\sigma) z_n) \\ = (\widetilde{f}_n - f_n) - (\widetilde{A} - A)\widetilde{y}_n^{(\sigma)}, \quad n = 0, 1, \ldots, \end{aligned} \tag{2.199}$$

$$z_n = \widetilde{u}_0 - u_0, \tag{2.200}$$

where $\widetilde{y}_n^{(\sigma)} = \sigma \widetilde{y}_{n+1} + (1-\sigma)\widetilde{y}_n$. Considering the inner product of equation (2.199) and $2(z_{n+1} - z_n)$, similarly to the inequality (2.189) we obtain the estimate

$$\|z_{n+1}\|_A^2 \leq \|z_0\|_A^2 + \sum_{k=0}^{n} \tau \|\widetilde{f}_k - f_k\|^2 + \sum_{k=0}^{n} \tau \|(\widetilde{A} - A)\widetilde{y}_k^{(\sigma)}\|^2. \tag{2.201}$$

Taking the inner product in H of the perturbed equation (2.195) and $2\tau\widetilde{A}\widetilde{y}_n^{(\sigma)}$, after certain transformations we obtain the equality

$$(\widetilde{A}\widetilde{y}_{n+1},\widetilde{y}_{n+1}) + 2\tau^2\left(\sigma-\frac{1}{2}\right)\left(\widetilde{A}\frac{\widetilde{y}_{n+1}-\widetilde{y}_n}{\tau},\frac{\widetilde{y}_{n+1}-\widetilde{y}_n}{\tau}\right)$$
$$+\,2\tau\|\widetilde{A}\widetilde{y}_n^{(\sigma)}\|^2 = 2\tau(\widetilde{A}\widetilde{y}_n^{(\sigma)},\widetilde{f}_n) + (\widetilde{A}\widetilde{y}_n,\widetilde{y}_n).$$

Hence we deduce the estimate

$$\sum_{k=0}^{n}\tau\|\widetilde{A}\widetilde{y}_k^{(\sigma)}\|^2 \le (\widetilde{A}\widetilde{y}_0,\widetilde{y}_0) + \sum_{k=0}^{n}\tau\|\widetilde{f}_k\|^2.$$

Substituting this into the inequality (2.201) we arrive at the estimate (2.197) for the difference scheme (2.199), (2.200).

As an example we consider the boundary value problem for the one-dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + f(x,t), \quad 0<x<l, \quad 0<t<T, \qquad (2.202)$$

$$u(0,t)=0, \quad u(l,t)=0, \quad t>0,$$
$$u(x,0)=u_0(x), \quad 0<x<l,$$

under the natural assumption $k(x)\ge c>0$.

We denote by H a set of mesh functions defined on the uniform grid

$$\overline{\omega}_h = \{x_i = ih, \quad i=0,1,\dots,N, \quad Nh=l\}$$

and equal to zero at $x=0,l$. Define the inner product and the norm respectively by the expressions

$$(u,v) = \sum_{i=1}^{N-1} u_i v_i h, \qquad \|u\| = \sqrt{(u,u)}.$$

Define the difference operator

$$Ay_i = -(ay_{\bar{x}})_{x,i}, \quad i=1,2,\dots,N-1, \quad y_0=y_N=0,$$

where a is some stencil functional. The properties of the operator A will be studied in detail in Chapter 6. In particular,

$$A=A^*\ge\delta E, \quad \delta=\frac{8c}{l^2}.$$

The operator

$$(\widetilde{A}-A)v = ((\widetilde{a}-a)v_{\bar{x}})_x$$

will satisfy condition (2.183) if

$$0 \le \widetilde{a}(x) - a(x) \le \alpha a(x).$$

In an analogous fashion, using the embedding (6.1.44)

$$\|v\|_A \le \frac{l}{2\sqrt{2c}}\|Av\|,$$

one can show that condition (2.182) is satisfied if

$$|\widetilde{a}(x) - a(x)| \le \alpha_1 \widetilde{a}(x), \quad |\widetilde{a}_x(x) - a_x(x)| \le \alpha_2,$$

so that $\alpha = c_0(\alpha_1 + \alpha_2)$, where $c_0 = \text{const} > 0$. As a result the estimation of the perturbation of the solution in the stronger norm requires more severe restrictions on the coefficients of the equation.

Chapter 3

DIFFERENCE SCHEMES WITH OPERATOR FACTORS

1. Introduction

When solving non-stationary problems of mathematical physics, a particular attention is paid to schemes with weighted factors. Assume that we solve the Cauchy problem for the first-order evolution equation

$$\frac{du}{dt} + Au = f(t), \quad 0 < t < T,$$

$$u(0) = u_0,$$

where $f(t)$, u_0 are given, whilst $u(t)$ is the unknown function with values in a finite-dimensional Hilbert space H.

We use the notation

$$v_n^{(\sigma)} = \sigma v_{n+1} + (1 - \sigma) v_n,$$

where $\sigma \geq 0$ is an arbitrary real parameter. Let us consider the following class of difference schemes with a constant weighted factor σ:

$$y_t + A y_n^{(\sigma)} = 0, \quad t_n \in \omega_\tau, \tag{3.1}$$

$$y_0 = u_0. \tag{3.2}$$

We transform scheme (3.1) into the canonical form

$$B \frac{y_{n+1} - y_n}{\tau} + A y_n = \varphi_n, \quad t_n \in \omega_\tau, \tag{3.3}$$

in which

$$B = E + \tau \sigma A, \quad \sigma = \text{const} \geq 0.$$

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In (3.1), (3.2) the weight parameter is constant. In a more general case variable weighted multipliers may be used, so we can talk about operator multipliers in a more global sense. Thus scheme (3.2), (3.3) is referred to such a class, in which

$$B = E + \tau GA.$$

One example of similar schemes is a 'chess board' difference scheme in which the weight parameter takes alternate values 0 and 1 in turn. The construction of adaptive computational algorithms on locally refined meshes in time or/and space, regionally additive difference schemes (iteration-free domain decomposition schemes), which are oriented on parallel computers, and of difference schemes for equations of mixed type is also based on the use of schemes with variable weighted factors.

Similar problems will be considered below in detail. The main complicating factor is related to the fact that the operator B is not self-adjoint and has no fixed sign, which does not allow us to apply directly the stability criteria for operator-difference schemes.

Let us give one simplest result on the stability of operator-difference schemes under consideration with respect to the initial data. Assume that in (3.3) A be a constant and positive operator, and

$$B = E + \tau GA, \quad GA \neq A^*G^*.$$

Then the condition

$$G \geq \frac{1}{2}E$$

is sufficient for the scheme to be stable in H_{A^*A}. More general necessary and sufficient conditions are stated in Theorem 3.2.

There is a correspondence of schemes with variable weighted factors to the choice of a diagonal discrete operator G in (3.3). When constructing schemes with variable weighted factors, one can average the solution between two time levels in different ways. This leads to different classes of schemes with operator factors, some of which are distinguished in this work.

Under formulated restrictions we establish the stability of difference scheme (3.3) in which

$$B = E + \tau AG, \quad GA \neq A^*G^*.$$

This choice of the operator B is associated, for example, with a new type of weighting when using conservative schemes with variable weighted factors. For a self-adjoint operator A we consider schemes, in which

$$B = E + \tau T^*GT, \quad A = T^*T.$$

For the mentioned classes of operator-difference schemes with operator factors we obtain *a priori* estimates of stability with respect to the initial data and the right hand in different norms.

2. Schemes with $B = E + \tau GA$

Operator difference schemes with operator weighted factors of the first kind are investigated. These schemes can be interpreted as the usual schemes with weights for which the weighting of the spatial operator applied to the solution of a problem is taken with respect to time.

2.1 Stability with Respect to the Initial Data

Consider again the canonical form of two-level operator-difference schemes

$$B\frac{y_{n+1} - y_n}{\tau} + Ay_n = 0, \quad t_n \in \omega_\tau, \tag{3.4}$$

$$y_0 = u_0, \tag{3.5}$$

where $y_n = y(t_n) \in H$, $u_0 \in H$ is given, H is a finite-dimensional Hilbert space with inner product $(\cdot,\cdot)$, A and B are linear operators acting in H.

So far we have considered two kind of schemes, either $A = A^* > 0$, $B \neq B^*$ or $A \neq A^*$, $B \neq B^*$, but with the operator B of a special form

$$B = D + \sigma\tau A, \quad D = D^* > 0, \quad \sigma = \text{const} > 0.$$

We consider below wider classes of operator-difference schemes with non-self-adjoint operators A and B.

If B^{-1} exists then the difference equation (3.4) can be written in the form resolved with respect to y_{n+1} as follows:

$$y_{n+1} = Sy_n,$$

with the transition operator $S = E - \tau B^{-1}A$.

Difference scheme (3.4), (3.5) is called *symmetrizable* if there is a nonsingular operator $K : H \to H$ such that the operator

$$\widetilde{S} = KSK^{-1}$$

is self-adjoint.

Throughout what follows we shall assume that

$$B = E + \tau GA, \quad GA \neq AG, \tag{3.6}$$

where $G = G(t)$ is a linear operator in H being non-commutative with A.

In the sequel the following Lemma will be needed.

LEMMA 3.1 *Let L and Q be operators acting in H, and also assume that the operator L^{-1} exists. The following operator inequalities are equivalent:*

$$Q \geq 0, \tag{3.7}$$

$$L^*QL \geq 0. \tag{3.8}$$

Proof. It suffices to consider the identity $(L^*QLy, y) = (Qv, v)$ with $v = Ly$ and $y = L^{-1}v$. Note that the inequality (3.7) follows from (3.8) without the reciprocity requirement for the operator L.

The following theorem gives necessary and sufficient conditions for stability in the spaces H_{B^*B} and H_{A^*A}.

THEOREM 3.1 *Let A and B be constant operators and B^{-1} do exist. Then the inequality*

$$A^*B + B^*A \geq \tau A^*A \tag{3.9}$$

is a necessary and sufficient condition for the scheme (3.4)–(3.5) to be stable in H_{B^*B}. Furthermore, if A^{-1} does exist, then condition (3.9) is necessary and sufficient also for stability in H_{A^*A}.

Proof. Stability with respect to the initial data in the space H_D, or the validity of the inequalities

$$\|y_{n+1}\|_D \leq \|Sy_n\|_D \leq \|y_n\|_D,$$

is obviously equivalent to the validity of the following operator inequality

$$D \geq S^*DS. \tag{3.10}$$

Substituting $S^* = E - \tau A^*(B^*)^{-1}$ and $S = E - \tau B^{-1}A$ into (3.10), respectively, we obtain the relation

$$A^*(B^*)^{-1}D + DB^{-1}A \geq \tau A^*(B^*)^{-1}DB^{-1}A. \tag{3.11}$$

If $D = B^*B$ then it implies the inequality (3.9). Then substituting the operator $D = A^*A$ into the relation (3.11) and using Lemma 3.1 with $L = A$ we obtain the operator inequality

$$(B^*)^{-1}A^* + AB^{-1} \geq \tau(B^*)^{-1}A^*AB^{-1}.$$

Using again Lemma 3.1 and multiplying the last inequality from the left and from the right by $L^* = B^*$ and $L = B$, respectively, after elementary transformations we come to the estimate (3.9).

Generally speaking, the operator inequality (3.9) is not necessary for the stability in norms of spaces that differs from H_{A^*A}, H_{B^*B}.

Let us formulate the corresponding necessary and sufficient conditions of stability for the scheme

$$(E + \tau GA)\, y_t + Ay = 0, \quad y_0 = u_0. \tag{3.12}$$

THEOREM 3.2 *Let the operator A of scheme (3.12) be constant, $G(t)A \neq AG(t)$ and let A^{-1} exist. Then the inequality*

$$A^* + \tau A^* \left(G - 0.5E\right) A \geq 0 \tag{3.13}$$

is a necessary and sufficient condition for the stability of the scheme (3.12) in the space H_{A^*A}.

Proof. Multiply equation (3.12) by the operator A^* from the left. We obtain another canonical form of the two-level operator-difference scheme (3.12)

$$\widetilde{B} y_t + \widetilde{A} y = 0, \quad y_0 = u_0, \tag{3.14}$$

where

$$\widetilde{B} = A^* + \tau A^* G A, \quad \widetilde{A} = A^* A, \quad \widetilde{A} = \widetilde{A}^* > 0.$$

Ascertaining the necessary and sufficient condition of stability with respect to the initial data for the scheme (3.14)

$$\widetilde{B} - 0.5\tau \widetilde{A} \geq 0$$

gives us the inequality (3.13)

REMARK 3.1 Theorem 3.2 could be proved on the basis of Theorem 3.1 by direct substitution of the operator $B = E + \tau G(t)A$, $t \in \omega_\tau$, into (3.9). Here, in the proof of stability in H_{A^*A}, the operator B can be variable.

COROLLARY 3.1 *The conditions $A > 0$, $G(t) > 0.5E$ are sufficient for the stability in H_{A^*A} of scheme (3.12) with respect to the initial data.*

The following theorem allows us to obtain *a priori* estimate under weaker conditions on the operator A than that in Theorem 3.2 (namely, it does not require the existence of the inverse operator A^{-1}).

THEOREM 3.3 *Let the operator A in scheme (3.12) be constant, and $G(t)A \neq AG(t)$. If the operator inequality (3.13) holds, then the solution of problem (3.12) satisfy the following* a priori *estimate*

$$\|Ay_{n+1}\| \leq \|Ay_0\|. \tag{3.15}$$

P r o o f. Since the conditions $\widetilde{B}(t) \geq 0.5\tau\widetilde{A}$, $\widetilde{A} = \widetilde{A}^*$ are satisfied for scheme (3.14), then we verify the validity of estimate (3.15) because of (2.34), (2.35).

REMARK 3.2 If the existence of the operator A^{-1} is additionally required in the hypotheses of Theorem 3.3, then the estimate (3.15) gives the stability of the scheme in the norm H_{A^*A}.

2.2 Stability with Respect to the Right Hand Side

Let us turn to obtaining *a priori* estimates of stability for a inhomogeneous two-level operator-difference scheme of the form

$$B\frac{y_{n+1} - y_n}{\tau} + Ay_n = \varphi_n, \quad t_n \in \omega_\tau, \quad y_0 = u_0 \tag{3.16}$$

with

$$B = E + \tau GA, \quad GA \neq AG. \tag{3.17}$$

THEOREM 3.4 *Let the operator $A > 0$ in the difference scheme (3.16), (3.17) be constant. Then the estimate*

$$\|Ay_{n+1}\| \leq \|Ay_0\| + \|\varphi_0\| + \|\varphi_n\| + \sum_{k=1}^{n} \tau\|\varphi_{\bar{t},k}\| \tag{3.18}$$

is valid provided that

$$G(t) \geq 0.5E. \tag{3.19}$$

P r o o f. As in the case of Theorem 3.2, we multiply the difference equation (3.16) by A^* from the left. We obtain the following operator-difference scheme

$$\widetilde{B}y_t + \widetilde{A}y = \widetilde{\varphi}, \quad \widetilde{\varphi} = A^*\varphi, \quad y_0 = u_0, \tag{3.20}$$

with the same operators $\widetilde{B}$, $\widetilde{A}$ as in equation (3.14). By the assumption (3.19) it follows that

$$\widetilde{B} - 0.5\tau\widetilde{A} = A^* + \tau A^*(G - 0.5E)A \geq 0,$$

and thus by Theorem 2.16 we obtain

$$\|y_{n+1}\|_{\widetilde{A}} \leq \|y_0\|_{\widetilde{A}} + \|\widetilde{\varphi}_0\|_{\widetilde{A}^{-1}} + \|\widetilde{\varphi}_n\|_{\widetilde{A}^{-1}} + \sum_{k=1}^{n} \tau\|\widetilde{\varphi}_{\bar{t},k}\|_{\widetilde{A}^{-1}}. \tag{3.21}$$

Going over to the original notation $\widetilde{A} = A^*A$ we obtain

$$\|y\|_{\widetilde{A}} = \sqrt{(A^*Ay, y)} = \|Ay\|,$$

$$\|\widetilde{\varphi}\|^2_{\widetilde{A}^{-1}} = (A^*\varphi, A^*\varphi)_{(A^*A)^{-1}} = \left((A^*A)^{-1}A^*AA^{-1}\varphi, A^*\varphi\right) = \|\varphi\|^2.$$

Replacing the terms in the inequality (3.21) with regard to the above relations, we prove the estimate (3.18).

REMARK 3.3 In the case of a constant self-adjoint operator A, Theorem 3.4 is valid under the much weaker conditions

$$G(t) \geq \sigma_0 E, \quad \sigma_0 = \frac{1}{2} - \frac{1}{\tau\|A\|}.$$

REMARK 3.4 Theorem 3.4 remains valid also in the case of a variable operator $A(t)$, $t \in \omega_\tau$.

In this case, to avoid the requirement of Lipschitz continuity of the operator $A(t)$, we transform scheme (3.16), (3.17) into another canonical form. Since the operator A^{-1} exists, introducing the change of variables $x = Ay$ we obtain the following difference problem:

$$\widetilde{B}x_t + \widetilde{A}x = \varphi, \quad x_0 = A_0 y_0,$$

where $\widetilde{B}(t) = A^{-1}(t) + \tau G(t) > 0$, $\widetilde{A} = E$. Because $\widetilde{A} = \widetilde{A}^* > 0$ is a constant operator and

$$\widetilde{B}(t) - 0.5\tau\widetilde{A} = A^{-1}(t) + \tau\left(G(t) - 0.5E\right) \geq 0,$$

under condition (3.19) we conclude by using estimate (2.112) that

$$\|x_{n+1}\|_{\widetilde{A}} \leq \|x_0\|_{\widetilde{A}} + \|\varphi_0\|_{\widetilde{A}^{-1}} + \|\varphi_n\|_{\widetilde{A}^{-1}} + \sum_{k=1}^{n} \tau\|\varphi_{\bar{t},k}\|_{\widetilde{A}^{-1}}.$$

Taking into account that $\widetilde{A} = E$ is the identity operator and $x = Ay$, from the last inequality we deduce the required estimate

$$\|A_{n+1}y_{n+1}\| \leq \|A_0 y_0\| + \|\varphi_0\| + \|\varphi_n\| + \sum_{k=1}^{n} \tau\|\varphi_{\bar{t},k}\|.$$

Difference schemes of the form (3.16), (3.17) are called schemes with operator factors or *schemes with operator factors* or *schemes with variable weighted factors* . Indeed, if $G = \sigma E$, $\sigma = \text{const} > 0$, E is the identity operator, then the scheme (3.16), (3.17) corresponds to the scheme with constant weights

$$y_t + Ay^{(\sigma)} = \varphi, \quad y_0 = u_0.$$

Two-level operator difference schemes with variable weighted factors (or operator multipliers) can be also defined in the following way:

$$y_t + (Ay)^{(\Sigma)} = \varphi, \quad y_0 = u_0, \tag{3.22}$$

where

$$v^{(\Sigma)} = \Sigma\hat{v} + (E - \Sigma)v, \tag{3.23}$$

with $v \in H$ and $\Sigma : H \longrightarrow H$ being a linear operator in H.

Using the identity

$$(Ay)^{(\Sigma)} = A^{(\Sigma)}y + \tau\Sigma\hat{A}y_t,$$

we transform problem (3.22) with $\varphi = \widetilde{\varphi}^{(\Sigma)}$ into the following canonical form

$$\left(E + \tau\Sigma\hat{A}\right)y_t + A^{(\Sigma)}y = \widetilde{\varphi}^{(\Sigma)}, \quad y_0 = u_0. \tag{3.24}$$

One can see from here that schemes (3.24) and (3.16), (3.17) are equivalent in the case of a constant operator $A^{(\Sigma)} = A$, $G(t) = \Sigma(t)$.

Let us study the stability of scheme (3.24) for the case of a variable non-self-adjoint operator $A(t)$ satisfying the conditions

$$A(t) \geq \delta E, \quad A\Sigma \neq \Sigma A, \tag{3.25}$$

$$\| \left(A(t+\tau) - A(t)\right) u\| \leq \tau c_0 \|Au\|, \tag{3.26}$$

where δ and c_0 are positive constants which do not depend on τ.

THEOREM 3.5 *Let the operator $A(t)$ in the difference scheme (3.24) satisfy conditions (3.25), (3.26), and assume that*

$$\Sigma(t) \geq \frac{1}{2}E.$$

Then for any $0 < \varepsilon \leq 2\delta$ we have the estimate

$$\|A_{n+1}y_{n+1}\| \leq M_1\|A_0y_0\| + M_2\left(\max_{0\leq k\leq n+1}\|\widetilde{\varphi}_k\| + \max_{0\leq k\leq n}\|\widetilde{\varphi}_{t,k}\|\right), \tag{3.27}$$

where

$$M_1 = \exp(0.5c_1T), \quad c_1 = 3c_0^2/(2\varepsilon), \quad M = \max\{c_1, 3/\varepsilon\},$$

$$M_2 = \left(1 + M_1(1 + MT)^{1/2}\right).$$

P r o o f. For convenience of our further considerations, we rewrite problem (3.24) in the form

$$y_t + (\widetilde{A}y)^{(\Sigma)} = 0, \quad \widetilde{A}y = Ay - \widetilde{\varphi}, \quad y_0 = u_0. \tag{3.28}$$

We now consider the inner product of equation (3.28) and $2\tau(\widetilde{A}y)_t$. Obviously,

$$2\tau\left((\widetilde{A}y)_t, (\widetilde{A}y)^{(\Sigma)}\right) = \|\hat{\widetilde{A}}\hat{y}\|^2 - \|\widetilde{A}y\|^2 \\ + 2\tau^2\left((\Sigma - 0.5E)(\widetilde{A}y)_t, (\widetilde{A}y)_t\right).$$

Using conditions (3.25), (3.26) and the inequality

$$\|Ay\| \le \|\widetilde{A}y\| + \|\widetilde{\varphi}\|,$$

we obtain the estimate

$$\begin{aligned} 2\tau\left((Ay)_t, y_t\right) &= 2\tau(\hat{A}y_t, y_t) + 2\left((\hat{A} - A)y, y_t\right) \\ &\ge 2\tau\delta\|y_t\|^2 - 2\tau c\|Ay\|\|y_t\| \\ &\ge 2\tau\delta\|y_t\|^2 - \frac{3\tau c_0^2}{2\varepsilon}\left(\|\widetilde{A}y\|^2 + \|\widetilde{\varphi}\|^2\right) - \frac{2}{3}\tau\varepsilon\|y_t\|^2. \end{aligned}$$

Applying the Cauchy inequality with ε we find

$$-2\tau(\widetilde{\varphi}_t, y_t) \ge -3\tau\varepsilon^{-1}\|\widetilde{\varphi}_t\|^2 - \frac{1}{3}\tau\varepsilon\|y_t\|^2.$$

Summing the inequalities obtained above we come to the relation

$$I^2 + \|\hat{\widetilde{A}}\hat{y}\|^2 \le (1 + \tau c_1)\|\widetilde{A}y\|^2 + \tau M\left(\|\widetilde{\varphi}\|^2 + \|\widetilde{\varphi}_t\|^2\right), \tag{3.29}$$

in which

$$I^2 = 2\tau^2\left((\Sigma - 0.5E)(\widetilde{A}y)_t, (\widetilde{A}y)_t\right) + 2\tau(\delta - 0.5\varepsilon)\|y_t\|^2 \ge 0.$$

From estimate (3.29) we deduce

$$\|\hat{A}\hat{y} - \hat{\widetilde{\varphi}}\| \le M_1\left(\|A_0y_0 - \widetilde{\varphi}_0\| + (TM)^{1/2}\max_{0\le k\le n}\left(\|\widetilde{\varphi}_k\| + \|\widetilde{\varphi}_{t,k}\|\right)\right). \tag{3.30}$$

Taking into account that

$$\|\hat{A}\hat{y} - \hat{\widetilde{\varphi}}\| \ge \|A_{n+1}y_{n+1}\| - \|\widetilde{\varphi}_{n+1}\|, \quad \|A_0y_0 - \widetilde{\varphi}_0\| \le \|A_0y_0\| + \|\widetilde{\varphi}_0\|$$

we obtain the required estimate (3.27) directly from the inequality (3.30).

2.3 The Stability in Other Norms

Let us study stability of the difference scheme

$$B\frac{y_{n+1}-y_n}{\tau}+Ay_n=\varphi_n,\quad t_n\in\omega_\tau,\quad y_0=u_0, \tag{3.16}$$

$$B=E+\tau GA,\quad GA\neq AG \tag{3.17}$$

in the energy space H_{B^*A}.

Let the constant operators A and G satisfy the conditions

$$A=A^*>0,\quad G=G^*\geq 0. \tag{3.31}$$

Because $GA\neq AG$ the operator $B\neq B^*$ is not self-adjoint.

We transform scheme (3.16) into the canonical form

$$\widetilde{B}\frac{y_{n+1}-y_n}{\tau}+\widetilde{A}y_n=\widetilde{\varphi}_n,\quad y_0=u_0, \tag{3.32}$$

where

$$\widetilde{B}=\widetilde{B}^*,\quad \widetilde{A}=\widetilde{A}^*>0. \tag{3.33}$$

To do this we multiply equation (3.16) from the left by $B^*=E+\tau AG$. Then we have in equation (3.32)

$$\widetilde{B}=B^*B,\quad \widetilde{A}=B^*A=A+\tau AGA,\quad \widetilde{\varphi}=B^*\varphi.$$

Now applying Theorem 2.18 one can prove the following assertion (we use below the notation $\|\cdot\|_D$ for $D^*=D\geq 0$ as well).

THEOREM 3.6 *Let conditions (3.31) and the inequality*

$$\widetilde{B}\geq\frac{1+\varepsilon}{2}\tau\widetilde{A}$$

be satisfied. Then the difference scheme (3.16), (3.17) is stable in H_{B^*A}, and the estimate

$$\|y_{n+1}\|_A^2+\tau\|Ay_{n+1}\|_G^2\leq\|y_0\|_A^2+\tau\|Ay_0\|_G^2+\frac{1+\varepsilon}{2\varepsilon}\sum_{k=0}^{n}\tau\|\varphi_k\|^2 \tag{3.34}$$

is valid. Here ε is a positive constant independent of τ.

Proof. The problem (3.32) satisfies all the conditions of Theorem 2.18, and on the basis of *a priori* estimate (2.131) we have

$$\|y_{n+1}\|_{\widetilde{A}}^2\leq\|y_0\|_{\widetilde{A}}^2+\frac{1+\varepsilon}{2\varepsilon}\sum_{k=0}^{n}\tau\|\widetilde{\varphi}\|_{\widetilde{B}^{-1}}^2. \tag{3.35}$$

Since

$$\|y\|^2_{\widetilde{A}} = \|y\|^2_A + \tau\|Ay\|^2_G, \quad \|\widetilde{\varphi}\|^2_{\widetilde{B}^{-1}} = \|\varphi\|^2,$$

the inequality (3.35) implies the required estimate (3.34).

Since the operators A and G are non-commutative, the condition of the theorem

$$\widetilde{B} - \frac{1+\varepsilon}{2}\tau\widetilde{A} = E + \tau\left(A\left(G - \frac{1+\varepsilon}{4}E\right) + \left(G - \frac{1+\varepsilon}{4}\right)A\right)$$
$$+\tau^2 AG\left(G - \frac{1+\varepsilon}{2}E\right)A \geq 0$$

is not so easily verifiable. Therefore we prove the theorem under more natural restrictions to the operators A and $G(t)$.

THEOREM 3.7 *Let the operator $A = A^* > 0$ be constant and $G(t)$ be a self-adjoint operator satisfying one of the inequalities*

$$\frac{1+\varepsilon}{2}E \leq G(t) \leq E, \tag{3.36}$$

$$E \leq G(t) \leq \frac{3-\varepsilon}{2}E \tag{3.37}$$

with $0 < \varepsilon \leq 1$. Then the solution of the difference scheme (3.16), (3.17) satisfies the estimate

$$\|y_{n+1}\|^2_A + \tau\|Ay_{n+1}\|^2_{G_n} \leq \|y_0\|^2_A + \tau\|Ay_0\|^2_{G_0} + \frac{1}{\varepsilon}\sum_{k=0}^{n}\tau\|\varphi_k\|^2. \tag{3.38}$$

Proof. We rewrite the scheme under consideration in the equivalent form

$$y_t + GA\hat{y} = (G-E)Ay + \varphi, \quad y_0 = u_0. \tag{3.39}$$

Taking the inner product of the operator equation (3.39) and $2\tau A\hat{y}$, we obtain

$$\begin{aligned}&\|\hat{y}\|^2_* + \tau^2\|y_t\|^2_A + \tau\|A\hat{y}\|^2_G \\ &\quad = \|y\|^2_A - 2\tau\,(A\hat{y}, (E-G)Ay) + 2\tau(A\hat{y}, \varphi),\end{aligned} \tag{3.40}$$

where

$$2\tau(A\hat{y}, \varphi) \leq \tau\varepsilon\|A\hat{y}\|^2 + \tau\varepsilon^{-1}\|\varphi\|^2, \quad \|y\|^2_* = \|y\|^2_A + \tau\|Ay\|^2_{\breve{G}}.$$

Using the generalized Cauchy–Bunyakovskii–Schwarz inequality we find

$$-2\tau\,(A\hat{y}, (E-G)Ay) \leq \tau\|A\hat{y}\|^2_D + \tau\|Ay\|^2_D,$$

where $D = E - G$ if the inequality (3.36) is satisfied, and $D = G - E$ when the operator $G(t)$ satisfies condition (3.37). Substituting these estimates into (3.40), we obtain the inequality

$$\|\hat{y}\|_*^2 + \tau(CA\hat{y}, A\hat{y}) \leq \|y\|_*^2 + \tau(\widetilde{C}Ay, Ay) + \tau\varepsilon^{-1}\|\varphi\|^2, \tag{3.41}$$

in which $C = G - D - \varepsilon E$, $\widetilde{C} = D - \check{G}$. Let us show that under the conditions of the theorem we have the following operator inequalities

$$C \geq 0, \quad \widetilde{C} \leq 0.$$

We first consider the case of the operator $G(t)$ satisfying condition (3.36). Then

$$C = 2G - (1+\varepsilon)E, \quad \widetilde{C} = E - (G + \check{G}) \leq E - (1+\varepsilon)E < 0.$$

If instead of (3.36) the inequality (3.37) is satisfied then

$$C = (1-\varepsilon)E \geq 0, \quad \widetilde{C} = G - E - \check{G} \leq -\frac{1+\varepsilon}{2}E < 0.$$

According to what has been stated above, the inequality (3.40) can be rewritten as

$$\|y_{k+1}\|_*^2 \leq \|y_k\|_*^2 + \frac{\tau}{\varepsilon}\|\varphi_k\|^2.$$

Summing this inequality over $k = 1, 2, ..., n$ we obtain

$$\|y_{n+1}\|_A^2 + \tau\|Ay_{n+1}\|_{G_n}^2 \leq \|y_1\|_A^2 + \tau\|Ay_1\|_{G_0}^2 + \frac{1}{\varepsilon}\sum_{k=1}^{n}\tau\|\varphi_k\|^2. \tag{3.42}$$

In order to estimate $\|y_1\|_*^2$ we make use of the inequality (3.41) for $n = 0$

$$\begin{aligned}\|y_1\|_*^2 + \tau(C_0Ay_1, Ay_1) \leq {} & \|y_0\|_A^2 + \tau\|Ay_0\|_{G_0}^2 \\ & + \tau\left((D_0 - G_0)Ay_0, Ay_0\right) + \tau\varepsilon^{-1}\|\varphi_0\|^2.\end{aligned}$$

Since $C_0 \geq 0$, $D_0 - G_0 \leq 0$, the last estimate can be rewritten in the form

$$\|y_1\|_A^2 + \tau\|Ay_1\|_{G_0}^2 \leq \|y_0\|_A^2 + \tau\|Ay_0\|_{G_0}^2 + \tau\varepsilon^{-1}\|\varphi_0\|^2.$$

Substituting this inequality into (3.42), we come to the estimate (3.38).

If A and G are constant self-adjoint operators, we can obtain more subtle *a priori* estimates. To this end we transform the original scheme into the canonical form (3.32) by multiplying equation (3.16) from the left not by the operator B^*, but by the operator A. In this case the operators

$$\widetilde{B} = A + \tau AGA, \quad \widetilde{A} = A^2, \quad \widetilde{\varphi} = A\varphi \tag{3.43}$$

satisfy conditions (3.33). We now show that the inequality

$$\widetilde{B} - 0.5\tau\widetilde{A} = A + \tau A(G - 0.5E)A \geq 0 \tag{3.44}$$

is satisfied for

$$G \geq \sigma_0 E, \quad \sigma_0 = \frac{1}{2} - \frac{1}{\tau\|A\|}.$$

First of all, note that the following identity is valid:

$$\widetilde{B} - 0.5\tau\widetilde{A} = A\left(A^{-1} - \frac{1}{\|A\|}E\right)A + \tau A(G - \sigma_0 E)A.$$

Since for the self-adjoint operator $A^{-1} \geq \dfrac{1}{\|A\|}E$ and $G - \sigma_0 E \geq 0$, the inequality $\widetilde{B} \geq 0.5\tau\widetilde{A}$ is satisfied by virtue of Lemma 3.1. Consequently, from Theorem 2.15 we obtain the following result.

THEOREM 3.8 *If the constant operators A and G in the difference scheme (3.16), (3.17) satisfy the conditions*

$$A = A^* > 0, \quad G = G^* \geq \sigma_0 E, \quad \sigma_0 = \frac{1}{2} - \frac{1}{\tau\|A\|}, \tag{3.45}$$

then the difference solution satisfies the following a priori estimate

$$(Ry_{n+1}, y_{n+1}) \leq (Ry_0, y_0) + \frac{1}{2}\sum_{k=0}^{n}\tau\|\varphi_k\|^2, \tag{3.46}$$

where

$$R = \widetilde{B} - 0.5\tau\widetilde{A} = A + \tau A(G - 0.5E)A.$$

Proof. The statement of this theorem follows from estimate (2.105) and from the identity

$$\|\widetilde{\varphi}\|^2_{\widetilde{A}^{-1}} = \|A\varphi\|^2_{A^{-2}} = \|\varphi\|^2.$$

COROLLARY 3.2 *If instead of condition (3.44) the strict inequality $\widetilde{B} > 0.5\tau\widetilde{A}$ is valid, then the estimate (3.46) gives the stability of the solution of the two-level difference scheme (3.16), (3.17) in the norm $H_{\widetilde{B}-0.5\widetilde{A}} = H_{A+\tau A(G-0.5E)A}$. Under the more severe restriction*

$$G > 0.5E$$

the estimate of stability takes the form

$$\|y_{n+1}\|_A^2 + \tau\|Ay_{n+1}\|^2_{G-0.5E}$$
$$\leq \|y_0\|_A^2 + \tau\|Ay_0\|^2_{G-0.5E} + \frac{1}{2}\sum_{k=0}^{n}\tau\|\varphi_k\|^2. \qquad (3.47)$$

Let us consider a question concerning stability in the space $H^{n+1}_{A^2}$.

THEOREM 3.9 *Let the constant operators A and G satisfy conditions*

$$A = A^* > 0, \quad G = G^* \geq \sigma_\varepsilon E, \quad \sigma_\varepsilon = \frac{1+\varepsilon}{2} - \frac{1}{\tau\|A\|}. \qquad (3.48)$$

Then the difference scheme (3.16), (3.17) with homogeneous initial condition $y_0 = 0$ is stable with respect to the right hand side, and for any $\tau > 0$ the estimate

$$\sum_{k=0}^{n}\tau\|Ay_k\|^2 \leq 2\sum_{k=0}^{n}\tau\|\varphi_k\|^2 \qquad (3.49)$$

is valid.

Proof. Similarly to condition (3.44), one can prove that under the assumptions (3.48) the inequality

$$\widetilde{B} \geq \frac{1+\varepsilon}{2}\tau\widetilde{A}$$

holds for the operators (3.43). Since all of the conditions of Theorem 2.19 are now satisfied, by virtue of Corollary 2.1 (for $\varepsilon = 1$) it follows that

$$\sum_{k=0}^{n}\tau\|y_k\|^2_{\widetilde{A}} \leq 2\sum_{k=0}^{n}\tau\|\widetilde{\varphi}_k\|^2_{\widetilde{A}^{-1}}.$$

Returning to the original notations, we obtain the required estimate (3.49).

3. Schemes with $B = E + \tau AG$

The second class of two-level operator-difference schemes is associated with the operator weighting of the solution itself. In this section we give conditions for stability of such schemes with respect to the initial data and right hand side in different norms.

3.1 Estimates of Stability with Respect to the Initial Data

We will consider schemes of the kind

$$(E + \tau AG)y_t + Ay = 0, \quad y_0 = u_0, \quad GA \neq AG. \tag{3.50}$$

Stability conditions are formulated in the following manner.

THEOREM 3.10 *If the operator $A \neq A^*$ is non-self-adjoint and constant, then for*

$$A^* + \tau A(G - 0.5E)A^* \geq 0 \tag{3.51}$$

the difference scheme (3.50) is stable in H with respect to the initial data, i.e., the estimate

$$\|y_{n+1}\| \leq \|y_0\|, \quad n = 0, 1, ..., N_0 - 1, \tag{3.52}$$

is valid.

Proof. Using the change $y = A^*v$ we transform scheme (3.50) into the form

$$\widetilde{B}v_t + \widetilde{A}v = 0, \quad A^*v_0 = u_0, \tag{3.53}$$

where

$$\widetilde{B} = A^* + \tau AGA^*, \quad \widetilde{A} = AA^*. \tag{3.54}$$

Obviously, the operator inequality $\widetilde{B} \geq 0.5\tau\widetilde{A}$ is satisfied under the condition (3.51). Consequently because of inequality (2.35) the solution of problem (3.53) satisfies the estimate

$$(\widetilde{A}v_{n+1}, v_{n+1}) \leq (\widetilde{A}v_0, v_0)$$

or

$$\|A^*v_{n+1}\| \leq \|A^*v_0\|.$$

Returning to the element $y = A^*v \in H$, we obtain the estimate (3.52) which was to be proved.

COROLLARY 3.3 *The sufficient condition for stability is satisfied if the constant operator A and the variable operator $G(t)$ satisfy the conditions*

$$A \geq 0, \quad G(t) \geq 0.5E.$$

THEOREM 3.11 *Let $A(t) > 0$ be a non-self-adjoint operator. Then the condition*

$$A^{-1}(t) + \tau\,(G(t) - 0.5E) \geq 0 \tag{3.55}$$

is necessary and sufficient for the stability of scheme (3.50) in H with respect to the initial data, i.e. for the validity of estimate (3.52).

Proof. Since the operator A^{-1} does exist, then acting with A^{-1} on equation (3.50) we obtain the difference problem

$$\widetilde{B}y_t + \widetilde{A}y = 0, \quad y_0 = u_0, \quad GA \neq AG \tag{3.56}$$

with operators

$$\widetilde{B}(t) = A^{-1}(t) + \tau G(t) > 0, \quad \widetilde{A} = E, \quad \widetilde{A} = \widetilde{A}^* > 0. \tag{3.57}$$

Ascertaining the necessary and sufficient condition of stability in $H_{\widetilde{A}}$ in the form $\widetilde{B} \geq 0.5\tau\widetilde{A}$ leads to the necessity of condition (3.55). Thus, by virtue of Theorem 2.1

$$\|y_n\|_{\widetilde{A}} \leq \|y_0\|_{\widetilde{A}}.$$

Since $\widetilde{A} = E$, this proves the theorem.

REMARK 3.5 If $A(t) = A^*(t) > 0$ then the inequality (3.55) is equivalent to the condition

$$G(t) \geq \sigma_0 E, \quad \sigma_0 = \frac{1}{2} - \frac{1}{\tau\|A\|}. \tag{3.58}$$

As a result the necessary and sufficient condition for stability of scheme (3.50) in H in the case of a self-adjoint positive operator A can be written in the form (3.58).

3.2 Stability with Respect to the Right Hand Side

Consider the two-level operator-difference scheme

$$B\frac{y_{n+1} - y_n}{\tau} + Ay_n = \varphi_n, \quad t_n \in \omega_\tau, \quad y_0 = u_0, \tag{3.59}$$

with

$$B = E + \tau AG, \quad AG \neq GA. \tag{3.60}$$

THEOREM 3.12 *Let $A = A(t) > 0$ in the difference scheme (3.59), (3.60). Then for*

$$G(t) \geq 0.5E \tag{3.61}$$

the following a priori *estimate of stability of the difference solution with respect to the initial data and right hand side is valid:*

$$\|y_{n+1}\| \leq \|y_0\| + \|(A^{-1}\varphi)_0\| + \|(A^{-1}\varphi)_n\| + \sum_{k=1}^{n} \tau\|(A^{-1}\varphi)_{\bar{t},k}\|. \tag{3.62}$$

Proof. As in the case of Theorem 3.11, we multiply the difference equation (3.59) from the left by A^{-1}. We obtain the operator-difference scheme

$$\widetilde{B} y_t + \widetilde{A} y = \widetilde{\varphi}, \quad \widetilde{\varphi} = A^{-1}\varphi, \quad y_0 = u_0, \tag{3.63}$$

where the operators $\widetilde{A}$ and $\widetilde{B}$ are defined by (3.57). Under condition (3.61), the operator inequality $\widetilde{B} \geq 0.5\tau\widetilde{A}$ is valid, then by virtue of Theorem 2.16

$$\|y_{n+1}\|_{\widetilde{A}} \leq \|y_0\|_{\widetilde{A}} + \|\widetilde{\varphi}_0\|_{\widetilde{A}^{-1}} + \|\widetilde{\varphi}_n\|_{\widetilde{A}^{-1}} + \sum_{k=1}^{n} \tau \|\widetilde{\varphi}_{\bar{t},k}\|_{\widetilde{A}^{-1}}.$$

Since $\widetilde{A} = E$, $\widetilde{\varphi} = A^{-1}\varphi$ then the required estimate (3.62) follows immediately from the last inequality.

THEOREM 3.13 *Let $A(t) = A^*(t) > 0$ and*

$$G(t) \geq \sigma_0 E, \quad \sigma_0 = \frac{1}{2} - \frac{1}{\tau\|A\|}\ .$$

Then the a priori *estimate (3.62) holds for scheme (3.59), (3.60).*

Proof. To prove this theorem it suffices to make use of the following sequence of inequalities

$$\begin{aligned} \widetilde{B} - 0.5\tau\widetilde{A} &= A^{-1} + \tau(G - 0.5E) \\ &\geq \frac{1}{\|A\|}E + \tau(G - 0.5E) = \tau(G - \sigma_0 E) \geq 0. \end{aligned}$$

REMARK 3.6 Two-level operator-difference schemes with operator weighted factors of the following form

$$y_t + Ay^{(\Sigma)} = \varphi, \quad y_0 = u_0 \tag{3.64}$$

can be reduced to schemes of the class (3.59), (3.60) considered above.

Indeed, by virtue of the identity

$$y^{(\Sigma)} = y + \tau\Sigma y_t$$

the scheme (3.64) can be transformed into the original scheme with $G = \Sigma$.

3.3 *A Priori* Estimates in Other Norms

We reduce the two-level scheme (3.59), (3.60) to the canonical form

$$\widetilde{B}\frac{y_{n+1}-y_n}{\tau}+\widetilde{A}y_n=\widetilde{\varphi}_n,\quad y_0=u_0, \tag{3.65}$$

in which the operators $\widetilde{A}$, $\widetilde{B}$ are self-adjoint. If A^{-1} exists then, multiplying equation (3.59) from the left by this operator, we obtain the scheme (3.65) with

$$\widetilde{B}=A^{-1}+\tau G,\quad \widetilde{A}=E,\quad \widetilde{\varphi}=A^{-1}\varphi. \tag{3.66}$$

THEOREM 3.14 *Let the constant operators A and G be self-adjoint and A^{-1} exist. Then the solution of problem (3.59), (3.60) satisfies the following* a priori *estimate*

$$(Ry_{n+1},y_{n+1})\le(Ry_0,y_0)+\frac{1}{2}\sum_{k=0}^{n}\tau\|A^{-1}\varphi_k\|^2 \tag{3.67}$$

with $R=A^{-1}+\tau(G-0.5E)$.

Proof. Since the difference scheme (3.65) satisfies all of the conditions of Theorem 2.15, then, according to the inequality (2.105), we have

$$\Big((\widetilde{B}-0.5\tau\widetilde{A})y_{n+1},y_{n+1}\Big)\le\Big((\widetilde{B}-0.5\tau\widetilde{A})y_0,y_0\Big)+\frac{1}{2}\sum_{k=0}^{n}\tau\|\widetilde{\varphi}_k\|^2_{\widetilde{A}^{-1}},$$

whence as $\widetilde{B}-0.5\tau\widetilde{A}=R$, $\|\widetilde{\varphi}\|_{\widetilde{A}^{-1}}=\|A^{-1}\varphi\|$, the estimate (3.67) follows.

REMARK 3.7 Provided $A>0$ and $G\ge 0.5E$, the operator $R=R^*>0$ is self-adjoint and positive, and the inequality (3.67) ensures the stability of the scheme in the energy space H_R:

$$\|y_{n+1}\|_R^2\le\|y_0\|_R^2+\frac{1}{2}\sum_{k=0}^{n}\tau\|A^{-1}\varphi_k\|^2. \tag{3.68}$$

Furthermore, if $G>0.5E$, then the estimate (3.68) can be also rewritten as

$$\begin{aligned}&\|y_{n+1}\|^2_{A^{-1}}+\tau\|y_{n+1}\|^2_{G-0.5E}\\&\qquad\le\|y_0\|^2_{A^{-1}}++\tau\|y_0\|^2_{G-0.5E}+\frac{1}{2}\sum_{k=0}^{n}\tau\|A^{-1}\varphi_k\|^2.\end{aligned} \tag{3.69}$$

THEOREM 3.15 *Let the constant operators A and G satisfy conditions*

$$A = A^* > 0, \quad G = G^*, \quad G \geq \sigma_\varepsilon E,$$
$$\sigma_\varepsilon = \frac{1+\varepsilon}{2} - \frac{1}{\tau\|A\|}, \quad \varepsilon > 0. \tag{3.70}$$

Then the solution of the difference problem

$$(E + \tau AG)\frac{y_{n+1} - y_n}{\tau} + Ay_n = \varphi_n, \quad t_n \in \omega_\tau, \quad y_0 = 0 \tag{3.71}$$

satisfies the following estimates

$$\|y_{n+1}\|^2_{A^{-1}} + \tau(Gy_{n+1}, y_{n+1}) \leq \frac{1+\varepsilon}{2\varepsilon}\sum_{k=0}^{n}\tau\|A^{-1}\varphi_k\|^2, \quad \varepsilon > 0, \tag{3.72}$$

$$\|y_{n+1}\|^2_{A^{-1}+\tau G} + \varepsilon\sum_{k=1}^{n}\tau\|y_k\|^2 \leq \frac{4}{\varepsilon(2-\varepsilon)}\sum_{k=0}^{n}\tau\|A^{-1}\varphi_k\|^2, \quad 0 < \varepsilon < 2. \tag{3.73}$$

Proof. Transform the difference problem (3.71) into the canonical form (3.65), (3.66). It is obvious that under conditions (3.70) we have

$$\widetilde{B} - \frac{1+\varepsilon}{2}\tau\widetilde{A} = A^{-1} + \tau(G - 0.5E) \geq \frac{1}{\|A\|}E + \tau(\sigma_0 - 0.5)E = 0.$$

Therefore all of the conditions of Theorem 2.19 are satisfied and then, using *a priori* estimates (2.135) and (2.136), we conclude that

$$\|y_{n+1}\|_{\widetilde{B}} \leq \sqrt{\frac{1+\varepsilon}{2\varepsilon}}\left(\sum_{k=0}^{n}\tau\|\widetilde{\varphi}_k\|^2_{\widetilde{A}^{-1}}\right)^{1/2}$$

and

$$\|y_{n+1}\|^2_{\widetilde{B}} + \varepsilon\sum_{k=1}^{n}\tau\|y_k\|^2_{\widetilde{A}} \leq \frac{4}{\varepsilon(2-\varepsilon)}\sum_{k=0}^{n}\tau\|\widetilde{\varphi}_k\|^2_{\widetilde{A}^{-1}}, \quad 0 < \varepsilon < 2.$$

Since $\widetilde{B} = A^{-1} + \tau G$, $\widetilde{A} = E$, $\widetilde{\varphi} = A^{-1}\varphi$, then the required inequalities (3.72), (3.73) follow directly from these last estimates.

COROLLARY 3.4 *Assuming that $\varepsilon = 1$, from the inequality (3.73) we deduce the following estimate of stability of the difference scheme (3.50) with respect to the right hand side in H^{n+1}:*

$$\sum_{k=0}^{n}\tau\|y_k\|^2_A \leq 4\sum_{k=0}^{n}\tau\|\varphi_k\|^2_{A^{-1}}.$$

4. Difference Schemes with $B = E + \tau T^*GT$

In this section we consider separately the class of two-level operator-difference schemes which can be interpreted as schemes with variable weighted factors when weighting not the solution itself but its fluxes.

4.1 Stability with Respect to the Initial Data

Consider operators of a special kind (divergent or conservative)

$$A = T^*T, \quad B = E + \tau T^*GT, \tag{3.74}$$

where T, T^* are linear bounded operators. Such operators are often used for discretization of differential equations with $Lu = \mathrm{div}(k\mathrm{grad}u)$.

Let H be a real finite-dimensional Hilbert space provided with the inner product (y, v) and the norm $\|y\| = (y, y)^{1/2}$. We define H^* to be a real finite-dimensional Hilbert space with the inner product $(y, v)^*$ and the norm $\|y\|^* = \sqrt{(y, y)^*}$. Assume that the operator T acts from H to H^*, that the operator G acts from H^* to H^*, and that the operator T^* acts from H^* to H. Then the operator A acts from H to H, i.e., $A : H \to H$.

The operators T and T^* are adjoint in the following sense:

$$(Ty, v)^* = (y, T^*v) \quad \text{for all} \quad y \in H, \quad v \in H^*. \tag{3.75}$$

Note that if the operator A is represented in the form

$$A = TST^*, \quad S = S^* > 0, \quad S : H^* \to H^* \tag{3.76}$$

then on defining the operators $\widetilde{T} : H \to H^*$, $\widetilde{T}^* : H^* \to H$ by

$$\widetilde{T} = S^{1/2}T, \quad \widetilde{T}^* = (S^{1/2}T)^* = T^*S^{1/2}$$

we arrive at the definition of operators

$$A = \widetilde{T}^*\widetilde{T}, \quad B = E + \tau\widetilde{T}^*G\widetilde{T}, \tag{3.77}$$

which is equivalent to (3.74).

Let us turn to the canonical form of two-level operator-difference schemes

$$B\frac{y_{n+1} - y_n}{\tau} + Ay_n = 0, \quad t_n \in \omega_\tau, \quad y_0 = u_0, \tag{3.78}$$

where the operators A and B are defined by (3.74).

THEOREM 3.16 *Let T be a constant operator and T^{-1} exist. For the stability of scheme (3.74), (3.78) in H_A, i.e., for the validity of the estimate*

$$\|y_{n+1}\|_A \le \|y_0\|_A, \quad n = 0, 1, ..., N_0 - 1, \tag{3.79}$$

it is necessary and sufficient to satisfy the inequality

$$G(t) \ge \sigma_0 E, \quad \sigma_0 = \frac{1}{2} - \frac{1}{\tau\|A\|}. \tag{3.80}$$

Proof. The desired conclusion follows from the inequality

$$\begin{aligned} B - 0.5\tau A &= E + \tau T^*(G - 0.5E)T \\ &\ge \frac{1}{\|A\|}T^*T + \tau T^*(G - 0.5E)T \\ &= \tau T^*\left(G - \left(0.5 - \frac{1}{\tau\|A\|}\right)E\right)T \ge 0, \end{aligned} \tag{3.81}$$

and from Lemma 3.1, Theorem 2.1, and the identity

$$\|y\|_A = \|y\|_{T^*T} = \sqrt{(T^*Ty, y)} = \|Ty\|^*. \tag{3.82}$$

If $A(t) = A^*(t) > 0$ (the operator $A(t)$ depends on t) then we shall require the fulfilment of the Lipschitz continuity condition for $A(t)$ (see (2.36)):

$$|\left(\left(A(t) - A(t-\tau)\right)v, v\right)| \le \tau c_0 \left(A(t-\tau)v, v\right) \tag{3.83}$$

for all $v \in H$, $0 < t \le T$, where c_0 is a positive constant independent of τ.

THEOREM 3.17 *Let the operator $A(t) = T^*(t)T(t)$ be Lipschitz continuous with respect to the variable t, T^{-1} exist, and the operator $G(t)$ satisfy conditions (3.80). Then the difference scheme (3.74), (3.78) is ρ-stable with respect to the initial data with $\rho = e^{0.5c_0\tau}$ $\left(M_1 = e^{0.5c_0T}\right)$, and the following estimates are valid:*

$$\|y_{n+1}\|_{A_n} \le \rho\|y_n\|_{A_{n-1}}, \quad n = 1, 2, ..., N_0 - 1, \tag{3.84}$$

$$\|y_{n+1}\|_{A_n} \le M_1\|y_0\|_{A_0}, \quad n = 0, 1, ..., N_0 - 1. \tag{3.85}$$

Proof. Since under condition (3.80) the inequality $B(t) \ge 0.5\tau A(t)$ is satisfied according to (3.81), then, applying Theorem 2.2, we verify that the solution of problem (3.78) satisfies the required estimates (3.84) and (3.85).

We are now in a position to formulate the stability results, similar to Theorem 2.3, for the case of operators A and B defined by (3.74).

THEOREM 3.18 *Let the operators T and G in the Cauchy problem (3.74), (3.78) be constant, T^{-1} exist, and assume that the operator $G = G^* > 0$ satisfies condition (3.80). Then the difference scheme (3.74), (3.78) is stable with respect to the initial data in H_B with $M_1 = 1$, and the following estimate is valid:*

$$\|y_{n+1}\|^2 + \tau\|y_{n+1}]|^2_{T^*GT} \leq \|y_0\|^2 + \tau\|y_0]|^2_{T^*GT}, \; n = 0, 1, ..., N_0 - 1. \tag{3.86}$$

Proof. To prove the theorem it suffices to use estimate (2.52) and the identity

$$\|y\|^2_B = \|y\|^2 + (\|Ty\|^*_G)^2 . \tag{3.87}$$

4.2 Estimates of Stability with Respect to the Right Hand Side

Let us turn now to the more general problem

$$B\frac{y_{n+1} - y_n}{\tau} + Ay_n = \varphi_n, \quad t_n \in \omega_\tau, \quad y_0 = u_0, \tag{3.88}$$

$$B = E + \tau T^*GT, \quad A = T^*T \tag{3.89}$$

with inhomogeneous right hand side and inhomogeneous initial conditions.

THEOREM 3.19 *Let the operator T in the difference scheme (3.88), (3.89) be constant, T^{-1} exist, and $G(t)$ be a linear operator acting from H^* into H^*. Then, if the operator $G(t)$ complies with condition (3.80), the solution of the difference problem satisfies the estimate*

$$\|y_{n+1}\|_A \leq \|y_0\|_A + \|\varphi_0\|_{A^{-1}} + \|\varphi_n\|_{A^{-1}} + \sum_{k=1}^{n} \tau\|\varphi_{\bar{t},k}\|_{A^{-1}}, \tag{3.90}$$

and also the a priori estimate

$$\|y_{n+1}\|^2_A \leq \|y_0\|^2_A + \frac{1}{2\varepsilon}\sum_{k=0}^{n} \tau\|\varphi\|^2, \tag{3.91}$$

if

$$G(t) \geq \sigma_\varepsilon E, \quad \sigma_\varepsilon = \frac{1}{2} - \frac{1-\varepsilon}{\tau\|A\|}, \quad 0 < \varepsilon \leq 1. \tag{3.92}$$

Proof. Since the assumption condition of Theorem 2.16 $B(t) \geq 0.5\tau A$ is satisfied, the estimate (3.90) is a simple consequence of the inequality (2.112). To prove the estimate (3.91) we apply Theorem 2.20. For this it suffices to verify whether the condition (2.142) is satisfied. Indeed, by virtue of the assumptions (3.92) and (3.81)

$$B - \varepsilon E - 0.5\tau A = (1-\varepsilon)E + \tau T^*(G - 0.5E)T$$
$$\geq \tau T^* \left(G - \left(0.5 - \frac{1-\varepsilon}{\tau \|A\|} \right) E \right) T \geq 0.$$

A similar estimate is valid also in the case of a variable operator $A(t)$.

THEOREM 3.20 *Let the hypotheses of Theorem 3.17 be satisfied. Then the difference scheme (3.88), (3.89) is stable with respect to the initial data and the right hand side, and also its solution satisfies the estimate*

$$\|y_{n+1}\|_{A_n} \leq M \left(\|y_0\|_{A_0} + \|\varphi_0\|_{A^{-1}} + \|\varphi_n\|_{A^{-1}} + \sum_{k=1}^{n} \tau \|\varphi_{\bar{t},k}\|_{A^{-1}} \right),$$

where $M > 0$ is a constant.

The *proof* is analogous to the proof of Theorem 2.17.

4.3 Some Other *A Priori* Estimates

We will obtain an estimate of stability in the energy space $H_{B-0.5\tau A}$.

THEOREM 3.21 *Let the hypotheses of Theorem 3.18 be satisfied. Then the solution of scheme (3.88), (3.89) satisfies the following* a priori *estimate*

$$(Ry_{n+1}, y_{n+1}) \leq (Ry_0, y_0) + \frac{1}{2} \sum_{k=0}^{n} \tau \|\varphi_k\|^2_{A^{-1}}, \tag{3.93}$$

where $R = B - 0.5\tau A = E + \tau T^*(G - 0.5E)T$.

Proof. Since by assumption $B = B^*$ and $A = A^* > 0$ are constant operators, and $B \geq 0.5\tau A$, then on the basis of Theorem 2.15 the estimate (3.93) follows from the inequality (2.105).

COROLLARY 3.5 *If the strong inequality $G(t) > \sigma_0 E$ is valid instead of (3.80) then $R > 0$, and hence the inequality (3.93) guarantees the stability of the scheme in the norm $H_{B-0.5\tau A}$. Under the more severe restriction $G > 0.5E$ the estimate (3.93) can be rewritten in the form*

$$\|y_{n+1}\|^2 + \tau \|y_{n+1}\|^2_{T^*(G-0.5E)T}$$
$$\leq \|y_0\|^2 + \tau \|y_0\|^2_{T^*(G-0.5E)T} + \frac{1}{2} \sum_{k=0}^{n} \tau \|\varphi_k\|^2. \tag{3.94}$$

The following theorem gives *a priori* estimate of stability for a problem with homogeneous initial data.

THEOREM 3.22 *Let the constant operators A and G satisfy conditions (3.70). Then the solution of the difference problem*

$$(E + \tau T^* G T) y_t + A y_n = \varphi_n, \quad t_n \in \omega_\tau, \quad y_0 = 0 \tag{3.95}$$

satisfies the following estimates:

$$\|y_{n+1}\|^2 + \tau \|y_{n+1}\|^2_{T^*GT} \le \frac{1+\varepsilon}{2\varepsilon} \sum_{k=0}^{n} \tau \|\varphi_k\|^2_{A^{-1}}, \quad \varepsilon > 0, \tag{3.96}$$

$$\|y_{n+1}\|^2_{E+\tau T^*GT} + \varepsilon \sum_{k=1}^{n} \tau \|y_{k+1}\|^2_A \le \frac{4}{\varepsilon(2-\varepsilon)} \sum_{k=0}^{n} \tau \|\varphi_k\|^2_{A^{-1}}, \ 0 < \varepsilon < 2. \tag{3.97}$$

Proof. The *a priori* estimates being proved follow from Theorem 2.19 and the inequality

$$B - \frac{1+\varepsilon}{2} \tau A = E + \tau T^* \left(G - \frac{1+\varepsilon}{2} E \right) T$$
$$\ge \tau T^* \left(G - \left(\frac{1+\varepsilon}{2} - \frac{1}{\tau \|A\|} \right) E \right) T \ge 0.$$

COROLLARY 3.6 *The inequality (3.97) implies the following estimate of stability of the difference scheme (3.95) with respect to the right hand side in H_A^{n+1}:*

$$\sum_{k=0}^{n} \tau \|y_{k+1}\|^2_A \le \frac{4}{\varepsilon^2(2-\varepsilon)} \sum_{k=0}^{n} \tau \|\varphi_k\|^2_{A^{-1}}, \quad 0 < \varepsilon < 2.$$

REMARK 3.8 Two-level operator-difference schemes with operator weighted factors of the form

$$y_t + T^*(Ty)^{(\Sigma)} = \varphi, \quad y_0 = u_0 \tag{3.98}$$

with $G = \Sigma$ can be reduced to schemes of the class (3.88), (3.89) considered above.

When considering the heat equation, we can say for such a scheme about the weighting of heat flow.

Chapter 4

THREE-LEVEL DIFFERENCE SCHEMES

1. Introduction

Along with two-level difference schemes, three-level schemes are often also used to solve numerically non-stationary problems of mathematical physics. Such difference schemes are typical if we consider second-order evolution equations, one example of which is the equation of oscillations.

We will consider real mesh functions y from a finite-dimensional real Hilbert space H, in which we use the following standard notations for a scalar product and a norm: $(\cdot,\cdot)$, $\|y\| = \sqrt{(y,y)}$. For an operator $D = D^* > 0$ we denote by H_D the space H provided with the scalar product $(y,w)_D = (Dy,w)$ and the norm $\|y\|_D = \sqrt{(Dy,y)}$.

Assume, for example, that we seek a solution $u(t) \in H$ of the Cauchy problem for the second-order evolution equation

$$\frac{d^2u}{dt^2} + Au = f(t), \quad 0 < t \le T, \tag{4.1}$$

$$u(0) = u_0, \tag{4.2}$$

$$\frac{du}{dt}(0) = u_1. \tag{4.3}$$

For simplicity we restrict ourselves to the simplest case of a positive, self-adjoint and stationary operator A, that is, $A \neq A(t) = A^* > 0$.

Let us give an estimate of stability with respect to the initial data and the right hand side for problem (4.1)–(4.3), on which we further rely when we will construct difference schemes. For problem (4.1)–(4.3)

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the *a priori* estimate

$$\|u(t)\|_* \le \|u_0\|_A + \|u_1\| + \int_0^t \|f(s)\| ds, \tag{4.4}$$

is true, where

$$\|u(t)\|_*^2 \equiv \|u\|_A^2 + \left\|\frac{du}{dt}\right\|^2.$$

As we see, in this case the solution is estimated in a sufficiently complex (composite) norm depending on the operator A. A similar situation occurs if we consider the corresponding difference schemes.

Three-level schemes are natural for consideration of problems for second-order equations. These schemes can be also used to find the approximate solution of the Cauchy problem for a first-order evolution equation for which two-level difference schemes have been studied earlier.

The stability criteria in the general theory of stability of difference scheme are formulated for difference schemes written down in canonical form. The following canonical form is usually used for homogeneous three-level difference schemes:

$$D\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} + B\frac{y_{n+1} - y_{n-1}}{2\tau} + Ay_n = 0, \quad n = 1, 2, \dots, \tag{4.5}$$

for given

$$y_0 = u_0, \quad y_1 = u_1. \tag{4.6}$$

Let us formulate conditions of stability with respect to the initial data for constant (independent of n) self-adjoint operators A, B, R. Let the operators R and A in the operator-difference scheme (4.5), (4.6) be self-adjoint. Then under the conditions

$$B \ge 0, \quad A > 0, \quad R = D - \frac{\tau^2}{4}A > 0$$

we have the required *a priori* estimate

$$\frac{1}{4}\|y_{n+1} + y_n\|_A^2 + \left\|\frac{y_{n+1} - y_n}{\tau}\right\|_R^2 \le \frac{1}{4}\|y_n + y_{n-1}\|_A^2 + \left\|\frac{y_n - y_{n-1}}{\tau}\right\|_R^2,$$

that is, the operator-difference scheme (4.5), (4.6) is stable with respect to the initial data.

The peculiarity of three-level schemes under consideration is just the complicated construction of the norm used, and it is not surprising. As has been mentioned above, this takes place even in the case of the simplest Cauchy problem for a second-order evolution equation. In certain important cases (for example, if we construct difference schemes for first-order evolution equations), under restriction of a class of difference schemes it is possible to pass to simpler norms.

It is convenient to study multi-level difference schemes by passage to an equivalent two-level scheme, because the most profound results (in particular, the coinciding necessary and sufficient conditions of stability) have been obtained for two-level schemes. The three-level operator-difference scheme (4.5) can be written in the form of a two-level vector schemes:

$$\mathcal{B}\frac{Y_{n+1} - Y_n}{\tau} + \mathcal{A}Y_n = 0, \quad n = 1, 2, \ldots,$$

where the vectors Y_n are defined by

$$Y_n = \left\{ \frac{1}{2}(y_n + y_{n-1}), y_n - y_{n-1} \right\}.$$

Based on this equivalence relation we derive *a priori* estimates of stability and ρ-stability of three-level difference schemes with respect to the initial data and the right hand side. We also obtain sufficient stability conditions for a wide class of three-level difference schemes with variable and not self-adjoint operators.

As meaningful examples, we consider three-level schemes with constant weighted factors for evolution equations of first and second orders. For example, to solve numerically problem (4.1)–(4.3) it is natural to apply the scheme with the second order of approximation

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} + A(\sigma y_{n+1} + (1 - 2\sigma)y_n + \sigma y_{n-1}) = f_n, \; n = 1, 2, \ldots, \tag{4.7}$$

for given y_0, y_1. For this difference scheme the *a priori* estimate

$$\|y_{n+1}\|_* \leq \|y_n\|_* + \tau\|f_n\| \tag{4.8}$$

is valid, where now

$$\|y_{n+1}\|_*^2 \equiv \left\| \frac{y_{n+1} - y_n}{\tau} \right\|^2_{E+(\sigma-\frac{1}{4})\tau^2 A} + \left\| \frac{y_{n+1} + y_n}{2} \right\|^2_A .$$

Estimate (4.8) is consistent with estimate (4.4) for the solution of the differential problem and, for $\sigma \geq 0.25$, provides the unconditional stability of the difference weighted scheme (4.7) with respect to the initial data and the right hand side.

2. Stability of Difference Schemes

The canonical form for three-level difference schemes is identified in this section. To derive necessary and sufficient conditions for stability in this class of schemes it is possible to consider a three-level difference scheme as a certain two-level vector scheme.

2.1 Canonical Form

Let H be a real finite-dimensional space and

$$\overline{\omega}_\tau = \{t_n = n\tau, \quad n = 0, 1, ..., N_0; \quad \tau N_0 = T\} = \omega_\tau \cup \{T\} \tag{4.9}$$

be a grid in time. We denote by $B_\alpha : H \to H$, $\alpha = 0, 1, 2$, linear operators in H which generally depend on τ, t_n. Consider the Cauchy problem for the operator-difference equation

$$B_2 y_{n+1} + B_1 y_n + B_0 y_{n-1} = \varphi_n, \quad n = 1, 2, ..., N_0 - 1, \tag{4.10}$$

$$y_0 = u_0, \quad y_1 = u_1, \tag{4.11}$$

where $y_n = y(t_n) \in H$ is the desired solution, φ_n, u_0, $u_1 \in H$ are given. We shall use the notation without indices

$$y = y_n, \quad \hat{y} = y_{n+1}, \quad \check{y} = y_{n-1}, \quad y_{\bar{t}} = \frac{y - \check{y}}{\tau}, \quad y_t = \frac{\hat{y} - y}{\tau},$$

$$y_{\overset{\circ}{t}} = \frac{\hat{y} - \check{y}}{2\tau}, \quad y_{\bar{t}t} = \frac{\hat{y} - 2y + \check{y}}{\tau^2}.$$

The scheme (4.10), (4.11) can be represented in the equivalent form

$$D y_{\bar{t}t} + B y_{\overset{\circ}{t}} + A y = \varphi, \quad y_0 = u_0, \quad y_1 = u_1, \tag{4.12}$$

where

$$A = B_0 + B_1 + B_2, \quad B = \tau(B_2 - B_0), \quad D = \frac{\tau^2}{2}(B_0 + B_2),$$

$$B_2 = \frac{1}{2\tau}B + \frac{1}{\tau^2}D, \quad B_1 = A - \frac{2}{\tau^2}D, \quad B_0 = -\frac{1}{2\tau}B + \frac{1}{\tau^2}D.$$

The set of Cauchy problems (4.12) depending on the parameter τ is regarded as a *three-level scheme* . We shall call the notation (4.12) the *canonical form of three-level schemes*. We shall call the notation (4.12) the *canonical form of three-level schemes*.

Rewrite the equation (4.12) in the form

$$\left(B + \frac{2}{\tau}D\right) y_{n+1} = \varphi_n^*,$$

$$\varphi_n^* = 2\left(\frac{2}{\tau^2}D - A\right)\tau y_n + \left(B - \frac{2}{\tau}D\right) y_{n-1} + 2\tau\varphi_n.$$

Here A, B and D are, generally speaking, variable operators which depend on t_n. It is obvious that the problem (4.12) is resolvable whenever the operator $\left(B + \frac{2}{\tau}D\right)^{-1}$ exists. In the sequel we always assume this condition to be satisfied.

2.2 Reduction to Two-Level Scheme

Define the space H^2 as a set of vectors

$$Y = \left\{Y^{(1)}, Y^{(2)}\right\}, \quad Y^{(\alpha)} \in H, \quad \alpha = 1, 2,$$

in which summing and multiplication by some number are performed by coordinates:

$$Y + V = \left\{Y^{(1)} + V^{(1)}, Y^{(2)} + V^{(2)}\right\}, \quad \alpha Y = \left\{\alpha Y^{(1)}, \alpha Y^{(2)}\right\}.$$

If an inner product $(\cdot, \cdot)$ is introduced in the space H, then the subordinate inner product can be also introduced in the space H^2 as

$$(Y, V) = \sum_{\alpha=1}^{2} \left(Y^{(\alpha)}, V^{(\alpha)}\right).$$

The three-level scheme (4.12) can be reduced to the following two-level one

$$\mathcal{B} Y_t + \mathcal{A} Y = \Phi, \quad Y_1 \in H^2, \tag{4.13}$$

where $Y = Y_n \in H^2$, $Y_t = (Y_{n+1} - Y_n)/\tau$, $\Phi = \Phi_n \in H^2$, $\mathcal{A}$ and $\mathcal{B}$ are operators acting in H^2. To do this it suffices to define the vectors

$$Y_n = \left\{\frac{1}{2}(y_n + y_{n-1}), y_n - y_{n-1}\right\}, \quad \Phi_n = \{\varphi_n, 0\} \tag{4.14}$$

and the operators $\mathcal{A}$ and $\mathcal{B}$ as operator matrices

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & \dfrac{1}{\tau^2}\left(D - \dfrac{\tau^2}{4}A\right) \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} B + 0.5\tau A & \dfrac{1}{\tau}\left(D - \dfrac{\tau^2}{4}A\right) \\ -\dfrac{1}{\tau}\left(D - \dfrac{\tau^2}{4}A\right) & \dfrac{1}{2\tau}\left(D - \dfrac{\tau^2}{4}A\right) \end{pmatrix} \tag{4.15}$$

with operator elements in H.

This representation of the three-level scheme (4.12) in the form of the two-level one (4.13)–(4.15) is not unique. To verify this we define a vector $\widetilde{Y}_n \in H^2$ and operator matrices $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{B}} : H^2 \to H^2$ in the following way:

$$\widetilde{Y} = \{y_n, y_n - y_{n-1}\}, \tag{4.16}$$

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A & 0 \\ 0 & \dfrac{1}{\tau^2}\left(D - \dfrac{\tau}{2}B\right) \end{pmatrix},$$

$$\widetilde{\mathcal{B}} = \begin{pmatrix} B & \dfrac{1}{\tau}\left(D - \dfrac{\tau}{2}B\right) \\ -\dfrac{1}{\tau}\left(D - \dfrac{\tau}{2}B\right) & \dfrac{1}{\tau}\left(D - \dfrac{\tau}{2}B\right) \end{pmatrix}. \tag{4.17}$$

Then the three-level scheme (4.12) can be transformed to one more two-level operator-difference scheme of the form

$$\widetilde{\mathcal{B}}\widetilde{Y}_t + \widetilde{\mathcal{A}}\widetilde{Y} = \Phi, \quad \widetilde{Y}_1 \in H^2. \tag{4.18}$$

Applying schemes (4.13), (4.18), one can investigate three-level difference schemes using the relevant results on stability of two-level operator-difference schemes stated in Chapter 2.

3. Stability with Respect to the Initial Data

Fundamental results on stability of three-level operator-difference schemes have been obtained in [Vabishchevich et al., 1993, Goolin, 1968, Samarskii, 1970]. The most complete exposition of results in this area is contained in the book [Samarskii and Goolin, 1973].

3.1 Necessary and Sufficient Conditions

If $A = A^*$, $D = D^*$ then the operator $\mathcal{A} : H^2 \to H^2$ is self-adjoint, whilst $\mathcal{B}$ is non-self-adjoint. In this case, for

$$A = A^* > 0, \quad D = D^* > \frac{\tau^2}{4} A \tag{4.19}$$

we have $\mathcal{A} > 0$, and so the expression

$$\|Y\|_{\mathcal{A}}^2 = (\mathcal{A}Y, Y) = \left(AY^{(1)}, Y^{(1)}\right) + \frac{1}{\tau^2}\left(\left(D - \frac{\tau^2}{4}A\right) Y^{(2)}, Y^{(2)}\right) \tag{4.20}$$

represents a norm in the energy space $H_{\mathcal{A}}^2$. For

$$Y_n^{(1)} = \frac{1}{2}(y_n + y_{n-1}), \quad Y_n^{(2)} = (y_n - y_{n-1}) = \tau y_{\bar{t}}$$

from the norm (4.20) we have

$$\|Y\|_{\mathcal{A}}^2 = \frac{1}{4}\|y + \check{y}\|_A^2 + \|y_{\bar{t}}\|_{D - \frac{\tau^2}{4}A}^2. \tag{4.21}$$

Under the additional condition $D > \dfrac{\tau^2}{2} A$ the equality (4.21) can be rewritten in the equivalent form

$$\|Y\|_{\mathcal{A}}^2 = \frac{1}{2}\left(\|y\|_A^2 + \|\check{y}\|_A^2 + \|y_{\bar{t}}\|_{D - \frac{\tau^2}{2}A}^2\right). \tag{4.22}$$

Let us consider the problem of stability with respect to the initial data for a homogeneous difference scheme

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = 0, \quad y_0 = u_0, \quad y_1 = u_1. \tag{4.23}$$

THEOREM 4.1 *Let the operators A and D do not depend on n and satisfy conditions (4.19). Then the condition*

$$B(t) \geq 0 \tag{4.24}$$

is necessary and sufficient for the stability of scheme (4.23) with respect to the initial data in $H_{\mathcal{A}}^2$ with constant $M_1 = 1$, i.e., for the fulfilment of the estimate

$$\|Y_n\|_{\mathcal{A}} \leq \|U_0\|_{\mathcal{A}}, \quad n = 1, 2, ..., N_0, \tag{4.25}$$

where

$$\|Y_n\|_{\mathcal{A}} = \frac{1}{4}\|y_n + y_{n-1}\|_A^2 + \|y_{\bar{t}}\|_{D - \frac{\tau^2}{4}A}^2,$$

$$U_0 = \left\{U^{(1)}, U^{(2)}\right\}, \quad U^{(1)} = \frac{1}{2}(u_0 + u_1), \quad U^{(2)} = u_1 - u_0.$$

Proof. Rewrite scheme (4.23) in the canonical form of two-level operator-difference schemes

$$\mathcal{B}Y_t + \mathcal{A}Y = 0, \quad Y_1 = U_0, \tag{4.26}$$

where $Y \in H^2$ and $\mathcal{A}$, $\mathcal{B} : H^2 \to H^2$ are defined by (4.14) and (4.15) respectively. Since $\mathcal{A} = \mathcal{A}^* > 0$ under conditions (4.19), it follows from Theorem 2.1 that the necessary and sufficient condition for stability of scheme (4.26) with respect to the initial data in $H^2_{\mathcal{A}}$ is expressed by the operator inequality

$$\mathcal{B} \geq 0.5\tau\mathcal{A}. \tag{4.27}$$

Let us show that for any element

$$Y = \left\{Y^{(1)}, Y^{(2)}\right\} \in H^2, \quad Y^{(\alpha)} \in H, \quad \alpha = 1, 2,$$

the identity

$$((\mathcal{B} - 0.5\tau\mathcal{A})\, Y, Y) = \left(BY^{(1)}, Y^{(1)}\right) \tag{4.28}$$

is valid. Indeed, since

$$\mathcal{B} - 0.5\tau\mathcal{A} = \begin{pmatrix} B & \dfrac{1}{\tau}\left(D - \dfrac{\tau^2}{4}A\right) \\ -\dfrac{1}{\tau}\left(D - \dfrac{\tau^2}{4}A\right) & 0 \end{pmatrix},$$

then by definition we have

$$(\mathcal{B}-0.5\tau\mathcal{A})\, Y = \left\{BY^{(1)} + \frac{1}{\tau}\left(D - \frac{\tau^2}{4}A\right)Y^{(2)}, -\frac{1}{\tau}\left(D - \frac{\tau^2}{4}A\right)Y^{(1)}\right\},$$

$$\begin{aligned} ((\mathcal{B} - 0.5\tau\mathcal{A})\, Y, Y) = \left(BY^{(1)}, Y^{(1)}\right) &+ \frac{1}{\tau}\left(\left(D - \frac{\tau^2}{4}A\right) Y^{(2)}, Y^{(1)}\right) \\ &- \frac{1}{\tau}\left(\left(D - \frac{\tau^2}{4}A\right) Y^{(1)}, Y^{(2)}\right). \end{aligned} \tag{4.29}$$

Hence because the operator $D - \dfrac{\tau^2}{4}A$ is self-adjoint, the identity (4.28) follows directly from (4.29). The inequalities (4.24) and (4.27) are therefore equivalent.

REMARK 4.1 If

$$D = D^*, \quad B(t) \geq 0, \quad A = A^*$$

(A, D are constant operators of not fixed sign), the solution of problem (4.23) satisfies the estimate

$$\begin{aligned}\frac{1}{4}(A(y_{n+1} + y_n), y_{n+1} + y_n) + \left(\left(D - \frac{\tau^2}{4}A\right) y_t, y_t\right) \\ \leq \frac{1}{4}\left(A(y_1 + y_0), y_1 + y_0\right) + \left(\left(D - \frac{\tau^2}{4}A\right) y_t(0), y_t(0)\right).\end{aligned} \tag{4.30}$$

The estimate (4.30) is a simple consequence of the identity (2.35). Really, because problems (4.23) and (4.26) are equivalent, for the solution of the two-level scheme (4.26), according to Remark 2.1, we have

$$(\mathcal{A}Y_{n+1}, Y_{n+1}) \leq (\mathcal{A}Y_1, Y_1). \tag{4.31}$$

Since

$$Y = \left\{\frac{1}{2}(y + \check{y}), y - \check{y}\right\}, \quad \mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & \frac{1}{\tau^2}\left(D - \frac{\tau^2}{4}A\right)\end{pmatrix}$$

the estimate (4.30) follows from the inequality (4.31).

3.2 ρ-Stability

As in the above considerations, we will study stability of the scheme (4.23) in the energy space $H^2_{\mathcal{D}}$ with the metric, generated by some self-adjoint positive operator $\mathcal{D} : H^2 \to H^2$,

$$(Y, V)_{\mathcal{D}} = (\mathcal{D}Y, V), \quad \|Y\|_{\mathcal{D}} = \sqrt{(\mathcal{D}Y, Y)}.$$

Assume that the operator $\mathcal{D}$ does not depend on n. We say that the scheme (4.23) is *ρ-stable* (*uniform stable*) with respect to the initial data if the following estimate

$$(\mathcal{D}Y_{n+1}, Y_{n+1}) \leq \rho^2(\mathcal{D}Y_n, Y_n) \tag{4.32}$$

is valid, where $\rho \leq M_1$, M_1 is a constant independent of τ, n. Let us also recall from the theory of difference schemes that one of the following magnitudes is usually chosen as the constant ρ: $\rho = 1$, $\rho = 1 + c\tau$, $c > 0$, $\rho = e^{c\tau}$, where the constant c is independent of τ, n. Consequently the condition $\mathcal{B} \geq 0.5\tau\mathcal{A}$ is a necessary and sufficient condition for ρ-stability of scheme (4.23) in the space $H^2_{\mathcal{A}}$ with the constant $\rho = 1$.

Up to now we have assumed that the operators A and D are constant. If $A(t) = A^*(t) > 0$, $D(t) = D^*(t) > 0$ (these operators depend on t), then we will require the fulfilment of the Lipschitz continuity conditions with respect to t:

$$|((A(t) - A(t-\tau))\, v, v)| \leq \tau c_1 \, (A(t-\tau)v, v)\,, \tag{4.33}$$

$$|((R(t) - R(t-\tau))\, v, v)| \leq \tau c_2 \, (R(t-\tau)v, v) \tag{4.34}$$

for all $v \in H$, $0 < t \leq T$, where $R(t) = D(t) - \dfrac{\tau^2}{4} A(t) \geq 0$, c_1, c_2 are positive constants.

THEOREM 4.2 *Let the operators of scheme (4.23) satisfy conditions (4.33), (4.34) and also*

$$B(t) \geq 0, \quad D(t) = D^*(t) > 0, \quad A(t) = A^*(t) > 0. \tag{4.35}$$

Then, if the inequality

$$D(t) > \frac{\tau^2}{4} A(t)$$

holds, the difference scheme (4.23) is ρ-stable with respect to the initial data with constants $\rho = e^{0.5c_0\tau}$, $c_0 = \max(c_1, c_2)$, $M_1 = \rho^{N_0} = e^{0.5c_0 T}$, and the following a priori *estimates are valid:*

$$\|y_n^{(0.5)}\|^2_{A_n} + \|y_{t,n}\|^2_{R_n} \leq \rho \left(\|y_{n-1}^{(0.5)}\|^2_{A_{n-1}} + \|y_{t,n-1}\|^2_{R_{n-1}} \right), \tag{4.36}$$

$$\|y_n^{(0.5)}\|^2_{A_n} + \|y_{t,n}\|^2_{R_n} \leq M_1 \left(\|y_0^{(0.5)}\|_{A_0} + \|y_{t,0}\|^2_{R_0} \right), \tag{4.37}$$

where $y^{(0.5)} = 0.5(y + \hat{y})$, $y_{t,n} = (y_{n+1} - y_n)/\tau$.

Proof. The estimate (4.36) is a simple consequence of Theorem 2.2. We have

$$\mathcal{A}(t) - \mathcal{A}(t-\tau) = \begin{pmatrix} A(t) - A(t-\tau) & 0 \\ 0 & R(t) - R(t-\tau) \end{pmatrix}$$

and, because of conditions (4.33) and (4.34), it follows that

$$\begin{aligned} &((\mathcal{A}(t) - \mathcal{A}(t-\tau))Y, Y) \\ &\quad = \left((A(t) - A(t-\tau))\, Y^{(1)}, Y^{(1)} \right) + \left((R(t) - R(t-\tau))\, Y^{(2)}, Y^{(2)} \right) \\ &\quad \leq \tau c_1 \left(A(t-\tau) Y^{(1)}, Y^{(1)} \right) + \tau c_2 \left(R(t-\tau) Y^{(2)}, Y^{(2)} \right) \\ &\quad \leq \tau c_0 \, (\mathcal{A}(t-\tau)Y, Y)\,. \end{aligned}$$

Therefore the operator $\mathcal{A}(t)$ is Lipschitz continuous in the sense of the definition (2.36), and on the basis of *a priori* estimates (2.39), (2.40), using the identity

$$\|Y_{n+1}\|^2_{\mathcal{A}_n} = \|y_n^{(0.5)}\|^2_{A_n} + \|y_{t,n}\|^2_{R_n} \tag{4.38}$$

we verify the validity of the required estimates (4.36), (4.37).

Assume that the operators A and D are both self-adjoint and independent of n. Then the operator $\mathcal{A}$ is a self-adjoint and constant operator as well. We apply Theorem 2.5 to scheme (4.23). According to this theorem, if $\mathcal{A} = \mathcal{A}^* > 0$, then the condition

$$\mathcal{B} \geq \frac{\tau}{1+\rho}\mathcal{A} \tag{4.39}$$

for $\rho \geq 1$ is sufficient for stability of scheme (4.23) in the space $H^2_{\mathcal{A}}$, i.e., for the following estimate to be satisfied:

$$\|Y_{n+1}\|_{\mathcal{A}} \leq \rho \|Y_n\|_{\mathcal{A}}, \quad \rho \geq 1.$$

Here we use the notation introduced above

$$Y_{n+1} = \left\{y_n^{(0.5)}, \tau y_{t,n}\right\}, \quad \mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & \dfrac{1}{\tau^2}R \end{pmatrix}, \quad R = D - \frac{\tau^2}{4}A.$$

Note that condition (4.39) for $\mathcal{A} > 0$ already guarantees the existence of the operator $\mathcal{B}^{-1}$. What is left to us is to formulate stability conditions for the three-level scheme (4.23) in terms of the original operators A, B, D. The operator

$$\mathcal{P} = \mathcal{B} - \frac{\tau}{1+\rho}\mathcal{A} \tag{4.40}$$

obviously has the following components:

$$\mathcal{P}_{11} = B + \frac{\tau}{2}\frac{\rho-1}{\rho+1}A, \quad \mathcal{P}_{12} = -\mathcal{P}_{21} = \frac{1}{\tau}\left(D - \frac{\tau^2}{4}A\right),$$

$$\mathcal{P}_{22} = \frac{1}{2\tau}\frac{\rho-1}{\rho+1}\left(D - \frac{\tau^2}{4}A\right).$$

Since the operators $\mathcal{P}_{12}$, $\mathcal{P}_{21}$ are self-adjoint the condition (4.39) for $\rho \geq 1$ is equivalent to the requirements

$$B + \frac{\tau}{2}\frac{\rho-1}{\rho+1}A \geq 0, \quad D \geq \frac{\tau^2}{4}A.$$

Thus the following theorem about stability of the scheme (4.23) has been proved.

THEOREM 4.3 *Let the operators A and D in scheme (4.23) be independent of n and self-adjoint. Then the conditions*

$$A > 0, \quad D > \frac{\tau^2}{4} A, \tag{4.41}$$

$$B + \frac{\tau}{2}\frac{\rho - 1}{\rho + 1} A \geq 0, \quad \rho \geq 1, \tag{4.42}$$

are sufficient for the fulfilment of the estimate

$$\|Y_{n+1}\|_{\mathcal{A}} \leq \rho \|Y_n\|_{\mathcal{A}}, \quad n = 0, 1, ..., N_0 - 1, \tag{4.43}$$

where

$$\|Y_{n+1}\|^2_{\mathcal{A}} = \frac{1}{4}\|y + \hat{y}\|^2_A + \|y_t\|^2_{D - \frac{\tau^2}{4}A}. \tag{4.44}$$

3.3 Stability in Simpler Norms

The norms used in estimates (4.25), (4.36), (4.37), (4.43) depend on the values of the solution of problem (4.23) at the n-th and $(n+1)$-th levels. It is possible to obtain estimates for the solution of problem (4.23) in somewhat simpler norms than the norm (4.44). Let us prove the following preliminary result.

LEMMA 4.1 *Let the self-adjoint operators A and D satisfy the conditions*

$$A > 0, \quad D \geq \frac{(1+\varepsilon)\tau^2}{4} A, \quad \varepsilon > 0. \tag{4.45}$$

Then the following inequalities hold:

$$\sqrt{\frac{\varepsilon}{1+\varepsilon}}\|y\|_A \leq \|Y\|_{\mathcal{A}} \leq \|\check{y}\|_A + \|y_{\bar{t}}\|_D, \tag{4.46}$$

$$\frac{1}{2}\sqrt{\frac{\varepsilon}{1+\varepsilon}}\left(\|y\|_A + \|y_{\bar{t}}\|_D\right) \leq \|Y\|_{\mathcal{A}}. \tag{4.47}$$

Proof. We first verify that the estimate (4.46) is valid. Writing the norm (4.44) in the form

$$\|Y\|^2_{\mathcal{A}} = \|y_{\bar{t}}\|^2_D + (Ay, \check{y}) \tag{4.48}$$

and applying the transformation

$$\check{y} = y - \tau y_{\bar{t}},$$

we have

$$\|Y\|_A^2 = \|y\|_A^2 - \tau\,(Ay, y_{\bar{t}}) + \|y_{\bar{t}}\|_D^2. \tag{4.49}$$

Then, using the generalized Cauchy–Bunyakovskii–Schwarz inequality, we obtain the estimate

$$|(Ay, y_{\bar{t}})| \le \|y\|_A \|y_{\bar{t}}\|_A.$$

Therefore the identity (4.49) implies the inequality

$$\|Y\|_A^2 \ge \|y\|_A^2 + \|y_{\bar{t}}\|_D^2 - \tau \|y\|_A \|y_{\bar{t}}\|_A. \tag{4.50}$$

We now make use of the inequality

$$\tau \|y\|_A \|y_{\bar{t}}\|_A \le \varepsilon_1 \|y\|_A^2 + \frac{\tau^2}{4\varepsilon_1} \|y_{\bar{t}}\|_A^2, \quad \varepsilon_1 > 0,$$

and take into account that

$$\|y_{\bar{t}}\|_A \le \frac{4}{(1+\varepsilon)\tau^2} \|y_{\bar{t}}\|_D^2$$

by virtue of condition (4.45). Then from the inequality (4.50) we obtain

$$\|Y\|_A^2 \ge (1-\varepsilon_1)\|y\|_A^2 + \left(1 - \frac{1}{\varepsilon_1(1+\varepsilon)}\right) \|y_{\bar{t}}\|_D^2. \tag{4.51}$$

Here ε_1 is an arbitrary parameter chosen in a way depending on ε. For instance, we assume that

$$1 - \frac{1}{\varepsilon_1(1+\varepsilon)} = 0,$$

from which $\varepsilon_1 = (1+\varepsilon)^{-1}$. Then the estimate (4.51) takes the form

$$\|Y\|_A^2 \ge \frac{\varepsilon}{1+\varepsilon} \|y\|_A^2.$$

To estimate the norm (4.44) from above we use again the identity (4.48). By the substitution

$$y = \check{y} + \tau y_{\bar{t}}$$

we find in much the same way as before that

$$\|Y\|_A^2 = \|\check{y}\|_A^2 + \tau\,(A\check{y}, y_{\bar{t}}) + \|y_{\bar{t}}\|_D^2.$$

From the Cauchy–Bunyakovskii–Schwarz inequality and from the condition $D > \dfrac{\tau^2}{4} A$ we obtain

$$|(A\check{y}, y_{\bar{t}})| \le \|\check{y}\|_A \|y_{\bar{t}}\|_A \le \frac{2}{\tau} \|\check{y}\|_A \|y_{\bar{t}}\|_D,$$

and therefore

$$\|Y\|_{\mathcal{A}}^2 \le \|\breve{y}\|_A^2 + 2\|\breve{y}\|_A\|y_{\bar{t}}\|_D + \|y_{\bar{t}}\|_D^2 = \left(\|\breve{y}\|_A + \|y_{\bar{t}}\|_D\right)^2,$$

i.e.,

$$\|Y\|_{\mathcal{A}} \le \|\breve{y}\|_A + \|y_{\bar{t}}\|_D.$$

To obtain the other estimate from below (4.47) for $\|Y\|_{\mathcal{A}}$ we turn to the inequality (4.51) in which we set

$$1 - \varepsilon_1 = 1 - \frac{1}{\varepsilon_1(1+\varepsilon)}.$$

In this case

$$\|Y\|_{\mathcal{A}}^2 \ge \frac{\sqrt{1+\varepsilon}-1}{\sqrt{1+\varepsilon}}\left(\|y\|_A^2 + \|y_{\bar{t}}\|_D^2\right). \tag{4.52}$$

Since $\sqrt{1+\varepsilon} \le 1 + 0.5\varepsilon$ and

$$\frac{\sqrt{1+\varepsilon}-1}{\sqrt{1+\varepsilon}} = \frac{\varepsilon + \left(1-\sqrt{1+\varepsilon}\right)}{1+\varepsilon} \ge \frac{\varepsilon}{2(1+\varepsilon)},$$

then the estimate (4.52) can be simplified as

$$\|Y\|_{\mathcal{A}}^2 \ge \frac{\varepsilon}{2(1+\varepsilon)}\left(\|y\|_A^2 + \|y_{\bar{t}}\|_D^2\right). \tag{4.53}$$

At last, taking account of the inequality $a^2 + b^2 \ge 0.5(a+b)^2$, from estimate (4.53) we obtain the conclusive inequality

$$\|Y\|_{\mathcal{A}} \ge \frac{1}{2}\sqrt{\frac{\varepsilon}{1+\varepsilon}}\left(\|y\|_A + \|y_{\bar{t}}\|_D\right).$$

Using inequalities (4.46) and (4.47) one can simplify the estimates from Theorems 4.2, 4.3.

THEOREM 4.4 *Let the operators A and D of scheme (4.23) be independent of n and self-adjoint. If conditions (4.45) and the inequalities*

$$B + \frac{\tau}{2}\frac{\rho-1}{\rho+1}A \ge 0, \quad \rho \ge 1, \tag{4.54}$$

are satisfied, then the following a priori *estimates are valid:*

$$\|y_{n+1}\|_A \le \sqrt{\frac{1+\varepsilon}{\varepsilon}}\rho^{n+1}\left(\|y_0\|_A + \|y_{t,0}\|_D\right), \tag{4.55}$$

$$\|y_{n+1}\|_A + \|y_{t,n}\|_D \le 2\sqrt{\frac{1+\varepsilon}{\varepsilon}}\rho^{n+1}\left(\|y_0\|_A + \|y_{t,0}\|_D\right). \tag{4.56}$$

Proof. It suffices to apply Theorem 4.3 and Lemma 4.1.

Let us formulate conditions of stability with respect to the initial data in $H^2_{\widetilde{\mathcal{A}}}$ for the scheme

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = 0, \quad y_0 = u_0, \quad y_1 = u_1. \tag{4.57}$$

We will assume that the operators of scheme (4.57) satisfy the conditions

$$A^* = A > 0, \quad B^* = B, \quad D^* = D > \frac{\tau}{2}B. \tag{4.58}$$

As has been shown in Subsection 4.1.2, the scheme (4.57) is equivalent to a two-level operator-difference scheme of the form

$$\widetilde{\mathcal{B}}\widetilde{Y}_t + \widetilde{\mathcal{A}}\widetilde{Y} = 0, \quad \widetilde{Y}_1 = U_0, \tag{4.59}$$

where

$$\widetilde{Y} = \{y, y - \check{y}\}, \quad U_0 = \{u_1, u_1 - u_0\}, \tag{4.60}$$

$$\widetilde{\mathcal{A}} = \begin{pmatrix} A & 0 \\ 0 & \dfrac{1}{\tau^2}\widetilde{R} \end{pmatrix}, \quad \widetilde{\mathcal{B}} = \begin{pmatrix} B & \dfrac{1}{\tau}\widetilde{R} \\ -\dfrac{1}{\tau}\widetilde{R} & \dfrac{1}{\tau}\widetilde{R} \end{pmatrix}, \tag{4.61}$$

$$\widetilde{R} = D - \frac{\tau}{2}B.$$

Under conditions (4.58) the operator $\widetilde{\mathcal{A}} : H^2 \to H^2$ is positive and self-adjoint. In this case the norm in the energy space $H^2_{\widetilde{\mathcal{A}}}$ for any element $\widetilde{Y} = \left\{\widetilde{Y}^{(1)}, \widetilde{Y}^{(2)}\right\}$, $\widetilde{Y}^{(\alpha)} \in H$, $\alpha = 1, 2$, is defined, according to (4.61), in the following manner:

$$\|\widetilde{Y}\|^2_{\widetilde{\mathcal{A}}} = \|\widetilde{Y}^{(1)}\|^2_A + \frac{1}{\tau^2}\|\widetilde{Y}^{(2)}\|^2_{\widetilde{R}}. \tag{4.62}$$

For the vector (4.60) we have

$$\|\widetilde{Y}\|^2_{\widetilde{\mathcal{A}}} = \|y\|^2_A + \|y_{\bar{t}}\|^2_{D-\frac{\tau}{2}B}. \tag{4.63}$$

THEOREM 4.5 *Let the constant operators A, B and D satisfy conditions (4.58). Then the condition*

$$B \geq 0.5\tau A \tag{4.64}$$

is sufficient for the stability of scheme (4.57) with respect to the initial data in $H^2_{\widetilde{\mathcal{A}}}$ with constant $M_1 = 1$, i.e., for the validity of the estimate

$$\|y_n\|^2_A + \|y_{\bar{t},n}\|^2_{D-\frac{\tau}{2}B} \leq \|y_1\|^2_A + \|y_{\bar{t},1}\|^2_{D-\frac{\tau}{2}B}. \tag{4.65}$$

P r o o f. Let us check the fulfilment of the condition for stability of the scheme (4.59) $\widetilde{\mathcal{B}} \geq 0.5\tau\widetilde{\mathcal{A}}$. By definition of the operators $\widetilde{\mathcal{B}}$, $\widetilde{\mathcal{A}}$, and according to the self-adjointness property of the operator $\widetilde{R} = D - \dfrac{\tau}{2}B$, we have

$$\left(\left(\widetilde{\mathcal{B}} - 0.5\tau\widetilde{\mathcal{A}}\right)\widetilde{Y}, \widetilde{Y}\right) = ((B - 0.5\tau A)y, y) + \frac{\tau}{2}\left(\left(D - \frac{\tau}{2}B\right)y_{\bar{t}}, y_{\bar{t}}\right). \tag{4.66}$$

By virtue of conditions (4.58), (4.64), the inequality

$$\widetilde{\mathcal{B}} \geq 0.5\tau\widetilde{\mathcal{A}} \tag{4.67}$$

follows from the relation (4.66). Using now the main theorem 2.1, we come to the required estimate (4.65).

The following assertion contains sufficient conditions of stability in the space H_A.

THEOREM 4.6 *Let the constant operators A, B, D satisfy the assumption*

$$A = A^* > 0, \quad B = B^*, \quad D = D^*.$$

Then the conditions

$$D \geq \frac{\tau}{2}B, \quad B \geq \frac{\tau}{2}A \tag{4.68}$$

are sufficient for the stability of scheme (4.57) with respect to the initial data in H_A with constant $M_1 = 1$, and the following estimate is valid:

$$\|y_{n+1}\|_A^2 \leq \|y_1\|_A^2 + \left(\left(D - \frac{\tau}{2}B\right)y_{\bar{t},1}, y_{\bar{t},1}\right). \tag{4.69}$$

P r o o f. Since the inequality (4.65) is valid under the assumptions of the theorem, then by virtue of estimate (2.35) we have for scheme (4.57) that

$$\left(\widetilde{\mathcal{A}}\widetilde{Y}_{n+1}, \widetilde{Y}_{n+1}\right) \leq \left(\widetilde{\mathcal{A}}\widetilde{Y}_1, \widetilde{Y}_1\right).$$

For the vector (4.60) we obtain

$$\left(\left(D - \frac{\tau}{2}B\right)y_{\bar{t},n+1}, y_{\bar{t},n+1}\right) + \|y_{n+1}\|_A^2 \leq \|y_1\|_A^2 + \left(\left(D - \frac{\tau}{2}B\right)y_{\bar{t},1}, y_{\bar{t},1}\right).$$

By virtue of $D \geq \dfrac{\tau}{2}B$, the conclusion follows from the last inequality.

REMARK 4.2 As

$$D = \frac{\tau}{2}B \tag{4.70}$$

the three-level scheme (4.12) degenerates into the two-level scheme

$$By_t + Ay = \varphi.$$

Hence all of the assertions stated in Chapter 2 are valid in this case for three-level schemes.

The following statement allows us to obtain an *a priori* estimate of the stability with respect to the initial data in H_B for the scheme (4.57) with variable and non-negative operator $A(t) \geq 0$, without requiring the Lipschitz continuity and self-adjointness properties of the operators A and D.

THEOREM 4.7 *Let the variable operators $A(t)$, $D(t)$ and the constant operator B satisfy the following conditions:*

$$A(t) \geq 0, \quad B = B^*, \quad D(t) = \frac{\tau^2}{4} A(t). \tag{4.71}$$

Then the solution of problem (4.57) satisfies the following a priori *estimate*

$$\left(By_n^{(0.5)}, y_n^{(0.5)}\right) \leq \left(By_0^{(0.5)}, y_0^{(0.5)}\right), \tag{4.72}$$

where $y_n^{(0.5)} = \dfrac{1}{2}(y_n + y_{n+1})$.

P r o o f. Using the identity

$$Y = Y^{(0.5)} - \frac{\tau}{2} Y_t, \quad Y^{(0.5)} = \frac{1}{2}(Y_n + Y_{n+1}),$$

we represent scheme (4.59) in the form

$$(\mathcal{B} - 0.5\tau\mathcal{A})\, Y_t + \mathcal{A} Y^{(0.5)} = 0. \tag{4.73}$$

As $D = \dfrac{\tau^2}{4} A$ the following representations are valid:

$$(\mathcal{B} - 0.5\tau\mathcal{A}) = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

and then the scheme (4.73) can be transformed into the form

$$By_{\overset{\circ}{t}} + Ay^{(1/4,1/4)} = 0. \tag{4.74}$$

Here

$$y^{(\sigma_1,\sigma_2)} = \sigma_1 \hat{y} + (1 - \sigma_1 - \sigma_2) y + \sigma_2 \check{y}.$$

Note that by using the identity

$$y^{(\sigma_1,\sigma_2)} = y + \tau(\sigma_1 - \sigma_2)y_{\overset{\circ}{t}} + \frac{\tau^2}{2}(\sigma_1 + \sigma_2)y_{\bar{t}t} \tag{4.75}$$

we can again transform scheme (4.74) into the form (4.57) with $D = \dfrac{\tau^2}{4}A$. By the change of variables

$$v_n = \frac{1}{2}(y_n + y_{n-1}) \tag{4.76}$$

the scheme (4.74) can be written in the form

$$Bv_t + Av^{(0.5)} = 0. \tag{4.77}$$

Multiplying the last equality by $2\tau v^{(0.5)}$ and taking into account that the operator A is non-negative, we obtain the inequality

$$(Bv_{n+1}, v_{n+1}) \leq (Bv_n, v_n), \tag{4.78}$$

which implies the required estimate (4.72).

COROLLARY 4.1 *Under the additional condition $B = B^* > 0$ the inequality (4.72) expresses the stability of the scheme in the space H_B:*

$$\left\| \frac{y_n + y_{n+1}}{2} \right\|_B \leq \left\| \frac{y_0 + y_1}{2} \right\|_B. \tag{4.79}$$

4. Stability with Respect to the Right Hand Side

Based on reducing three-level difference schemes to two-level vector schemes, we establish here *a priori* estimates of stability with respect to the right hand side.

4.1 *A Priori* Estimates

Consider a three-level difference scheme in the canonical form

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad y_0 = u_0, \quad y_1 = u_1. \tag{4.80}$$

To obtain specific *a priori* estimates, we start from an equivalent form of a two-level operator-difference scheme (see. (4.13)):

$$\mathcal{B}Y_t + \mathcal{A}Y = \Phi, \quad Y_1 = U_0. \tag{4.81}$$

We remind that

$$Y = \left\{ \frac{1}{2}(y + \check{y}), y - \check{y} \right\}, \quad \Phi = \{\varphi, 0\},$$
$$U_0 = \left\{ \frac{1}{2}(u_1 + u_0), u_1 - u_0 \right\}, \tag{4.82}$$

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & \dfrac{1}{\tau^2}R \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B + 0.5\tau A & \dfrac{1}{\tau}R \\ -\dfrac{1}{\tau}R & \dfrac{1}{2\tau}R \end{pmatrix}, \tag{4.83}$$
$$R = D - \frac{\tau^2}{4}A.$$

The basic energy identity for a two-level scheme has the form (2.132). Applying this identity to (4.81) in the case of self-adjoint operators A and D, we obtain

$$2\tau\left((\mathcal{B} - 0.5\tau\mathcal{A})Y_t, Y_t\right) + \left(\mathcal{A}\hat{Y}, \hat{Y}\right) = (\mathcal{A}Y, Y) + 2\tau(\Phi, Y_t). \tag{4.84}$$

Since

$$\mathcal{B} - 0.5\tau\mathcal{A} = \begin{pmatrix} B & \dfrac{1}{\tau}R \\ -\dfrac{1}{\tau}R & 0 \end{pmatrix}, \tag{4.85}$$

then from the identity (4.84) we obtain

$$2\tau\left(By_{\overset{\circ}{t}}, y_{\overset{\circ}{t}}\right) + \left(A\hat{Y}, \hat{Y}\right) = (AY, Y) + 2\tau\left(\varphi, y_{\overset{\circ}{t}}\right), \tag{4.86}$$

where

$$(AY, Y) = \frac{1}{4}\left(A(y + \check{y}), y + \check{y}\right) + \left(\left(D - \frac{\tau^2}{4}A\right)y_{\bar{t}}, y_{\bar{t}}\right). \tag{4.87}$$

THEOREM 4.8 *Let $A = A^* > 0$, $D = D^* > 0$ be constant operators. Then, under the conditions*

$$B \geq \varepsilon E, \quad D > \frac{\tau^2}{4}A, \quad \varepsilon = \text{const}, \tag{4.88}$$

the following a priori *estimate for problem (4.80) is valid:*

$$\|Y_{n+1}\|_{\mathcal{A}}^2 \leq \|Y_1\|_{\mathcal{A}}^2 + \frac{1}{2\varepsilon}\sum_{k=1}^{n} \tau\|\varphi(t_k)\|^2. \tag{4.89}$$

P r o o f. The estimation of the functional $2\tau\left(\varphi, y_{\overset{\circ}{t}}\right)$ plays the key role in deriving *a priori* estimates of the kind (4.89).

First of all we note that the obvious inequality

$$2\tau\left(\varphi, y_{\overset{\circ}{t}}\right) \leq 2\tau\varepsilon\|y_{\overset{\circ}{t}}\|^2 + \frac{\tau}{2\varepsilon}\|\varphi\|^2 \tag{4.90}$$

is valid. Substituting the estimate (4.90) into the identity (4.86) and taking into account that $2\tau\left((B-\varepsilon E)\, y_{\overset{\circ}{t}}, y_{\overset{\circ}{t}}\right) \geq 0$, we obtain the estimate

$$\|Y_{k+1}\|_{\mathcal{A}}^2 \leq \|Y_k\|_{\mathcal{A}}^2 + \frac{\tau}{2\varepsilon}\|\varphi(t_k)\|^2.$$

It remains to sum this inequality over all $k = 1, 2, ..., n$.

THEOREM 4.9 *Let the conditions of theorem 4.8 be satisfied and assume that*

$$D \geq \frac{1+\varepsilon}{4}\tau^2 A.$$

Then the difference scheme is stable with respect to the initial data and the right hand side, and its solution satisfies the following estimate:

$$\|y_{n+1}\|_A \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}}\left(\|y_0\|_A + \|y_{t,0}\|_D + \frac{1}{2\sqrt{\varepsilon}}\left[\sum_{k=1}^{n}\tau\|\varphi(t_k)\|^2\right]^{1/2}\right). \tag{4.91}$$

P r o o f. The conclusion of the theorem follows from estimate (4.89) and Lemma 4.1.

In Section 4.2 we have shown that the conditions

$$A = A^* > 0, \quad D = D^* > \frac{\tau^2}{4}A, \quad B(t) \geq 0 \tag{4.92}$$

provide the fulfilment of the operator inequality

$$\mathcal{B}(t) \geq 0.5\tau\mathcal{A}. \tag{4.93}$$

Consequently, because of Theorem 2.16, we have

THEOREM 4.10 *Let the constant operators A, D and the variable operator $B(t)$ satisfy conditions (4.92). Then the difference scheme is stable with respect to the initial data and the right hand side, and the solution of the problem satisfies the following* a priori *estimate:*

$$\|Y_{n+1}\|_{\mathcal{A}} \leq \|Y_1\|_{\mathcal{A}} + \|\varphi_1\|_{A^{-1}} + \|\varphi_n\|_{A^{-1}} + \sum_{k=2}^{n}\tau\|\varphi_{\bar{t},k}\|_{A^{-1}}. \tag{4.94}$$

In the case of a self-adjoint operator B we have

THEOREM 4.11 *If the conditions $A = A^* > 0$, $D = D^* > 0$, $B = B^* > 0$ are satisfied, the operators A and D are constant, and*

$$D > \frac{\tau^2}{4} A,$$

then the following estimate for scheme (4.80) is valid:

$$\|Y_{n+1}\|_{\mathcal{A}}^2 \leq \|Y_1\|_{\mathcal{A}}^2 + \frac{1}{2} \sum_{k=1}^{n} \tau \|\varphi_k\|_{B_k^{-1}}^2. \qquad (4.95)$$

Proof. The fact that the operator B is self-adjoint is used when we estimate the term of identity (4.86) as follows

$$2\tau \left(\varphi, y_{\overset{\circ}{t}}\right) \leq 2\tau \|y_{\overset{\circ}{t}}\|_B \|\varphi\|_{B^{-1}} \leq \tau\varepsilon \|y_{\overset{\circ}{t}}\|_B^2 + \frac{\tau}{\varepsilon} \|\varphi\|_{B^{-1}}^2.$$

Assuming $\varepsilon = 2$ and taking account of the last estimate in the identity (4.86), we obtain the inequality

$$\|Y_{n+1}\|^2 \leq \|Y_n\|^2 + \frac{\tau}{2} \|\varphi_n\|_{B^{-1}}^2,$$

from which the required estimate follows immediately.

4.2 Stability for Homogeneous Initial Data

Together with scheme (4.80) we will consider the problems

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = 0, \quad y(0) = y_0, \quad y_t(0) = \bar{y}_0, \qquad (4.96)$$

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad y(0) = y_t(0) = 0. \qquad (4.97)$$

THEOREM 4.12 *Let $A = A^* \geq 0$, $D = D^* \geq 0$ be constant operators and*

$$B = B(t) \geq \varepsilon E, \quad \varepsilon = \text{const} > 0, \qquad (4.98)$$

and assume that the condition

$$D \geq \frac{\tau^2}{4} A \qquad (4.99)$$

is satisfied. Then the solution of problem (4.97) satisfies the following a priori *estimate*

$$\|y_{n+1}\| \leq \frac{2\sqrt{t}}{\varepsilon} \left[\sum_{k=1}^{n} \tau \|\varphi(t_k)\|^2 \right]^{1/2}. \qquad (4.100)$$

Proof. Consider the identity (4.86). Using, instead of (4.90), the inequality

$$2\tau\left(\varphi, y_{\overset{\circ}{t}}\right) \le \tau\varepsilon\|y_{\overset{\circ}{t}}\|^2 + \frac{\tau}{\varepsilon}\|\varphi\|^2$$

and condition (4.98) we deduce from the identity (4.86) that

$$\tau\varepsilon\|y_{\overset{\circ}{t},k}\|^2 + (\mathcal{A}Y_{k+1}, Y_{k+1}) \le (\mathcal{A}Y_k, Y_k) + \frac{\tau}{\varepsilon}\|\varphi(t_k)\|.$$

Summing over $k = 1, 2, ..., n$ and taking into account the equality $(\mathcal{A}Y_1, Y_1) = 0$, we obtain

$$\varepsilon\sum_{k=1}^{n}\tau\|y_{\overset{\circ}{t},k}\|^2 + (\mathcal{A}Y_{k+1}, Y_{k+1}) \le \frac{1}{\varepsilon}\sum_{k=1}^{n}\tau\|\varphi(t_k)\|^2 \tag{4.101}$$

or

$$\sum_{k=1}^{n}\tau\|y_{\overset{\circ}{t},k}\|^2 \le \frac{1}{\varepsilon^2}\sum_{k=1}^{n}\tau\|\varphi(t_k)\|^2, \tag{4.102}$$

since $(\mathcal{A}Y, Y) \ge 0$.

To derive the estimate (4.100) from the inequality (4.102) we need the following lemma.

LEMMA 4.2 *If* $y(0) = y(\tau) = 0$ *then*

$$\|y\|^2 + \|\hat{y}\|^2 \le 4t\sum_{k=1}^{n}\tau\|y_{\overset{\circ}{t},k}\|^2. \tag{4.103}$$

Proof. In fact, $y + \hat{y} = 2\sum_{k=1}^{n}\tau y_{\overset{\circ}{t},k}$. Then we have

$$\|y + \hat{y}\|^2 \le 4t\sum_{k=1}^{n}\tau\|y_{\overset{\circ}{t},k}\|^2. \tag{4.104}$$

Furthermore, having denoted $w = y - \check{y}$ we obtain

$$\hat{w} = 2\tau y_{\overset{\circ}{t}} - w, \quad w_1 = 0,$$

from which

$$\|w_{k+1}\| \le 2\tau\|y_{\overset{\circ}{t},k}\| + \|w_k\|, \quad k = 1, 2, ..., n.$$

Summing this inequality over $k = 1, 2, ..., n$ yields

$$\|w_{n+1}\| \le 2\sum_{k=1}^{n}\tau\|y_{\overset{\circ}{t},k}\|$$

or

$$\|y_{n+1} - y_n\| \le 2\sum_{k=1}^{n} \tau \|y_{\overset{\circ}{t},k}\|,$$

$$\|y_{n+1} - y_n\|^2 \le 4t\sum_{k=1}^{n} \tau \|y_{\overset{\circ}{t},k}\|^2. \tag{4.105}$$

Making use of the obvious identity

$$\|y + \hat{y}\|^2 + \|\hat{y} - y\|^2 = 2\left(\|y\|^2 + \|\hat{y}\|^2\right), \tag{4.106}$$

from inequalities (4.104), (4.105) and identity (4.106) we deduce the required estimate (4.103).

To complete the proof of the theorem we substitute the relation (4.103) into inequality (4.102) and obtain the desired estimate (4.100).

For the mesh function $y(t) \in H$ depending on the parameter $t \in \overline{\omega}_\tau$, Furthermore $\|y(t)\| = \sqrt{(y(t), y(t))}$, $\|y(t)\|_{A^{-1}} = \sqrt{(A^{-1}y(t), y(t))}$ with $A = A^* > 0$, we will also use norms of the form

$$\|y(t)\|_{A^{-1},0} = \sum_{t'=0}^{T} \tau \|y(t')\|_{A^{-1}}, \quad \|y(t)\|_{0,-1}^2 = \sum_{t'=\tau}^{T} \tau \|\xi(t')\|^2, \tag{4.107}$$

where $y(t) = \xi_{\bar{t}}$, $\xi(0) = 0$.

The next theorem establishes *a priori* estimate for the difference solution with respect to the right hand side in a 'negative' with respect to t norm [Moskal'kov, 1974].

THEOREM 4.13 *Let the constant operators A, B, D satisfy the conditions*

$$A^* = A \ge 0, \quad D^* = D > 0, \tag{4.108}$$

$$D - \frac{\tau^2}{4}A \ge 2\delta E, \quad \delta > 0, \quad B \ge 0. \tag{4.109}$$

Then the scheme (4.97) is stable with respect to the right hand side, and also the estimate

$$\|y(t)\| \le M\|\varphi\|_{0,-1}, \quad M = \frac{1}{\delta}e^{T/2} \tag{4.110}$$

is valid.

Proof. Let us first derive the energy inequality. To this end we consider the inner product of equation (4.97) and $\tau v(t)$, where $v(t)$ is a function of a discrete argument $t \in \overline{\omega}_\tau$ which will be defined later. Taking into account the obvious identities

$$\tau\left(Dy_{\bar{t}t}, v\right) = \left(Dy_t, v\right) - \left(Dy_{\bar{t}}, \check{v}\right) - \tau\left(Dy_{\bar{t}}, v_{\bar{t}}\right),$$
$$\tau\left(By_{\overset{\circ}{t}}, v\right) = \frac{1}{2}\left(B\left(\hat{y}+y\right), v\right) - \frac{1}{2}\left(B\left(y+\check{y}\right), \check{v}\right) - \frac{\tau}{2}\left(B\left(y+\check{y}\right), v_{\bar{t}}\right),$$

after multiplying by $\tau v(t)$ we come to the equality

$$\begin{aligned}\left(Dy_t, v\right) + \frac{1}{2}\left(B\left(\hat{y}+y\right), v\right) - \tau\left(Dy_{\bar{t}}, v_{\bar{t}}\right) - \frac{\tau}{2}\left(B\left(y+\check{y}\right), v_{\bar{t}}\right) + \tau\left(Ay, v\right) \\ = \left(Dy_{\bar{t}}, \check{v}\right) + \frac{1}{2}\left(B\left(y+\check{y}\right), \check{v}\right) + \tau\left(\varphi, v\right).\end{aligned} \tag{4.111}$$

After summing over t from τ to some $t_1 \le T$, from (4.111) we obtain

$$\begin{aligned}\left(Dy_t, v\right)\left(t_1\right) + \frac{1}{2}\left(B\left(\hat{y}+y\right), v\right)\left(t_1\right) \\ - \sum_{t'=\tau}^{t_1} \tau\left(Dy_{\bar{t}}, v_{\bar{t}}\right)\left(t'\right) - \frac{1}{2}\sum_{t'=\tau}^{t_1} \tau\left(B\left(y+\check{y}\right), v_{\bar{t}}\right)\left(t'\right) \\ + \sum_{t'=\tau}^{t_1} \tau\left(Ay, v\right)\left(t'\right) = \sum_{t'=\tau}^{t_1} \tau\left(\varphi, v\right)\left(t'\right).\end{aligned} \tag{4.112}$$

Here $(u,v)(t) = (u(t), v(t))$.

We choose the function $v(t)$ so that

$$v_{\bar{t}} = -\left(y+\check{y}\right), \quad 0 \le t < t_1, \quad v(t) = 0, \quad t_1 \le t \le T.$$

It is not difficult to see that

$$v(t) = -\sum_{t'=t+\tau}^{t_1} \tau\left[y(t') + y(t'-\tau)\right], \quad 0 \le t < t_1.$$

By virtue of the choice of $v(t)$ it follows from equality (4.112) that

$$\begin{aligned}\sum_{t'=\tau}^{t_1} \tau\left(Dy_{\bar{t}}, y+\check{y}\right)\left(t'\right) + \frac{1}{2}\sum_{t'=\tau}^{t_1} \tau\left(B\left(y+\check{y}\right), y+\check{y}\right)\left(t'\right) \\ + \sum_{t'=\tau}^{t_1} \tau\left(Ay, v\right)\left(t'\right) = \sum_{t'=\tau}^{t_1} \tau\left(\varphi, v\right)\left(t'\right).\end{aligned} \tag{4.113}$$

Let us introduce the function $g(t) = (v(t) - \tau y(t))/2$. Then

$$y = \frac{1}{2}(y + \check{y}) + \frac{\tau}{2} y_{\bar{t}} = -\frac{1}{2} v_{\bar{t}} + \frac{\tau}{2} y_{\bar{t}} = -g_{\bar{t}},$$
$$v = \frac{1}{2}(v + \check{v}) + \frac{\tau}{2} v_{\bar{t}} = \frac{1}{2}(v + \check{v}) - \frac{\tau}{2}(y + \check{y}) = g + \check{g}.$$

Hence, because the operator A is self-adjoint, we have

$$\begin{aligned}\sum_{t'=\tau}^{t_1} \tau\,(Ay, v)\,(t') &= -\sum_{t'=\tau}^{t_1} \tau\,(Ag_{\bar{t}}, g + \check{g})\,(t') \\ &= (Ag, g)\,(t_1) + (Ag, g)\,(0) \\ &= -\frac{\tau^2}{4}\,(Ay, y)\,(t_1) + (Ag, g)(0).\end{aligned}$$

After simple transformations, from the relations (4.113) we deduce the main energy identity, which is valid for any $t_1 \in (0, T]$:

$$\begin{aligned}\left(\left(D - \frac{\tau^2}{4} A\right) y, y\right)(t_1) + \frac{1}{2} \sum_{t'=\tau}^{t_1} \left(B(y+\check{y})\,, y+\check{y}\right)(t') \\ + (Ag, g)(0) = \sum_{t'=\tau}^{t_1} \tau(\varphi, v)(t').\end{aligned} \tag{4.114}$$

We now estimate the right hand side of the identity (4.114). Assume that $\varphi = \xi_t$, $\varphi(0) = 0$. Since $v(t_1) = 0$ and also $y(0) = y(\tau) = 0$, then we have

$$\begin{aligned}\sum_{t'=\tau}^{t_1} \tau(\varphi, v)(t') &= (\xi, v)(t_1) - (\xi, v)(0) - \sum_{t'=\tau}^{t_1} \tau\,(\check{\xi}, v_{\bar{t}})\,(t') \\ &= \sum_{t'=\tau}^{t_1} \tau\,(\check{\xi}, y + \check{y})\,(t') \\ &\le 2c_0 \sum_{t'=\tau}^{t_1} \tau \|y(t')\|^2 + \frac{1}{2c_0} \|\varphi(t_1)\|_{0,-1}^2,\end{aligned}$$

where $c_0 > 0$ is arbitrary.

Using the last estimate for $c_0 = \delta/2$, conditions (4.108), (4.109) when $\tau \le 1$, from (4.114) we find

$$\|y(t_1)\|^2 \le \sum_{t'=\tau}^{t_1-\tau} \tau \|y(t')\|^2 + \frac{1}{\delta^2} \|\varphi(t_1)\|_{0,-1}^2. \tag{4.115}$$

To prove the conclusive estimate (4.110), we need

LEMMA 4.3 *Let $g_n \geq 0$, $n = 1, 2, \dots$, and $f_n \geq 0$, $n = 0, 1, \dots$, be non-negative functions. If f_n is a non-decreasing function ($f_{n+1} \geq f_n$), then the inequality*

$$g_{n+1} \leq c_0 \sum_{k=1}^{n} \tau g_k + f_n, \quad n=1,2,\dots, \qquad g_1 \leq f_0, \qquad c_0 = \text{const} > 0, \tag{4.116}$$

implies the following estimate

$$g_{n+1} \leq e^{c_0 t_n} f_n. \tag{4.117}$$

Proof. Let φ_n, $n = 1, 2, \dots$, be a solution of the system of equations

$$\varphi_{n+1} = c_0 \sum_{k=1}^{n} \tau \varphi_k + f_n, \qquad n \geq 1, \qquad \varphi_1 = f_0. \tag{4.118}$$

It is easy to observe that $g_n \leq \varphi_n$ for all $n > 0$. Indeed,

$$g_1 \leq \varphi_1, \quad g_2 \leq c_0 \tau g_1 + f_1 \leq c_0 \tau \varphi_1 + f_1 = \varphi_2, \quad \text{etc.}$$

It is seen from (4.116) and (4.118) that the inequalities $g_k \leq \varphi_k$ for $k \leq n$ implies $g_{n+1} \leq \varphi_{n+1}$. We replace n in (4.118) by $n-1$ and subtract the obtained equation from (4.118). Then for φ_n we have the difference equation

$$\varphi_{n+1} = q\varphi_n + \psi_n, \quad \psi_n = f_n - f_{n-1} \geq 0, \quad q = 1 + c_0\tau \leq e^{c_0\tau}, \ c_0 > 0.$$

Hence

$$\varphi_{n+1} = q^2 \varphi_{n-1} + q\psi_{n-1} + \psi_n = q^n \varphi_1 + q^{n-1}\psi_1 + \dots + q\psi_{n-1} + \psi_n.$$

Since $q \geq 1$ and $\psi_n \geq 0$, and also $\varphi_1 = f_0$, then $\varphi_{n+1} \leq q^n(\varphi_1 + \psi_1 + \dots + \psi_n) = q^n(\varphi_1 + f_1 - f_0) = q^n f_n$ and therefore $q_{n+1} \leq \varphi_{n+1} \leq q^n f_n \leq e^{c_0 t_n} f_n$.

Now, applying Lemma 4.3 to the inequality (4.115) we verify the validity of estimate (4.110) with $M = \dfrac{1}{\delta} e^{T/2}$.

4.3 Schemes with Variable Operators

If A and D depend on t, then it usually requires the fulfilment of the additional Lipschitz continuity condition for the operators A and

$R = D - \frac{\tau^2}{4}A$, in the form (4.33) and (4.34) respectively. In this case the operator

$$\mathcal{A}(t) = \begin{pmatrix} A(t) & \frac{1}{\tau}R(t) \\ -\frac{1}{\tau}R(t) & \frac{1}{2\tau}R(t) \end{pmatrix}$$

satisfies the relation

$$((\mathcal{A}(t) - \mathcal{A}(t-\tau))\,Y, Y) \le \tau c_0 \,(\mathcal{A}(t-\tau)Y, Y)\,, \tag{4.119}$$

where $Y \in H^2$, $c_0 = \max\{c_1, c_2\}$.

Using the equivalent notation for three-level schemes in the two-level form (4.81), Theorems 2.17, 4.8 and Lemma 4.1, it is not difficult to prove the following assertion.

THEOREM 4.14 *Let the operators of scheme (4.80) satisfy the conditions (4.33), (4.34) and*

$$A^*(t) = A(t) > 0, \quad D^*(t) = D(t) \ge \frac{(1+\varepsilon)\tau^2}{4}A(t). \tag{4.120}$$

Then the difference scheme (4.80) is stable with respect to the initial data and right hand side, and the following estimates are valid:

$$\begin{aligned} \|y_{n+1}\|_{A(t_n)} \le M_1\sqrt{\frac{1+\varepsilon}{\varepsilon}}\Big(&\|y_0\|_{A(\tau)} + \|y_t(0)\|_{D(\tau)} \\ &+ \max_{\tau < t' \le t_n} \left[\|\varphi(t')\|_{A^{-1}(t')} + \|\varphi_{\bar{t}}(t')\|_{A^{-1}(t')}\right]\Big), \end{aligned} \tag{4.121}$$

$$\begin{aligned} \|y_{n+1}\|_{A(t_n)} &+ \|y_{t,n}\|_{D(t_n)} \\ &\le 2M_1\sqrt{\frac{1+\varepsilon}{\varepsilon}}\Big(\|y_0\|_{A(\tau)} + \|y_t(0)\|_{D(\tau)} \\ &\quad + \max_{\tau < t' \le t_n} \left[\|\varphi(t')\|_{A^{-1}(t')} + \|\varphi_{\bar{t}}(t')\|_{A^{-1}(t')}\right]\Big) \end{aligned} \tag{4.122}$$

when $B(t) \ge 0,\ \ 0 < t = n\tau < T$,

$$\begin{aligned} \|y_{n+1}\|_{A(t_n)} \le M_2\sqrt{\frac{1+\varepsilon}{\varepsilon}}\big(&\|y_0\|_{A(\tau)} + \|y_t(0)\|_{D(\tau)} \\ &+ \max_{\tau \le t' \le t_n} \|\varphi(t')\|\big), \end{aligned} \tag{4.123}$$

$$\|y_{n+1}\|_{A(t_n)} + \|y_{t,n}\|_{D(t_n)} \leq 2M_2\sqrt{\frac{1+\varepsilon}{\varepsilon}}\Big(\|y_0\|_{A(\tau)} + \|y_t(0)\|_{D(\tau)} + \max_{\tau \leq t' \leq t_n} \|\varphi(t')\|\Big) \tag{4.124}$$

when $B(t) \geq \bar{\varepsilon}E$, *where* $\bar{\varepsilon} = \text{const} > 0$, *and the constants* M_1, $M_2 > 0$ *do not depend on* τ.

To prove the stability of scheme (4.80) with respect to the right hand side in the norm $\|\cdot\|_{D^{-1}(t)}$, we use the superposition principle and seek a solution of problem (4.97) as a sum

$$y_n = \sum_{s=1}^{n} \tau g_{n,s}, \quad n = 1, 2, ..., \quad y_0 = 0, \tag{4.125}$$

where $g_{n,s}$ is a function of n for any fixed $s = 1, 2$. This function satisfies equation (4.96) and the initial conditions

$$(0.5\tau B(t_s) + D(t_s))\,\frac{g_{s+1,s} - g_{s,s}}{\tau} = \varphi_s, \quad g_{s,s} = 0.$$

Since $D > 0$ and H is a finite-dimensional space, then $D \geq \delta E$ and the operator D^{-1} exists ($\delta > 0$). Because $B \geq 0$ and $D^* = D \geq \delta E$, then the solution of the equation $(0.5\tau B + D)w = \varphi$ satisfies the estimate $\|w\|_D \leq \|\varphi\|_{D^{-1}}$. Indeed, taking the inner product of the last equation and w, we have

$$((0.5\tau B + D)\,w, w) = (\varphi, w)\,. \tag{4.126}$$

Since $B \geq 0$, then, applying the generalized Cauchy–Bunyakovskii–Schwarz inequality and recalling the relation (4.126), we find $\|w\|_D^2 \leq \|\varphi\|_{D^{-1}}\|w$ Consequently

$$\|\,(g_t)_{s,s}\,\|_{D(t_s)} \leq \|\varphi_s\|_{D^{-1}(t_s)}.$$

By virtue of the estimate (4.123) we obtain

$$\|g_{n+1,s}\|_{A(t_n)} \leq M_1\sqrt{\frac{1+\varepsilon}{\varepsilon}}\|\,(g_t)_{s,s}\,\|_{D(t_s)} \leq M_1\sqrt{\frac{1+\varepsilon}{\varepsilon}}\|\varphi_s\|_{D^{-1}(t_s)}.$$

Using then the representation (4.125) and the triangle inequality, we obtain the following estimate for the solution of problem (4.97):

$$\|y_{n+1}\|_{A(t_n)} \leq M_1\sqrt{\frac{1+\varepsilon}{\varepsilon}}\sum_{s=1}^{n} \tau\|\varphi_s\|_{D^{-1}(t_s)}. \tag{4.127}$$

We summarize the last results in the following statement.

THEOREM 4.15 *Assume that the operators A and D satisfy the Lipschitz continuity conditions (4.33) and (4.34) respectively, and let*

$$A(t) = A^*(t) > 0, \quad B(t) \geq 0, \quad D(t) \geq \frac{(1+\varepsilon)\tau^2}{4} A. \tag{4.128}$$

Then the scheme (4.80) is stable with respect to the initial data and the right hand side, and the following a priori *estimate of the solution holds:*

$$\begin{aligned} \|y_{n+1}\|_{A(t_n)} \leq M_1 \sqrt{\frac{1+\varepsilon}{\varepsilon}} \Big(& \|y(0)\|_{A(\tau)} + \|y_t(0)\|_{D(\tau)} \\ & + \sum_{s=1}^{n} \tau \|\varphi_s\|_{D^{-1}(t_s)} \Big). \end{aligned} \tag{4.129}$$

COROLLARY 4.2 *If $D^{-1} \leq E$, then the inequality $\|\varphi_s\|_{D^{-1}} \leq \|\varphi_s\|$ is valid, and the solution of problem (4.97) satisfies the estimate*

$$\|y_{n+1}\|_{A(t_n)} \leq M_1 \sqrt{\frac{1+\varepsilon}{\varepsilon}} \sum_{s=1}^{n} \tau \|\varphi_s\|. \tag{4.130}$$

4.4 Stability in Other Norms

More subtle estimates can be obtained for the following class of three-level operator-difference schemes:

$$Dy_{\bar{t}t} + Ay = \varphi, \quad 0 < t = n\tau < T, \quad y(0) = u_0, \quad y_t(0) = \overline{u}_0. \tag{4.131}$$

Assume that A and D are constant positive and self-adjoint operators

$$A = A^* > 0, \quad D = D^* > 0, \quad D \geq \frac{1+\varepsilon}{4}\tau^2 A. \tag{4.132}$$

Then for scheme (4.131) the estimate (4.129) with $M_1 = 1$ is valid. Assuming

$$x = D^{1/2} y, \quad C = D^{-1/2} A D^{-1/2} \tag{4.133}$$

we transform scheme (4.131) to the form

$$x_{\bar{t}t} + Cx = \widetilde{\varphi}, \quad x(0) = x_0, \quad x_t(0) = \overline{x}_0. \tag{4.134}$$

Applying the operator C^{-1} to equation (4.134) we obtain the scheme

$$C^{-1} x_{\bar{t}t} + x = C^{-1}\widetilde{\varphi}, \quad x(0) = x_0, \quad x_t(0) = \overline{x}_0. \tag{4.135}$$

Comparing it with the scheme (4.131) we establish the correspondence

$$C^{-1} \sim D, \quad E \sim A, \quad C^{-1}\widetilde{\varphi} \sim \varphi.$$

The condition (4.128) takes the form

$$C^{-1} \geq \frac{1+\varepsilon}{4}\tau^2 E \quad \text{or} \quad E \geq \frac{1+\varepsilon}{4}\tau^2 C.$$

Use now the estimate (4.129). Since C is a constant operator, then $M_1 = 1$ and

$$\|x_{n+1}\| \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}}\left(\|x(0)\| + \|x_t(0)\|_{C^{-1}} + \sum_{s=1}^{n} \tau \|C^{-1}\widetilde{\varphi}\|_C\right). \tag{4.136}$$

Taking into account that $x = D^{1/2}y$, $\widetilde{\varphi} = D^{-1/2}\varphi$ and

$$\begin{aligned}
\|x_t(0)\|^2_{C^{-1}} &= (C^{-1}x_t(0), x_t(0)) \\
&= \left(D^{1/2}A^{-1}D^{1/2}D^{1/2}y_t(0), D^{1/2}y_t(0)\right) \\
&= \|Dy_t(0)\|^2_{A^{-1}}, \\
\|C^{-1}\widetilde{\varphi}\|^2_C &= (C^{-1}\widetilde{\varphi}, \widetilde{\varphi}) \\
&= \left(D^{1/2}A^{-1}D^{1/2}D^{-1/2}\varphi, D^{-1/2}\varphi\right) \\
&= (A^{-1}\varphi, \varphi) = \|\varphi\|^2_{A^{-1}},
\end{aligned}$$

we write estimate (4.136) in the original coordinates

$$\begin{aligned}
\|y_{n+1}\|_D \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}}\Bigg(&\|y(0)\|_D + \|Dy_t(0)\|_{A^{-1}} \\
&+ \sum_{s=1}^{n} \tau \|\varphi(t_s)\|_{A^{-1}}\Bigg).
\end{aligned} \tag{4.137}$$

Thus we have proved:

THEOREM 4.16 *If conditions (4.132) are satisfied, then for scheme (4.131) a priori estimate (4.137) is valid. In particular, for the scheme (4.131) with $D = E$ and $u_0 = \overline{u}_0 = 0$ the estimate of stability has the form*

$$\|y_{n+1}\| \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \sum_{s=1}^{n} \tau \|\varphi(t_s)\|_{A^{-1}}. \tag{4.138}$$

Several new *a priori* estimates can be obtained for the three-level scheme

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad y_0 = u_0, \quad y_1 = u_1, \tag{4.139}$$

if we make use another equivalent form of its representation as a two-level scheme (see (4.18))

$$\widetilde{\mathcal{B}}\widetilde{Y}_t + \widetilde{\mathcal{A}}\widetilde{Y} = \Phi, \quad Y_1 = U_0 \tag{4.140}$$

with operators (4.17)

$$\begin{aligned} \widetilde{\mathcal{A}} &= \begin{pmatrix} A & 0 \\ 0 & \dfrac{1}{\tau^2}\left(D - \dfrac{\tau}{2}B\right) \end{pmatrix}, \\ \widetilde{\mathcal{B}} &= \begin{pmatrix} B & \dfrac{1}{\tau}\left(D - \dfrac{\tau}{2}B\right) \\ -\dfrac{1}{\tau}\left(D - \dfrac{\tau}{2}B\right) & \dfrac{1}{\tau}\left(D - \dfrac{\tau}{2}B\right) \end{pmatrix}, \end{aligned} \tag{4.141}$$

Here we define the vector $\widetilde{Y}_n \in H^2$ by

$$\widetilde{Y}_n = \{y_n, y_n - y_{n-1}\}, \quad \Phi_n = \{\varphi_n, 0\}. \tag{4.142}$$

Note that under conditions (4.59) it is possible to define the norm

$$\|\widetilde{Y}\|^2_{\widetilde{\mathcal{A}}} = \|y\|^2_A + \|y_{\bar{t}}\|^2_{D-\frac{\tau}{2}B} \tag{4.143}$$

in $H^2_{\widetilde{\mathcal{A}}}$.

As has been done in Subsection 4.3.1 for the scheme (4.80), we first derive the basic energy identity. Recall that for a two-level scheme it has the form (2.3.36). Applying this identity to the self-adjoint operators

$$A = A^*, \quad B = B^*, \quad D = D^*,$$

we obtain

$$2\tau\left(\left(\widetilde{\mathcal{B}} - 0.5\tau\widetilde{\mathcal{A}}\right)\widetilde{Y}_t, \widetilde{Y}_t\right) + \left(\widetilde{\mathcal{A}}\hat{\widetilde{Y}}, \hat{\widetilde{Y}}\right) = \left(\widetilde{\mathcal{A}}\widetilde{Y}, \widetilde{Y}\right) + 2\tau\left(\Phi, \widetilde{Y}_t\right). \tag{4.144}$$

Since

$$\begin{aligned} 2\tau\left(\left(\widetilde{\mathcal{B}} - 0.5\tau\widetilde{\mathcal{A}}\right)\widetilde{Y}, \widetilde{Y}\right) &= ((B - 0.5\tau A)\, y, y) \\ &\quad + \frac{\tau}{2}\left(\left(D - \frac{\tau}{2}B\right) y_{\bar{t}}, y_{\bar{t}}\right), \end{aligned} \tag{4.145}$$

then the equality (4.144) implies the basic energy identity for the three-level scheme (4.139)

$$\begin{aligned} \tau^2\left(\left(D - \frac{\tau}{2}B\right) y_{\bar{t}t}, y_{\bar{t}t}\right) &+ 2\tau\left(\left(B - \frac{\tau}{2}A\right) y_t, y_t\right) \\ &+ \left(\widetilde{\mathcal{A}}\widetilde{Y}_{n+1}, \widetilde{Y}_{n+1}\right) = \left(\widetilde{\mathcal{A}}\widetilde{Y}_n, \widetilde{Y}_n\right) + 2\tau(\varphi, y_t), \end{aligned} \tag{4.146}$$

where

$$\left(\widetilde{\mathcal{A}}\widetilde{Y},\widetilde{Y}\right) = (Ay,y) + \left(\left(D - \frac{\tau}{2}B\right) y_{\bar{t}}, y_{\bar{t}}\right). \tag{4.147}$$

For further simplicity we assume that the operators A, B, D are constant and satisfy the conditions

$$A = A^* > 0, \quad B = B^*, \quad D = D^*, \tag{4.148}$$

$$D > \frac{\tau}{2}B, \quad B \geq 0.5\tau A. \tag{4.149}$$

Note that the inequality $D > \dfrac{\tau^2}{4}A$ follows directly from conditions (4.149). It has been shown in Theorem 4.5 that conditions (4.148), (4.149) are sufficient for the validity of the operator inequality

$$\widetilde{\mathcal{B}} \geq 0.5\tau\widetilde{\mathcal{A}}. \tag{4.150}$$

Therefore by Theorem 2.16 we have the following *a priori* estimate of the solution of the difference scheme (4.139):

$$\|\widetilde{Y}_{n+1}\|_{\widetilde{\mathcal{A}}} \leq \|\widetilde{Y}_1\|_{\widetilde{\mathcal{A}}} + \|\varphi_1\|_{A^{-1}} + \|\varphi_n\|_{A^{-1}} + \sum_{k=2}^{n} \tau\|\varphi_{\bar{t},k}\|_{A^{-1}}, \tag{4.151}$$

where $\|\widetilde{Y}\|_{\widetilde{\mathcal{A}}}$ is defined by (4.143).

THEOREM 4.17 *Let conditions (4.148) be satisfied and* $D \geq \dfrac{\tau}{2}B$, $B \geq \varepsilon E + 0.5\tau A$. *Then the solution of problem (4.139) satisfies the estimate*

$$\left(\widetilde{\mathcal{A}}\widetilde{Y}_{n+1},\widetilde{Y}_{n+1}\right) \leq \left(\widetilde{\mathcal{A}}\widetilde{Y}_1,\widetilde{Y}_1\right) + \frac{1}{2\varepsilon}\sum_{k=1}^{n} \tau\|\varphi(t_k)\|^2. \tag{4.152}$$

Proof. When estimating the functional $2\tau(\varphi, y_t)$ in (4.146), we use the inequality (4.90)

$$2\tau(\varphi, y_t) \leq 2\tau\varepsilon\|y_t\|^2 + \frac{\tau}{2\varepsilon}\|\varphi\|^2.$$

Substituting this estimate into (4.146) and using the assumptions of the theorem, we come to the desired result.

REMARK 4.3 Having applied the results of Theorem 2.20 for the proof of Theorem 4.17, we would see that the requirement $\widetilde{\mathcal{B}} \geq \varepsilon E + 0.5\tau\widetilde{\mathcal{A}}$ leads to an additional condition imposed on the operator D.

The following theorem gives a stability estimate for three-level difference schemes with respect to the right hand side in the energy space $H_{(B-0.5\tau A)^{-1}}$.

THEOREM 4.18 *Let conditions (4.148) be satisfied and*

$$D > \frac{\tau}{2}B, \quad B > \frac{\tau}{2}A. \tag{4.153}$$

Then the difference scheme (4.139) is stable with respect to the initial data and right hand side, and its solution satisfies the following a priori *estimate*

$$\begin{aligned} &\|\widetilde{Y}_{n+1}\|^2_{\widetilde{A}} + \sum_{k=1}^{n} \tau \|y_{t,k}\|^2_{B-0.5\tau A} \\ &\qquad \le \|\widetilde{Y}_1\|^2_{\widetilde{A}} + \sum_{k=1}^{n} \tau \|\varphi(t_k)\|^2_{(B-0.5\tau A)^{-1}}. \end{aligned} \tag{4.154}$$

P r o o f. To prove this estimate it is necessary to use the generalized Cauchy–Bunyakovskii–Schwarz inequality, i.e.,

$$2\tau(\varphi, y_t) \le \tau \|y_t\|^2_{B-0.5\tau A} + \tau \|\varphi\|^2_{(B-0.5\tau A)^{-1}} \tag{4.155}$$

and the identity (4.146).

COROLLARY 4.3 *If the initial data are homogeneous, $y(0) = y(\tau) = 0$, the inequality (4.154) implies the following estimate for the solution of problem (4.97) in the integral with respect to time norm:*

$$\sum_{k=1}^{n+1} \tau \|y_{\bar{t},k}\|^2_{B-0.5\tau A} \le \sum_{k=1}^{n} \tau \|\varphi(t_k)\|^2_{(B-0.5\tau A)^{-1}}. \tag{4.156}$$

In the sequel we will need the following lemma.

LEMMA 4.4 *Let $R = R^* > 0$ be a self-adjoint positive operator. Then for any function $y(t) \in H$ the inequalities*

$$\|y_{n+1}\|^2_R \le 2t_{n+1} \sum_{k=1}^{n+1} \tau \|y_{\bar{t},k}\|^2_R + 2\|y_0\|^2_R, \tag{4.157}$$

$$\sum_{k=1}^{n+1} \tau \|y_k\|^2_R \le 2t^2_{n+1} \sum_{k=1}^{n+1} \tau \|y_{\bar{t},k}\|^2_R + 2t_{n+1}\|y_0\|^2_R \tag{4.158}$$

are valid.

Proof. Denote $x = R^{1/2}y$ and let

$$v(t) = \|y(t)\|_R = \|x\|.$$

We use the embeddings for the function $v(t)$ as follows

$$\max_{\tau \le t \le t_{n+1}} |v(t)|^2 \le 2\left(t_{n+1}\sum_{k=1}^{n+1}\tau v_{\bar{t},k}^2 + v_0^2\right), \tag{4.159}$$

$$\sum_{k=1}^{n+1}\tau v^2(t_k) \le 2t_{n+1}\left(t_{n+1}\sum_{k=1}^{n+1}\tau v_{\bar{t},k}^2 + v_0^2\right). \tag{4.160}$$

By the triangle inequality

$$v_{\bar{t}}^2 = \left(\|x\|_{\bar{t}}^2\right) = \frac{1}{\tau^2}\left(\|x\| - \|\check{x}\|\right)^2 \le \frac{1}{\tau^2}\|x - \check{x}\|^2 = \|x_{\bar{t}}\|^2, \tag{4.161}$$

from relations (4.159), (4.160) we obtain

$$\|x_{n+1}\|^2 \le 2t_{n+1}\sum_{k=1}^{n+1}\tau\|x_{\bar{t},k}\|^2 + 2\|x_0\|^2,$$

$$\sum_{k=1}^{n+1}\tau\|x_k\|^2 \le 2t_{n+1}^2\sum_{k=1}^{n+1}\|x_{\bar{t},k}\|^2 + 2t_{n+1}\|x_0\|^2.$$

Recalling that $\|x\| = \|y\|_R$, the last inequalities imply the required estimates (4.157), (4.158).

It is possible to prove the following assertion with the help of Lemma 4.4.

THEOREM 4.19 *Let the hypotheses of Theorem 4.18 be satisfied. Then the solution of problem (4.139) satisfies the following* a priori *estimates:*

$$\begin{aligned}\|y_{n+1}\|_{B-0.5\tau A}^2 \le & 2t_{n+1}\|\widetilde{Y}_1\|_{\widetilde{A}} + 2\|y_0\|_{B-0.5\tau A}^2 \\ & + 2t_{n+1}\sum_{k=1}^{n}\tau\|\varphi_k\|_{(B-0.5\tau A)^{-1}}^2,\end{aligned} \tag{4.162}$$

$$\begin{aligned}\sum_{k=1}^{n+1}\tau\|y_k\|_{B-0.5\tau A}^2 \le & 2t_{n+1}^2\|\widetilde{Y}_1\|_{\widetilde{A}} + 2t_{n+1}\|y_0\|_{B-0.5\tau A}^2 \\ & + 2t_{n+1}^2\sum_{k=1}^{n}\tau\|\varphi_k\|_{(B-0.5\tau A)^{-1}}^2.\end{aligned} \tag{4.163}$$

P r o o f. The required estimates (4.162), (4.163) follow from the inequality (4.154) and the embeddings (4.157), (4.158).

REMARK 4.4 If the initial data of scheme (4.139) are homogeneous, $y(0) = y(\tau) = 0$, then the estimates (4.162), (4.163) can be slightly improved

$$\|y_{n+1}\|^2_{B-0.5\tau A} \leq t_{n+1} \sum_{k=1}^{n} \tau \|\varphi_k\|^2_{(B-0.5\tau A)^{-1}}, \tag{4.164}$$

$$\sum_{k=1}^{n+1} \tau \|y_k\|^2_{B-0.5\tau A} \leq t^2_{n+1} \sum_{k=1}^{n} \tau \|\varphi_k\|^2_{(B-0.5\tau A)^{-1}}. \tag{4.165}$$

5. Schemes with Weights

In this section we render concrete estimates of stability with respect to the initial data for classical three-level operator-difference schemes with constant weighted factors.

5.1 Schemes for First-Order Evolutionary Equations

Considering an example of the weighted scheme

$$y_{\overset{\circ}{t}} + Ay^{(\sigma_1,\sigma_2)} = \varphi, \quad y_0 = u_0, \quad y_1 = u_1, \tag{4.166}$$

where, as usual,

$$\begin{aligned} y^{(\sigma_1,\sigma_2)} &= \sigma_1 \hat{y} + (1 - \sigma_1 - \sigma_2) y + \sigma_2 \check{y} \\ &= y + \tau(\sigma_1 - \sigma_2) y_{\overset{\circ}{t}} + \frac{\tau^2}{2} (\sigma_1 + \sigma_2) y_{\bar{t}t}, \end{aligned} \tag{4.167}$$

we will show how to apply the theorems proved above.

Using identity (4.167) we transform scheme (4.166) into the canonical form of three-level schemes

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad y_0 = u_0, \quad y_1 = u_1, \tag{4.168}$$

with operators

$$D = \frac{(\sigma_1 + \sigma_2)\tau^2}{2} A, \quad B = E + \tau(\sigma_1 - \sigma_2) A. \tag{4.169}$$

Assume that the operators $A(t)$, $D(t)$ satisfy the Lipschitz continuity conditions (4.33), (4.34), and

$$\begin{gathered} A^*(t) = A(t) > 0, \quad B(t) \geq \overline{\varepsilon} E, \\ D^*(t) = D(t) \geq \frac{(1+\varepsilon)\tau^2}{4} A(t). \end{gathered} \tag{4.170}$$

Notice that A is a self-adjoint operator and therefore conditions (4.170) are satisfied whenever

$$\sigma_1 - \sigma_2 \geq -\frac{(1-\overline{\varepsilon})}{\tau \|A\|}, \quad \sigma_1 + \sigma_2 \geq \frac{1+\varepsilon}{2}, \quad 0 \leq \overline{\varepsilon} \leq 1. \tag{4.171}$$

It should be also observed that as a consequence of conditions (4.171) we have the inequality

$$\sigma_1 \geq \frac{1+\varepsilon}{4} - \frac{1-\overline{\varepsilon}}{2\tau\|A\|}, \quad \varepsilon > 0, \quad 0 \leq \overline{\varepsilon} \leq 1. \tag{4.172}$$

The following result holds on the basis of Theorem 4.14.

THEOREM 4.20 *Let conditions (4.33), (4.34), (4.171) be satisfied. Then the solution of the three-level scheme (4.166) satisfies the following* a priori *estimate for* $\overline{\varepsilon} = 0$*:*

$$\begin{aligned} \|y_{n+1}\|_{A_n} \leq M_1 \sqrt{\frac{1+\varepsilon}{\varepsilon}} \Big(& \|y_0\|_{A_1} + 0.5\tau^2(\sigma_1+\sigma_2)\|y_t(0)\|_{A_1} \\ & + \max_{1<k\leq n} \Big[\|\varphi_k\|_{A_k^{-1}} + \|\varphi_{\bar{t},k}\|_{A_k^{-1}} \Big] \Big), \end{aligned} \tag{4.173}$$

where $A_n = A(t_n)$, $\varphi_k = \varphi(t_k)$. *If* $0 < \overline{\varepsilon} \leq 1$ *then*

$$\begin{aligned} \|y_{n+1}\|_{A_n} \leq M_2 \sqrt{\frac{1+\varepsilon}{\varepsilon}} \Big(& \|y_0\|_{A_1} + 0.5\tau^2(\sigma_1+\sigma_2)\|y_t(0)\|_{A_1} \\ & + \max_{1<k\leq n} \|\varphi_k\| \Big). \end{aligned} \tag{4.174}$$

Up to this point we have considered the case of self-adjoint operator $A(t)$. Let $A(t) > 0$ be not self-adjoint operator. Acting by the operator A^{-1} on scheme (4.168), (4.169), we obtain

$$\widetilde{D} y_{\bar{t}t} + \widetilde{B} y_{\overset{\circ}{t}} + \widetilde{A} y = \widetilde{\varphi}, \quad y_0 = u_0, \quad y_1 = u_1, \tag{4.175}$$

where

$$\begin{gathered} \widetilde{B} = A^{-1} + \tau(\sigma_1 - \sigma_2)E, \quad \widetilde{D} = \frac{(\sigma_1+\sigma_2)\tau^2}{2} E, \\ \widetilde{A} = E, \quad \widetilde{\varphi} = A^{-1}\varphi. \end{gathered} \tag{4.176}$$

Applying Theorem 4.1 to scheme (4.175), we see that the following operator inequalities are valid:

$$\widetilde{D} - \frac{\tau^2}{4}\widetilde{A} = \left(\frac{\sigma_1 + \sigma_2}{2} - \frac{1}{4}\right)\tau^2 E > 0 \quad \text{for } \sigma_1 + \sigma_2 > 0.5, \tag{4.177}$$

$$\widetilde{B} = A^{-1} + (\sigma_1 - \sigma_2)\tau E \geq 0 \quad \text{for} \quad \sigma_1 \geq \sigma_2. \tag{4.178}$$

THEOREM 4.21 *If $A(t)$ is a variable positive operator and the conditions*

$$\sigma_1 \geq \sigma_2, \quad \sigma_1 + \sigma_2 > 0.5, \tag{4.179}$$

are satisfied, then the scheme (4.166) is stable, and for it the following estimate is valid:

$$\|Y_{n+1}\| \leq \|Y_1\| + \sqrt{2(\sigma_1 + \sigma_2)} \sum_{k=1}^{n} \tau \|\varphi_k\|, \tag{4.180}$$

where

$$\|Y\|^2 = \frac{1}{4}\|y + \check{y}\|^2 + \frac{1}{2}\left(\sigma_1 + \sigma_2 - \frac{1}{2}\right)\|y - \check{y}\|^2. \tag{4.181}$$

Proof. *Stability with respect to the initial data.* Since the assumptions $\widetilde{D} > \frac{\tau^2}{4}\widetilde{A}$, $\widetilde{B} \geq 0$ of Theorem 4.1 are satisfied, then the solution of problem (4.175) with $\varphi = 0$ satisfies

$$\|Y_{n+1}\| \leq \|Y(\tau)\|, \tag{4.182}$$

where $\|Y\|$ is defined by the formula (4.181), which is a particular case of the formula (4.44) when $\widetilde{A} = E$, $\widetilde{D} = \frac{\tau^2}{2}\left(\sigma_1 + \sigma_2 - \frac{1}{2}\right)E$.

Stability with respect to the right hand side. Consider problem (4.166) when $y_0 = y_1 = 0$. We will look for its solution into the form

$$y_{n+1} = \sum_{s=1}^{n} \tau g_{n+1,s}, \quad y_0 = 0, \tag{4.183}$$

where $g_{n+1,s}$, as a function of n for any fixed $s = 1, 2, ..., n$, satisfies equation (4.166) with $\varphi = 0$ whenever $n > s + 1$ and the initial data

$$g_{s+1,s} + 2\sigma_1 \tau A_{s+1,s} = 2\varphi_s, \quad g_{s,s} = 0. \tag{4.184}$$

Substituting (4.183) into equation (4.166) and taking into account the initial data (4.184), we verify that the expression (4.183) is the solution of problem (4.166). By virtue of the stability condition (4.182) with respect to the initial data, we have for $g_{n,s}$ that

$$\|G_{n+1,s}\| \le \|G_{s+1,s}\| \quad \text{for fixed } s = 1, 2, \ldots, \tag{4.185}$$

where $\|G_{n+1,s}\|$ is expressed by (4.181) in terms of $g_{n,s}$ and $g_{n+1,s}$. From (4.184) we find

$$g_{s+1,s} = 2(E + 2\sigma_1 \tau A)^{-1} \varphi_s,$$

and since $E + 2\sigma_1 \tau A \ge E$ for $\sigma_1 \ge 0$, then

$$\|(E + 2\sigma_1 \tau A)^{-1}\| \le 1 \quad \text{and} \quad \|g_{s+1,s}\| \le 2\|\varphi_s\|.$$

We have $g_{s,s} = 0$ and therefore

$$\begin{aligned} \|G_{s+1,s}\|^2 &= \frac{1}{4}\|g_{s+1,s}\|^2 + \frac{1}{2}\left(\sigma_1 + \sigma_2 - \frac{1}{2}\right)\|g_{s+1,s}\|^2 \\ &= \frac{1}{2}(\sigma_1 + \sigma_2)\|g_{s+1,s}\|^2 \le 2(\sigma_1 + \sigma_2)\|\varphi_s\|^2, \end{aligned}$$

from where

$$\|G_{n+1,s}\| \le \|G_{s+1,s}\| \le \sqrt{2(\sigma_1 + \sigma_2)}\|\varphi_s\|. \tag{4.186}$$

Substituting (4.186) into the right hand side of the inequality

$$\|Y_{n+1}\| \le \sum_{s=1}^{n} \tau \|G_{n+1,s}\|$$

we obtain the following estimate for the solution of problem (4.166) with $y_0 = y_1 = 0$:

$$\|Y_{n+1}\| \le \sqrt{2(\sigma_1 + \sigma_2)} \sum_{k=1}^{n} \tau \|\varphi_k\|. \tag{4.187}$$

From here and from the inequality (4.182), the estimate (4.180) follows.

THEOREM 4.22 *If $A(t) = A^*(t)$ is a positive operator and conditions (4.179) are satisfied, then the solution of problem (4.166) satisfies the inequality*

$$\|Y_{n+1}\| \le \|Y_1\| + \frac{1}{\sqrt{2}}\left[\sum_{k=1}^{n} \tau \|\varphi_k\|^2_{A_k^{-1}}\right]^{1/2}, \tag{4.188}$$

where $\|Y\|$ is given by (4.181).

P r o o f. Since all of the conditions of Theorem 4.11 are satisfied for scheme (4.166), then using estimate (4.95) we conclude that

$$\|Y_{n+1}\| \leq \|Y_1\| + \frac{1}{\sqrt{2}} \left[\sum_{k=1}^{n} \tau \|\widetilde{\varphi}_k\|^2_{B_k^{-1}}\right]^{1/2}. \tag{4.189}$$

Since for $\sigma_1 \geq \sigma_2$ we have

$$B_k^{-1} = \left[A_k^{-1} + \tau(\sigma_1 - \sigma_2)E\right]^{-1} \leq A_k,$$

then

$$\|\widetilde{\varphi}_k\|_{B_k^{-1}} \leq \|(A^{-1}\varphi)_k\|_{A_k} = \|\varphi_k\|_{A_k^{-1}}.$$

Substituting the last estimate into the inequality (4.189) completes the proof of the theorem.

Consider now a scheme of the form (4.166) with a non-self-adjoint and non-negative operator A:

$$By_{\overset{\circ}{t}} + Ay^{(\sigma_1,\sigma_2)} = \varphi, \quad y_0 = u_0, \quad y_1 = u_1. \tag{4.190}$$

THEOREM 4.23 *Let the constant operator B and the variable operator A in scheme (4.190) satisfy the conditions*

$$A(t) \geq 0, \quad B^* = B > 0. \tag{4.191}$$

Then for $\sigma_1 = \sigma_2 = 0.25$ the difference scheme is stable with respect to the initial data and the right hand side, and the following estimate is valid:

$$\left\|\frac{y_n + y_{n+1}}{2}\right\|_B \leq \left\|\frac{y_0 + y_1}{2}\right\|_B + \sum_{k=1}^{n} \tau \|\varphi\|_{B-1}. \tag{4.192}$$

P r o o f. Using the identity (4.167) we transform the original scheme with $\sigma_1 = \sigma_2 = 0.25$ into the form

$$Bv_t + Av^{(0.5)} = \varphi, \quad v = \frac{1}{2}(\check{y} + y). \tag{4.193}$$

Consider first a question of stability with respect to the initial data. Since the operator B is self-adjoint, then for $\varphi = 0$ (4.79) is satisfied, from which we have

$$\|v_{n+1}\|_B \leq \|v_1\|_B. \tag{4.194}$$

To prove stability of the scheme (4.190) with respect to the right hand side, we rewrite (4.193) in the canonical form of two-level operator difference schemes

$$\widetilde{B}v_t + \widetilde{A}v = \varphi, \quad v_1 = 0.5(u_0 + u_1), \tag{4.195}$$

where $\widetilde{B} = B + 0.5\tau A$, $\widetilde{A} = E$. The scheme (4.195) can be also written in the form

$$v_{n+1} = Sv_n, \quad S = E - \tau\widetilde{B}^{-1}A, \quad \widetilde{\varphi}_n = \widetilde{B}^{-1}\varphi_n. \tag{4.196}$$

From Theorem 2.14 we conclude that ρ-stability of the scheme with respect to the initial data (in our case $\rho = 1$) in H_B implies the stability with respect to the right hand side under the compatibility condition for norms $\|\varphi\|_* = \|\widetilde{B}^{-1}\varphi\|_B$.

Consequently from estimate (2.100) we have

$$\|v_{n+1}\|_B \le \|v_1\|_B + \sum_{k=1}^{n} \tau \|\widetilde{B}^{-1}\varphi\|_B.$$

Since $A \ge 0$ it follows that $\widetilde{B}^{-1} \le B^{-1}$ and $\|\widetilde{B}^{-1}\varphi\|_B \le \|\varphi\|_{B^{-1}}$.

REMARK 4.5 If $B = E$, $A(t) \ge 0$, $\sigma_1 = \sigma_2 = 0.25$, then for the scheme (4.166), according to (4.192) we have

$$\left\|\frac{y_n + y_{n+1}}{2}\right\| \le \left\|\frac{y_0 + y_1}{2}\right\| + \sum_{k=1}^{n} \tau \|\varphi\|. \tag{4.197}$$

5.2 Schemes with Weights for Second-Order Evolutionary Equations

As an example, we now consider the scheme with weighted factors

$$y_{\bar{t}t} + Ay^{(\sigma_1,\sigma_2)} = \varphi, \quad y_0 = u_0, \quad y_1 = u_1. \tag{4.198}$$

Substituting here $y^{(\sigma_1,\sigma_2)} = y + \tau(\sigma_1 - \sigma_2)y_{\overset{\circ}{t}} + \dfrac{\tau^2}{2}(\sigma_1 + \sigma_2)y_{\bar{t}t}$, we obtain

$$\left(E + \frac{\tau^2}{2}(\sigma_1 + \sigma_2)A\right) y_{\bar{t}t} + \tau(\sigma_1 - \sigma_2)Ay_{\overset{\circ}{t}} + Ay = \varphi, \tag{4.199}$$

that is,

$$D = E + \frac{\tau^2}{2}(\sigma_1 + \sigma_2)A, \quad B = \tau(\sigma_1 - \sigma_2)A. \tag{4.200}$$

The stability conditions

$$B(t) \ge 0, \quad D(t) \ge \frac{1+\varepsilon}{4}\tau^2 A$$

are satisfied for

$$\sigma_1 \geq \sigma_2, \quad \sigma_1 + \sigma_2 \geq \frac{1+\varepsilon}{2} - \frac{2}{\tau^2 \|A\|}.$$

Note that for $\sigma_1 + \sigma_2 \geq 0$ we have $D^{-1} \leq E$, and the next theorem easily follows from Theorem 4.15.

THEOREM 4.24 *Let the operator $A^*(t) = A(t) > 0$ be Lipschitz continuous with respect to the variable t. Then for*

$$\sigma_1 \geq \sigma_2, \quad \sigma_1 + \sigma_2 \geq 0, \quad \sigma_1 + \sigma_2 \geq \frac{1+\varepsilon}{2} - \frac{2}{\tau^2 \|A\|} \tag{4.201}$$

the solution of problem (4.198) satisfies the following a priori *estimate:*

$$\|y_{n+1}\|_{A_n} \leq M_1 \sqrt{\frac{1+\varepsilon}{\varepsilon}} \left(\|y_0\|_{A_1} + \|y_{t,0}\|_{D_1} + \sum_{k=1}^{n} \tau \|\varphi_k\| \right). \tag{4.202}$$

In the energy space H it is possible to obtain the corresponding stability estimates in 'negative' norms with respect to the right hand side. To do this we act with the operator A^{-1} on equation (4.199) and as a result we obtain one more canonical form of three-level schemes

$$\widetilde{D} y_{\bar{t}t} + \widetilde{B} y_{\overset{\circ}{t}} + \widetilde{A} y = \widetilde{\varphi}, \quad \widetilde{\varphi} = A^{-1}\varphi, \tag{4.203}$$

with

$$\widetilde{D}(t) = A^{-1}(t) + \frac{\tau^2}{2}(\sigma_1 + \sigma_2)E, \quad \widetilde{B} = \tau(\sigma_1 - \sigma_2)E, \quad \widetilde{A} = E.$$

Since we have $A^{-1} \geq \dfrac{E}{\|A\|}$ for the self-adjoint operator, then

$$\widetilde{D} - \frac{1+\varepsilon}{4}\tau^2 \widetilde{A} = A^{-1} + \frac{\tau^2}{2}\left(\sigma_1 + \sigma_2 - \frac{1+\varepsilon}{2}\right) E \geq 0$$

for

$$\sigma_1 + \sigma_2 \geq \frac{1+\varepsilon}{2} - \frac{2}{\tau^2 \|A\|}.$$

In addition, for $\sigma_1 + \sigma_2 \geq 0$ we have

$$\|\widetilde{\varphi}\|_{\widetilde{D}^{-1}} \leq \|\widetilde{\varphi}\|_A = \|\varphi\|_{A^{-1}}.$$

Consequently if the conditions of Theorem 4.24 are satisfied, then the solution of problem (4.198) satisfies the estimate

$$\|y_{n+1}\| \leq M_1 \sqrt{\frac{1+\varepsilon}{\varepsilon}} \left(\|y_0\| + \|y_{t,0}\|_{\widetilde{D}_1} + \sum_{k=1}^{n} \tau \|\varphi_k\|_{A_k^{-1}} \right). \tag{4.204}$$

Another type of sufficient conditions for stability of three-level schemes with respect to the initial data and right hand side are the conditions (see relations (4.149))

$$D > \frac{\tau}{2}B, \quad B \geq \frac{\tau}{2}A. \tag{4.205}$$

It is obvious that the inequalities

$$D - \frac{\tau}{2}B = E + \frac{\tau^2}{2}(\sigma_1 + \sigma_2 - \sigma_1 + \sigma_2)A = E + \tau^2\sigma_2 A > 0, \tag{4.206}$$

$$B - \frac{\tau}{2}A = \tau(\sigma_1 - (\sigma_2 + 0.5))A \geq 0 \tag{4.207}$$

are satisfied if

$$\sigma_2 \geq -\frac{1}{\tau^2\|A\|}, \quad \sigma_1 \geq \sigma_2 + \frac{1}{2}. \tag{4.208}$$

Therefore on the basis of (4.151) in the case where A is a constant operator and conditions (4.208) are satisfied, we have the following estimate for the solution of the difference scheme (4.198):

$$\begin{aligned} \|y_{n+1}\|_A \leq & \|y_1\|_A + \|y_{t,0}\|_{E+\tau^2\sigma_2 A} + \|\varphi_1\|_{A^{-1}} \\ & + \|\varphi_n\|_{A^{-1}} + \sum_{k=1}^{n-1} \tau\|\varphi_{t,k}\|_{A^{-1}}. \end{aligned} \tag{4.209}$$

In the case where the operator $A^*(t) = A(t) > 0$ is Lipschitz continuous we verify, by reducing scheme (4.198) to the form (4.203), that because of the relations (4.208) the conditions

$$\widetilde{D} > \frac{\tau}{2}\widetilde{B}, \quad \widetilde{B} \geq \frac{\tau}{2}\widetilde{A}$$

are also satisfied. Now, taking into account that

$$\|\widetilde{Y}_{n+1}\|_{\widetilde{A}} = \left\{\|y_{n+1}\|_{\widetilde{A}} + \|y_{t,n}\|^2_{\widetilde{D}-\frac{\tau}{2}\widetilde{B}}\right\}^{1/2} \geq \|y_{n+1}\|,$$

$$\|\widetilde{Y}_1\|_{\widetilde{A}} \leq \|y_1\| + \|y_{t,0}\|_{A^{-1}(\tau)+0.5\tau^2(\sigma_1+\sigma_2)E},$$

from (4.151) we obtain the estimate

$$\|y_{n+1}\| \leq \|y_1\| + \|y_{t,0}\|_{R(\tau)} + \|\varphi_1\| + \|\varphi_n\| + \sum_{k=1}^{n-1} \tau\|\varphi_{t,k}\| \tag{4.210}$$

with the operator $R = A^{-1} + 0.5\tau^2(\sigma_1 + \sigma_2)E$.

Chapter 5

THREE-LEVEL SCHEMES WITH OPERATOR FACTORS

1. Introduction

Three-level schemes with weights are very often used for finding the numerical solution of the Cauchy problem for the second-order evolution equation

$$\frac{d^2u}{dt^2} + Au = f(t), \quad 0 < t \le T, \tag{5.1}$$

$$u(0) = u_0, \tag{5.2}$$

$$\frac{du}{dt}(0) = u_1 \tag{5.3}$$

with a constant operator $A = A^* > 0$. As an example, we consider a two-parameter family of difference schemes (with the first and second order of approximation)

$$\begin{aligned} \frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} &+ A(\sigma_1 y_{n+1} + (1 - \sigma_1 - \sigma_2)y_n + \sigma_2 y_{n-1}) \\ &= f_n, \quad n = 1, 2, \dots, \end{aligned} \tag{5.4}$$

for given y_0, y_1.

We transform scheme (5.4) into the canonical form

$$\begin{aligned} D\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} &+ B\frac{y_{n+1} - y_{n-1}}{2\tau} \\ &+ Ay_n = 0, \quad n = 1, 2, \dots, \end{aligned} \tag{5.5}$$

where

$$D = E + \frac{\tau^2}{2}(\sigma_1 + \sigma_2)A, \quad B = \tau(\sigma_1 - \sigma_2)A. \tag{5.6}$$

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For constant weight parameters σ_1 and σ_2 the operators D and B will be self-adjoint as well as the operator A. In many cases we have to make use of schemes with variable (in space and/or time) weighted factors σ_1 and σ_2. In these cases the direct application of general results on stability of three-level operator-difference schemes is practically impossible. In this chapter we formulate stability conditions for schemes with operator factors, the special case of which are schemes with variable weighted factors. For an important class of three-level difference schemes we give *a priori* estimates which reflect the stability of the difference scheme with respect to the initial data and the right hand side under different conditions.

Among schemes with operator factors, we distinguish and consider in detail three-level operator-difference schemes of the form (5.5), where

$$D = E + \frac{\tau^2}{2} G_1 A, \quad B = \tau G_2 A. \tag{5.7}$$

This is the form in which classical schemes with variable weighted factors are written down in the usual way (see (5.6)).

The second class of examined schemes with operator factors consists of schemes with

$$D = E + \frac{\tau^2}{2} A G_1, \quad B = \tau A G_2.$$

Their distinction from (5.7) concerns the choice of a certain way of weighting (for the solution, operator and so on).

If the problem operator is represented in the form $A = T^*T$, one can use the difference schemes (5.5) with

$$D = E + \frac{\tau^2}{2} T^* G_1 T, \quad B = \tau T^* G_2 T.$$

We come to such schemes in the case of flow weighing, if the operator A is a discrete analog of a second-order elliptic operator.

The classes of three-level difference schemes under consideration belong to a class of symmetrizable difference schemes. In this case the original difference scheme transforms in an appropriate manner into the three-level difference scheme (5.5) with self-adjoint operators.

The generic result on stability of difference schemes with operator factors are formulated in the following way (see Theorem 5.1 for a more complete statement of this stability result). Let the constant operators A and G_1 in scheme (5.5), (5.7) satisfy the conditions

$$A = A^* > 0, \quad G_1 = G_1^* \geq \frac{1}{2} E.$$

Then the condition $G_2(t) \geq 0$ is necessary and sufficient for stability with respect to the initial data.

2. Schemes with $D = E + 0.5\tau^2 G_1 A$, $B = \tau G_2 A$

We present a class of difference schemes with operator multipliers which are related to schemes with variable weighted factors when weighting the term with spatial discrete operators.

2.1 Stability with Respect to the Initial Data

Consider again the canonical form of three-level operator-difference schemes

$$D y_{\bar{t}t} + B y_{\overset{\circ}{t}} + A y = 0, \quad y_0 = u_0, \quad y_1 = u_1. \tag{5.8}$$

Up to this point in the book we studied schemes for which the following conditions (in the case of constant operators A and D) are satisfied:

$$A^* = A > 0, \quad B(t) \geq 0, \quad D^* = D > \frac{\tau^2}{4} A, \tag{5.9}$$

or

$$B \geq \frac{\tau}{2} A, \quad D > \frac{\tau}{2} B \tag{5.10}$$

whenever the operators A, B, D are self-adjoint.

Below we will consider wider classes of operator-difference schemes with non-self-adjoint operators D, A.

Here we will assume that

$$D = E + 0.5\tau^2 G_1 A, \quad B = \tau G_2 A, \quad G_k A \neq A G_k, \ k = 1, 2\ . \tag{5.11}$$

Then we can rewrite scheme (5.8)–(5.11) in the form

$$(E + 0.5\tau^2 G_1 A) y_{\bar{t}t} + \tau G_2 A y_{\overset{\circ}{t}} + A y = 0, \quad y_0 = u_0, \ y_1 = u_1. \tag{5.12}$$

Multiplying equation (5.12) from the left by the operator A, we arrive at the representation

$$\widetilde{D} y_{\bar{t}t} + \widetilde{B} y_{\overset{\circ}{t}} + \widetilde{A} y = 0, \quad y_0 = u_0, \quad y_1 = u_1, \tag{5.13}$$

with operators

$$\widetilde{D} = A + 0.5\tau^2 A G_1 A, \quad \widetilde{B} = \tau A G_2 A, \quad \widetilde{A} = A^2. \tag{5.14}$$

Note that for $A = A^* > 0$, $G_1^* = G_1$, $G_2 \geq 0$ we have

$$\widetilde{D}^* = \widetilde{D} > 0, \quad \widetilde{B}(t) \geq 0, \quad \widetilde{A}^* = \widetilde{A} > 0. \tag{5.15}$$

We now check under what conditions for the operator G_1 the inequality

$$\widetilde{D} > \frac{\tau^2}{4}\widetilde{A} \tag{5.16}$$

is valid. As the operator A is self-adjoint and positive, then by virtue of Lemma 3.1 we see that

$$\begin{aligned} \widetilde{D} - \frac{\tau^2}{4}\widetilde{A} &= A + 0.5\tau^2 A(G_1 - 0.5E)A \\ &= A\left(A^{-1} - \frac{1}{\|A\|}E\right)A + 0.5\tau^2 A(G_1 - \sigma_0 E)A \geq 0 \end{aligned} \tag{5.17}$$

when

$$G_1 > \sigma_0 E, \quad \sigma_0 = \frac{1}{2} - \frac{1}{0.5\tau^2\|A\|}. \tag{5.18}$$

Let us formulate the corresponding necessary and sufficient conditions of stability with respect to the initial data.

THEOREM 5.1 *Let the constant operators A, G_1 in the difference scheme (5.8), (5.11) satisfy conditions (5.18) and*

$$A^* = A > 0, \quad G_1^* = G_1. \tag{5.19}$$

Then the condition

$$G_2(t) \geq 0 \tag{5.20}$$

is necessary and sufficient for stability of the scheme with respect to the initial data, and for any τ the estimate

$$\|Y_{n+1}\|_{\mathcal{A}} \leq \|Y_1\|_{\mathcal{A}} \tag{5.21}$$

is valid, where

$$\|Y_{n+1}\|_{\mathcal{A}}^2 = \|Ay_n^{(0.5)}\|^2 + \|y_{t,n}\|_{\widetilde{R}}^2, \quad \widetilde{R} = A + 0.5\tau^2 A(G_1 - 0.5E)A.$$

P r o o f. Since all of the conditions of Theorem 4.1 are satisfied for scheme (5.13), then recalling (4.25) we conclude that for

$$Y_n = \left\{\frac{1}{2}(y_n + y_{n-1}), y_n - y_{n-1}\right\}, \quad \mathcal{A} = \begin{pmatrix} A^2 & 0 \\ 0 & \widetilde{R} \end{pmatrix} \tag{5.22}$$

the estimate (5.21) is valid.

REMARK 5.1 If we require in Theorem 5.1 that instead of (5.18) the inequality

$$G_1 \geq \sigma_\varepsilon E, \quad \sigma_\varepsilon = \frac{1+\varepsilon}{2} - \frac{1}{0.5\tau^2\|A\|}, \quad \varepsilon > 0, \tag{5.23}$$

is satisfied, then condition (5.20) is sufficient for the fulfilment of the estimate

$$\|Ay_{n+1}\| \le \sqrt{\frac{1+\varepsilon}{\varepsilon}} \left(\|Ay_0\| + \left(\widetilde{D} y_{t,0}, y_{t,0} \right) \right), \tag{5.24}$$

where $\widetilde{D} = A + 0.5\tau^2 A G_1 A$.

In the case of a non-negative operator $A = A^* \ge 0$ the condition

$$\widetilde{D} - \frac{(1+\varepsilon)\tau^2}{4} \widetilde{A} = A + 0.5\tau^2 A \left(G_1 - \frac{1+\varepsilon}{2} E \right) A \ge 0, \tag{5.25}$$

and consequently the estimate (5.24) is satisfied for

$$G_1 \ge \frac{1+\varepsilon}{2} E, \quad \varepsilon > 0. \tag{5.26}$$

The proof of estimate (5.24) obviously follows from (4.46).

As an example, we consider a weighted scheme for a second-order evolutionary equation. In the real finite-dimensional Hilbert space H endowed with an inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, we consider the following Cauchy problem

$$\frac{d^2u}{dt^2} + Au = \varphi, \quad 0 < t \le T, \quad u(0) = u_0, \quad \frac{du}{dt}(0) = \overline{u}_0. \tag{5.27}$$

Here $A : H \to H$ is a linear self-adjoint operator; $u(t)$, $\varphi(t)$ is an abstract functions with values in H.

Let us introduce the notation

$$v^{(\Sigma_1,\Sigma_2)} = \Sigma_1 v_{n+1} + (E - \Sigma_1 - \Sigma_2) v_n + \Sigma_2 v_{n-1}, \tag{5.28}$$

where Σ_k, $k = 1, 2$, are linear operators in H, $A\Sigma_k \ne \Sigma_k A$.

On the grid $\overline{\omega}_\tau$ we put the following difference problem into correspondence to the Cauchy problem (5.27):

$$y_{\bar{t}t} + (Ay)^{(\Sigma_1,\Sigma_2)} = \varphi(t), \quad t \in \omega_\tau, \quad y_0 = u_0, \quad y_1 = u_1. \tag{5.29}$$

Using the identity

$$v^{(\Sigma_1,\Sigma_2)} = v + \tau(\Sigma_1 - \Sigma_2) v_{\overset{\circ}{t}} + \frac{\tau^2}{2} (\Sigma_1 + \Sigma_2) v_{\bar{t}t} \tag{5.30}$$

we transform the three-level scheme (5.29) to the form (5.12) with $\varphi = 0$ and

$$G_1 = \Sigma_1 + \Sigma_2, \quad G_2 = \Sigma_1 - \Sigma_2. \tag{5.31}$$

Thus, *a priori* estimate (5.24) for the operator-difference scheme (5.29) is valid for constant operators A, Σ_k, $k = 1, 2$, $\varphi = 0$ and under the following conditions:

$$A = A^* > 0, \quad (\Sigma_1 + \Sigma_2) = (\Sigma_1 + \Sigma_2)^* > \sigma_\varepsilon E,$$
$$\sigma_\varepsilon = \frac{1+\varepsilon}{2} - \frac{1}{0.5\tau^2\|A\|}, \tag{5.32}$$

$$\Sigma_1 \geq \Sigma_2. \tag{5.33}$$

Note that under condition (5.32) the inequality (5.33) is necessary and sufficient for the stability of scheme (5.29) with respect to the initial data.

2.2 Other *A Priori* Estimates

We state and prove the corresponding analogues of Theorems 4.5, 4.6. Taking into account the requirements (4.58), one can see that for scheme (5.13) the conditions

$$\widetilde{A}^* = \widetilde{A} > 0, \quad \widetilde{B}^* = \widetilde{B}, \quad \widetilde{D}^* = \widetilde{D} > \frac{\tau}{2}\widetilde{B} \tag{5.34}$$

are obviously satisfied whenever

$$A^* = A > 0, \quad G_k^* = G_k, \ k = 1, 2, \tag{5.35}$$

$$G_1 > G_2 - \frac{1}{0.5\tau^2\|A\|}E. \tag{5.36}$$

Indeed, using (5.14), (5.36), Lemma 3.1 and recalling that A is a self-adjoint operator, we have

$$\widetilde{R} = \widetilde{D} - \frac{\tau}{2}\widetilde{B} = A + 0.5\tau^2 A(G_1 - G_2)A$$

$$= A\left(A^{-1} - \frac{1}{\|A\|}E\right)A \tag{5.37}$$

$$+ 0.5\tau^2 A\left(G_1 - \left(G_2 - \frac{1}{0.5\tau^2\|A\|}E\right)\right)A > 0.$$

THEOREM 5.2 *Let the constant operators A, G_1, G_2 satisfy conditions (5.35), (5.36). Then the condition*

$$G_2 \geq 0.5E \tag{5.38}$$

is sufficient for the stability of scheme (5.8), (5.11) with respect to the initial data, and for any τ the following estimate is valid:

$$\|Ay_n\|^2 + \left(\widetilde{R}_1 y_{\bar{t},n}, y_{\bar{t},n}\right) \le \|Ay_1\|^2 + \left(\widetilde{R}_1 y_{\bar{t},1}, y_{\bar{t},1}\right), \tag{5.39}$$

where $\widetilde{R}_1 = \widetilde{D} - \frac{\tau}{2}\widetilde{B} = A + 0.5\tau A(G_1 - G_2)A.$

Proof. Since the inequality

$$\widetilde{B} - 0.5\tau\widetilde{A} = \tau A(G_2 - 0.5E)A \ge 0 \tag{5.40}$$

is valid under (5.38), then recalling (4.65) we verify that the required estimate (5.39) holds for scheme (5.13).

In the case of a non-negative operator A we have

THEOREM 5.3 *Let the constant operators A, G_1, G_2 satisfy the conditions*

$$A^* = A \ge 0, \quad G_k^* = G_k, \ k = 1, 2, \tag{5.41}$$

$$G_1 \ge G_2, \quad G_2 \ge 0.5E. \tag{5.42}$$

Then for the difference scheme (5.8), (5.11), for any τ we have the estimate

$$\|Ay_n\|^2 \le \|Ay_1\|^2 + \left(\widetilde{R}_1 y_{\bar{t},1}, y_{\bar{t},1}\right). \tag{5.43}$$

Proof. Write the transformed scheme (5.13) in the canonical form (4.59) of two-level operator-difference schemes

$$\widetilde{\mathcal{B}}\widetilde{Y}_t + \widetilde{\mathcal{A}}\widetilde{Y} = 0, \quad \widetilde{Y}_1 = U_0,$$

where

$$\widetilde{Y} = \{y, y - \check{y}\}, \quad \widetilde{\mathcal{A}} = \begin{pmatrix} A^2 & 0 \\ 0 & \dfrac{1}{\tau^2}\widetilde{R}_1 \end{pmatrix},$$

$$\widetilde{\mathcal{B}} = \begin{pmatrix} \widetilde{B} & \dfrac{1}{\tau}\widetilde{R}_1 \\ -\dfrac{1}{\tau}\widetilde{R}_1 & \dfrac{1}{\tau}\widetilde{R}_1 \end{pmatrix}, \quad \widetilde{R}_1 = A + 0.5\tau^2 A(G_1 - G_2)A.$$

By the definition of the inner product in H^2 (see (4.67)) we have

$$\left(\left(\widetilde{\mathcal{B}} - 0.5\tau\widetilde{\mathcal{A}}\right)\widetilde{Y}, \widetilde{Y}\right) = \left(\left(\widetilde{B} - 0.5\tau\widetilde{A}\right)y, y\right) + \frac{1}{2\tau}\left(\widetilde{R}_1 y_{\bar{t}}, y_{\bar{t}}\right).$$

Since $A = A^* \geq 0$, $G_1 \geq G_2$, it is obvious that under assumption (5.42), according to (5.40), we have $\widetilde{\mathcal{B}} \geq 0.5\tau\widetilde{\mathcal{A}}$ and $\widetilde{R}_1 \geq 0$. Thus the following operator relations are valid:

$$\widetilde{\mathcal{B}} \geq 0.5\tau\widetilde{\mathcal{A}}, \quad \widetilde{\mathcal{A}}^* = \widetilde{\mathcal{A}}.$$

Applying now the inequality (2.35) (see Remark 2.1) we find

$$\left(\widetilde{\mathcal{A}}\widetilde{Y}_{n+1}, \widetilde{Y}_{n+1}\right) \leq \left(\widetilde{\mathcal{A}}\widetilde{Y}_1, \widetilde{Y}_1\right).$$

Furthermore, we have the upper bound

$$\left(\widetilde{\mathcal{A}}\widetilde{Y}, \widetilde{Y}\right) = \left(\widetilde{A}y, y\right) + \left(\widetilde{R}_1 y_{\bar{t}}, y_{\bar{t}}\right) \geq (A^2 y, y) = \|Ay\|^2,$$

which completes the proof.

REMARK 5.2 For the difference scheme with operator factors (5.29) considered above, because of (5.31) we conclude that the estimate (5.43) is valid provided the operator inequalities

$$\Sigma_2 \geq 0, \quad \Sigma_1 \geq \Sigma_2 + 0.5E \tag{5.44}$$

are satisfied.

Recall that, as has been already shown above, the estimate, similar to (5.24), for problem (5.29) in the case of self-adjoint operators Σ_1, Σ_2 is also valid for

$$\Sigma_1 \geq \Sigma_2, \quad \Sigma_1 + \Sigma_2 \geq 0.5E.$$

2.3 Stability with Respect to the Right Hand Side

We now deduce *a priori* estimates of stability with respect to the initial data and right hand side for a inhomogeneous three-level operator-difference scheme of the form

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad y_0 = u_0, \quad y_1 = u_1 \tag{5.45}$$

with

$$D = E + 0.5\tau^2 G_1 A, \quad B = \tau G_2 A, \quad G_k A \neq A G_k, \ k = 1, 2. \tag{5.46}$$

THEOREM 5.4 *Let the constant operators A, G_1 and the variable operator $G_2(t)$ satisfy the conditions*

$$A^* = A > 0, \quad G_1^* = G_1, \quad G_2(t) \geq 0, \tag{5.47}$$

$$G_1 \geq \sigma_\varepsilon E, \quad \sigma_\varepsilon = \frac{1+\varepsilon}{2} - \frac{1}{0.5\tau^2\|A\|}, \quad \varepsilon > 0. \tag{5.48}$$

Then the difference scheme (5.45), (5.46) is stable with respect to the right hand side, and its solution satisfies the estimate

$$\begin{aligned} \|Ay_{n+1}\| \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \Bigg(\|Ay_0\| + \left(\widetilde{D}y_{t,0}, y_{t,0}\right) \\ + \|\varphi_1\| + \|\varphi_n\| + \sum_{k=1}^{n-1} \|\varphi_{t,k}\| \Bigg), \end{aligned} \tag{5.49}$$

where $\widetilde{D} = A + 0.5\tau^2 AG_1A$.

P r o o f. Since the operators G_k do not commute with A, then we can say nothing about such properties of the operator D as self-adjointness and the property of having fixed sign, and also about non-negativity of the operator $B(t)$. In order to use the known results of Chapter 4, we transform the original problem, by multiplication from the left by the operator A, into the convenient canonical form of three-level operator-difference schemes

$$\widetilde{D}y_{\bar{t}t} + \widetilde{B}y_{\overset{\circ}{t}} + \widetilde{A}y = \widetilde{\varphi}, \quad y_0 = u_0, \quad y_1 = u_1, \tag{5.50}$$

in which $\widetilde{D}$, $\widetilde{B}$, $\widetilde{A}$ are defined by (5.14), and

$$\widetilde{\varphi} = A\varphi.$$

It is easy to verify that under assumption (5.48) we have

$$\widetilde{D} \geq \frac{(1+\varepsilon)\tau^2}{4}\widetilde{A}, \quad \widetilde{B}(t) \geq 0, \quad t \in \omega_\tau. \tag{5.51}$$

Consequently by Theorem 4.10 we obtain

$$\begin{aligned} \|Y_{n+1}\|_{\mathcal{A}} \leq & \|Y_1\|_{\mathcal{A}} + \|\widetilde{\varphi}_1\|_{\widetilde{A}^{-1}} + \|\widetilde{\varphi}_n\|_{\widetilde{A}^{-1}} \\ & + \sum_{k=1}^{n-1} \tau \|\widetilde{\varphi}_{t,k}\|_{\widetilde{A}^{-1}}, \end{aligned} \tag{5.52}$$

where

$$\|Y\|^2_{\mathcal{A}} = \frac{1}{4}\|y + \check{y}\|^2_{\widetilde{A}} + \|y_{\bar{t}}\|^2_{\widetilde{D} - \frac{\tau^2}{4}\widetilde{A}}.$$

Taking into account that

$$\widetilde{A} = A^2, \quad \widetilde{R} = \widetilde{D} - \frac{\tau^2}{4}\widetilde{A}, \quad \|\widetilde{\varphi}\|_{\widetilde{A}^{-1}} = \|\varphi\|,$$

and reinforcing the inequality (5.52), in accordance with (4.46), we arrive at the estimate (5.49).

REMARK 5.3 The inequality (5.48) can be replaced by the more rough requirement

$$G_1 \geq \frac{1+\varepsilon}{2} E. \tag{5.53}$$

In this case the check of conditions of Theorem 5.4 requires no knowledge about the norm of the operator A.

Let us give without proofs some *a priori* estimates of stability in the case of variable operators $A(t)$, $G_1(t)$ satisfying the condition

$$\| \left(\left(C(t) - C(t-\tau) \right) v, v \right) \| \leq \tau c_0 \left(C(t-\tau)v, v \right) \tag{5.54}$$

for all $v \in H$, $0 < t \leq T$. Here $C = \widetilde{A} = A^2$ or $C = \widetilde{R} = A + 0.5\tau^2 A(G_1 - 0.5E)A$.

THEOREM 5.5 *Let the operators of scheme (5.45), (5.46) satisfy conditions (5.47), (5.48) and (5.54). Then the scheme is stable with respect to the initial data and right hand side, and the following estimate is true:*

$$\begin{aligned} \|A_{n+1} y_{n+1}\| \leq M_1 \sqrt{\frac{1+\varepsilon}{\varepsilon}} \Big(& \|A_0 y_0\| + \|y_{t,0}\|_{\widetilde{D}(\tau)} \\ & + \max_{1<k\leq n} \left[\|\varphi_k\| + \|\varphi_{\bar{t},k}\| \right] \Big), \end{aligned} \tag{5.55}$$

where $\widetilde{D} = A + 0.5\tau^2 A G_1 A$, $M_1 = \text{const} > 0$.

To derive the estimate (5.55) one must beforehand reduce the difference problem (5.45), (5.46) to the form (5.50) and then use Theorem 4.14.

When investigating stability of inhomogeneous difference schemes with variable and discontinuous in time weighted factors for the wave equation, we often arrive atscheme (5.45), (5.46) in which (as a consequence) the operators $G_1(t)$, $G_2(t)$ are not Lipschitz continuous. This gives rise to additional difficulties because of the lack of this property of the operator $D(t)$ or $\widetilde{D}(t)$.

Let us obtain the relevant result for the above described case. Using the identities

$$y_{\bar{t}t} = \frac{y_t - y_{\bar{t}}}{\tau}, \quad y_{\overset{\circ}{t}} = \frac{y_t + y_{\bar{t}}}{2},$$

we transform problem (5.45), (5.46) into the form

$$y_{\bar{t}t} + \tau \left(\Sigma_1 A y_t - \Sigma_2 A y_{\bar{t}} \right) + Ay = \varphi, \quad y_0 = u_0, \quad y_1 = u_1, \tag{5.56}$$

where

$$\Sigma_1 = \frac{1}{2}(G_1 + G_2), \quad \Sigma_2 = \frac{1}{2}(G_1 - G_2). \tag{5.57}$$

THEOREM 5.6 *Assume that in the difference scheme (5.45), (5.46) $A^* = A > 0$ is a constant operator and $G_k(t)$ satisfy the operator relations*

$$\Sigma_2^*(t) = \Sigma_2(t) > -\frac{1}{\tau^2\|A\|}E, \quad \Sigma_1(t) > \Sigma_2^{(0.5)}(t) + 0.5E. \tag{5.58}$$

Then the following estimate holds:

$$\begin{aligned} \|Ay_{n+1}\| \le & M_1\left(\|Ay_0\| + \|Ay_{t,0}\|_{R(\tau)} + \|\varphi_0\|\right) + \|\varphi_n\| \\ & + M_2 \max_{0<k\le n} \|\varphi_{\bar{t},k}\|, \end{aligned} \tag{5.59}$$

where $R = A^{-1} + \tau^2(\Sigma_2 + 2E)$, and the constants M_1, M_2 take the values $M_1 = \sqrt{2}e^{0.5T}$, $M_2 = M_1 T^{1/2}$.

P r o o f. Taking the inner product of the transformed equation (5.56) and $2\tau Ay_t$, we arrive atthe following estimates

$$\begin{aligned} 2\tau\left(Ay_t, y_{\bar{t}t}\right) &= \tau^2\|y_{\bar{t}t}\|_A^2 + \|y_t\|_A^2 - \|y_{\bar{t}}\|_A^2, \\ 2\tau\left(Ay_t, Ay\right) &= \|A\hat{y}\|^2 - \|Ay\|^2 - \tau^2\|Ay_t\|^2, \\ 2\tau^2\left(Ay_t, \Sigma_1 Ay_t\right) &= 2\tau^2\left(\Sigma_1 Ay_t, Ay_t\right). \end{aligned}$$

On the basis of self-adjointness of the operator Σ_2 and the identity

$$2(u, v) = -\|u - v\|^2 + \|u\|^2 + \|v\|^2, \tag{5.60}$$

the following inner product can be written as

$$\begin{aligned} -2\tau^2\left(Ay_t, \Sigma_2 Ay_{\bar{t}}\right) = & -\tau^2\left(\Sigma_2 Ay_t, Ay_t\right) - \tau^2\left(\Sigma_2 Ay_{\bar{t}}, Ay_{\bar{t}}\right) \\ & + \tau^4\left(\Sigma_2 Ay_{\bar{t}t}, Ay_{\bar{t}t}\right). \end{aligned}$$

Summing all of the estimates obtained, we find that

$$\begin{aligned} 2\tau^2(\Sigma_3 Ay_t, Ay_t) + \tau^2\left(QAy_{\bar{t}t}, Ay_{\bar{t}t}\right) + \left(\hat{Q}A\hat{y}_{\bar{t}}, A\hat{y}_{\bar{t}}\right) & \\ = \left(QAy_{\bar{t}}, Ay_{\bar{t}}\right) - \|A\hat{y}\|^2 + \|Ay\|^2 + 2\tau\left(Ay_t, \varphi\right). & \end{aligned} \tag{5.61}$$

Here $\Sigma_3(t) = \Sigma_1(t) - \Sigma_2^{(0.5)}(t) - 0.5E \ge 0$, $Q = A^{-1} + \tau^2\Sigma_2 > 0$.
Using (5.60) and the identity

$$\|\varphi\|^2 - 2\tau\left(\check{\varphi}, \varphi_{\bar{t}}\right) - \|\check{\varphi}\|^2 = \tau^2\|\varphi_{\bar{t}}\|^2, \tag{5.62}$$

we obtain the inequality

$$
\begin{aligned}
-\|A\hat{y}\|^2 &+ \|Ay\|^2 + 2\tau\,(Ay_t, \varphi) \\
&= \|Ay\|^2 - \|A\hat{y}\|^2 + 2\,(A\hat{y}, \varphi) - 2\,(Ay, \check{\varphi}) \\
&- 2\,(Ay - \check{\varphi}, \tau\varphi_{\bar{t}}) - 2\tau\,(\check{\varphi}, \varphi_{\bar{t}}) \\
&\leq -\|A\hat{y} - \varphi\|^2 + (1+\tau)\|Ay - \check{\varphi}\|^2 + \tau(1+\tau)\|\varphi_{\bar{t}}\|^2.
\end{aligned}
\tag{5.63}
$$

Taking account of inequality (5.63) in the energy identity (5.61), we arrive atthe following estimate

$$
\begin{aligned}
\|\hat{y}\|^2_{(1)} &\leq (1+\tau)\left(\|y\|^2_{(1)} + \tau\|\varphi_{\bar{t}}\|^2\right) \leq \dots \\
&\leq e^{t_n}\left(\|y_1\|^2_{(1)} + t_n \max_{0<k\leq n} \|\varphi_{\bar{t},k}\|^2\right)
\end{aligned}
$$

or

$$
\|y_{n+1}\|_{(1)} \leq e^{0.5t_n}\left(\|y_1\|_{(1)} + \sqrt{t_n} \max_{0<k\leq n} \|\varphi_{\bar{t},k}\|\right), \tag{5.64}
$$

where

$$
\|y\|^2_{(1)} = \|Ay - \check{\varphi}\|^2 + \|Ay_{\bar{t}}\|^2_Q. \tag{5.65}
$$

As

$$
\begin{aligned}
\|y_{n+1}\|_{(1)} &\geq \|Ay_{n+1} - \varphi_n\| \geq \|Ay_{n+1}\| - \|\varphi_n\|, \\
\|y_1\|_{(1)} &\leq \sqrt{2}\|Ay_0\| + \|Ay_{t,0}\|_{R(\tau)} + \sqrt{2}\|\varphi_0\|,
\end{aligned}
$$

then the desired estimate (5.59) follows from inequality (5.64).

REMARK 5.4 Theorem 5.6 remains valid under the following restrictions on the operators A and $\Sigma_2(t)$:

$$
A^* = A \geq 0, \quad \Sigma_2^*(t) = \Sigma_2(t) \geq 0. \tag{5.66}
$$

In this case the estimate (5.59) takes the form

$$
\begin{aligned}
\|Ay_{n+1}\| \leq M_1 \Big(& \|Ay_0\| + (Ay_{t,0}, y_{t,0}) \\
& + \tau^2\left((\Sigma_2(\tau) + 2E)\, Ay_{t,0}, Ay_{t,0}\right) + \|\varphi_0\|\Big) \\
& + \|\varphi_n\| + M_2 \max_{0<k\leq n} \|\varphi_{\bar{t},k}\|.
\end{aligned}
\tag{5.67}
$$

REMARK 5.5 Since relations (5.57) and (5.31) are equivalent, then estimates (5.59), (5.67) remain valid also for three-level schemes with operator factors of the form (5.29).

3. Schemes with $D = E + 0.5\tau^2 AG_1$, $B = \tau AG_2$

In this section we consider three-level schemes with operator factors which correspond to schemes with a weighting of the solution.

3.1 Estimates of Stability with Respect to Initial Data

Let us formulate the corresponding stability conditions for the scheme

$$\left(E + 0.5\tau^2 AG_1\right) y_{\bar{t}t} + \tau AG_2 y_{\overset{\circ}{t}} + Ay = 0, \quad t \in \omega_\tau, \tag{5.68}$$

$$y_0 = u_0, \quad y_1 = u_1. \tag{5.69}$$

Here we assume that

$$AG_k \neq G_k A, \quad k = 1, 2. \tag{5.70}$$

Note that in a simpler case, when A and G_k are self-adjoint positive and commutative operators, we have

$$D = D^* > 0, \quad B(t) \geq 0, \quad A = A^* > 0. \tag{5.71}$$

For this class of problems one can apply directly the results of Chapter 4.

We transform problem (5.68) so that conditions of the kind (5.71) are satisfied, assuming $y = Ax$. Then the original problem takes the form

$$\widetilde{D} x_{\bar{t}t} + \widetilde{B} x_{\overset{\circ}{t}} + \widetilde{A} x = 0, \quad x_0 = A^{-1} u_0, \quad x_1 = A^{-1} u_1, \tag{5.72}$$

where the operators

$$\widetilde{D} = A + 0.5\tau^2 AG_1 A, \quad \widetilde{B} = \tau AG_2 A, \quad \widetilde{A} = A^2$$

are defined as in scheme (5.13), but now for the element $x(t) \in H$, $t \in \omega_\tau$.

THEOREM 5.7 *Let the hypothesis of Theorem 5.1 be satisfied, i.e., the constant operators A, G_1 satisfy the conditions*

$$A = A^* > 0, \quad G_1 = G_1^*, \tag{5.73}$$

$$G_1 \geq \sigma_0 E, \quad \sigma_0 = \frac{1}{2} - \frac{1}{0.5\tau^2 \|A\|}. \tag{5.74}$$

Then the condition

$$G_2(t) \geq 0 \tag{5.75}$$

is necessary and sufficient for stability of the scheme (5.68) with respect to initial data, and the following estimate for each $\tau > 0$ *is valid:*

$$\|y_n^{(0.5)}\|^2 + \|y_{t,n}\|_R^2 \leq \|y_0^{(0.5)}\|^2 + \|y_{t,0}\|_R^2, \tag{5.76}$$

where $y^{(0.5)} = 0.5\,(\hat{y} + y)$, $R = A^{-1} + 0.5\tau^2(G_1 - 0.5E)$.

P r o o f. By (5.21) the following estimate is valid for the solution of problem (5.72):

$$\|Ax_n^{(0.5)}\|^2 + \|x_{t,n}\|_{\widetilde{R}}^2 \leq \|Ax_0^{(0.5)}\|^2 + \|x_{t,0}\|_{\widetilde{R}}^2,$$

where $\widetilde{R} = A + 0.5\tau^2 A(G_1 - 0.5E)A$.

Returning to the original notation

$$\|Ax_n^{(0.5)}\| = \|y_n^{(0.5)}\|, \quad \|x_{t,0}\|_{\widetilde{R}} = \|y_{t,0}\|_R$$

we obtain the assertion of the theorem.

REMARK 5.6 If we require in Theorem 5.7 that, instead of (5.74), conditions (5.26) are satisfied, then the inequality (5.75) is sufficient for the validity of the estimate

$$\|y_{n+1}\| \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}}\,\left(\|y_0\| + \|y_{t,0}\|_{\widetilde{D}}\right) \tag{5.77}$$

with $\widetilde{D} = A^{-1} + 0.5\tau^2 G_1$.

As in the case of (5.24), the inequality (5.77) follows from estimate (4.46).

To approximate the abstract Cauchy problem (5.27) on a uniform grid ω_τ, one can apply *difference schemes with operator weighted factors* of the form

$$y_{\bar{t}t} + Ay^{(\Sigma_1,\Sigma_2)} = \varphi(t), \quad t \in \omega_\tau, \quad y_0 = u_0, \quad y_1 = u_1. \tag{5.78}$$

Here $y^{(\Sigma_1,\Sigma_2)}$ is defined by (5.30), G_1, G_2 are defined according to (5.31) as

$$G_1 = \Sigma_1 + \Sigma_2, \quad G_2 = \Sigma_1 - \Sigma_2.$$

Then for scheme (5.78), under conditions (5.32) and (5.33), i.e.:

$$A^* = A > 0, \quad (\Sigma_1 + \Sigma_2)^* = (\Sigma_1 + \Sigma_2), \tag{5.79}$$

$$\Sigma_1 \geq \Sigma_2, \quad \Sigma_1 + \Sigma_2 \geq \sigma_\varepsilon E, \quad \sigma_\varepsilon = \frac{1+\varepsilon}{2} - \frac{1}{0.5\tau^2\|A\|} \tag{5.80}$$

the estimate (5.77) is valid.

3.2 Other *A Priori* Estimates

In the energy space H it is possible to derive estimates of the form (5.77), but under rather different conditions on the operators G_k, $k = 1, 2$. To do this, according to requirements (4.58), it is necessary to check for scheme (5.72) whether or not the following conditions are satisfied:

$$\widetilde{A}^* = \widetilde{A} > 0, \quad \widetilde{B}^* = \widetilde{B}, \quad \widetilde{D}^* = \widetilde{D} > \frac{\tau}{2}\widetilde{B}, \tag{5.81}$$

and then to use the conclusions of Theorems 4.5, 4.6.

REMARK 5.7 As has been already noted above, the canonical forms (5.72) and (5.13) differ only by elements $x(t)$, $y(t) \in H$, $t \in \omega_\tau$. Therefore in order to obtain *a priori* estimates in our case it is necessary to replace formally Ay by y in estimates (5.39).

In accordance with this remark we state without proofs the analogues of Theorems 5.2, 5.3.

THEOREM 5.8 *Let the constant operators A, G_1, G_2 satisfy the conditions*

$$A^* = A > 0, \quad G_k^* = G_k, \quad k = 1, 2, \tag{5.82}$$

$$G_1 > G_2 - \frac{1}{0.5\tau^2\|A\|}E. \tag{5.83}$$

Then the condition

$$G_2 \geq 0.5E \tag{5.84}$$

is sufficient for the stability of scheme (5.68), (5.69) with respect to the initial data, and the following estimate is valid:

$$\|y_n\|^2 + \|y_{\bar{t},n}\|^2_{\widetilde{R}} \leq \|y_1\|^2 + \|y_{\bar{t},1}\|^2_{\widetilde{R}}, \tag{5.85}$$

where

$$\widetilde{R} = \widetilde{D} - \frac{\tau}{2}\widetilde{B} = A^{-1} + 0.5\tau^2(G_1 - G_2). \tag{5.86}$$

THEOREM 5.9 *Let the constant operators A, G_1, G_2 satisfy conditions (5.82) and*

$$G_1 \geq G_2, \quad G_2 \geq 0.5E. \tag{5.87}$$

Then the difference scheme (5.68), (5.69) is stable with respect to the initial data, and the estimate

$$\|y_{n+1}\|^2 \leq \|y_1\|^2 + \|y_{\bar{t},1}\|^2_{\widetilde{R}} \tag{5.88}$$

holds.

REMARK 5.8 Conditions (5.41) for the three-level difference scheme (5.78) with operator weighted factors are satisfied when

$$\Sigma_2 \geq 0, \quad \Sigma_1 \geq \Sigma_2 + 0.5E. \tag{5.89}$$

Thus if $A^* = A > 0$ and if the operators Σ_k, $k = 1, 2$, are self-adjoint and satisfy conditions (5.89), then the difference scheme (5.78) is stable with respect to the initial data, and its solution satisfies the estimate (5.88).

3.3 Stability with Respect to the Right Hand Side

In this subsection we derive the corresponding *a priori* estimates for the three-level operator-difference scheme

$$(E + 0.5\tau^2 AG_1)y_{\bar{t}t} + \tau AG_2 y_{\overset{\circ}{t}} + Ay = \varphi, \quad t \in \omega_\tau, \tag{5.90}$$

$$y_0 = u_0, \quad y_1 = u_1 \tag{5.91}$$

under the assumption that the operators A and G_k, $k = 1, 2$ are non-commutative. Assuming $y = Ax$, from (5.90), (5.91) we deduce the canonical form of three-level schemes as

$$\widetilde{D}x_{\bar{t}t} + \widetilde{B}x_{\overset{\circ}{t}} + \widetilde{A}x = \varphi, \quad x_0 = A^{-1}u_0, \quad x_1 = A^{-1}u_1, \tag{5.92}$$

where

$$\widetilde{D} = A + 0.5\tau^2 AG_1 A, \quad \widetilde{B} = \tau AG_2 A, \quad \widetilde{A} = A^2. \tag{5.93}$$

THEOREM 5.10 *Let conditions (5.47), (5.48) are satisfied. Then the difference scheme (5.90), (5.91) is stable with respect to the initial data and right hand side, and its solution satisfies the estimate*

$$\begin{aligned} \|y_{n+1}\| \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \Bigg(& \|y_0\| + \|y_{t,0}\|_{\widetilde{R}} + \|(A^{-1}\varphi)_1\| \\ & + \|(A^{-1}\varphi)_n\| + \sum_{k=1}^{n-1} \tau \|(A^{-1}\varphi_t)_k\| \Bigg), \end{aligned} \tag{5.94}$$

where $\widetilde{R} = A^{-1} + 0.5\tau^2 G_1$.

Proof. Using inequality (5.52) and Lemma 4.1 applied to scheme (5.92), we obtain the estimate

$$\|x_{n+1}\|_{\tilde{A}} \le \sqrt{\frac{1+\varepsilon}{\varepsilon}} \Big(\|x_0\|_{\tilde{A}} + \|x_{t,0}\|_{\tilde{D}} + \|\varphi_1\|_{\tilde{A}^{-1}} + \|\varphi_n\|_{\tilde{A}^{-1}} + \sum_{k=1}^{n-1} \tau \|\varphi_{t,k}\|_{\tilde{A}^{-1}} \Big). \tag{5.95}$$

Since $\|x\|_{\tilde{A}} = \|x\|_{A^2} = \|Ax\| = \|y\|$, $\|x_{t,0}\|_{\tilde{D}} = \|Ax_{t,0}\|_{\tilde{R}}$, then the estimate (5.94) follows directly from (5.95).

By analogous arguments, according to Theorem 5.5, it is possible to prove stability of the scheme also in the case of variable operators $A(t)$, $G_1(t)$ satisfying the Lipschitz continuity condition (5.54).

THEOREM 5.11 *Let the operators of scheme (5.90), (5.91) satisfy conditions (5.47), (5.48), (5.54). Then the scheme is stable with respect to initial data and right hand side, and the following estimate is valid:*

$$\|y_{n+1}\| \le M_1 \sqrt{\frac{1+\varepsilon}{\varepsilon}} \Big(\|y_0\| + \|y_{t,0}\|_{\tilde{D}(\tau)} + \max_{1<k\le n} \left[\|(A^{-1}\varphi)_k\| + \|(A^{-1}\varphi)_{\bar{t},k}\| \right] \Big). \tag{5.96}$$

If the operators $G_k(t)$, $k = 1,2$, do not satisfy the Lipschitz continuity conditions with respect to the variable t, then we have the following stability result.

THEOREM 5.12 *Assume that in the difference scheme (5.90), (5.91) $A^* = A > 0$ is a constant operator and $G_k(t)$ satisfy the operator relations*

$$\Sigma_2^*(t) = \Sigma_2(t) > -\frac{1}{\tau^2\|A\|}E, \quad \Sigma_1(t) \ge \Sigma_2^{(0.5)}(t) + 0.5E, \tag{5.97}$$

$$\Sigma_1 = 0.5(G_1 + G_2), \quad \Sigma_2 = 0.5(G_1 - G_2). \tag{5.98}$$

Then the following estimate is true:

$$\begin{aligned} \|y_{n+1}\| \le & M_1 \left(\|y_0\| + \|y_{t,0}\|_{R(\tau)} + \|A^{-1}\varphi_0\| \right) \\ & + \|A^{-1}\varphi_n\| + M_2 \max_{0<k\le n} \|A^{-1}\varphi_{\bar{t},k}\|, \end{aligned} \tag{5.99}$$

where $R = A^{-1} + \tau^2 (\Sigma_2 + 2E)$ and $M_1 = \sqrt{2}e^{0.5T}$, $M_2 = M_1 T^{1/2}$.

P r o o f. Using the difference formulae

$$y_{\bar{t}t} = \frac{y_t - y_{\bar{t}}}{\tau}, \quad y_{\overset{\circ}{t}} = 0.5(y_t + y_{\bar{t}}), \tag{5.100}$$

we rewrite equation (5.90) in the form

$$y_{\bar{t}t} + Ay + \tau\left(A\Sigma_1 y_t - A\Sigma_2 y_{\bar{t}}\right) = \varphi.$$

Considering the inner product of the last equation and $2\tau A^{-1}y_t$, we make transformations in much the same way as before for deriving estimate (5.59):

$$\begin{aligned} 2\tau\left(A^{-1}y_t, y_{\bar{t}t}\right) &= \tau^2\|y_{\bar{t}t}\|^2_{A^{-1}} + \|y_t\|^2_{A^{-1}} - \|y_{\bar{t}}\|^2_{A^{-1}}, \\ 2\tau\left(A^{-1}y_t, Ay\right) &= \|\hat{y}\|^2 - \|y\|^2 - \tau^2\|y_t\|^2, \\ 2\tau^2\left(A^{-1}y_t, A\Sigma_1 y_t\right) &= 2\tau^2(\Sigma_1 y_t, y_t). \end{aligned}$$

Because of self-adjointness of the operators A, Σ_2 and the identity (5.60), we have

$$-2\tau^2\left(A^{-1}y_t, A\Sigma_2 y_{\bar{t}}\right) = -\tau^2(\Sigma_2 y_t, y_t) - \tau^2(\Sigma_2 y_{\bar{t}}, y_{\bar{t}}) + \tau^4(\Sigma_2 y_{\bar{t}t}, y_{\bar{t}t}).$$

Summing the estimates obtained, we find that

$$\begin{aligned} &2\tau^2\|y_t\|^2_{\Sigma_3} + \tau^2\|y_{\bar{t}t}\|^2_Q + \|\hat{y}_{\bar{t}}\|^2_{\hat{Q}} \\ &\quad = \|y_{\bar{t}}\|^2_Q - \|\hat{y}\|^2 + \|y\|^2 + 2\tau(A^{-1}y_t, \varphi), \end{aligned} \tag{5.101}$$

where

$$\Sigma_3(t) = \Sigma_1(t) - \Sigma_2^{(0.5)}(t) - 0.5E > 0, \quad Q = A^{-1} + \tau^2\Sigma_2 > 0.$$

Using the inequality (5.63), we estimate the expression from the equality (5.101) as

$$\begin{aligned} &-\|\hat{y}\|^2 + \|y\|^2 + 2\tau(A^{-1}y_t, \varphi) \\ &\quad = -\|\hat{y}\|^2 + \|y\|^2 + 2\tau(y_t, A^{-1}\varphi) \\ &\quad \le -\|\hat{y} - A^{-1}\varphi\|^2 + (1+\tau)\|y - A^{-1}\check{\varphi}\|^2 \\ &\qquad + \tau(1+\tau)\|A^{-1}\varphi_{\bar{t}}\|^2. \end{aligned} \tag{5.102}$$

Substituting (5.102) into the energy identity (5.101), we arrive at the estimate

$$\begin{aligned} \|\hat{y}\|^2_{(1)} &\le (1+\tau)\left(\|y\|^2_{(1)} + \tau\|A^{-1}\varphi_{\bar{t}}\|^2\right) \le \dots \\ &\le e^{t_n}\left(\|y_1\|^2_{(1)} + t_n \max_{0<k\le n}\|A^{-1}\varphi_{\bar{t},k}\|^2\right), \end{aligned} \tag{5.103}$$

in which

$$\|y\|_{(1)}^2 = \|y - A^{-1}\check{\varphi}\|^2 + \|y_{\bar{t}}\|_Q^2. \qquad (5.104)$$

The inequality (5.103) implies the estimate

$$\|y_{n+1}\|_{(1)} \le e^{0.5t_n}\left(\|y_1\|_{(1)} + \sqrt{t_n}\max_{0<k\le n}\|\varphi_{\bar{t},k}\|\right). \qquad (5.105)$$

Taking into account that

$$\|y_{n+1}\|_{(1)} \ge \|y_{n+1} - A^{-1}\varphi_n\| \ge \|y_{n+1}\| - \|A^{-1}\varphi_n\|,$$
$$\|y_1\|_{(1)} \le \sqrt{2}\|y_0\| + \|y_{t,0}\|_{R(\tau)} + \sqrt{2}\|A^{-1}\varphi_0\|,$$

we obtain the estimate

$$\|y_{n+1}\| \le M_1\left(\|y_0\| + \|y_{\bar{t},0}\|_{R(\tau)}\right) + \|\varphi_n\|_{(2)},$$

where

$$\|\varphi_n\|_{(2)} = \|A^{-1}\varphi_n\| + M_1\|A^{-1}\varphi_0\| + M_2\max_{0<k\le n}\|A^{-1}\varphi_{\bar{t},k}\|.$$

REMARK 5.9 Theorem 5.12 is valid also for a three-level scheme of the form (5.78):

$$y_{\bar{t}t} + Ay^{(\Sigma_1,\Sigma_2)} = \varphi, \quad y_0 = u_0, \quad y_1 = u_1. \qquad (5.106)$$

Indeed, using the identity

$$y^{(\Sigma_1,\Sigma_2)} = y + \tau(\Sigma_1 - \Sigma_2)y_{\overset{\circ}{t}} + 0.5\tau^2(\Sigma_1 + \Sigma_2)y_{\bar{t}t}$$

the scheme can be transformed into the form

$$\left(E + 0.5\tau^2 A(\Sigma_1 + \Sigma_2)\right)y_{\bar{t}t} + \tau A(\Sigma_1 - \Sigma_2)y_{\overset{\circ}{t}} + Ay = \varphi. \qquad (5.107)$$

Comparing scheme (5.107) with the original scheme (5.90), we observe that the operators Σ_k and G_k, $k = 1, 2$, are connected by (5.99).

4. Difference Schemes of Divergent Form

In this section an important class of three-level difference schemes with operator factors is distinguished. This class arises, in particular, when one uses schemes with a weighting of flows for solving an initial boundary value problem for the oscillation equation.

4.1 Stability with Respect to the Initial Data

Consider operators of the special form

$$D = E + 0.5\tau^2 T^* G_1 T, \quad B = \tau T^* G_2 T, \quad A = T^* T, \tag{5.108}$$

which occurs in approximations of second-order hyperbolic equations. According to Section 3.3, we will consider a real finite-dimensional space H endowed with an inner product (y, v) and a norm $\|y\| = (y, y)^{1/2}$. We denote by H^* a real finite-dimensional Hilbert space with an inner product $(y, v)^*$ and a norm $\|y\|^* = \sqrt{(y, y)^*}$. Let

$$T : H \to H^*, \quad G_k : H^* \to H^*, \quad T^* : H^* \to H. \tag{5.109}$$

Then the operator A acts from H into H, i.e.,

$$A : H \to H.$$

The operators T and T^* are conjugate in the sense that

$$(Ty, v)^* = (y, T^* v) \quad \text{for every} \quad y \in H, \quad v \in H^*.$$

If the operator $A = A^* > 0$ then the norm in the energy space $H_A = H_{T^*T}$ is defined as

$$\|y\|_A = (T^* T y, y)^{1/2} = \|Ty\|^2. \tag{5.110}$$

We will use below the following analog of Lemma 3.1.

LEMMA 5.1 *Let T^{-1} exist. Then the following operator inequalities are equivalent:*

$$G_k \geq 0, \qquad T^* G_k T \geq 0, \quad k = 1, 2. \tag{5.111}$$

Moreover, the latter follows from the former even without the requirement of reversibility of the operator T.

Let us consider three-level operator-difference schemes in the canonical form

$$D y_{\bar{t}t} + B y_{\overset{\circ}{t}} + A y = 0, \quad t_n \in \omega_\tau, \quad y_0 = u_0, \quad y_1 = u_1, \tag{5.112}$$

where the operators D, B, A are defined by (5.108).

In order to use Theorem 4.1, we check the assumptions

$$A^* = A > 0, \quad D^* = D > \frac{\tau^2}{4} A. \tag{5.113}$$

It is obvious that by definition $A = A^*$, and self-adjointness of the operator D will follow from self-adjointness of the operator G_1. On the other hand, $A > 0$ if T^{-1} exists. It remains to consider the operator

$$R = D - \frac{\tau^2}{4} A. \tag{5.114}$$

By the definition of operators (5.108) and in virtue of Lemma 5.1, we have

$$\begin{aligned} R &= D - \frac{\tau^2}{4} A = E + 0.5\tau^2 T^*(G_1 - 0.5E)T \\ &\geq \frac{1}{\|A\|} T^*T + 0.5\tau^2 T^*(G_1 - 0.5E)T \\ &= 0.5\tau^2 T^* \left(G_1 - \left(0.5 - \frac{1}{0.5\tau^2 \|A\|} \right) E \right) T > 0 \end{aligned} \tag{5.115}$$

for

$$G_1 > \sigma_0 E, \quad \sigma_0 = \frac{1}{2} - \frac{1}{0.5\tau^2 \|A\|}. \tag{5.116}$$

THEOREM 5.13 *Let the operators $A > 0$ and $G_1^* = G_1$ in the difference scheme (5.112), (5.108) be constant, and let the inequality (5.116) be satisfied. Then the condition*

$$G_2(t) \geq 0 \tag{5.117}$$

is necessary and sufficient for stability of the scheme with respect to the initial data, and for any $\tau > 0$ the following estimate is valid:

$$\|y_n^{(0.5)}\|_A^2 + \|y_{t,n}\|_R^2 \leq \|y_0^{(0.5)}\|_A^2 + \|y_{t,0}\|_R^2. \tag{5.118}$$

Proof. The necessary and sufficient condition of stability $B(t) \geq 0$ is equivalent to the inequality

$$T^* G_2(t) T \geq 0. \tag{5.119}$$

Since this estimate (5.119) follows from condition (5.117) by Lemma 5.1, the theorem is thus proved.

REMARK 5.10 It is seen from (5.115) that the estimate (5.118) is valid under more easily verifiable conditions on the operators G_1, $G_2(t)$:

$$G_1^* = G_1 \geq \frac{1}{2} E, \quad G_2(t) \geq 0.$$

REMARK 5.11 In order to obtain an estimate of stability with respect to the initial data, according to Lemma 4.1, in the more natural norm H_A, we require that the operator

$$R_\varepsilon = D - \frac{(1+\varepsilon)\tau^2}{4}A, \quad \varepsilon > 0 \tag{5.120}$$

is non-negative. It is possible if

$$G_1 \geq \left(\frac{1+\varepsilon}{2} - \frac{1}{0.5\tau^2\|A\|}\right)E. \tag{5.121}$$

Then the condition $G_2(t) \geq 0$ is necessary and sufficient for the fulfilment of the estimate

$$\|y_{n+1}\|_A \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}}\left(\|y_0\|_A + \|y_{t,0}\|_D\right), \tag{5.122}$$

where

$$D = E + 0.5\tau^2 T^* G_1 T.$$

REMARK 5.12 If the variable operators A, G_k, $k = 1, 2$ satisfy the Lipschitz conditions (4.33) and (4.34), then according to Theorem 4.2 one can obtain, instead of (5.122), the inequality

$$\|y_{n+1}\|_{A_n} \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} M\left(\|y_0\|_{A_0} + \|y_{t,0}\|_{D(\tau)}\right), \ M = \text{const} > 0. \tag{5.123}$$

To difference schemes of the form (5.112), (5.108), we can reduce the following three-level schemes with operator weighted factors

$$y_{\bar{t}t} + T^*(Ty)^{(\Sigma_1,\Sigma_2)} = \varphi(t), \quad y_0 = u_0, \quad y_1 = u_1, \tag{5.124}$$

where the operators G_k, Σ_k, $k = 1, 2$ are coupled by

$$G_1 = \Sigma_1 + \Sigma_2, \quad G_2 = \Sigma_1 - \Sigma_2. \tag{5.125}$$

In the energy space H_A it is possible to derive *a priori* estimates by using sufficient conditions for stability of three-level difference schemes, namely

$$A^* = A > 0, \quad B^* = B, \quad D^* = D > \frac{\tau}{2}B. \tag{5.126}$$

Let us denote

$$R = D - \frac{\tau}{2}B = E + 0.5\tau^2 T^*(G_1 - G_2)T \tag{5.127}$$

and verify the veracity of the following assertion.

THEOREM 5.14 *Assume that in the difference scheme (5.112), (5.108) T and $G_k^* = G_k$, $k = 1, 2$ are constant operators, T^{-1} exists, and the inequality*

$$G_1 > G_2 - \frac{1}{0.5\tau^2\|A\|}E \tag{5.128}$$

is satisfied. Then the condition

$$G_2 \geq 0.5E \tag{5.129}$$

is sufficient for the stability of scheme (5.112), (5.108) with respect to the initial data, and the following estimate is true:

$$\|y_n\|_A^2 + \|y_{\bar{t},n}\|_R^2 \leq \|y_1\|_A^2 + \|y_{\bar{t},1}\|_R^2 \tag{5.130}$$

Proof. Since under the condition

$$G_1 > G_2 - \frac{1}{0.5\tau^2\|A\|}E$$

the inequality $D > \dfrac{\tau}{2}B$ holds, then the desired estimate (5.130) follows from estimate (4.65) and Theorem 4.5.

Let us formulate the corresponding analogue of Theorem 4.6 for the scheme with variable weighted factors (5.124).

THEOREM 5.15 *Let the operators T, Σ_1, Σ_2 be constant, T^{-1} exist and*

$$\Sigma_k^* = \Sigma_k, \quad k = 1, 2; \qquad \Sigma_2 \geq 0, \quad \Sigma_1 \geq \Sigma_2 + 0.5E. \tag{5.131}$$

Then the difference scheme (5.124) is stable with respect to the initial data, and the following estimate holds:

$$\|y_{n+1}\|_A \leq \|y_1\|_A + \|y_{\bar{t},1}\|_R. \tag{5.132}$$

4.2 Stability with Respect to the Right Hand Side

Consider the inhomogeneous three-level operator-difference scheme in divergent form

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad y_0 = u_0, \quad y_1 = u_1, \tag{5.133}$$

$$D = E + 0.5\tau^2 T^* G_1 T, \quad B = \tau T^* G_2 T, \quad A = T^* T. \tag{5.134}$$

At first we assume that the operators A, G_1, G_2 do not depend on n, the operator T^{-1} exists and that $G_k^* = G_k$ are self-adjoint operators. Then the following theorem is valid.

THEOREM 5.16 *Let the conditions*

$$G_1 \geq G_2, \quad G_2 \geq 0.5E \tag{5.135}$$

be satisfied. Then the difference scheme (5.133), (5.134) is stable with respect to the initial data and right hand side, and its solution for any $\tau > 0$ *satisfies the estimate*

$$\|y_n\|_A \leq \|y_1\|_A + \|y_{\bar{t},1}\|_R + \|\varphi_1\|_{A^{-1}} + \|\varphi_n\|_{A^{-1}} + \sum_{k=1}^{n-1} \tau \|\varphi_{t,k}\|_{A^{-1}} \tag{5.136}$$

in which

$$R = D - \frac{\tau}{2}B = E + 0.5\tau^2 T^*(G_1 - G_2)T.$$

Proof. We will show that if conditions (5.135) are satisfied, then the inequalities

$$D > \frac{\tau}{2}B, \quad B \geq 0.5\tau A \tag{5.137}$$

are valid. Really, by virtue of Lemma 5.1 and under the assumptions of the theorem, we have

$$D - \frac{\tau}{2}B = E + 0.5\tau^2 T^*(G_1 - G_2)T > 0,$$

$$B - 0.5\tau A = \tau T^*\left(G_2 - 0.5E\right)T \geq 0.$$

Under the mentioned conditions the estimate (4.151) holds. Since in our case

$$\|\widetilde{Y}_{n+1}\|_{\widetilde{A}} = \left\{\|y\|_A^2 + \|y_{\bar{t}}\|_R^2\right\}^{1/2} \geq \|y\|_A,$$

the desired conclusion follows from estimate (4.151).

It may turn out that the operators $G_1(t)$, $G_2(t)$ of inhomogeneous difference schemes are not Lipschitz continuous with respect to the variable t. As a result of this the operator D is not Lipschitz continuous too. Taking into account the practical importance of such schemes (for example, in the theory of adaptive grids), we will study them in the case of a constant operator A.

Using the identities

$$y_{\bar{t}t} = (y_t - y_{\bar{t}})/\tau, \quad y_{\overset{\circ}{t}} = \frac{1}{2}(y_t + y_{\bar{t}}), \quad y = y^{(0.5)} - \frac{\tau}{2}y_t,$$

we rewrite scheme (5.133), (5.134) in the form

$$y_{\bar{t}t} + \frac{\tau}{2}T^*G_1T(y_t - y_{\bar{t}}) + \frac{\tau}{2}T^*G_2T(y_t + y_{\bar{t}}) + Ay^{(0.5)} - \frac{\tau}{2}T^*Ty_t = \varphi.$$

Using the notations

$$\Sigma_1 = \frac{1}{2}(G_1 + G_2), \quad \Sigma_2 = \frac{1}{2}(G_1 - G_2), \tag{5.138}$$

$$A_1 = T^*(\Sigma_1 - 0.5E)T, \quad A_2 = T^*\Sigma_2 T, \tag{5.139}$$

the last equation in turn can be rewritten as

$$y_{\bar{t}t} + Ay^{(0.5)} + \tau(A_1 y_t - A_2 y_{\bar{t}}) = \varphi. \tag{5.140}$$

Let, in addition,

$$\|y\|_1^2 = \|y\|_{\check{A}}^2 + \|y_{\bar{t}}\|_R^2, \quad \|\hat{y}\|_1^2 = \|\hat{y}\|_A^2 + \|y_t\|_{\hat{R}}^2, \tag{5.141}$$

$$R = E + \tau^2 A_2, \quad B = R + \tau^2 \check{A}, \quad \check{A} = A(t_{n-1}). \tag{5.142}$$

Note that the notations (5.141), (5.142) are introduced at once also for the case of a variable operator $A(t) = T^*T$, which we will consider below.

THEOREM 5.17 *Let the operator T be constant, T^{-1} exist, and the operators $G_1(t)$, $G_2(t)$ be self-adjoint. Then if*

$$\Sigma_1(t) \geq \Sigma_2^{(0.5)} + 0.5E, \quad \Sigma_2(t) > 0 \tag{5.143}$$

the difference scheme 5.133), (5.134) is stable with respect to the initial data and right hand side, and its solution for any $\tau > 0$ satisfies the estimate

$$\|y_{n+1}\|_A \leq M\left(\|y_0\|_A + \|y_{t,0}\|_{B(\tau)} + \left(\sum_{k=1}^{n} \tau\|\varphi_k\|^2\right)^{1/2}\right), \tag{5.144}$$

where $M = e^{0.5T}$, $y_n = y(t_n)$, $\varphi_k = \varphi(t_k)$, $t_n = n\tau$.

P r o o f. Consider the inner product of the operator equation (5.140) and $2\tau y_t$ term by term. As the operator A is self-adjoint, then we have

$$2\tau\left(y_t, y_{\bar{t}t} + Ay^{(0.5)} + \tau A_1 y_t\right)$$
$$= \tau\left(\|y_t\|^2 + \|\hat{y}\|_A^2\right)_{\bar{t}} + \tau^2\left(\|y_{\bar{t}t}\|^2 + 2\|y_t\|_{A_1}^2\right).$$

Applying the generalized Cauchy–Bunyakovskii–Schwarz inequality with $\varepsilon = 1$, we arrive at the estimates

$$-2\tau^2(y_t, A_2 y_{\bar{t}}) \leq \tau^2\left(\|y_t\|_{A_2}^2 + \|y_{\bar{t}}\|_{A_2}^2\right),$$

$$2\tau(y_t, \varphi) = 2\tau(y_{\bar{t}} + \tau y_{\bar{t}t}, \varphi) \leq \tau\|y_{\bar{t}}\|^2 + \tau^2\|y_{\bar{t}t}\|^2 + \tau(1+\tau)\|\varphi\|^2.$$

Summing all of the above estimates yields

$$\|\hat{y}\|_A^2 + \|y_t\|^2 + \tau^2\|y_t\|_{\hat{A}_2}^2 - \tau^2\|y_t\|_{\hat{A}_2}^2 + 2\tau^2\|y_t\|_{A_1}^2 - \tau^2\|y_t\|_{A_2}^2 \\ \leq \|y\|_A^2 + \|y_{\bar{t}}\|^2 + \tau^2\|y_{\bar{t}}\|_{A_2}^2 + \tau\|y_{\bar{t}}\|^2 + \tau(1+\tau)\|\varphi\|^2.$$

Taking into account that $A_1 - 0.5(A_2 + \hat{A}_2) \geq 0$ by the assumption of the theorem, the last inequality can be rewritten in the form

$$\|y_{n+1}\|_A^2 + \|y_{\bar{t},n+1}\|_{E+\tau^2 A_{2n+1}}^2 \\ \leq (1+\tau)\left(\|y_n\|_A^2 + \|y_{\bar{t},n}\|_{E+\tau^2 A_{2n}}^2 + \tau\|\varphi_n\|^2\right), \tag{5.145}$$

or, by virtue of (5.141), we have

$$\|\hat{y}\|_1^2 \leq (1+\tau)\left(\|y\|_1^2 + \tau\|\varphi\|^2\right). \tag{5.146}$$

Hence we obtain the inequality

$$\|y_{n+1}\|_A \leq M\left(\|y_1\|_A + \|y_{t,0}\|_R + \left(\sum_{k=1}^{n}\tau\|\varphi_k\|^2\right)^{1/2}\right). \tag{5.147}$$

Taking into account the estimate

$$\|y_1\|_A = \|y_0 + \tau y_{t,0}\|_A \leq \|y_0\|_A + \|y_{t,0}\|_{\tau^2 A} \tag{5.148}$$

and substituting this into the inequality (5.147), we arrive at the required estimate.

REMARK 5.13 Since the relations (5.138) and (5.125) are equivalent, then the estimate (5.144) remains true also for the scheme with operator weighted factors (5.124).

We now formulate the corresponding result for the variable operator $A(t)$, $t \in \omega_\tau$.

THEOREM 5.18 *Let the hypotheses of Theorem 5.17 be satisfied, and the variable operator $A(t)$ satisfy the Lipschitz condition (2.36)*

$$((A_n - A_{n-1})v_n, v_n) \leq \tau c_0(A_{n-1}v_n, v_n) \tag{5.149}$$

for all $v \in H$, $0 < t \leq T$, where c_0 is a positive constant independent of τ. Then the solution of the difference scheme (5.133), (5.134) satisfies the a priori estimate (5.144) with $M = e^{0.5cT}$, $c = \max\{c_0, 1\}$.

P r o o f. Considering the inner product of equation (5.140) and $2\tau y_t$, similarly to the relation (5.145) we obtain the energy inequality

$$\|y_{n+1}\|^2_{A_n} + \|y_{\bar{t},n+1}\|^2_{R_{n+1}} \le \|y_n\|^2_{A_n} + (1+\tau)\left(\|y_{\bar{t},n}\|^2_{R_n} + \tau\|\varphi_n\|^2\right). \tag{5.150}$$

Because the operator $A(t)$ is Lipschitz continuous we obtain the estimate

$$\|y_n\|^2_{A_n} \le (1+\tau c_0)\|y_n\|^2_{A_{n-1}}. \tag{5.151}$$

Substituting the above estimate into the inequality (5.150) we arrive at the estimate

$$\begin{aligned} \|y_{n+1}\|^2_{\bar{1}} &\le (1+\tau c)\left(\|y_n\|^2_{\bar{1}} + \tau\|\varphi_n\|^2\right) \le \dots \\ &\le M^2\left(\|y_1\|^2_{\bar{1}} + \sum_{k=1}^{n}\tau\|\varphi_k\|^2\right). \end{aligned}$$

From here the desired conclusion follows.

Chapter 6

DIFFERENCE SCHEMES FOR NON-STATIONARY EQUATIONS

1. Introduction

The general stability theory for operator-difference schemes serves as a theoretical foundation for solving the principal problems which arise in the analysis of numerical methods. The stability of difference schemes for linear problems with respect to the initial data and the right hand side, and also coefficient (strong) stability ensure the well posedness of a discrete problem, in other words, its right to exist.

The most important question is the analysis of the proximity of the approximate solution to the exact one, that is, *convergence* . In this case we consider an inhomogenous problem for the error which is formed by the errors of the equation's approximation, boundary, and initial conditions. The error is evaluated in different norms on the basis of the corresponding *a priori* estimates of the numerical solution (first of all, with respect to the right hand side).

The accuracy of finite difference methods applied to base problems of mathematical physics is investigated in the textbooks on difference methods under the simplest assumptions (constant coefficient problems, smooth solution, one-dimensional problems in space, problems with self-adjoint operators and so on). Under more general conditions, more complicated mathematical techniques and more subtle results about stability of the difference schemes studied are required. For example, for the problems with low smoothness of the exact solution we should use bounds on the solution of a difference problem in weaker norms.

In this chapter the classical information dealing with the analysis of difference schemes for non-stationary boundary value problems of mathematical physics based on the general theory of stability of operator-

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difference schemes is supplemented with a new theoretical material. These are the boundary value problems for non-stationary partial differential equations, which will be considered below in detail.

We begin with consideration of the boundary value problem for the one-dimensional parabolic equation

$$\frac{\partial u}{\partial t} = Lu + f(x,t), \quad 0<x<l, \quad 0<t\leq T, \tag{6.1}$$

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad 0 < t \leq T, \tag{6.2}$$

$$u(x,0) = u_0(x), \quad 0 \leq x \leq l. \tag{6.3}$$

In equation (6.1)

$$Lu = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right),$$

i.e., we assume the dependence of the coefficient k on the space variable x.

To obtain an approximate solution of problem (6.1) - (6.3), we use a two-level scheme with weights. We establish the convergence of this difference scheme under different conditions. In particular, we distinguish one practical important case of a piecewise smooth coefficient $k(x)$. Analogous questions are considered for a three-dimensional problem in a regular space domain (parallelepiped). In this case

$$Lu = \sum_{\alpha=1}^{3} L_\alpha u, \quad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right),$$

i.e., the operator of this problem is represented by the sum of one-dimensional operators.

To investigate difference schemes for problems with low smoothness, we apply *a priori* estimates of stability in norms that are integral with respect to time. This technique is naturally more complex than the traditional one. In order not to complicate the presentation, we confine ourselves to an illustrative example of the analysis of a difference scheme for one-dimensional problem (6.1) - (6.3) under the additional assumption about the constancy of the coefficient k.

A class of non-stationary problems, important for applications, is related to parabolic problems when the elliptic operator is not self-adjoint. The following convection–diffusion equation acts as a model problem in continuum mechanics:

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{2} v_\alpha(x)\frac{\partial u}{\partial x_\alpha} - k\sum_{\alpha=1}^{2}\frac{\partial^2 u}{\partial x_\alpha^2} = 0, \quad x \in \Omega, \quad t > 0.$$

To obtain an appropriate solution, we use difference schemes with separate weighting factors for the convection and diffusion terms.

We also investigate difference schemes for approximate solution of the Cauchy problem with periodic in space conditions for the Korteweg–de Vries equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial t^3} = 0, \quad \beta > 0.$$

The consideration of a linearized equation resulted in the construction of nonlinear difference schemes that relate to the class of conservative schemes.

Particular attention in this chapter is given to problems for second-order hyperbolic equations. Standard techniques can be applied to these problems only under practically unjustifiable assumptions about the smoothness of an exact solution. It is more reasonable to assume that the solutions of such problems are piecewise smooth. Under these conditions,the convergence of three-level weighted schemes for a hyperbolic equation with variable coefficients is studied.

An example of a nonclassical boundary value problem is a hyperbolo parabolic problem. In this case the processes involved are described by a hyperbolic equation in one part of the computational domain, and by a parabolic equation in the other part. Of special interest are the conjugation conditions on the interior (interface) boundary where the differential equation changes its type. We construct and study difference schemes for one-dimensional (in space) hyperbolo parabolic problems. In this case we have a scheme with weighting factors discontinuous in space, and for such problems it is most natural to use the general results concerning the stability of difference schemes with operator factors that have been considered above.

2. Boundary Problems for Parabolic Equations

Accuracy of standard two-level difference schemes for parabolic equations with a self-adjoint elliptic operator of the second order is investigated. The case of one-dimensional equation is considered in detail, its generalization to multi-dimensional problems is given, and a class of problems with discontinuous coefficients is singled out.

2.1 Difference–Differential Problem

To illustrate the methods of construction of difference schemes for non-stationary problems and the methods as for their investigation we consider in a rectangle

$$\overline{Q}_T = \overline{\Omega} \times [0,T], \quad \overline{\Omega} = \{x : \; 0 \leq x \leq l\}, \quad 0 \leq t \leq T, \tag{6.4}$$

the first boundary value problem for the *one-dimensional parabolic equation*

$$\frac{\partial u}{\partial t} = Lu + f(x,t), \quad Lu = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right), \quad 0<x<l,\ 0<t\le T, \tag{6.5}$$

$$u(0,t) = 0, \quad u(l,t) = 0, \quad 0 < t \le T, \tag{6.6}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le l. \tag{6.7}$$

The dependence of the coefficient k only on the spatial variable is assumed in order to simplify the discussion. We assume also that it is bounded from above and below, i.e.,

$$0 < c_1 \le k(x) \le c_2, \quad x \in \overline{\Omega}, \tag{6.8}$$

where c_1, c_2 are constants.

Assume that the problem (6.5)–(6.7) has a unique classical solution with all necessary derivatives with respect to spatial and time variable.

We seek approximate solution of the problem (6.5)–(6.7) using the method of finite differences. For this, we consider the following *uniform spatial grid*

$$\overline{\omega}_h = \{\ x_i = ih, \quad i = 0,1,...,N; \quad h = l/N\ \} \tag{6.9}$$

and approximate the initial problem with respect to the variable x with the second order. We obtain the difference–differential problem

$$\frac{dv_i}{dt} = \Lambda v_i + f_i(t), \quad i = 1,2,...,N-1, \quad 0 < t \le T, \tag{6.10}$$

$$v_0 = 0, \quad v_N = 0, \tag{6.11}$$

$$v_i^0 = u_0(x_i), \quad i = 1,2,...,N-1. \tag{6.12}$$

Here

$$\begin{aligned}(\Lambda v)_i &= (a v_{\bar{x}})_{x,i} = \frac{a_{i+1}v_{\bar{x},i+1} - a_i v_{\bar{x},i}}{h} = \\ &= \frac{1}{h}\left(a_{i+1}\frac{v_{i+1}-v_i}{h} - a_i\frac{v_i - v_{i-1}}{h}\right)\end{aligned} \tag{6.13}$$

is a grid operator with respect to space and $a_i = a(x_i)$, $x_i \in \omega_h^+$ is a *stencil functional* . The grid

$$\omega_h^+ = \omega_h \cup \{x_N = l\}, \quad \omega_h = \{\ x_i = ih, \quad i = 1,2,...,N-1\ \},$$

is constructed using the condition

$$(\Lambda u)_i - (Lu)_i = (a u_{\bar{x}})_{x,i} - \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)\bigg|_{x=x_i} = O(h^2). \tag{6.14}$$

As this functional one can choose

$$a_i = k_{i-1/2} = k(x_{i-1/2}), \quad x_{i-1/2} = x_i - \frac{h}{2}, \tag{6.15}$$

$$a_i = 0.5(k_{i-1} + k_i), \tag{6.16}$$

$$a_i = \left[\frac{1}{h}\int\limits_{x_{i-1}}^{x_i} \frac{dx}{k(x)}\right]^{-1}. \tag{6.17}$$

Note that the stencil functional (6.17) is obtained using the integro-interpolational method (IIM) [Samarskii, 1989].

Let us define the grid operator A and vector φ by the formulae

$$(Av)_i = -(av_{\bar{x}})_{x,i}, \quad i=1,2,...,N-1, \quad v_0=0, \ v_N=0, \tag{6.18}$$

$$\varphi(t) = (f_1(t), f_2(t), ..., f_{N-1}(t))^T. \tag{6.19}$$

Then the difference–differential problem (6.10)–(6.12) can be rewritten in an equivalent vector-matrix form

$$\frac{dv}{dt} + Av = \varphi(t), \quad v(0) = u_0. \tag{6.20}$$

Here $v(t) = (v_1, v_2, ..., v_{N-1})^T$, $t > 0$, $\varphi(t)$, $u_0 = (u_1, u_2, ..., u_{N-1})^T$ is the sought and given vectors respectively.

Consider in more detail the properties of the operator A. Let $\overset{\circ}{\Omega}_h$ be a set of grid functions $y_i(t) = y(x_i, t)$ defined for every t on $\overline{\omega}_h$ and satisfying the condition $y_0(t) = y_N(t) = 0$. Introduce the vector $y(t) = (y_1(t), y_2(t), ..., y_{N-1}(t))^T$ and the linear space $H = \Omega_h$ as a set of vectors with ordinary addition and multiplication operations, and with the inner product

$$(y, v) = \sum_{i=1}^{N-1} y_i v_i h$$

and norm

$$\|y\| = \sqrt{(y, y)}.$$

The operator $A : H \to H$ introduced is linear and constant (since the stencil functional $a \neq a(t)$ does not depend on the time variable). It maps Ω_h onto Ω_h (its domain of definition and range of values coincide with the whole space Ω_h). Note that for the real linear space $\Omega_h = R^{N-1}$ for every fixed t. It is obvious that $A = A_h$, $v = v_h$, $\varphi = \varphi_h$ depend on the grid step. Therefore we consider the whole family of the Cauchy difference–differential problems.

Assume that for the stencil functional $a(x)$ (as for the coefficient $k(x)$) the inequalities

$$0 < c_1 \leq a(x) \leq c_2, \quad x \in \omega_h^+ \tag{6.21}$$

are satisfied. The properties of the operator A have been well studied (see [Samarskii, 1989, Samarskii and Andreev, 1976, Samarskii and Goolin, 1973, Samarskii and Nikolaev, 1978]). In particular, it is self-adjoint and positive:

$$A^* = A > 0, \tag{6.22}$$

i.e., for any y, $v \in H$

$$(Ay, v) = (y, Av). \tag{6.23}$$

Moreover, the operator A is self-adjoint, and we have the estimate from below satisfied:

$$(Ay, y) \geq \frac{8c_1}{l^2} \|y\|^2. \tag{6.24}$$

This follows from *Green's formulae*

$$(y, (ay_{\bar{x}})_x) = -\left(a, y_{\bar{x}}^2\right], \quad (v, (ay_{\bar{x}})_x) = (y, (av_{\bar{x}})_x) \tag{6.25}$$

and imbeddings

$$\frac{h^2}{4}\|y_{\bar{x}}]|^2 \leq \|y\|^2 \leq \frac{l^2}{8}\|y_{\bar{x}}]|^2, \tag{6.26}$$

$$\|y\|_A^2 = \left(a, y_{\bar{x}}^2\right] \geq c_1 \|y_{\bar{x}}]|^2, \tag{6.27}$$

valid for any y, $v \in \overset{\circ}{\Omega}_h$. Here we use the notation

$$(y, v] = \sum_{i=1}^{N} y_i v_i h, \quad \|y]| = \sqrt{(y, y]}. \tag{6.28}$$

From here we conclude that

$$(Ay, y) \leq \|A\| \|y\|, \quad \|A\| < \frac{4c_2}{h^2}, \tag{6.29}$$

$$\delta_1 E \leq A < \delta_2 E, \quad \delta_1 = \frac{8c_1}{l^2}, \quad \delta_2 = \frac{4c_2}{h^2}. \tag{6.30}$$

Note also that the inverse operator A^{-1} exists, $(A^{-1})^* = A^{-1} > 0$, and we have the estimates from above and below:

$$\frac{1}{\|A\|} E \leq A^{-1} \leq \frac{1}{\delta_1} E \tag{6.31}$$

with constants defined above.

2.2 Two-Level Difference Schemes

On the segment $[0, T]$ let us introduce the *uniform grid with a time step* τ

$$\overline{\omega}_\tau = \{\ t_n = n\tau, \quad n = 0, 1, ..., N_0; \quad \tau N_0 = T\ \} = \omega_\tau \cup \{T\}. \qquad (6.32)$$

For numerical solution of difference–differential problem (6.10), consider the following class of difference schemes with constant weights:

$$y_t + Ay^{(\sigma)} = \varphi, \quad t \in \omega_\tau, \quad y_0 = u_0, \qquad (6.33)$$

where $y^{(\sigma)} = \sigma y_{n+1} + (1 - \sigma)y_n$ and $y_n = y(t_n) \in \Omega_h$ is the sought function, and the operator $A : H \to H$, $H = \Omega_h$ is defined by formula (6.18), $\varphi_n = \left(\varphi_1^n, \varphi_2^n, ..., \varphi_{N-1}^n\right)^T \in H$, $\varphi_i^n = f_i^{(\sigma)}$.

Using the identity $y^{(\sigma)} = y + \sigma\tau y_t$, the operator-difference scheme (6.33) can be reduce to the canonical form of a two-level scheme:

$$By_t + Ay = \varphi, \quad y_0 = u_0 \qquad (6.34)$$

with the operator

$$B = E + \sigma\tau A.$$

Note, that by virtue of the construction of the operator A the condition

$$B^* = B > 0, \qquad (6.35)$$

is satisfied, where A and B do not depend on n, i.e., they are constant operators.

2.3 Stability Conditions

In Section 2.2.4 it is shown, that the necessary and sufficient condition for the stability of scheme (6.33) with respect to the initial data

$$B \geq 0.5\tau A$$

is satisfied when

$$\sigma \geq \sigma_0, \quad \sigma_0 = \frac{1}{2} - \frac{1}{\tau\|A\|}. \qquad (6.36)$$

Since in our case $\|A\| < \dfrac{4c_2}{h^2}$, the explicit scheme (i.e.,$\sigma = 0$) is stable in the space H_A if

$$\tau \leq \frac{h^2}{2c_2}. \qquad (6.37)$$

The scheme with $\sigma \geq 0.5$ is absolute (i.e., for any τ) stable in the space H_A.

Using *a priori* estimates in the space H_A, one can obtain the corresponding estimates in a uniform metric:

$$\|y_n\|_C = \max_{x\in\omega_h} |y(x,t_n)|, \quad t_n \in \omega_\tau. \tag{6.38}$$

This is done with the use of

LEMMA 6.1 *For any grid function $y \in \overset{\circ}{\Omega}_h$ the inequality*

$$\|y\|_C^2 \le \frac{l}{4c_1}\|y\|_A^2 \tag{6.39}$$

is valid.

Proof. It is sufficient to use the following embedding

$$\|y\|_C \le \frac{\sqrt{l}}{2}\|y_{\bar{x}}]| \tag{6.40}$$

and inequality (6.27).

2.4 Convergence of Difference Schemes

Consider the problem for error of the method. Substituting $y = z + u$ into the operator difference scheme (6.33), we obtain the problem for the approximation error of the method

$$Bz_t + Az = \psi, \quad z_0 = 0. \tag{6.41}$$

Here $\psi_n = \left(\psi_1^n, ..., \psi_{N-1}^n\right)^T$, and

$$\psi_i = \left[\Lambda\left(\sigma\hat{u} + (1-\sigma)u\right)\right]_i + \varphi_i - u_{t,i}, \quad i = 1,2,...,N-1, \tag{6.42}$$

is an approximation error for the scheme in solution of the initial problem (6.5)–(6.7). Substituting

$$\hat{u} = \frac{\hat{u}+u}{2} + \frac{\tau}{2}u_t = \overline{u} + \frac{\tau}{2}\overline{\dot{u}} + O(\tau^2),$$

$$u = \frac{\hat{u}+u}{2} - \frac{\tau}{2}u_t = \overline{u} - \frac{\tau}{2}\overline{\dot{u}} + O(\tau^2),$$

$$u_t = \overline{\dot{u}} + O(\tau^2), \quad \varphi = f^{(\sigma)} = f^{(0.5)} + \tau(\sigma - 0.5)f_{\bar{t}},$$

where $\overline{u} = u(x_i, t_{n+1/2})$, $\dot{u} = \dfrac{\partial u}{\partial t}$, we obtain

$$\psi_i = \left(\Lambda\frac{\hat{u}+u}{2} + (\sigma-0.5)\tau\left(\Lambda u_t + f_t\right)\right)_i + f_i^{(0.5)} - u_{t,i} =$$

$$= \left(\Lambda\overline{u} + \overline{f} - \overline{\dot{u}}\right)_i + \left(\Lambda\overline{u} - L\overline{u}\right)_i + \left(f_i^{(0.5)} - \overline{f}_i\right) \tag{6.43}$$

$$+ (\sigma - 0.5)\tau\left(\Lambda\overline{\dot{u}} + \overline{\dot{f}}\right)_i + O(\tau^2).$$

Since $Lu + f - \dot{u} = 0$ and, in accordance with (6.14),

$$\Lambda u = Lu + O(h^2), \quad f^{(0.5)} - \overline{f} = O(\tau^2),$$

then

$$\psi = (\sigma - 0.5)\tau L\dot{u} + O(\tau^2 + h^2).$$

One can see that for fairly smooth solutions

$$\psi,\ \psi_t = \begin{cases} O(h^2 + \tau^2) & \text{for} \quad \sigma = 0.5, \\ O(h^2 + \tau) & \text{for} \quad \sigma \neq 0.5. \end{cases} \tag{6.44}$$

To obtain an estimate for the convergence rate of the difference method in a class of fairly smooth solutions, it is natural to use the strongest norms. Since the problem for the error method of the (6.41) satisfies all the conditions of Theorem 3.4 when

$$\sigma \geq 0.5, \tag{6.45}$$

then from estimate (3.1.16) we obtain

$$\|z\|_{A^2} \leq \|\psi_n\| + \sum_{k=1}^{n} \tau \|\psi_{\bar{t},k}\|.$$

Hence from this inequality and relation (6.44) we arrive at the estimate

$$\|Az\| \leq c\,(h^2 + \tau^m), \tag{6.46}$$

where $c = \text{const} > 0$, $m = m(\sigma) = 2$, when $\sigma = 0.5$, and $m = 1$, when $\sigma \neq 0.5$.

From inequalities (6.46) on the basis of the corresponding embedding theorems one can find the estimate for the rate of convergence of the solution y with respect to the exact solution u in other norm [Matus, 1993c] too.

LEMMA 6.2 *For any grid function $z \in \overset{\circ}{\Omega}_h$ the following inequalities*

$$\|z\|_A \leq \frac{l}{2\sqrt{2c_1}} \|z\|_{A^2}, \tag{6.47}$$

$$\|z_{\bar{x}}\|_C \leq M_1 \|z\|_{A^2}, \tag{6.48}$$

are valid, where $\|z_{\bar{x}}\|_C = \max\limits_{x \in \omega_h^+} |z_{\bar{x}}|, \quad M_1^2 = \dfrac{1}{c_1^2}\left(\varepsilon + \left(\dfrac{1}{\varepsilon} + \dfrac{1}{l}\right)\dfrac{l^2 c_2}{8c_1}\right),$

$\varepsilon = \text{const} > 0$.

P r o o f. We obtain the first estimate (6.47) using the Cauchy inequality

$$\|y\|_A^2 = (Ay, y) \leq \|Ay\| \, \|y\| \tag{6.49}$$

and the estimate (see (6.24))

$$\|y\| \leq \frac{l}{2\sqrt{2c_1}} \|y\|_A. \tag{6.50}$$

For error estimate of the first derivative $z_{\bar{x}} = y_{\bar{x}} - u_{\bar{x}}$ in a uniform metric one can use the embedding [Samarskii and Andreev, 1976]

$$\max_{x \in \omega_h^+} |v(x)|^2 \leq \varepsilon \|v_x\|^2 + (1/\varepsilon + 1/l) \, \|v\|^2, \tag{6.51}$$

which is valid for any grid function, given on the uniform mesh ω_h^+ and not necessarily vanishing on the boundary, i.e., when $v = az_{\bar{x}}$, $z \in \overset{\circ}{\Omega}_h$, $0 < c_1 \leq a(x) \leq c_2$. We obtain

$$\|z_{\bar{x}}\|_C^2 \leq \frac{\varepsilon}{c_1^2} \|z\|_{A^2}^2 + \frac{1}{c_1^2} \left(\frac{1}{\varepsilon} + \frac{1}{l} \right) \|az_{\bar{x}}]|^2, \tag{6.52}$$

and since

$$\|az_{\bar{x}}]|^2 \leq c_2 \|z\|_A^2 \leq \frac{l^2 c_2}{8c_1} \|z\|_{A^2}^2,$$

then from inequality (6.52)the second estimate (6.48)follows.

Now using the identity

$$u_{\bar{x},i} = \frac{\partial u}{\partial x}(x_{i-1/2}, t_n) + O(h^2),$$

we find the estimate

$$\left\| y_{\bar{x}} - \frac{\partial u}{\partial x}(x_{i-1/2}, t_n) \right\|_C \leq c \, (h^2 + \tau^{m_\sigma})$$

with $m = m(\sigma)$.

2.5 Equation with Discontinuous Coefficients

We investigate the convergence of a uniform scheme with weights (6.33) presuming that the coefficient $k(x)$ has discontinuity of the first kind on the line $x = x_k$, $x_k \in \omega_h$, in the plane (x, t). On the discontinuity line the usual conjugation conditions are satisfied:

$$[u] = 0, \quad \left[k \frac{\partial u}{\partial x} \right] = 0 \quad \text{when} \quad x = x_k, \quad t \geq 0, \tag{6.53}$$

where

$$[v] = v(x_{k+0}, t) - v(x_{k-0}, t).$$

Assume that $k(x)$ and the solution $u(x,t)$ are rather continuous functions outside the discontinuous line. For the sake of calculation simplicity we will assume that $f(x,t) = 0$.

Integrating equation (6.5) over the segment $[x_{i-1/2}, x_{i+1/2}]$ we obtain

$$0 = -\frac{1}{h}\left[(k\overline{u}')_{i+1/2} - (k\overline{u}')_{i-1/2}\right] + \frac{1}{h}\int\limits_{x_{i-1/2}}^{x_{i+1/2}} \overline{\dot{u}}dx, \quad t = t_{n+1/2}. \tag{6.54}$$

Now combining this result with the approximation error (6.42) we obtain ψ in the form

$$\psi_i = \eta_{x,i} + \psi_i^*, \quad i = 1, 2, ..., N-1. \tag{6.55}$$

In the case of (6.55) we have

$$\eta_i = au_{\bar{x},i}^{(\sigma)} - (k\overline{u}')_{i-1/2} = \mathrm{O}\left(h^2 + \tau^{m(\sigma)}\right), \tag{6.56}$$

$$\begin{gathered} \psi_i^* = -(u_t - \overline{\dot{u}})_i = O(\tau^2), \quad \eta_t = O(h^2 + \tau^{m(\sigma)}), \\ \psi_t^* = O(\tau^2). \end{gathered} \tag{6.57}$$

To obtain an estimate for the convergence rate we use an *a priori* estimate (see relation (2.3.16))

$$\|z_{n+1}\|_A \le \|\psi_n\|_{A^{-1}} + \sum_{k=1}^{n} \tau \|\psi_{\bar{t},k}\|_{A^{-1}}, \tag{6.58}$$

where the operator $A : \Omega_h \to \Omega_h$ is defined by formula (6.18).

Let us show that for $\psi = (\psi_1, ..., \psi_{N-1})^T$ the following estimate

$$\|\psi\|_{A^{-1}} \le \frac{1}{\sqrt{c_1}}\left(\|\eta]\| + \frac{l}{2\sqrt{2}}\|\psi^*\|\right) \tag{6.59}$$

is true. In fact, by definition we have

$$\|\psi\|_{A^{-1}} = \sup_{\|v\|_A \ne 0,\ v \in \overset{\circ}{\Omega}_h} \frac{|(\psi, v)|}{\|v\|_A}. \tag{6.60}$$

Using Green's formula and the Cauchy inequality we find

$$|(\psi, v)| = |-(\eta, v_x] + (\psi^*, v)| \le \frac{1}{\sqrt{c_1}}\|\eta]\|\|v\|_A + \|\psi^*\|\|v\|.$$

Substituting the above estimate into identity (6.60) and using the embedding

$$\|v\| \le \frac{l}{2\sqrt{2c_1}}\|v\|_A,$$

we arrive at inequality (6.59). Similarly we can find the estimate for the derivative of the approximation error:

$$\|\psi_t\|_{A^{-1}} \le \frac{1}{\sqrt{c_1}}\left(\|\eta_t]\| + \frac{l}{2\sqrt{2}}\|\psi_t^*\|\right). \tag{6.61}$$

Then taking into account the relations (6.56), (6.57) in inequalities (6.59), (6.61) from the *a priori* estimate (6.58) we find that

$$\|y-u\|_A \le c\,(h^2+\tau^{m(\sigma)}). \tag{6.62}$$

Here, as pointed out above, $m(\sigma) = 1$, where $\sigma \ne 0.5$; $m(\sigma) = 2$, if $\sigma = 0.5$, $c = \text{const} > 0$. Moreover, from Lemma 6.1 we have

$$\|z\|_C \le \frac{\sqrt{l}}{2\sqrt{c_1}}c\,(h^2+\tau^{m_\sigma}). \tag{6.63}$$

Thus the convergence of difference scheme (6.33) in the class of discontinuous coefficients is proved, and we have:

THEOREM 6.1 *Let $k(x)$ have a discontinuity of the first kind at the grid point $x = x_k$, $\sigma \ge 0.5$, and conditions (6.53), (6.55)–(6.57) be satisfied. Then the solution of the difference scheme converges in uniform metrics to an exact one (also in the energy space H_A) with the rate $O(h^2+\tau^{m(\sigma)})$; $m(\sigma) = 2$ for $\sigma = 0.5$, $m(\sigma) = 1$, if $\sigma \ne 0.5$.*

2.6 Multi-dimensional Problems

The convergence of difference schemes in a multi-dimensional case in investigated in the same way. Let $\Omega = \{\ 0 < x_\alpha < l_\alpha, \quad \alpha = 1,2,3\ \}$ be a rectangle with the boundary Γ, $x = (x_1, x_2, x_3)$. We are to find the function $u(x,t)$ satisfying in the area $\overline{Q}_T = \overline{\Omega} \times [0,T]$ the conditions

$$\frac{\partial u}{\partial t} = Lu + f(x,t), \quad x \in \Omega, \quad t \in (0,T], \tag{6.64}$$

$$Lu = \sum_{\alpha=1}^{3} L_\alpha u, \quad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right), \qquad 0 < c_1 \le k_\alpha(x) \le c_2, \quad \alpha = 1,2,3, \tag{6.65}$$

$$u(x,0) = u_0(x),\ x \in \overline{\Omega}, \quad u|_\Gamma = 0,\ x \in \Gamma, \quad t \in (0,T]. \tag{6.66}$$

As in the one-dimensional case, we consider on the segment $[0, T]$ a uniform grid with the step τ with respect to time $\overline{\omega}_\tau$, and in the parallelepiped $\overline{\Omega}$ we introduce a grid $\overline{\omega}_h = \omega_h \cup \gamma_h$, uniform in every direction x_α, γ_h is a set of boundary nodes such that

$$\left.\begin{aligned} \omega_h = \Big\{ x_i = \left(x_1^{(i_1)}, x_2^{(i_2)}, x_3^{(i_3)} \right), \quad x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha, \\ i_\alpha = 1, 2, ..., N_\alpha - 1, \quad h_\alpha N_\alpha = l_\alpha, \ \alpha = 1, 2, 3 \Big\}. \end{aligned}\right\} \tag{6.67}$$

We approximate the operators L_α by difference operators such that

$$\Lambda_\alpha y = (a_\alpha y_{\bar{x}_\alpha})_{x_\alpha}, \quad 0 < c_1 \leq a_\alpha \leq c_2, \tag{6.68}$$

with the stencil functionals being defined by relations (6.15)–(6.17). We write a two-level difference scheme with constant weighting factors

$$y_t = \Lambda y^{(\sigma)} + f^{(\sigma)}, \quad x \in \omega_h, \quad t \in \omega_\tau, \tag{6.69}$$

$$y = 0, \quad x \in \gamma_h, \quad t \in \omega_\tau; \quad y(x, 0) = u_0(x), \quad x \in \overline{\omega}_h, \tag{6.70}$$

such that

$$\Lambda y = \sum_{\alpha=1}^{3} \Lambda_\alpha y, \quad y = y(x, t_n), \quad x \in \omega_h.$$

Let $H = \overset{\circ}{\Omega}_h$ be a space of grid functions given on $\overline{\omega}_h$ and vanishing at the boundary nodes γ_h. To use the corresponding *a priori* estimates on the right hand side for the rate of convergence of our difference scheme, we introduce the following operator:

$$Av = \sum_{\alpha=1}^{3} A_\alpha v, \quad A_\alpha = -\Lambda_\alpha, \quad v \in \overset{\circ}{\Omega}_h, \tag{6.71}$$

$$A_\alpha v = -(a_\alpha v_{\bar{x}_\alpha})_{x_\alpha}, \quad x \in \omega_h, \quad v(x) = 0, \quad x \in \gamma_h. \tag{6.72}$$

On the set $\overset{\circ}{\Omega}_h$ we also introduce inner products and norms:

$$
\begin{aligned}
&(u,v) = \sum_{x\in\omega_h} u(x,t_n)v(x,t_n)h_1h_2h_3, \quad u,v\in\overset{\circ}{\Omega}_h,\\
&\|v\| = \sqrt{(v,v)}, \quad \|v\|_A^2 = \sum_{\alpha=1}^{3}\|v\|_{A_\alpha}^2,\\
&\|v\|_{A_\alpha}^2 = \sum_{i_\alpha=1}^{N_\alpha}\sum_{\substack{i_\beta=1\\ \beta\neq\alpha}}^{N_\beta-1} a_\alpha v_{\bar{x}_\alpha}^2 h_1h_2h_3,\\
&\|Av\|^2 = \sum_{x\in\omega_h} h_1h_2h_3\left(\sum_{\alpha=1}^{3}(a_\alpha v_{\bar{x}_\alpha})_{x_\alpha}\right)^2.
\end{aligned}
\tag{6.73}
$$

Since the operators A_α are positive and self-adjoint it follows that $A^* = A > 0$. Moreover,by virtue of Friedrichs' inequality and embedding (see estimate (6.24))

$$
\|v\| \le M\|v\|_A, \quad M^2 = \frac{l^2}{24c_1}, \quad l^2 = \max_\alpha\left\{l_\alpha^2\right\}, \quad v\in\overset{\circ}{\Omega}_h, \tag{6.74}
$$

we note the positiveness of the operator A:

$$
A \ge \delta_1 E, \quad \delta_1 = M^{-1}.
$$

Substituting $y = z+u$ into difference scheme (6.69), (6.70), we write the problem for the error of the method as the two-level operator-difference scheme:

$$
(E+\sigma\tau A)z_t + Az = \psi, \quad z_0 = 0, \tag{6.75}
$$

$$
\psi = \Lambda u^{(\sigma)} + f^{(\sigma)} - u_t, \quad x\in\omega_h, \quad t\in\omega_\tau. \tag{6.76}
$$

If $\sigma \ge 0.5$ then the conditions of Theorem 3.4 are satisfied, and for the solution of problem (6.75), we have the *a priori* estimate

$$
\|Az\| \le \|\psi_n\| + \sum_{k=1}^{n}\tau\|\psi_{\bar{t},k}\|. \tag{6.77}
$$

Since for smooth enough solutions

$$
\psi,\ \psi_{\bar{t}} = O(h^2+\tau^{m(\sigma)}),
$$

it follows from the estimate (6.77) that

$$
\|Az\| \le c\,(h^2+\tau^{m(\sigma)}). \tag{6.78}
$$

To obtain an *a priori* information about the order of accuracy in the norm $\|\cdot\|_C = \max_{x\in\omega_h} |\cdot|$, we need the following [Matus, 1993c]

LEMMA 6.3 *For any grid function* $v \in \overset{\circ}{\Omega}_h$ *the following embedding*

$$\|v\|_C \le M_0 \|Av\|, \quad M_0 = \text{const} > 0 \tag{6.79}$$

is valid.

Proof. Let

$$c_1 \le a_\alpha \le c_2, \quad \|a_{\bar{x}_\alpha}\|_{C_{+\alpha}} \le c_3, \quad \alpha = 1,2,3, \tag{6.80}$$

$$\|v\|_{C_{+\alpha}} = \|v\|_{C(\omega_h^{+\alpha})}, \quad \omega_h^{+\alpha} = \omega_h \cup \{\, x_{N_\alpha} = l_\alpha \,\},$$

$$\|v\|_{(\alpha)} = \sqrt{\sum_{x\in\omega_h^{+\alpha}} v^2(x) h_1 h_2 h_3}, \quad \|v\|_1 = \sqrt{\sum_{\alpha=1}^{3} \|v_{\bar{x}}\|^2_{(\alpha)}}, \tag{6.81}$$

$$\|v\|_2 = \left\| \sum_{\alpha=1}^{3} v_{\bar{x}_\alpha x_\alpha} \right\|.$$

We show that the expression $\|Av\|$ defines the norm equivalent to $\|v\|_2$, i.e., the following estimate

$$c_4 \|v\|_2 \le \|Av\| \le c_5 \|v\|_2 \tag{6.82}$$

is valid with the positive constants c_4, c_5, depending only on c_1, c_2, c_3 and l_α. To do this, we make use of the estimate [Andreev, 1966]

$$0.5c_1^2 \|v\|_2^2 - M_1 \|v\|_1^2 \le \|Av\|^2 \le M_2 \|v\|_2^2. \tag{6.83}$$

Applying the embedding (6.74) we can prove the inequality

$$\|v\|_1 \le M \|Av\|. \tag{6.84}$$

This follows from the chain of inequalities

$$c_1 \|v\|_1^2 \le \sum_{\alpha=1}^{3} (a_\alpha v_{\bar{x}_\alpha}, v_{\bar{x}_\alpha}]_{(\alpha)} = -\sum_{\alpha=1}^{3} \left((a_\alpha v_{\bar{x}_\alpha})_{x_\alpha}, v\right)$$

$$= (Av, v) \le \|Av\| \|v\| \le \frac{l^2}{24} \|Av\| \|v\|_1.$$

Using now estimate (6.84), from estimate (6.83) we find that

$$0.5c_1^2 \|v\|_2^2 \le \|Av\|^2 + M_1 \|v\|_1^2 \le (1 + M_1 M^2) \|Av\|^2,$$

and then

$$c_4\|v\|_2 \le \|Av\|, \quad c_4^2 = \frac{c_1^2}{2(1+M_1M^2)}. \tag{6.85}$$

Subordination of the norm $\|Av\|$ to the norm $\|v\|_2$ follows from estimate (6.83). It is known [Samarskii, 1989] that for any grid function $v \in \overset{\circ}{\Omega}_h$ the embedding

$$\|v\|_C \le M_3\|v\|_2, \quad M_3 = l_0^2/\sqrt{\bar{l}}, \quad l_0 = \max_\alpha l_\alpha, \quad \bar{l} = \prod_{\alpha=1}^{3} l_\alpha \tag{6.86}$$

is valid, taking account of which and of inequality (6.85), we obtain the desired estimate (6.79).

From Lemma 6.3 and inequality (6.78) we conclude that for $\sigma \ge 0.5$ a solution of the difference scheme (6.69), (6.70) converges unconditionally in a uniform norm to the solution of initial problem (6.64), (6.65). Moreover, for any $\tau > 0$ we have the estimate

$$\|z\|_C \le cM_0\,(h^2 + \tau^{m(\sigma)}), \tag{6.87}$$

where $m(\sigma) = 2$, when $\sigma = 0.5$ and $m(\sigma) = 1$, when $\sigma \ne 0.5$.

3. Problems with Generalized Solutions

Convergence of difference schemes is established in different norms which are to be coordinated with the class of smoothness for solutions of a differential problem. In view of this it is preferable to have a range of estimates for difference solutions. Estimates of a difference solution in time integral norms (see Theorem 2.25) are valuable for the investigation of difference schemes for non-stationary boundary value problems with generalized solutions [Ladyzhenskaya, 1973]. Estimates of the theory of stability [Samarskii, 1989, Samarskii and Goolin, 1973] are given in time-uniform norms. The account below is based on [Samarskii, 1984, Samarskii et al., 1997c, Samarskii et al., 1987, Samarskii et al., 1997a].

3.1 Stability in Integral with Respect to Time Norms

Below we will prove *a priori* estimates in the integral in time norms for two-level difference schemes written in a canonical form. The fundamental point is obtaining an estimate for a difference solution at half-integer time nodes which is determined by linear interpolation using the grid functions at the nodes. We consider the difference–differential equation

$$D\frac{du}{dt} + Au = f(t), \quad 0 < t < T. \tag{6.88}$$

The operators of this equation are assumed to be constant ($D \neq D(t)$, $A \neq A(t)$) in H and satisfying the conditions

$$D^* = D > 0, \quad A^* = A > 0. \tag{6.89}$$

To find convergence estimates for a difference solution,we consider the Cauchy problem for equation (6.88) with a homogeneous initial condition:

$$u(0) = 0. \tag{6.90}$$

Taking the dot product in H of equation (6.88) with $u(t)$, integrating over time from 0 to t, and taking into account conditions (6.89), (6.90) we obtain the *a priori* estimate

$$\|u(t)\|_D^2 + \int_0^t \|u(\theta)\|_A^2 d\theta \leq \int_0^t \|f(\theta)\|_{A^{-1}}^2 d\theta. \tag{6.91}$$

In obtaining *a priori* estimates for difference analogs of problem (6.88)–(6.90), in the foregoing we used as a starting point the following simple estimate:

$$\|u(t)\|_D^2 \leq \int_0^t \|f(\theta)\|_{A^{-1}}^2 d\theta,$$

which follows immediately from equation (6.91). Investigating problems with generalized solutions, attention should be paid to the estimate

$$\int_0^t \|u(\theta)\|_A^2 d\theta \leq \int_0^t \|f(\theta)\|_{A^{-1}}^2 d\theta \tag{6.92}$$

of the solution of the problem in the norms integral in time.

Similarly, multiplying equation (6.88) by du/dt, we obtain the inequality

$$\|u(t)\|_A^2 + \int_0^t \left\| \frac{du(\theta)}{d\theta} \right\|_D^2 d\theta \leq \int_0^t \|f(\theta)\|_{D^{-1}}^2 d\theta, \tag{6.93}$$

which yields a standard estimate of the solution of problem (6.88)–(6.90):

$$\|u(t)\|_A^2 \leq \int_0^t \|f(\theta)\|_{D^{-1}}^2 d\theta.$$

Moreover, the following inequality is worthy of special note:

$$\int_0^t \left\| \frac{du(\theta)}{d\theta} \right\|_D^2 d\theta \le \int_0^t \|f(\theta)\|_{D^{-1}}^2 d\theta. \tag{6.94}$$

The following are some results associated with the difference analogs of *a priori* estimates (6.92), (6.94).

Let $\tau > 0$ be a grid step with respect to time, and $y_n = y(t_n)$, $t_n = n\tau$. We consider a two-level operator-difference scheme written in the canonical form:

$$B\frac{y_{n+1} - y_n}{\tau} + Ay_n = \varphi_n, \quad n = 0, 1, \ldots. \tag{6.95}$$

We consider the stability of the difference scheme on the right hand side, i.e., for

$$y_0 = 0. \tag{6.96}$$

THEOREM 6.2 *For the difference scheme (6.95), (6.96) with the operators $A^* = A > 0$, $B^* = B > 0$ under the condition*

$$B \ge \frac{\tau}{2}A \tag{6.97}$$

the following a priori *estimate*

$$\sum_{k=0}^{n} \tau \left\| \frac{1}{2}(y_{k+1} + y_k) \right\|_A^2 \le \sum_{k=0}^{n} \tau \|\varphi_k\|_{A^{-1}}^2 \tag{6.98}$$

is valid.

Proof. We rewrite the operator equation (6.95) for $t = t_k$ in the form

$$\left(B - \frac{\tau}{2}A\right) y_{t,k} + Ay_k^{(0.5)} = \varphi_k. \tag{6.99}$$

Taking the dot product of equation (6.99) with $2\tau y_k^{(0.5)}$ applying the general Cauchy inequality to the right hand side, and taking into account that the operator $R = \left(B - \dfrac{\tau}{2}A\right)$ is self-adjoint, we find from formula (6.99) that

$$(Ry_{k+1}, y_{k+1}) + \tau\|y_k^{(0.5)}\|_A^2 \le (Ry_k, y_k) + \tau\|\varphi_k\|_{A^{-1}}^2.$$

Summing up this inequality over $k = 0, 1, ..., n$ and using condition (6.96), we obtain the statement of the Theorem.

The estimate (6.98) is a difference analog of *a priori* estimate (6.92). It was obtained for the difference solution at half-integer nodes, where the solution is determined by formula $y_{k+1/2} = (y_{k+1} + y_k)/2$.

Note that the stability in time integral norm have been established under conditions (6.97), that are necessary and sufficient for stability in uniform in time norms.

For a difference solution on integer time steps one can obtain an *a priori* estimate of stability on to the right hand side in time integral norms under more strong assumptions (see Theorem 2.19 and estimate (6.131)).

THEOREM 6.3 *For difference scheme (6.95), (6.96) with the operators $A^* = A > 0$, $B^* = B > 0$ under the condition*

$$B \geq (1+\varepsilon)\frac{\tau}{2}A, \quad 0 < \varepsilon < 2, \tag{6.100}$$

the following a priori *estimate*

$$\sum_{k=0}^{n} \tau \|y_k\|_A^2 \leq \frac{2}{\varepsilon^2(2-\varepsilon)} \sum_{k=0}^{n} \tau \|\varphi_k\|_{A^{-1}}^2 \tag{6.101}$$

is valid.

Proof. Estimate (6.101) follows from Theorem 7 [Samarskii and Goolin, 1973, p. 175].

We prove now a difference analog of estimate (6.94).

THEOREM 6.4 *For difference scheme (6.95), (6.96) with the operators $A^* = A^* \geq 0$, $B^* = B$ under the condition*

$$G = B - \frac{\tau}{2}A > 0 \tag{6.102}$$

the following a priori *estimate*

$$\sum_{k=0}^{n} \tau \left\| \frac{y_{k+1} - y_k}{\tau} \right\|_G^2 \leq \sum_{k=0}^{n} \tau \|\varphi_k\|_{G^{-1}}^2 \tag{6.103}$$

is valid.

Proof. Taking the dot product of equation (6.99) with $2\tau y_{t,k}$. and taking into account that the operator A is self-adjoint, we obtain the energy identity

$$\|y_{k+1}\|_A^2 + 2\tau \|y_{t,k}\|_G^2 = \|y_k\|_A^2 + 2\tau \left(y_{t,k}, \varphi_k\right). \tag{6.104}$$

Using the generalized Cauchy inequality, we find for the last term in the right hand side the estimate

$$2\tau\,(y_{t,k}, \varphi_k) \le \tau\|y_{t,k}\|_G^2 + \tau\|\varphi_k\|_{G^{-1}}^2. \tag{6.105}$$

Substituting the obtained estimate into identity (6.104) and summing then over all $k = 0, 1, ..., n$, we arrive at the required estimate (6.103).

Let us apply the formulated conditions for stability to the scheme with weights

$$D\frac{y_{n+1} - y_n}{\tau} + A(\sigma y_{n+1} + (1-\sigma)y_n) = \varphi_n, \quad n = 0, 1, ..., \tag{6.106}$$

with $y_0 = 0$ which approximates problem (6.88), (6.89). Scheme (6.106) can be rewritten in the canonical form (6.95) with the operator

$$B = D + \sigma\tau A. \tag{6.107}$$

The estimate (6.98) for the difference scheme (6.95), (6.96), (6.107) is valid by virtue of conditions (6.97) when $A \le \Delta D$ if

$$\sigma \ge \frac{1}{2} - \frac{1}{\Delta\tau}.$$

Similarly, from condition (6.100) with operator (6.107) it follows that if for scheme (6.95), (6.96), (6.107) the condition

$$\sigma \ge \frac{1+\varepsilon}{2} - \frac{1}{\Delta\tau}$$

is satisfied then the following estimate (6.101) is valid.

On the basis of Theorem 6.4 under ordinary restriction that $\sigma \ge 0.5$ from estimate (6.103) one can obtain the inequality

$$\sum_{k=0}^{n} \tau\left\|\frac{y_{k+1} - y_k}{\tau}\right\|_D^2 \le \sum_{k=0}^{n} \tau\|\varphi_k\|_{D^{-1}}^2, \tag{6.108}$$

which is the difference analogue of inequality (6.94) for the differential-difference problem.

According to the embedding theorem [Samarskii and Goolin, 1973] relation (6.108) yields also the estimates for the solution both in the uniform metric

$$\|y_{n+1}\|_D^2 \le t_{n+1}\sum_{k=0}^{n} \tau\|\varphi_k\|_{D^{-1}}^2, \tag{6.109}$$

and in the integral in time norm :

$$\sum_{k=0}^{n+1} \tau\|y_k\|_D^2 \le t_{n+1}^2\sum_{k=0}^{n} \tau\|\varphi_k\|_{D^{-1}}^2. \tag{6.110}$$

We apply the obtained estimates of stability on the right hand side to investigation of convergence to a generalized solution of the simplest boundary value problem for a parabolic equation.

3.2 A Differential Problem

In the rectangle

$$Q_T = \{\ (x,t): \quad x \in \Omega = (0,1), \quad 0 < t < T\ \}$$

we consider the following heat conduction equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad (x,t) \in Q_T, \tag{6.111}$$

with the initial conditions

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{6.112}$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 < t < T. \tag{6.113}$$

For the functions $u = u(x)$ set on Ω, we introduce the Sobolev space $W_2^1(\Omega)$ with the norm

$$\|u\|^2_{W_2^1(\Omega)} = \|u\|^2_{L_2(\Omega)} + \|u'\|^2_{L_2(\Omega)}, \quad \|u\|^2_{L_2(\Omega)} = \int\limits_{\Omega} u^2(x)dx,$$

and for the functions $u = u(x,t)$ set on the rectangle Q_T we determine the space $W_2^{\alpha,\beta}(Q_T)$ with the norm

$$\|u\|^2_{W_2^{\alpha,\beta}(Q_T)} = \|u\|^2_{L_2(Q_T)} + \sum_{s=1}^{\alpha} \left\| \frac{\partial^s u}{\partial x^s} \right\|^2_{L_2(Q_T)} + \sum_{s=1}^{\beta} \left\| \frac{\partial^s u}{\partial t^s} \right\|^2_{L_2(Q_T)},$$

$$\|u\|^2_{L_2(Q_T)} = \int\limits_0^T \|u(x,t)\|^2_{L_2(\Omega)} dt.$$

The quantities $\overset{\circ}{W_2^1}(\Omega)$ and $W_{2,0}^{1,1}(Q_T)$ denote the subspaces of $W_2^1(\Omega)$ and $W_2^{1,1}(Q_T)$, respectively, in which the smooth functions are compact sets vanishing at $x = 0$ and $x = 1$ [Ladyzhenskaya, 1973].

DEFINITION 6.1 ([LADYZHENSKAYA, 1973]) *The* generalized solution *of problem (6.111)–(6.113) in the space $W_2^{2,1}(Q_T)$ is the element u of a* *space* $W_{2,0}^{2,1} \equiv W_2^{2,1}(Q_T) \cap W_{2,0}^{1,1}(Q_T)$ *satisfying almost everywhere in the rectangle* Q_T *equation (6.111) and being equal to* $u_0(x)$ *when* $t = 0$.

LEMMA 6.4 ([LADYZHENSKAYA, 1973, MIKHAILOV, 1983]) *The problem (6.111)–(6.113) is uniquely solvable in* $W_{2,0}^{2,1}(Q_T)$ *if* $f \in L_2(Q_T)$, $u_0 \in \overset{\circ}{W}{}_2^1(0,1)$, *with the inequality*

$$\|u\|_{W_2^{2,1}(Q_T)} \leq M\left(\|u_0\|_{W_2^1(0,1)} + \|f\|_{L_2(Q_T)}\right),$$

being valid and M, *a positive constant, being independent of* u_0, f.

3.3 Difference Scheme

In Q_T we introduce the uniform grids

$$\omega_{\tau} = \left\{\ t_j = j\tau, \quad j = \overline{0, j_0 - 1}, \quad j_0\tau = T\ \right\},$$

$$\omega_h = \left\{\ x_i = ih, \quad i = \overline{1, N-1}, \quad Nh = 1\ \right\}, \quad \omega_{h\tau} = \omega_h \times \omega_{\tau}.$$

Below we use Steklov's averaging operators. First, we define *the one-dimensional averaging operators* acting in every direction x and t:

$$S_x v(x,t) = \frac{1}{h}\int_{x-0.5h}^{x+0.5h} v(\xi, t)d\xi,$$

$$S_t v(x,t) = \frac{1}{\tau}\int_{t}^{t+\tau} v(x, t')dt' = \int_0^1 f(x, t+\theta\tau)d\theta.$$

We introduce *the operator of iterated averaging* over the direction x by the relation

$$S_x^2 v(x,t) = \frac{1}{h^2}\int_x^{x+h}\int_{\xi-h}^{\xi} v(\eta, t)d\eta d\xi.$$

From here, using the formula of integration by parts, we can obtain the following representation: [Samarskii et al., 1987, p. 57]

$$\begin{aligned} S_x^2 v(x,t) &= \frac{1}{h}\int_{x-h}^{x+h}\left(1 - \frac{|x'-x|}{h}\right)v(x', t)dx' \\ &= \int_{-1}^{1}(1-|s|)v(x+sh, t)ds. \end{aligned} \tag{6.114}$$

For the averaging operators the following basic properties hold:

$$S_x^2 \frac{\partial^2 u}{\partial x^2} = u_{\bar{x}x}, \quad S_t \frac{\partial u}{\partial t} = u_t. \tag{6.115}$$

We approximate problem (6.111)–(6.113) by the following difference scheme with the weighting factors

$$y_t = y_{\bar{x}x}^{(\sigma)} + S_x^2 S_t f, \quad (x,t) \in \omega_{h\tau}, \tag{6.116}$$

$$y(x,0) = S_x^2 u_0(x), \quad x \in \omega_h, \quad y(0,\hat{t}) = y(1,\hat{t}) = 0, \quad t \in \omega_\tau. \tag{6.117}$$

3.4 Approximation Error and Convergence

We define $L_2(\omega_{h\tau})$ as a grid analog of $L_2(Q_T)$ with the norm

$$\|v\|_{h\tau} = \left(\sum_{k=0}^{N_0-1} \tau \|v_k\|^2 \right)^{1/2}, \quad \|v_k\|^2 = \sum_{i=1}^{N-1} h v^2(x_i, t_k). \tag{6.118}$$

Here the question can be posed of the convergence difference scheme in the grid norm $L_{2,h}(\omega_{h\tau})$. We use $\overline{u} = S_x^2 u$ to denote the averaging of the exact solution of problem (6.111)–(6.113), with the solution being extended oddly through the lines $x = 0$, $x = 1$:

$$\widetilde{u}(x,t) = \begin{cases} -u(-x,t), & x \in (-1,0], \\ u(x,t), & x \in [0,1], \\ -u(2-x,t), & x \in [1,2). \end{cases} \tag{6.119}$$

It is easy to show that

$$\|\widetilde{u}\|^2_{W_2^{2,1}(\widetilde{Q}_T)} = 3\|u\|^2_{W_2^{2,1}(Q_T)},$$

$$\widetilde{Q}_T = \{ (x,t) : x \in (-1,2), \ 0 < t < T \}.$$

Thus $\widetilde{u}(x,t) \in W_2^{2,1}$.

We compare the approximate solution y with the averaging $\overline{u}$. Substituting $y = z + \overline{u}$ into equation (6.116), we obtain the following difference problem:

$$z_t = z_{\bar{x}x}^{(\sigma)} + \psi(x,t), \quad (x,t) \in \omega_{h\tau}, \tag{6.120}$$

$$z(x,0) = 0, \quad x \in \overline{\omega}_h, \quad z(0,\hat{t}) = z(1,\hat{t}) = 0, \quad t \in \omega_\tau, \tag{6.121}$$

where

$$\psi = S_x^2 S_t f - \overline{u}_t + \overline{u}_{\bar{x}x}^{(\sigma)} \tag{6.122}$$

is an approximation error. We transform it to a divergent form. For this purpose we apply the operator $S_x^2 S_t$ to the differential equation (6.111) and taking into account the properties (6.115) we obtain

$$\overline{u}_t = S_t u_{\bar{x}x} + S_x^2 S_t f.$$

Substituting now an expression from the latter inequality for $S_x^2 S_t f$ into transformed relation (6.122) we obtain

$$\psi = \eta_{\bar{x}x}, \quad \eta = S_t u - \overline{u}^{(\sigma)}. \tag{6.123}$$

We show that $\eta \in \overset{\circ}{\Omega}_h$, where $\overset{\circ}{\Omega}_h$ is the space of grid functions defined on the grid $\overline{\omega}_h$ and equal to zero at the boundary nodes $x = 0$, $x = 1$.

Taking into account that the integral of the antisymmetric function over the symmetric interval is equal to zero, we obtain

$$\eta(0,t) = \int\limits_0^1 \widetilde{u}(0, t + \theta\tau) d\theta - \sigma \int\limits_{-1}^1 (1 - |s|) \widetilde{u}(sh, t + \tau) ds$$
$$- (1 - \sigma) \int\limits_{-1}^1 (1 - |s|) \widetilde{u}(sh, t) ds = 0.$$

Similarly we can show that $\eta(1,t) = 0$. Because of this the functions z and η belong to the same space of the grid functions $\overset{\circ}{\Omega}_h$. We rewrite the problem for the error of a difference scheme in an operator form

$$z_t + A z^{(\sigma)} = A\eta, \quad t \in \omega_\tau, \quad z_0 = 0, \tag{6.124}$$

where the operator $A : \Omega_h \longrightarrow \Omega_h$ is defined by formula (6.18) with $a(x) \equiv 1$, $z_n = (z_1^n, ..., z_{N-1}^n)^T$, $\eta_n = (\eta_1^n, ..., \eta_{N-1}^n)^T$. Since $A = A^* >$ > 0 and A^{-1} does exist, then, using the identity $z^{(\sigma)} = z + \sigma\tau z_t$ and acting by operator A^{-1} onto the equation (6.124), we transform the problem for error to the canonical form

$$\widetilde{B} z_t + \widetilde{A} z = \eta, \quad z_0 = 0, \tag{6.125}$$

with $\widetilde{B} = A^{-1} + \sigma\tau E$, $\widetilde{A} = E$, $\widetilde{B}^* = \widetilde{B}$.

To apply the *a priori* estimate in the integral in time norm (6.101), we show that

$$\begin{aligned}\widetilde{B} - \frac{1+\varepsilon}{2}\tau\widetilde{A} &= A^{-1} + \left(\sigma - \frac{1+\varepsilon}{2}\right)\tau E \\ &\geq \left(\frac{1}{\|A\|} - \left(\sigma - \frac{1+\varepsilon}{2}\right)\tau\right) E \geq 0.\end{aligned} \tag{6.126}$$

Since by virtue of relations (6.29) $\|A\| < 4/h^2$ it follows that inequality (6.126) is satisfied whenever the constant weighting factor satisfies the condition

$$\sigma \geq \sigma_\varepsilon, \quad \sigma_\varepsilon = \frac{1+\varepsilon}{2} - \frac{h^2}{4\tau}. \tag{6.127}$$

Further, by virtue of Theorem 6.3 we have

$$\|z\|_{h\tau} \leq \frac{2}{\varepsilon\sqrt{2-\varepsilon}}\|\eta\|_{h\tau}, \quad 0 < \varepsilon < 2. \tag{6.128}$$

It now remains to estimate the error η in the norm $\|\cdot\|_{h\tau}$. For this we need

LEMMA 6.5 *For the grid function (6.123) the following* a priori *estimate*

$$|\eta(x,t)| \leq \frac{M}{\sqrt{h\tau}}\left(\tau\left\|\frac{\partial u}{\partial t}\right\|_{L_2(e)} + h^2\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L_2(e)}\right) \tag{6.129}$$

is valid, where $e = (x-h, x+h) \times (t, t+\tau)$, and the constant $M > 0$ does not depend on τ, h.

Proof. Using the transformation

$$s = (x'-x)/h, \quad \theta = (t'-t)/\tau, \quad (x',t') \in e,$$

we map the cell e onto the domain

$$\widetilde{e} = \{\,(s,\theta): \quad -1 < s < 1, \quad 0 < \theta < 1\,\}.$$

We denote $v(s,\theta) = u(x+sh, t+\theta\tau)$ and rewrite the grid function η in the form

$$\begin{aligned}
\eta &= S_t u - \overline{u}^{(\sigma)} = S_t u - \overline{u} - \sigma\tau\overline{u}_t \\
&= \int_0^1 u(x, t+\theta\tau)d\theta - \int_{-1}^1 (1-|s|)u(x+sh,t)ds \\
&\quad - \sigma\tau\int_{-1}^1 (1-|s|)u_t(x+sh,t)ds \\
&= \int_0^1 v(0,\theta)d\theta - \int_{-1}^1 (1-|s|)v(s,0)ds - \sigma\int_{-1}^1 (1-|s|)\left(\int_0^1 \frac{\partial v(s,\eta)}{\partial\eta}d\eta\right)ds.
\end{aligned}$$

Since $\int\limits_0^1 d\theta = 1$, $\int\limits_{-1}^1 (1-|s|)ds = 1$, then the latter expression can be represented also in the form

$$\begin{aligned}\eta = \int\limits_0^1\int\limits_{-1}^1 (1-|s|)\left[v(0,\theta)-v(s,\theta)+v(s,\theta)-v(s,0)\right]dsd\theta \\ -\sigma\tau\int\limits_{-1}^1 (1-|s|)v_t(s,0)ds = I_1+I_2+I_3.\end{aligned} \tag{6.130}$$

Here for the integrals we use the notation

$$I_1 = \int\limits_0^1\int\limits_{-1}^1 (1-|s|)\left[v(0,\theta)-v(s,\theta)\right]dsd\theta,$$

$$I_2 = \int\limits_0^1\int\limits_{-1}^1 (1-|s|)\left[v(s,\theta)-v(s,0)\right]dsd\theta,$$

$$I_3 = -\sigma\int\limits_{-1}^1 (1-|s|)\left(\int\limits_0^1 \frac{\partial v(s,\eta)}{\partial\eta}d\eta\right)ds.$$

Using Taylor's formula with the reminder term in the integral form

$$\begin{aligned}f(x) = f(x_0) + (x-x_0)f'(x_0) + \ldots \\ + \frac{(x-x_0)^m}{m!}f^{(m)}(x_0) + R_{m+1}(x),\end{aligned} \tag{6.131}$$

$$\begin{aligned}R_{m+1}(x) &= \frac{1}{m!}\int\limits_{x_0}^x (x-\xi)^m f^{(m+1)}(\xi)d\xi \\ &= \frac{(x-x_0)^{m+1}}{m!}\int\limits_0^1 (1-\eta)^m f^{(m+1)}(x_0+\eta(x-x_0))d\eta,\end{aligned}$$

we obtain

$$v(s,\theta) = v(0,\theta) + s\frac{\partial v}{\partial s}(0,\theta) + \int\limits_0^s (s-\xi)\frac{\partial^2 v}{\partial\xi^2}(\xi,\theta)d\xi. \tag{6.132}$$

Applying the Cauchy inequality, the identity

$$\int_{-1}^{1}(1-|s|)s\,ds=0,\qquad \left|\int_{s}^{0}(s-\xi)^2d\xi\right|^{1/2}=\frac{1}{\sqrt{3}}|s|^{3/2},$$

$$\int_{-1}^{1}\frac{1}{\sqrt{3}}(1-|s|)|s|^{3/2}ds=\frac{4}{5\sqrt{3}},$$

and relation (6.132), we estimate the integral

$$\begin{aligned}
I_1&=\int_0^1\int_{-1}^{1}(1-|s|)\left(-s\frac{\partial v}{\partial s}(0,\theta)+\int_s^0(s-\xi)\frac{\partial^2 v(\xi,\theta)}{\partial\xi^2}d\xi\right)ds\,d\theta\\
&\le\int_0^1\int_{-1}^{1}(1-|s|)\left|\int_s^0(s-\xi)^2d\xi\right|^{1/2}\left(\int_{-1}^{1}\left(\frac{\partial^2 v(\xi,\theta)}{\partial\xi^2}\right)^2d\xi\right)^{1/2}ds\,d\theta\\
&=\frac{4}{5\sqrt{3}}\int_0^1\left(\int_{-1}^{1}\left(\frac{\partial^2 v}{\partial\xi^2}\right)^2d\xi\right)^{1/2}d\theta \qquad (6.133)\\
&\le\frac{4}{5\sqrt{3}}\left[\int_0^1\int_{-1}^{1}\left(\frac{\partial^2 v(\xi,\theta)}{\partial\xi^2}\right)^2d\xi\,d\theta\right]^{1/2}\\
&=\frac{4}{5\sqrt{3}}\left\|\frac{\partial^2 v(s,\theta)}{\partial s^2}\right\|_{L_2(\tilde e)}=\frac{4}{5\sqrt{3}}\frac{h^2}{\sqrt{h\tau}}\left\|\frac{\partial^2 u(x,t)}{\partial x^2}\right\|_{L_2(e)}.
\end{aligned}$$

The functionals I_2, I_3 can also be estimated using the Cauchy inequality and Taylor's formula (6.131):

$$\begin{aligned}
I_2&=\int_0^1\int_{-1}^{1}(1-|s|)\left(\int_0^\theta\frac{\partial v(s,\eta)}{\partial\eta}d\eta\right)ds\,d\theta\\
&\le\int_0^1\sqrt{\theta}\int_{-1}^{1}(1-|s|)\left(\int_0^1\left(\frac{\partial v(s,\eta)}{\partial\eta}\right)^2d\eta\right)^{1/2}ds\,d\theta \qquad (6.134)\\
&=\frac{2}{3}\left\|\frac{\partial v}{\partial\theta}\right\|_{L_2(\tilde e)}=\frac{2}{3}\frac{\tau}{\sqrt{h\tau}}\left\|\frac{\partial u}{\partial t}\right\|_{L_2(e)},
\end{aligned}$$

$$
\begin{aligned}
I_3 &\le \sigma \int_{-1}^{1} (1-|s|) \left(\int_0^1 \left(\frac{\partial v(s,\eta)}{\partial \eta} \right)^2 d\eta \right)^{1/2} ds \\
&\le \frac{\sqrt{2}}{\sqrt{3}} \sigma \left\| \frac{\partial v}{\partial \eta} \right\|_{L_2(\tilde{e})} = \frac{\sqrt{2}}{\sqrt{3}} \frac{\sigma \tau}{\sqrt{h\tau}} \left\| \frac{\partial u(x,t)}{\partial t} \right\|_{L_2(e)} .
\end{aligned}
\tag{6.135}
$$

Combining estimates (6.133)–(6.135), we prove Lemma.

We show now that

$$
\|\eta\|_{h\tau} \le \sqrt{2} M \left(\tau \left\| \frac{\partial u}{\partial t} \right\|_{L_2(Q_T)} + h^2 \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L_2(Q_T)} \right). \tag{6.136}
$$

Indeed, by definition

$$
\|v\|^2_{L_2(e)} = \int_t^{t+\tau} \int_{x-h}^{x+h} v^2(x,t) dx dt,
$$

therefore from the estimate (6.129) we have

$$
|\eta|^2 \le \frac{2M^2}{h\tau} \left(\tau^2 \int_t^{t+\tau} \int_{x-h}^{x+h} \left(\frac{\partial u}{\partial t} \right)^2 dx dt + h^4 \int_t^{t+\tau} \int_{x-h}^{x+h} \left(\frac{\partial^2 u}{\partial x^2} \right)^2 dx dt \right).
$$

Summing the latter inequality over all the grid nodes $\omega_{h\tau}$ we find that

$$
\sum_{t \in \omega_\tau} \sum_{x \in \omega_h} h\tau\, \eta^2(x,t) \le 2M^2 \left(\tau^2 \left\| \frac{\partial u}{\partial t} \right\|^2_{L_2(Q_T)} + h^4 \left\| \frac{\partial^2 u}{\partial x^2} \right\|^2_{L_2(Q_T)} \right).
$$

Consequently inequality (6.136) is satisfied. Taking it into account in the error estimate (6.128), we obtain

$$
\|z\|_{h\tau} = \|y - S_x^2 u\|_{h\tau} \le c \left(\tau \left\| \frac{\partial u}{\partial t} \right\|_{L_2(Q_T)} + h^2 \left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L_2(Q_T)} \right), \tag{6.137}
$$

where $c = M\sqrt{2}$.

Thus, we have

THEOREM 6.5 *Let $u(x,t) \in W_{2,0}^{2,1}$, $f \in L_2(Q_T)$, $u_0 \in \overset{\circ}{W}{}_2^1\,(0,1)$. Then under the condition that*

$$
\sigma \ge \frac{1+\varepsilon}{2} - \frac{h^2}{4\tau} \tag{6.138}
$$

the difference scheme with weighting factors (6.116), (6.117) converges to a generalized solution of problem (6.111)–(6.113), and for any τ, h we have estimate (6.137).

In accordance with Theorem 6.2 the following result is valid.

THEOREM 6.6 *Let $u(x,t) \in W_{2,0}^{2,1}$, $f(x,t) \in L_2(Q_T)$, $u_0 \in \overset{\circ}{W}{}_2^1(0,1)$. Then under the condition that*

$$\sigma \geq \frac{1}{2} - \frac{h^2}{4\tau}$$

the difference solution of the scheme with weights (6.116), (6.117) at half-integer nodes $y_{n+1/2} = \frac{1}{2}(y^{n+1} + y^n)$ converges to a generalized solution of the initial problem, and the estimate

$$\|z^*\|_{h\tau} \leq c\left(\tau \left\|\frac{\partial u}{\partial t}\right\|_{L_2(Q_T)} + h^2 \left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L_2(Q_T)}\right) \tag{6.139}$$

is valid. Here $z^* = y_{n+1/2} - S_x^2 u_{n+1/2}$.

Proof. Since in the problem for the error (6.125) $\widetilde{A}^* = \widetilde{A} > 0$, $\widetilde{B}^* = \widetilde{B}$ and the condition of Theorem 6.2 $\widetilde{B} \geq 0.5\tau\widetilde{A}$ is satisfied for $\sigma \geq \dfrac{1}{2} - \dfrac{h^2}{4\tau}$, then from estimate (6.98) we have

$$\sum_{k=0}^{n} \tau \left\|\frac{1}{2}(z_{k+1} + z_k)\right\|^2 \leq \|\eta\|_{h\tau}^2. \tag{6.140}$$

Substituting estimate (6.136) into the latter inequality we obtain the desired result.

4. Difference Schemes for Non-stationary Convection–Diffusion Problems

The stability of difference schemes for a model two-dimensional convection–diffusion problem in a rectangular area is investigated. Symmetric approximations of convective terms by means of central differences are used. The unconditional stability of classical schemes with weighting factors is shown. Attention is specially paid to schemes with explicit convective and implicit diffusive transfer.

4.1 Introduction

Convection–diffusion problems are basic in modeling the problems of hydrodynamics and heat transfer [Roache, 1982, Patankar, 1980, Temam, 1979]. When solving numerically attention is specially paid to approximation of convective terms. Schemes with directed differences, hybrid schemes, and schemes with directed differences of higher order are widely used. The main drawback of the classical schemes of the second order of approximation with central differences is associated with stability violation.

At the present time the characteristics of specific difference schemes are studied mainly experimentally on the basis of numerical calculations in solving some test problems [Patel and Marcatos, 1986, Shyy, 1985]. Theoretical investigation is usually performed using the principle of maximum. The methods are also used which are based on the Fourier transformation and the Neumann method [Siemieniuch and Gladwell, 1978, Morton, 1980, Richtmyer and Morton, 1967]. This allows consideration of difference schemes only for the simplest one-dimensional convection–diffusion problems with constant coefficients.

This section (see also [Samarskii and Vabishchevich, 1995a, Vabishchevich and Samarskii, 1997, Vabishchevich, 1994a, Vabishchevich and Samarskii, 1998]) is devoted to investigation of difference schemes for convection–diffusion problems on the basis of the general theory of stability of difference schemes. We consider the model Dirichlet boundary value problem for a non-stationary convective diffusion problem in a rectangle, with the convective terms being taken in a symmetrical form (a half-sum of the convective terms in divergent and non-divergent forms),and approximated by central differences with the second order of approximation.The main properties of difference operators of convective transfer are determined; on the basis of which the stability of various difference schemes for the convection–diffusion problem is investigated.

4.2 Model Convection-Diffusion Problems

It is convenient to consider the problems of construction, investigation and solution of *convection–diffusion problems* on elementary cases that retain the main specifics of general multi-dimensional problems of continuum mechanics.

In the parallelepiped $\overline{Q}_T = \overline{\Omega} \times [0, T]$

$$\Omega = \{x = (x_1, x_2), \quad 0 < x_\alpha < l_\alpha, \quad \alpha = 1, 2\}, \quad \overline{\Omega} = \Omega \cup \Gamma,$$

a solution is sought for the parabolic equation

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{2} v_\alpha(x) \frac{\partial u}{\partial x_\alpha} - k \sum_{\alpha=1}^{2} \frac{\partial^2 u}{\partial x_\alpha^2} = 0, \quad x \in \Omega, \quad t > 0. \qquad (6.141)$$

Here $k = \text{const}$ is the diffusion coefficient, $v_\alpha(x), \alpha = 1, 2$ are the velocity components that determine stationary convective transfer in a non-divergent form. We supplement equation (6.141) with elementary homogeneous boundary conditions of the first kind:

$$u(x,t) = 0, \quad x \in \Gamma, \quad t > 0 \tag{6.142}$$

and with the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{6.143}$$

Equation (6.141) with non-divergent convective terms is the basic one in investigation of boundary value problems for parabolic equations of the second kind. From the viewpoint of applied mathematical simulation greater attention is to be paid to the problems with convective terms in a divergent form:

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{2} \frac{\partial (v_\alpha(x) u)}{\partial x_\alpha} - k \sum_{\alpha=1}^{2} \frac{\partial^2 u}{\partial x_\alpha^2} = 0, \quad x \in \Omega, \quad t > 0. \tag{6.144}$$

In the divergent form (6.144) the equation of convection–diffusion explicitly represents the corresponding conservation law.

In considering the problems of hydrodynamics, one may often restrict ourselves to the case of an incompressible medium. Using the incompressibility condition

$$\operatorname{div} \mathbf{v} \equiv \sum_{\alpha=1}^{2} \frac{\partial v_\alpha}{\partial x_\alpha} = 0, \quad x \in \Omega,$$

we may rewrite equation (6.141) in the equivalent form (6.144) and vice versa. For such problems, the representation of convective terms in a symmetrical form is especially important when the convection–diffusion equation has the form

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{\alpha=1}^{2} \left(v_\alpha(x) \frac{\partial u}{\partial x_\alpha} + \frac{\partial (v_\alpha(x) u)}{\partial x_\alpha} \right) - k \sum_{\alpha=1}^{2} \frac{\partial^2 u}{\partial x_\alpha^2} = 0, \tag{6.145}$$

$$x \in \Omega, \quad t > 0.$$

Here the convective terms are combined in both divergent and non-divergent forms.

A solution of difference problems must inherit the basic properties of the set differential problem (6.141)–(6.143) (or the problem (6.142)–(6.144) and (6.142), (6.143), (6.145)); the difference operators must have the same basic properties as the differential ones. Above all, this applies to the stability in the corresponding norms. Therefore we begin our considerations from obtaining ordinary *a priori* estimates for the differential problem.

4.3 The Stability of the Solution for the Continuous Problem

Let $\mathcal{H} = L_2(\Omega)$ be the Hilbert space with the inner product

$$(v, w) = \int_{\Omega} v(x)w(x)dx$$

for the functions $v(x)$ and $w(x)$ vanishing on Γ. Let $\|v\| = \sqrt{(v, v)}$ be the norm in $\mathcal{H}$.

We write the convection–diffusion problem (6.141)–(6.143) as the Cauchy problem for the evolution equation of the first order:

$$\frac{du}{dt} + \mathcal{C}u + \mathcal{D}u = 0. \tag{6.146}$$

The *operator of diffusive transfer*

$$\mathcal{D}u = -k \sum_{\alpha=1}^{2} \frac{\partial^2 u}{\partial x_\alpha^2} \tag{6.147}$$

is self-adjoint and positively defined in $\mathcal{H}$, i.e.,

$$\mathcal{D} = \mathcal{D}^* \geq k\lambda_0 E, \tag{6.148}$$

where E is the identity operator and $\lambda_0 > 0$ is the smallest eigenvalue of the Laplace operator.

For the *operator of convective transfer* we have

$$\mathcal{C}u = \mathcal{C}_1 u = \sum_{\alpha=1}^{2} v_\alpha(x) \frac{\partial u}{\partial x_\alpha}. \tag{6.149}$$

Using the convection–diffusion equation in the form of (6.144) we obtain

$$\mathcal{C}u = \mathcal{C}_2 u = \sum_{\alpha=1}^{2} \frac{\partial(v_\alpha(x)u)}{\partial x_\alpha}. \tag{6.150}$$

Therefore, for the operator of convective transfer in a symmetrical form (see equation (6.145)), we have

$$\mathcal{C} = \mathcal{C}_0 = \frac{1}{2}(\mathcal{C}_1 + \mathcal{C}_2). \tag{6.151}$$

Taking into account the uniform boundary conditions (6.142) we obtain

$$(\mathcal{C}_1 u, w) = \sum_{\alpha=1}^{2} \int\limits_{\Omega} v_\alpha \frac{\partial u}{\partial x_\alpha} w dx = -\sum_{\alpha=1}^{2} \int\limits_{\Omega} \frac{\partial (v_\alpha w)}{\partial x_\alpha} u dx = -(u, \mathcal{C}_2 w).$$

Thus the conjugation is established with accuracy to the sign of the operators of convective transfer in divergent and non-divergent forms, and the skewness of the operator of convective transfer a in symmetric form, i.e.,

$$\mathcal{C}_1^* = -\mathcal{C}_2, \quad \mathcal{C}_0 = -\mathcal{C}_0^*. \tag{6.152}$$

If the incompressibility condition is satisfied, the operators of convective transfer in a non-divergent (6.149) and divergent (6.150) form are also skew-symmetrical. A fundamental moment in constructing discrete approximations for operators of convective transfer is that of the property of skew-symmetry for the operators $\mathcal{C}_0$ holds for any $v_\alpha(x)$ and $\alpha = 1, 2$, including those which do not satisfy the incompressibility condition.

For the operators (6.149) and (6.150) of the convective transfer we have

$$(\mathcal{C}_1 u, u) = -(\mathcal{C}_2 u, u) = \frac{1}{2} \sum_{\alpha=1}^{2} \int\limits_{\Omega} v_\alpha \frac{\partial u^2}{\partial x_\alpha} dx = -\frac{1}{2} \int\limits_{\Omega} \mathrm{div}\mathbf{v} u^2 dx$$

and therefore

$$|(\mathcal{C}_\alpha u, u)| \leq \frac{1}{2} \|\mathrm{div}\,\mathbf{v}\|_C \|u\|^2, \quad \alpha = 1, 2, \tag{6.153}$$

where

$$\|w\|_C = \max_{x \in \Omega} |w(x)|.$$

Then for the operators of convective transfer defined in (6.149), (6.150) ($\mathcal{C} = \mathcal{C}_\alpha, \alpha = 1, 2$), we have

$$|(\mathcal{C} u, u)| \leq \mathcal{M}_1 \|u\|^2, \tag{6.154}$$

where the constant $\mathcal{M}_1$ depends only on div$\mathbf{v}$ and is defined in (6.153).

We also show that the operator of convective transfer is subordinate to the diffusion operator, i.e.,

$$\|\mathcal{C} u\|^2 \leq \mathcal{M}_2 (\mathcal{D} u, u), \tag{6.155}$$

where the constant $\mathcal{M}_2$ depends on the velocity. For the non-divergent operator of convection (6.149), we have

$$\|\mathcal{C}_1 u\|^2 = \int\limits_\Omega \left(\sum_{\alpha=1}^{2} v_\alpha \frac{\partial u}{\partial x_\alpha} \right)^2 dx \leq 2 \sum_{\alpha=1}^{2} \int\limits_\Omega v_\alpha^2 \left(\frac{\partial u}{\partial x_\alpha} \right)^2 dx$$

$$\leq 2 \max_\alpha \{\|v_\alpha^2\|_C\} \frac{1}{k} \sum_{\alpha=1}^{2} \int\limits_\Omega k \left(\frac{\partial u}{\partial x_\alpha} \right)^2 dx$$

$$\leq \frac{2}{k} \max_\alpha \{\|v_\alpha^2\|_C\} (\mathcal{D}u, u),$$

i.e., for $\mathcal{C} = \mathcal{C}_1$ in inequality (6.155) one can put

$$\mathcal{M}_2 = \frac{2}{k} \max_\alpha \{\|v_\alpha^2\|_C\}.$$

For $\mathcal{C} = \mathcal{C}_2$ (see formula (6.150)) we obtain

$$\|\mathcal{C}_2 u\|^2 = \int\limits_\Omega \left(\sum_{\alpha=1}^{2} v_\alpha \frac{\partial u}{\partial x_\alpha} + \operatorname{div} \mathbf{v} u \right)^2 dx$$

$$\leq 2 \int\limits_\Omega \left(\sum_{\alpha=1}^{2} v_\alpha \frac{\partial u}{\partial x_\alpha} \right)^2 + 2 \int\limits_\Omega (\operatorname{div} \mathbf{v})^2 u^2 dx.$$

Taking into account the Fridrichs inequality

$$\int\limits_\Omega u^2 dx \leq \mathcal{M}_0 \sum_{\alpha=1}^{2} \int\limits_\Omega \left(\frac{\partial u}{\partial x_\alpha} \right)^2 dx,$$

with the constant $\mathcal{M}_0$ depending only on the domain and equal to

$$\mathcal{M}_0^{-1} = \pi^2 (l_1^{-2} + l_2^{-2}),$$

in the case of a rectangle Ω we obtain the estimate (6.155) with $\mathcal{C} = \mathcal{C}_2$ and

$$\mathcal{M}_2 = \frac{2}{k} (2 \max_\alpha \{\|v_\alpha^2\|_C\} + \mathcal{M}_0 \|\operatorname{div} \mathbf{v}\|_C^2).$$

Similarly, for $\mathcal{C} = \mathcal{C}_0$ we have

$$\mathcal{M}_2 = \frac{1}{k} (3 \max_\alpha \{\|v_\alpha^2\|_C\} + \mathcal{M}_0 \|\operatorname{div} \mathbf{v}\|_C^2).$$

Estimates (6.154) and (6.155) serve as a reference point in investigation of difference analogs of the operators of convection–diffusion.

To obtain an *a priori* estimate, in $\mathcal{H}$ we take a dot product of equation (6.146) with u, taking into account the above estimates (6.148) and (6.154) and obtain

$$\frac{d\|u\|}{dt} + (k\lambda_0 - \mathcal{M}_1)\|u\| \leq 0, \tag{6.156}$$

whence we have the estimate

$$\|u(x,t)\| \leq \exp((\mathcal{M}_1 - k\lambda_0)t)\|u(x,0)\|, \tag{6.157}$$

which guarantees the stability of the solution of the convection–diffusion problem with respect to the initial data. To avoid unwieldness in the presentation, we restrict our discussion to the case of

$$\mathcal{M}_1 - k\lambda_0 \leq 0.$$

Taking as a reference point inequality (6.155) and taking into account that

$$-(\mathcal{C}u, u) \leq \frac{\mathcal{M}_2}{4}\|u\|^2 + \frac{1}{\mathcal{M}_2}\|\mathcal{C}u\|^2$$

we obtain

$$\frac{d\|u\|}{dt} \leq \frac{\mathcal{M}_2}{4}\|u\|$$

instead inequality (6.156) and thus the estimate

$$\|u(x,t)\| \leq \exp(\frac{\mathcal{M}_2}{4}t)\|u(x,0)\| \tag{6.158}$$

is valid.

Estimate (6.158) can also be obtained in a stronger norm. For this purpose, we take the dot product of equation (6.146) with du/dt. Taking into account the independence of the operators $\mathcal{C}$ and $\mathcal{D}$ of time and of the inequality

$$-(\mathcal{C}u, \frac{du}{dt}) \leq \|\frac{du}{dt}\|^2 + \frac{\mathcal{M}_2}{4}(\mathcal{D}u, u),$$

we have

$$\frac{d\|u\|^2_{\mathcal{D}}}{dt} \leq \frac{\mathcal{M}_2}{2}\|u\|^2_{\mathcal{D}}. \quad \|u\|^2_{\mathcal{D}} = (\mathcal{D}u, u).$$

Thus the estimate

$$\|u(x,t)\|_{\mathcal{D}} \leq \exp(\frac{\mathcal{M}_2}{4}t)\|u(x,0)\|_{\mathcal{D}} \tag{6.159}$$

is proved.

Other forms of the operators of convective transfer can be used. One of them is the representation of the operator $\mathcal{C}_0$ in the form [Temam, 1979]

$$\mathcal{C}_0 = \mathcal{C}_1 + \frac{1}{2}\operatorname{div}\mathbf{v}. \tag{6.160}$$

Analogously we may put

$$\mathcal{C}_0 = \mathcal{C}_2 - \frac{1}{2}\operatorname{div}\mathbf{v}. \tag{6.161}$$

In this case the operator $\mathcal{C}_0$ is also skew-symmetric for any $\mathbf{v}$ and the corresponding estimates of the solution of the Cauchy problem for equation (6.146) are simplified. Nevertheless, the symmetric form (6.151) is more convenient for the construction of difference operators of the convective transfer with the skew-symmetric property than representations (6.160), (6.161).

4.4 Difference Operators of Convection and Diffusion

In the rectangle Ω we introduce a grid with the steps h_α, $\alpha = 1, 2$, i.e., uniform one with respect to both variables. Let ω_h be the set of the inner nodes of the grid:

$$\omega_h = \{x = (x_1, x_2),\ x_\alpha = i_\alpha h_\alpha, \quad i_\alpha = 1, 2, ..., N_\alpha - 1,$$
$$N_\alpha h_\alpha = l_\alpha, \quad \alpha = 1, 2\},$$

and γ_h be the set of boundary nodes. We denote a difference solution of the convection–diffusion problem for the moment t as $y(x, t)$, $x \in \omega_h \cup \gamma_h$, $t > 0$.

For grid functions vanishing at γ_h we define the Hilbert space H. The inner product and the norm in it are introduced by the following relations:

$$(y, w) = \sum_{x \in \omega} y w h_1 h_2, \quad \|y\| = \sqrt{(y, y)}.$$

For $R = R^* > 0$ H_R denotes the space H with the inner product $(y, w)_R = (Ry, w)$ and the norm $\|y\|_R = \sqrt{(Ry, y)}$.

The *difference operator of diffusive transfer* on the set of the functions $y \in H$ is defined by the expression

$$Dy = -k \sum_{\alpha=1}^{2} y_{\bar{x}_\alpha x_\alpha}, \quad y = 0, \quad \text{if} \quad x \in \gamma_h. \tag{6.162}$$

The quantity D is the self-adjoint operator in H and the following estimate holds (6.148):

$$D = D^* \geq k\delta E, \tag{6.163}$$

where

$$\delta = \sum_{\alpha=1}^{2} \frac{4}{h_\alpha^2} \sin^2 \left(\frac{\pi h_\alpha}{2l_\alpha} \right).$$

The upper estimate [Samarskii and Nikolaev, 1978] for the difference operator D

$$D \leq k\Delta E, \quad \Delta = \sum_{\alpha=1}^{2} \frac{4}{h_\alpha^2} \cos^2 \left(\frac{\pi h_\alpha}{2l_\alpha} \right) \tag{6.164}$$

is useful too.

Let us approximate the convective terms in the convection–diffusion equation with the second order using central differences. Expression (6.149) infers that

$$Cy = C_1 y = \sum_{\alpha=1}^{2} b_\alpha y_{\overset{\circ}{x}_\alpha}. \tag{6.165}$$

In the simplest case of smooth enough speed components and solution of the differential problem, we select $b_\alpha(x) = v_\alpha(x)$, $\alpha = 1, 2$. The difference operator

$$Cy = C_2 y = \sum_{\alpha=1}^{2} (b_\alpha y)_{\overset{\circ}{x}_\alpha} \tag{6.166}$$

is used for the convection–diffusion equation with the convective terms in a divergent form (see representation (6.150)).

Similarly to (6.151) we obtain

$$C = C_0 = \frac{1}{2}(C_1 + C_2),$$

i.e.,

$$C_0 y = \frac{1}{2} \sum_{\alpha=1}^{2} (b_\alpha y_{\overset{\circ}{x}_\alpha} + (b_\alpha y)_{\overset{\circ}{x}_\alpha}). \tag{6.167}$$

We must be more cautious in constructing of *difference convective operators* using formulas (6.160), (6.161). In particular, this becomes clear if we investigate the properties of the skew-symmetry of the difference operator

$$Cy = \sum_{\alpha=1}^{2} \left(b_\alpha y_{\overset{\circ}{x}_\alpha} + \frac{1}{2} (b_\alpha)_{\overset{\circ}{x}_\alpha} y \right).$$

We note now the main properties of the introduced difference operators of convective transfer in H. To find an operator adjoint to C_1 we consider the expression

$$(C_1 y, w) = \sum_{\alpha=1}^{2} \sum_{x \in \omega_h} b_\alpha(x) y_{\overset{\circ}{x}_\alpha} w(x) h_1 h_2.$$

On the set of grid functions vanishing on γ_h, we have the equality

$$\begin{aligned}
&\sum_{x \in \omega_h} b_1(x) y_{\overset{\circ}{x}_1} w(x) h_1 h_2 \\
&\quad = \frac{1}{2} \sum_{x \in \omega_h} b_1(x)(y(x_1 + h_1, x_2) - y(x_1 - h_1, x_2)) w(x) h_2 \\
&\quad = \frac{1}{2} \sum_{x \in \omega_h} y(x)(b_1(x_1 - h_1, x_2) w(x_1 - h_1, x_2) \\
&\quad\quad - b_1(x_1 + h_1, x_2) w(x_1 + h_1, x_2)) h_2 \\
&\quad = - \sum_{x \in \omega_h} y(x)(b_1(x) w)_{\overset{\circ}{x}_1} h_1 h_2.
\end{aligned}$$

Then

$$C_1^* y = - \sum_{\alpha=1}^{2} (b_\alpha(x) y)_{\overset{\circ}{x}_\alpha},$$

and, just as in the continuous case, for representations (6.165) and (6.166) we obtain

$$C_1^* = -C_2, \quad C_0 = -C_0^*. \tag{6.168}$$

Direct calculations easily show that there is no difference analog of estimate (6.153) for the difference operators considered. Therefore the use of symmetrical approximations (6.167) of convective transfer seems to be more preferable. One has to limit ourselves to rougher upper bounds of the convective transfer operator in a divergent and a non-divergent form. Taking into account the inequality

$$\begin{aligned}
(b_1(x) y_{\overset{\circ}{x}_1}, y) &= \frac{1}{2h_1}(b_1(x)(y(x_1 + h_1, x_2) - y(x_1 - h_1, x_2)), y) \\
&\le \frac{1}{h_1} \max_{x \in \omega_h \cup \gamma_h} |b_1(x)| \, \|y\|^2,
\end{aligned}$$

we obtain the estimate

$$|(C_\alpha y, y)| \le \sum_{\beta=1}^{2} \frac{1}{h_\beta} \|b_\beta(x)\|_C \|y\|^2, \quad \alpha = 1, 2, \tag{6.169}$$

with

$$\|w(x)\|_C = \max_{x\in\omega_h\cup\gamma_h} |w(x)|.$$

Thus we have the difference analog of inequality (6.154):

$$|(Cy,y)| \le M_1\|y\|^2, \tag{6.170}$$

where the constant M_1 is defined in accordance with inequalities (6.169).
We also deduce a difference analog of inequality (6.155):

$$\|Cy\|^2 \le M_2(Dy,y). \tag{6.171}$$

For the non-divergent difference operator of convective transfer we have

$$\begin{aligned}\|C_1y\|^2 &= \sum_{x\in\omega_h}\left(\sum_{\alpha=1}^{2} b_\alpha(x)y_{\overset{\circ}{x}_\alpha}\right)^2 h_1h_2 \\ &\le 2\sum_{\alpha=1}^{2}\sum_{x\in\omega_h} b_\alpha^2(x)(y_{\overset{\circ}{x}_\alpha})^2h_1h_2 \\ &\le 2\max_\alpha\{\|b_\alpha^2\|_C\}\sum_{\alpha=1}^{2}\sum_{x\in\omega_h}\frac{1}{4}(y_{x_\alpha}+y_{\bar{x}_\alpha})^2h_1h_2 \\ &\le \max_\alpha\{\|b_\alpha^2\|_C\}\sum_{\alpha=1}^{2}\sum_{x\in\omega_h}((y_{x_\alpha})^2+(y_{\bar{x}_\alpha})^2)h_1h_2 \\ &\le \frac{2}{k}\max_\alpha\{\|b_\alpha^2\|_C\}(Dy,y).\end{aligned}$$

Therefore, when $C=C_1$, in the subordination inequality (6.171) we may put

$$M_2 = \frac{2}{k}\max_\alpha\{\|b_\alpha^2\|_C\}.$$

With $C = C_2$ we obtain the estimate (6.171) on the basis of the inequality

$$(b_1(x)y)_{\overset{\circ}{x}_1} = \frac{1}{2}b_1(x_1+h_1,x_2)y_{x_1} + \frac{1}{2}b_1(x_1-h_1,x_2)y_{\bar{x}_1} + y(x)(b_1(x))_{\overset{\circ}{x}_1}.$$

Just as in the continuous case, we have

$$\begin{aligned}\|C_2y\|^2 &= \sum_{x\in\omega_h}\left(\sum_{\alpha=1}^{2}(b_\alpha(x)y)_{\overset{\circ}{x}_\alpha}\right)^2 h_1h_2 \\ &\le \frac{4}{k}\max_\alpha\|b_\alpha^2(x)\|_C(Dy,y) + 2\|\sum_{\alpha=1}^{2}(b_\alpha(x))_{\overset{\circ}{x}_\alpha}\|_C^2\|y\|^2.\end{aligned}$$

There is the well known [Samarskii and Andreev, 1976] *Fridrichs inequality*

$$\|y\|^2 \le M_0(-\sum_{\alpha=1}^{2} y_{\bar{x}_\alpha x_\alpha}, y),$$

where the constant M_0 is independent of the grid steps:

$$M_0^{-1} = 8(l_1^{-2} + l_2^{-2}).$$

Thus, we obtain the desired inequality (6.171) for the operator $C = C_2$ with

$$M_2 = \frac{2}{k}\left(2\max_{\alpha}\{\|b_\alpha^2\|_C\} + M_0\|\sum_{\alpha=1}^{2}(b_\alpha(x))_{\overset{\circ}{x}_\alpha}\|_C^2\right).$$

Similarly, for the operator $C = C_0$ in inequality (6.171), we have

$$M_2 = \frac{1}{k}\left(3\max_{\alpha}\{\|b_\alpha^2\|_C\} + M_0\|\sum_{\alpha=1}^{2}(b_\alpha(x))_{\overset{\circ}{x}_\alpha}\|_C^2\right).$$

Thus for the difference operators of convective transfer the subordination estimates (6.171) are satisfied and the constants M_2 are completely coordinated with those of the continuous case provided that for the divergence approximations central differences are used.

For the approximation of convective terms in a divergent and a non-divergent form, the constant M_1 depends on a grid (see inequality (6.169)). Because of this the problem of approximation of convective term in a divergent and non-divergent forms requires additional consideration. It is natural to construct approximations with the constant M_1 in (6.170) to be coordinated with the constant $\mathcal{M}_1$ in inequality (6.153).

Taking into account relations (6.160) and (6.161) we define the convective operators in the divergent and the non-divergent form by the following relations:

$$\mathcal{C}_1 = \mathcal{C}_0 - \frac{1}{2}\operatorname{div}\mathbf{v}, \qquad (6.172)$$

$$\mathcal{C}_2 = \mathcal{C}_0 + \frac{1}{2}\operatorname{div}\mathbf{v}. \qquad (6.173)$$

Using expressions (6.172), (6.173)we assume that

$$C_1 = C_0 - \frac{1}{2}\sum_{\alpha=1}^{2}(b_\alpha(x))_{\overset{\circ}{x}_\alpha}, \qquad (6.174)$$

$$C_2 = C_0 + \frac{1}{2}\sum_{\alpha=1}^{2}(b_\alpha(x))_{\overset{\circ}{x}_\alpha}. \qquad (6.175)$$

For operators (6.174), (6.175) the inequality (6.170) with the constant

$$M_1 = \frac{1}{2} \| \sum_{\alpha=1}^{2} (b_\alpha(x))_{\overset{\circ}{x}_\alpha} \|_C^2$$

is valid. Here M_1 does not already depend on the computational grid and it does completely coordinated with the constant $\mathcal{M}_1$ in the continuous case.

4.5 Difference Schemes for Non-Stationary Problems

After sampling with respect to space we obtain the solution of the difference–differential equation

$$\frac{dv}{dt} + Cv + Dv = 0, \quad x \in \omega_h, \quad t > 0, \tag{6.176}$$

with the initial condition

$$v(x, 0) = u_0(x), \quad x \in \omega_h. \tag{6.177}$$

For solution of problem (6.176), (6.177) the following estimates, of the type of (6.157), (6.159), are valid, i.e.,

$$\|v(t)\| \leq \exp((M_1 - k\delta)t) \|v(0)\|, \tag{6.178}$$

$$\|v(t)\|_D \leq \exp\left(\frac{M_2}{4} t\right) \|v(0)\|_D, \tag{6.179}$$

where M_1, M_2 are constants in the bounds (6.170) and (6.171), respectively.

The constant M_1 in inequality (6.170) is equal to zero whenever the symmetric approximation of convective terms (6.167) is used. For problems with divergent and non-divergent operators of convective transfer the following approximations (6.174), (6.175) are the reference points in optimization of the constant M_1. Let us construct difference schemes for approximative solution of problem (6.176), (6.177) with $C = C_0$ and investigate the stability of this schemes considering the problem with convective terms in a symmetrical form.

With $\tau > 0$ being a time step, we take y_n to denote the difference solution at the moment $t_n = n\tau$. For problem (6.176), (6.177) we consider the following two-level difference scheme with weighting factors:

$$\begin{aligned} &\frac{y_{n+1} - y_n}{\tau} + C_0(\sigma_1 y_{n+1} + (1 - \sigma_1) y_n) \\ &\qquad + D(\sigma_2 y_{n+1} + (1 - \sigma_2) y_n) = 0, \\ &x \in \omega_h, \quad n = 0, 1, \ldots, \end{aligned} \tag{6.180}$$

and the complementary initial condition

$$y_0(x) = u_0(x), \quad x \in \omega_h. \tag{6.181}$$

To investigate the stability, we rewrite scheme (6.180) in the canonical form

$$B\frac{y_{n+1} - y_n}{\tau} + Ay_n = 0, \quad n = 0, 1, \dots . \tag{6.182}$$

The grid operators B and A here have the form

$$B = E + \sigma_1 \tau C_0 + \sigma_2 \tau D, \quad A = C_0 + D. \tag{6.183}$$

The operators A and B in the previous difference scheme with weighting factors are non-self-adjoint.

We consider the class of schemes (6.182), (6.183) with equal weighting factors:

$$\sigma_1 = \sigma_2 = \sigma. \tag{6.184}$$

THEOREM 6.7 *The difference scheme (6.180), (6.181), (6.184) is stable in H provided that*

$$\tau \le \tau_0 = \frac{1}{\nu(0.5 - \sigma)}, \tag{6.185}$$

where

$$\nu = 2M_2 + 2k\Delta, \tag{6.186}$$

and the constants M_2 and Δ are defined by formulas (6.171) and (6.164).

Proof. By virtue of representations (6.183) and condition (6.184) we have the relations

$$B = E + \sigma\tau A, \quad A \ne A^*. \tag{6.187}$$

The difference scheme with weighting factors (6.182), (6.187) and operator $A > 0$ is stable in H (see estimate (6.178)) and in H_R, $R = BB^*$, provided that the following operator inequality

$$A + \left(\sigma - \frac{1}{2}\right)\tau A^* A \ge 0 \tag{6.188}$$

is valid.

The inequality (6.188) is obviously satisfied for $\sigma \ge 0.5$, and the difference scheme is absolutely stable. It is conditionally stable for $\sigma < 0.5$. Assume that the estimate

$$\|Ay\|^2 \le \nu(Ay, y) \tag{6.189}$$

is valid. Then inequality (6.188) is valid on condition that

$$\sigma \geq \frac{1}{2} - \frac{1}{\nu\tau},$$

or when estimate (6.185) is valid. Let us define more specifically the conditions (6.189) when scheme (6.182), (6.187) is used. Taking into account estimate (6.164) for the diffusion operator and subordination inequality (6.171), we have

$$\begin{aligned} \|Ay\|^2 = \|(C_0 + D)y\|^2 &\leq 2(\|C_0 y\|^2 + \|Dy\|^2) \\ &\leq 2M_2(Dy, y) + 2k\Delta(Dy, y). \end{aligned}$$

Taking into consideration that the operator C_0 is skew-symmetric, we obtain for the constant ν expression (6.186), i.e., $\nu = O(h_1^{-2} + h_2^{-2})$ and it depends on the velocity of convective transfer. Expressions (6.185), (6.186) yield, for instance, sufficient conditions for the stability of the explicit scheme ($\sigma = 0$).

Among the schemes with weighting factors (6.180), (6.181) attention should specially be paid to the scheme with convective terms taken from the previous level, i.e.,

$$\sigma_1 = 0, \quad \sigma_2 = \sigma. \tag{6.190}$$

In the implicit–explicit scheme (6.180), (6.190) only a symmetric operator of the diffusion

$$B = E + \sigma\tau D, \quad A = C_0 + D \tag{6.191}$$

is considered on the upper level. Consider the conditions of stability for scheme (6.182), (6.191).

THEOREM 6.8 *For difference scheme (6.180) with conditions (6.181), (6.190) and $\sigma \geq 0.5$, the following estimate*

$$\|y_{n+1}\|_D \leq \left(1 + \frac{M_2}{4}\tau\right) \|y_n\|_D \tag{6.192}$$

is valid.

Proof. By virtue of the skew-symmetry of the operator C_0 and symmetry of the operator D, scheme (6.180), (6.181) corresponds to the scheme (6.182) with the operator

$$B = E + \sigma\tau A_0, \quad A = A_0 + A_1, \tag{6.193}$$

where A_0 and A_1 are symmetrical and skew-symmetric parts of the operator A, i.e.,

$$A_0 = \frac{1}{2}(A + A^*), \quad A_1 = \frac{1}{2}(A - A^*).$$

Let us consider the stability of scheme (6.182), (6.193) in sufficiently general conditions of subordination of the skew-symmetric part of the operator A:

$$\|A_1 y\|^2 \leq M(A_0 y, y). \tag{6.194}$$

When $A_0 > 0$

$$B - \frac{\tau}{2}A \geq \varepsilon E, \quad \varepsilon > 0, \tag{6.195}$$

and for inequality (6.194) satisfied, the estimate

$$\|y_{n+1}\|_{A_0} \leq \left(1 + \frac{M}{4\varepsilon}\tau\right) \|y_n\|_{A_0} \tag{6.196}$$

for difference scheme (6.182), (6.193) is valid.

Taking the dot product of (6.182) with $2\tau y_t = 2(y_{n+1} - y_n)$ we obtain the energy identity

$$\tau((2B - \tau A)y_t, y_t) + (A_0 y_{n+1}, y_{n+1}) - (A_0 y_n, y_n) + 2\tau(A_1 y_n, y_t) = 0. \tag{6.197}$$

Taking into account inequality (6.197) and condition (6.195) we have the inequality

$$2\tau\varepsilon(y_t, y_t) + (A_0 y_{n+1}, y_{n+1}) - (A_0 y_n, y_n) \leq 2\tau| - (A_1 y_n, y_t)|. \tag{6.198}$$

The right hand side can be estimated with allowance for condition (6.194) as follows:

$$\begin{aligned} | - (A_1 y_n, y_t)| &\leq \varepsilon\|y_t\|^2 + \frac{1}{4\varepsilon}\|A_1 y_n\|^2 \\ &\leq \varepsilon\|y_t\|^2 + \frac{M}{4\varepsilon}(A_0 y_n, y_n). \end{aligned} \tag{6.199}$$

On substitution of the latter estimate into inequality (6.198) we obtain the relation

$$(A_0 y_{n+1}, y_{n+1}) \leq (1 + \frac{M}{2\varepsilon}\tau)(A_0 y_n, y_n).$$

From this estimate and the inequality

$$\left(1 + \frac{M}{2\varepsilon}\tau\right)^{1/2} \leq 1 + \frac{M}{4\varepsilon}\tau$$

the desired estimate (6.196) of the stability follows.

The proved statement now will be used to investigate the implicit–explicit difference scheme (6.180), (6.181), (6.190). It can be rewritten in the canonical form (6.182), (6.193). Then $A_0 = D$, $A_1 = C_0$ and from condition (6.171) we have $M = M_2$ in inequality (6.194). For $\sigma \geq 0.5$ in inequality (6.195) one can put $\varepsilon = 1$. Then, estimate (6.192) with $\sigma \geq 0.5$ is valid for the implicit–explicit scheme (6.180), (6.181), (6.190). Estimate (6.192) expresses unconditional ρ-stability of the implicit–explicit scheme and is completely consistent with estimate (6.158) for the solution of the differential-difference problem.

Various other classes of difference schemes for the problems of convective diffusion are considered likewise. In particular, this refers to the problems of convection–diffusion with the convective operator in a divergent and a non-divergent form, to construction of economic schemes for multi-dimensional problems with splitting by spatial variables, to three-level schemes, and so on.

5. Korteweg–de Vries Equation

We carry out an analysis of difference schemes for the Korteweg–de Vries (KdV) equation to verify the fulfilment of conservation laws. We use the notion of L_2-conservative difference scheme [Akrivis, 1993, Akrivis et al., 1997] whose solution satisfies the grid analog of the conservative law:

$$\int_0^l u^2(x,t)dx = \int_0^l u^2(x,0)dx, \quad t \in (0,T),$$

which is valid for the initial differential problem. This principle is used to construct new classes of three level difference schemes with weighting factors. Using the general theory of the stability of operator difference schemes, a linear analysis of the schemes constructed is performed [Samarskii et al., 1997b].

5.1 Introduction

At the present time nonlinear wave processes are comprehensively investigated in various branches of science and technology (optics, plasma physics, radiophysics, hydrodynamics).

For the investigation of waves of small but finite amplitude the KdV equation is frequently used. The equations which for the first time was derived in 1895 [Korteweg and de Vries, 1895] by expanding of an ideal incompressible fluid in small parameters. However, it was only in the sixties of this last century that the KdV equation attracted great attention. The pioneers in this field were Zabusky and Kruskal [Zabusky and Kruskal, 1965] who constructed an explicit three-level scheme with the order of approximation $O(h^2 + \tau^2)$. A fundamental result in the study of the properties of the KdV equation was the joint paper by Gardner, Green, Kruskal and Miura [Gardner et al., 1967] who found an exact solution of the KdV equation on a real axis. In their subsequent papers [Gardner, 1971, Kruskal et al., 1970, Miura, 1968, Miura et al., 1968, Su and Gardner, 1969] many of the properties of the KdV equation were revealed that were typical of the whole class of solvable equations in partial derivatives. Complete enough analysis of the results pertaining to the KdV equation was performed by Miura [Miura, 1976]. In 1976 Greig and Moris suggested a 'black and white' scheme which has a less strong stability condition in comparison with the scheme of [Zabusky and Kruskal, 1965]. Further investigation and stability analysis of implicit and explicit difference schemes for the KdV equation were undertaken by U.A.Berezin [Berezin, 1972, Berezin, 1982]. Conditions of nonlinear stability and convergence of difference schemes were investigated, for instance, by Newell [Newell, 1997] and I.D.Turetaev [Turetaev, 1992].

5.2 Formulation of the Problem and Basic Properties of Its Solution

In the rectangle $\overline{Q}_T = \{ (x,t) : \ 0 \le x \le l, \ 0 \le t \le T \}$ we consider the *Korteweg–de Vries equation*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \quad \beta = \text{const} > 0, \tag{6.200}$$

with the periodic, with respect to space, conditions

$$u(x,0) = u_0(x), \quad 0 \le x \le l, \tag{6.201}$$

$$u(x,t) = u(x+l,t), \quad t \ge 0. \tag{6.202}$$

To construct effective numerical methods it is desirable that the grid analogs of the properties of the differential problem were satisfied in them.

LEMMA 6.6 *Let condition (6.202) be satisfied. Then for the nonlinear operator*

$$Lu = L_1 u + \beta L_2 u, \quad L_1 u = u\frac{\partial u}{\partial x}, \quad L_2 u = \frac{\partial^3 u}{\partial x^3}, \tag{6.203}$$

the relations

$$(L_k u, u) = 0, \quad k = 1,2; \quad (Lu, u) = 0, \tag{6.204}$$

are valid. Here

$$(u,v) = \int_0^l u(x,t)v(x,t)dx.$$

P r o o f. In fact, taking the dot product of the operator Lu (6.203) with $2u$, and applying the formula of integration by parts we have

$$(Lu,u) = I(l,t) - I(0,t).$$

Here

$$I(x,t) = \frac{2}{3}u^3 - \beta\left(\frac{\partial u}{\partial x}\right)^2 + 2\beta u\frac{\partial^2 u}{\partial x^2}.$$

For solutions periodic in space (see conditions (6.202)), we have $I(l,t) = I(0,t)$ and $(Lu,u) = 0$. The equalities $(L_k u,u) = 0$, $k = 1,2$ can be proved likewise.

For construction of conservative and completely conservative difference schemes [Samarskii, 1989, Samarskii and Popov, 1980], it is very important to know the integral characteristics of the differential problem which do not vary with time.

LEMMA 6.7 *For the solution of problem (6.200)–(6.202), the following conservative laws hold*

$$E_1(t) = E_1(0), \quad E_1(t) = \int_0^l u(x,t)dx, \tag{6.205}$$

$$E_2(t) = E_2(0), \quad E_2(t) = \int_0^l u^2(x,t)dx. \tag{6.206}$$

P r o o f. Relation (6.205) is obvious. To validate equality (6.206) we take the dot product of equation (6.200) with $2u$, and applying the periodic condition (6.202),and we obtain the identity

$$\frac{d}{dt}E_2(t) = 0. \tag{6.207}$$

Integrating it over t we have proved the statement of the lemma.

5.3 A Model Equation

Major moments in construction of effective algorithms for the nonlinear KdV equation will be illustrated on the model equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \quad a, \beta = \text{const} > 0, \tag{6.208}$$

with conditions (6.201), (6.202). In selecting a model problem we demand first of all the linearity of the equation and then the preservation of the basic properties of the initial nonlinear problem. For instance, following Lemma 6.6, for the linear analog of the operator Lu

$$L_0 u = L_{10} u + \beta L_2 u, \quad L_{10} u = a\frac{\partial u}{\partial x},$$

one can easily show that the property (6.205) is valid. In other words, the operator L_0 is a skew-symmetrical one, and

$$L_0 = -L_0^*. \tag{6.209}$$

For the model problem select as a reference point for our considerations of numerical methods the following explicit difference scheme:

$$y_t + a y_{\overset{\circ}{x}} + \beta y_{\bar{x}x\overset{\circ}{x}} = 0, \quad i = \overline{0, N}, \quad t \in \omega_\tau, \tag{6.210}$$

with periodicity conditions on the previous level

$$y_{i+N} = y_i, \quad y(x,0) = u_0(x), \quad x \in \overline{\omega}_h, \tag{6.211}$$

where $\overline{\omega}_h$ and $\overline{\omega}_\tau$ are the spatially and temporarily uniform grids with the steps h and τ, respectively.

Note that under conditions (6.211) we obtain an approximation of differential conjugation conditions:

$$y_0 = y_N, \quad y_{\overset{\circ}{x},0} = y_{\overset{\circ}{x},N}, \quad y_{\bar{x}x,0} = y_{\bar{x}x,N}. \tag{6.212}$$

We show that in the explicit scheme (6.210) the conditions (6.212) are valid also for $t = t_{n+1}$. Since $y_{\overset{\circ}{x},0} = y_{\overset{\circ}{x},N}$,

$$y_{\bar{x}x\overset{\circ}{x},0} = \frac{y_2 - 2y_1 + 2y_{N-1} - y_{N-2}}{2h^3} = y_{\bar{x}x\overset{\circ}{x},N},$$

equation (6.210) implies the equality $\hat{y}_0 = \hat{y}_N$. Assuming now that

$$\hat{y}_{-2} = \hat{y}_{N-2}, \quad \hat{y}_{-1} = \hat{y}_{N-1}, \quad \hat{y}_{N+1} = \hat{y}_1, \quad \hat{y}_{N+2} = \hat{y}_2, \tag{6.213}$$

we obtain the approximation of conjugation conditions (6.212) on the level $(n+1)$.

To analyze the properties of the difference scheme (6.210) we bring it into the canonical form

$$B\frac{y_{n+1} - y_n}{\tau} + A y_n = 0, \quad y_0 = u_0, \tag{6.214}$$

where $y_n = (y_0^n, y_1^n, ..., y_N^n)^T \in H$, $u_0 \in H$ and $H = \Omega_h$ is the space of the grid functions defined on the grid $\overline{\omega}_h$ and satisfying the periodicity condition $y_i = y_{i+N}$. Then in equation (6.214) we have

$$B = E, \quad A : \Omega_h \to \Omega_h, \quad A = aA_1 + \beta A_2, \tag{6.215}$$

$$(A_1 y)_i = \begin{cases} \dfrac{y_1 - y_{N-1}}{2h} & \text{for} i = 0, \\ y_{\overset{\circ}{x},i} & \text{for} i = 1, 2, ..., N-1, \\ \dfrac{y_1 - y_{N-1}}{2h} & \text{for} i = N, \end{cases} \tag{6.216}$$

$$(A_2 y)_i = \begin{cases} \dfrac{y_{\bar{x}x,1}}{2h} - \dfrac{y_{\bar{x}x,N-1}}{2h} & \text{for} i = 0, \\ \dfrac{y_{\bar{x}x,2}}{2h} - \dfrac{y_{\bar{x},1}}{2h^2} + \dfrac{y_0 - y_{N-1}}{2h^3} & \text{for} i = 1, \\ y_{\bar{x}x\overset{\circ}{x},i} & \text{for} i = 2, 3, ..., N-2, \\ \dfrac{y_1 - y_N}{2h^3} - \dfrac{y_{\bar{x},N}}{2h^2} - \dfrac{y_{\bar{x}x,N-2}}{2h} & \text{for} i = N-1, \\ \frac{y_{\bar{x}x,1}}{2h} - \frac{y_{\bar{x}x,N-1}}{2h} & \text{for} i = N. \end{cases} \tag{6.217}$$

We supply the linear space Ω_h with the inner product

$$(u, v) = \frac{h}{2} u_0 v_0 + \sum_{i=1}^{N-1} h u_i v_i + \frac{h}{2} u_N v_N \tag{6.218}$$

and the norm $\|y\| = \sqrt{(y, y)}$.

To investigate the properties of the operator A, we need the following Lemma.

LEMMA 6.8 *For the arbitrary grid functions $u, v \in \Omega_h$, the equality*

$$(u_{\overset{\circ}{x}}, v) = -(u, v_{\overset{\circ}{x}}). \tag{6.219}$$

is valid.

Proof. Equality (6.219) follows from the formulae of summing by parts

$$\sum_{i=1}^{N-1} h u_{x,i} v_i = -\sum_{i=1}^{N} h u_i v_{\bar{x},i} + u_N v_N - u_1 v_0,$$

$$\sum_{i=1}^{N-1} h u_{\bar{x},i} v_i = -\sum_{i=0}^{N-1} h u_i v_{x,i} + u_{N-1} v_N - u_0 v_0$$

and from the identities

$$\frac{h}{2}\left(u_N v_{\bar{x},N} + u_0 v_{x,0}\right) = \frac{h}{2}\left(u_N v_{\overset{\circ}{x},N} + u_0 v_{\overset{\circ}{x},0}\right), \quad u_N v_N = u_0 v_0,$$
$$\frac{1}{2}(u_{N-1} v_N - u_1 v_0) = -\frac{h}{2}\left(u_{\overset{\circ}{x},0} v_0 + u_{\overset{\circ}{x},N} v_N\right).$$

Similarly, it can be proved that for any grid function $v \in \Omega_h$ we have

$$\left(v_{\bar{x}x\overset{\circ}{x}}, v\right) = 0. \tag{6.220}$$

Indeed, assuming in relation (6.219) that $u = v_{\bar{x}x} \in \Omega_h$ we obtain

$$\left(v_{\bar{x}x\overset{\circ}{x}}, v\right) = -\left(v_{\bar{x}x}, v_{\overset{\circ}{x}}\right) = -\frac{1}{4}\left(v_{x,0}^2 - v_{\bar{x},0}^2\right) - \frac{1}{4}\left(v_{x,N}^2 - v_{\bar{x},N}^2\right)$$
$$- \sum_{i=1}^{N-1} \frac{h}{2}\left(v_{\bar{x},i+1}^2 - v_{\bar{x},i}^2\right) = 0.$$

We show now that the equality

$$(Ay, y) = 0, \tag{6.221}$$

is valid for arbitrary $y \in \Omega_h$, i.e., A is the skew-symmetrical operator.

Indeed, in accordance with formulae (6.219), (6.220), we have identities

$$(A_1 y, y) = 0, \quad (A_2 y, y) = 0. \tag{6.222}$$

Thus equality (6.221) is proved.

Owing to the divergence of the scheme, a grid analog of the differential conservation law (6.205) is valid and the scheme is conservative.

We call a difference scheme L_2-conservative if the scheme satisfies the grid analog of the integral conservation law (6.206) which is valid for the initial differential problem.

We check now whether scheme (6.210) is L_2-conservative. For this purpose we take the dot product of the operator equation (6.214) with $2\tau y$. By virtue of equality (6.221) we have the following energy identity:

$$\|y(t)\|^2 - \sum_{t'=\tau}^{t} \tau^2 \|y_{\bar{t}}(t')\|^2 = \|y(0)\|^2, \quad t \in \omega_\tau.$$

The negative disbalance points out to the violation of the corresponding conservation law and generally calls in question the stability of the scheme in the $L_2(\omega_h)$ norm of the space H.

In fact, since the operators A and B are constant and the operator $B^{-1} = E$ exists, then in accordance with Theorem 3.1, the necessary and sufficient condition of the stability of scheme (6.214) in H_{B^*B} is equivalent to the inequality

$$BA^* + AB^* \geq \tau A^* A.$$

Since the operator A is skew-symmetric and $A = -A^*$ $(B = E)$, the latter inequality is equivalent to $A^*A \leq 0$. Thus the necessary condition of stability in H is not satisfied, and both of schemes (6.214) and (6.210) are absolutely unstable in the L_2 norm.

5.4 Three-Level Difference Schemes

To approximate equation (6.208) let us use the simplest explicit three-level difference scheme

$$y_{\overset{\circ}{t}} + a y_{\overset{\circ}{x}} + \beta y_{\bar{x}x\overset{\circ}{x}} = 0, \quad (x,t) \in \omega_{h\tau}, \tag{6.223}$$

with the initial conditions

$$y(x,0) = u_0(x), \quad y_t(x,0) = \overline{u}_0(x) \tag{6.224}$$

and the periodic conditions $y_{i+N} = y_i$. For the second initial condition (6.224), one can use, for instance, the differential equation (6.208) for $t = 0$. We rewrite scheme (6.223), (6.224) in the operator form

$$y_{\overset{\circ}{t}} + Ay = 0, \quad t \in \omega_\tau, \quad y_0 = u_0, \quad y_1 = \overline{u}_0, \tag{6.225}$$

where $y = y_n \in \Omega_h$, and A is the skew-symmetric operator defined by formulae (6.215)–(6.217). We show that this scheme is stable provided that

$$\tau \|A\| < 1,$$

and the following energy identity [Samarskii, 1989]

$$E(t) = E(0), \quad t \in \omega_\tau, \tag{6.226}$$

with

$$E(t) = \|\hat{y}\|^2 + 2\tau(\hat{y}, Ay) + \|y\|^2 > 0, \quad \hat{y} = y(t+\tau)$$

is valid for it. To prove this we rewrite the differential equation (6.225) in the form

$$\hat{y} + \tau Ay = \check{y} - \tau Ay$$

and compute the square degrees of the left hand and right hand sides:

$$\|\hat{y}\|^2 + 2\tau(\hat{y}, Ay) + \tau^2\|Ay\|^2 = \|\check{y}\|^2 - 2\tau(Ay, \check{y}) + \tau^2\|Ay\|^2. \tag{6.227}$$

Adding $\|y\|^2$ to both parts and taking into account that A is the skew-symmetric operator

$$(Ay, \check{y}) = -(y, A\check{y}),$$

we have $E(t) = E(t-\tau) = ... = E(0)$. We show that $E(t) > 0$. Indeed,

$$E(t) \geq \|\hat{y}\|^2 - 2\tau\|\hat{y}\|\|Ay\| + \|y\|^2 \geq \|y\|^2 - \tau^2\|Ay\|^2 > 0,$$

provided that $\tau\|A\| < 1$. For the stronger condition

$$\tau^2\|A\|^2 \leq 1 - \varepsilon, \quad 0 < \varepsilon < 1, \tag{6.228}$$

equalities (6.226) and (6.227) imply the stability of scheme (6.223), (6.224) in the grid L_2-norm, i.e., for any $t \in \omega_\tau$ the *a priori* estimate

$$\|y\|^2 \leq \frac{1}{1-\varepsilon} E(0) \tag{6.229}$$

is valid. Thus the principal distinction of the three-level scheme (6.223) from the two-level one (6.210) consists in its stability in the space $H = \Omega_h$ provided that condition (6.228) is satisfied.

To obtain the conditions of stability which could be useful in numerical practice let us estimate a norm of the operator $A = aA_1 + \beta A_2$ from the above. From definition 6.216), it follows that

$$\begin{aligned}
\|A_1 y\|^2 &= \sum_{i=1}^{N-1} h y^2_{\overset{\circ}{x},i} + \frac{1}{4h}(y_1 - y_{N-1})^2 \\
&= \frac{1}{4h^2}\left(\sum_{i=1}^{N-1} h(y_{i+1} - y_{i-1})^2 + h(y_1 - y_{N-1})^2\right) \\
&\leq \frac{1}{h^2}\left[\frac{1}{2}\left(\sum_{i=1}^{N-1} h y^2_{i+1} + \sum_{i=1}^{N-1} h y^2_{i-1}\right) + \frac{h}{2}(y_1^2 + y_{N-1}^2)\right] = \frac{1}{h^2}\|y\|^2.
\end{aligned}$$

Since A_1 is a linear operator in the Hilbert space and

$$\|A_1\| = \sup_{y \in \Omega_h} \sqrt{\frac{(A_1 y, A_1 y)}{(y, y)}},$$

the inequality $\|A_1 y\| \leq \|y\|/h$ implies that

$$\|A_1\| \leq \frac{1}{h}.$$

To estimate the norm of the operator A_2 we consider the quantity

$$\|A_2 y\|^2 = \frac{h}{2} l_0^2 + h l_1^2 + \sum_{i=2}^{N-2} h y^2_{\bar{x}x\overset{\circ}{x},i} + h l_{N-1}^2 + \frac{h}{2} l_N^2, \tag{6.230}$$

where in accordance with Definition (6.217) we have

$$l_0 = l_N = \frac{1}{2h}\left(y_{\bar{x}x,1} - y_{\bar{x}x,N-1}\right),$$
$$l_1 = \frac{1}{2h}\left(y_{\bar{x}x,2} - \frac{y_{\bar{x},1}}{h} + \frac{y_0 - y_{N-1}}{h^2}\right)$$
$$l_{N-1} = -\frac{1}{2h}\left(y_{\bar{x}x,N-2} - \frac{y_{\bar{x},N}}{h} + \frac{y_1 - y_N}{h^2}\right).$$

Using the algebraic inequality

$$\left(\sum_{k=1}^{p} a_k\right)^2 \le p\sum_{k=1}^{p} a_k^2,$$

we consider the third summand on the right hand side of equality (6.230):

$$\sum_{i=2}^{N-2} h y^2_{\bar{x}x\overset{\circ}{x}} = \frac{1}{4h^6}\sum_{i=2}^{N-2} h(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2})^2$$
$$\le \frac{1}{h^6}\sum_{i=2}^{N-2} h\left(y_{i+2}^2 + 4y_{i+1}^2 + 4y_{i-1}^2 + y_{i-2}^2\right).$$

Similarly,

$$\frac{h}{2}(l_0^2 + l_N^2) \le \frac{1}{h^6}\left[h\left(4y_1^2 + y_2^2 + y_{N-2}^2 + 4y_{N-1}^2\right)\right],$$
$$hl_1^2 \le \frac{1}{h^6}\left[h\left(2y_0^2 + 4y_2^2 + y_3^2 + y_{N-1}^2 + 2y_N^2\right)\right],$$
$$hl_{N-1}^2 \le \frac{1}{h^6}\left[h\left(2y_0^2 + y_1^2 + 4y_{N-2}^2 + y_{N-3}^2 + 2y_N^2\right)\right].$$

Summing the estimates obtained we arrive at the following relation:

$$\|A_2\| \le \sqrt{10}/h^3.$$

From the triangle inequality it follows that

$$\|A\| \le \varphi(h), \quad \varphi(h) = \frac{a}{h} + \frac{\beta\sqrt{10}}{h^3}. \tag{6.231}$$

The stability of the explicit three-level difference scheme (6.223)–(6.225) with respect to the initial data have been proved when $\tau\|A\| < 1$. Using the estimate (6.231) of the operator norm we obtain the equivalent requirement

$$\tau \le \tau_k, \quad \tau_k = \frac{h^3}{\beta\sqrt{10} + ah^2}. \tag{6.232}$$

In other words, the explicit scheme (6.223), (6.224) is conditionally stable. Inequality (6.232) is often called *'Courant's criterion of stability'* and the quantity τ_k is *'Courant's number'*.

We show now that the solution of the difference scheme (6.225) satisfies a grid analogue of the integral conservation law (6.206):

$$E_{2_h}(t) = E_{2_h}(0), \quad t \in \omega_\tau, \tag{6.233}$$

where $E_{2_h}(t) = (y(t+\tau), y(t))$, $y(t) \in \Omega_h$. Indeed, taking the dot product of equation (6.225) with $2\tau y$ and using the identities

$$2\tau \left(y_{\overset{\circ}{t}}, y\right) = E_{2_h}(t_n) - E_{2_h}(t_{n-1}), \quad 2\tau(Ay, y) = 0,$$

we obtain equality (6.233).

5.5 Schemes with Weighting Factors

In the case of the model equation consider a multi-parameter set of schemes

$$y_{\overset{\circ}{t}} + Ay^{(\sigma_1,\sigma_2)} = 0, \quad t \in \omega_\tau, \quad y_0 = u_0, \quad y_{t,0} = \overline{u}_0, \tag{6.234}$$

where σ_1 and σ_2 are real parameters, and

$$y^{(\sigma_1,\sigma_2)} = \sigma_1 \hat{y} + (1 - \sigma_1 - \sigma_2)y + \sigma_2 \check{y}.$$

THEOREM 6.9 *Let for the scheme (6.234) $A \neq A(t)$,*

$$\sigma_1 \geq \sigma_2, \quad \sigma_1 + \sigma_2 \geq 1, \quad A = -A^*, \quad A : \Omega_h \to \Omega_h. \tag{6.235}$$

Then the difference scheme is stable and the following inequality is valid:

$$\|y_{n+1}\|_1^2 + \sum_{k=1}^{n} 2\tau^2(\sigma_1 - \sigma_2)\|y_{\overset{\circ}{t},k}\|^2 = \|y_1\|_1^2, \tag{6.236}$$

where

$$\|y\|_1^2 = 0.5\left(\|y\|^2 + \|\check{y}\|^2\right) + 0.5\tau^2(\sigma_1 + \sigma_2 - 1)\|y_{\bar{t}}\|^2,$$

$$y_{\overset{\circ}{t}} = 0.5(y_t + y_{\bar{t}}) = (y_{n+1} - y_{n-1})/(2\tau), \quad y_{\bar{t}} = (y_n - y_{n-1})/\tau.$$

Proof. Since A is a skew-symmetrical operator and $y^{(\sigma_1,\sigma_2)} \in \Omega_h$, from the inner product of equation (6.234) with $2\tau y^{(\sigma_1,\sigma_2)}$ we have

$$2\tau \left(y_{\overset{\circ}{t}}, y^{(\sigma_1,\sigma_2)}\right) = 0. \tag{6.237}$$

Substituting the equality (6.237) into the identity

$$y^{(\sigma_1,\sigma_2)} = \frac{\hat{y}+\check{y}}{2} + \tau(\sigma_1-\sigma_2)y_{\overset{\circ}{t}} + \frac{\tau^2}{2}(\sigma_1+\sigma_2-1)y_{\bar{t}t}, \qquad (6.238)$$

we obtain the relation

$$\|y_{n+1}\|_1^2 + 2\tau^2(\sigma_1-\sigma_2)\|y_{\overset{\circ}{t}}\|^2 = \|y_n\|_1^2.$$

This completes the proof of the statement.

Thus, we have proved the absolute stability of scheme (6.234) (i.e., without restrictions on the relations between the steps τ and h), provided that the weighting factors σ_1, σ_2 satisfy conditions (6.235).

THEOREM 6.10 *Let $\sigma_1 = \sigma_2 = 0.25$. Then scheme (6.234) is L_2-conservative and the following energy identity is valid:*

$$\left\|\frac{y_n+y_{n+1}}{2}\right\| = \left\|\frac{y_0+y_1}{2}\right\|. \qquad (6.239)$$

P r o o f. Using the identity

$$y^{(\sigma_1,\sigma_2)} = y + \tau(\sigma_1-\sigma_2)y_{\overset{\circ}{t}} + 0.5\tau^2(\sigma_1+\sigma_2)y_{\bar{t}t},$$

the three-level difference scheme (6.234) can be transformed into the following two-level one:

$$v_t + Av^{(0.5)} = 0, \quad v_0 = 0.5(y_0+y_1), \qquad (6.240)$$

where $v_n = (y_n + y_{n+1})/2$.

Taking the dot product of equation (6.240) with $2\tau v^{(0.5)}$ in Ω_h, and taking into account the equalities

$$2\tau\left(v_t, v^{(0.5)}\right) = \|\hat{v}\|^2 - \|v\|^2, \quad \left(Av^{(0.5)}, v^{(0.5)}\right) = 0,$$

we arrive at the discrete analog (6.239) of the integral conservation law (6.206). It is valid for the grid solution on the half-integer level $y_{n+1/2} = 0.5(y_n + y_{n+1})$.

REMARK 6.1 Scheme (6.234) with any σ_1, σ_2 is conservative with regard to the solution $y_{n+1/2}$ too.

Indeed, summing scheme (6.234) for every $x \in \omega_h$ and taking into account that $y \in \Omega_h$ we obtain

$$\left(\frac{y_n+y_{n+1}}{2}, 1\right) = \left(\frac{y_0+y_1}{2}, 1\right).$$

5.6 Nonlinear Schemes

In this Section completely conservative (conservative and L_2-conservative) difference schemes with weighting factors are constructed. These schemes preserve the main properties of a differential problem. Corresponding estimates for a nonlinear case are also obtained.

A difference scheme must reflect the basic properties of a continuous medium. Therefore it is natural to expect that a scheme satisfies the difference analogs of conservative laws (6.205), (6.206). This kind of schemes are customarily called conservative. Importance of the requirement that the schemes be conservative was noted in the 50s by A.N. Tikhonov and A.A. Samarskii [Tikhonov and Samarskii, 1959]. They formulated the integro-interpolational method (IIM) for constructing conservative difference schemes and constructed an example of a nonconservative scheme that had the second order of accuracy for the smooth enough coefficients and diverging for discontinuous ones [Tikhonov and Samarskii, 1961].

To construct a conservative difference scheme for the nonlinear KdV equation, we, along with IIM, use Steklov's averaging of the function u^2 [Abrashin, 1986, Samarskii et al., 1997b]:

$$u^2 \sim \frac{1}{u_+ - u}\int_u^{u_+} u^2 du = \frac{1}{3}\frac{(u^3)_x}{u_x} = \frac{1}{3}(u_+^2 + uu_+ + u^2), \qquad (6.241)$$

where $u_\pm = u(x_{i\pm 1}, t)$, $t \in \omega_\tau$. Applying formula (6.241) we construct an explicit three-level scheme of the second order of approximation

$$y_{\overset{\circ}{t}} + \frac{1}{6}\left(\frac{(y^3)_{\bar{x}}}{y_{\bar{x}}}\right)_x + \beta y_{\bar{x}x\overset{\circ}{x}} = 0, \qquad (6.242)$$

Algebraically it is equivalent to the grid equation [Zabusky and Kruskal, 1965]

$$y_{\overset{\circ}{t}} + \bar{y}y_{\overset{\circ}{x}} + \beta y_{\bar{x}x\overset{\circ}{x}} = 0, \quad i = \overline{0, N}. \qquad (6.243)$$

The initial and periodic conditions are approximated exactly:

$$y(x,0) = u_0(x), \quad x \in \overline{\omega}_h, \quad \hat{y}_{i+N} = \hat{y}_i, \qquad (6.244)$$

where $\overline{y} = (y_+ + y + y_-)/3$. We note that by the virtue of the divergence of the difference equation (6.242) for any $n = 0, 1, ..., N_0 - 1$

$$\sum_{x\in\overline{\omega}_h} h\frac{y_n + y_{n+1}}{2} = \sum_{x\in\overline{\omega}_h} h\frac{y_0 + y_1}{2},$$

i.e., the difference scheme (6.243) is conservative.

THEOREM 6.11 *The non-linear difference scheme (6.243) is L_2-conservative and it is solution satisfies the grid analog (6.233) of the integral conservation law (6.206):*

$$E_{2_h}(t) = E_{2_h}(0), \quad t \in \omega_\tau. \tag{6.245}$$

P r o o f. We rewrite scheme (6.243) in the operator form

$$y_{\overset{\circ}{t}} + Ay = 0, \tag{6.246}$$

where the nonlinear operator

$$Ay = \overline{y}A_1 y + \beta A_2 y$$

is defined by formulae (6.216), (6.217). By analogy with the differential problem (see equalities (6.204)) we show that

$$(Ay, y) = 0, \tag{6.247}$$

provided that $y \in \Omega_h$. It is enough to prove that $(\overline{y}A_1 y, y) = 0$. Using the identity

$$\overline{y}y_{\overset{\circ}{x}} = \frac{1}{3}yy_{\overset{\circ}{x}} + \frac{1}{3}(y^2)_{\overset{\circ}{x}}$$

and Lemma 6.8, we obtain

$$(\overline{y}A_1 y, y) = (\overline{y}y_{\overset{\circ}{x}}, y) = \frac{1}{3}(yy_{\overset{\circ}{x}}, y) - \frac{1}{3}(y^2, y_{\overset{\circ}{x}}) = 0.$$

Hence equality (6.247) is satisfied. Now taking in Ω_h the dot product of equation (6.246) with $2\tau y$, and using the property (6.247), we obtain the grid analog of the integral conservation law (6.245).

REMARK 6.2 The drawback of the grid law (6.245) is that the expression of E_{2_h} does not represent any grid norm that approximates $\int\limits_0^l u^2(x,t)dx$. In this sense, the relation (6.245) cannot be called the *a priori* estimate.

We also note that the expressions $E_{2_h}(t)$, $t \in \omega_\tau$, can be rewritten in the form

$$(\hat{y}, y) = \frac{1}{2}\left(\|\hat{y}\|^2 + \|y\|^2\right) - \frac{\tau^2}{2}\|y_t\|^2,$$

$$(\hat{y}, y) = \left\|\frac{\hat{y}+y}{2}\right\|^2 - \frac{\tau^2}{4}\|y_t\|^2.$$

5.7 Implicit Conservative Schemes

We consider the following class of three-level schemes with weighting factors:

$$y_{\overset{\circ}{t}} + \overline{y}^{(\sigma,\sigma)} y_{\overset{\circ}{x}}^{(\sigma,\sigma)} + \beta y_{\bar{x}x\overset{\circ}{x}}^{(\sigma,\sigma)} = 0. \tag{6.248}$$

The grid equation (6.248) approximates the initial one with the order $O(h^2 + \tau^2)$.

THEOREM 6.12 *Difference scheme (6.248) with spatially periodic solutions is conservative, L_2-conservative and for its solution there are the following grid analogs of differential conservation laws (6.205), (6.206):*

$$E_{k_h}^{(\sigma)}(t) = E_{k_h}^{(\sigma)}(0), \quad k = 1, 2, 3, \quad t \in \omega_\tau, \tag{6.249}$$

where

$$E_{2_h}^{(\sigma)} = E_{3_h}^{(\sigma)},$$

$$E_{1_h}^{(\sigma)} = \left(\frac{\hat{y} + y}{2}, 1\right), \; E_{2_h}^{(\sigma)} = \frac{\|\hat{y}\|^2 + \|y\|^2}{2} + \tau^2 \left(\sigma - \frac{1}{2}\right) \|y_t\|^2, \tag{6.250}$$

$$E_{3_h}^{(\sigma)} = \left\|\frac{\hat{y} + y}{2}\right\|^2 + \tau^2 \left(\sigma - \frac{1}{4}\right) \|y_t\|^2. \tag{6.251}$$

Proof. Just as in the proof of Theorem 6.11 one has to take the dot product of the difference equation (6.248) with $2\tau y^{(\sigma,\sigma)} = 2\tau(y + \sigma\tau^2 y_{\bar{t}t})$. Then, the L_2-conservative nature of the scheme follows from formulae (6.249)–(6.251) at $\sigma = 1/2$ and $\sigma = 1/4$, respectively

It is interesting to note that energy relations (6.249) are also *a priori* estimate for $\sigma \geq 1/2$ and $\sigma \geq 1/4$, respectively. Equality (6.249) for $k = 3$ can be interpreted as the *a priori* estimate for difference equation at the half-integer nodes $y_{n+1/2} = (y_n + y_{n+1})/2$.

Using the representation of convective terms in a divergent and a non-divergent form (see [Vabishchevich and Samarskii, 1998]), one can construct the following class of L_2-conservative difference schemes:

$$y_{\overset{\circ}{t}} + \frac{1}{3}\left(y y_{\overset{\circ}{x}}^{(\sigma_1,\sigma_2)} + \left(y y^{(\sigma_1,\sigma_2)}\right)_{\overset{\circ}{x}}\right) + \beta y_{\bar{x}x\overset{\circ}{x}}^{(\sigma_1,\sigma_2)} = 0. \tag{6.252}$$

This is now already the nonlinear scheme. For $\sigma_1 = \sigma_2 = 0$ it coincides with the previously investigated explicit scheme (6.243). Using the identity (6.219) we have

$$\left(y y_{\overset{\circ}{x}}^{(\sigma_1,\sigma_2)} + \left(y y^{(\sigma_1,\sigma_2)}\right)_{\overset{\circ}{x}}, y^{(\sigma_1,\sigma_2)}\right) = 0.$$

Hence for the solution of the difference scheme (6.252), (6.224) Theorem 6.9 with $\sigma_1 \geq \sigma_2$, $\sigma_1 + \sigma_2 \geq 1$ gives the *a priori* estimate (6.236). In accordance with Theorem 6.12, scheme (6.252) with $\sigma_1 = \sigma_2 = 0.5$ is L_2-conservative with respect to functional (6.249). For $\sigma_1 = \sigma_2 = 0.25$ the solution at a half-integer level satisfies the conservation law

$$E_{3_h}^{(0.25)}(t) = E_{3_h}^{(0.25)}(0), \quad t \in \omega_\tau.$$

REMARK 6.3 Unfortunately the scheme (6.252) has lost its conservative nature. It can be considered as a linearization method

$$\overset{k+1}{v}_{\overset{\circ}{t}} + \frac{1}{3}\left(\overset{k}{v}\overset{k+1}{v}_{\overset{\circ}{x}} + \left(\overset{k}{v}\overset{k+1}{v}\right)_{\overset{\circ}{x}}\right) + \beta\overset{k+1}{v}_{\bar{x}x\overset{\circ}{x}} = 0 \qquad (6.253)$$

of the nonlinear scheme

$$y_{\overset{\circ}{t}} + \overline{y}^{(\sigma_1,\sigma_2)}y_{\overset{\circ}{x}}^{(\sigma_1,\sigma_2)} + \beta y_{\bar{x}x\overset{\circ}{x}}^{(\sigma_1,\sigma_2)} = 0, \qquad (6.254)$$

with

$$\overset{k+1}{y}_{\overset{\circ}{t}} = \left(\overset{k+1}{y} - \overset{\vee}{y}\right)\Big/(2\tau), \quad v = y^{(\sigma_1,\sigma_2)}, \quad \overset{k+1}{v}_{\overset{\circ}{x}} = \left(\overset{k+1}{v}_+ - \overset{k+1}{v}_-\right)\Big/(2h).$$

If the iterative process converges $\left(\lim\limits_{k\to\infty} \overset{k}{y} = \hat{y}\right)$ then it gives the solution of the conservative and L_2-conservative scheme (6.248).

6. A Boundary Value Problem for a Hyperbolic Equation of the Second Order

Difference schemes for a wave equation are constructed and investigated. In particular, the convergence of difference schemes in a class of piecewise smooth solutions is considered.

6.1 Difference-Differential Problem

We consider the first boundary value problem for the *hyperbolic equation*

$$\frac{\partial^2 u}{\partial t^2} = Lu + f(x,t), \; Lu = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right), \; 0<x<l, \; 0<t\le T, \qquad (6.255)$$

$$u(0,t) = u(l,t) = 0, \quad 0 < t \le T, \qquad (6.256)$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \overline{u}_0(x), \qquad (6.257)$$

$$0 < c_1 \le k(x,t) \le c_2, \quad (x,t) \in \overline{Q}_T; \quad c_1,\, c_2 = \text{const}, \qquad (6.258)$$

in the rectangle

$$\overline{Q}_T = \overline{\Omega} \times [0,T], \quad \overline{\Omega} = \{\, x : \; 0 \le x \le l\}, \quad 0 \le t \le T. \qquad (6.259)$$

Since the solution to be found is continuous, the corresponding conditions for the coordination of the solution are assumed to be satisfied. In particular,

$$u_0(x) = \overline{u}_0(x) = 0, \quad x \in \Gamma; \quad f(x,t) = 0, \quad x \in \Gamma_T, \tag{6.260}$$

where Γ is a boundary of Ω and Γ_T is a lateral surface of Q_T.

On the uniform spatial grid $\overline{\omega}_h$ we replace the initial problem by the difference-differential one:

$$\frac{d^2v}{dt^2} + Av = \varphi(t), \quad v_0 = u_0, \quad \frac{dv(0)}{dt} = \overline{u}_0. \tag{6.261}$$

In accordance with formulae (6.18), (6.19) we have

$$\begin{aligned} (Av)_i = -(av_{\bar{x}})_{x,i}, \quad i = 1,2,...,N-1, \quad v_0 = v_N = 0, \\ \varphi = (\varphi_1(t), \varphi_2(t), ..., \varphi_{N-1}(t))^T, \quad \varphi_i(t) = f(x_i,t), \end{aligned} \tag{6.262}$$

where $a = a(x)$, $x \in \omega_h^+$ is the standard stencil functional defined by one of the formulae (6.15)–(6.17); $A : H \to H$, where the linear space $H = \Omega_h$ consists of a set of the vectors of the form $(v_1(t), v_2(t), ..., v_{N-1}(t))^T$. The inner product and norm in H are specified in the usual fashion:

$$(y,v) = \sum_{i=1}^{N-1} y_i v_i h, \quad \|y\| = \sqrt{(y,y)}.$$

We note that according the condition (6.14) the difference operator approximates the initial differential operator with respect to space with the second order $O(h^2)$ at every inner node $x \in \omega_h$.

6.2 Difference Schemes

Over the segment $[0,T]$ we introduce the temporarily uniform grid:

$$\overline{\omega}_\tau = \{\ t_n = n\tau, \quad n = 0,1,...,N_0; \quad \tau N_0 = T\ \} = \omega_\tau \cup \{T\}. \tag{6.263}$$

For numerical solution of the difference–differential problem (6.260), we consider a class of difference schemes with the weighting factors

$$y_{\bar{t}t} + Ay^{(\sigma,\sigma)} = \varphi, \quad t \in \omega_\tau, \quad y_0 = u_0, \quad y_1 = u_1, \tag{6.264}$$

where

$$A^*(t) = A(t) > 0, \quad t \in \omega_\tau,$$

$$y^{(\sigma,\sigma)} = \sigma\hat{y} + (1-2\sigma)y + \sigma\check{y},$$

$$y_1 = y_0 + \tau\widetilde{u}_0, \quad \widetilde{u}_0 = \{u_1, ..., u_{N-1}\},$$

$$\widetilde{u}_0(x) = \overline{u}_0(x) + 0.5\tau(Lu_0 + f(x,0)), \quad x \in \overline{\omega}_h.$$

From Taylor's formula it follows that

$$\widetilde{u}_0(x) - u_t(x,0) = O(\tau^2). \tag{6.265}$$

Thus the operator-difference problem (6.264) is formulated.

We represent it in the canonical form

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad y_0 = u_0, \quad y_1 = u_1, \tag{6.266}$$

with

$$D = E + \tau^2\sigma A, \quad B = 0, \quad A^*(t) = A(t) > 0. \tag{6.267}$$

We consider the conditions of the stability of the difference scheme with respect to the initial data. To simplify our investigations we assume that the coefficient k is independent of t, or that the operator A is a constant operator. We prove the stability of the explicit difference scheme. From (6.264) with $\sigma = 0$ it follows that

$$y_{\bar{t}t} + Ay = 0, \quad y_0 = u_0, \quad y_1 = u_1. \tag{6.268}$$

THEOREM 6.13 *Let the coefficient k be independent of t ($k = k(x)$). If*

$$\tau \le h\sqrt{c_2} \tag{6.269}$$

then the explicit difference scheme (6.268) is stable in H_A and the following energy identity is valid:

$$\left\|\frac{y_n + y_{n-1}}{2}\right\|_A^2 + \|y_{\bar{t},n}\|^2_{E-\frac{\tau^2}{4}A} = \left\|\frac{y_1 + y_0}{2}\right\|_A^2 + \|y_{\bar{t},1}\|^2_{E-\frac{\tau^2}{4}A} \tag{6.270}$$

Proof. To apply Theorem 4.1 one has to show (see inequalities (4.2.1)) that the condition

$$D > \frac{\tau^2}{4}A \tag{6.271}$$

is valid. Since we have

$$D = E \ge \frac{A}{\|A\|}, \quad \|A\| < \frac{4c_2}{h^2},$$

it is obvious that the condition (6.271) holds, provided that $\tau \le h/\sqrt{c_2}$. The necessary and sufficient conditions of the stability of scheme (6.268) with respect to the initial data are also satisfied, since in the scheme (6.268) the operator $B \equiv 0$. Thus the inequality (4.2.11) yields the required relation (6.270).

REMARK 6.4 For a more restrictive constraint on time step τ of the form

$$\tau \leq \frac{\sqrt{2}}{\sqrt{c_2}} h, \tag{6.272}$$

we obtain $D = E > \frac{\tau^2}{2} A$. The following energy identity

$$E(t) = E(0), \quad t \in \omega_\tau, \tag{6.273}$$

with

$$E(t) = \frac{1}{2} \left(\|y\|_A^2 + \|\hat{y}\|_A^2 \right) + \|y_t\|^2_{E - \frac{\tau^2}{2} A}, \tag{6.274}$$

is valid.

The conservation law (6.273) follows from the equality (6.270) and from the identity

$$E(t_n) = \left\| \frac{y_n + y_{n+1}}{2} \right\|_A^2 + \|y_{t,n}\|^2_{E - \frac{\tau^2}{4} A}. \tag{6.275}$$

We pass to studying the problem of stability with respect to the right hand side. Up to now we have assumed for simplicity that the coefficient k depends only on x. We consider now the case of the coefficient $k = k(x,t)$ being Lipschitz continuous with respect to t, i.e.,

$$|k(x,t) - k(x,t-\tau)| \leq \tau c k(x, t-\tau). \tag{6.276}$$

We specify the value of the stencil functional a:

$$a = k(x_{i-1/2}, t). \tag{6.277}$$

We show that in conditions (6.276) the operator $A = A(t)$ is Lipschitz continuous with respect to the variable t. Indeed, using the definition of the operator A and taking into account Green's formula, we have

$$\begin{aligned} ((A(t) - A(t-\tau))y, y) &= ((k - \check{k}), y_{\bar{x}}^2] \\ &\leq \tau c(\check{a}, y_{\bar{x}}^2] = \tau c (A(t-\tau)y, y). \end{aligned} \tag{6.278}$$

We use now Theorem 4.24. It is obvious that conditions (4.4.36) for the scheme with weighting factors (6.264) are satisfied when

$$\sigma \geq \frac{1+\varepsilon}{4} - \frac{h^2}{\tau^2 c_2}, \quad \varepsilon > 0 \text{is any number.} \tag{6.279}$$

Then the *a priori* estimate (4.4.37) yields the inequality

$$\|y_{n+1}\|_{A_n} \leq M_1\sqrt{\frac{1+\varepsilon}{\varepsilon}}\left(\|y_0\|_{A_1} + \|y_{t,0}\|_{E+\sigma\tau^2 A} + \sum_{k=1}^{n} \tau\|\varphi_k\|\right). \quad (6.280)$$

REMARK 6.5 It follows from inequality (6.279) that if

$$\tau \leq 2h/\sqrt{(1+\varepsilon)c_2}, \quad \varepsilon > 0, \quad (6.281)$$

the explicit difference scheme ($\sigma = 0$) is stable with respect to the initial data, the right hand side in the Hilbert space H_A, and that for its solution the *a priori* estimate (6.280) is valid.

6.3 An Approximation Error and Convergence

Assuming $y = z + u$ in the operator-difference scheme (6.264) we obtain the problem for the error

$$(E + \sigma\tau^2 A)z_{\bar{t}t} + Az = \psi(t), \quad t \in \omega_\tau, \quad (6.282)$$

$$z_0 = 0, \quad z_{t,0} = \nu, \quad (6.283)$$

where

$$\psi_i^n = -u_{\bar{t}t,i} + \left(au_{\bar{x}}^{(\sigma,\sigma)}\right)_{x,i}, \quad \nu_i = \widetilde{u}_0(x_i) - u_t(x_i, 0), \ i = 1, 2, ..., N-1,$$

are the approximation errors for the difference scheme (6.264) and for the second initial condition on the solution of the differential problem (6.255)–(6.258). Everywhere below we assume that there are all necessary bounded derivatives with respect to the variables x and t. Then from formulae (6.265) we have

$$\begin{aligned} v^{(\sigma,\sigma)} &= v + \sigma\tau^2 v_{\bar{t}t}, \\ u_{\bar{t}t,i} &= \frac{\partial^2 u(x_i, t_n)}{\partial t^2} + O(\tau^2), \\ (au_{\bar{x}})_{x,i} &= (Lu)_i^n + O(h^2) \end{aligned}$$

and hence the inequalities

$$\sum_{k=1}^{n} \tau\|\psi_k\|? \leq t_n l^{1/2} \max_{1\leq i\leq N-1} |\psi_i| \leq c\,(h^2 + \tau^2), \quad (6.284)$$

$$\|z_{t,0}\|_{E+\sigma\tau^2 A}? \leq c\,(h^2 + \tau^2) \quad (6.285)$$

hold. Here and hereafter c is a constant independent of h, τ and of the approximative solution.

Now using the *a priori* estimate (6.280) for a solution of the problem (6.282), (6.283) we find that

$$\max_{t\in\omega_\tau} \|z(t)\|_{A(t-\tau)} \le M_1\sqrt{\frac{1+\varepsilon}{\varepsilon}}\left(\|z_{t,0}\|_{E+\sigma\tau^2 A} + \sum_{k=1}^{n} \tau\|\psi_k\|\right).$$

Substituting the estimates (6.284), (6.285) into the latter inequality one can see that the difference scheme with weighting factors (6.264) converges as $O(h^2+\tau^2)$. Thus for the solution of problem (6.282), (6.283) the estimate

$$\max_{t\in\omega_\tau} \|z(t)\|_{A(t-\tau)} = \max_{t\in\omega_\tau} \|y(t)-u(t)\|_{A(t-\tau)} \le c\,(h^2+\tau^2) \qquad (6.286)$$

is valid. From Lemma 6.1 and from (6.286) a convergence follows in a uniform norm too.

6.4 Problems with Piecewise Smooth Solutions

Many problems of mathematical physics that describe shock processes in gases, liquids, and solids lead to the problem of seeking non-smooth solutions for hyperbolic type equations of the second order. The simplest representative of this class of equations is the following equation:

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) + f(x,t), \quad (x,t)\in Q_T, \qquad (6.287)$$

with a given boundary and initial conditions

$$u(0,t) = u(l,t) = 0, \quad t>0, \qquad (6.288)$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \overline{u}_0(x), \quad x\in[0,l]. \qquad (6.289)$$

For the simplicity of further representation we assume that k and ρ are positive constants and that $k/\rho = a^2$. Below we restrict our consideration to piecewise smooth solutions, i.e., we assume that the first derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}$ and the initial data of the problem $f(x,t)$, $\overline{u}_0(x)$ suffer discontinuous of the first kind such that in the neighbourhood of the finite number of discontinuity lines $x_k = \nu_k(t)$, $k=1,2,...,m$, and outside these lines they are sufficiently smooth functions; $u_0(x)$ is also piecewise smooth function. These lines can be represented only by the characteristics $x\pm at = \text{const}$, along which the solutions are weakly discontinuous [Tikhonov and Samarskii, 1972].

As a typical example the following model problem can be considered [Moskal'kov, 1974]:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad 0 < t \le 0.5,$$

$$u(0,t) = t, \quad u(1,t) = 1 - t, \quad u(x,0) = x, \quad \frac{\partial u}{\partial t}(x,0) = -1;$$

$u = |x - t|$ is the exact (generalized) solution of this problem. In the vicinity of the characteristic $x = t$ the first derivatives of the solution are discontinuous.

For the above described class of piecewise smooth solutions that satisfy problem (6.287), (6.288), (6.289) in the entire domain Q_T, in the generalized sense the following integral conservation law holds [Tikhonov and Samarskii, 1972, Godunov, 1971]:

$$\oint_C \left(\frac{\partial u}{\partial t} dx + a^2 \frac{\partial u}{\partial x} dt \right) = \iint_G f dx dt, \tag{6.290}$$

where $G \subset \overline{Q}_T$ is the arbitrary domain bounded by the piecewise smooth curve C.

In particular, from relation (6.290) it follows that the problem stated above can be formulated in the following form. We are to find the function $u(x,t)$ continuous in $\overline{Q}_T$ and satisfying the differential equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad (x,t) \in Q_T, \quad x \ne \nu_k(t) \tag{6.291}$$

and initial and boundary conditions (6.288), (6.289). On the lines of weak discontinuity $x_k = \nu_k(t)$, $k = 1, 2, ..., m$, the following conjugation conditions must be satisfied:

$$[u] = 0, \quad \left[\frac{\partial u}{\partial t} \right] = \pm a \left[\frac{\partial u}{\partial x} \right], \tag{6.292}$$

where $[g] = g(\nu + 0) - g(\nu - 0)$ for $\nu = x \pm at$.

We introduce the uniform grids

$$\overline{\omega}_h = \{x_i = ih, \ i = 0, 1, ..., N; \ h = l/N\}$$

and

$$\overline{\omega}_\tau = \{t_n = n\tau, \ n = 0, 1, ..., N_0; \ \tau = T/N_0\}.$$

We substitute the differential problem by the difference one:

$$y_{\bar{t}t} = \Lambda y^{(\sigma_1, \sigma_2)} + \varphi, \quad x \in \omega_h, \quad t \in \omega_\tau, \quad t > 0, \tag{6.293}$$

$$y_0 = y_N = 0, \quad t \geq \tau, \tag{6.294}$$

$$y^0 = u_0(x), \quad x \in \overline{\omega}_h, \tag{6.295}$$

$$\left(E - \tau^2 \frac{\sigma_1 + \sigma_2}{2} \Lambda\right) y_t(x, 0) = \overline{u}_0(x) + \frac{\tau}{2}[\Lambda u_0 + f(x, 0)], \quad x \in \omega_h. \tag{6.296}$$

Here $\Lambda y = (a y_{\bar{x}})_x$, and the right hand side is to be selected, for instance, as follows:

$$\varphi = \frac{1}{\tau h} \int\limits_{x-\frac{h}{2}}^{x+\frac{h}{2}} \int\limits_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} f(\xi, \eta) d\xi d\eta.$$

In the space of grid functions $\overset{\circ}{\Omega}_h$ vanishing for $i = 0, N$ we introduce the operator $A = -\Lambda$. As shown above, $A^* = A > 0$.

We cite some *a priori* estimates for the difference solution. We reduce scheme (6.293)–(6.296) to the canonical form of the three-level difference schemes:

$$D y_{\bar{t}t} + B y_{\overset{\circ}{t}} + A y = \varphi, \quad y_0 = u_0, \quad D y_{t,0} = \nu. \tag{6.297}$$

Furthermore, investigating the convergence rate of the difference scheme (6.293)–(6.296) that approximates a differential problem with piecewise smooth solutions, we need an *a priori* estimate in a negative norm both with respect to space and time [Moskal'kov, 1974].

THEOREM 6.14 *Assume the operators of scheme (6.297) independent of n and satisfy the conditions*

$$A^* = A > 0, \quad D^* = D \geq \beta E, \quad \beta > 0, \tag{6.298}$$

$$B \geq 0, \quad B D^{-1} A \geq 0, \quad D \geq \frac{1+\varepsilon}{4} \tau^2 A, \quad \varepsilon > 0. \tag{6.299}$$

Then for the solution of problem (6.297) the estimate

$$\|y(t)\| \leq M \left\{ \|y(0)\|_D + \|D y_t(0)\|_{A^{-1}} + \|\varphi_1\|_{0,-1} + \|\varphi_2\|_{A^{-1},0} \right\}, \tag{6.300}$$

is valid. Here

$$M = \max\left\{ \frac{2(1+\varepsilon)}{\varepsilon\beta} e^{T/2}, \sqrt{\frac{1+\varepsilon}{\varepsilon}} \right\}, \quad \varphi_1 = \varphi_2 = \varphi$$

and

$$\|y(t)\|_{A^{-1},0} = \sum_{t'=0}^{T} \tau \|y(t')\|_{A^{-1}}, \quad \|y(t)\|^2_{0,-1} = \sum_{t'=\tau}^{T} \tau \|\xi(t')\|^2, \tag{6.301}$$

$$y(t) = \xi_{\bar{t}}, \quad \xi(0) = 0.$$

Proof. Since the scheme (6.297) is linear, it is sufficient to consider three problems for which: a) $\varphi = 0$ (stability with respect to initial data); b) $\varphi = \varphi_1$, $y_0 = y_1 = 0$; c) $\varphi = \varphi_2$, $y_0 = y_1 = 0$.

Since conditions (4.3.29), (4.3.30) are satisfied the validity of the estimate (6.300) for the problems with the conditions (b) follows from Theorem 4.13. Indeed, the inequality (6.299) can be rewritten in the equivalent form

$$(1+\varepsilon)\left(D - \frac{\tau^2}{4}A\right) \geq \varepsilon D. \tag{6.302}$$

From the conditions (6.298) we have $D \geq \beta E$. Thus from the relation (6.302) the inequality

$$D - \frac{\tau^2}{4}A \geq 2\delta E, \quad \delta = \frac{\beta\varepsilon}{2(1+\varepsilon)}$$

follows.

Since for $B \geq 0$ the conditions of Theorem 4.4 are satisfied, from the *a priori* estimate (4.2.37) with $\rho = 1$ it follows that

$$\|y_{n+1}\|_A \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}}\left(\|y_0\|_A + \|y_{t,0}\|_D\right). \tag{6.303}$$

To apply the above estimate to the problem with the condition (a) we use substitution $w = D^{1/2}y$ and multiply the resulting equation by $D^{1/2}A^{-1}$ from the left. As a result we transform the operator scheme (6.297) to the following form:

$$\begin{gathered} \widetilde{D}w_{\bar{t}t} + \widetilde{B}w_{\overset{\circ}{t}} + \widetilde{A}w = \widetilde{\varphi}, \\ w_0 = D^{1/2}y_0, \quad w_{t,0} = D^{1/2}y_{t,0}, \end{gathered} \tag{6.304}$$

where

$$\begin{gathered} \widetilde{D} = D^{1/2}A^{-1}D^{1/2}, \quad \widetilde{B} = D^{1/2}A^{-1}BD^{-1/2}, \\ \widetilde{A} = E, \quad \widetilde{\varphi} = D^{1/2}A^{-1}\varphi. \end{gathered} \tag{6.305}$$

Now applying the estimate (6.303) to problem (6.304) we obtain

$$\|w(t)\| \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}}\left(\|w_0\| + \|w_{t,0}\|_{\widetilde{D}}\right). \tag{6.306}$$

Since $\|w\| = \|y\|_D$ and

$$\|w_t\|^2_{\widetilde{D}} = \left(D^{1/2}A^{-1}D^{1/2}w_t, w_t\right) = \|D^{1/2}w_t\|^2_{A^{-1}} = \|y_t\|^2_{A^{-1}},$$

then the required estimate for $\varphi = 0$ follows from inequality (6.306).

Similarly, from Theorem 4.15 we have

$$\|y(t)\|_D \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \sum_{t'=\tau}^{t-\tau} \tau \|\varphi_2(t')\|_{A^{-1}}.$$

Applying the triangle inequality one can ensure the validity of estimate (6.300).

REMARK 6.6 In the case of the commutative operators A and D, the condition $BD^{-1}A \geq 0$ is satisfied for $A > 0$, $D > 0$, $B \geq 0$.

For the error $z = y - u$ we have a problem

$$z_{\bar{t}t} = \Lambda z^{(\sigma_1,\sigma_2)} + \psi, \tag{6.307}$$

$$z_0 = z_N = 0, \tag{6.308}$$

$$z^0 = 0, \quad \left(E - \tau^2 \frac{\sigma_1 + \sigma_2}{2} \Lambda\right) z_t(x, 0) = \psi^0, \tag{6.309}$$

where $\psi = \psi^n = -u_{\bar{t}t} + \Lambda u^{(\sigma_1,\sigma_2)} + \varphi$, $n = 1, 2, ..., N_0 - 1$, is an error of approximation for the difference equation (6.293) on the solution of problem (6.287)–(6.289), and

$$\psi^0 = \overline{u}_0 - \left(E - \tau^2 \frac{\sigma_1 + \sigma_2}{2} \Lambda\right) u_t(x, 0) + \frac{\tau}{2}[\Lambda u_0 + f(x, 0)] \tag{6.310}$$

is an error of approximation for the initial condition. Transforming problem (6.307)–(6.309) to the form of (6.297) one can see that

$$D = E + \tau^2 \frac{\sigma_1 + \sigma_2}{2} A, \ B = \tau(\sigma_1 - \sigma_2)A, \quad z_0 = 0, \ Dz_{t,0} = \psi^0. \tag{6.311}$$

Let us estimate ψ^n. Letting

$$G = G_{in} = \{\overline{x}_i \leq x \leq \overline{x}_{i+1}, \ \bar{t}_n \leq t \leq \bar{t}_{n+1}\},$$
$$i = \overline{1, N-1}; \ n = \overline{1, N_0 - 1},$$

in identity (6.290) and dividing by τh, after obvious transformations, we obtain

$$0 = \left[\frac{1}{h} \int_{\overline{x}_i}^{\overline{x}_{i+1}} \frac{\partial u}{\partial t}(x, \bar{t}) dx\right]_{\bar{t}} - \left[\frac{1}{\tau} \int_{\bar{t}_n}^{\bar{t}_{n+1}} a^2 \frac{\partial u}{\partial x}(\overline{x}_i, t) dt\right]_x - \frac{1}{h\tau} \iint_{G_{in}} f dx dt, \tag{6.312}$$

where $\overline{x}_i = x_i - h/2$, $\overline{t}_n = t_n - \tau/2$.

Add the expression for ψ^n to the equality (6.312) and represent the error of approximation in the form $\psi = \psi_1 + \psi_2$, where

$$\psi_1^n = \xi_{\overline{t}}, \quad \xi = \xi(x_i, t_n) = \frac{1}{h}\int\limits_{\overline{x}_i}^{\overline{x}_{i+1}} \frac{\partial u}{\partial t}(x, \overline{t}_{n+1})dx - u_t, \tag{6.313}$$

$$\psi_2^n = \eta_x, \quad \eta = \eta(x_i, t_n) = \frac{1}{\tau}\int\limits_{\overline{t}_n}^{\overline{t}_{n+1}} a^2\frac{\partial u}{\partial x}(\overline{x}_i, t)dt - a^2 u_{\overline{x}}^{(\sigma_1,\sigma_2)} \tag{6.314}$$

We assume that the solution satisfies the condition

$$\xi(x,0) = \frac{1}{h}\int\limits_{x-h/2}^{x+h/2} \frac{\partial u}{\partial t}(x, \tau/2)dx - u_{t,0} = 0, \quad x \in \omega_h. \tag{6.315}$$

We need the following:

LEMMA 6.9 *If $f(x)$ is a piecewise continuous function over the segment $[a,b]$, then*

$$\frac{1}{h}\int\limits_{x-h}^{x} f(\xi)d\xi = O(1), \quad x \in [a+h, b],$$

but if $f(x)$ is a smooth function then

$$\frac{1}{h}\int\limits_{x-h}^{x} f(\xi)d\xi = f\left(x - \frac{h}{2}\right) + \Phi(x), \quad \Phi(x) = O(h^2).$$

Using the Lemma we estimate

$$\xi = \frac{1}{h}\int\limits_{\overline{x}_i}^{\overline{x}_{i+1}} \frac{\partial u}{\partial t}(x, \overline{t}_{n+1})dx - \frac{1}{\tau}\int\limits_{\overline{t}_n}^{\overline{t}_{n+1}} \frac{\partial u}{\partial t}(x_i, t)dt.$$

At the points of the smoothness of integrand expressions we have $\xi = O(h^2 + +\tau^2)$.

If the line $x = \nu(t)$ of the discontinuity of the derivative $\dfrac{\partial u}{\partial t}$ intersects the rectangle $G_{i,n+1/2}$, then from the Lemma we have $\xi = O(1)$. The number of points for any fixed $t = t_n$, such that $\xi = O(1)$ in the vicinity

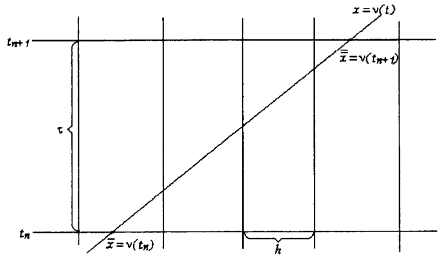

Figure 6.1.

of one of the lines of discontinuity of derivatives can be estimated as below (see. Fig. 6.1):

$$m_1 \leq \frac{\nu(t_{n+1}) - \nu(t_n)}{h} + 1 = a\frac{\tau}{h} + 1.$$

Since the number of the lines of the discontinuity of derivatives points is assumed to be finite, than the total number of the points ω_p on the level $t = t_n$, with $\xi = O(1)$ is equal in order to $O(h+\tau)/h$.

From the above we have

$$\begin{aligned}
\|\psi_1\|^2_{0,-1} &= \sum_{t'=\tau}^{T} \tau \sum_{x' \in \omega_p} h\xi^2(x',t') + \sum_{t'=\tau}^{T} \tau \sum_{x' \in \overline{\omega}_h \backslash \omega_p} h\xi^2(x't') \\
&= \sum_{t'=\tau}^{T} \tau h \frac{1}{h} O(\tau + h) + \sum_{t'=\tau}^{T} \tau \sum_{x' \in \overline{\omega}_h \backslash \omega_p} hO(h^2 + \tau^2)^2 \\
&= O(\tau + h).
\end{aligned}$$

Similarly, with the use of (6.59), (6.301) the term ψ_2 is estimated as:

$$\|\psi_2\|^2_{A^{-1},0} = O(\tau + h).$$

To estimate ψ^0 assume $\psi^0 = \psi^0_1 + \psi^0_2 + \psi^0_3 + \psi^0_4$, where

$$\psi^0_1 = \overline{u}_0(x) - u_t(x,0) = \frac{\partial u}{\partial t}(x,0) - \frac{1}{\tau}\int_0^{\tau} \frac{\partial u}{\partial t}(x,t)dt,$$

$$\psi^0_2 = -\tau\frac{\sigma_1 + \sigma_2}{2}Au(x,\tau), \quad \psi^0_3 = \tau\frac{\sigma_1 + \sigma_2 - 1}{2}Au(x,0),$$

$$\psi^0_4 = \frac{\tau}{2}f(x,0).$$

Just as in the estimation of ξ, we have $\psi_1^0 = O(1)$, provided that the discontinuity line for the derivative $\partial u/\partial t$ intersects the segment $0 \leq t \leq \tau$, $x = x_i$. But if $\left|\dfrac{\partial^2 u}{\partial t^2}\right| \leq M$ then $\psi_1^0 = O(\tau)$, and therefore,

$$\|\psi_1^0\|_{A^{-1},0} = O(\tau + h).$$

Now estimate

$$\|\psi_2^0\|_{A^{-1}}^2 = \tau^2 \left(\frac{\sigma_1 + \sigma_2}{2}\right)^2 (Au^1, u^1) = \tau^2 \left(\frac{\sigma_1 + \sigma_2}{2}\right)^2 \sum_{x=h}^{l} ha^2 (u_{\bar{x}}^1)^2.$$

If the discontinuity line $\partial u/\partial x$ intersects the segment $x_{i-1} \leq x \leq x_i$, $t = \tau$, then $u_{\bar{x}}^1 = u_{\bar{x}}(x,\tau) = O(1)$, and therefore $\|\psi_2^0\|_{A^{-1}}^2 = O(\tau^2 h)$. A similar reasoning gives $\|\psi_3^0\|_{A^{-1}}^2 = O(\tau^2 h)$. Moreover, $\|\psi_4^0\|_{A^{-1}}^2 = O(\tau^2)$ and then

$$\|\psi^0\|_{A^{-1}} = O(\sqrt{h} + \sqrt{\tau}).$$

Thus we have proved:

THEOREM 6.15 *With the assumptions made concerning the solution of problem (6.287)–(6.289) and with the conditions*

$$\sigma_1 \geq \sigma_2, \quad \frac{\sigma_1 + \sigma_2}{2} \geq \frac{1+\varepsilon}{4} - \frac{1-\beta}{\tau^2 \|A\|}, \quad \varepsilon > 0, \quad 0 < \beta \leq 1,$$

being satisfied, the solution of the difference problem (6.293)–(6.296) converges on the average to a generalized solution of the differential problem (6.291), (6.288), (6.289), and for any $t \in \overline{\omega}_\tau$, the following estimate is valid:

$$\|y(t) - u(t)\| \leq M(\sqrt{h} + \sqrt{\tau}),$$

Here $M = \text{const} > 0$ is the constant independent of h and τ.

6.5 A Multi-dimensional Degenerative Equation with Dissipation

Assume now that $\Omega = \{0 < x_\alpha < l_\alpha,\ \alpha = \overline{1,p}\}$ is a p-dimensional parallelepiped with a boundary Γ, $x = (x_1, x_2, ..., x_p)$. We need to find continuous function $u(x,t)$ satisfying in $\overline{Q}_T = \overline{\Omega} \times [0,T]$ the following initial-boundary problem:

$$\frac{\partial}{\partial t}\left(\rho_1 \frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial t}(\rho_2 u) = Lu + f(x,t), \quad x \in \Omega, \quad t \in (0,T], \quad (6.316)$$

$$Lu = \sum_{\alpha=1}^{p} L_\alpha u, \quad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right),$$

$$0 < c_1 \le k_\alpha(x) \le c_2, \ \alpha = 1,2,...,p; \quad \rho_m(x) \ge 0, \ m = 1,2, \tag{6.317}$$

$$\begin{gathered} u(x,0) = u_0(x), \quad \rho_1 \frac{\partial u}{\partial t}(x,0) = \overline{u}_0(x), \quad x \in \Omega, \\ u\big|_\Gamma = 0, \quad t \in (0,T]. \end{gathered} \tag{6.318}$$

Just as in Section 6.1, we consider over the segment $[0,T]$ a uniform grid $\overline{\omega}_\tau$ with a step τ in time. In the parallelepiped $\overline{\Omega}$ we introduce the grid $\overline{\omega}_h = \omega_h \cup \gamma_h$ which is uniform in every direction x_α. Let γ_h be a set of boundary nodes

$$\omega_h = \Big\{x_i = (x_1^{(i_1)}, x_2^{(i_2)}, ..., x_p^{(i_p)}), \quad x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha, \\ i_\alpha = 1,2,...,N_{\alpha-1}, \quad h_\alpha N_\alpha = l_\alpha, \quad \alpha = 1,2,...,p\Big\}.$$

We approximate the operators $L_\alpha u$ by the following difference operators:

$$\Lambda_\alpha y = (a_\alpha y_{\bar{x}_\alpha})_{x_\alpha}, \quad 0 < c_1 \le a_\alpha \le c_2,$$

where the stencil functionals are defined by formulae (6.15)–(6.17).

To approximate problems (6.316)–(6.318) we apply a three-level difference scheme with constant weighting factors $0 \le \sigma_m \le 1$, $m = 1,2$:

$$D_1 y_{\bar{t}t} + B_1 y_{\overset{\circ}{t}} + Ay^{(\sigma_1,\sigma_2)} = \varphi, \tag{6.319}$$

$$y(0) = y_0, \quad D_1 y_t(0) = \overline{y}_0. \tag{6.320}$$

Here y_0, $\overline{y}_0$, $y(t)$, $\varphi(t) \in \overset{\circ}{\Omega}_h$, $t \in \omega_\tau$, and $\overset{\circ}{\Omega}_h$ is a set of grid functions defined on $\overline{\omega}_h$ and vanishing at the boundary nodes

$$\begin{aligned} y^{(\sigma_1,\sigma_2)} &= \sigma_1 \hat{y} + (1 - \sigma_1 - \sigma_2)y + \sigma_2 \check{y}, \\ D_1 &= \operatorname{diag}\{\rho_{11}, \rho_{12}, ..., \rho_{1N-1}\}, \\ B_1 &= \operatorname{diag}\{\rho_{21}, \rho_{22}, ..., \rho_{2N-1}\} \\ Ay &= \sum_{\alpha=1}^{p} A_\alpha y, \quad A_\alpha = -\Lambda_\alpha, \end{aligned} \tag{6.321}$$

$$A_\alpha y = -(a_\alpha y_{\bar{x}_\alpha})_{x_\alpha}, \quad x \in \omega_h, \quad y(x) = 0, \quad x \in \gamma_h,$$

$$\overline{y}_0 = \overline{u}_0 + 0.5\tau\left(Lu + f - \rho_2 \frac{\partial u}{\partial t}\right)(x,0).$$

By analogy with formulae (6.73), we introduce, as usual, the following inner products and norms:

$$(u,v) = \sum_{x\in\omega_h} u(x,t_n)v(x,t_n)h_1...h_p, \quad \|v\| = \sqrt{(v,v)},$$

$$\|v\|_A^2 = \sum_{\alpha=1}^{p} \|v\|_{A_\alpha}^2, \quad \|v\|_{A_\alpha}^2 = \sum_{i_\alpha=1}^{N_\alpha} \sum_{\substack{i_\beta=1\\ \beta\neq\alpha}}^{N_\beta-1} a_\alpha v_{\bar{x}_\alpha}^2 h_1...h_p, \tag{6.322}$$

$$\|Av\|^2 = \sum_{x\in\omega_h} h_1...h_p \left(\sum_{\alpha=1}^{p} (a_\alpha v_{\bar{x}_\alpha})_{x_\alpha}\right)^2.$$

The most natural reference point in investigation of three-level difference schemes with a degenerating operator $D \geq 0$ is the canonical form

$$Dy_{\bar{t}t} + By_t + Ay = \varphi, \quad y(0) = y_0, \quad y_t(0) = \overline{y}_0. \tag{6.323}$$

THEOREM 6.16 *Let the constant operators A, B, D satisfy the conditions*

$$A = A^* > 0, \quad B \geq 0, \quad D \geq 0. \tag{6.324}$$

If $\varphi = 0$ then the condition

$$B \geq \frac{\tau}{2} A \tag{6.325}$$

is sufficient for the stability of the scheme (6.323) with respect to the initial data in H_A, and the estimate

$$\|y_{n+1}\|_A^2 \leq \|y(\tau)\|_A^2 + (Dy_t(0), y_t(0)) \tag{6.326}$$

is valid.

Proof. We take the dot product of the operator equation with $2\tau y_t$. For any $t = t_k$ we obtain the energy identity

$$\big((B - 0.5\tau A)y_{\bar{t},k+1}, y_{\bar{t},k+1}\big) + \big(Dy_{\bar{t},k+1}, y_{\bar{t},k+1}\big) + \|y_{k+1}\|_A^2$$
$$= \|y_k\|_A^2 + \big(Dy_{\bar{t},k}, y_{\bar{t},k}\big).$$

Summing this identity over all $k = 1, 2, ..., n$ and taking into account the condition (6.325) and identity $y_{\bar{t},1} = y_t(0)$ we obtain the statement of Theorem.

REMARK 6.7 We note that canonical form (6.323) follows formally from (4.1.4) provided that one has used the identity

$$y_{\overset{\circ}{t}} = y_t - \frac{\tau}{2} y_{\bar{t}t}.$$

Nevertheless, unlike Theorem 4.6, the condition of self-adjointness for the operator B is not used in the investigation of the stability of scheme (6.323) with respect to the initial data.

To prove the stability of the explicit scheme

$$D_1 y_{\bar{t}t} + B_1 y_t + Ay = \varphi, \quad \rho_2 = \text{const} > 0 \tag{6.327}$$

that follows from (6.319) with $\sigma_1 = \sigma_2 = 0$, we apply Theorem 6.16. Obviously, in the conditions (6.317) the relations (6.324) are satisfied. Moreover, the inequality

$$B_1 - \frac{\tau}{2}A = \rho_2 E - \frac{\tau}{2}A \geq \left(\frac{\rho_2}{\|A\|} - \frac{\tau}{2}\right) A \geq 0 \tag{6.328}$$

is valid for $\tau \leq 2\rho_2/\|A\|$. Since $A = \sum\limits_{\alpha=1}^{p} A_\alpha$, $\|A_\alpha\| \leq \dfrac{4c_2}{h_\alpha^2}$, the explicit difference scheme (6.327), (6.320), for the initial differential problem (6.316)–(6.318) with the coefficients

$$\rho_1(x) \geq 0, \quad \rho_2(x) = \rho_2 = \text{const} > 0, \tag{6.329}$$

is stable in the space H_A with respect to the initial data, provided that $\varphi = 0$,

$$\tau \leq \frac{\rho_2 h^2}{2pc_2}, \quad h = \min_\alpha h_\alpha. \tag{6.330}$$

For the solution of the difference scheme the *a priori* estimate (6.326) is valid.

If the coefficient ρ_2 degenerates, i.e.,

$$\rho_1(x) \geq \rho_1 = \text{const} > 0, \quad \rho_2(x) \geq 0,$$

one has to apply Theorem 4.4. Conditions (4.2.27)

$$A = A^* > 0, \quad D \geq \frac{(1+\varepsilon)\tau^2}{4} A, \quad \varepsilon > 0,$$

that ensure the stability of the explicit scheme in H_A are obviously satisfied for

$$\tau \leq \sqrt{\frac{\rho_1}{c_2 p(1+\varepsilon)}}\, h. \tag{6.331}$$

Moreover, in accordance with inequality (4.2.37) the *a priori* estimate

$$\|y_{n+1}\|_A \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}}\, (\|y(0)\| + \|y_t(0)\|_D)$$

is true.

If both coefficients can be degenerated, i.e.,

$$\rho_1(x) \geq 0, \quad \rho_2(x) \geq 0,$$

then one has to use the schemes with weighting factors (6.319). From Theorem 4.14 one can infer that under the conditions

$$\sigma_1 \geq \sigma_2, \quad \sigma_1 + \sigma_2 \geq 0.5(1+\varepsilon)$$

the difference scheme (6.319), (6.320) is stable with respect to the initial data and right hand side, and for the solution of the scheme the estimate

$$\|y_{n+1}\|_A \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \Big(\|y_0\|_A + \|y_t(0)\|_D + \max_{0 \leq k \leq n} \left(\|\varphi_k\|_{A^{-1}} + \|\varphi_{t,k}\|_{A^{-1}} \right) \Big)$$

is valid.

The *a priori* estimates of stability with respect to the initial data and right hand side are used in investigations of the stability of difference schemes for degenerating equations in proper classes of smoothness.

7. Hyperbolo Parabolic Problems

In the mathematical simulation of physico-chemical processes in composite solids it is often required to use mathematical models based on different types of equations in separate parts of a computational domain. Here attention is specially paid to conjugation conditions on the interfaces of sub-domains. The questions of the existence and uniqueness of the solution of the boundary value problems for mixed type of equations are actively studied in the literature (see, for example, [Bitsadze, 1981, Godunov, 1971]).

Among the noted classes of problems we single out boundary value problems for hyperbolo parabolic equations, whose single-valued solvability in the class of generalized solutions was established in [Korzjuk, 1968, Korzjuk, 1970]. In this Section the questions of numerically solving these problems are considered in relation to the simplest one-dimensional boundary value problem. A uniform difference scheme relating to a class of schemes with variable (discontinuous) weight factors is constructed. The classes of unconditionally stable schemes are singled out, and investigation of the convergence rate of an approximate solution to the exact one is carried out [Samarskii et al., 1998a].

7.1 Statement of the Problem

In the rectangle (see Fig. 6.2)

$$\overline{Q}_T = \overline{Q}_1 \cup \overline{Q}_2, \quad Q_1 = \{(x,t) : \ 0 < x < \xi, \ 0 < t \le T\},$$

$$Q_2 = \{(x,t) : \ \xi < x < l, \ 0 < t \le T\}$$

we consider the boundary *problem on the conjugation of the equations*

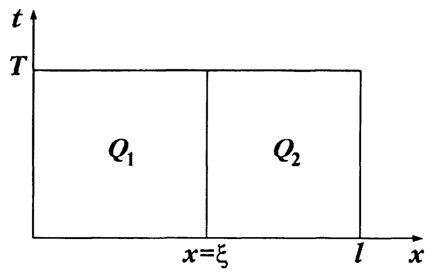

Figure 6.2.

of hyperbolic and parabolic types:

$$\rho_1(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k_1(x)\frac{\partial u}{\partial x}\right) + f_1(x,t), \quad (x,t) \in Q_1, \tag{6.332}$$

$$\rho_2(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k_2(x)\frac{\partial u}{\partial x}\right) + f_2(x,t), \quad (x,t) \in Q_2, \tag{6.333}$$

where $\rho_m(x)$ are strictly positive functions in $\overline{Q}_m$, $0 < c_1 \le k_m(x) \le c_2$ $(m = 1,2)$. We supplement these equations with the boundary and initial conditions:

$$u(0,t) = u(l,t) = 0, \quad t \ge 0, \tag{6.334}$$

$$\begin{aligned} u(x,0) &= u_0(x), && 0 \le x \le l, \\ \frac{\partial u}{\partial t}(x,0) &= u_1(x), && \xi \le x \le l. \end{aligned} \tag{6.335}$$

On the interface between two sub-domains, $x = \xi$, the following conjugation conditions hold:

$$[u] = 0, \quad \left[k\frac{\partial u}{\partial x}\right] = 0, \quad x = \xi, \quad t \ge 0, \tag{6.336}$$

where

$$[u] = u(\xi + 0, t) - u(\xi - 0, t),$$
$$k(x) = \begin{cases} k_1(x), & 0 < x < \xi, \\ k_2(x), & \xi < x < l. \end{cases}$$

The construction and investigation of the difference schemes for the conjugation problem formulated will be carried out using the following assumption: the coefficients $\rho_m(x)$, $k_m(x)$, $f_m(x,t)$ $(m = 1,2)$ suffer a discontinuity of the first kind on the straight line $x = \xi$, and outside the discontinuity line this functions are sufficiently smooth. Under such assumptions about the properties of the input data the questions of the existence and uniqueness of the strong solution were studied in [Korzjuk, 1968, Korzjuk, 1970]. Furthermore, we shall assume that the solution $u(x,t)$ of the problem (6.332)–(6.336) is a piecewise smooth function: outside the line $x = \xi$ the solution has all the necessary (for the analysis) bounded continuous derivatives, and on the straight line $x = \xi$ satisfies the conjugation conditions (6.336). We note that the dependence of the coefficient ρ_m, k_m on one variable x, and also the uniformity of the boundary conditions (6.334) are assumed only for the simplicity of arguments.

7.2 Difference Schemes

We introduce the uniform grids

$$\overline{\omega}_h = \{x_i = ih, \quad i = 0, 1, ..., N; \quad hN = l\},$$
$$\omega_\tau = \{t_n = n\tau, \quad n = 0, 1, ..., N_0 - 1; \quad \tau N_0 = T\}.$$

Let the point of the discontinuity of the coefficients

$$\xi = x_p = ph \in \omega_h, \quad 2 \le p \le N - 2,$$

be a node of the uniform spatial grid. In addition, in the domains Q_1 and Q_2 we shall consider the following grids:

$$\omega_1 = \omega_{1h} \times \omega_\tau, \quad \omega_2 = \omega_{2h} \times \omega_\tau,$$

where

$$\omega_{1h} = \{x_i = ih, \quad i = 1, 2, ..., p - 1\},$$
$$\omega_{2h} = \{x_i = ih, \quad i = p + 1, p + 2, ..., N - 1\}.$$

The differential problem is approximated by the three-level difference scheme

$$\rho_1 y_t = \left((a y_{\bar{x}})^{(\sigma_1,\sigma_2)} \right)_x + \varphi, \quad (x,t) \in \omega_1, \tag{6.337}$$

$$\rho_2 y_{\bar{t}t} = \left((a y_{\bar{x}})^{(\sigma_1,\sigma_2)} \right)_x + \varphi, \quad (x,t) \in \omega_2, \tag{6.338}$$

$$\hat{y}_0 = \hat{y}_N = 0, \tag{6.339}$$

$$y(x,0) = u_0(x), \quad y_t(x,0) = \overline{u}_0(x), \quad x \in \omega_{1h} \cup \{0\}, \tag{6.340}$$

$$y(x,0) = u_0(x), \quad y_t(x,0) = \overline{u}_1(x), \quad x \in \overline{\omega}_{2h}, \tag{6.341}$$

with the constant weights σ_α, $\alpha = 1,2$.

Just as in the case of the third boundary value problem for a one-dimensional heat equation [Samarskii, 1989, c. 95], the conjugation conditions (6.336)is approximated with the second order of approximation with respect to the spatial variable:

$$(a y_{\bar{x}})^{(\sigma_1,\sigma_2)} + \frac{h}{2} (\rho_1 y_t + \varphi) = (a_+ y_x)^{(\sigma_1,\sigma_2)} - \frac{h}{2} (\rho_2 y_{\bar{t}t} + \varphi), \ x = \xi. \tag{6.342}$$

Here

$$\begin{gathered} \rho_k = \rho_k(x_i), \quad a = k(x_{i-0.5}), \\ a_+ = a = k(x_{i+0.5}), \quad \varphi = 0.5(f_{i-0.5} + f_{i+0.5}), \end{gathered} \tag{6.343}$$

$$\overline{u}_0 = \rho_1^{-1}(x)(Lu_0(x) + f(x,0)), \tag{6.344}$$

$$\overline{u}_1 = u_1(x) + 0.5\tau\rho_2^{-1}(x)(Lu_0(x) + f(x,0)),$$

$$Lu = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right), \quad f(x,t) = \begin{cases} f_1(x,t), & (x,t) \in Q_1, \\ f_2(x,t), & (x,t) \in Q_2. \end{cases}$$

It is convenient to write the approximation of the conjugation conditions (6.342) in the form

$$0.5\,(\rho_1 y_t + \rho_2 y_{\bar{t}t}) = \left((a y_{\bar{x}})^{(\sigma_1,\sigma_2)} \right)_x + \varphi, \quad x = \xi. \tag{6.345}$$

We also note that the second initial condition $y_t(x,0) = \overline{u}_0$ for the parabolic equation have been obtained on the following grounds. Since for the use of the three-level scheme it is required to assign one more initial condition,$y(x,\tau)$ it is natural to approximate it with the second order of approximation $\mathrm{O}\,(h^2 + \tau^2)$. The idea is that we seek the value $y(x,\tau)$ in the form [Samarskii, 1989, c. 97]

$$y(x,\tau) = u_0(x) + \tau\mu(x).$$

The term μ is selected so that the error $y(x,\tau) - u(x,\tau)$ has the order $\mathrm{O}(\tau^2 + h^2)$. We substitute the value $\left.\dfrac{\partial u}{\partial t}\right|_{t=0}$ into the formulas

$$u(x,\tau) - u_0(x) = \tau \left.\frac{\partial u}{\partial t}\right|_{t=0} + \frac{\tau^2}{2}\left.\frac{\partial^2 u}{\partial t^2}\right|_{t=0} + \mathrm{O}(\tau^3)$$

using the differential equation

$$\rho_1 \left.\frac{\partial u}{\partial t}\right|_{t=0} = Lu_0 + f_1(x,0), \quad 0 < x < \xi.$$

Then we obtain

$$\mu = \left(Lu_0 + f_1(x,0)\right)/\rho_1(x).$$

Whence we obtain the approximation (6.340).

We note also the second possible way of the representation of the value $y(x,\tau)$ with the accuracy order $\mathrm{O}(h^2 + \tau^2)$. The first step in the domain $(x,t) \in \omega_1$ we implement following the two-level scheme

$$\rho_1 \frac{y^1 - y^0}{\tau} = \left((ay_{\bar{x}})^{(1/2,0)}\right)_x + \varphi^0, \quad 0 < x < \xi, \tag{6.346}$$

$$y(x,0) = u_0(x), \quad y(0,\tau) = 0, \quad y(x_p,\tau) = u_0(x_p) + \tau\overline{u}_1(x_p). \tag{6.347}$$

We note that for the determination of the second boundary condition $y(x_p,\tau)$ at the conjugation point of the two problems we take into account the given initial condition for the wave equation $y_t(x,0) = \overline{u}_1$ from formulae (6.341) at the point $\xi = x_p$.

Using the mentioned approximation for the parabolic equation in the domain ω_1 the scheme (6.337), (6.346), (6.347) is reduced to the scheme with the variable weight factors. This question is discussed in more detail below. The implementation of the difference schemes suggested above (like in the case of inhomogeneous conditions) is carried out provided that $\sigma_1 \neq 0$ in a standard way using the sweep method.

7.3 Stability of the Difference Schemes with Constant Weights

In the study of stability we shall use the canonical form of the three-level operator-difference scheme (4.1.4)

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad 0 < t \in \omega_\tau, \tag{6.348}$$

$$y(0) = y_0, \quad y_t(0) = \overline{y}_0, \tag{6.349}$$

where $y \in H$, and $H = \Omega_h$ is the real finite-dimensional Hilbert space with ordinary inner product and norm (see 6.1); A, B, $D : H \to H$ is the linear operator in H. As before, by H_D (here $D = D^* > 0$) we denote the Hilbert space with the inner product $(y, v)_D = (Dy, v)$ and the norm $\|y\|_D^2 = (y, y)_D$.

We note that taking into account Theorem 4.14 under the conditions

$$A^* = A > 0, \quad D^* = D \geq \frac{(1+\varepsilon)\tau^2}{4} A, \quad B \geq 0, \quad \varepsilon > 0, \tag{6.350}$$

for the solution of the difference scheme (6.348), (6.349) the following *a priori* estimate holds:

$$\begin{aligned} \|y_{n+1}\|_A \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}} \Big(& \|y(0)\|_A + \|y_t(0)\|_D \\ & + \max_{0 \leq k \leq n} \left(\|\varphi_k\|_{A^{-1}} + \|\varphi_{t,k}\|_{A^{-1}} \right) \Big). \end{aligned} \tag{6.351}$$

In order to reduce scheme (6.337)–(6.341), (6.345) to a canonical form, let us define the operator A in a standard way according to expression (6.262), and the operators B, D as follows:

$$D = S + \frac{\tau^2}{2} (\sigma_1 + \sigma_2) A, \quad S = \operatorname{diag}\{s_1, s_2, ..., s_{N-1}\}, \tag{6.352}$$

$$s(x) = \begin{cases} 0.5\tau\rho_1(x), & x \in \omega_{1h}, \\ 0.25\tau\rho_1(x) + 0.5\rho_2(x), & x = \xi, \\ \rho_2(x), & x \in \omega_{2h}; \end{cases} \tag{6.353}$$

$$B = G + (\sigma_1 - \sigma_2)\tau A, \quad G = \operatorname{diag}\{g_1, g_2, ..., g_{N-1}\}, \tag{6.354}$$

$$g(x) = \begin{cases} \rho_1(x), & x \in \omega_{1h}, \\ 0.5\rho_1(x), & x = \xi, \\ 0, & x \in \omega_{2h}. \end{cases}$$

Also we define the initial conditions by the relations

$$y_0 = u_0, \tag{6.355}$$

$$y_t(0) = \overline{y}_0, \quad \overline{y}(x, 0) = \begin{cases} \overline{u}_0(x), & x \in \omega_{1h}, \\ \overline{u}_1(x), & x \in \omega_{2h}^+, \end{cases} \quad \omega_{2h}^+ = \omega_{2h} \cup \{\xi\}. \tag{6.356}$$

Then using the identity

$$v^{(\sigma_1,\sigma_2)} = v + \tau(\sigma_1 - \sigma_2) v_{\overset{\circ}{t}} + \frac{\tau^2}{2} (\sigma_1 + \sigma_2) v_{\bar{t}t} \tag{6.357}$$

we reduce difference schemes (6.337)–(6.341), (6.345) to the canonical form of three-level operator-difference schemes with the operators A, B, D, the right hand side φ and the initial data y_0, $\overline{y}_0$, defined by the corresponding relations (6.262), (6.352)–(6.356).

The following statement holds:

THEOREM 6.17 *Let for the difference scheme (6.337)–(6.341), (6.345) the following conditions be satisfied:*

$$\sigma_1 \geq \sigma_2, \quad \sigma_1 + \sigma_2 \geq 0.5(1+\varepsilon). \tag{6.358}$$

Then the scheme (6.348), (6.349), (6.352)–(6.356) is stable with respect to initial data and the right hand side, and for its solution the estimate (6.351) holds.

Proof. Taking into account the conditions of the theorem $\sigma_1 \geq \sigma_2$, we obtain $A = A^* > 0$, $D = D^* > 0$, and $B \geq 0$. Consequently, to prove the theorem it is sufficient to be convinced in the correctness of the inequality

$$D - \frac{1+\varepsilon}{4}\tau^2 A = S + \left(\frac{\sigma_1+\sigma_2}{2} - \frac{1+\varepsilon}{4}\right)\tau^2 A \geq 0. \tag{6.359}$$

Obviously it holds under the conditions (6.358).

7.4 Difference Schemes with Variable Weighting Factors

We have already note above that difference schemes with constant weights for the considered problem on conjugation of polytypic equations arise in the case of the three-level scheme (6.337) for parabolic equation with the initial conditions (6.346), (6.347). In addition, in approximating a parabolic equation computing practice usually employs a two-level scheme (in this case $\sigma_2 = 0$), and a three-level scheme is used for approximating a hyperbolic equation.

We define the variable weights σ_1, σ_2 by the following relations

$$\sigma_1(x) = \begin{cases} \sigma, & x \in \omega_{1h}, \\ 0.5(\sigma + \sigma_1^*), & x = \xi, \\ \sigma_1^*, & x \in \overline{\omega}_{2h}, \end{cases}$$

$$\sigma_2(x) = \begin{cases} 0, & x \in \omega_{1h}, \\ 0.5\sigma_2^*, & x = \xi, \\ \sigma_2^*, & x \in \overline{\omega}_{2h}. \end{cases}$$

Here σ, σ_1^*, σ_2^* are constants. Then the problem (6.337)–(6.341), (6.355) can be written in the form

$$\tilde{\rho}_1 y_t + \tilde{\rho}_2 y_{\bar{t}t} = \left((a y_{\bar{x}})^{(\sigma_1(x),\sigma_2(x))}\right)_x + \varphi, \quad (x,t) \in \omega, \tag{6.360}$$

$$y(0,\hat{t}) = y(l,\hat{t}) = 0, \quad t \in \omega_\tau, \tag{6.361}$$

$$y(x,0) = u_0(x), \quad x \in \overline{\omega}_h, \quad y_t(x,0) = \overline{u}_1(x), \quad x \in \omega_{2h}^+, \tag{6.362}$$

where $\omega = \omega_h \times \omega_\tau$ and the coefficients $\tilde{\rho}_k$ are defined by the expressions

$$\tilde{\rho}_1(x) = \begin{cases} \rho_1(x), & x \in \omega_{1h}, \\ 0.5\rho_1(\xi), & x = \xi, \\ 0, & x \in \omega_{2h}, \end{cases} \tag{6.363}$$

$$\tilde{\rho}_2(x) = \begin{cases} 0, & x \in \omega_{1h}, \\ 0.5\rho_2(\xi), & x = \xi, \\ \rho_2(x), & x \in \omega_{2h}. \end{cases} \tag{6.364}$$

The study stability of the scheme is carried out according to Subsection 5.3. Let $\overset{1}{\Omega}{}_h^n$ be the space of the grid function defined on $\overline{\omega}_h$ and satisfying the condition $v_0^n = 0$. Along with the Hilbert space H let us introduce into consideration the space H_1 as the set of vectors in the form $v_n = (v_1^n, v_2^n, ..., v_N^n)^T$. In the space H_1 we give the inner product and the norm

$$(y,v] = \sum_{i=1}^{N} y_i v_i h, \quad \|y]| = \sqrt{(y,y]}. \tag{6.365}$$

We show that the operator $A = \Omega_h \to \Omega_h$,

$$(Ay)_i = -(a y_{\bar{x}})_{x,i}, \quad i = 1,2,...,N-1, \quad y_0 = y_N = 0, \tag{6.366}$$

allows their presentation in the form $A = T^*T$. Let $y = y_n = (y_1^n, y_2^n, ..., y_{N-1}^n) \in H$, $\dim H = N-1$. The linear operators $T : H \to H_1$, $T^* : H_1 \to H$ are defined by the expressions, respectively:

$$(Ty)_i = \sqrt{a_i} y_{\bar{x},i}, \quad i = 1,2,...,N, \quad y_0 = y_N = 0, \tag{6.367}$$

$$(T^*v)_i = -(\sqrt{a}v)_{x,i}, \quad i = 1,2,...,N-1. \tag{6.368}$$

Note that the operators T and T^* are joined with each other as regards the inner product (6.365). Actually, in accordance with the formula for summing by parts for any $y \in H$, $v \in H_1$, we have

$$(v,Ty] = \sum_{i=1}^{N} v_i \sqrt{a_i} y_{\bar{x},i} h = \sum_{i=1}^{N-1} -(\sqrt{a}v)_{x,i} y_i h = (T^*v, y). \tag{6.369}$$

We now define the operators B and D in the following way:

$$B = G + \tau T^* (\Sigma_1 - \Sigma_2) T, \tag{6.370}$$

$$G = \mathrm{diag}\{\tilde{\rho}_{11}, \tilde{\rho}_{12}, ..., \tilde{\rho}_{1N-1}\}, \tag{6.371}$$

$$D = C + 0.5\tau^2 T^* (\Sigma_1 + \Sigma_2) T, \tag{6.372}$$

$$C = \mathrm{diag}\{c_1, c_2, ..., c_{N-1}\}, \quad c_i = 0.5\tilde{\rho}_{1i}\tau + \tilde{\rho}_{2i},$$

$$\Sigma_k : H_1 \to H_1, \quad \Sigma_k = \mathrm{diag}\{\sigma_{k1}, \sigma_{k2}, ..., \sigma_{kN-1}\}. \tag{6.373}$$

Obviously the difference scheme with the variable weighting factors (6.360)–(6.362) is reduced to the canonical form (6.348), (6.349) with the operators $A = T^*T$, B, D defined by (6.366)–(6.373) and with

$$y_t(0) = \overline{y}_0, \quad \overline{y}_0 = \begin{cases} \rho_1^{-1}(Ay^{(\sigma)} + \varphi), & \text{if} x \in \omega_{1h}, \\ \overline{u}_1(x), & \text{if} x \in \omega_{2h}^+. \end{cases} \tag{6.374}$$

It has been shown that the operator A is positive and self-adjoint (see 6.1). Since σ, σ_1^*, σ_2^* are positive constants, we obtain that the operators Σ_1, Σ_2 are non-negative and $D = D^* > 0$. By the structure $B \geq 0$ provided that $\Sigma_1 \geq \Sigma_2$. In order to use the *a priori* estimate (6.351) we verify the fulfilment of the sufficient stability condition (6.350). We note that

$$D - \frac{1+\varepsilon}{4}\tau^2 A = C + \frac{\tau^2}{2} T^* \left(\Sigma_1 + \Sigma_2 - \frac{1+\varepsilon}{2} E\right) T \geq 0$$

provided that $\Sigma_1 + \Sigma_2 \geq \dfrac{1+\varepsilon}{2} E$. The latter inequality and also the condition $\Sigma_1 \geq \Sigma_2$ is satisfied when $\sigma_1(x) + \sigma_2(x) \geq 0.5(1+\varepsilon)$ and $\sigma_1(x) \geq \sigma_2(x)$, respectively. Consequently we have proved the following statement.

THEOREM 6.18 *Let for the difference scheme (6.360)–(6.362) the following conditions be correct:*

$$\sigma_1(x) \geq \sigma_2(x), \quad \sigma_1(x) + \sigma_2(x) \geq \frac{1+\varepsilon}{2}, \quad x \in \omega_h^+ = \omega_h \cup \{x_N\}.$$

Then the scheme is stable with respect to initial data, the right hand side, and for the solution the estimate (6.351) holds.

Taking into account the imbedding (see (6.27), (6.39))

$$\|y_{\bar{x}}]| \leq \frac{1}{\sqrt{c_1}} \|y\|_A, \quad \|y\|_C \leq \frac{\sqrt{l}}{2\sqrt{c_1}} \|y\|_A$$

from inequality (6.107) the corresponding *a priori* estimates in the seminorm W_2^1 and in the uniform metric follow.

7.5 Truncation Error and Convergence

We study the convergence of the difference scheme with the variable weighting factors (6.360). By analogy with [Samarskii and Fryazinov, 1961] let us formulate the conditions used below.

A. The functions k'_m, k''_m, ρ'_m, f'_m, u''_m, $(k_m u')''$ $(m = 1, 2)$ satisfies the Lipschitz condition with respect to x in both sub-domains Q_m, $\dfrac{\partial u}{\partial t}$ satisfy the Lipschitz condition with respect to t in Q_1, and $\dfrac{\partial^2 u}{\partial t^2}$ satisfies the Lipschitz condition with respect to t in Q_2. Here we use the notation $v' = \partial v/\partial x$.

B. The limiting values on the left and right of the functions f_m, f'_m, f''_m $(m = 1, 2)$, u', u'', u''' satisfy the Lipschitz condition along the straight line $x = \xi$ with respect to t provided that $0 \le t \le T$.

Let $y \in H$ be the solution of problem (6.348), (6.349), (6.367)–(6.373), $u(x,t)$ the solution of the differential problem (6.332)–(6.336). We write equation for the error $z = y - u$. Substituting $y = z + u$ into (6.348), (6.349) we obtain

$$Dz_{\bar{t}t} + Bz_{\overset{\circ}{t}} + Az = \psi, \quad z(0) = 0, \quad z_t(0) = \nu(x). \tag{6.375}$$

Here z, ψ, $\nu \in H$, where

$$\psi_i = -\tilde{\rho}_1 u_{t,i} - \tilde{\rho}_2 u_{\bar{t}t,i} + \left((a u_{\bar{x}})^{(\sigma_1,\sigma_2)}\right)_{x,i} + \varphi_i \tag{6.376}$$

is the truncation error of equation (6.332), (6.333) and the conjugation conditions (6.336), and

$$\nu = \begin{cases} -\rho_1 u_t + (a u_{\bar{x}})_x^{(\sigma)} + \varphi, & x \in \omega_{1h}, \\ \overline{u}_1(x) - u_t(x,0), & x \in \omega_{2h}^{+}, \end{cases} \tag{6.377}$$

defines the truncation error of the second initial condition.

By analogy with Section 6.1 we transform the expression for disparity. For this we take equations (6.332), (6.333) at the time moment $t = t_n$ and integrate the results with respect to x within the limits from $x_{i-0.5}$ to $x_{i+0.5}$:

$$W_{i+0.5} - W_{i-0.5} + \int\limits_{x_{i-0.5}}^{x_{i+0.5}} \left(f_1(x,t) - \rho_1(x) \frac{\partial u(x,t)}{\partial t} \right) dx = 0, \quad (x,t) \in \omega_1,$$

$$W_{p+0.5}-W_{p-0.5}+\int\limits_{x_{p-0.5}}^{\xi}\left(f_1(x,t)-\rho_1(x)\frac{\partial u(x,t)}{\partial t}\right)dx$$
$$+\int\limits_{\xi}^{x_{p+0.5}}\left(f_2(x,t)-\rho_2(x)\frac{\partial^2 u(x,t)}{\partial t^2}\right)dx=0,\quad x=\xi, \tag{6.378}$$

$$W_{i+0.5}-W_{i-0.5}+\int\limits_{x_{i-0.5}}^{x_{i+0.5}}\left(f_2(x,t)-\rho_2(x)\frac{\partial^2 u(x,t)}{\partial t^2}\right)dx=0,\quad (x,t)\in\omega_2,$$

where $W=ku'$.

We divide each of identities (6.378) by h and subtract results from the equality (6.376). Then we obtain

$$\psi=\eta_{1x}+\psi_1,\quad \eta_1=(au_{\bar{x}})^{(\sigma_1,\sigma_2)}-\overline{W}=\mathrm{O}\left(h^2+\tau\right),\quad \overline{v}=v(x_{i-0.5},t),$$

$$\psi_1=\begin{cases}-\rho_1 u_t+\varphi-\dfrac{1}{h}\displaystyle\int\limits_{x_{i-0.5}}^{x_{i+0.5}}\left(f_1(x,t)-\rho_1\frac{\partial u(x,t)}{\partial t}\right)dx, & (x,t)\in\omega_1,\\[2ex] -0.5\left(\rho_1 u_t+\rho_2 u_{\bar{t}t}\right)+\varphi- & \\ -\dfrac{1}{h}\displaystyle\int\limits_{x_{p-0.5}}^{\xi}\left(f_1(x,t)-\rho_1\frac{\partial u(x,t)}{\partial t}\right)dx- & \\ -\dfrac{1}{h}\displaystyle\int\limits_{\xi}^{x_{p+0.5}}\left(f_2(x,t)-\rho_2\frac{\partial^2 u(x,t)}{\partial t^2}\right)dx, & x=\xi,\\[2ex] -\rho_2 u_{\bar{t}t}+\varphi-\dfrac{1}{h}\displaystyle\int\limits_{x_{i-0.5}}^{x_{i+0.5}}\left(f_2(x,t)-\rho_2\frac{\partial^2 u(x,t)}{\partial t^2}\right)dx, & (x,t)\in\omega_2.\end{cases}$$

By virtue of the smoothness conditions we have

$$\frac{1}{h}\int\limits_{x_{i-0.5}}^{x_{i+0.5}}f(x,t)dx=\varphi+\frac{h^2}{8}\overline{f'}_{x,i}+\mathrm{O}\left(h^2\right),\quad x\in\omega_h,$$

$$\frac{1}{h}\int\limits_{x_{i-0.5}}^{\xi}\rho_1\frac{\partial u}{\partial t}dx+\frac{1}{h}\int\limits_{\xi}^{x_{i+0.5}}\rho_2\frac{\partial^2 u}{\partial t^2}dx = 0.5\left(\rho_1\frac{\partial u}{\partial t}+\rho_2\frac{\partial^2 u}{\partial t^2}\right)_{x=\xi}$$
$$+\frac{h^2}{8}\overline{p'}_{x,p}+\mathrm{O}\,(h^2),$$
$$p(x,t)=\begin{cases}\rho_1(x)\dfrac{\partial u(x,t)}{\partial t}, & (x,t)\in Q_1,\\ \rho_2(x)\dfrac{\partial^2 u(x,t)}{\partial t^2}, & (x,t)\in Q_2.\end{cases}$$

Consequently the truncation error can be rewritten in the form

$$\psi=\eta_x+\psi^*,\quad \eta=\eta_1+\frac{h^2}{8}\left(\overline{p'}-\overline{f'}\right),\quad \psi^*,\,\psi_t^*=\mathrm{O}\,(h^2+\tau).$$

We turn to the determination of the accuracy order of the scheme. For the solution of problem (6.375) on the basis of relation (6.351) the following estimate holds:

$$\|z_{n+1}\|_A\le\sqrt{\frac{1+\varepsilon}{\varepsilon}}\left(\|z_t(0)\|_D+\max_{0\le k\le n}\left(\|\psi_k\|_{A^{-1}}+\|\psi_{t,k}\|_{A^{-1}}\right)\right).\quad(6.379)$$

By direct computations we obtain

$$\|z_t(0)\|_D^2=\|\nu(x)\|_D^2=(C\nu,\nu)+0.5\tau^2\left((\Sigma_1+\Sigma_2)\,T\nu,T\nu\right).$$

Since C, Σ_1, Σ_2 are bounded operators it follows that

$$\|\nu(x)\|_D\le c_0\left(\|\nu\|^2+\tau^2\|\nu_{\bar x}]|^2\right)^{1/2}\le c\,(h^2+\tau),\quad(6.380)$$

where c_0, c are constants independent on h, τ. Similarly, according to the estimate (6.59) we obtain

$$\begin{aligned}&\|\psi\|_{A^{-1}}\le\frac{1}{\sqrt{c_1}}\left(\|\eta]|+\frac{l}{2\sqrt{2}}\|\psi^*\|\right)\le c\,(h^2+\tau),\\ &\|\psi_t\|_{A^{-1}}\le c\,(h^2+\tau).\end{aligned}\quad(6.381)$$

THEOREM 6.19 *In both sub-domains Q_m $(m=1,2)$ let the conditions A and B be satisfied. Then, under the conditions*

$$\sigma_1(x)\ge\sigma_2(x),\quad \sigma_1(x)+\sigma_2(x)\ge\frac{1+\varepsilon}{2},\quad x\in\overline{\omega}_h,$$

the solution of the difference scheme (6.360)–(6.362) converges to the solution of the differential problem (6.332)–(6.336), so that the following estimate holds

$$\max_{t\in\omega_\tau}\|z(t)\|_C\le c\,(h^2+\tau).$$

Proof. The statement of the theorem follows from inequalities (6.379)–(6.381) and the imbedding (6.39).

Chapter 7

SCHEMES ON ADAPTIVE GRIDS

1. Introduction

At the present time various methodical techniques can be used for improving the accuracy of the approximate solution. Often an exact solution has singularities in a part of the computational domain. An example can be furnished by the behaviour of the solution near the boundary between media with different characteristics. It is necessary to make additional effort not to lose accuracy because of this kind of singularities of the solution.

The traditional approach uses a finer computational grid in the region in which the singularity of the solution occurs, for example, in that part of the computational domain in which the solution is not sufficiently smooth. There are well known difficulties of using these approaches. One of them is that because of the inhomogeneity of the grid used the homogeneity of the computational algorithm can be violated (the problem of computational implementation).

The second, not so obvious, problem is related to a possible loss of accuracy in the approximate solution if we use such non-uniform grids. In fact, we use these kind of computational grids in a part of the domain in order to increase the accuracy of a numerical method, but often apply inappropriate tools. Here the main topic of research concerns the questions of theoretical and methodical investigations of the accuracy of such adaptive computational algorithms.

Unfortunately the problem of complete theoretical justification of computational algorithms on adaptive meshes applied to the approximate solution of problems in mathematical physics is far from being resolved. There are few strong results on the accuracy of difference

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schemes on locally refined computational grids. Some questions of theoretical analysis of numerical methods on locally refined grids are considered in this chapter. This research is based, first of all, on the use of the results of the theory of stability of difference schemes with operator multipliers.

In solving non-stationary problems of mathematical physics one can use grids locally refined in space (the so called space adaptation), grids locally refined in time (adaptation in time), and, in a more general case, grids which are locally refined in both space and time. It is necessary also to emphasize the following. The adaptation zone in non-stationary problems often changes in time (the singularities of the solution are localized in different parts of the computational domain at different time moments). Therefore in these cases we must speak about dynamic adaptation, that is, about changes in the space of the subregions where the grid is refined in space and (or) in time.

First we consider difference schemes, with the use of adaptive grids in time, for solving boundary value problems for second-order parabolic equations. In a part of the computational domain we find the solution by using finer grids in time. The computational implementation of such techniques is based on special organization of computations for inversion of one-dimensional (in space) grid operators. To solve many-dimensional problems additive difference schemes with splitting into space variables are used.

In considering schemes on locally refined meshes attention is mainly paid to the approximation of equations on the boundaries of the adaptation zone, precisely, to the formulation of the interface conditions. The difference schemes under consideration are interpreted as operator-difference schemes with variable weighted factors. Based on the previously obtained results on the stability of schemes with operator factors, we investigate the accuracy of difference schemes on locally refined grids, amongst others, for problems with low order smoothness of an exact solution.

A similar range of problems is considered in solving boundary value problems for a second-order hyperbolic equation with a self-adjoint space operator on locally refined grids in time. Separate research is carried out for conservative and non-conservative schemes, which are related to different types of difference schemes with operator factors. As before, for a parabolic problem we obtain estimates of the unconditional stability of the considered difference schemes in different norms.

We construct also the second class of difference schemes on locally refined grids in time, which are more convenient for implementation on a computer. However, the accuracy of these difference schemes depends on the width of the transition zone (from a coarse to a finer mesh). For schemes without overlapping (without this transition zone) we arrive at schemes for which the convergence rate falls; the schemes of this class are referred as to conditionally convergent schemes.

Furthermore, we touch upon the questions of constructing and studying difference schemes on dynamic locally refined grids in space. The essence of the problems arising is discussed in detail with an example of the simplest problem for a one-dimensional parabolic equational with constant coefficients. One of the key questions is about the approximation of the equation in an additional node on a new time level (dynamical adaptation, in this case the adaptation zone changes). We obtain *a priori* estimates of the conditional convergence of such schemes under different conditions.

We have great possibilities for increasing the accuracy of the approximate solution if we use the usual difference schemes on non-uniform grids. For problems with static adaptation (the adaptation zone is fixed) it is most natural to use simple non-uniform grids, for example, with constant step sizes in the adaptation zone and outside it). We note that it is possible to construct difference schemes on non-uniform grids in a non-standard way, by approximating the equation not at a mesh point, but at some interior point. By using this methodical technique we construct schemes for some one-dimensional and many-dimensional boundary value problems for parabolic and hyperbolic equations of second order.

2. Difference Schemes on Grids Adaptive in Time for a Parabolic Equation

In part of the computational domain in which a solution has singularity we can use difference schemes with a smaller step in time to increase the accuracy of approximate solution. One possible approach to the development of the difference schemes on *grids locally refined in time* is suggested [Matus, 1990, Matus, 1991, Matus, 1993b, Matus, 1994].

2.1 Non-Conservative Schemes

The concept of the suggested approach to the construction and implementation of the unconditionally stable schemes on time adaptive grids will be illustrated by an example of numerical solution of the first-kind boundary value problem for the heat conduction equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) + f(x,t), \quad (x,t) \in Q_T, \tag{7.1}$$

$$0 < c_1 \le k(x,t) \le c_2, \quad (x,t) \in \overline{Q}_T,$$

$$u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \tag{7.2}$$

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad t > 0. \tag{7.3}$$

Let us bring into consideration the uniform spatial grid with the step h

$$\overline{\omega}_h = \left\{x_i = ih, \ i = \overline{0,N}, \ hN = l\right\}$$

and time grids with the steps τ and $\tau_0 = \tau/p$ ($p \ge 1$ is an integer), respectively:

$$\overline{\omega}_\tau = \{t_n = n\tau, \ n = 0,1,\dots,N_0; \ \tau N_0 = T\} = \omega_\tau \cup \{T\},$$
$$\overline{\omega}_{\tau_0} = \left\{t_\alpha = t_{n+\alpha/p} = (n+\alpha/p)\tau, \ \alpha = \overline{0,p}, \ n = \overline{0, N_0 - 1}\right\}. \tag{7.4}$$

We represent the domain $Q^n = \Omega \times [t_n, t_{n+1}]$ for every fixed n in the form

$$Q^n = Q_1^n \cup Q_2^n, \quad Q_1^n = Q^n \backslash Q_2^n,$$

$$Q_2^n = \left\{(x,t) : x_{m_1^n} < x < x_{m_2^n}, \ t_{\alpha-1} \le t \le t_\alpha, \ \alpha = \overline{1,p}\right\}.$$

The set of the nodes of the grid $\omega_{h\tau_0} = \omega_h \times \omega_{\tau_0}$, lying in the domain Q_2^n, will be denoted by ω_2^n, where

$$\omega_2^n = \left\{(x_i, t_{n+\alpha/p}) : m_1^n < i < m_2^n, \ \alpha = \overline{0,p}, \ m_1^n \ge 1, m_2^n \le N-1\right\}.$$

Then $\omega_1^n = \omega_{h\tau_0} \backslash \omega_2^n$ is a set of nodes of the grid $\omega_{h\tau_0}$ that belong to the domain Q_1^n. The interior nodes

$$x_{m_1^n} = m_1^n h, \quad x_{m_2^n} = m_2^n h$$

also pertain to this set. The grid described is presented in Fig. 7.1.

A priori we assume that the solution $u(x,t)$ is a sufficiently smooth function in the domain Q_1^n, and has a singularity in Q_2^n moving with respect to time. The latter leads to the necessity of using a certain sufficiently small time step τ_0 in calculations.

We set the problem of constructing unconditionally stable difference schemes which would allow one to find a solution of a difference problem provided that $\alpha < p$ only in the domain of the existence of the assumed solution singularity $\overline{\omega}_2^n$, and to use a sufficiently large step τ in ω_1^n. Solution of this problem will make it possible to substantially reduce

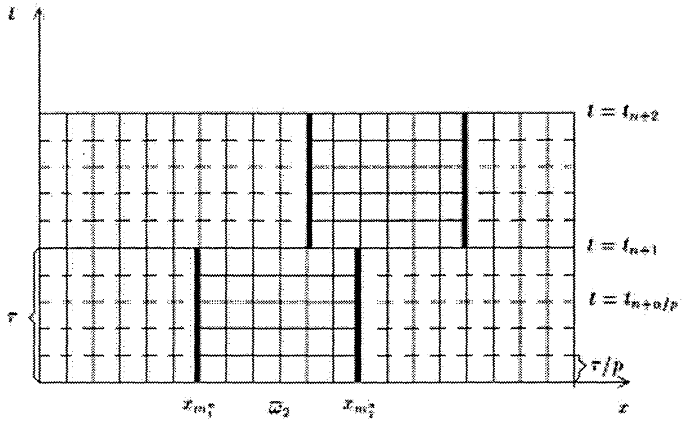

Figure 7.1. Adaptive grid

the amount of necessary calculations in mathematical simulation of the problems which admit the singularities of solutions.

To make the discussion clear we limit ourselves to a detailed description of the calculating process in determining the solution on the fixed $(n+1)$-th time level. At the nodes x_i $(i = \overline{0, N})$, provided that $t = t_n$, let us find the values of the approximate solution y_i^n. On the grid introduced for the original problem we construct the following difference scheme:

$$\frac{y_i^{n+1} - y_i^n}{\tau} = (\Lambda y)_i^{n+1} + f_i^{n+1}, \quad (x,t) \in \omega_1^n, \tag{7.5}$$

$$\frac{y_i^{n+\alpha/p} - y_i^{n+(\alpha-1)/p}}{\tau_0} = (\Lambda y + f)^{(\sigma)}, \quad (x,t) \in \omega_2^n, \tag{7.6}$$

where

$$\Lambda y = (a y_{\bar{x}})_x, \quad v^{(\sigma)} = \sigma v^{n+\alpha/p} + (1-\sigma) v^{n+(\alpha-1)/p},$$

$\sigma \geq 0$, a is the pattern functional determined by the values $k(x,t)$ [Samarskii, 1989].

Thus it is logical to use a purely implicit scheme with the step in the domains of assumed smoothness, and an ordinary scheme with weights in the domain $\overline{\omega}_2^n$. If we know the boundary conditions at the nodes $(x_{m_1^n}, t_{n+\alpha/p})$ and $(x_{m_2^n}, t_{n+\alpha/p})$, $\alpha = \overline{1, p-1}$, then the scheme (7.5), (7.6) in conjunction with the approximation of boundary conditions (7.3) is implemented by the usual scalar sweep method. The basic difficulty consists in exact (but not approximate) obtaining of these boundary conditions. We note that the techniques of interpolation and extrapolation sometimes used in practice for obtaining conditions on the interior boundaries of the sub-domains can lead to restrictions on the grid steps (conditional stability).

We turn to the determination of the grid solution in the domain $\overline{\omega}_2^n$ which contains the assumed singularity. For this we use an idea of the opposite direction sweep method for the ordinary three-point equation and special approximation of the derivative with respect to time outside the adaptation zone:

$$\frac{\partial u}{\partial t} \sim \frac{u_i^{n+\alpha/p} - u_i^n}{\alpha\tau/p}, \quad t_n \le t \le t_{n+1}. \tag{7.7}$$

On the whole grid of the nodes $\omega_h \times \omega_{\tau_\alpha}$ we approximate the differential problem (7.1)–(7.3) by the difference problem:

$$\frac{y_i^{n+\alpha/p} - y_i^n}{\alpha\tau_0} = (\Lambda y)_i^{n+\alpha/p} + f_i^{n+\alpha/p}, \quad (x,t) \in \omega_1^n, \tag{7.8}$$

$$\frac{y_i^{n+\alpha/p} - y_i^{n+(\alpha-1)/p}}{\tau_0} = (\Lambda y + f)^{(\sigma)}, \quad (x,t) \in \omega_2^n, \tag{7.9}$$

$$y_i^0 = u_0(x_i), \quad y_0^{n+\alpha/p} = \mu_1^{n+\alpha/p}, \quad y_N^{n+\alpha/p} = \mu_2^{n+\alpha/p}. \tag{7.10}$$

We note that if there is no need to use an adaptive grid, then setting $\sigma = p = 1$ we obtain a usual purely implicit scheme whose properties is well known.

We consider in detail the implementation of the difference scheme. Obviously the difference equations (7.8), (7.9) on every fractional level $t = t_{n+\alpha/p}$ are reduced to the three-point equation

$$A_i y_{i-1}^{n+\alpha/p} - C_i y_i^{n+\alpha/p} + B_i y_{i+1}^{n+\alpha/p} = -F_i, \quad i = \overline{1, N-1}, \tag{7.11}$$

where the operators and right hand side are determined as follows:

$$C_i = 1 + A_i + B_i, \quad A_i = \frac{\tau_0 \sigma_{\alpha,i}^n}{h^2} a_i^{n+\alpha/p}, \quad B_i = \frac{\tau_0 \sigma_{\alpha,i}^n}{h^2} a_{i+1}^{n+\alpha/p},$$

$$F_i = \begin{cases} y_i^n + \tau_0 \alpha f_i^{n+\alpha/p}, & i = 1, \dots, m_1^n, m_2^n, \dots, N-1, \\ y_i^{n+(\alpha-1)/p} + \tau_0 \left(f_i^{(\sigma)} + (1-\sigma)\, (\Lambda y)_i^{n+(\alpha-1)/p} \right), & \\ \qquad i = m_1^n + 1, \dots, m_2^n - 1, & \end{cases}$$

$$\sigma_{\alpha,i}^n = \begin{cases} \alpha, & \text{if} \quad i=1,\ldots,m_1^n, m_2^n, m_2^n+1,\ldots,N-1, \\ \sigma, & \text{if} \quad i=m_1^n+1,\ldots,m_2^n-1. \end{cases} \tag{7.12}$$

The coefficients defined in this way are independent of the values of the desired grid function at the 'fictitious' points of the domain ω_1^n. Let us calculate the necessary sweep coefficients for determining the boundary value $y_{m_1}^{n+\alpha/p}$ (in what follows,the superscript n for m_1, m_2 will be omitted):

$$\alpha_{i+1} = \frac{B_i}{C_i - \alpha_i A_i}, \qquad i=\overline{1,m_1}; \quad \alpha_1 = 0, \tag{7.13}$$

$$\beta_{i+1} = \frac{F_i + A_i\beta_i}{C_i - \alpha_i A_i}, \qquad i=\overline{1,m_1}; \quad \beta_1 = \mu_1^{n+\alpha/p}, \tag{7.14}$$

$$\xi_i = \frac{A_i}{C_i - \xi_{i+1}B_i}, \quad i=N-1,N-2,\ldots,m_1+1; \quad \xi_N = 0, \tag{7.15}$$

$$\eta_i = \frac{F_i + B_i\eta_{i+1}}{C_i - \xi_{i+1}B_i}, \quad i=N-1,N-2,\ldots,m_1+1; \quad \eta_N = \mu_2^{n+\alpha/p}. \tag{7.16}$$

Following the opposite direction sweep method we obtain

$$y_{m_1}^{n+\alpha/p} = \frac{\beta_{m_1+1} + \alpha_{m_1+1}\eta_{m_1+1}}{1-\alpha_{m_1+1}\xi_{m_1+1}}. \tag{7.17}$$

Now the solution at the points $(x_i, t_{n+\alpha/p})$, $i = m_1+1,\ldots,m_2$, is determined by the left sweep formulas:

$$y_{i+1}^{n+\alpha/p} = \xi_{i+1}y_i^{n+\alpha/p} + \eta_{i+1}, \quad i=m_1,\ldots,m_2-1. \tag{7.18}$$

Letting consecutively $\alpha = 1,2,\ldots,p$, from formulas (7.13)–(7.18) we find the solution in the domain $\overline{\omega}_2^n$, i.e., at the nodes $(x_i, t_{n+\alpha/p})$, $i=\overline{m_1,m_2}$. The solution at the points (x_i, t_{n+1}), $i = m_2+1, \ldots, N-1$ provided that $\alpha = p$ is calculated by formula (7.18), where $i = m_2,\ldots,N-2$, and at the nodes (x_i, t_{n+1}), $i=\overline{1,m_1}$, by the right sweep formulas

$$y_i^{n+1} = \alpha_{i+1}y_{i+1}^{n+1} + \beta_{i+1}, \quad i=m_1-1,m_1-2,\ldots,1. \tag{7.19}$$

By analogous reasoning, at first we can determine $y_{m_2}^{n+\alpha/p}$, and for finding the solution at the points $(x,t) \in \omega_2^n$ we use the right sweep formulas (7.19).

Actually, the algorithm stated above generalizes in a sense the opposite direction sweep method on the non-stationary case. Note some merits of the method suggested. Since in the domain of the smooth solution ω_1^n it is not required to calculate the values of the grid solution, we have an economy of $2(N-(m_2-m_1)-3)$ arithmetical operations on every fractional level $t_{n+\alpha/p}$. This fact allows one to save 12—40% of machine time provided that $m_2-m_1-3 \ll N$ (depending on the structure of the equation coefficients [Hockney and Eastwood, 1981]). In the nonlinear case, when it is necessary to use an iterative process, the economy can be more significant [Matus, 1993a]. Moreover, during the transition from the level t_{n+1} to t_{n+2} position of the adaptation domain can be changed depending on motion of the singularity of the solution in the calculation process (see Fig. 7.1).

Further, during the study of stability the following lemma about the equivalence of the difference schemes which allow the reduction of the scheme (7.8) to the form (7.9) will often be used.

LEMMA 7.1 *The difference scheme*

$$\frac{y_{(\alpha)}-y^n}{\alpha\tau_0}=(\Lambda y+f)_{(\alpha)}, \tag{7.20}$$

for any $\alpha=1,2,\dots,p$, *is algebraically equivalent to the scheme*

$$y_{\bar{t},\alpha}=(\Lambda y+f)^{(\alpha)}, \tag{7.21}$$

where

$$y_{(\alpha)}=y_i^{n+\alpha/p}, \quad y_{\bar{t},\alpha}=\left(y_{(\alpha)}-y_{(\alpha-1)}\right)/\tau_0. \tag{7.22}$$

Proof. We consider the grid equation (7.20) provided that $\alpha=2$:

$$\frac{y_{(2)}-y^n}{\tau_0}=2\,(\Lambda y+f)_{(2)}\,. \tag{7.23}$$

Using equation (7.20), provided $\alpha=1$, let us reduce the left-hand side of the last equality:

$$\frac{y_{(2)}-y_{(1)}}{\tau_0}+\frac{y_{(1)}-y^n}{\tau_0}\equiv y_{\bar{t},2}+(\Lambda y+f)_{(2)}\,.$$

Substituting the latter into equation (7.23), we obtain

$$y_{\bar{t},2}=(\Lambda y+f)^{(2)}\,.$$

The proof is by induction over α. Assume the lemma holds for all $\alpha=3,\dots,p-1$. We demonstrate the algebraic equivalence of the schemes also for $\alpha=p$. For equation (7.20) $\alpha=p$ can be written in the form

$$\frac{y_{(p)}-y^n}{\tau_0}=p\,(\Lambda y+f)_{(p)}\,. \tag{7.24}$$

By the induction assumption we have

$$\frac{y^{n+1}-y^n}{\tau_0} = \sum_{\alpha=1}^{p} y_{\bar{t},\alpha} = y_{\bar{t},p} + \sum_{\alpha=1}^{p-1} (\Lambda y + f)^{(\alpha)}$$
$$= y_{\bar{t},p} + (p-1)\,(\Lambda y + f)_{(p-1)}\,.$$

Consequently equation (7.24) can be written in the form

$$y_{\bar{t},p} = p\,(\Lambda y + f)_{(p)} + (1-p)\,(\Lambda y + f)_{(p-1)} = (\Lambda y + f)^{(p)}\,.$$

This completes the proof of the lemma.

By virtue of Lemma 7.1 the difference scheme (7.8), (7.9) can be transformed into the scheme with a variable weight

$$y_{\bar{t},\alpha} = (\Lambda y + f)^{(\sigma_\alpha)}, \quad (x,t) \in \omega_{h\tau_0}, \tag{7.25}$$

where the weight $\sigma_\alpha = \sigma_\alpha(x_i, t_{n+\alpha/p})$ is determined by formula (7.12). If the process of changing the weighting factor $\alpha > 1$ has already been fixed, then we may select order of the parameter $\sigma \geq 0$. Moreover, in the domain $\overline{\omega}_2^n$ we can obtain the scheme of the prescribed properties.

To simplify the study the boundary conditions will be assumed homogeneous ($\mu_1 = \mu_2 = 0$). Defining the self-adjoint operator $A : H \to H$ in a standard way with respect to formula (6.1.15) in the space H of the grid functions that vanish when $x = 0, l,$, with the inner product and the norm

$$(y, v) = \sum_{i=1}^{N-1} y_i v_i h, \quad \|y\| = \sqrt{(y,y)}. \tag{7.26}$$

We transform the difference scheme (7.25) into a scheme with operator weighting (see (3.1.19)):

$$y_{\bar{t},\alpha} + \left(Ay_{(\alpha)}\right)^{(\Sigma)} = \varphi(t), \quad t = t_{n+\alpha/p}, \tag{7.27}$$

$$y(0) = u_0, \tag{7.28}$$

where

$$\varphi = f^{(\Sigma)}, \quad \Sigma(t) = \mathrm{diag}\left\{\sigma^n_{\alpha+1,1}, \sigma^n_{\alpha+1,2}, \ldots, \sigma^n_{\alpha+1,N-1}\right\}.$$

In order to use Theorem 3.5 it is necessary to show that

$$A(t) \geq \delta E, \quad \delta > 0, \quad A\Sigma \neq \Sigma A, \quad t \in \omega_{\tau_0}, \tag{7.29}$$

$$\|(A(t+\tau) - A(t))\,u\| \leq \tau c_0\,\|Au\|, \quad t \in \omega_{\tau_0}. \tag{7.30}$$

Note that the positiveness of the operator $A(t)$ follows from the imbedding (6.1.21) provided that we have $\delta = \frac{8c_1}{l^2}$. By their structure the operators $A(t)$ and $\Sigma(t)$ are not commutative. Now show that for the operator $Ay = -\left(ay_{\bar{x}}\right)_x$ the Lipschitz continuity condition of the operator $A(t)$ with respect to variable t (7.30) holds provided that

$$\left|\left(a_t a^{-1}\right)_x a_+\right| \le c_3, \quad \left|a_t a^{-1}\right| \le c_4, \quad a \ge c_1 > 0, \tag{7.31}$$

for all $(x,t) \in \omega_h \times \omega_{\tau_0}$, $a_+ = a(x+h,t)$, where c_3, c_4 are positive constants. Really,

$$\left\|\left(\hat{A} - A\right) y\right\| = \tau \left\|(a_t y_{\bar{x}})_x\right\| = \tau \left\|\left(a_t a^{-1}\right)_x a_+ y_x + a_t a^{-1}\left(a y_{\bar{x}}\right)_x\right\|$$
$$\le \tau\left(c_3 \; \|y_{\bar{x}}]| + c_4 \|Ay\|\right).$$

From the sequence of inequalities

$$c_1 \; \|y_{\bar{x}}]|^2 \le \left(a, y_{\bar{x}}^2\right] = -\left(y, (ay_{\bar{x}})_x\right) \le \|y\| \, \|Ay\| \le \frac{l^2}{8} \; \|y_{\bar{x}}]| \; \|Ay\|$$

it follows that

$$\|y_{\bar{x}}]| \le \frac{l^2}{8c_1} \|Ay\|. \tag{7.32}$$

Consequently the Lipschitz condition (7.30) is satisfied with the constant

$$c_0 = \frac{c_3 l^2}{8c_1} + c_4.$$

Let us examine the correctness of the sufficient condition of stability

$$\Sigma(t) \ge \frac{1}{2} E.$$

Using (7.12), (3.1.24) we conclude that the given operator inequality is correct for $\sigma \ge 0.5$, and for any τ, h for the solution of the difference scheme (7.8), (7.9) with the homogeneous boundary conditions the following *a priori* estimate holds:

$$\max_{t \in \omega_{\tau_0}} \|Ay\| \le M_1 \|A_0 y_0\| + M_2 \max_{t \in \omega_{\tau_0}} \left(\|\varphi\| + \|\varphi_t\|\right). \tag{7.33}$$

The earlier investigated scheme (7.8), (7.9) has only the first order of approximation on the variable t. To solve of the original differential problem, we apply the difference scheme of the form

$$\begin{aligned} \frac{y_{(\alpha)} - y^n}{\alpha\tau_0} &= \sigma_1 (\Lambda_1 y)_{(\alpha)} + (1-\sigma_1)\left(\Lambda_1 y\right)^n, \quad (x,t) \in \omega_1^n, \\ \frac{y_{(\alpha)} - y_{(\alpha-1)}}{\tau_0} &= \sigma (\Lambda_1 y)_{(\alpha)} + (1-\sigma)\left(\Lambda_1 y\right)_{(\alpha-1)}, \quad (x,t) \in \omega_2^n, \\ \Lambda_1 y &= \Lambda y + \varphi, \end{aligned} \tag{7.34}$$

which on an adaptive-time grid (see Fig. 7.1), provided $\sigma_1 = \sigma = 0.5$, has the second order of approximation $\psi = O\left(h^2 + \tau_0^2\right)$. Below, the corresponding accuracy estimates of the given scheme will performed. Everywhere further we assume that the solution $u(x,t)$ and the coefficients of equation (7.1) have the bounded derivatives necessary in the course of consideration of bounded derivatives. In investigating the rate of convergence of a solution for the difference scheme it would be natural to use the *a priori* estimate (7.33) that expresses unconditional stability with respect to both initial data and the right hand side. However, as is shown below, the difference scheme (7.34) even for smooth solutions does not approximate the original problem in the norm (7.33). The existence of this paradox significantly complicates the study of the convergence of the above-analyzed schemes with variable weights.

Using Lemma 7.1, we write scheme (7.34) in the form

$$
\begin{aligned}
y_{\bar{t},\alpha} &= \sigma_1 \left(\Lambda_1 y\right)^{(\alpha)} + (1-\sigma_1)\left(\Lambda_1 y\right)^n, \quad (x,t) \in \omega_1^n, \\
y_{\bar{t},\alpha} &= \sigma(\Lambda_1 y)_{(\alpha)} + (1-\sigma)\left(\Lambda_1 y\right)_{(\alpha-1)}, \quad (x,t) \in \omega_2^n,
\end{aligned}
\tag{7.35}
$$

$$
y(x_i,0) = u_0(x_i), \quad y_{(\alpha),0} = \mu_1^{n+\alpha/p}, \quad y_{(\alpha),N} = \mu_2^{n+\alpha/p}. \tag{7.36}
$$

To simplify calculations, in what follows we assume that

$$
\sigma_1 = \sigma, \quad a = k(x_{i-0.5},t), \quad \varphi = f, \quad t \in \omega_{\tau_0}. \tag{7.37}
$$

Substituting $y = z + u$ into the scheme (7.35), (7.36) we obtain the problem for the error of the method:

$$
\begin{aligned}
z_{\bar{t},\alpha} =& \sigma\alpha(\Lambda z)_{(\alpha)} + \sigma(1-\alpha)\left(\Lambda z\right)_{(\alpha-1)} \\
&+ (1-\sigma)\Lambda z^n + \psi_{1(\alpha)}, \quad (x,t) \in \omega_1^n,
\end{aligned}
\tag{7.38}
$$

$$
z_{\bar{t},\alpha} = \sigma(\Lambda z)_{(\alpha)} + (1-\sigma)\left(\Lambda z\right)_{(\alpha-1)} + \psi_{2(\alpha)}, \quad (x,t) \in \omega_2^n, \tag{7.39}
$$

$$
z_{(\alpha),0} = z_{(\alpha),N} = z(x,0), \quad x \in \overline{\omega}_h. \tag{7.40}
$$

It can be easily seen that

$$
\psi_{k(\alpha)} = \begin{cases} \mathrm{O}(h^2 + \tau_0), & \text{if} \sigma \neq 0.5, \\ O(h^2 + \tau_0^2), & \text{if} \sigma = 0.5. \end{cases} \tag{7.41}
$$

However, it will now be shown that $\psi_{1\bar{t},\alpha} = O(1)$. To simplify the reasoning we assume in the scheme (7.38) that $\sigma = 1$. Then $\psi_{1(\alpha)}$ can be represented in the form

$$
\psi_{1(\alpha)} = -u_{\bar{t},\alpha} + \left(\Lambda_1 u\right)^{(0.5)} + \eta_{1(\alpha)}, \quad \eta_{1(\alpha)} = (\alpha - 0.5)\tau_0 \left(\Lambda_1 u\right)_{\bar{t},\alpha}
$$

and since $(\alpha)_{\bar{t},\alpha} = \tau_0^{-1}$, we obtain $\eta_{1\bar{t},\alpha} = O(1)$. Consequently in the norm (7.33) the difference scheme does not approximate the original problem. The latter does not allow the use of the *a priori* estimates, obtained above, for investigating the convergence rate of the difference scheme.

For the grid functions prescribed on $\overline{\omega}_h$, we define the inner products and the norms:

$$(u,v)_1 = \sum_{\substack{i=1 \\ i\neq m_1+1,\dots,m_2-1}}^{N-1} hu_iv_i, \quad (u,v)_2 = \sum_{i=m_1+1}^{m_2-1} hu_iv_i, \quad m_k = m_k^n,$$

$$\|u\|^2 = \|u\|_1^2 + \|u\|_2^2 = \sum_{i=1}^{N-1} hu_i^2, \quad \left\|z_{(\alpha)}\right\|_A^2 = \left(a_{(\alpha)}, z_{(\alpha)\bar{x}}^2\right],$$

$$Q_{(\alpha)}^2 = \left\|z_{(\alpha)}\right\|_A^2 + \tau_0\sigma\alpha\left\|(\Lambda z)_{(\alpha)}\right\|_1^2 + \tau\,|1-\sigma|\,\left\|(\Lambda z)_{(\alpha)}\right\|_2^2.$$

We prove the following statement:

THEOREM 7.1 *The difference scheme (7.35), (7.36) converges unconditionally in the norm C with the rate $O(h^2+\tau_0)$ provided that*

$$1 - \frac{2(1-\varepsilon)}{p+3} \le \sigma \le 1 + \frac{2(1-\varepsilon)}{p-1}, \quad 0 < \varepsilon \le 1,$$

so that under sufficiently small $\tau_0 < \tau_0^*$ for all $\alpha = 1,2,\dots,p$, $n = 0,1,\dots, N_0-1$ the following estimates hold:

$$\left\|z_{(\alpha)}\right\|_A^2 + \tau_0\,|1-\sigma|\,\left\|(\Lambda z)_{(\alpha)}\right\|^2 \le c\left(h^2+\tau_0\right)^2,$$
$$\left\|y_{(\alpha)} - u_{(\alpha)}\right\|_{C(\overline{\omega}_h)} \le c\left(h^2+\tau_0\right).$$

Here and hereafter $c > 0$ is a constant independent of h and τ_0, $y_{(\alpha)}$ and every specific case has its own constant.

Proof. Without restricting generality we assume that the heat conduction coefficient depends only on variable the x. Taking the dot products of equations (7.38), (7.39) with $-2\tau_0\Lambda z_{(\alpha)}$ in the corresponding domains and summing both results, we obtain the energy equality

$$\begin{aligned}\left\|z_{(\alpha)}\right\|_A^2 - \left\|z_{(\alpha-1)}\right\|_A^2 + \tau_0^2\left\|z_{\bar{t},\alpha}\right\|_A^2 + 2\tau_0\sigma\left(\alpha\left\|\Lambda z_{(\alpha)}\right\|_1^2 + \left\|\Lambda z_{(\alpha)}\right\|_2^2\right) \\ = -2\tau_0\left\{\left(\Lambda z_{(\alpha)}, \sigma(1-\alpha)\Lambda z_{(\alpha-1)} + (1-\sigma)\Lambda z^n + \psi_{1(\alpha)}\right)_1\right. \\ \left. + \left(\Lambda z_{(\alpha)}, (1-\sigma)\Lambda z_{(\alpha-1)} + \psi_{2(\alpha)}\right)_2\right\}.\end{aligned} \tag{7.42}$$

We apply the Cauchy inequality with ε to the inner products from the right hand side (7.42):

$$-2\tau_0\sigma(1-\alpha)\big(\Lambda z_{(\alpha)},\Lambda z_{(\alpha-1)}\big)_1 \le \tau_0\sigma(\alpha-1)\left(\left\|\Lambda z_{(\alpha)}\right\|_1^2+\left\|\Lambda z_{(\alpha-1)}\right\|_1^2\right),$$

$$-2\tau_0(1-\sigma)\big(\Lambda z_{(\alpha)},\Lambda z^n\big)_1 \le \tau_0\left|1-\sigma\right|\left(\left\|\Lambda z_{(\alpha)}\right\|_1^2+\left\|\Lambda z^n\right\|_1^2\right),$$

$$-2\tau_0(1-\sigma)\big(\Lambda z_{(\alpha)},\Lambda z_{(\alpha-1)}\big)_2 \le \tau_0\left|1-\sigma\right|\left(\left\|\Lambda z_{(\alpha)}\right\|_2^2+\left\|\Lambda z_{(\alpha-1)}\right\|_2^2\right),$$

$$-2\tau_0\Big(\big(\Lambda z_{(\alpha)},\psi_{1(\alpha)}\big)_1+\big(\Lambda z_{(\alpha)},\psi_{2(\alpha)}\big)_2\Big)$$
$$\le \tau_0\varepsilon\left(\left\|\Lambda z_{(\alpha)}\right\|_1^2+2\left\|\Lambda z_{(\alpha)}\right\|_2^2\right)+\tau_0\left\|\psi_{(\alpha)}\right\|^2,$$

$$\left\|\psi_{(\alpha)}\right\|^2=\varepsilon^{-1}\left(\left\|\psi_{1(\alpha)}\right\|_1^2+0.5\left\|\psi_{2(\alpha)}\right\|_2^2\right)\le c\left(h^2+\tau_0\right)^2.$$

Substituting the estimates obtained into equality (7.42) and taking into account the fact that on satisfying the condition of the theorem

$$\sigma-|1-\sigma|-\varepsilon>0,\quad 2\sigma-|1-\sigma|-2\varepsilon\ge|1-\sigma|\,p,$$

we obtain the recurrence relation

$$Q_{(\alpha)}^2\le Q_{(\alpha-1)}^2+\tau_0\left|1-\sigma\right|\left\|\Lambda z^n\right\|_1^2+\tau_0\left\|\psi_{(\alpha)}\right\|^2. \tag{7.43}$$

Summing the latter expression over $\alpha=1,2,\dots,p$ and using the inequality $\sigma\ge|1-\sigma|$, we obtain

$$\|\hat z\|_A^2+\tau\left|1-\sigma\right|\left\|\Lambda\hat z\right\|^2\le\|z\|_A^2+\tau\left|1-\sigma\right|\left\|\Lambda z\right\|^2+\tau\max_{1\le\alpha\le p}\left\|\psi_{(\alpha)}\right\|^2. \tag{7.44}$$

It is from relations (7.43), (7.44) that the required estimates of accuracy follow.

2.2 Conservative Schemes

We have considered above the method of construction and implementation of unconditionally stable difference schemes in the sub-domains by using an adaptive grid with respect to a time variable. The given method guarantees fulfilment of grid analogies of conservation laws in every computational domains. However, in the whole of the grid the property of conservatism is violated, because in the case of the variable weight $\sigma=\sigma(x)$, $x\in\omega_h$ we have

$$\left(\left(y_{\bar x}\right)^{(\sigma)}\right)_x\ne\left(y_{\bar xx}\right)^{(\sigma)}.$$

Here conservative unconditionally stable difference schemes on adaptive grids of composite type will be constructed and investigated. It is of interest to note that despite the presence of conventional approximation, at the points of joining the calculated sub-domains one succeeds in proving the unconditional convergence of the considered schemes in the metric C with the rate $O(h^2 + \sqrt{\tau_0})$.

We turn again to the first-kind boundary value problem for the heat conduction equation with variable coefficients (7.1)–(7.3). Employing an integro-interpolational method [Samarskii, 1989], the original problem on the grid $\omega_{h\tau_0} = \omega_h \times \omega_{\tau_0}$ introduce above is approximated by the difference one:

$$y_{\bar{t},\alpha} = \left((ay_{\bar{x}})^{(\sigma_\alpha)}\right)_x + f^{(\sigma_\alpha)}, \quad (x,t) \in \omega_{h\tau_0}, \tag{7.45}$$

$$y_i^0 = u_0(x_i), \quad y_0^{n+\alpha/p} = \mu_1(t_{n+\alpha/p}), \quad y_N^{n+\alpha/p} = \mu_2(t_{n+\alpha/p}), \tag{7.46}$$

where

$$\sigma_{\alpha,i}^n = \begin{cases} \alpha, & \text{if} i = 1, 2, \ldots, m_1^n + 1, m_2^n, \ldots, N, \\ \sigma, & \text{if} i = m_1^n + 2, \ldots, m_2^n - 1. \end{cases} \tag{7.47}$$

Here $\sigma = \text{const} > 0$ is a variable numerical parameter selected on of stability grounds, $a = a(x_i, t_n)$ is a certain pattern functional. The conservative scheme constructed relates to a class of difference scheme with a variable weight σ_α depending on the node of the grid $(x_i, t_{n+\alpha/p})$.

We show how by means of the indicated difference scheme the solution can be found on fractional levels $t_{n+\alpha/p}$, $0 < \alpha < p$ only in the domain ω_2^n (see Fig. 7.1). We note that provided $(x,t) \in \omega_1^n$

$$\Lambda y^{(\sigma_\alpha)} = \left((ay_{\bar{x}})^{(\sigma_\alpha)}\right)_x = ((ay_{\bar{x}})_x)^{(\alpha)} = (\Lambda y)^{(\alpha)},$$

and by virtue of Lemma 7.1, scheme (7.45) can be transformed as

$$\begin{aligned} &\frac{y_{(\alpha)} - y^n}{\alpha\tau_0} = (\Lambda y + f)_{(\alpha)}, && (x,t) \in \omega_1^n, \\ &\frac{y_{(\alpha)} - y_{(\alpha-1)}}{\tau_0} = \Lambda y^{(\sigma_\alpha)} + f^{(\sigma_\alpha)}, && (x,t) \in \omega_2^n. \end{aligned} \tag{7.48}$$

On every fractional level the difference equations (7.48) are reduced to the system of three-points equations:

$$A_i y_{i-1}^{n+\alpha/p} - C_i y_i^{n+\alpha/p} + B_i y_{i+1}^{n+\alpha/p} = -F_i, \quad i = \overline{1, N-1}; \tag{7.49}$$

moreover, the condition of the stability of the sweep method ($C_i = 1 + A_i + B_i$) and requirement of scheme conservatism $A_i = B_{i-1}$ are satisfied. It is easy to see that the right hand side of equation (7.49) does not depend on the values of the grid function $y_{(\alpha-1)}$ provided that $(x,t) \in \omega_1^n$. Consequently to find the solution for $\alpha < p$ only in the domain ω_2^n $(i = m_1^n, \dots, m_2^n)$ we can use the opposite direction sweep method. For $\alpha = p$ it is required to find the solution y^{n+1} for all $x \in \omega_h$, which is done in a standard way.

To study the stability of the difference scheme (7.45) with homogeneous boundary conditions, we write it in the canonical form of two-level difference schemes:

$$B_{\alpha-1} y_{\bar{t},\alpha} + (Ay)_{\alpha-1} = \varphi_{(\alpha-1)}, \quad t = t_{\alpha-1}, \quad y(0) = u_0. \tag{7.50}$$

In the case considered we have

$$B = B(t) = E + \tau_0 R, \quad (Ry)_{\alpha-1} = -\left(\sigma_\alpha a_{(\alpha)} y_{(\alpha-1)\bar{x}}\right)_x, \quad x \in \omega_h,$$

$$(Ay)_{\alpha-1} = A_{\alpha-1} y_{\alpha-1} = -\left(a^{(\sigma_\alpha)} y_{(\alpha-1)\bar{x}}\right)_x, \quad x \in \omega_h,$$

$$A_{\alpha-1} = A(t_{\alpha-1}), \quad y = 0, \quad x \in \gamma_h, \quad \varphi_{(\alpha-1)} = f^{(\sigma_\alpha)}.$$

To use Theorem 2.17 it is necessary to show that

1° $A(t) = A^*(t)$, $\quad B(t) > 0$ for all $t \in \omega_{\tau 0}$;

2° the operator $A(t)$ is Lipschitz continuous with respect to t.

To be explicit, we assume later that the stencil functional $a(x_i, t)$, $t \in \omega_{\tau 0}$ is defined by one of the simplest formulas

$$a = k(x_{i-0.5}, t) \quad \text{or} \quad a = 0.5\left(k(x_{i-1}, t) + k(x_i, t)\right), \tag{7.51}$$

and the heat conduction coefficient $k(x,t)$ is Lipschitz continuous in t with a constant c_0:

$$|k(x,t) - k(x, t-\tau_0)| \le \tau_0 c_0 k(x, t-\tau_0). \tag{7.52}$$

Then self-adjoint character of the operator $A = A^*$ follows from the difference analogy of the second Green's formula. Let us prove the positiveness of the operator $A(t)$. Previously we note that for $\tau_0 < (\sigma_\alpha c_0)^{-1}$ we have the relation

$$a^{(\sigma_\alpha)} = a_{\alpha-1} + \tau_0 \sigma_\alpha a_{\bar{t},\alpha} \ge (1 - \tau_0 c_0 \sigma_\alpha)\, c_1 \ge c_3 > 0.$$

Consequently, on the basis of the imbedding $\|y_{\bar{x}}]| \ge 2\sqrt{2}\,\|y\|/l$ we obtain the estimate

$$\left((Ay)_{(\alpha-1)}, y_{(\alpha-1)}\right) = \left(a^{(\sigma_\alpha)}, y^2_{(\alpha-1)\bar{x}}\right] \ge 8c_3 \left\|y_{(\alpha-1)}\right\|^2 / l^2,$$

i.e., $A(t) > 0$. Since σ_α, $a_{(\alpha)} > 0$ we have that the positiveness of the operator $B(t)$ follows from the inequality

$$\left((By)_{(\alpha-1)}, y_{(\alpha-1)}\right) = \left\|y_{(\alpha-1)}\right\|^2 + \tau_0 \left(\sigma_\alpha a_{(\alpha)}, y^2_{(\alpha-1)\bar{x}}\right] \geq (1 + \tau_0 c_4) \left\|y_{(\alpha-1)}\right\|^2,$$

where $c_4 = 8\sigma^* c_1/l^2 > 0$, $\sigma^* = \min\{1, \sigma\}$.

The proof of the Lipschitz continuity for the operator $A(t)$ is based on application of inequality (7.52) and the identity

$$a^{(\sigma_\alpha)} - a^{(\sigma_{\alpha-1})} = \tau_0 \left(\sigma_\alpha a_{\bar{t},\alpha} + (1 - \sigma_{\alpha-1})\, a_{\bar{t},\alpha-1}\right).$$

So the conditions 1°, 2° are satisfied. In the case of variable operators $A(t)$, $B(t)$ let us define the norms

$$\left\|y_{(\alpha-1)}\right\|_{A_{\alpha-1}} = \left(a^{(\sigma_\alpha)}, y^2_{(\alpha-1)\bar{x}}\right]^{1/2},$$
$$\left\|\varphi_{(\alpha-1)}\right\|_{A^{-1}_{\alpha-1}} = \left(\left(A^{-1}\varphi\right)_{\alpha-1}, \varphi_{(\alpha-1)}\right)^{1/2}.$$

We check the correctness of the following sufficient condition of stability:

$$B(t) \geq \varepsilon E + 0.5\tau_0 A(t), \quad t \in \omega_{\tau_0}, \quad 0 < \varepsilon \leq 1.$$

Employing the Green difference formula, we obtain that

$$\begin{aligned}
&((B - \varepsilon E - 0.5\tau_0 A)_{\alpha-1} y_{(\alpha-1)}, y_{(\alpha-1)}) \\
&= (1-\varepsilon)\left\|y_{(\alpha-1)}\right\|^2 + \tau_0 \left(\sigma_\alpha a_{(\alpha)} - 0.5 a^{(\sigma_\alpha)}, y^2_{(\alpha-1)\bar{x}}\right] \\
&= (1-\varepsilon)\left\|y_{(\alpha-1)}\right\|^2 \\
&\quad + \tau_0 \left(a^{(0.5)}_{(\alpha)} \left(\sigma_\alpha - 1/\left(2 + \tau_0 a^{-1}_{(\alpha-1)} a_{\bar{t},\alpha}\right)\right), y^2_{(\alpha-1)\bar{x}}\right].
\end{aligned}$$

Consequently, for $\sigma_\alpha \geq \sigma_0$, $\sigma_0 = 1/(2 - \tau_0 c_0)$, $\tau_0 \leq c_0^{-1}$, the sufficient condition of stability of the difference scheme (7.50) is satisfied, and on the basis of Theorems 2.17, 2.21 the *a priori* estimates hold

$$\left\|y_{(\alpha-1)}\right\|^2_{A_{\alpha-1}} \leq M_1^2 \left\{ \left\|y_0\right\|^2_{A_{0,0}} + \frac{1}{2\varepsilon} \sum_{k=0}^{pn+\alpha-1} \tau_0 \left\|\varphi(\tau_0 k)\right\|^2 \right\}, \tag{7.53}$$

$$\left\|y_{(\alpha-1)}\right\|_{A_{\alpha-1}} \leq M_1 \left(\left\|y_0\right\|_{A_{0,0}} + \max_{0 \leq k \leq pn+\alpha-1} \left(\left\|\varphi_k\right\|_{A_k^{-1}} + \left\|\varphi_{\bar{t},k}\right\|_{A_k^{-1}} \right)\right), \tag{7.54}$$

where

$$A_\alpha = A(t_\alpha) = A(t_{n+\alpha/p}), \quad A_{0,0} = A(0), \quad M_1 = e^{0.5cT}, \quad 0 < \varepsilon \le 1.$$

REMARK 7.1 In the case of $A \neq A(t)$ (constant operator) the operators A and B in a scheme (7.50) can be represented in the conservative form (3.80):

$$A = T^*T, \quad B = E + \tau_0 T^* G T, \tag{7.55}$$

where T^*, T are defined in Section 6.6, and

$$G(t) = \operatorname{diag}\{\sigma_1(t), \sigma_2(t), \dots, \sigma_N(t)\}, \quad t \in \omega_{\tau_0}.$$

Sufficient conditions of stability with respect to initial the data and right hand side (see (3.3.7)):

$$G(t) \ge \sigma_0 E, \quad \sigma_0 = \frac{1}{2} - \frac{1}{\tau_0 \|A\|}, \quad \|A\| < \frac{4c_2}{h^2},$$

and (see (3.3.18))

$$G(t) \ge \sigma_\varepsilon E, \quad \sigma_\varepsilon = \frac{1}{2} - \frac{1-\varepsilon}{\tau_0 \|A\|}, \quad 0 < \varepsilon \le 1,$$

are satisfied provided that

$$\sigma \ge \frac{1}{2} - \frac{h^2}{4\tau_0 c_2}, \tag{7.56}$$

$$\sigma \ge \frac{1}{2} - \frac{1-\varepsilon}{4c_2}\frac{h^2}{\tau_0}, \quad 0 < \varepsilon \le 1, \tag{7.57}$$

respectively.

Taking into account the *a priori* estimates (3.3.16), (3.3.17) we conclude that the conservative difference scheme on the adaptive time grid (7.45), (7.46), provided $\mu_1(t) = \mu_2(t) = 0$, $t \in \omega_{\tau_0}$, is absolutely (without restrictions on the sufficient smallness of the steps τ_0, h) stable with respect to the initial data, right hand side, and for its solution under condition (7.56) the following estimates hold:

$$\|y_{n+1}\|_A \le \|y_0\|_A + \|\varphi_0\|_{A^{-1}} + \|\varphi_n\|_{A^{-1}} + \sum_{t\in\omega_{\tau_0}} \tau_0 \|\varphi_t\|_{A^{-1}},$$

$$\|y_{n+1}\|_A^2 \le \|y_0\|_A^2 + \frac{1}{2\varepsilon} \sum_{t\in\omega_{\tau_0}} \tau_0 \|\varphi\|^2,$$

when the parameter σ satisfies restriction (7.57).

Let us study the error of approximation and convergence of the difference scheme considered. Substituting $y = z + u$ into equations (7.45), (7.46), we obtain the problem for an error

$$\begin{aligned} z_{\bar{t},\alpha} &= \Lambda z^{(\sigma_\alpha)} + \psi_{(\alpha-1)}, \quad x \in \omega_h, \quad t \in \omega_{\tau_0}, \\ z_{(\alpha),0} &= z_{(\alpha),N} = 0, \quad z(x,0) = 0, \quad x \in \overline{\omega}_h, \end{aligned} \tag{7.58}$$

where

$$\Lambda z^{(\sigma_\alpha)} = \left((a z_{\bar{x}})^{(\sigma_\alpha)}\right)_x, \quad \psi_{(\alpha-1)} = -u_{\bar{t},\alpha} + \Lambda u^{(\sigma_\alpha)} + f^{(\sigma_\alpha)}.$$

Obviously, under the sufficient smoothness of the solution $u(x,t)$ and initial data we have

$$\psi_{(\alpha-1)} = \begin{cases} O(h^2 + \tau_0), & \text{if} i \neq m_1^n + 1, m_2^n - 1, \\ O(h^2 + \tau_0 h^{-1}), & \text{if} i = m_1^n + 1, m_2^n - 1, \end{cases}$$

i.e., at the points of joining of the grid domains with the different time steps the truncation error has a conditional character. In order to employ the estimate of the truncation error in the negative norm, containing no differentiation with respect to t, we use the balance equation on the interval $x_{i-0.5} \le x \le x_{i+0.5}$ for $\bar{t} = t_{n+(\alpha-1)/p}$ which takes the form

$$0 = \frac{1}{h} \int\limits_{x_{i-0.5}}^{x_{i+0.5}} \frac{\partial u}{\partial t}(x,\bar{t})dx - \left(k \frac{\partial u(x_{i-0.5},\bar{t})}{\partial x}\right)_x - \frac{1}{h} \int\limits_{x_{i-0.5}}^{x_{i+0.5}} f(x,\bar{t})dx,$$

and the identity $v^{(\sigma_\alpha)} = v_{(\alpha-1)} + \tau_0 \sigma_\alpha v_{\bar{t},\alpha}$. The truncation error $\psi_{(\alpha-1)}$ can then be written as follows:

$$\psi_{(\alpha-1)} = \eta_{(\alpha-1)x} + \psi_{1(\alpha-1)}, \quad \eta_{(\alpha-1)} = \tau_0 \sigma_\alpha (a u_{\bar{x}})_{\bar{t},\alpha},$$

$$\begin{aligned} \psi_{1(\alpha-1)} = &\left((a u_{\bar{x}})_{\alpha-1} - \left(k \frac{\partial u}{\partial x}\right)\Big|_{t=\bar{t}}\right)_x + f^{(\sigma_\alpha)} - \frac{1}{h} \int\limits_{x_{i-0.5}}^{x_{i+0.5}} f(x,\bar{t})dx \\ &+ \frac{1}{h} \int\limits_{x_{i-0.5}}^{x_{i+0.5}} \frac{\partial u}{\partial t}(x,\bar{t})dx - u_{\bar{t},\alpha}. \end{aligned}$$

Obviously, for all $(x,t) \in \omega_{h\tau_0}$ we have

$$\eta_{(\alpha-1)} = O(\tau_0), \quad \psi_{1(\alpha-1)} = O(h^2 + \tau_0). \tag{7.59}$$

THEOREM 7.2 *Let*

$$u(x,t) \in C_2^4(\overline{Q}_T), \quad k(x,t) \in C_1^2(\overline{Q}_T), \quad f(x,t) \in C_1^2(\overline{Q}_T).$$

Then under the conditions

$$\sigma \geq 0.5(1+\varepsilon), \quad 0 < \varepsilon \leq 1, \tag{7.60}$$

the difference scheme (7.45), (7.46) unconditionally converges in the metric C such that for sufficient small $\tau_0 < \tau_0^*$ *and for all* $\alpha = 1, 2, \ldots, p$; $n = 0, 1, \ldots, N_0 - 1$ *the following estimate is valid:*

$$\left\| z^{n+\alpha/p} \right\|_{C(\overline{\omega}_h)} \leq c\left(h^2 + \sqrt{\tau_0}\right). \tag{7.61}$$

P r o o f. Unfortunately, to obtain the estimates of accuracy without limitations on the grid steps we cannot use the *a priori* estimate in the negative norm, because by virtue of dependence the of the weight σ_α on the variable $t \in \omega_{\tau_0}$ the relation $\eta_{\bar{t},\alpha} = O(1)$ holds.

We take the dot product of the equation for scheme error (7.58) with $2\tau_0 z_{\bar{t},\alpha}$. It is readily seen that

$$\begin{aligned} 2\tau_0\left(z_{\bar{t},\alpha}, \Lambda z^{(\sigma_\alpha)}\right) = &- \tau_0\left(\|z\|_1^2\right)_{\bar{t},\alpha} + \tau_0\left(a_{\bar{t},\alpha}, z^2_{(\alpha-1)\bar{x}}\right] \\ &- 2\tau_0^2\left(\left(\sigma_\alpha - 0.5\right)a_{(\alpha)}, z^2_{\bar{x}\bar{t},\alpha}\right] \\ &+ 2\tau_0^2\left(z_{\bar{x}\bar{t},\alpha}, (1-\sigma_\alpha)a_{\bar{t},\alpha}z_{(\alpha-1)\bar{x}}\right], \end{aligned}$$

where $\|z_{(\alpha)}\|_1 = \left\|a^{1/2}_{(\alpha)} z_{(\alpha)\bar{x}}\right]\Big|$. Since $\left|a_{\bar{t},\alpha}\right| \leq c_0 a_{\alpha-1}$, then applying the ε-inequality to the last term of the previous energy relation and summing the estimates obtained, we obtain the inequality

$$\begin{aligned} &2\tau_0\left\|z_{\bar{t},\alpha}\right\|^2 + 2\tau_0\left(a_{(\alpha)}\left(\sigma_\alpha - 0.5 - \varepsilon/4\right), z^2_{\bar{x}\bar{t},\alpha}\right] + \left\|z_{(\alpha)}\right\|_1^2 \\ &\qquad \leq \left(1 + \tau_0 c_1\right)\left\|z_{(\alpha-1)}\right\|_1^2 + 2\tau_0\left(z_{\bar{t},\alpha}, \psi_{(\alpha-1)}\right), \end{aligned} \tag{7.62}$$

where $c_1 = c_0\left(1 + 4c_0\tau_0^*\varepsilon^{-1}(p-1)^2\right)$. Using the expression for $\psi_{(\alpha-1)}$, the formula of summation by parts, the Cauchy inequality and ε-inequality, we estimate the last term an the right hand side of (7.62):

$$\begin{aligned} 2\tau_0\left(z_{\bar{t},\alpha}, \psi_{(\alpha-1)}\right) = &-2\tau_0\left(z_{\bar{x}\bar{t},\alpha}, \eta_{(\alpha-1)}\right] + 2\tau_0\left(z_{\bar{t},\alpha}, \psi_{1(\alpha-1)}\right) \\ &\leq 0.5\tau_0^2\varepsilon\left(a_{(\alpha)}, z^2_{\bar{x}\bar{t},\alpha}\right] + 2\tau_0\left\|z_{\bar{t},\alpha}\right\|^2 + \tau_0\left\|\psi_{2(\alpha-1)}\right\|^2, \end{aligned}$$

where $\left\|\psi_{2(\alpha-1)}\right\|^2 = 2\left(\tau_0\varepsilon c_1\right)^{-1}\left\|\eta_{(\alpha-1)}\right]\Big|^2 + 0.5\left\|\psi_{1(\alpha-1)}\right\|^2 = O\left(h^2 + \sqrt{\tau_0}\right)^2$. Substituting the estimate obtained into (7.62) and taking into account the theorem condition (7.60), we obtain the recurrence relation

$$\left\|z_{(\alpha)}\right\|_1^2 \leq (1+\tau_0 c_1)\left\|z_{(\alpha-1)}\right\|_1^2 + \tau_0 \left\|\psi_{2(\alpha-1)}\right\|^2. \tag{7.63}$$

Summing the last inequality over all $\alpha = 1, 2, \dots, p$, we obtain the estimate

$$\left\|z^{n+1}\right\|_1^2 \leq e^{\tau c_1}\left(\left\|z^n\right\|_1^2 + \tau \max_{1\leq\alpha\leq p}\left\|\psi_{2(\alpha-1)}\right\|^2\right).$$

By virtue of the arbitrariness of n we have

$$\left\|z^n\right\|_1 \leq t_n^{1/2} e^{0.5 c_1 t_n} \max_{0\leq k\leq n-1}\max_{1\leq\alpha\leq p}\left\|\psi_2^{n+\alpha/p}\right\|.$$

Consequently from relation (7.63) we immediately obtain

$$\left\|z_{(\alpha)}\right\|_1^2 \leq e^{c_1\tau\alpha/p}\left(\left\|z^n\right\|_1^2 + \tau\alpha/p \max_{1\leq k\leq\alpha}\left\|\psi_{2(k-1)}\right\|^2\right) \leq c_2\left(h^2+\sqrt{\tau_0}\right)^2.$$

From the imbedding $\left\|z_{(\alpha)}\right\|_1 \geq 2\sqrt{c_1 l^{-1}}\left\|z_{(\alpha)}\right\|_{C(\overline{\omega}_h)}$ it is follows that the required estimate of accuracy of (7.61).

REMARK 7.2 Theorem 7.2 is valid also for the difference schemes of more general form

$$\frac{y_{(\alpha)} - y^n}{\alpha\tau_0} = \sigma(\Lambda y + f)_{(\alpha)} + (1-\sigma)\left(\Lambda y + f\right)^n, \quad (x,t) \in \omega_1^n,$$
$$\frac{y_{(\alpha)} - y_{(\alpha-1)}}{\tau_0} = \Lambda y^{(\sigma_\alpha)} + f^{(\sigma_\alpha)}, \qquad (x,t) \in \omega_2^n,$$

that approximate the original problem with the order $O(h^2+\tau_0)$ provided that $\sigma \neq 0.5$ and with the order $O(h^2 + \tau_0^2)$ provided that $\sigma = 0.5$, $i \neq m_1^n + 1, m_2^n - 1$.

It is shown above that the non-conservative difference schemes converge unconditionally in the uniform metric with the rate $O(h^2 + \tau_0)$, and the conservative ones have the accuracy order $O\left(h^2 + \sqrt{\tau_0}\right)$. The loss of accuracy in the latter case is owed to the presence of conditional approximation at the points of joining the grid domains, which in turn is explained by discontinuity of the weighting function σ_α determined by formula (7.47).

REMARK 7.3 The convergence rate of conservative difference schemes on adaptive grids can be raised by smoothing the function $\sigma_\alpha(x)$, for example, in the following way:

$$\sigma_\alpha = \begin{cases} \alpha, & 0 < x \le x_{m_1+1}, x_{m_2} \le x < l, \\ \alpha + \dfrac{\sigma - \alpha}{\varepsilon}(x - x_{m_1+1}), & x_{m_1+1} \le x \le x_{m_1+1} + \varepsilon, \\ \sigma, & x_{m_1+1} + \varepsilon \le x \le x_{m_2} - \varepsilon, \\ \sigma + \dfrac{\alpha - \sigma}{\varepsilon}(x - (x_{m_2} - \varepsilon)), & x_{m_2} - \varepsilon \le x \le x_{m_2}. \end{cases} \tag{7.64}$$

In this case it is assumed that the singularity of the solution at $t = t_n$ is localized in the domain

$$\omega_3^n = \left\{(x, t_{n+\alpha/p}),\, x_{m_1+1} + \varepsilon \le x \le x_{m_2} - \varepsilon,\, \alpha = \overline{0,p}\right\}.$$

The difference solution of problem (7.45), (7.46), (7.64) can be found by the algorithm described above. In calculations it is reasonable to select the 'spreading' zone width ε in proportion to the quantity ph. The present difference schemes will be more effective at large p's. The conservative schemes with the smoothed weight (7.64), like in the non-conservative case, converge in the metric C with the rate $O(h^2 + \tau_0)$.

2.3 Difference Schemes for a Problem with Weak Solutions

For the non-conservative (7.8)–(7.10) and conservative (7.45)–(7.47) difference schemes on the adaptive time grids (see Fig. 7.1) $\omega_{h\tau_0}$ constructed above the corresponding unconditional estimates of the rate of precision are obtained on the assumption of sufficient smoothness of the solution and input data. Below, a new class of the schemes is constructed. For these schemes the accuracy estimates on the weak solutions $u \in W_2^{2,1}$ (see Section 6.2) are established. To simplify the presentation we consider the differential problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < l, \quad 0 < t < T, \\ &u(x,0) = u_0(x),\ x \in \Omega,\ u(0,t) = u(l,t) = 0,\ 0 < t < T. \end{aligned} \tag{7.65}$$

The given differential problem (7.65) will be approximated by the difference one:

$$y_{\bar{t},\alpha} = \left(y_{(\alpha)}^{(\sigma_\alpha)}\right)_{\bar{x}x} + \varphi_{(\alpha)},\ (x,t) \in \omega_{h\tau_0}, \quad \varphi_{(\alpha)} = \frac{1}{\tau_0} \int\limits_{t+(\alpha-1)\tau_0}^{t+\alpha\tau_0} S_x^2 f\, dt', \tag{7.66}$$

$$y(x,0) = S_x^2 u_0(x),\ x \in \omega_h, \quad y(0, t_{n+\alpha/p}) = y(l, t_{n+\alpha/p}) = 0,\ t \in \overline{\omega}_{\tau_0}, \tag{7.67}$$

$$\sigma_\alpha = \begin{cases} \alpha, & \text{if} 0 < x \le x_{m_1^n+1},\ x_{m_2^n-1} \le x < l, \\ \sigma, & \text{if} x_{m_1^n+1} < x < x_{m_2^n-1}, \end{cases}$$

$$S_x^2 f(x,t') = \frac{1}{h} \int\limits_{x-h}^{x+h} \left(1 - \frac{|x'-x|}{h}\right) f(x',t')dx',$$

where $\sigma > 0$ is a variable numerical parameter selected so as to ensure stability.

The difference scheme constructed is related to the class of schemes with variable weighting factors, where the grid function σ_α depends on both the number of the fractional level $t_{n+\alpha/p}$ ($\alpha = \overline{1,p}$) and the node $x \in \omega_h$.

We consider the question about the organization of a calculational process [Samarskii et al., 1997a]. We show in which way the solution on the fractional levels can be obtained only in the domain of the non-smooth solution $\overline{\omega}_2^n$. For this, in formula (7.66) it is necessary to exclude the values of the approximate solution on the level $t_{n+(\alpha-1)/p}$. For this we need the following:

LEMMA 7.2 *The difference scheme*

$$\frac{y_{(\alpha)} - y_{(0)}}{\alpha\tau_0} + Ay_{(\alpha)} = \overline{\varphi}_{(\alpha)}, \quad \overline{\varphi}_{(\alpha)} = \frac{1}{\alpha\tau_0} \int\limits_{t}^{t+\alpha\tau_0} S_x^2 f dt' \tag{7.68}$$

is algebraically equivalent to the equation

$$y_{\bar{t},\alpha} + \alpha A y_{(\alpha)} + (1-\alpha)Ay_{(\alpha-1)} = \varphi_{(\alpha)}. \tag{7.69}$$

Proof. We write the difference scheme (7.68) in the form

$$\frac{y_{(\alpha)} - y_{(\alpha-1)}}{\tau_0} + \frac{y_{(\alpha-1)} - y_{(0)}}{\tau_0} + \alpha A y_{(\alpha)} = \alpha\overline{\varphi}_{(\alpha)}. \tag{7.70}$$

From the considered scheme (7.68) it also follows that

$$\frac{y_{(\alpha-1)} - y_{(0)}}{\tau_0} = -(\alpha-1)Ay_{(\alpha-1)} + (\alpha-1)\overline{\varphi}_{(\alpha-1)}. \tag{7.71}$$

Substituting expression (7.71) into equation (7.70), we obtain

$$\begin{aligned} \frac{y_{(\alpha)} - y_{(\alpha-1)}}{\tau_0} + \frac{y_{(\alpha-1)} - y_{(0)}}{\tau_0} + \alpha A y_{(\alpha)} + (1-\alpha)Ay_{(\alpha-1)} \\ = \alpha\overline{\varphi}_{(\alpha)} - (\alpha-1)\overline{\varphi}_{(\alpha-1)}. \end{aligned} \tag{7.72}$$

It remains to estimate the expression on the right hand side of (7.72):

$$\alpha\overline{\varphi}_{(\alpha)} - (\alpha-1)\overline{\varphi}_{(\alpha-1)} = \frac{1}{\tau_0}\int\limits_{t}^{t+\alpha\tau_0} S_x^2 f\,dt' - \frac{1}{\tau_0}\int\limits_{t}^{t+(\alpha-1)\tau_0} S_x^2 f\,dt'$$
$$= \frac{1}{\tau_0}\int\limits_{t+(\alpha-1)\tau_0}^{t+\alpha\tau_0} S_x^2 f\,dt' = \varphi_{(\alpha)}. \tag{7.73}$$

Then from formulas (7.73) and (7.72) the statement required follows.

By virtue of the proved lemma about the equivalence of the difference schemes, equation (7.66) can be reduced to the form

$$\frac{y_{(\alpha)} - y_{(0)}}{\alpha\tau_0} + Ay_{(\alpha)} = \overline{\varphi}_{(\alpha)}, \quad (x,t) \in \omega_1^n,$$
$$\frac{y_{(\alpha)} - y_{(\alpha-1)}}{\tau_0} + Ay_{(\alpha)}^{(\sigma_\alpha)} = \varphi_{(\alpha)}, \quad (x,t) \in \omega_2^n.$$

The present system of equations for the function $v = \sigma_\alpha y_{(\alpha)}$ is reduced to the three-point difference scheme and realized by the opposite direction sweep method.

Following the paper [Samarskii et al., 1997a] it can be shown that under the condition

$$\sigma \geq \sigma_\varepsilon > 0, \quad \sigma_\varepsilon = \frac{1+\varepsilon}{2} - \frac{h^2}{4\tau}, \quad \frac{2}{3} < \varepsilon < 2, \tag{7.74}$$

for the difference scheme 7.66), (7.67) the *a priori* estimates are correct

$$\|y(t)\| \leq \|y_0\| + \left\|A^{-1}\varphi(0)\right\| + \left\|A^{-1}\varphi(t)\right\| + \sum_{k=1}^{n}\tau\left\|A^{-1}\varphi_{\bar{t},k}\right\|,$$
$$\sum_{t\in\omega_{\tau_0}}\tau\|y(t)\| \leq M_1\left(\|y_0\|_{A^{-1}} + \|y_0\|\right) + M_2\sum_{t\in\omega_{\tau_0}}\tau\left\|A^{-1}\varphi\right\|, \tag{7.75}$$

which express stability with respect to the right hand side and initial data.

Similarly to estimate (6.2.50) under conditions (7.74) for the error of the difference scheme the following inequality, which expresses convergence of the scheme, holds

$$\sum_{t\in\omega_{\tau_0}}\tau\left\|y - S_x^2 u\right\| \leq \frac{c}{\varepsilon\sqrt{2-\varepsilon}}\left(\tau_0\left\|\frac{\partial u}{\partial t}\right\|_{L_2(Q_T)} + h^2\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L_2(Q_T)}\right).$$

2.4 Difference Schemes for Multi-dimensional Equations

Assume it is required to find the function $u(x,t)$ satisfying in $\overline{Q}_T = \overline{\Omega} \times [0 \le t \le T]$, where

$$\Omega = \{x = (x_1, x_2),\ 0 < x_\beta < l_\beta,\ \beta = 1,2\},$$

the initially boundary value problem

$$\frac{\partial u}{\partial t} = Lu + f(x,t), \quad x \in \Omega, \quad t \in (0,T], \tag{7.76}$$

$$L = L_1 + L_2, \quad L_\beta u = \frac{\partial}{\partial x_\beta}\left(k_\beta(x,t)\frac{\partial u}{\partial x_\beta}\right), \quad \beta = 1,2,$$

$$0 < c_1 \le k_\beta(x,t) \le c_2,$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{7.77}$$

$$u|_\Gamma = \mu(x,t), \quad x \in \Gamma, \quad t \in (0,T]. \tag{7.78}$$

Just as in the one-dimensional case, on $[0,T]$ we shall consider uniform grids with the steps τ and $\tau_0 = \tau/p$, respectively:

$$\overline{\omega}_{\tau_0} = \left\{t_{n+\alpha/p} = (n + \alpha/p)\tau,\ \alpha = \overline{0,p},\ n = \overline{0, N_0 - 1}\right\}.$$

The domain $\overline{Q}^n = \overline{\Omega} \times [t_n, t_{n+1}]$ will be represented as a sum of two domains:

$$\overline{Q}^n = \overline{Q}^{n_1} \cup Q^{n_2}, \quad Q_2^n = \Omega_2^n \times [t_n, t_{n+1}],$$

$$\Omega_2^n = \left\{x : x_{m_{\beta 1}}^n \le x_\beta \le x_{m_{\beta 2}}^n,\ \beta = 1,2\right\}.$$

Let it be known *a priori* that the solution $u(x,t)$ in the domain $\overline{Q}_1^n$ is a sufficiently smooth function, and in Q_2^n it has the singularity moving with time. To simplify the calculations, in the rectangle $\overline{\Omega}$ we introduce into consideration the grid $\overline{\omega}_h = \omega_h \cup \gamma_h$, uniform with respect to each direction x_β, where

$$\omega_h = \Big\{x_i = (x_1^{(i_1)}, x_2^{(i_2)}),\ x_\beta^{(i_\beta)} = i_\beta h_\beta,\ i_\beta = 1,2,\ldots,N_\beta - 1,$$
$$h_\beta N_\beta = l_\beta,\ \beta = 1,2\Big\}.$$

The set of the nodes $x \in \overline{\Omega}_2^n$ will be denoted by $\overline{\omega}_{2h}^n$, with

$$\overline{\omega}_{2h}^n = \Big\{x_i = (x_1^{(i_1)}, x_2^{(i_2)}),\ x_\beta^{(i_\beta)} = i_\beta h_\beta,\ m_{\beta 1}^n \le i_\beta \le m_{\beta 2}^n,$$

$$\beta = 1, 2,\ m_{\beta 1}^n \geq 1,\ m_{\beta 2}^n \leq N_\beta - 1\} .$$

Then $\omega_h \backslash \omega_{2h}^n$ is the set of interior nodes lying in the zone of the smooth solution $\overline{Q}_2^n$, with the nodes situated on the interior boundaries belonging to this set. Using the integro-interpolation method and special approximation of time derivative outside the adaptation domain

$$\frac{\partial u}{\partial t} \sim \frac{u^{n+\alpha/p} - u^n}{\alpha\tau/p}, \quad \alpha = 1, 2, \ldots, p,$$

on the whole grid $\omega = \omega_h \times \omega_{\tau_0}$ we construct the following difference scheme:

$$\frac{y_{(\alpha)} - y^n}{\alpha\tau_0} = (\Lambda y + f)_{(\alpha)}, \quad x \in \omega_h \backslash \omega_{2h}^n,\ t \in \omega_{\tau_0}, \tag{7.79}$$

$$\frac{y_{(\alpha)} - y_{(\alpha-1)}}{\tau_0} = (\Lambda y + f)^{(\sigma)}, \quad x \in \omega_{2h}^n,\ t \in \omega_{\tau_0}, \tag{7.80}$$

$$y(x, 0) = u_0(x), \quad x \in \omega_h, \quad y_{(\alpha)}\big|_{\gamma_h} = \mu(x, t_{n+\alpha/p}), \quad x \in \gamma_h, \tag{7.81}$$

where

$$\Lambda y = \sum_{\beta=1}^{2} \left(a_\beta y_{\bar{x}_\beta}\right)_{x_\beta} = -Ay, \quad a_\beta = k_\beta(x^{(-0.5\beta)}, t_n),$$

$$x^{(-0.5\beta)} = \left(x_1^{(i_1)}, \ldots, x_{\beta-1}^{(i_\beta - 1)}, x_\beta^{(i_\beta - 0.5)}, x_{\beta+1}^{(i_\beta + 1)}, \ldots, x_p^{(i_p)}\right).$$

For the realization of the given scheme, analogous to the one-dimensional case, we may use both direct algorithms (the opposite direction sweep method) and different iterative methods [Matus, 1991].

To study the stability of the difference scheme on an adaptive grid we assume in (7.81) that $\mu(x, t) = 0$, $x \in \gamma_h$, $t \in \omega_{\tau_0}$. Then, using Lemma 7.1, similarly to (7.27), (7.28), we obtain the problem

$$y_{\bar{t},\alpha} + \left(Ay_{(\alpha)}\right)^{(\Sigma)} = \varphi(t), \quad t = t_{n+\alpha/p}, \quad y(0) = u_0, \tag{7.82}$$

where

$$Ay = -\Lambda y,\ x \in \omega_h, \qquad y = 0 \quad \text{if} \quad x \in \gamma_h,$$

$$\Sigma v = \sigma(x, t) v(x, t), \quad x \in \omega_h, \quad t \in \omega_{\tau_0},$$

$$\sigma(x, t) = \begin{cases} \alpha, & \text{if} \quad x \in \omega_h \backslash \omega_{2h}^n, \\ \sigma, & \text{if} \quad x \in \omega_{2h}^n. \end{cases}$$

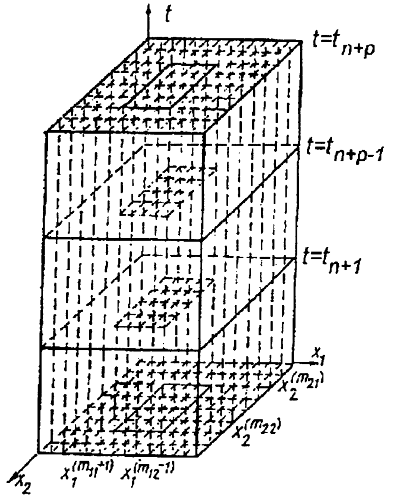

Figure 7.2.

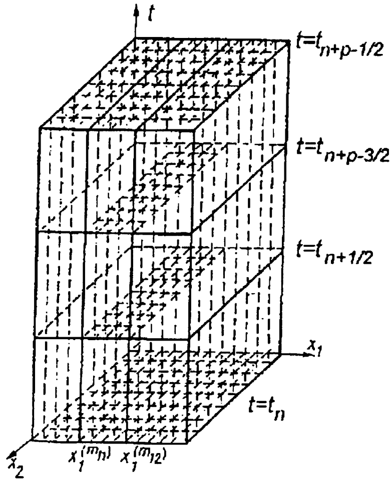

Figure 7.3.

By analogy with (7.33) it is easy to prove that, provided $\sigma \geq 0.5$, the difference scheme (7.79)–(7.81) is stable in the norm H_{A^2} with respect to both the initial data and the right hand side.

Let us study the opportunity of constructing economical difference schemes. To simplify calculations, we consider the heat conduction equation of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \quad x \in \Omega, \quad t \in (0, T], \tag{7.83}$$

with the initial and boundary conditions (7.77), (7.78). As previously, in the domain Ω we introduce the grid ω_h uniform with respect to x_β and having the steps $h_1 = l_1/N_1$, $h_2 = l_2/N_2$ on the interval $[0, T]$ we shall use the uniform grid $\overline{\omega}_\tau$ with the step τ. In addition, we consider the grids (see Fig. 7.2)

$$\overline{\omega}_1 = \overline{\omega} \backslash \omega_2, \qquad \overline{\omega} = \overline{\omega}_h \times \overline{\omega}_\tau,$$

$$\omega_2 = \Big\{ (x_1^{(i_1)}, x_2^{(i_2)}, t_{n+\alpha}), \ m_{11} \leq i_1 \leq m_{12},$$

$$m_{21} \leq i_2 \leq m_{22}, \ \alpha = \overline{0, p} \Big\}.$$

To avoid confusion in the notation, here the fractional levels $t_{n+\alpha/p}$ are replaced by the integral ones $t_{n+\alpha}$. Furthermore, we shall use the standard notation. In the domain of the smooth solution $\overline{\omega}_1$, to construct the difference scheme we use the coarse step $\tau_p = t_{n+p} - t_n = p\tau$, and in the zone $\overline{\omega}_2$, containing the assumed singularity of the solution, the smaller step $t_{n+1} - t_n = \tau$. Let us construct economical difference schemes with the aid of which it will be possible to determine the solution only at the nodes $(x, t) \in \omega_2$, $0 < \alpha < p$, and for t_{n+p} at the points $x \in \omega_h \cup \gamma_h$.

On the basis of the method of alternating directions, on the grids represented in Figs. 7.2, 7.3 the differential problem (7.83), (7.77), (7.78) is approximated by the difference one:

$$\frac{y^{n+\alpha+1/2} - y^n}{(\alpha+1)\tau} = \sigma_1 \Lambda_1 y^{n+\alpha+1/2} + (1-\sigma_2)\Lambda_2 y^n, \tag{7.84}$$

$$\frac{y^{n+\alpha+1} - y^{n+\alpha+1/2}}{(\alpha+1)\tau} = (1-\sigma_1)\Lambda_1 y^{n+\alpha+1/2} + \sigma_2 \Lambda_2 y^{n+\alpha+1}, \tag{7.85}$$

$$(x,t) \in \omega \backslash \omega_2, \quad \alpha = \overline{0, p-1}, \quad \Lambda_k y = y_{\bar{x}_k x_k},$$

$$\frac{y^{n+\alpha+1/2} - y^{n+\alpha}}{\tau} = \sigma_1 \Lambda_1 y^{n+\alpha+1/2} + (1-\sigma_2)\Lambda_2 y^{n+\alpha}, \tag{7.86}$$

$$\frac{y^{n+\alpha+1} - y^{n+\alpha+1/2}}{\tau} = (1-\sigma_1)\Lambda_1 y^{n+\alpha+1/2} + \sigma_2 \Lambda_2 y^{n+\alpha+1}, \tag{7.87}$$

$$(x,t) \in \omega_2, \quad \alpha = \overline{0, p-1},$$

$$y^{n+\alpha+1} = \mu^{n+\alpha+1} \quad \text{if} \quad i_2 = 0 \quad \text{and} \quad i_2 = N_2, \quad \alpha = \overline{0, p-1},$$

$$y^{n+\alpha+1/2} = \overline{\mu} \quad \text{if} \quad i_1 = 0 \quad \text{and} \quad i_1 = N_1, \quad \alpha = \overline{0, p-1},$$

$$\overline{\mu} = \sigma_1 \mu^{n+\alpha+1} + (1-\sigma_1)\mu^{n+\alpha} - \tau\Lambda_2(\sigma_1\sigma_2\mu^{n+\alpha+1} - \\ -(1-\sigma_1)(1-\sigma_2)\mu^{n+\alpha}).$$

For $t = t_n$ let the values of the grid function $y^n = y^n_{i_1 i_2}$ be known at all the nodes of the grid $\omega_h \cup \gamma_h$. We consider the way in which using scheme (7.84)–(7.87) it is possible to pass from the level t_n to the level t_{n+p}. When $\alpha = 0$ equations (7.84), (7.86) can be written in the form

$$\frac{y^{n+1/2} - y^n}{\tau} = \sigma_1 \Lambda_1 y^{n+1/2} + (1-\sigma_2)\Lambda_2 y^n, \quad x \in \omega_h.$$

Since y^n is determined for all $x \in \omega_h \cup \gamma_h$, it follows that we can find the difference solution by means of the opposite direction sweep method in the direction Ox_1:

$$y^{n+1/2}_{i_1 i_2} \text{ for } i_1 = m_{11}, m_{11}+1, \ldots, m_{12}, \quad i_2 = 1, 2, \ldots, N_2 - 1. \tag{7.88}$$

We now consider equations (7.85), (7.87):

$$\frac{y^{n+1}-y^{n+1/2}}{\tau}=(1-\sigma_1)\Lambda_1 y^{n+1/2}+\sigma_2\Lambda_2 y^{n+1}$$

only for $i_1=m_{11}+1,\dots,m_{12}-1$; $i_2=\overline{1,N_2-1}$. According to (7.88), applying the opposite direction sweep method in the direction Ox_2 for each fixed $i_1=m_{11}+1,\dots,m_{12}-1$, we can determine the grid function

$$y^{n+1}_{i_1i_2} \quad \text{for} \quad i_1=m_{11}+1,\dots,m_{12}-1, \quad i_2=m_{21},\dots,m_{22}. \tag{7.89}$$

For $\alpha=1$ from system (7.84), (7.86) we obtain

$$\frac{y^{n+3/2}-y^{n}}{2\tau}=\sigma_1\Lambda_1 y^{n+3/2}+(1-\sigma_2)\Lambda_2 y^{n},$$

$$i_1=1,\dots,m_{11},m_{12},\dots,N_1-1, \quad i_2=1,\dots,m_{21},m_{22},\dots,N_2-1,$$

$$\frac{y^{n+3/2}-y^{n+1}}{\tau}=\sigma_1\Lambda_1 y^{n+3/2}+(1-\sigma_2)\Lambda_2 y^{n+1},$$

$$i_1=m_{11}+1,\dots,m_{12}-1, \quad i_2=m_{21}+1,\dots,m_{22}-1.$$

According to (7.89) the functions $y^{n+1}_{i_1i_2}$, $\Lambda_2 y^{n+1}$ for the i_1, i_2 mentioned are defined. Consequently by analogy with (7.88) we find $y^{n+3/2}$ for all $i_1=m_{11},\dots,m_{12}$, $i_2=1,2,\dots,\dots,N_2-1$. For the same reason we successively determine the values of the desired function $y^{n+\alpha}$ for all $i_1=m_{11}+1,\dots,m_{12}-1$; $i_2=m_{21},\dots,m_{22}$; $\alpha=0,1,\dots,p-1$. We consider now the passage from the level t_{n+p-1} to the t_{n+p}th level when it is necessary to define the values already for all $x\in\omega_h$. For $\alpha=p-1$ the corresponding equations on the fractional level can be written in the form

$$\frac{y^{n+p-1/2}-y^{n}}{\tau p}=\sigma_1\Lambda_1 y^{n+p-1/2}+(1-\sigma_2)\Lambda_2 y^{n}, \quad (x,t)\in\omega\backslash\omega_2,$$

$$\frac{y^{n+p-1/2}-y^{n+p-1}}{\tau}=\sigma_1\Lambda_1 y^{n+p-1/2}+(1-\sigma_2)\Lambda_2 y^{n+p-1}, \quad (x,t)\in\omega_2.$$

Since the grid function y^n is defined for all the nodes of ω_h and the function y^{n+p-1} for $i_1=m_{11}+1,\dots,m_{12}-1$; $i_2=m_{21},\dots,m_{22}$, then from the latter system of equations we determine $y^{n+p-1/2}$ for all $x\in\omega_h$ by sweep method. To find y^{n+p}, $x\in\omega_h$ in (7.85), (7.87), we set $\alpha=p-1$. Solving the obtained system of equations by one-dimensional sweep methods in the direction x_2 for each fixed $i_1=1,2,\dots,N_1-1$, we find the solution on the level t_{n+p} for all $x\in\omega_h\cup\gamma_h$. We note that to determine the solution on the integral levels $t_{n+\alpha}$, $\alpha=1,2,\dots,p-1$, at the grid nodes belonging to ω_2 (Fig. 7.2), we used the solution from the fractional level $t_{n+\alpha-1/2}$, calculated in the wider domain (see Fig. 7.3). In addition, to find the solution $y^{n+\alpha}$, $\alpha = 1, 2, \ldots, p-1$, at the integral nodes, in contrast to the one-dimensional case, for $m_{12} - m_{11} \ll N_1$ only $O(N_2)$ is required instead of the $O(N_1 N_2)$ operations .

Using the summary approximation method, on the grid domains introduced above (see Fig. 7.2, 7.3) it is not difficult to construct the economical additive schemes for equation (7.65) of the following form:

$$\frac{y^{n+\alpha+1/2} - y^n}{(\alpha+1)\tau} = \Lambda_1 y^{n+\alpha+1/2}, \quad \frac{y^{n+\alpha+1} - y^{n+\alpha+1/2}}{(\alpha+1)\tau} = \Lambda_2 y^{n+\alpha+1},$$
$$(x,t) \in \omega \backslash \omega_2, \quad \alpha = \overline{0, p-1}, \tag{7.90}$$

$$\frac{y^{n+\alpha+1/2} - y^{n+\alpha}}{\tau} = \Lambda_1 y^{n+\alpha+1/2}, \quad \frac{y^{n+\alpha+1} - y^{n+\alpha+1/2}}{\tau} = \Lambda_2 y^{n+\alpha+1},$$
$$(x,t) \in \omega_2, \quad \alpha = \overline{0, p-1}, \tag{7.91}$$

$$y^0 = u_0(x), \quad x \in \omega_h, \quad y^{n+m/2}\Big|_{\gamma_h} = \mu(x, t_{n+m/2}), \quad m = \overline{1, 2p}. \tag{7.92}$$

Just as in the case of the method of alternating directions, the solution on the intermediate levels $y^{n+\alpha+1/2}$ is determined on the grid represented in Fig. 7.3, and on the integral levels $t_{n+\alpha}$, $\alpha = \overline{0, p}$ it is fined at the nodes of the grid ω_2 (see Fig. 7.2). The locally one-dimensional scheme (LOS) mentioned can be obtained from (7.84)–(7.87) provided that $\sigma_1 = \sigma_2 = 1$.

The following statement holds [Matus, 1991]

THEOREM 7.3 *Let problem (7.83), (7.77), (7.78) in the domain $\overline{Q}_T$ has the unique solution $u = u(x,t)$ and the derivatives $\dfrac{\partial^2 u}{\partial t^2}$, $\dfrac{\partial^4 u}{\partial x_1^2 \partial x_2^2}$, $\dfrac{\partial^3 u}{\partial t \partial x_\beta^2}$ exist continous in Q_T . The LOS (7.90)–(7.92) converges uniformly. Moreover, for the error $z = y - u$ the following estimate holds:*

$$\max_{t \in \omega_\tau} \|z(t)\|_{C(\overline{\omega}_h)} \le c\left(h_1^2 + h_2^2 + \tau\right), \quad c = \text{const} > 0.$$

3. Schemes with Adaptation with Respect to Time for a Wave Equation

The schemes with local refinement with respect to time for the hyperbolic equation of the second order are considered. The presentation is based on the papers [Matus, 1993b, Matus, 1994].

3.1 Non-Conservative Schemes

In the rectangle $\overline{Q}_T$ we consider the first-kind boundary problem for the equation of the second order of hyperbolic type:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) + f(x,t), \quad (x,t) \in Q_T, \qquad (7.93)$$
$$0 < c_1 \le k(x,t) \le c_2, \quad c_1,\ c_2 = \text{const},$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \bar{u}_0(x), \quad x \in \overline{\Omega},$$
$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad t > 0. \qquad (7.94)$$

On the grid $\overline{\omega} = \overline{\omega}_h \times \overline{\omega}_{\tau_0}$, introduced in 7.1.1, we approximate the problem (7.93), (7.94) by the difference problem

$$y_{\bar{t}t,\alpha} = \left((a y_{\bar{x}})_x + f\right)^{(\sigma_{1\alpha},\sigma_{2\alpha})},$$
$$y_{(\alpha+1),0} = \mu_{1(\alpha+1)}, \quad y_{(\alpha+1),N} = \mu_{2(\alpha+1)}, \quad \overset{0}{y_i} = u_0(x_i), \qquad (7.95)$$

$$y_{t,0}(x,0) = \tilde{u}_0(x), \quad \tilde{u}_0(x) = \bar{u}_0(x) + 0.5\tau_0 (Lu + f)\,|_{t=0}\,. \qquad (7.96)$$

Here the following notation is used:

$$y_{\bar{t}t,\alpha} = \frac{y_{(\alpha+1)} - 2y_{(\alpha)} + y_{(\alpha-1)}}{\tau_0^2}, \quad y_{t,\alpha} = \frac{y_{(\alpha+1)} - y_{(\alpha)}}{\tau_0},$$

$$v^{(\sigma_{1\alpha},\sigma_{2\alpha})} = \sigma_{1\alpha} v_{(\alpha+1)} + (1 - \sigma_{1\alpha} - \sigma_{2\alpha}) v_{(\alpha)} + \sigma_2 v_{(\alpha-1)},$$
$$a(x_i,t) = 0.5(k(x_{i-1},t) + k(x_i,t)), \quad t \in \omega_{\tau_0},$$
$$\sigma_{1\alpha} = \begin{cases} \overline{\sigma}_{\alpha+1}, & \text{if } i \in I_{1n}, \\ \sigma_1, & \text{if } i \in I_{2n}, \end{cases}$$
$$\sigma_{2\alpha} = \begin{cases} \overline{\sigma}_{\alpha-1}, & \text{if } i \in I_{1n}, \\ \sigma_2, & \text{if } i \in I_{2n}, \end{cases} \qquad (7.97)$$
$$\alpha = \overline{1, p-1},$$

$$\overline{\sigma}_\alpha = 0.5\alpha(\alpha-1), \quad \sigma_{1p} = \sigma_1, \quad \sigma_{2p} = \sigma_2, \quad \sigma_1, \sigma_2 = \text{const} > 0,$$
$$I_{1n} = \{1, ..., m_1^n, m_2^n, ..., N-1\}, \quad I_{2n} = \{m_1^n + 1, ..., m_2^n - 1\},$$
$$y_{(\alpha)} = y^{n+\alpha/p} = y(x_i, t_{n+\alpha/p}), \quad y_{(0)} = y^n, \quad y_{(p)} = y^{n+1}.$$

From the construction of the difference scheme (7.95) it is seen that for $(x,t) \in w_2^n$ the usual three-level scheme with the constant weights σ_1, σ_2 is used. The form of the weights $\bar{\sigma}_{\alpha+1}$ and $\bar{\sigma}_{\alpha-1}$, defined outside of the adaptation domain, will be written below, when realization of the given

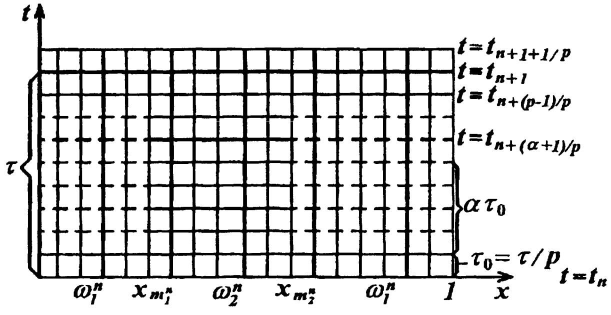

Figure 7.4.

scheme will be considered. We turn to finding the approximate solution $y^{n+\alpha/p}$ provided that $\alpha = 2, 3, ..., p-2$ only in the domain of assumed singularity of the solution (Fig. 7.4).

Let for $t = t_n$, $t = t_{n+1/p}$ the values of the desired function y^n, $y^{n+1/p}$ are known for all $x \in \bar{\omega}_h$. We note that scheme (7.95) can be written in the form ($\alpha = \overline{1, p-1}$)

$$y_{\bar{t}t,\alpha} = (\Lambda y + f)^{(\bar{\sigma}_{\alpha+1}, \bar{\sigma}_{\alpha-1})}, \quad (x,t) \in w_1^n, \tag{7.98}$$

$$y_{\bar{t}t,\alpha} = (\Lambda y + f)^{(\sigma_1, \sigma_2)}, \quad (x,t) \in w_2^n. \tag{7.99}$$

In order to avoid computation of the approximate solution in the domain $\omega \backslash \overline{\omega}_2^n$ it is necessary in the formulas (7.98) to exclude the values of the approximate solution on the levels $n + \alpha/p$ and $n + (\alpha-1)/p$. It can be realized by means of the following lemma about equivalence of the difference schemes.

LEMMA 7.3 *Scheme (7.98) for any $\alpha = 1, 2, ..., p-1$ can be reduced to the form*

$$\frac{y_{(\alpha+1)} - (\alpha+1)y_{(1)} + \alpha y_{(0)}}{\bar{\sigma}_{\alpha+1}\tau_0^2} = (\Lambda y + f)_{(\alpha+1)}. \tag{7.100}$$

Proof. Let $\alpha = 1$, then $\bar{\sigma}_2 = 1$, $\bar{\sigma}_0 = 0$ and expression (7.100) can be rewritten in the following form:

$$y_{\bar{t}t,1} = (\bar{\Lambda} y)^{n+2/p}, \quad \bar{\Lambda} y = \Lambda y + f. \tag{7.101}$$

Consequently, provided $\alpha = 1$, the difference schemes (7.98) and (7.100) are equivalent, and provided $\alpha = 2$, equation (7.100) is reduced to the form

$$y_{\bar{t}t,2} + 2y_{\bar{t}t,1} = 3(\bar{\Lambda} y)_{(3)}.$$

According to formula (7.101), we have

$$y_{\bar{t}t,2} = 3(\bar{\Lambda}y)^{n+3/p} - 2(\bar{\Lambda}y)^{n+2/p} = (\bar{\Lambda}y)^{(\bar{\sigma}_3,\bar{\sigma}_1)}. \tag{7.102}$$

We consider now scheme (7.100) provided that $\alpha = 3$:

$$\left(y_{(4)} - 4y_{(1)} + 3y_{(0)}\right)/\tau_0^2 = y_{\bar{t}t,3} + 2y_{\bar{t}t,2} + 3y_{\bar{t}t,1} = 6(\bar{\Lambda}y)_{(4)}.$$

If we replace $y_{\bar{t}t,2}$, $y_{\bar{t}t,1}$ by their values from equations (7.101), (7.102) in the latter expression we obtain the equality

$$y_{\bar{t}t,3} = 6(\bar{\Lambda}y)_{(4)} - 6(\bar{\Lambda}y)_{(3)} + (\bar{\Lambda}y)_{(2)} = (\bar{\Lambda}y)_{(3)}^{(6,1)}.$$

The latter equation can be rewritten as follows:

$$y_{\bar{t}t,3} = (\bar{\Lambda}y)^{(\bar{\sigma}_4,\bar{\sigma}_2)}.$$

This completes the proof of the theorem for $\alpha = 3$. Furthermore, the proof is by induction on α. For $\alpha = 4, 5, ..., p-2$ there is nothing to prove. Let us demonstrate the equivalence of scheme (7.98) to (7.100) for $\alpha = p-1$. By direct testing we may convince ourselves of the correctness of the identity

$$\left(y^{n+1} - py^{n+1/p} + (p-1)y^n\right)/\tau_0^2 = y_{\bar{t}t,p-1} + 2y_{\bar{t}t,p-2} + ... + (p-1)y_{\bar{t}t,1}.$$

On the basis of the latter equality and on the inductive assumption, from expression (7.100) we obtain

$$\begin{aligned} y_{\bar{t}t,p-1} =& \bar{\sigma}_p(\bar{\Lambda}y)_{(p)} - 2\bar{\sigma}_{p-1}(\bar{\Lambda}y)_{(p-1)} \\ & - \left(2(1-\bar{\sigma}_{p-1}-\bar{\sigma}_{p-3}) + 3\bar{\sigma}_{p-2}\right)(\bar{\Lambda}y)_{(p-2)} - \sum_{\alpha=2}^{p-3}[(p-(\alpha+1))\bar{\sigma}_\alpha \\ & + (p-\alpha)(1-\bar{\sigma}_{\alpha+1}-\bar{\sigma}_{\alpha-1}) + (p-(\alpha+1))\bar{\sigma}_\alpha](\bar{\Lambda}y)_{(\alpha)}. \end{aligned}$$

Using the identity $-2\bar{\sigma}_\alpha = 1 - \bar{\sigma}_{\alpha+1} - \bar{\sigma}_{\alpha-1}$ we can write the previous equation in the form

$$y_{\bar{t}t,p-1} = (\bar{\Lambda}y)^{(\bar{\sigma}_p,\bar{\sigma}_{p-2})}.$$

This completes the proof of the theorem.

Taking into account Lemma 7.3, we reduce the difference equations (7.99), (7.100) to the system of three-point equations

$$A_i y_{(\alpha+1),i-1} - C_i y_{(\alpha+1),i} + B_i y_{(\alpha+1),i+1} = -F_i, \quad i = \overline{1, N-1},$$

Figure 7.5.

where

$$A_i = \tau_0^2 \sigma_{1\alpha} a_{(\alpha+1)i}/h^2, \quad B_i = \tau_0^2 \sigma_{1\alpha} a_{(\alpha+1)i+1}/h^2, \quad C_i = 1 + A_i + B_i,$$

and the right hand side F_i depends for $(x,t) \in \omega_1^n$ only on $y_{(0)}$, $y_{(1)}$. Applying now the opposite direction sweep method, we define the boundary condition $y(m_1 h, t_{\alpha+1})$ and find the solution in the domain ω_2^n if $i = \overline{m_1, m_2}$ on the levels $n + \alpha/p$, $\alpha = 2, 3, ..., p-2$ by means of the formulas of the left sweep method. For $t = t_{(p-1)}$, $t = t_{n+1}$, $t = t_{n+1+1/p}$ it is necessary to calculate the solution for all $x \in \bar{\omega}_h$. Thereafter the process described is repeated.

REMARK 7.4 It is of interest to note that the term on the left hand side of equation (7.100)

$$\frac{y_{(\alpha+1)} - (\alpha+1)y_{(1)} + \alpha y_{(0)}}{\bar{\sigma}_{\alpha+1}\tau_0} = \frac{1}{0.5(\tau_0+\tau_1)}\left(\frac{y_{(\alpha+1)} - y_{(1)}}{\tau_1} - \frac{y_{(1)} - y_{(0)}}{\tau_0}\right)$$

represents approximation of the second derivative $\dfrac{\partial^2 u}{\partial t^2}$ on the non-uniform grid represented in Fig. 7.5.

3.2 Stability and Convergence

Defining in the ordinary way the operator $A : H \to H$ and assuming, for the simplicity of calculations, the homogeneity of the boundary conditions and independence of the coefficient $k(x)$ from the variable t, we write scheme (7.95), (7.96) in the operator form

$$y_{\bar{t}t} + (Ay)^{(\Sigma_1, \Sigma_2)} = \varphi(t), \quad t \in \omega_{\tau_0}, \tag{7.103}$$

$$y(0) = u_0, \quad y_t(0) = \tilde{u}_0. \tag{7.104}$$

In this case we have

$$y_t = (y(t+\tau_0) - y(t))/\tau_0, \quad y_{\bar{t}} = (y(t) - y(t-\tau_0))/\tau_0,$$

$$y_{\bar{t}t} = (y_t - y_{\bar{t}})/\tau_0.$$

Here we use the notation $y = (y_1, ..., y_{N-1})^T \in H$; moreover,

$$\varphi \in H, \quad \varphi(x,t) = f^{(\sigma_1(x,t),\sigma_2(x,t))}, \quad x \in \omega_h, t \in \omega_{\tau_0},$$

and $\sigma_\beta(x,t)$, $\beta = 1, 2$, are defined by formulas (7.97), and let

$$\Sigma_\beta(t) = \mathrm{diag}\{\sigma_{\beta 1}(t), ..., \sigma_{\beta N-1}(t)\}, \quad \beta = 1, 2, \quad t \in \omega_{\tau_0}.$$

The scheme with the operator weighting (7.103) is reduced to the canonical form of three-level schemes:

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi$$

with the operators

$$D = E + 0.5\tau^2 AG_1, \quad B = \tau AG_2, \quad G_1 = \Sigma_1 + \Sigma_2, \quad G_2 = \Sigma_1 - \Sigma_2,$$

which have been considered in Section 5.2.

We obtain the estimate with respect to the right hand side in the norm such that it should yield convergence of the corresponding difference scheme in the case of variable and non-differentiable weighting factors.

THEOREM 7.4 *Let in the difference scheme (7.103), (7.104) $A = A^* > 0$ be a constant operator, $\varphi = \varphi_1 + \tau_0^{1/2}\varphi_2$ and the following conditions be satisfied:*

$$\Sigma_2(t) = \Sigma_2^*(t) \geq -\frac{1}{\tau^2\|A\|}E, \quad \Sigma_1(t) \geq \Sigma_2^{(0.5)} + \frac{1+\varepsilon}{2}E, \ \varepsilon > 0. \quad (7.105)$$

Then for the difference solution the a priori estimate is correct:

$$\max_{t\in\omega_{\tau_0}} \|Ay(t)\| \leq M_1 \big(\|Ay(0)\| + \|Ay_t(0)\|_{R(\tau_0)} + \|\varphi_1(0)\|\big)$$
$$+ \|\varphi_1(t)\| + M_2 \max_{0<t'\leq t-\tau_0} \left(\|\varphi_{1\bar{t}}(t')\| + \frac{1}{\sqrt{\varepsilon}}\|\varphi_2(t')\| \right), \quad (7.106)$$

where $R = A^{-1} + \tau_0^2(\Sigma_2 + 2E)$ and $M_1 = \sqrt{2}e^{0.5T}$, $M_2 = M_1\sqrt{T}$.

P r o o f. By analogy with the proof of Theorem 5.6, using the identity

$$v^{(\Sigma_1,\Sigma_2)} = v + \tau(\Sigma_1 v_t - \Sigma_2 v_{\bar{t}}),$$

we rewrite the original scheme in the form (see problem (5.1.49))

$$y_{\bar{t}t} = Ay + \tau(\Sigma_1 Ay_t - \Sigma_2 Ay_{\bar{t}}) = \varphi. \quad (7.107)$$

We take the dot product of the operator equation (7.107) with $2\tau_0 Ay_t$. Similarly to equality (5.1.54) we obtain the energy identity

$$\begin{aligned} 2\tau_0^2\|Ay_t\|_{\Sigma_3}^2 + \tau^2(QAy_{\bar{t}t}, Ay_{\bar{t}t}) + (\hat{Q}A\hat{y}_{\bar{t}}, A\hat{y}_{\bar{t}}) \\ = (QAy_{\bar{t}}, Ay_{\bar{t}}) - \|A\hat{y}\|^2 + \|Ay\|^2 + 2\tau_0(Ay_t, \varphi), \end{aligned} \tag{7.108}$$

where

$$\Sigma_3(t) = \Sigma_1(t) - \Sigma_2^{(0.5)} - 0.5E > 0, \quad Q = A^{-1} + \tau_0^2\Sigma_2 \geq 0.$$

Since $\varphi = \varphi_1 + \tau_0^{1/2}\varphi_2$, then according to inequality (5.1.56) it follows that

$$\begin{aligned} &\|Ay\|^2 - \|A\hat{y}\|^2 + 2\tau_0(\varphi_1, Ay_t) \\ &\quad\leq -\|A\hat{y} - \varphi_1\|^2 + (1+\tau_0)\|Ay - \check{\varphi}_1\|^2 + \tau_0(1+\tau_0)\|\varphi_{1\bar{t}}\|^2. \end{aligned}$$

Applying the Cauchy inequality with ε, we have

$$2\tau_0(Ay_t, \tau_0^{1/2}\varphi_2) \leq \tau_0^2\varepsilon\|Ay_t\|^2 + \frac{\tau_0}{\varepsilon}\|\varphi_2\|^2.$$

Substituting the estimates mentioned into formula (7.108) and rejecting the positive terms on the left-right side of the inequality obtained, taking into account the conditions of the theorem, we derive

$$\begin{aligned} \|\hat{y}\|_{(1)}^2 &\leq (1+\tau_0)\left(\|y\|_{(1)}^2 + \tau_0(\|\varphi_{1\bar{t}}\|^2 + \frac{1}{\varepsilon}\|\varphi_2\|^2)\right) \leq \cdots \\ &\leq e^t\left(\|y(\tau_0)\|_{(1)}^2 + t\max_{t\in\omega_{\tau_0}}(\|\varphi_{1\bar{t}}\|^2 + \frac{1}{\varepsilon}\|\varphi_2\|^2)\right) \end{aligned}$$

or

$$\|y_{n+1}\|_{(1)} \leq e^{0.5t_n}\left(\|y(\tau_0)\|_{(1)} + \sqrt{t}\max_{t\in\omega_{\tau_0}}(\|\varphi_{1\bar{t}}\| + \frac{1}{\sqrt{\varepsilon}}\|\varphi_2\|)\right). \tag{7.109}$$

Here we have used the notation $\|y\|_{(1)}^2 = \|Ay - \check{\varphi}_1\|^2 + (QAy_{\bar{t}}, Ay_{\bar{t}})$.

Taking into account the inequalities

$$\begin{aligned} &\|y_{n+1}\|_{(1)} \geq \|Ay_{n+1}\| - \|\varphi_{1n}\|, \\ &\|y(\tau_0)\|_{(1)} \leq \sqrt{2}\|Ay(0)\| + \|Ay_t(0)\|_R + \sqrt{2}\|\varphi_1(0)\|, \end{aligned}$$

we obtain the estimate required from (7.109).

We apply Theorem 7.4 to the study of the stability and convergence of the difference scheme (7.95), (7.96). According to the definition of the weighting factors (7.97) and of the estimate of the operator norm (6.1.27)

$$\bar{\sigma}_{\alpha-1} \geq 0, \quad \bar{\sigma}_{\alpha+1} \geq 1, \quad \alpha = 1, ..., p-1; \quad \|A\| \leq \frac{4c_2}{h^2}, \quad y \in \overset{\circ}{\Omega}_h,$$

we conclude that the operator inequalities (7.105) are satisfied provided the constant weights σ_1, σ_2 in the domain $\bar{\omega}_2^n$ satisfy the inequalities

$$\sigma_1 \geq \sigma_2 + \frac{1+\varepsilon}{2}, \quad \sigma_2 \geq -\frac{h^2}{4\tau^2 c_2}, \quad 0 < \varepsilon \leq 1. \tag{7.110}$$

Consequently under conditions (7.110) the scheme with variable weighting factors (7.95), (7.96) is stable in H_{A^2} and, provided $\varphi = \varphi_1$, homogeneous boundary conditions for the solution of the scheme, the *a priori* estimate (7.106) holds.

We now study the convergence of the solution of the difference scheme to the solution of the differential problem (7.93), (7.94). Substituting $y = z + u$ into equations (7.95), (7.96) we obtain the problem for error of the method of the form (7.103), (7.104):

$$z_{\bar{t}t} + (Az)^{(\Sigma_1, \Sigma_2)} = \psi(t), \quad t \in \omega_{\tau_0}, \tag{7.111}$$

$$z(0) = 0, \quad z_t(0) = \psi^0, \tag{7.112}$$

where

$$\psi^0(x) = \bar{u}_0(x) + 0.5\tau_0 \left((k(x)u_0'(x))' + f(x,0) - u_t(x,0)\right),$$

$$\psi = \psi_1 + \tau_0^{1/2}\psi_2, \quad \psi_1 = ((au_{\bar{x}})_x + f) - u_{\bar{t}t},$$

$$\psi_2 = \tau_0^{1/2} \left(\sigma_1 (au_{\bar{x}})_x + f\right)_t - \sigma_2 \left((au_{\bar{x}})_x + f\right)_{\bar{t}}.$$

To derive the given problem we have used the identity

$$v^{(\sigma_1, \sigma_2)} = v + \tau(\sigma_1 v_t - \sigma_2 v_{\bar{t}}). \tag{7.113}$$

In the case where

$$\begin{gathered} u(x,t) \in C^{4,4}(Q_T), \quad u_0(x) \in C^4(\Omega), \quad \bar{u}_0(x) \in C^2(\Omega), \\ k(x) \in C^3(\Omega), \quad f(x,t) \in C^{2,1}(Q_T), \end{gathered} \tag{7.114}$$

it is not difficult to show that the following estimates are valid:

$$\|A\psi^0\|_{R(\tau_0)} = O(\tau_0^2), \quad \|\psi_1\| = O(h^2 + \tau_0^2), \quad \|\psi_{1\bar{t}}\| = O(h^2 + \tau_0^2),$$

$$\|\psi_2\| = O(\tau_0^{1/2}).$$

Using the *a priori* estimate (7.14) we conclude that for any $t \in \omega_{\tau_0}$ the inequality holds

$$\|Az(t)\| \leq c(h^2 + \tau_0^{1/2}), \quad c = \text{const} > 0.$$

THEOREM 7.5 *For the differential problem (7.93), (7.94) let the conditions (7.114) be valid. Then on satisfaction of inequalities (7.110) the solution of the difference scheme (7.95), (7.96) converges unconditionally to the exact solution $u(x,t)$ and the following estimate of the convergence rate in the grid space $W_2^2(\omega_h)$ holds:*

$$\max_{t \in \omega_{\tau_0}} \|y - u\|_{W_2^2(\omega_h)} \leq c(h^2 + \tau_0^{1/2}).$$

REMARK 7.5 By virtue of the imbedding $W_2^2(\omega_h)$ into $C(\omega_h)$ the following estimate for the error of the method is valid:

$$\max_{t \in \omega_{\tau_0}} \|y - u\|_{C(\omega_h)} \leq c(h^2 + \tau_0^{1/2}).$$

3.3 Conservative Schemes

We have constructed and investigated above the non-conservative difference schemes for the hyperbolic second order equation which retain the conservatism property in separate sub-domains. And it is shown that on smooth solutions the difference scheme converges unconditionally in W_2^2 with the rate $O(h^2 + \tau_0^{1/2})$. Loss of the method's accuracy is explained by the impossibility of using the difference Green's formulas (and, consequently of bringing the norm W_2^1 into the investigation), and, on the other hand, by virtue of the dependence weight on the number of the fractional level by the lack of the scheme approximation (expression for $\|\psi_t\|$) even on smooth solutions.

Below we construct the difference schemes for a wave equation that preserve the conservatism property in ω. However, at the points of joining of grid domains the approximation has only a conventional character (both in the norm C and in the norm L_2). Nevertheless, it can be proved that the scheme has unconditional convergence in W_2^1 also with the rate $O(h^2 + \tau_0^{1/2})$.

On the grid $\overline{\omega}$ we replace the differential problem (7.93)–(7.95) by the difference one

$$y_{\bar{t}t,\alpha} = \left((a y_{\bar{x}})^{(\sigma_{1\alpha}, \sigma_{2\alpha})} \right)_x + f^{(\sigma_{1\alpha}, \sigma_{2\alpha})} \tag{7.115}$$

with approximation of the boundary and initial conditions in the form of (7.96). The weights $\sigma_{1\alpha}$, $\sigma_{2\alpha}$ are determined by formulas (7.97); moreover,

$$
\begin{aligned}
I_{1n} &= \{1, 2, ..., m_1^n + 1, m_2^n, ..., N-1\},\\
I_{2n} &= \{m_1^n + 2, ..., m_2^n - 1\}.
\end{aligned}
\tag{7.116}
$$

The three-level scheme (7.115) with the variable weighting factors $\sigma_{1\alpha}$, $\sigma_{2\alpha}$ is a conservative scheme.

Now let us find the approximate solution $y^{n+\alpha/p}$ provided that $\alpha = 2, 3, ..., p-2$ only in the domain $\bar{\omega}_2^n$ (see Fig. 7.4). Let at $t = t_n$, $t = t_{n+1/p}$ values of the grid function $y_{(0)}$, $y_{(1)}$ be known for all $x \in \omega_h$. We consider equation (7.115) at $(x,t) \in \bar{\omega}_1^n$

$$
y_{\bar{t}t,\alpha} = (\Lambda y + f)^{(\bar{\sigma}_{\alpha+1}, \bar{\sigma}_{\alpha-1})}, \quad i = 1, ..., m_1^n, m_2^n, m_2^n + 1, ..., N-1.
$$

In accordance with Lemma 7.3, the latter equation is transformed like

$$
\frac{y_{\alpha+1} - (\alpha+1)y_{(1)} + \alpha y_0}{\tau_0^2} = \bar{\sigma}_{\alpha+1}(\Lambda y + f)_{(\alpha+1))}. \tag{7.117}
$$

Equations (7.115) ($i = m_1^n + 1, ..., m_2^n - 1$), (7.117) are reduced to the system of three-point equations (7.11), where the right hand side does not depend on $y_{(\alpha-1)}$, $y_{(\alpha)}$ for the values $i = 1, 2, ..., m_1^n, m_2^n, ..., N-1$ from the domain of the smooth solution. Using the opposite direction sweep method we find the boundary condition $y(m_1^n h, t_{n+(\alpha+1)/p})$, and also a solution in the domain ω_2^n on the levels $n + \alpha/p$, $\alpha = 2, 3, ..., p-2$, by the formulas of the left sweep method. Provided that $\alpha = p-1, p$ the solution $y_{(p-1)}$, $\hat{y}$, $\hat{y}_{(1)}$ is computed for all $x \in \omega_h$. Then the procedure described is repeated.

Now we consider stability of the conservative difference scheme. To simplify calculations we assume that the coefficient $k(x) \neq k(x,t)$ in equation (7.93) does not depend on t, and $\mu_1(t) = \mu_2(t) = 0$. Defining in an ordinary way the grid function $y(t) = (y_1(t), ..., y_{N-1}(t))^T$, provided that $y_0(t) = y_N(t) = 0$, $t \in \omega_{\tau_0}$, i.e., $y(t) \in \overset{\circ}{\Omega}_h$, and introducing the Hilbert space $H = \Omega_h$, we write the difference scheme (7.115), (7.96) in the operator form

$$
y_{\bar{t}t} + T^*(Ty)^{(\Sigma_1, \Sigma_2)} = \varphi, \quad t \in \omega_{\tau_0}, \tag{7.118}
$$

$$
y(0) = u_0, \quad y_t(0) = \tilde{u}_0, \tag{7.119}
$$

where $y_{\bar{t}t} = (y(t+\tau_0) - 2y(t) + y(t-\tau_0))/\tau_0^2$.

The operators T^* and T are given by formulas (6.6.36), (6.6.37), and the operators $\Sigma_\beta(t)$, $\beta = 1, 2$, are defined as in scheme (6.6.42).

To investigate stability of the scheme with operator weighting (7.118) we could apply the results obtained in Section 5.3 to three-level schemes (see (5.3.26), (5.3.27))

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi \tag{7.120}$$

with operators of conservative type

$$D = E + 0.5\tau_0^2 T^* G_1 T, \quad B = \tau_0 T^* G_2 T, \quad A = T^* T. \tag{7.121}$$

In fact, using the identity

$$v^{(\Sigma_1, \Sigma_2)} = v + \tau_0(\Sigma_1 - \Sigma_2) v_{\overset{\circ}{t}} + \frac{\tau_0^2}{2}(\Sigma_1 + \Sigma_2) v_{\bar{t}t},$$

we reduce scheme (7.118) to the form of (7.120), (7.121) with

$$G_1 = \Sigma_1 - \Sigma_2, \quad G_2 = \Sigma_1 + \Sigma_2.$$

To study convergence of the scheme with discontinuous, with respect to time and space, weighting factors of the form (7.115), we obtain the *a priori* estimates for the operator scheme (7.118), when the operators $\Sigma_\beta(t)$ are not Lipschitz continuous in the variable t and they do not commute with the operator A.

Below we need the notation

$$A_1 = T^*(\Sigma_1 - 0.5E)T, \quad A_2 = T^* \Sigma_2 T, \quad M_n = \exp(t_n),$$

$$\Sigma_3 = \Sigma_1 - \Sigma_2^{(0.5)} - 0.5E, \quad Q = E + \tau_0^2 A_2, \quad R = Q + \tau_0^2 A,$$

$$\|y\|_1^2 = \|y\|_A^2 + \|y_{\bar{t}}\|_Q^2, \quad \|\varphi\|_2^2 = \|\varphi_1\|^2 + \frac{1}{\varepsilon}\|\varphi_2]\|^2,$$

$$\Sigma_2^{(0.5)} = 0.5(\Sigma(t) + \Sigma(t + \tau_0)), \quad \|\cdot\|_D = \sqrt{(D\cdot, \cdot)}, \quad D = D^* \geq 0.$$

The following statement is valid:

THEOREM 7.6 *Let in scheme (7.118), (7.119) T be a constant operator and T^{-1} exist, and variable weighting operators satisfy the inequalities*

$$\begin{aligned} &\Sigma_1(t) \geq \Sigma_2^{(0.5)} + \frac{1+\varepsilon}{2} E, && \varepsilon > 0, \\ &\Sigma_2^*(t) = \Sigma_2(t) \geq -\frac{1}{\tau_0^2 \|A\|} E, && t \in \omega_\tau, \end{aligned} \tag{7.122}$$

$$\varphi = \varphi_1 + \tau_0^{1/2} T^* \varphi_2. \tag{7.123}$$

Then for its solution for any τ_0 the estimate holds

$$\|y_n\|_A^2 \leq M_n \left(\|y(0)\|_A^2 + \|y_t(0)\|_{R(\tau_0)}^2 + \sum_{t \in \omega_{\tau_0}} \tau_0 \|\varphi\|_2^2 \right). \tag{7.124}$$

Proof. First let us show that under the conditions of the theorem the operator $Q(t) \geq 0$. In fact, since for the self-adjoint operator $E \geq A/\|A\|$, it follows that

$$Q \equiv T^* \left(\Sigma_2 + \frac{1}{\tau_0^2\|A\|}E\right)T + \left(E - \frac{A}{\|A\|}\right) \geq 0.$$

Now using the identity

$$v^{(\Sigma_1,\Sigma_2)} = v^{(0.5)} + \tau_0(\Sigma_1 - 0.5E)v_t - \tau_0\Sigma_2 v_{\bar{t}},$$

we rewrite the difference equation (7.118) in the form

$$y_{\bar{t}t} + Ay^{(0.5)} + \tau_0 A_1 y_t - \tau_0 A_2 y_{\bar{t}} = \varphi_1 + \tau_0^{1/2}T^*\varphi_2. \tag{7.125}$$

We take the dot product of the latter equation with $2\tau_0 y_t$ and consider individual inner products

$$\begin{aligned}
2\tau_0(y_t, y_{\bar{t}t} + Ay^{(0.5)} + \tau A_1 y_t)& \\
= \tau_0(\|y_t\|^2 + \|\hat{y}\|_A^2)_{\bar{t}} &+ \tau_0^2(\|y_{\bar{t}t}\|^2 + 2\|y_t\|_{A_1}^2),\\
2\tau_0(y_t, \varphi_1) \leq \tau_0\|y_{\bar{t}}\|^2 &+ \tau_0^2\|y_{\bar{t}t}\|^2 + \tau_0(1+\tau_0)\|\varphi_1\|^2,\\
2\tau_0(\tau_0^{1/2}T^*\varphi_2, y_t) = 2\tau_0(\tau_0^{1/2}\varphi_2, Ty_t] &\leq \tau_0^2\varepsilon\|y_t\|_A^2 + \frac{\tau_0}{\varepsilon}\|\varphi_2]|^2.
\end{aligned}$$

Adding the estimates obtained, similarly to (5.3.39) we obtain the recurrence relation

$$\|y_{k+1}\|_1^2 \leq (1+\tau_0)\left(\|y_k\|_1^2 + \tau_0\|\varphi_k\|_2^2\right). \tag{7.126}$$

Summing it over $k = 1, 2, ..., n-1$ and taking into account that

$$\|y(\tau_0)\|_1^2 = \|y(0)\|_A^2 + \|y_{\bar{t}}\|_{R(\tau_0)}^2,$$

we obtain the theorem's statement.

According to the inequalities (7.110) we conclude that the conservative difference scheme on the adaptive time grid (7.115), (7.116) under the conditions

$$\sigma_1 \geq \sigma_2 + \frac{1+\varepsilon}{2}, \quad \sigma_2 \geq -\frac{h^2}{4\tau_0^2 c_2}, \quad 0 < \varepsilon \leq 1, \tag{7.127}$$

is stable with respect to the initial data and right hand side in H_A, and for its solution for any τ_0 the estimate (7.126) is correct with $\varphi_1 = f^{(\sigma_{1\alpha},\sigma_{2\alpha})}$, $\varphi_2 = 0$.

It now remains to consider the truncation error and investigate convergence of the conservative difference scheme. Substituting $y = z + u$

into equations (7.118), (7.119), similarly to equalities (7.111), (7.112) we obtain the problem

$$z_{\bar{t}t} + T^*(Ty)^{(\Sigma_1,\Sigma_2)} = \psi, \quad t \in \omega_{\tau_0}, \tag{7.128}$$

$$z(0) = 0, \quad z_t(0) = \psi^0. \tag{7.129}$$

By means of identity (7.113) the truncation error can be written in the form

$$\psi = \psi_1 + \tau_0^{1/2} T^* \psi_2,$$
$$\psi_1 = -u_{\bar{t}t} + (au_{\bar{x}})_x + f^{(\sigma_1,\sigma_2)} = O(h^2 + \tau_0),$$
$$\psi_2 = -\tau_0^{1/2}\left(\sigma_1(au_{\bar{x}})_t - \sigma_2(au_{\bar{x}})_{\bar{t}}\right) = O(\tau_0^{1/2}).$$

The application of Theorem 7.6 for the solution of problem (7.128), (7.129) yields

$$\|z_n\|_A^2 \le M\left(\|\psi^0\|^2_{R(\tau_0)} + \sum_{t\in\omega_{\tau_0}} \tau_0\left(\|\psi_1\|^2 + \frac{1}{\varepsilon}\|\psi_2]|^2\right)\right). \tag{7.130}$$

Let the smoothness conditions (7.114) be satisfied. Then obviously

$$\|\psi^0\|^2_{R(\tau_0)} = O(\tau_0^4), \quad \|\psi_1\|^2 = O(h^4 + \tau_0^2), \quad \|\psi_2]|^2 = O(\tau_0).$$

Consequently the following statement holds.

THEOREM 7.7 *For the differential problem (7.93), (7.94) let the conditions (7.114) be valid. Then under conditions (7.127) the solution of the difference scheme (7.115), (7.96) converges unconditionally to the exact solution $u(x,t)$ and the estimate of convergence rate holds:*

$$\max_{t\in\omega_{\tau_0}} \|y-u\|_A \le c(h^2 + \tau_0^{1/2}), \quad \|z\|_{C(\omega_{h\tau_0})} \le c(h^2 + \tau_0^{1/2}).$$

Among the more important generalizations of the given fundamental result we note the possibility of obtaining similar estimates of 7.6 for the multi-dimensional wave equation.

4. Difference Schemes of Domain Decomposition on the Grids Locally Refined with Respect to Time

The schemes on the grids locally refined with respect to time for parabolic equations on the basis of decomposition of a computational domain on each time level are investigated [Vabishchevich, 1995]. The schemes constructed can be interpreted as the schemes with interpolation of solution on the adaptation zone boundary.

4.1 Introduction

An increase in the accuracy of an approximate solution in the problems of mathematical physics is often achieved by means of local refinement of a grid with respect to space [Babuska et al., 1986, Flaherty et al., 1989, McCormick, 1989]. In many non-stationary problems an increase in the accuracy of an approximate solution can be achieved by refining a grid with respect to time. This kind of time-adaptive numerical methods constructed for example, in [Davis and Flaherty, 1982, Osher and Sanders, 1983, Revilla, 1986]. These methods use a time step in the adaptation zone (in a part of the computational domain) substantially smaller than outside of this zone. Let us recall the main approaches to construction of difference schemes on grids locally refined with respect to time.

The unconditionally convergent difference schemes with adaptation with respect to time have been constructed above. In this case a computational effect is achieved owing to the special order of calculations outside of the zone of adaptation.

The schemes with adaptation with respect to time can be considered as the schemes on substantially non-uniform grids with respect to time. In this interpretation we have a scheme with fictitious (equal to zero) time steps. On this methodological basis the time-adaptive schemes for parabolic problems were considered in [Abrashin and Lapko, 1993, Vabishchevich and Matus, 1994, Matus and Vabishchevich, 1994].

Traditionally, wide use is made of the algorithms with interpolation on the boundary of the zone of adaptation (see, e.g., [Drobyshevich, 1995, Ewing et al., 1989]). The same approach is often used for refining grids with respect both to time and space [Ewing et al., 1992, Ewing et al., 1990]. In this case, only conditional convergence of these difference schemes can be determined [Ewing and Lazarov, 1994].

Here we follow the paper [Vabishchevich, 1995] where the schemes on grids locally refined with respect to time for parabolic problems were investigated on the basis of decomposition of a computational domain on each time level. The schemes constructed can be interpreted as the schemes with interpolation of solution on the boundary of the zone of adaptation. Different types of exchange boundary conditions are used. The investigation of stability and convergence in Hilbert grid spaces is based on the general theory of difference schemes [Samarskii, 1989, Samarskii and Vabishchevich, 1995a].

4.2 Model Problem

As an illustrative example we will consider the boundary a value problem for one-dimensional parabolic. Since basic to this is the operator formulation of the problem, the conversion to more general problems is in great part editorial. In the domain $\Omega = (0,1)$ we find a solution of the parabolic equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) = f(x,t), \quad x \in \Omega, \quad t > 0 \tag{7.131}$$

supplemented with the simplest homogeneous boundary conditions of the first kind

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t > 0 \tag{7.132}$$

and the initial condition

$$u(x,0) = u_0(x), \quad x \in \overline{\Omega}. \tag{7.133}$$

In the domain Ω we introduce a uniform grid with the step h. By ω we denote a set of interior nodes. Using the standard notation of the theory of difference schemes, on a set of the grid functions $v \in H$ such that $v(x) = 0$, $x \notin \omega$ we define the grid operator A

$$Av = -(a(x)v_{\bar{x}})_x, \quad x \in \omega. \tag{7.134}$$

Here, e.g., $a(x) = k(x - 0.5h)$. In the Hilbert space H we introduce the scalar product and the norm by the relations

$$(y,v) = \sum_{x\in\omega} y(x)v(x)h, \quad \|y\| = \sqrt{(y,y)}.$$

In the space H we have $A = A^* > 0$ for the operator A defined in accordance with relation (7.134). From problem (7.131)–(7.133) we pass to the equation

$$\frac{dv}{dt} + Av = \varphi, \quad x \in \omega \tag{7.135}$$

with prescribed $v(x,0)$, $x \in \omega$. Let us assume that in a certain selected sub-domain $\Omega^* \subset \Omega$ (the zone of adaptation), for some reason, it is necessary to use substantially smaller time step (equaled to τ/p, $p > 1$) than outside this zone where we can use the time step τ.

4.3 Decomposition Operators

Let $t_n = nt$ $(n = 0,1,\ldots)$ be the grid with respect to time. To bring into effect the passage from time level t_n to t_{n+1} we introduce a finer grid in the sub-domain Ω^*. Typical of non-stationary problems is the situation with dynamical singularities, i.e., the zone of adaptation unique in each time moment ($\Omega^* = \Omega^*(t)$). For formal selection of the adaptation sub-domain we introduce the function $\chi(x,t)$ such that $\chi(x,t) = 1$ in $\bar{\Omega}^*(t) = \Omega^* \cup \partial\Omega^*$. We select the transition domain $\Omega^{**}(t)$ outside of the zone of adaptation. This domain is adjoined to the boundary $\partial\Omega^*$. The function $\chi(x,t)$ is changed from unity to zero in the domain Ω^{**} and $\chi(x,t) = 0$ outside of it (at a certain distance from the zone of adaptation $\Omega^*(t)$). In the important limiting case of the absence of the transition domain we have $\chi(x,t) = 0,\ x \notin \Omega^*(t)$.

Let us construct difference schemes for equation (7.135) on the basis of additive splitting of the operator (7.134):

$$A = A^{(1)} + A^{(2)}. \tag{7.136}$$

Let us couple the zone of adaptation (the function $\chi(x,t)$) with the operator $A^{(1)}$. We shall use three main methods of construction of the operator $A^{(1)}$ in (7.136) by analogy with the schemes of domain decomposition for parabolic problems (a more detailed discussion is suggested in the next chapter).

Two general classes of the decomposition method are connected with the definition of the decomposition operator $A^{(1)}$ in accordance with the conditions

$$A^{(1)} = \chi(x,t)A, \tag{7.137}$$

$$A^{(1)} = A\chi(x,t). \tag{7.138}$$

Moreover, in the case of a second-order parabolic equation we can define the operator $A^{(1)}$ in accordance with the condition

$$A^{(1)}y = -(a_1(x)y_{\bar{x}})_x, \quad x \in \omega, \tag{7.139}$$

where $a_1(x) = \chi(x-0.5h,t)k(x-0.5h))$. In this case $A^{(\beta)} = (A^{(\beta)})^* \geq 0,\ \beta = 1,2$, but for (7.137), (7.138) the operators $A^{(\beta)},\ \beta = 1,2$ are not non-negative and self-adjoint.

We shall construct the schemes on grids locally refined with respect to time on the basis of the splitting of the form (7.136) with the decomposition operator defined in accordance with (7.137)–(7.139).

To construct schemes with additional steps, we introduce the following notation. Let $y_{n+\alpha/p}$ be an approximate solution at the time moment $t_{n+\alpha/p} = t_n + \alpha\tau/p,\ \alpha = 1,2,\dots,p$. In the zone of adaptation we shall use the simplest purely implicit scheme

$$\frac{y_{n+\alpha/p} - y_{n+(\alpha-1)/p}}{\tau/p} + Ay_{n+\alpha/p} = \varphi_{n+\alpha/p}. \tag{7.140}$$

It may be noted at the outset that one could also turn to using a more general class of two-level schemes with weights.

Far from the zone of adaptation (outside of the transition domain $\Omega^{**}(t)$) calculations are carried out with a large step, and therefore by analogy with the implicit scheme (7.140) we have

$$\frac{y_{n+1} - y_n}{\tau} + Ay_{n+1} = \varphi_{n+1}. \tag{7.141}$$

Here the solutions on intermediate levels can be obtained by interpolation of the grid functions y_n and y_{n+1}. Linear interpolation seems to be the most natural. In this case we have

$$y_{n+\alpha/p} = y_n + \frac{\alpha}{p}(y_{n+1} - y_n), \quad \alpha = 1, 2, \ldots, p-1. \tag{7.142}$$

It is convenient to use a representation like that of equations (7.141), (7.142) in the form of the scheme

$$\frac{y_{n+\alpha/p} - y_{n+(\alpha-1)/p}}{\tau/p} + Ay_{n+1} = \varphi_{n+1}. \tag{7.143}$$

Thus on the basis of the schemes (7.140), (7.143) it is possible to try to construct a scheme on grids locally refined in time. Sewing of solutions in the transition zone is carried out by selecting of another one operator of decomposition. Define the approximate solution on the whole spatial grid by means of the difference scheme

$$\frac{y_{n+\alpha/p} - y_{n+(\alpha-1)/p}}{\tau/p} + A_n^{(1)} y_{n+\alpha/p} + A_n^{(2)} y_{n+1} = \phi_{n+\alpha/p}, \tag{7.144}$$

where

$$\phi_{n+\alpha/p} = \chi(x, t_n)\varphi_{n+\alpha/p} + (1 - \chi(x, t_n))\varphi_{n+1}$$

and $A_n^{(\beta)} = A^{(\beta)}(t_n)$, $\beta = 1, 2$. Taking into account the relations (7.136)–(7.139), we have $A_n^{(2)} = A$ far from of the zone of adaptation. This, in accordance with scheme (7.143), defines the approximate solution on intermediate levels by means of linear interpolation of solutions on the integral time levels. The choice of different decomposition operators corresponds to specification of different boundary conditions on the interface boundary $\partial\Omega^*(t)$. In the case of $\chi(x, t) = 0$, $x \notin \bar{\Omega}^*$, the boundary value problem in the zone of adaptation is solved subject to these or other boundary conditions on the boundary $\partial\Omega^*$ that are calculated using interpolated solution outside of the zone of adaptation. In particular, if $A^{(1)}$ is specified in accordance with formula (7.137), then we obtain the ordinary boundary conditions of the first kind. The boundary conditions of the second kind are actually implemented when operator (7.139) is used.

Scheme (7.144) is not quite convenient for computation since, it is necessary to know the solution at the time moment t_{n+1} to find the solution on the transition step. The questions of construction of iterative algorithms for this class of problems were discussed, e.g., in the papers [Ewing and Lazarov, 1994, Ewing et al., 1993].

4.4 Stability

Let us show the unconditional stability of scheme (7.144) in the corresponding Hilbert spaces H_D, $D = D^* > 0$. In these spaces we define the scalar product by $(u, v)_D = (Du, v)$. We will carry out detailed study on specification of the operators $A^{(\beta)}$, $\beta = 1, 2$ according to formulas (7.136), (7.137). In this case, scheme (7.144) can be written in the form

$$\begin{aligned}\frac{y_{n+\alpha/p} - y_{n+(\alpha-1)/p}}{\tau/p} &+ \chi(x, t_n) A y_{n+\alpha/p} \\ &+ (1-\chi(x, t_n)) A y_{n+1} = \phi_{n+\alpha/p}, \quad x \in \omega, \quad \alpha = 1, 2, \dots, p.\end{aligned} \tag{7.145}$$

Let us consider scheme (7.145) as a scheme with variable weights. It belongs to a class of symmetrizable difference schemes. Before obtaining the corresponding *a priori* estimate we introduce the following notation for the difference scheme (7.145):

$$v_\alpha = y_{n+\alpha/p}, \quad v_{\bar{t},\alpha} = \frac{v_\alpha - v_{\alpha-1}}{\tau/p},$$

where $v_0 = y_n$, $v_p = y_{n+1}$. We rewrite the difference scheme (7.145) in the form

$$\begin{gathered}\left(E + \frac{\tau}{2p}\chi_n A\right) v_{\bar{t},\alpha} + \frac{1}{2}\chi_n A(v_\alpha + v_{\alpha-1}) + (1 - \chi_n) A v_p = \phi_{n+\alpha/p}, \\ x \in \omega, \quad \alpha = 1, 2, \dots, p.\end{gathered} \tag{7.146}$$

Taking a dot product of each equation (7.146) with $(2\tau/p) A v_{\bar{t},\alpha} = 2A(v_\alpha - v_{\alpha-1})$ and summing them, we obtain the equality

$$\begin{aligned}&\frac{2\tau}{p}\sum_{\alpha=1}^{p}(A v_{\bar{t},\alpha}, v_{\bar{t},\alpha}) + \frac{\tau^2}{p^2}\sum_{\alpha=1}^{p}(\chi_n A v_{\bar{t},\alpha}, A v_{\bar{t},\alpha}) \\ &+ (\chi_n A v_p, A v_p) - (\chi_n A v_0, A v_0) \\ &+ 2((1 - \chi_n) A v_p, A v_p) - 2((1 - \chi_n) A v_p, A v_0) \\ &\qquad = \frac{2\tau}{p}\sum_{\alpha=1}^{p}(A v_{\bar{t},\alpha}, \phi_{n+\alpha/p}).\end{aligned} \tag{7.147}$$

Taking into account the inequalities

$$2((1-\chi_n)Av_p, Av_p) - 2((1-\chi_n)Av_p, Av_0)$$
$$\geq ((1-\chi_n)Av_p, Av_p) - ((1-\chi_n)Av_0, Av_0),$$
$$(Av_{\bar{t},\alpha}, \phi_{n+\alpha/p}) \leq (Av_{\bar{t},\alpha}, v_{\bar{t},\alpha}) + \frac{1}{4}(A\phi_{n+\alpha/p}, \phi_{n+\alpha/p}),$$

from equality (7.147) we obtain

$$(Av_p, Av_p) \leq (Av_0, Av_0) + \frac{\tau}{2p}\sum_{\alpha=1}^{p}(A\phi_{n+\alpha/p}, \phi_{n+\alpha/p}).$$

Thus we obtain the desired *a priori* estimate for the scheme (7.145):

$$\|Ay_{n+1}\|^2 \leq \|Ay_0\|^2 + \frac{1}{2p}\sum_{k=0}^{n}\tau\sum_{\alpha=1}^{p}(A\phi_{k+\alpha/p}, \phi_{k+\alpha/p}). \qquad (7.148)$$

We consider similarly scheme (7.144) on specifying the operators of decomposition according to formulas (7.138), (7.139). It allows us to formulate the following statement concerning the stability of scheme (7.144) with respect to initial conditions and right hand side.

Thus we have proved the following statement:

THEOREM 7.8 *The difference scheme (7.136), (7.144) is unconditionally stable in H_D where $D = A^2$ with the decomposition operator defined in accordance with relation (7.137), with $D = E$ under condition (7.138) and $D = A$ under condition (7.139). The following estimate for the difference solution is correct:*

$$\|y_{n+1}\|_D^2 \leq \|y_0\|_D^2 + \frac{1}{2p}\sum_{k=0}^{n}\tau\sum_{\alpha=1}^{p}(DA^{-1}\phi_{k+\alpha/p}, \phi_{k+\alpha/p}). \qquad (7.149)$$

Thus with the different ways of defining decomposition operators we obtain the schemes which have stability in different norms.

4.5 Convergence of Difference Schemes

The dependence of the accuracy of an approximate solution on interface boundary conditions is important for investigation of convergence of schemes on locally refined grids. In our case, we talk about the influence of the function $\chi(x,t)$ on the accuracy of the difference scheme (7.144).

Let $z_{n+\alpha/p} = y_{n+\alpha/p} - u_{n+\alpha/p}$ be an error of approximate solution at the corresponding time moment. From scheme (7.144) we arrive at the following difference problem for the error:

$$\frac{z_{n+\alpha/p} - z_{n+(\alpha-1)/p}}{\tau/p} + A_n^{(1)}z_{n+\alpha/p} + A_n^{(2)}z_{n+1} = \psi_{n+\alpha/p},$$

where $\psi_{n+\alpha/p}$ is the truncation error. Let us limit ourselves again to consideration of the decomposition operators (7.136), (7.137):

$$\begin{aligned}&\frac{z_{n+\alpha/p} - z_{n+(\alpha-1)/p}}{\tau/p} + \chi(x,t_n)Az_{n+\alpha/p}\\&+(1-\chi(x,t_n))Az_{n+1} = \psi_{n+\alpha/p}, \quad x \in \omega, \quad \alpha = 1,2,\dots,p.\end{aligned} \tag{7.150}$$

Taking into account that $z_0(x) = 0$, $x \in \omega$, we obtain the following estimate for the difference scheme (7.150) (see (7.148)):

$$\|Az_{n+1}\|^2 \le \frac{1}{2p}\sum_{k=0}^{n}\tau\sum_{\alpha=1}^{p}(A\psi_{k+\alpha/p}, \psi_{k+\alpha/p}). \tag{7.151}$$

For the truncation error we have

$$\begin{aligned}\psi_{k+\alpha/p} =& \phi_{k+\alpha/p} - \frac{u_{n+\alpha/p} - u_{n+(\alpha-1)/p}}{\tau/p} - Au_{n+1}\\&+\left(1-\frac{\alpha}{p}\right)\chi(x,t_n)\frac{u_{n+1} - u_{n+(\alpha-1)/p}}{(1-\alpha/p)\tau}.\end{aligned}$$

Taking into account this statement on sufficiently smooth solutions of the problem (7.131)–(7.133) we obtain

$$\psi_{k+\alpha/p} = O(\tau + h^2) + \left(1-\frac{\alpha}{p}\right)\tau\chi(x,t_n)A\frac{\partial u}{\partial t}. \tag{7.152}$$

Substituting the expression (7.152) into inequality (7.151) we obtain the desired estimate for the error

$$\|Az_{n+1}\| \le M_1(\tau + h^2) + M_2\tau\left\|\chi A\frac{\partial u}{\partial t}\right\|_A. \tag{7.153}$$

The first item on the right hand side of estimate (7.153) is the usual error of a purely implicit scheme, and the second one is conditioned by the use of the grid locally refined in time.

We can similarly consider variants of the decomposition operator (7.136), (7.138) and (7.136), (7.139).

Thus the following statement is correct:

THEOREM 7.9 *For the difference scheme (7.136), (7.144) the estimate of the convergence rate has the form*

$$\|z_{n+1}\|_D \le M_1(\tau + h^2) + M_2\tau\|\chi A\frac{\partial u}{\partial t}\|_A, \tag{7.154}$$

where $D = A^2, E, A$ with the use of the operators (7.137)–(7.139), respectively.

In the case of the absence of the transition zone ($\chi(x,t) = 1,\ x \in \bar{\Omega}^*(t)$, $\chi(x,t) = 0,\ x \notin \bar{\Omega}^*(t)$), from estimate (7.154) it follows that the rate of convergence is conditional and determined by the value $O(\tau + h^2 + \tau h^{-1/2})$.

In the above the investigation of different classes of schemes on grids locally refined in time has been carried out on the operator level (for the evolution equation of the first order (7.135)). Therefore the results are extended to far more general problems than (7.131)–(7.133). The boundary value problems for the evolution equations of the second order (in particular, for hyperbolic equation) can be considered in the same way. Mention might also be made of the schemes with grids locally refined in time for equations (7.135) with a non-self-adjoint positive operator A. As an independent object we can single out problems with the subordinate skew-symmetric part of the operator A. As regards computational realization, the schemes in which calculations in the zone of adaptation and outside of it are different deserve particular attention.

5. Difference Schemes on Dynamical Grids Locally Refined in Space

The questions of construction of difference schemes with local refinement of a grid in space are considered. Investigation of convergence of difference schemes for the simplest non-stationary problems of mathematical physics is carried out.

5.1 Introduction

The correct choice of a computational grid in the problems of mathematical physics has always been the most important component of a numerical solution. A coarser grid can be used in the areas of a quite smooth solution and coefficients and a more detailed one can be used near various singularities [Vabishchevich, 1989a, Dar'in et al., 1988, Matus, 1993b].

At the present time two main trends in constructing adaptive grids for non-stationary problems can be singled out: the method of dynamical locally refined grids (it consists in local addition of grid nodes in the areas of low precision of solution and their possible elimination in other areas) and the method of moving grids (see, e.g., the paper [Dar'in et al., 1988]). The advantages and disadvantages of moving and rectangular grids were discussed, e.g., in [Gropp, 1987, Hedstrom and Rodrique, 1960].

With the first approach certain difficulties arise in approximation of time derivatives at new nodes. In [Bayarunas and Čiegis, 1991] this problem is solved by using a difference scheme of the 'predictor-corrector' type and interpolation of solution on different grids.

In the present section, using as an example the simplest parabolic equation, we study the problems of construction and investigation of difference schemes on locally refined grids. On the basis of the general theory of operator-difference schemes the estimates of stability and convergence of the grid methods in strong energy norms are obtained. These *a priori* estimates show that it is necessary to apply the interpolation carefully to avoid the instability of a difference scheme or the divergence of difference solution [Vabishchevich and Matus, 1995, Vabishchevich et al., 1995].

5.2 Statement of the Problem

Let us illustrate the problem of construction and investigation of difference schemes on *dynamical locally refined rectangular grids* by means of the simplest one-dimensional problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in \Omega, \quad t > 0, \tag{7.155}$$

supplemented with the simplest homogeneous boundary conditions of the first kind

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t > 0 \tag{7.156}$$

and the initial condition

$$u(x,0) = u_0(x), \quad x \in \overline{\Omega}. \tag{7.157}$$

Let us introduce into consideration the uniform grids

$$\overline{\omega}_h = \{x_i = ih, \quad i = \overline{0,N}, \quad h = l/N\},$$
$$\overline{\omega}_{h/2} = \{x_{i+\alpha/2} = (i+\alpha/2)h, \quad i = \overline{0,N-1}; \quad \alpha = 0,1; \quad x_N = l\}$$

with the steps h and $h/2$ with respect to the spatial variable and the step τ with respect to the time variable:

$$\omega_\tau = \{t_n = n\tau, \quad n = 0,1,...,N_0 - 1, \quad \tau N_0 = T\}.$$

We shall use a more detailed grid in the part of the computational domain where the desired solution has a singularity.

5.3 Construction of a Scheme with New Nodes on the Upper Level

In the mathematical simulation of elliptic problems with singularities we can use non-uniform grids to achieve the necessary accuracy by refining them in the domains with high gradients. When we use locally

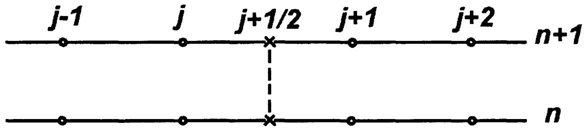

Figure 7.6.

refined grids for non-stationary problems the situation is complicated, since the area of irregularity usually, changes in time. This makes it necessary to use own non-uniform grids with respect to a spatial variable on each time level,i.e., to the disappearence and appearence of new nodes. This circumstance leads to certain problems in approximation at the new nodes, since on the previous time level the approximate solution was not computed at those nodes.

For the simplificity of discussion let us assume that on the level $t_n = n\tau$ we know the approximate solution at all the nodes of the grid ω_h: $x_i = ih$, $i = 0, 1, ..., N$. It is necessary to find the solution on the level t_{n+1} at the additional node (the minimal zone of adaptation)

$$x_{j+1/2} = (j + 1/2)h.$$

We note the main moments which arise in construction of a difference scheme with new nodes on the time level t_{n+1}:

1) interpolation at the node $(x_{j+1/2}, t_n)$;

2) scheme at the node $(x_{j+1/2}, t_{n+1})$;

3) scheme outside of the zone of adaptation $(i \leq j - 1,\ i \geq j + 2)$;

4) scheme at the boundary of adaptation $(i = j,\ j + 1)$.

Let us consider the problems 1)–4) separately.

1) To determinate the solution $y_{j+1/2} = y(x_{j+1/2}, t_n)$ on the level t_n we apply the natural interpolation

$$y_{j+1/2} = 0.5(y_j + y_{j+1}). \tag{7.158}$$

This solution is used to find the solution by means of the difference scheme at the additional node. Let us note that this expression can be written in the form

$$Ay_{j+1/2} = 0, \tag{7.159}$$

$$Ay_{j+1/2} = \frac{y_{j+1} - 2y_{j+1/2} + y_j}{(h/2)^2}, \tag{7.160}$$

It defines the second difference derivative on the grid $\omega_{h/2}$. The solution at the fictitious nodes $(x_{i+1/2}, t_{n+1})$, $i = \overline{0, N-1}$, $i \neq j$, on the level $t = t_{n+1}$ will also be defined by means of interpolation like (7.158):

$$Ay^{n+1} = 0. \tag{7.161}$$

2) At the node $(x_{j+1/2}, t_{n+1})$ we use the following purely implicit schemes to determine the approximate solution

$$\frac{y^{n+1}_{j+1/2} - 0.5(y^n_j + y^n_{j+1})}{\tau} + Ay^{n+1}_{j+1/2} = f(x_{j+1/2}, t_{n+1}). \tag{7.162}$$

3) Outside the zone of adaptation at integer nodes we shall use the approximation

$$\frac{y^{n+1}_i - y^n_i}{\tau} - \frac{y^{n+1}_{i+1} - 2y^{n+1}_i + y^{n+1}_{i-1}}{h^2} = f^{n+1}_i. \tag{7.163}$$

In the scheme (7.163) we transform the operator

$$A_0 y^{n+1}_i = -\frac{y^{n+1}_{i+1} - 2y^{n+1}_i + y^{n+1}_{i-1}}{h^2}$$

taking into account interpolation in accordance with formula (7.161) at the neighboring nodes. We obtain

$$\begin{aligned}(A_0 y)^{n+1}_i &= -\frac{0.5(y^{n+1}_{i+1} + y^{n+1}_i) - 2y^{n+1}_i + 0.5(y^{n+1}_i + y^{n+1}_{i-1})}{h^2/2} \\ &= -\frac{y^{n+1}_{i+1/2} - 2y^{n+1}_i + y^{n+1}_{i-1/2}}{h^2/2} = 0.5Ay^{n+1}_i.\end{aligned} \tag{7.164}$$

It allows us to write the difference scheme (7.163) in the form

$$\frac{y^{n+1}_i - y^n_i}{\tau} + 0.5Ay^{n+1}_i = f^{n+1}_i. \tag{7.165}$$

4) We consider the approximation of the difference equation (7.155) at the boundary of adaptation $i = j$:

$$\frac{y^{n+1}_j - y^n_j}{\tau} - \frac{1}{\hbar}\left(\frac{y^{n+1}_{j+1/2} - y^{n+1}_j}{h/2} - \frac{y^{n+1}_j - y^{n+1}_{j-1}}{h}\right) = f^{n+1}_j, \tag{7.166}$$

where $\hbar = 0.5(h/2 + h) = 3h/4$. Let us separately investigate the operator with respect to space:

$$A_1 y_j = -\frac{4}{3h}\left(\frac{y_{j+1/2} - y_j}{h/2} - \frac{y_j - y_{j-1}}{h}\right).$$

Taking into account interpolation of the solution at the half-integer node $j - 1/2$ $(y_{j-1/2} = 0.5(y_j + y_{j-1}))$ we have

$$\frac{y_j - y_{j-1}}{h} = \frac{2}{h}\left(y_j - \frac{1}{2}(y_j + y_{j-1})\right) = \frac{y_j - y_{j-1/2}}{(h/2)}.$$

Therefore

$$A_1 y_j = -\frac{4}{3h}\left(\frac{y_{j+1/2} - y_j}{h/2} - \frac{y_j - y_{j-1/2}}{h/2}\right) = \frac{2}{3}Ay_j.$$

Thus we obtain the following difference equation at the boundary of adaptation:

$$\frac{y_j^{n+1} - y_j^n}{\tau} + \frac{2}{3}Ay_j^{n+1} = f_j^{n+1}. \tag{7.167}$$

The scheme for $i = j + 1$ can be written similarly:

$$\frac{y_{j+1}^{n+1} - y_{j+1}^n}{\tau} + \frac{2}{3}Ay_{j+1}^{n+1} = f_{j+1}^{n+1}. \tag{7.168}$$

The foregoing arguments allow us to use the unified notation of difference schemes on a grid locally refined in space.

5.4 *A Priori* Estimates

Let $\overset{\circ}{\Omega}_{h/2}$ be a set of grid functions $y(x, t)$ defined for each $t \in \overline{\omega}_\tau$ on the grid $\overline{\omega}_{h/2}$ and be equal to zero for $x = 0$, $x = l$. The operator A can be defined like formula (6.1.5):

$$\begin{gathered}(Ay)_{i+\alpha/2} = -y_{\bar{x}x,i+\alpha/2} = \frac{y_{i+(\alpha+1)/2} - 2y_{i+\alpha/2} + y_{i+(\alpha-1)/2}}{(h/2)^2},\\ i = 0,\ \alpha = 1; \quad i = 1, 2, \ldots, N-1,\ \alpha = 0, 1;\\ y_0(t) = y_N(t) = 0, \qquad t \in \omega_\tau.\end{gathered} \tag{7.169}$$

Furthermore, let $y_n = (y_{1/2}^n, y_1^n, \ldots, y_{N-1/2}^n)^T$ be the desired vector. We define the Euclidean space $H = \Omega_{h/2}$ by a set of these vectors with the scalar product and the norm

$$(u, v) = \sum_{i=1}^{N-1} \frac{h}{2} u_i v_i + \sum_{i=0}^{N-1} \frac{h}{2} u_{i+1/2} v_{i+1/2}, \quad \|u\| = \sqrt{(u, u)},$$

respectively. Then the difference schemes (7.161), (7.162), (7.166)–(7.168) can be written in the canonical form

$$By_t + Ay = \varphi, \quad y_0 = u_0, \tag{7.170}$$

where

$$B = S + \tau A, \quad S = \mathrm{diag}\{s_{1/2}, s_1, \dots, s_{N-1/2}\}, \tag{7.171}$$

$$s_{i+\alpha/2} = \begin{cases} 0 & \text{for } \alpha = 1,\ i \neq j, \\ 1 & \text{for } \alpha = 1,\ i = j, \\ 3/2 & \text{for } \alpha = 0,\ i = j, j+1, \\ 2 & \text{for } \alpha = 0,\ i \neq j, j+1, \end{cases}$$

$$\varphi = \varphi_n = \left(\varphi^n_{1/2}, \varphi^n_1, \dots, \varphi^n_{N-1/2}\right)^T, \quad \varphi_{i+\alpha/2} = s_{i+\alpha/2} f^{n+1}_{i+\alpha/2}. \tag{7.172}$$

Clearly the following relations are correct:

$$A = A^* > 0, \quad B = B^* > 0. \tag{7.173}$$

Let us control the fulfilment of the condition

$$B \geq 0.5\tau A. \tag{7.174}$$

This condition is necessary and sufficient for stability of the scheme with respect to the initial data in the energy space H_A (see Section 2.2). In fact, we have

$$B - 0.5\tau A = S + 0.5\tau A > 0$$

and, on the basis of Theorem 2.16, scheme (7.170) is stable with respect to the initial data and the right hand side, and the following *a priori* estimate for the solution of the problem holds:

$$\|y_{n+1}\|_A \leq \|y_0\| + \|\varphi_0\|_{A^{-1}} + \|\varphi_n\|_{A^{-1}} + \sum_{k=1}^{n} \tau \|\varphi_{\bar{t},k}\|_{A^{-1}}.$$

Let us find an estimate of the stability with respect to the right hand side which does not involve the difference differentiation by t necessary in studying the accuracy on locally refined grids. In this case we can not also use the well known inequality (see (2.3.47))

$$\|y_{n+1}\|^2_A \leq \|y_0\|^2_A + \frac{1}{2\varepsilon} \sum_{k=1}^{n} \tau \|\varphi_k\|^2. \tag{7.175}$$

The condition (2.3.46) $B \geq \varepsilon E + 0.5\tau A$ under which estimate (7.175) was obtained is not satisfied for scheme (7.170) for an arbitrary $\varepsilon > 0$ (which is independent of τ and $\|A\|$).

Assume that the operators in the operator-difference scheme (7.170) satisfy the conditions

$$B(t) > 0, \quad t \in \omega_\tau, \qquad A \neq A(t), \quad A = A^* > 0. \tag{7.176}$$

Furthermore, let H be the Euclidean space with the scalar product

$$(y, v) = \sum_{\alpha=1}^{2} \left(y^{(\alpha)}, v^{(\alpha)}\right) \tag{7.177}$$

and the norm $\|y\| = \sqrt{(y,y)}$, i.e., the space $H = H_1 \oplus H_2$ can be represented as a direct sum of the Euclidean spaces $H_1(t_n)$ and $H_2(t_n)$. In other words, the space H is defined as a set of the vectors $v = \left\{v^{(1)}, v^{(2)}\right\}$, $v^{(\alpha)} \in H_\alpha$, $\alpha = 1, 2$; the addition and multiplication by a number are defined coordinatewise. In the difference scheme (7.170) we consider that the operators $B(t_n)$, A act from H to H and

$$\varphi = \left(\varphi^{(1)}, 0\right), \quad \varphi^{(1)} \in H_1, \tag{7.178}$$

is the defined function of the discrete argument t_n with values in H. Let us also define the operator (projector) P in the following way:

$$Py = \left(y^{(1)}, 0\right), \quad y^{(1)} \in H_1. \tag{7.179}$$

Thus the following statement is correct:

THEOREM 7.10 *Let conditions (7.176)–(7.178) be satisfied and*

$$B \geq \varepsilon P + 0.5\tau A, \quad \varepsilon \text{ be an arbitrary real number.} \tag{7.180}$$

Then the following a priori estimate for solution of problem (7.170) holds:

$$\|y_{n+1}\|_A^2 \leq \|y_0\|_A^2 + \frac{1}{2\varepsilon} \sum_{k=0}^{n} \tau \|\varphi_k\|^2. \tag{7.181}$$

Proof. Taking the scalar product of (7.170) with $2\tau y_t$, we obtain the energy identity

$$2\tau\left((B - 0.5\tau A)\, y_t, y_t\right) + \|y_{n+1}\|_A^2 = \|y_n\|_A^2 + 2\tau(\varphi, y_t).$$

Let us estimate the second item of the right hand side $2\tau(\varphi, y_t)$ using the generalized Cauchy inequality and the ε-inequality. Taking into account formulas (7.177)–(7.180) we obtain

$$2\tau(\varphi, y_t) = 2\tau\left(\varphi^{(1)}, y_t^{(1)}\right) \le 2\tau\varepsilon \left\|y_t^{(1)}\right\|^2 + \frac{\tau}{2\varepsilon}\left\|\varphi^{(1)}\right\|^2.$$

Substituting this estimate into the energy identity, we obtain estimate (7.181).

We show that for $0 < \varepsilon \le 1$ the estimate of stability with respect to the right hand side (7.181) holds for scheme (7.170), where the operators B, A and the right hand side are defined by formulas (7.171), (7.169), (7.172), respectively. Let us consider the expression

$$B - \varepsilon P - 0.5\tau A = S - \varepsilon P + 0.5\tau A \ge (1-\varepsilon)P + 0.5\tau A.$$

Since the operator $P \ge 0$ is non-positive, the following statement holds:

THEOREM 7.11 *The difference scheme (7.170), (7.171), (7.160), (7.172) is absolutely stable with respect to the initial data and the right hand side, and for* $0 < \varepsilon \le 1$ *the following* a priori *estimate for the solution of problem is correct: (7.181).*

5.5 Convergence

Further we shall assume that the solution of the differential problem (7.155)–(7.157) exists and is unique and that it has limited derivatives necessary in the course of discussion. We shall carry out the investigation on the basis of the method of energy inequalities. For this purpose we determine the error of the method in a standard way $z = y - u$ at all computational nodes and by the formula

$$z_{i+1/2} = y_{i+1/2} - \overline{u}_{i+1/2}, \quad \overline{u}_{i+1/2} = 0.5(u_i + u_{i+1})$$

at fictitious nodes where interpolation is realized in accordance with (7.158). Substituting $y = z + u$ or $y = z + \overline{u}$ into the corresponding equations like (7.170), we obtain the problem for the error of the method:

$$Bz_t + Az = \psi, \quad z_0 = 0,$$

where the operator B is determined by (7.171), and the truncation error is determined in the following way. At the fictitious nodes $(x_{i+1/2}, t_{n+1})$, $i = 0, 1, ..., N-1$, $i \ne j$ we assume

$$\psi_{i+1/2} = 0; \tag{7.182}$$

at the node $(x_{j+1/2}, t_{n+1})$ by the formula

$$\begin{aligned}\psi_{j+1/2}^{n+1} &= -u_{t,j+1/2}^{n} + u_{\bar{x}x,j+1/2}^{n+1} + f_{j+1/2}^{n+1} + \frac{h^2}{8\tau} u_{\bar{x}x,j+1/2}^{n+1} \\ &= O\left(h^2 + \tau + \frac{h^2}{\tau}\right);\end{aligned} \tag{7.183}$$

for $i = j,\ j+1$ by

$$\begin{aligned}\psi_j &= \frac{3}{2}\left(-u_{t,j}^{n} + \frac{1}{\hbar}\left(\frac{u_{j+1/2}^{n+1} - u_j^{n+1}}{h/2} - \frac{u_j^{n+1} - u_{j-1}^{n+1}}{h}\right) + f_j^{n+1}\right) \\ &= O(h + \tau),\end{aligned} \tag{7.184}$$

$$\begin{aligned}\psi_{j+1} &= \frac{3}{2}\left(-u_{t,j+1}^{n} + \frac{1}{\hbar}\left(\frac{u_{j+2}^{n+1} - u_{j+1}^{n+1}}{h} - \frac{u_{j+1}^{n+1} - u_{j+1/2}^{n+1}}{h/2}\right) + f_{j+1}^{n+1}\right) \\ &= O(h + \tau);\end{aligned} \tag{7.185}$$

and at the integer nodes $i \neq j,\ j+1$ by

$$\psi_i = 2\left(-u_{t,i}^{n} + \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + f_i^{n+1}\right) = O(h^2 + \tau). \tag{7.186}$$

From formulas (7.182)–(7.186) we determine that

$$\|\psi\| \leq M\left(\tau + h^{3/2} + \frac{h^{5/2}}{\tau}\right), \tag{7.187}$$

where $M > 0$ is the constant of approximation.

From the *a priori* estimate (7.181) for $0 < \varepsilon \leq 1$ we arrive at the following inequality:

$$\|z_{n+1}\|_A^2 \leq \frac{1}{2\varepsilon} \sum_{k=0}^{n} \tau \|\psi_k\|^2.$$

Substituting the estimate (7.187) into this inequality we obtain the relation

$$\|z_{n+1}\|_A \leq c\left(\tau + h^{3/2} + \frac{h^{5/2}}{\tau}\right).$$

The optimal estimate is obtained for $\tau \sim h^{5/4}$, and

$$\max_n \|z_n\|_A = O(h^{5/4}). \tag{7.188}$$

Thus the essential loss of accuracy arises from the use of a non-uniform adaptive grid.

5.6 Other Type of Interpolation

We note that scheme (7.162) which is used to obtain the solution on the level t_{n+1} for the unknown $y^n_{j+1/2}$ from the lower level is equivalent to the well known scheme

$$\frac{\hat{y}_i - 0.5(y_{i-1} + y_i)}{\tau} = \hat{y}_{\bar{x}x,i} + \varphi_i. \tag{7.189}$$

By means of the principle of maximum it can be easily shown that this scheme is absolutely stable in the norm C and can be reduced to the equivalent form

$$y_t = \hat{y}_{\bar{x}x} + \frac{h^2}{2\tau} y_{\bar{x}x} + \varphi,$$

i.e., the local approximation holds only for $\tau > h^2$. Since this scheme is absolutely stable it can converge to an incorrect solution rather than to a solution of the original problem. This means divergence for $\tau \ll h^2$.

We remove this drawback. In the above mentioned scheme, instead of equation (7.162) we use the following approximation:

$$\frac{1}{2}\left(y_{t,j} + y_{t,j+1}\right) = \hat{y}_{\bar{x}x,j+1/2} + \hat{f}_{j+1/2}, \tag{7.190}$$

where

$$y_{\bar{x}x,j+1/2} = \frac{y_{j+1} - 2y_{j+1/2} + y_{j-1}}{(h/2)^2}.$$

By virtue of the identity

$$\frac{1}{2}\left(y_{t,j} + y_{t,j+1}\right) = y_{t,j+1/2} + \frac{h^2}{8} y_{t\bar{x}x,j+1/2}$$

equation (7.190) can be reduced to the scheme

$$y_{t,j+1/2} = y^{(\sigma)}_{\bar{x}x,j+1/2} + \hat{f}_{j+1/2} \tag{7.191}$$

with the weight $\sigma = 1 - \dfrac{h^2}{8\tau}$. Now, if we define the operator of the second difference derivative on the non-uniform grid $\hat{\omega}_h = \omega_h \cup \{x_{j+1/2}\}$ in the standard way

$$Ay = -y_{\bar{x}\hat{x}}, \quad x \in \hat{\omega}_h, \quad A = A^* > 0,$$

we can reduce the difference scheme (7.163), (7.166), (7.168), (7.191) to the scheme with operator weighting

$$y_t + (Ay)^{(\Sigma)} = \varphi, \quad y_0 = u_0,$$

where

$$\Sigma = \mathrm{diag}\{\sigma_1, ..., \sigma_j, \sigma_{j+1/2}, \sigma_{j+1}, ..., \sigma_{N-1}\},$$

$$\sigma_i = \begin{cases} 1 & \text{for } i = 1, 2, ..., N-1, \\ 1 - \frac{h^2}{8\tau} & \text{for } i = j + \frac{1}{2}. \end{cases}$$

Checking the sufficient conditions of stability in H_{A^2}

$$\Sigma \geq \frac{1}{2} E,$$

we see that it is necessary to fulfill the following inequality:

$$\tau \geq \frac{h^2}{4}.$$

Thus the difference scheme is unconditionally stable and converges for not very small steps in time.

5.7 The Case of Variable Coefficients

Let us consider the boundary value problem for the parabolic equation with variable coefficients

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + f(x,t), \quad x \in \Omega, \quad t > 0, \tag{7.192}$$

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t > 0, \tag{7.193}$$

$$u(x,0) = u_0(x), \quad x \in \overline{\Omega}. \tag{7.194}$$

As before, the main stages in the construction and investigation of the difference schemes with locally refined of grid with respect to the spatial variable will be stated on the assumption that on the level $t_n = n\tau$ the approximate solution is known at the all the grid nodes ω_h: $x_i = ih$, $i = \overline{0, N}$. It is necessary to find the solution on the level t_{n+1} at the additional node $x_{j+1/2} = (j + 1/2)/h$ (Fig. 7.6). Let us consider the following questions in detail:

1. Interpolation at the node $(x_{j+1/2}, t_n)$.

2. The difference scheme at the node $(x_{j+1/2}, t_{n+1})$.

3. The scheme outside of the neighborhood of the additional node ($i \leq j-1$, $i \geq j+2$).

4. The scheme near the additional node ($i = j$, $j+1$).

Let us approximate the flux $W = -k\dfrac{\partial u}{\partial x}$ in the interval $[x_j, x_{j+1}]$ by

$$q_{j,j+1} = -a_{j+1/2}\frac{y_{j+1} - y_j}{h}, \tag{7.195}$$

and in the intervals $[x_j, x_{j+1/2}]$, $[x_{j+1/2}, x_{j+1}]$ by the relations

$$q_{j,j+1/2} = -a_{j+1/4}\frac{y_{j+1/2} - y_j}{h/2}, \tag{7.196}$$

$$q_{j+1/2,j+1} = -a_{j+3/4}\frac{y_{j+1} - y_{j+1/2}}{h/2}, \tag{7.197}$$

respectively.

To determine the value $y_{j+1/2} = y(x_{j+1/2}, t_n)$ on the level t_n, which is used for finding the solution by means of the difference scheme in the area of singularity of the solution, let us apply the natural conditions of continuity of the flux:

$$q_{j,j+1/2} = q_{j+1/2,j+1}, \quad q_{j+1/2,j+1} = q_{j,j+1}. \tag{7.198}$$

From the first relation of (7.198) we obtain

$$y_{j+1/2} = \frac{a_{j+1/4}y_j + a_{j+3/4}y_{j+1}}{a_{j+1/4} + a_{j+3/4}}. \tag{7.199}$$

Note expression (7.199) can be written in the form

$$Ay_{j+1/2} = 0, \tag{7.200}$$

where the grid operator

$$Ay_i = -\frac{1}{h/2}\left(a_{i+3/4}\frac{y_{i+1} - y_{i+1/2}}{h/2} - a_{i+1/4}\frac{y_{i+1/2} - y_i}{h/2}\right) \tag{7.201}$$

is an approximation of the differential operator

$$Lu = -\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) \tag{7.202}$$

on the uniform grid $\omega_{h/2}$.

From the second condition of (7.198) we obtain

$$2a_{j+1/4}(y_{j+1/2} - y_j) = a_{j+1/2}(y_{j+1} - y_j). \tag{7.203}$$

Substituting expression (7.199) into formula (7.203), we may note that the given equality is satisfied only when the following condition is imposed on coefficients of the difference operator (7.201):

$$a_{j+1/2} = \frac{2a_{j+1/4}a_{j+3/4}}{a_{j+1/4} + a_{j+3/4}}. \tag{7.204}$$

Note that the stencil functional [Samarskii, 1989] of the form

$$a_{j+1/p} = \left(\frac{p}{2h} \int\limits_{x_j}^{x_{j+2/p}} \frac{dx}{k(x)} \right)^{-1}, \; p = 2, 4; \quad a_{j+3/4} = \left(\frac{2}{h} \int\limits_{x_{j+1/2}}^{x_{j+1}} \frac{dx}{k(x)} \right)^{-1}$$

satisfies relation (7.204). We will determine the solution at the fictitious nodes $(x_{i+1/2}, t_{n+1})$ for $i = 0, 1, ..., N-1$, $i \neq j$ on the level $t = t_{n+1}$ also with the aid means of the interpolation (7.199), (7.204), i.e., as a solution of the difference equation:

$$Ay^{n+1} = 0. \tag{7.205}$$

At the additional node $(x_{j+1/2}, t_{n+1})$ we shall find the approximate solution by means of the clearly implicit scheme

$$\frac{y^{n+1}_{j+1/2} - y^n_{j+1/2}}{\tau} + Ay^{n+1}_{j+1/2} = \widetilde{\varphi}^n_{j+1/2}, \tag{7.206}$$

where $\widetilde{\varphi}$ is some stencil functional of f.

Outside of the neighborhood of additional node the approximate solution at the integral nodes is determined from the difference equation

$$\frac{y^{n+1}_i - y^n_i}{\tau} = \frac{1}{h} \left(a_{i+1/2} \frac{y^{n+1}_{i+1} - y^{n+1}_i}{h} - a_{i-1/2} \frac{y^{n+1}_i - y^{n+1}_{i-1}}{h} \right) = \widetilde{\varphi}^n_i. \tag{7.207}$$

Here we transform the difference operator

$$A_0 y_i = \frac{1}{h^2}(q_{i,i+1} - q_{i,i-1})$$

using the second equation of (7.198). Taking into account the representation (7.201) we have

$$A_0 y_i = \frac{1}{2} A y_i. \tag{7.208}$$

Using equality (7.208), we rewrite the difference scheme (7.207) in the form

$$\frac{y^{n+1}_i - y^n_i}{\tau} + \frac{1}{2} A y_i = \widetilde{\varphi}^n_i. \tag{7.209}$$

Let us return to the approximation of the differential equation (7.192) near the additional node. For $i = j$ (see Fig. 7.6) we have

$$\frac{y^{n+1}_j - y^n_j}{\tau} - \frac{1}{\hbar} \left(a_{j+1/4} \frac{y^{n+1}_{j+1/2} - y^{n+1}_j}{h/2} - a_{j+1/2} \frac{y^{n+1}_j - y^{n+1}_{j-1}}{h} \right) = \widetilde{\varphi}^n_j,$$

where $\hbar = \frac{1}{2}\left(\frac{h}{2} + h\right) = \frac{3}{4}h$. Let us study the operator with respect to space

$$A_1 y = \frac{4}{3h}(q_{j,j+1/2} - q_{j-1,j})$$

separately. Since at the half-integer node $j - 1/2$ the flux is interpolated by the rule

$$q_{j-1,j-1/2} = q_{j-1,j},$$

we obtain the relation

$$A_1 y = \frac{2}{3} A y.$$

Thus, for $i = j,\ j+1$ the difference scheme takes the form

$$\frac{y_i^{n+1} - y_i^n}{\tau} + \frac{2}{3} A y_i^{n+1} = \widetilde{\varphi}_i^n. \tag{7.210}$$

The difference scheme (7.205), (7.206), (7.209), (7.210) is written in the canonical form (7.170)–(7.172) with the operator A defined by formula (7.201) like the case of constant coefficients. Since

$$B - \varepsilon P - 0.5\tau A \geq (1-\varepsilon)P + 0.5\tau A, \quad P \geq 0, \quad A = A^* > 0,$$

the following *a priori* estimate for solution of the difference problem holds in accordance with Theorem 7.11:

$$\|y_{n+1}\|_A^2 \leq \|y_0\|_A^2 + \frac{1}{2\varepsilon}\sum_{k=0}^{n} \tau \|\varphi_k\|^2, \quad 0 < \varepsilon \leq 1. \tag{7.211}$$

To obtain the estimates of accuracy let us define the error of the difference scheme at all computational nodes by the standard way $z = y - u$. Taking into account (7.199) we find the error at the fictitious nodes in accordance with the rule

$$z_{i+1/2} = y_{i+1/2} - \overline{u}_{i+1/2},$$
$$\overline{u}_{i+1/2} = \frac{a_{i+1/4}u_i + a_{i+3/4}u_{i+1}}{a_{i+1/4} + a_{i+3/4}}.$$

Similarly to scheme (7.170) we obtain the problem for z:

$$B z_t + A z = \psi, \quad z_0 = 0,$$

where

$$\psi = \begin{cases} 0 & \text{at the fictitious nodes,} \\ O\left(h^2 + \tau + \dfrac{h^2}{\tau}\right) & \text{at the node } (x_{j+1/2}, t_{n+1}), \\ O(h+\tau) & \text{at the boundary of adaptation,} \\ O(h^2+\tau) & \text{at the integer nodes.} \end{cases} \tag{7.212}$$

Hence it follows that

$$\|\psi\| \leq M \left(\tau + h^{3/2} + \frac{h^{5/2}}{\tau} \right), \quad M = \text{const} > 0. \tag{7.213}$$

From the *a priori* estimate (7.211) the inequality follows

$$\|z_{n+1}\|_A^2 \leq \frac{1}{2\varepsilon} \sum_{k=0}^{n} \tau \|\psi_k\|^2.$$

Taking into account estimate (7.213) we obtain the estimate required for the error

$$\|z_{n+1}\|_A \leq c \left(\tau + h^{3/2} + \frac{h^{5/2}}{\tau} \right).$$

The optimal estimate is again achieved for $\tau \sim h^{5/4}$.

6. Schemes of High Order of Approximation on Grids Non-Uniform with Respect to Space

We shall consider difference schemes on ordinary non-uniform grids with a high order of local approximation. The most important moment of investigation is connected with the approximation of equations at the non-calculating nodes of the computational domain.

Furthermore, we shall follow the papers [Ananich, 1998, Samarskii et al., 1998c, Samarskii et al., 1996a, Samarskii et al., 1996b, Samarskii et al., 1998b].

6.1 Introduction

In constructing adaptive numerical algorithms for approximate solution of the problems of mathematical physics it is often necessary to use non-uniform grids. In transition from a uniform grid to a non-uniform one the order of local approximation is usually decreased. For example, the approximation of the second derivative on an ordinary three-point stencil [Samarskii, 1989] has only first order in a uniform norm and in the grid norm L_2. Only using a negative norm we can prove the second order of accuracy of corresponding difference schemes on non-uniform grids.

The accuracy of approximation can be very easy increased by means of expanded stencils or on a narrower class of functions (on the solutions of differential problem) [Samarskii, 1989].

We note that the accuracy of the method can be increased with the help of the approximation of the original differential equation at some intermediate nodes of a computational domain rather than at the nodes of a calculating grid. Such an approach has the common points with the superconvergence in the theory of the method of finite elements [Krizer and Neittaanmaki, 1987]. An increase in accuracy on approximation of the second derivative on a non-uniform grid at a special node has been noted even in [Berkovsky and Polevikov, 1988].

In the present section we consider different schemes of high order of approximation for one-dimensional and multi-dimensional equations. We prove convergence with the second order with respect to spatial variable in stronger norms than the well known ones in the theory of the difference schemes [Samarskii, 1989].

6.2 Difference Schemes for a Parabolic Equation

Let us consider the boundary problem of the first kind for the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad (x,t) \in Q_T, \tag{7.214}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{7.215}$$

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t > 0, \tag{7.216}$$

in the rectangle

$$\overline{Q}_T = \overline{\Omega} \times [0,T], \quad \overline{\Omega} = \{x : \ 0 \le x \le l\}.$$

Let us define the arbitrary *non-uniform spatial grid*

$$\begin{aligned} \hat{\overline{\omega}}_h &= \{x_i = x_{i-1} + h_i, \ i = 1,2,...,N, \quad x_0 = 0, \ x_N = l\} \\ &= \hat{\omega}_h \cup \{x_0 = 0, \ x_N = l\} \end{aligned} \tag{7.217}$$

and the time grid with the constant step τ

$$\overline{\omega}_\tau = \{t_n = n\tau, \ n = 0,1,...,N_0; \quad \tau N_0 = T\} = \omega_\tau \cup \{T\}.$$

We define the grid operator with respect to space

$$\Lambda u = u_{\bar{x}\hat{x}} = \frac{1}{\hbar_i}\left(\frac{u_{i+1} - u_i}{h_{i+1}} - \frac{u_i - u_{i-1}}{h_i}\right), \quad \hbar_i = 0.5(h_i + h_{i+1}). \tag{7.218}$$

Let us approximate the differential problem (7.214)–(7.216) on the grid $\hat{\overline{\omega}}_h \times \overline{\omega}_\tau$ by absolutely implicit difference scheme (for the simplicity of further investigation we assume that $h_{i+1} \ge h_i$):

$$y_t + \frac{h_+ - h}{3} y_{t\bar{x}} = \hat{y}_{\bar{x}\hat{x}} + f(\overline{x}, \hat{t}), \tag{7.219}$$

$$y(x,0) = u_0(x), \quad x \in \hat{\omega}_h; \qquad \hat{y}_0 = \hat{y}_N = 0. \tag{7.220}$$

Here we use the notation

$$h_+ = h_{i+1}, \quad h = h_i, \qquad y_{t\bar{x}} = (y_{t,i} - y_{t,i-1})/h_i, \quad \hat{t} = t_{n+1},$$

$$\overline{x} = \overline{x}_i = \frac{1}{3}(x_{i+1} + x_i + x_{i-1}) = x_i + \frac{h_{i+1} - h_i}{3} = x + \frac{h_+ - h}{3}. \quad (7.221)$$

Let us note that in the case of a uniform grid ($h_{i+1} = h_i$) the difference equation (7.219) is degenerated to a usual scheme. Using Taylor's formula we obtain the following expansions:

$$\begin{gathered}\hat{u}_{\bar{x}\hat{x},i} - \frac{\partial^2 u}{\partial x^2}(\overline{x}_i, t_{n+1}) = \frac{\partial^2 u}{\partial x^2}(x_i, t_{n+1}) \\ + \frac{h_{i+1} - h_i}{3}\frac{\partial^3 u}{\partial x^3}(x_i, t_{n+1}) - \frac{\partial^2 u}{\partial x^2}(\overline{x}_i, t_{n+1}) + O(\hbar_i^2) = O(\hbar_i^2),\end{gathered} \quad (7.222)$$

$$\left(u_{t,i} + \frac{h_{i+1} - h_i}{3} u_{t\bar{x},i}\right) - \frac{\partial u}{\partial t}(\overline{x}_i, t_{n+1}) = O(\hbar_i^2 + \tau), \quad (7.223)$$

owing to which the discrepancy of the scheme has the form

$$\psi(\overline{x}_i, t_{n+1}) = -u_{t,i} - \frac{h_{i+1} - h_i}{3} u_{t\bar{x},i} + \hat{u}_{\bar{x}\hat{x},i} + f(\overline{x}_i, \hat{t}) = O(\hbar_i^2 + \tau). \quad (7.224)$$

Consequently the difference scheme (7.219), (7.220) approximates an initial differential problem on an arbitrary non-uniform grid with respect to space with the second order

$$\|\psi\|_C \le M(h^2 + \tau), \quad h = \max_i h_i. \quad (7.225)$$

Here $\|\cdot\|_C = \max\limits_{x \in \hat{\omega}_h} |\cdot|$.

Let us give some *a priori* estimates for a difference solution. We define the scalar products and the norms on a non-uniform grid as

$$\begin{gathered}(y, v)_* = \sum_{i=1}^{N-1} \hbar_i y_i v_i, \quad (y, v) = \sum_{i=1}^{N-1} h_i y_i v_i, \\ (y, v] = \sum_{i=1}^{N} h_i y_i v_i, \quad \|v_{\bar{x}}\|_C = \max_{x \in \omega^+} |v_{\bar{x}}(x)|, \quad \omega^+ = \hat{\omega}_h \cup \{x_N = l\},\end{gathered} \quad (7.226)$$

respectively.

LEMMA 7.4 *For any grid function $y(x)$ defined on the non-uniform grid (7.217) and equaled to zero for $x = 0$ and $x = l$ the following inequalities are correct:*

$$\|y\|_* \le \frac{l^2}{4}\|y_{\bar{x}\hat{x}}\|_*, \quad \|y_{\bar{x}}]| \le \frac{l}{2}\|y_{\bar{x}\hat{x}}\|_*, \quad \|y_{\bar{x}}\|_C \le M_1\|y_{\bar{x}\hat{x}}\|_*, \quad (7.227)$$

where $M_1 = \varepsilon + l/4 + l^2(1+c_1)/(8\varepsilon)$, $\varepsilon > 0$ *is an arbitrary real number and the constant* c_1 *satisfies the inequality*

$$c_1^{-1} \le \max_i (h_i/h_{i+1}) \le c_1. \tag{7.228}$$

P r o o f. Using the first Green's difference formula and the imbedding [Samarskii and Goolin, 1973, p. 37]

$$\|y\|_* \le \frac{l}{2} \|y_{\bar{x}}]|, \tag{7.229}$$

we obtain the inequality

$$\|y_{\bar{x}}]|^2 = -\left(y_{\bar{x}\hat{x}}, y\right)_* \le \|y_{\bar{x}\hat{x}}\|_* \|y\|_* \le \frac{l}{2} \|y_{\bar{x}}]| \|y_{\bar{x}\hat{x}}\|_*.$$

Hence using estimate (7.229), the first inequality from (7.227) follows. The third inequality is a consequence of the imbedding [Samarskii and Andreev, 1976]

$$\|y_{\bar{x}}\|_C \le \varepsilon \|y_{\bar{x}\hat{x}}\|_*^2 + \left((1+c_1)/(2\varepsilon) + 1/l\right) \|y_{\bar{x}}]|^2$$

and the estimates (7.227).

Let us discuss the stability of difference scheme in the energy norm of W_2^2. We shall not consider the weaker norms, because such estimates of stability and accuracy can be obtained for ordinary conservative methods of the first order of local approximation with the use of the technique of negative norms [Samarskii, 1989].

THEOREM 7.12 *The difference scheme (7.219), (7.220) is stable with respect to the initial data and the right hand side, and the following estimate holds*

$$\|y_{\bar{x}\hat{x}}\|_* \le M_2 \left(\|u_{0\bar{x}\hat{x}}\|_* + \|\overline{f}_0\|_*\right) + \|\overline{f}_n\|_* + M_3 \max_{1\le k\le n} \|\overline{f}_{\bar{t},k}\|_*, \tag{7.230}$$

where the constants $M_2 = e^{0.5T}$, $M_3 = M_2 T^{1/2}$, *and* $\overline{f}_k = f(\overline{x}, t_k)$.

P r o o f. Taking the dot product of equation (7.219) with $-2\tau\hbar_i y_{t\bar{x}\hat{x},i}$ and summing over all the nodes of the grid $\hat{\omega}_h$, we obtain the energy identity

$$\begin{aligned} &2\tau\left(\|y_{t\bar{x}}]|^2 - \left(\frac{h_+ - h}{3} y_{t\bar{x}}, y_{t\bar{x}\hat{x}}\right)_*\right) \\ &\qquad = -\|\hat{y}_{\bar{x}\hat{x}}\|_*^2 + \|y_{\bar{x}\hat{x}}\|_*^2 - \tau^2 \|y_{t\bar{x}\hat{x}}\|_*^2 - 2\tau\left(y_{t\bar{x}\hat{x}}, \hat{\overline{f}}\right). \end{aligned} \tag{7.231}$$

Using the ε-inequality $-ab \geq -\varepsilon a^2 - \dfrac{1}{4\varepsilon}b^2$ with $\varepsilon = 1$, we transform the expression entering into identity (7.231)

$$
\begin{aligned}
-\left(\frac{h_+ - h}{3\hbar}, y_{t\bar{x}}y_{tx} - y_{t\bar{x}}^2\right)_* &\geq \left(\frac{h_+ - h}{3\hbar}, y_{t\bar{x}}^2\right)_* - \left(\frac{h_+ - h}{3\hbar}, y_{t\bar{x}}^2 + \frac{1}{4}y_{tx}^2\right)_* \\
&= -\frac{1}{12}\sum_{i=1}^{N-1}(h_+ - h)y_{tx,i}^2 \\
&= -\frac{1}{12}\sum_{i=2}^{N}(h - h_-)y_{t\bar{x},i}^2 \\
&= -\frac{1}{12}\left(\frac{h-h_-}{h}, y_{t\bar{x}}^2\right], \quad h_0 = h_1.
\end{aligned}
\tag{7.232}
$$

Substituting the expression obtained into identity (7.231) we obtain the inequality

$$
0 \leq -\|\hat{y}_{\bar{x}\hat{x}}\|_*^2 + \|y_{\bar{x}\hat{x}}\|_*^2 - 2\tau(y_{t\bar{x}\hat{x}}, \hat{\bar{f}})_*. \tag{7.233}
$$

Using estimate (5.1.56) and the latter inequality, we have

$$
\begin{aligned}
\|\hat{y}_{\bar{x}\hat{x}} + \hat{\bar{f}}\|_*^2 &\leq (1+\tau)\left(\|y_{\bar{x}\hat{x}} + \bar{f}\|_*^2 + \tau\|\bar{f}_t\|_*^2\right) \leq \dots \\
&\leq e^{t_n}\left(\|u_{0\bar{x}\hat{x}} + \bar{f}_0\|_*^2 + t_n \max_{0\leq k\leq n}\|f_{t,k}\|_*^2\right).
\end{aligned}
$$

Hence the *a priori* estimate required follows.

To obtain the estimates of accuracy let us substitute $y = z + u$ into equations (7.219), (7.220). We obtain the problem for error

$$
z_t + \frac{h_+ - h}{3}z_{t\bar{x}} = \hat{z}_{\bar{x}\hat{x}} + \psi(\bar{x}, \hat{t}),
$$

$$
z(x,0) = 0, \quad \hat{z}_0 = \hat{z}_N = 0,
$$

where the truncation error $\overline{\psi}$ is defined in accordance with formula (7.224). Using the *a priori* estimates (7.230), (7.225), (7.227) for the solution of this problem and assuming the existence of the corresponding limited derivatives, we obtain the following estimates of the rate of convergence:

$$
\|z_{\bar{x}\hat{x}}\|_* \leq c(h^2 + \tau), \quad \|z_{\bar{x}}\|_C \leq c(h^2 + \tau), \quad c = \text{const} > 0.
$$

Using Lemma 7.4 we can obtain *a priori* estimates in other norms.

REMARK 7.6 The difference scheme (7.219) was constructed and investigated on refinement of the spatial grid $\hat{\overline{\omega}}_h$ to the beginning of the interval. On arbitrary refinement the approximation of the derivative $\dfrac{\partial u}{\partial t}$ must by carried out taking into account the directed differences

$$y_t + 0.5\left(\widetilde{h} + |\widetilde{h}|\right) y_{t\overline{x}} + 0.5\left(\widetilde{h} - |\widetilde{h}|\right) y_{tx} = \hat{y}_{\overline{x}\hat{x}} + f(\overline{x}, \hat{t}), \qquad (7.234)$$

where $\widetilde{h} = (h_+ - h)/3$.

We consider the following scheme of the second order of approximation (we assume that $h_{i+1} \geq h_i$)

$$y_t + \frac{h_+ - h}{3} y_{tx} = y^{(\sigma)}_{\overline{x}\hat{x}} + \varphi, \quad \varphi = f^{(\sigma)}(\overline{x}, t). \qquad (7.235)$$

We shall say that this difference scheme is called the 'downstream' scheme. By the formulas

$$(Ay)_i = -y_{\overline{x}\hat{x},i}, \quad i = 1, 2, ..., N-1, \qquad y_0 = y_N = 0, \qquad (7.236)$$

$$(A_1 y)_i = \frac{h_{i+1} - h_i}{3} y_{x,i}, \quad i = 1, 2, ..., N-1, \quad y_0 = y_N = 0, \qquad (7.237)$$

we define the grid operators A and A_1, respectively. Let $\overset{\circ}{\Omega}_h$ be a set of the grid functions defined on $\hat{\overline{\omega}}_h$ and equal to zero on the boundary. We define the vector $y = y(t) = (y_1(t), y_2(t), ..., y_{N-1}(t))^T$ and the linear space $H = \Omega_h$ as a set of such vectors with the scalar product and the norms determined by formula (7.226). Since $A = A^* > 0$, then H_A will denote the Hilbert space which consists of the elements of the space H and has the scalar product and the norm

$$\|y\|^2_A = (Ay, y)_* = \|y_{\overline{x}}]|^2 = \sum_{i=1}^{N} h_i y^2_{\overline{x},i}. \qquad (7.238)$$

Then the difference scheme (7.235), (7.220) can be written in the canonical form

$$By_t + Ay = \varphi, \qquad y_0 = u_0, \qquad (7.239)$$

$$B = D + \sigma\tau A, \qquad D = E + A_1. \qquad (7.240)$$

LEMMA 7.5 *For arbitrary relations on the grid steps τ and h $(h_+ \geq h)$ the following operator inequality holds:*

$$A_1 \geq -\frac{2}{3} E.$$

P r o o f. We consider the scalar product

$$(Dy, y)_* = \|y\|_*^2 + \left(\frac{h_+ - h}{3h_+}, y_+ y - y^2\right)_* = \left(\frac{2h_+ + h}{3h_+}, y^2\right)_* - \left(\frac{h_+ - h}{3h_+} y_+, y\right)_*. \tag{7.241}$$

Since $h_+ \geq h$ and

$$-\left(\frac{h_+ - h}{3h_+} y_+, y\right)_* \geq -\left(\frac{h_+ - h}{6h_+}, y^2\right)_* - \left(\frac{h - h_-}{6h}, y^2\right)_*,$$

from equality (7.241) we obtain

$$(Dy, y)_* \geq \left(\frac{2h_+ h + 3h^2 + h_+ h_-}{6hh_+}, y^2\right)_* \geq \frac{1}{3}\|y\|_*^2.$$

Taking into account the relation $D = E + A_1$ we obtain the inequality required.

Checking the sufficient condition of stability

$$B - 0.5\tau A \geq \frac{1}{3}E + (\sigma - 0.5)\tau A \geq \left(\frac{1}{3\|A\|} + (\sigma - 0.5)\tau\right) A \geq 0,$$

we have the following limitation on σ:

$$\sigma \geq \frac{1}{2} - \frac{1}{3\tau\|A\|}.$$

Taking into account [Samarskii and Goolin, 1973]

$$\|A\| \leq \max_{1 \leq i \leq N-1} \frac{4}{h_i h_{i+1}} \leq \frac{4}{h^2}, \quad h = \min_i h_i,$$

the following statement is correct in accordance with Theorem 2.16.

THEOREM 7.13 *The difference scheme (7.235), (7.220) is stable with respect to the initial data, the right hand side, and under the condition*

$$\sigma \geq \frac{1}{2} - \frac{h^2}{12\tau} \tag{7.242}$$

the following a priori *estimate holds:*

$$\|y_{n+1}\|_A \leq \|y_0\|_A + \|\varphi_0\|_{A^{-1}} + \|\varphi_n\|_{A^{-1}} + \sum_{k=1}^{n} \tau \|\varphi_{\bar{t},k}\|_{A^{-1}}. \tag{7.243}$$

From condition (7.242) it follows that the explicit scheme ($\sigma = 0$) is stable in H_A for $\tau \leq \dfrac{h^2}{6}$.

Let us now consider conservative schemes. On the non-uniform grid $\omega = \hat{\omega}_h \times \omega_\tau$ we consider a class of difference schemes with the weights

$$y_t + \left(\frac{h^2}{6} y_{t\bar{x}}\right)_{\hat{x}} = y_{\bar{x}\hat{x}}^{(\sigma)} + f^{(\sigma)}(\overline{x}, t). \tag{7.244}$$

Since for smooth solutions

$$u_t + \left(\frac{h^2}{6} u_{t\bar{x}}\right)_{\hat{x}} = u_t + \frac{h_+ - h}{3} u_{tx} + \frac{h^2}{6} u_{t\bar{x}\hat{x}} = \frac{\partial u}{\partial t}(\overline{x}, t) + O(\hbar^2 + \tau),$$

then on the basis of relation (7.224) we conclude that scheme (7.244) has the second order of approximation with respect to the spatial variable $\overline{\psi} = O(\hbar^2 + \tau)$.

We define the grid operator

$$(A_2 y)_i = (h^2 y_{\bar{x}})_{\hat{x},i}, \quad i = 1, 2, ..., N-1, \qquad y_0 = y_N = 0.$$

Scheme (7.244) is then reduced to the canonical form (7.239) with self-adjoint operators B and A:

$$B = D + \sigma\tau A, \qquad D = E + A_2. \tag{7.245}$$

We show that the operator $D > 0$ for $h_{i+1} \geq h_i$, $i = 1, 2, ..., N-1$. In fact,

$$(Dy, y)_* = \|y\|_*^2 - \left(\frac{h^2}{6}, y_{\bar{x}}^2\right] = \|y\|_*^2 - \frac{1}{6}\|y - y_-]|^2. \tag{7.246}$$

Since

$$-\frac{1}{6}\|y - y_-]|^2 \geq -\frac{2}{3}\|y\|^2,$$

from equality (7.246) the inequality required follows:

$$(Dy, y)_* = \left(\frac{3h_+ - h}{6\hbar}, y^2\right)_* > 0.$$

Consequently the operator inequality

$$B - 0.5\tau A = D + \tau(\sigma - 0.5)A \geq 0$$

holds for all $\sigma \geq 0.5$ and the conservative difference scheme (7.244) is unconditionally stable for $h_+ \geq h$ in the energy norm of H_A.

It is necessary to specially pay attention to the case of variable coefficients. Instead (7.214) we consider the more general equation

$$\frac{\partial u}{\partial t} = k(x)\frac{\partial^2 u}{\partial x^2} + r(x)\frac{\partial u}{\partial x} + f(x,t). \tag{7.247}$$

On the non-uniform grid $\omega = \hat{\omega}_h \times \omega_\tau$ let us approximate the differential equation (7.247) by a difference one (we consider that $h_+ \geq h$)

$$y_t + \frac{h_+ - h}{3} y_{t\bar{x}} = k(\overline{x})\hat{y}_{\bar{x}\hat{x}} + r(\overline{x})\hat{y}_{\overset{\diamond}{x}} + f(\overline{x},\hat{t}), \tag{7.248}$$

where

$$y_{\overset{\diamond}{x}} = \frac{1}{3}\left(y_{\bar{x}} + y_x + y_{\overset{\circ}{x}}\right),$$

$$y_{\bar{x}} = \frac{y_i - y_{i-1}}{h_i}, \quad y_x = \frac{y_{i+1} - y_i}{h_{i+1}}, \quad y_{\overset{\circ}{x}} = \frac{y_{i+1} - y_{i-1}}{h_{i+1} + h_i}.$$

We show that on an arbitrary grid non-uniform with respect to space the difference scheme (7.248) approximates the initial differential equation (7.247) with the second order $O(\hbar^2 + \tau)$. It is sufficiently to make certain that the following relation holds:

$$r(\overline{x})u_{\overset{\diamond}{x}} - r(\overline{x})\frac{\partial u}{\partial x}(\overline{x},t) = O(\hbar^2). \tag{7.249}$$

In fact, expanding the difference derivatives $u_{\bar{x}}$, u_x, $u_{\overset{\circ}{x}}$ in the Taylor series in the neighborhood of the node $(\overline{x}_i, t)$, we obtain

$$u_{\bar{x},i} = \frac{\partial u}{\partial x}(\overline{x}_i,t) - \frac{2h_{i+1} + h_i}{6}\frac{\partial^2 u}{\partial x^2}(\overline{x}_i,t) + O(\hbar_i^2),$$
$$u_{x,i} = \frac{\partial u}{\partial x}(\overline{x}_i,t) + \frac{h_{i+1} + 2h_i}{6}\frac{\partial^2 u}{\partial x^2}(\overline{x}_i,t) + O(\hbar_i^2),$$
$$u_{\overset{\circ}{x},i} = \frac{\partial u}{\partial x}(\overline{x}_i,t) + \frac{h_{i+1} - h_i}{6}\frac{\partial^2 u}{\partial x^2}(\overline{x}_i,t) + O(\hbar_i^2).$$

It follows that

$$u_{\overset{\diamond}{x},i} - \frac{\partial u}{\partial x}(\overline{x}_i,t) = \frac{u_{\bar{x},i} + u_{x,i} + u_{\overset{\circ}{x},i}}{3} - \frac{\partial u}{\partial x}(\overline{x}_i,t) = O(\hbar_i^2).$$

Consequently by virtue of relations (7.222), (7.223), (7.249) the truncation error of scheme (7.248)

$$\psi(\overline{x},\hat{t}) = -\left(u_t + \frac{h_+ - h}{3}u_{t\bar{x}}\right) + k(\overline{x})\hat{u}_{\bar{x}\hat{x}}$$
$$+ r(\overline{x})\hat{u}_{\overset{\diamond}{x}} + f(\overline{x},\hat{t}) = O(\hbar^2 + \tau)$$

has the second order with respect to the spatial variable.

6.3 Difference Schemes for a Hyperbolic Equation

Let us consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < l, \quad 0 < t \le T. \tag{7.250}$$

At the initial time the following conditions are defined:

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \overline{u}_0(x). \tag{7.251}$$

For the simplicity of the discussion we shall consider the boundary conditions to be homogeneous:

$$u(0,t) = 0, \quad u(l,t) = 0. \tag{7.252}$$

Using the notation

$$y = y^n, \quad \hat{y} = y^{n+1}, \quad \check{y} = y^{n-1}, \quad y_t = (\hat{y} - y)/\tau, \quad y_{\bar{t}} = (y - \check{y})/\tau,$$

$$\Lambda y = y_{\bar{x}\hat{x}}, \quad y_{\bar{x}\hat{x}} = \frac{1}{\hbar}\left(\frac{y_+ - y}{h_+} - \frac{y - y_-}{h}\right), \quad \hbar = 0.5(h + h_+),$$

$$v_\pm = v_{i\pm 1}, \quad y_{\bar{t}t} = (y_t - y_{\bar{t}})/\tau = (\hat{y} - 2y + \check{y})/\tau^2,$$

we substitute the following expressions for derivatives in (7.250):

$$\frac{\partial^2 u}{\partial t^2} \sim u_{\bar{t}t} + \left(\frac{h^2}{6} u_{\bar{t}t\bar{x}}\right)_{\hat{x}}, \quad \frac{\partial^2 u}{\partial x^2} \sim u_{\bar{x}\hat{x}}^{(\sigma,\sigma)}, \quad f \sim \varphi,$$

$$\varphi = f^{(\sigma,\sigma)}(\overline{x}, t), \quad t \in \omega_\tau, \quad \overline{x}_i = x_i + \frac{h_{i+1} - h_i}{3},$$

$$v^{(\sigma_1,\sigma_2)} = \sigma_1 \hat{v} + (1 - \sigma_1 - \sigma_2) v + \sigma_2 \check{v}.$$

We consider the family of the schemes with constant weights

$$y_{\bar{t}t} + \left(\frac{h^2}{6} y_{\bar{t}t\bar{x}}\right)_{\hat{x}} = y_{\bar{x}\hat{x}}^{(\sigma,\sigma)} + \varphi, \tag{7.253}$$

$$y(0,t+\tau) = y(l,t+\tau) = 0, \quad y(x,0) = u_0(x), \quad y_t(x,0) = \widetilde{u}_0(x). \tag{7.254}$$

We note that $\widetilde{u}_0(x)$ is chosen so that the approximation error of the second initial condition is $O(\tau^2)$:

$$\widetilde{u}_0(x) = \overline{u}_0(x) + 0.5\tau(u_0''(x) + f(x,0)).$$

Since

$$u_{\bar{t}t} + \left(\frac{h^2}{6}u_{\bar{t}t\bar{x}}\right)_{\hat{x}} = u_{\bar{t}t} + \frac{h_+ - h}{3}u_{\bar{t}tx} + \frac{h^2}{6}u_{\bar{t}t\bar{x}\hat{x}}$$
$$= \frac{\partial^2 u}{\partial t^2}(\overline{x}, t) + \frac{h^2}{6}u_{\bar{t}t\bar{x}\hat{x}} + O(\hbar^2 + \tau^2),$$

for the sufficiently smooth function it follows that

$$u_{\bar{t}t,i} + \left(\frac{h^2}{6}u_{\bar{t}t\bar{x}}\right)_{\hat{x},i} - \frac{\partial^2 u}{\partial t^2}(\overline{x}_i, t_n) = O(\hbar_i^2 + \tau^2).$$

Using the identity

$$v^{(\sigma,\sigma)} = v + \sigma\tau^2 v_{\bar{t}t} \tag{7.255}$$

and formula (7.222), we conclude that the following relation is correct:

$$u^{(\sigma,\sigma)}_{\bar{x}\hat{x},i} - \frac{\partial^2 u}{\partial x^2}(\overline{x}_i, t) + \varphi_i^n - f(\overline{x}_i, t_n) = O(\hbar_i^2 + \tau^2).$$

Thus the difference scheme (7.253), (7.254) approximates the differential problem with the second order, i.e.,

$$\psi(\overline{x}_i, t_n) = O(\hbar_i^2 + \tau^2).$$

To obtain the *a priori* estimates of stability let us reduce the difference equation (7.253) to a form convenient for further investigation. Using identity (7.255) we have the representation

$$\left(y_{\bar{x}}^{(\sigma,\sigma)} - \frac{h^2}{6}y_{\bar{t}t\bar{x}}\right)_{\hat{x}} = \left(y_{\bar{x}} + \tau^2\left(\sigma - \frac{h^2}{6\tau^2}\right)y_{\bar{t}t\bar{x}}\right)_{\hat{x}} = \left(y_{\bar{x}}^{(\sigma_1,\sigma_1)}\right)_{\hat{x}},$$
$$\sigma_{1i} = \sigma - \frac{h_i^2}{6\tau^2}, \quad \sigma = \text{const} \geq 0.$$

Thus the difference scheme (7.253) is reduced to the scheme

$$y_{\bar{t}t,i} = \left(y_{\bar{x}}^{(\sigma_{1i},\sigma_{1i})}\right)_{\hat{x}} + \varphi_i \tag{7.256}$$

with the spatially variable weights.

Let $\overset{1}{\Omega}{}_h^n$ be a set of the grid functions $v_i^n = v(x_i, t_n)$ defined on the grid $\hat{\overline{\omega}}_h$ and satisfying the condition $v_0^n = 0$. Side by side with the Hilbert space H let us introduce the space H^* by a set of vectors of the form $v_n = (v_1^n, v_2^n, ..., v_N^n)$. In the space H^* we define the scalar product

$$(v_n, w_n] = \sum_{i=1}^{N} h_i v_i^n w_i^n. \tag{7.257}$$

Obviously, the operator $A : H \to H$,

$$(Ay)_i = \begin{cases} -\frac{1}{\hbar_1}\left(\frac{y_2-y_1}{h_2} - \frac{y_1}{h_1}\right) & \text{for } i = 1, \\ -y_{\bar{x}\hat{x},i} & \text{for } i = 2, 3, ..., N-2, \\ -\frac{1}{\hbar_{N-1}}\left(\frac{-y_{N-1}}{h_N} - \frac{y_{N-1}-y_{N-2}}{h_{N-1}}\right) & \text{for } i = N-1, \end{cases} \tag{7.258}$$

defined in accordance with formula (7.246) can be represented in the form

$$A = T^*T, \tag{7.259}$$

where the linear operators $T : H \to H^*$, $T^* : H^* \to H$ have the form

$$(Ty)_i = y_{\bar{x},i} = \frac{y_i - y_{i-1}}{h_i}, \quad i = 1, 2, ..., N, \quad y_0 = y_N = 0, \tag{7.260}$$

$$(T^*v)_i = -v_{\hat{x},i} = -\frac{v_{i+1} - v_i}{\hbar_i}, \quad i = 1, 2, ..., N-1. \tag{7.261}$$

In fact, using the definition (7.260), (7.261) we have

$$(T^*Ty)_i = -\frac{(Ty)_{i+1} - (Ty)_i}{\hbar_i} = -y_{\bar{x}\hat{x},i} = (Ay)_i, \quad i = 1, 2, ..., N-1.$$

We show that the operators T and T^* are conjugate to each other. In fact, in accordance with the formula of summation by parts for any $y \in H$, $v \in H^*$, we have

$$(v, Ty] = \sum_{i=1}^{N} h_i v_i y_{\bar{x},i} = -\sum_{i=1}^{N-1} \hbar_i v_{\hat{x},i} y_i = (T^*v, y)_*.$$

Let us write scheme (7.256) in the canonical form of three-level operator-difference schemes of conservative type (5.3.1)

$$Dy_{\bar{t}t} + Ay = \varphi, \quad t_n \in \omega_\tau, \quad y_0 = u_0, \quad y_1 = u_1,$$
$$D = E + \tau^2 T^* \Sigma T, \quad \Sigma = \mathrm{diag}\{\sigma_{11}, \sigma_{12}, ..., \sigma_{1N}\}, \quad \Sigma : H^* \to H^*. \tag{7.262}$$

Let us control the fulfilment of the sufficient condition of stability

$$D \geq \frac{1+\varepsilon}{4}\tau^2 A, \quad \varepsilon > 0 \text{ is any real number.}$$

Note that

$$D - \frac{1+\varepsilon}{4}\tau^2 A = E + \tau^2 T^* \left(\Sigma - \frac{1+\varepsilon}{4}E\right) T \geq 0$$

under the condition

$$\Sigma \geq \frac{1+\varepsilon}{4}E.$$

Since Σ is the diagonal operator, the latter inequality is satisfied for $\sigma_{1i} \geq \dfrac{1+\varepsilon}{4}$ or

$$\sigma \geq \frac{h_i^2}{6\tau^2} + \frac{1+\varepsilon}{4}, \quad i = 1, 2, ..., N-1. \tag{7.263}$$

Thus by virtue of estimates (6.6.19), (6.6.20) the following statement is correct:

THEOREM 7.14 *If condition (7.263) holds then the difference scheme (7.253), (7.254) is stable with respect to the initial data and the right hand side and the following* a priori *estimate for its solution is correct:*

$$\|y_{n+1}\|_A \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}}\left(\|y(0)\|_A + \|y_t(0)\|_D + \sum_{k=1}^{n}\tau\|\varphi_n\|_{D^{-1}}\right).$$

Note that in the case of a uniform grid the conservative scheme (7.253) can not be reduced to the classic unconditionally stable scheme for $\sigma < 1$. It is the drawback of this scheme.

Let us consider the class of the schemes with spatially variable weights

$$y_{(\omega_1,\omega_2)\bar{t}t} = y_{\bar{x}\hat{x}}^{(\sigma_1,\sigma_2)} + \varphi, \quad y_{(\omega_1,\omega_2)} = \omega_{1i}y_{i+1} + (1-\omega_{1i}-\omega_{2i})y_i + \omega_{2i}y_{i-1},$$

where the weights ω_{1i}, ω_{2i} have been chosen from the condition

$$v(\overline{x}_i) - v_{(\omega_1,\omega_2)} = O(\hbar^n), \quad n \geq 2.$$

To simplify the later investigation we confine ourselves to consideration of the purely implicit scheme $(\sigma_1 = 1, \sigma_2 = 0)$

$$y_{(\omega_1,\omega_2)\bar{t}t} = \hat{y}_{\bar{x}\hat{x}} + \varphi, \quad \varphi = f(\overline{x}, \hat{t}). \tag{7.264}$$

Let us give examples of difference schemes of the second order of approximation with respect to the spatial variable. For $\omega_1 = \omega_2 = \dfrac{1}{3}$ we obtain the simplest scheme

$$\frac{y_{\bar{t}t,i-1} + y_{\bar{t}t,i} + y_{\bar{t}t,i+1}}{3} = \hat{y}_{\bar{x}\hat{x},i} + \varphi_i. \tag{7.265}$$

Assuming that $\omega_1 = 0$, $\omega_2 = -\dfrac{h_+ - h}{3h}$, from the implicit scheme (7.264) we obtain for $h_+ \geq h$ the scheme with a directed difference (see (7.219))

$$y_{\bar{t}t} + \frac{h_+ - h}{3}y_{\bar{t}t\bar{x}} = \hat{y}_{\bar{x}\hat{x}} + \varphi. \tag{7.266}$$

Varying the variable weights ω_{1i}, ω_{2i} we can obtain an approximation of the derivative $\dfrac{\partial u}{\partial t}$ with regard to the non-computational node $(\overline{x}_i, t_j)$ and the order higher then 2. For example, for

$$\omega_1 = \frac{2}{9}\frac{(2h+h_+)(h_+-h)}{\hbar h_+}, \quad \omega_2 = -\frac{2}{9}\frac{(h+2h_+)(h_+-h)}{\hbar h}, \tag{7.267}$$

by representing the interpolation residual in the Lagrange form it is easy to show that

$$\frac{\partial u(\overline{x}_i, t)}{\partial t} - u_{(\omega_1,\omega_2)t,i} = O(\hbar_i^3 + \tau).$$

The local truncation error of schemes (7.265), (7.266), (7.264), (7.267) is the value $O(\hbar^2 + \tau)$. In the case of the uniform grid ω_h the difference equation (7.266) degenerates to the classical approximation of a wave equation.

Let us formulate the conditions of stability with respect to initial data. If $\varphi = 0$, then scheme (7.264) is reduced to the equation

$$y_{\bar{t}t} + \omega_1 h_+ y_{\bar{t}tx} - \omega_2 h y_{\bar{t}t\bar{x}} = \hat{y}_{\bar{x}\hat{x}}.$$

Taking the dot product of this equation with $-2\tau y_{t\bar{x}\hat{x}}$ we obtain

$$-2\tau\,(y_{t\bar{x}\hat{x}}, y_{\bar{t}t})_* = \|\hat{y}_{\bar{t}\bar{x}}]|^2 + \tau^2\|y_{\bar{t}t\bar{x}}]|^2 - \|y_{\bar{t}\bar{x}}]|^2.$$

To estimate the scalar product we shall use the inequality

$$\|v\|_* = \left(\sum_{i=1}^{N-1} \hbar_i v_i^2\right)^{1/2} \le \max_i \left(\frac{\hbar_i}{h_i}\right)^{1/2} \|v]| = \sqrt{c_2}\|v]|, \quad c_2 = \frac{1}{2}(1 + c_1^{-1}).$$

Using the Cauchy inequality with ε we have

$$\begin{aligned} &-2\tau\,(y_{t\bar{x}\hat{x}}, \omega_1 h_+ y_{\bar{t}tx} - \omega_2 h y_{\bar{t}t\bar{x}})_* \\ &\qquad \ge -\tau^2\|y_{\bar{t}t\bar{x}}]|^2 - 2h^2 c_2\left(\|\omega_1\|_C^2 + \|\omega_2\|_C^2\right)\|y_{t\bar{x}\hat{x}}\|_*^2. \end{aligned}$$

Clearly,

$$-2\tau\,(y_{t\bar{x}\hat{x}}, \hat{y}_{\bar{x}\hat{x}})_* = -\|\hat{y}_{\bar{x}\hat{x}}\|_*^2 - \tau^2\|y_{t\bar{x}\hat{x}}\|_*^2 + \|y_{\bar{x}\hat{x}}\|_*^2.$$

Summing the estimates obtained we arrive at the recurrence inequality

$$\|y_{n+1}\|_1^2 + I(y_n) \le \|y_n\|_1^2, \tag{7.268}$$

where

$$\|y\|_1^2 = \|y_{\bar{t}\bar{x}}]|^2 + \|y_{\bar{x}\hat{x}}\|_*^2,$$

$$I(y) = \left(\tau^2 - 2h^2 c_2 \left(\|\omega_1\|_C^2 + \|\omega_2\|_C^2\right)\right) \|y_{t\bar{x}\hat{x}}\|_*^2.$$

Note that $I(y) \geq 0$ under the condition

$$\tau \geq c_3 h, \tag{7.269}$$

where $c_3 = \sqrt{2c_2}\left(\|\omega_1\|_C^2 + \|\omega_2\|_C^2\right)^{1/2}$, $h = \max_i h_i$.

Thus if condition (7.269) is satisfied the difference scheme with variable weights (7.264), (7.254) is stable with respect to the initial data, and the following *a priori* estimate holds:

$$\|y_{n\bar{x}\hat{x}}\|_* \leq \|u_{0\bar{x}\hat{x}}\|_* + \tau\|\widetilde{u}_{0\bar{x}\hat{x}}\|_* + \|\widetilde{u}_{0\bar{x}}]|.$$

Let us consider the conditions of stability with respect to initial data for specific difference schemes of the second order of approximation. For the scheme (7.265) ($\omega_1 = \omega_2 = 1/3$) it follows from the condition (7.269) that the step τ must satisfy the inequality

$$\tau \geq (2\sqrt{c_2}/3)h, \tag{7.270}$$

and the non-conservative scheme (7.266) is stable with respect to the initial data under the condition

$$\tau \geq \frac{\sqrt{2c_2}}{3} \max_i |h_{i+1} - h_i|.$$

When obtaining above the *a priori* estimates of stability in the energy semi-norm W_2^2, we saw the necessity of satisfying relations on the grid steps τ and h of the form (7.269), (7.270), i.e., we have obtained the inverse Currant condition. We show that under these conditions the corresponding estimates of accuracy hold. To simplify the investigation we confine ourselves to consideration of scheme (7.266). The corresponding problem for error of the method $z = y - u$ has the form

$$z_{\bar{t}t} + \frac{h_+ - h}{3} z_{\bar{t}t\bar{x}} = \hat{z}_{\bar{x}\hat{x}} + \hat{\bar{\psi}}, \tag{7.271}$$

$$z(0, t+\tau) = z(l, t+\tau) = 0, \quad z(x, 0) = 0, \quad z_t(x, 0) = \nu(x). \tag{7.272}$$

We recall that the truncation error of scheme ψ, $\nu(x)$ satisfies the condition

$$\psi(\overline{x}_i, t) = O(\hbar_i^2 + \tau), \quad \nu(x) = \widetilde{u}(x) - \overline{u}_0(x) = O(\tau^2).$$

THEOREM 7.15 *If a unique solution of problem (7.250)–(7.252) $u(x,t) \in C_5^4(Q_T)$ exists, then the solution of the difference scheme (7.266), (7.254)* *is conditionally convergent to the solution of the difference problem, and under the condition*

$$\tau \geq \frac{\sqrt{c_2}}{3} \max_{1 \leq i \leq N-1} |h_{i+1} - h_i| \tag{7.273}$$

the following estimates of accuracy hold:

$$\max_{t\in\omega_\tau} \|z_{\bar{x}\hat{x}}\|_* \leq c_4(h^2+\tau), \ c_4 = \text{const} > 0, \ \max_{t\in\omega_\tau} \|z_{\bar{x}}\|_C \leq M_1 c_4(h^2+\tau).$$

P r o o f. Taking the dot product of the equation for error of the method (7.271) with $-2\tau z_{t\bar{x}\hat{x}}$, similarly to inequality (7.268) we have the energy relation

$$\|\hat{z}_{\bar{t}\bar{x}}]|^2 + \|\hat{z}_{\bar{x}\hat{x}}\|_*^2 + I_1(z) \leq \|z_{\bar{t}\bar{x}}]|^2 + \|z_{\bar{x}\hat{x}}\|_*^2 - 2\tau(z_{t\bar{x}\hat{x}}, \psi)_*. \tag{7.274}$$

By virtue of condition (7.273)

$$I_1(z) = \left(\tau^2 - h^2 c_2 \|h_+ - h\|_C^2/9\right) \|z_{t\bar{x}\hat{x}}\|_*^2 \geq 0.$$

Now let us estimate the scalar product with the truncation error:

$$\begin{aligned} -2\tau(z_{t\bar{x}\hat{x}}, \hat{\psi})_* &= -2(\hat{z}_{\bar{x}\hat{x}}, \hat{\psi})_* + 2(z_{\bar{x}\hat{x}}, \psi)_* + 2\tau(z_{\bar{x}\hat{x}}, \psi_t)_* \\ &\leq -2\tau\left((z_{\bar{x}\hat{x}}, \psi)_*\right)_t + \tau\|z_{\bar{x}\hat{x}} + \psi\|_*^2 \\ &\quad + \tau\|\psi_t\|_*^2 - 2\tau(\psi, \psi_t)_*. \end{aligned}$$

Substituting this estimate into inequality (7.274) and taking into account the identity

$$\|\hat{\psi}\|_*^2 = \|\psi\|_*^2 + \tau(\psi + \hat{\psi}, \psi_t)_*,$$

which is correct for an arbitrary grid function, we obtain the recurrent relation

$$\|\hat{z}\|_2^2 \leq (1+\tau)\|z\|_2^2 + \tau r(\hbar^2 + \tau),$$

where $r > 0$ is the constant of approximation,

$$\|z\|_2^2 = \|z_{t\bar{x}}]|^2 + \|z_{\bar{x}\hat{x}} + \psi\|_*^2.$$

By virtue of the arbitrariness of n, from latter inequality the estimates required follow.

6.4 Difference Schemes for a Two-Dimensional Parabolic Equation

Let it be required to find the function $u(x,t)$, $x = (x_1, x_2)$ which satisfies the boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + f(x,t), \quad x \in \Omega, \quad t \in (0,T], \tag{7.275}$$

$$u(x,0) = u_0(x), \quad x \in \Omega; \quad u\big|_\Gamma = 0, \quad t \in (0,T] \tag{7.276}$$

in the domain

$$\overline{Q}_T = \overline{\Omega} \times [0 \le t \le T],$$

where $\overline{\Omega} = \{0 \le x_1 \le l_1,\ 0 \le x_2 \le l_2\}$ is the rectangle with the boundary Γ. In the rectangle $\overline{\Omega}$ let us introduce the arbitrary non-uniform grid

$$\hat{\overline{\omega}}_h = \Big\{ x = x_{i_1 i_2} = \left(x_1^{i_1}, x_2^{i_2}\right); \quad x_\alpha^{i_\alpha} = x_\alpha^{i_\alpha - 1} + h_\alpha^{i_\alpha},$$

$$i_\alpha = 1, 2, ..., N_\alpha - 1, \quad x_\alpha^0 = 0, \quad x_\alpha^{N_\alpha} = l_\alpha,\ \alpha = 1, 2 \Big\},$$

where $\sum\limits_{i_\alpha = 1}^{N_\alpha} h_\alpha^{i_\alpha} = l_\alpha$, $\alpha = 1, 2$. By $\hat{\omega}_h$ denote a set of interior nodes of the grid $\hat{\overline{\omega}}_h$. By γ_h we denote a set of the boundary nodes. In the domain $\overline{Q}_T$ we introduce the spatial-time grid in the standard way:

$$\overline{\omega} = \overline{\omega}_h \times \overline{\omega}_\tau, \quad \overline{\omega}_\tau = \{t_n = n\tau,\ n = 0, 1, ..., N_0;\ \tau N_0 = T\} = \omega_\tau \cup \{T\}.$$

The simplest difference scheme of the first order of approximation on the non-uniform stencil 'cross' has the form [Samarskii, 1989]

$$y_t = \hat{y}_{\bar{x}_1 \hat{x}_1} + \hat{y}_{\bar{x}_2 \hat{x}_2} + f(x, \hat{t}), \quad (x,t) \in \omega, \tag{7.277}$$

$$y(x,0) = u_0(x), \quad x \in \overline{\omega}_h; \qquad \hat{y}\big|_{\gamma_h} = 0, \quad t \in \omega_\tau. \tag{7.278}$$

Here we use the following indices-free notation of the theory of difference schemes:

$$y_{\bar{x}_\alpha \hat{x}_\alpha} = \frac{1}{\hbar_\alpha}\left(y_{x_\alpha} - y_{\bar{x}_\alpha}\right), \quad y_{x_\alpha} = \frac{1}{h_{\alpha+}}\left(y^{(+1_\alpha)} - y\right), \quad h_{\alpha\pm} = h_\alpha^{i_\alpha \pm 1},$$

$$y_{\bar{x}_\alpha} = \frac{1}{h_\alpha}\left(y - y^{(-1_\alpha)}\right), \quad \hbar_\alpha = 0.5(h_\alpha + h_{\alpha+}),$$

$$v^{(\pm 1_1)} = v_{i_1 \pm 1 i_2}, \quad v^{(\pm 1_2)} = v_{i_1 i_2 \pm 1}.$$

The difference scheme of the second order of approximation on the non-uniform grid $\overline{\omega}$ has the form

$$y_t + \widetilde{h}_1 y_{t\bar{x}_1} + \widetilde{h}_2 y_{t\bar{x}_2} = \hat{y}_{(2)\bar{x}_1 \hat{x}_1} + \hat{y}_{(1)\bar{x}_2 \hat{x}_2} + f(\overline{x}, \hat{t}), \tag{7.279}$$

where

$$\overline{x} = (\overline{x}_1, \overline{x}_2), \quad \overline{x}_\alpha = x_\alpha + \widetilde{h}_\alpha, \quad \widetilde{h}_\alpha = \frac{h_{\alpha+} - h_\alpha}{3},$$

$$y_{(1)} = y(\overline{x}_1, x_2) = y + h_1^+ y_{x_1} + h_1^- y_{\bar{x}_1},$$

$$y_{(2)} = y(x_1, \overline{x}_2) = y + h_2^+ y_{x_2} + h_2^- y_{\overline{x}_2},$$
$$h_\alpha^\pm = 0.5 \left(\widetilde{h}_\alpha \pm |\widetilde{h}_\alpha|\right).$$

We note that the values of the grid function y at the non-calculating nodes $(\overline{x}_1, x_2)$, $(x_1, \overline{x}_2)$ are averaged by means of formulas of the second order of approximation taking into account the directed differences. A derivative with respect to time may be approximated similarly. In the case of a uniform grid ($h_\alpha^\pm = 0$), the scheme (7.279) is reduced to the classical absolutely implicit scheme of the second order of accuracy with respect to spatial variables

$$y_t = \hat{y}_{\overline{x}_1 x_1} + \hat{y}_{\overline{x}_2 x_2} + f(x, \hat{t}).$$

To investigate the approximation error we assume that there exists a unique essentially smooth solution of the problem (7.275), (7.276). Let us write the discrepancy of the scheme in the form

$$\psi = \psi_1 + \psi_2, \tag{7.280}$$

$$\psi_1 = \frac{\partial u}{\partial t}(\overline{x}, \hat{t}) - \left(u_t + \widetilde{h}_1 u_{t\overline{x}_1} + \widetilde{h}_2 u_{t\overline{x}_2}\right),$$
$$\psi_2 = \left(\hat{u}_{(2)} - u(x_1, \overline{x}_2, \hat{t})\right)_{\overline{x}_1 \hat{x}_1} + \left(u(x_1, \overline{x}_2, \hat{t})\right)_{\overline{x}_1 \hat{x}_1} - \frac{\partial^2 u}{\partial x_1^2}(\overline{x}, \hat{t})$$
$$+ \left(\hat{u}_{(1)} - u(\overline{x}_1, x_2, \hat{t})\right)_{\overline{x}_2 \hat{x}_2} + \left(u(\overline{x}_1, x_2, \hat{t})\right)_{\overline{x}_2 \hat{x}_2} - \frac{\partial^2 u}{\partial x_2^2}(\overline{x}, \hat{t}).$$

Taking into account the relations

$$\psi_3 = u_t(\overline{x}, t) - \left(u_t + \widetilde{h}_1 u_{t\overline{x}_1} + \widetilde{h}_2 u_{t\overline{x}_2}\right) = O(\hbar_1^2 + \hbar_2^2),$$
$$\psi_4 = \frac{\partial u}{\partial t}(\overline{x}, \hat{t}) - u_t(\overline{x}, t) = O(\tau),$$

we obtain

$$\psi_1 = O(\hbar_1^2 + \hbar_2^2 + \tau). \tag{7.281}$$

To estimate the expression $\xi_1 = \left(\hat{u}_{(2)} - u(x_1, \overline{x}_2, \hat{t})\right)_{\overline{x}_1 \hat{x}_1}$ we use the following expansion:

$$u_{x_2} = \frac{\partial u}{\partial x_2} + \frac{h_{2+}}{2} \overline{\frac{\partial^2 u}{\partial x_2^2}}, \quad u_{\overline{x}_2} = \frac{\partial u}{\partial x_2} - \frac{h_2}{2} \overline{\frac{\partial^2 u}{\partial x_2^2}}.$$

Here the bar denotes that the values of arguments are taken at the corresponding intermediate nodes (in the given case in the intervals

$(x_2, x_2 + h_{2+})$ and $(x_2 - h_2, x_2)$). Therefore

$$\hat{u}_{(2)} = \hat{u} + \widetilde{h}_2 \frac{\partial \hat{u}}{\partial x_2} + r_0, \quad r_0 = \frac{h_2^+ h_{2+}}{2} \overline{\frac{\partial^2 \hat{u}}{\partial x_2^2}} - \frac{h_2^- h_{2-}}{2} \overline{\frac{\partial^2 \hat{u}}{\partial x_2^2}} = O(\hbar_2^2).$$

Taking into account that

$$u(x_1, \overline{x}_2, \hat{t}) = \hat{u} + \widetilde{h}_2 \frac{\partial \hat{u}}{\partial x_2} + r_1, \quad r_1 = \frac{\widetilde{h}_2^2}{2} \overline{\frac{\partial^2 \hat{u}}{\partial x_2^2}},$$

we obtain the following representation of ξ_1:

$$\xi_1 = r_{2\bar{x}_1\hat{x}_1}, \quad r_2 = r_0 - r_1 = O(\hbar_2^2).$$

Since

$$|r_{2\bar{x}_1\hat{x}_1}| = \left| \frac{1}{\hbar_1} \int\limits_0^1 \int\limits_{x_1 - \xi h_1}^{x_1 + \xi h_{1+}} \frac{\partial^2 r_2(\eta, x_2, \hat{t})}{\partial \eta^2} d\eta d\xi \right| \le \hbar_2^2 \left\| \frac{\partial^4 \hat{u}}{\partial x_1^2 \partial x_2^2} \right\|_{C(\overline{Q}_T)}$$

it follows that $\xi_1 = O(\hbar_2^2)$. Similarly we have

$$|\xi_2| = \left| \left(\hat{u}_{(1)} - u(\overline{x}_1, x_2, \hat{t}) \right)_{\bar{x}_2\hat{x}_2} \right| \le \hbar_1^2 \left\| \frac{\partial^4 \hat{u}}{\partial x_1^2 \partial x_2^2} \right\|_{C(\overline{Q}_T)}.$$

Furthermore, let us use expansion (7.222) in the form

$$u(x_1, \overline{x}_2, \hat{t})_{\bar{x}_1\hat{x}_1} - \frac{\partial^2 u(\overline{x}_1, \overline{x}_2, \hat{t})}{\partial x_1^2} = O(\hbar_1^2),$$

$$u(\overline{x}_1, x_2, \hat{t})_{\bar{x}_2\hat{x}_2} - \frac{\partial^2 u(\overline{x}_1, \overline{x}_2, \hat{t})}{\partial x_2^2} = O(\hbar_2^2).$$

On the basis of the foregoing we conclude that if the fourth derivatives of the solution $u(x, t)$ are bounded then the error of approximation has the first order of smallness with respect to τ and the second order with respect to

$$|h| = \left(h_1^2 + h_2^2 \right)^{1/2}, \quad h_\alpha = \max_{i_\alpha} h_\alpha^{i_\alpha}, \quad \alpha = 1, 2, \tag{7.282}$$

i.e., such constant M_1 exists that

$$\|\psi\|_{C(\omega)} \le M_1 \left(h_1^2 + h_2^2 + \tau \right), \tag{7.283}$$

where M_1 is independent of h_1, h_2 and τ.

To simplify further investigation, let us consider grids with respect to variables x_1 and x_2 which are refined to the beginning of the corresponding segment:

$$h_\alpha^{i_\alpha+1} - h_\alpha^{i_\alpha} \geq 0, \quad \alpha = 1,2 \tag{7.284}$$

Therefore, the scheme (7.279) has the simpler form

$$y_t + \frac{h_{1+} - h_1}{3} y_{t\bar{x}_1} + \frac{h_{2+} - h_2}{3} y_{t\bar{x}_2} + \widetilde{A}\hat{y} = \varphi, \tag{7.285}$$

where

$$\widetilde{A} = A + A_0, \quad A = A_1 + A_2, \quad A_k y = -y_{\bar{x}_k\hat{x}_k}, \tag{7.286}$$

$$A_0 y = \frac{h_{2+} - h_2}{3} A_1 y_{x_2} + \frac{h_{1+} - h_1}{3} A_2 y_{x_1}, \quad \varphi = f(\overline{x}, \hat{t}). \tag{7.287}$$

For scalar products and norms we shall use the standard notation of the theory of difference schemes:

$$(v, y)_* = \sum_{i_1=1}^{N_1-1} \sum_{i_2=1}^{N_2-1} \hbar_1^{i_1} \hbar_2^{i_2} v_{i_1 i_2} y_{i_1 i_2} = \sum_{i_1=1}^{N_1-1} \sum_{i_2=1}^{N_2-1} \hbar_1 \hbar_2 v y,$$

$$\|y\|_*^2 = (y, y)_*, \quad \|y\|_C = \max_{x \in \hat{\omega}_h} |y(x)|.$$

For the difference Laplacian on a non-uniform grid we define the following norms:

$$\|y\|_{A_1}^2 = (A_1 y, y) = \|y_{\bar{x}_1}]|^2 = \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2-1} h_1^{i_1} \hbar_2^{i_2} y_{\bar{x}_1, i_1 i_2}^2,$$

$$\|y\|_{A_2}^2 = \|y_{\bar{x}_2}]|^2 = \sum_{i_1=1}^{N_1-1} \sum_{i_2=1}^{N_2} \hbar_1^{i_1} h_2^{i_2} y_{\bar{x}_2, i_1 i_2}^2,$$

$$\|y\|_A^2 = \|y\|_{A_1}^2 + \|y\|_{A_2}^2 = \|y_{\bar{x}_1}]|^2 + \|y_{\bar{x}_2}]|^2,$$

$$\|y_{\bar{x}_1\bar{x}_2}]|^2 = \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} h_1^{i_1} h_2^{i_2} y_{\bar{x}_1\bar{x}_2, i_1 i_2}^2.$$

Let us produce the subsidiary results.

LEMMA 7.6 *Let the arbitrary grid function $y(x)$ be defined on the non-uniform rectangular grid $\hat{\overline{\omega}}_h$ and be equal to zero on the boundary γ_h. Then the following relations are correct:*

$$(A_1 y, y) = \|y_{\bar{x}_1}]|^2, \quad (A_2 y, y) = \|y_{\bar{x}_2}]|^2, \quad \|y\|_A^2 = \|y_{\bar{x}_1}]|^2 + \|y_{\bar{x}_2}]|^2; \tag{7.288}$$

$$\|Ay\|_*^2 = \|y_{\bar{x}_1\hat{x}_1} + y_{\bar{x}_2\hat{x}_2}\|_*^2 = \|y_{\bar{x}_1\hat{x}_1}\|_*^2 + \|y_{\bar{x}_2\hat{x}_2}\|_*^2 + 2\|y_{\bar{x}_1\bar{x}_2}]|^2. \quad (7.289)$$

Proof. Using the first Green's difference formula in the one-dimensional case [Samarskii and Goolin, 1973, p. 35]

$$(y, v_{\bar{x}\hat{x}})_* = -(y_{\bar{x}}, v_{\bar{x}}] + y_N v_{\bar{x},N} - y_0 v_{\bar{x},1}, \quad (7.290)$$

we have the expression

$$-\sum_{i_1=1}^{N_1-1} \hbar_1 y_{\bar{x}_1\hat{x}_1} y = \sum_{i_1=1}^{N_1} h_1 y_{\bar{x}_1}^2.$$

Multiplying it by $\hbar_2$ and summing for all $i_2 = 1, 2, ..., N_2 - 1$, we obtain the first identity of (7.288). The second equality can be proved similarly. The last equality of (7.288) is the algebraic corollary of the first two.

Furthermore, by definition of the norm $\|\cdot\|_*$ we have

$$\|y_{\bar{x}_1\hat{x}_1} + y_{\bar{x}_2\hat{x}_2}\|_*^2 = \|y_{\bar{x}_1\hat{x}_1}\|_*^2 + \|y_{\bar{x}_2\hat{x}_2}\|_*^2 + 2(y_{\bar{x}_1\hat{x}_1}, y_{\bar{x}_2\hat{x}_2})_*. \quad (7.291)$$

Using the Green's formula (7.290) with respect to the variable x_1 and taking into account that the grid function $y_{\bar{x}_2\hat{x}_2}$ is equal to zero for $i_1 = 0, N_1$, we obtain

$$(y_{\bar{x}_1\hat{x}_1}, y_{\bar{x}_2\hat{x}_2})_* = -\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2-1} h_1\hbar_2 y_{\bar{x}_1} y_{\bar{x}_1\bar{x}_2\hat{x}_2}.$$

Using Green's formula with respect to the variable x_2 and taking into account that $y_{\bar{x}_1} = 0$ for $i_2 = 0, N_2$, we obtain energy relation

$$(y_{\bar{x}_1\hat{x}_1}, y_{\bar{x}_2\hat{x}_2})_* = \|y_{\bar{x}_1\bar{x}_2}]|^2.$$

Substituting this relation into equality (7.291), we obtain identity (7.289).

LEMMA 7.7 *Let conditions (7.284) hold. Then the following inequality for the arbitrary grid function $y(x_1, x_2)$ defined on the non-uniform rectangular grid $\hat{\bar{\omega}}_h$ and equal to zero on the boundary γ_h is correct:*

$$\left(\frac{h_{2+} - h_2}{3} A_1 y_{x_2}, A_1 y\right)_* + \left(\frac{h_{1+} - h_1}{3} A_2 y_{x_1}, A_2 y\right)_* \geq -\frac{2}{3}\left(\|A_1 y\|_*^2 + \|A_2 y\|_*^2\right). \quad (7.292)$$

Under the additional assumption

$$\frac{h_k - h_{k-}}{2h_k} + \frac{h_{k+} - h_k}{6h_k} \leq \frac{2}{3}, \quad k=1,2;\ i_k=1,2,...,N_k,\ h_k^1 = h_k^0, \quad (7.293)$$

the following estimate holds

$$\left(\frac{h_{2+}-h_2}{3}A_1y_{x_2},A_2y\right)_*+\left(\frac{h_{1+}-h_1}{3}A_2y_{x_1},A_1y\right)_*\geq-\frac{4}{3}\|y_{\bar{x}_1\bar{x}_2}]|^2. \quad (7.294)$$

P r o o f. Inequality (7.292) can be proved similarly to the one-dimensional case (see Lemma 7.5). Now let us prove estimate (7.294). Using the Green difference formula in the direction x_1 and the algebraic inequality

$$-(a-b)a\geq-\frac{3}{2}a^2-\frac{1}{2}b^2,$$

we obtain

$$\begin{aligned}\left(\frac{h_{2+}-h_2}{3}A_1y_{x_2},A_2y\right)&=-\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2-1}h_1\hbar_2y_{\bar{x}_1x_2}y_{\bar{x}_1\bar{x}_2\hat{x}_2}\\&=-\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2-1}h_1\frac{h_{2+}-h_2}{3}y_{\bar{x}_1\bar{x}_2}^{(+1_2)}\left(y_{\bar{x}_1\bar{x}_2}^{(+1_2)}-y_{\bar{x}_1\bar{x}_2}\right)\\&\geq-\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2-1}h_1\frac{h_{2+}-h_2}{3}\left(\frac{3}{2}\left(y_{\bar{x}_1\bar{x}_2}^{(+1_2)}\right)^2+\frac{1}{2}y_{\bar{x}_1\bar{x}_2}^2\right)\\&=-\sum_{i_1=1}^{N_1}\sum_{i_2=2}^{N_2}h_1h_2\frac{h_2-h_{2-}}{2h_2}y_{\bar{x}_1\bar{x}_2}^2\\&\quad-\sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2-1}h_1h_2\frac{h_{2+}-h_2}{6h_2}y_{\bar{x}_1\bar{x}_2}^2.\end{aligned}$$

Now using condition (7.293) for $k=2$, we obtain

$$\left(\frac{h_{2+}-h_2}{3}A_1y_{x_2},A_2y\right)_*\geq-\frac{2}{3}\|y_{\bar{x}_1\bar{x}_2}]|^2.$$

Similarly we can show that

$$\left(\frac{h_{1+}-h_1}{3}A_2y_{x_1},A_1y\right)_*\geq-\frac{2}{3}\|y_{\bar{x}_1\bar{x}_2}]|^2.$$

Summing the last two inequalities we obtain the estimate required (7.294).

REMARK 7.7 Let a grid be refined in the direction x_k in accordance with the geometric progression $h_k^{i_k+1}=q_kh_k^{i_k}$ with the constant $q_k\geq1$.

Let us determine such q_k that it satisfies condition (7.293). Taking into account the assumption formulated, we have

$$\frac{h_k - h_{k-}}{2h_k} + \frac{h_{k+} - h_k}{6h_k} - \frac{2}{3} = \frac{q_k^2 - 2q_k - 3}{6q_k}.$$

Thus inequality (7.293) is satisfied for all $1 \le q_k \le 3$.

We apply Lemmas 7.6, 7.7 proved above to the investigation of the stability of the difference scheme with respect to the initial data.

THEOREM 7.16 *Let inequalities (7.284), (7.293) be satisfied and*

$$\tau \ge c_4 \max_k \|h_{k+} - h_k\|_C^2, \qquad \max_{1 \le i_k \le N_k - 1} \frac{\hbar_k^{i_k}}{h_k^{i_k}} \le c_4. \tag{7.295}$$

Then the difference scheme (7.285), (7.278) is stable with respect to the initial data and the right hand side and the following estimate holds:

$$\max_{t \in \overline{\omega}_\tau} \|y\|_1 \le \|u_0\|_1 + 3\sqrt{T} \max_{t \in \omega_\tau} \|f\|_*, \tag{7.296}$$

where

$$\|y\|_1^2 = \|y\|_A^2 + \frac{\tau}{3}\|Ay\|_*^2.$$

Proof. Taking the dot product of equation (7.285) with $2\tau A\hat{y}$ and taking into account Lemma 7.6 and the identity

$$2\tau(v_t, \hat{v})_* = \|\hat{v}\|_*^2 - \|v\|_*^2 + \tau^2\|v_t\|_*^2$$

we obtain the energy relation

$$\begin{aligned} &\|\hat{y}\|_A^2 - \|y\|_A^2 + \tau^2\|y_t\|_A^2 \\ &\quad + 2\tau\left(\frac{h_{1+} - h_1}{3} y_{t\bar{x}_1} + \frac{h_{2+} - h_2}{3} y_{t\bar{x}_2}, A\hat{y}\right)_* \\ &\quad + 2\tau\left(\widetilde{A}\hat{y}, A\hat{y}\right)_* = 2\tau\left(\varphi, A\hat{y}\right)_*. \end{aligned} \tag{7.297}$$

Using the Cauchy inequality with ε and the second condition (7.295), we estimate the item on the left hand side of a relation (7.297)

$$\begin{aligned} &2\tau\left(\frac{h_{1+} - h_1}{3} y_{t\bar{x}_1} + \frac{h_{2+} - h_2}{3} y_{t\bar{x}_2}, A\hat{y}\right)_* \\ &\qquad \ge -\frac{2}{9}\tau\|A\hat{y}\|_*^2 - \tau\left(\|(h_{1+} - h_1)y_{t\bar{x}_1}\|_*^2 + \|(h_{2+} - h_2)y_{t\bar{x}_2}\|_*^2\right) \\ &\qquad \ge -\frac{2}{9}\tau\|A\hat{y}\|_*^2 - \tau c_4 \max_k \|h_{k+} - h_k\|_C^2 \|y_t\|_A^2. \end{aligned}$$

Taking into account identity (7.289) and inequalities (7.292), (7.294) we obtain

$$
\begin{aligned}
2\tau\left(\widetilde{A}\hat{y}, A\hat{y}\right) =& 2\tau\Bigg\{ \|A\hat{y}\|_*^2 + \left(\frac{h_{2+}-h_2}{3}A_1\hat{y}_{x_2}, A_1\hat{y}\right)_* \\
&+ \left(\frac{h_{1+}-h_1}{3}A_2\hat{y}_{x_1}, A_2\hat{y}\right)_* \\
&+ \left(\frac{h_{2+}-h_2}{3}A_1\hat{y}_{x_2}, A_2\hat{y}\right)_* \\
&+ \left(\frac{h_{1+}-h_1}{3}A_2\hat{y}_{x_1}, A_1\hat{y}\right)_* \Bigg\} \\
\geq& \frac{2}{3}\tau\|A\hat{y}\|_*^2.
\end{aligned}
$$

Now let us estimate the scalar product on the right hand side:

$$2\tau(\varphi, A\hat{y})_* \leq \frac{\tau}{9}\|A\hat{y}\|_*^2 + 9\tau\|\varphi\|_*^2.$$

Substituting the estimates obtained into identity (7.297) we obtain the inequality

$$\|y_{n+1}\|_1^2 \leq \|y_n\|_1^2 + 9\tau\|\overline{f}_{n+1}\|_*^2.$$

From this inequality there follows the required estimate of stability (7.296).

To obtain the estimates of accuracy let us substitute $y = z + u$ into equation (7.285), (7.278). We obtain the problem for error

$$z_t + \frac{h_{1+}-h_1}{3}z_{t\bar{x}_1} + \frac{h_{2+}-h_2}{3}z_{t\bar{x}_2} + \widetilde{A}\hat{z} = \psi, \tag{7.298}$$

$$z(x,0) = 0, \quad x \in \hat{\omega}_h, \qquad \hat{z}\Big|_{\gamma_h} = 0, \quad t \in \omega_\tau. \tag{7.299}$$

Here the truncation error is defined in accordance with (7.280). On satisfying conditions of Theorem 7.16 we have the *a priori* estimate for the solution of problem (7.298), (7.299):

$$\|z\|_1 \leq 3\sqrt{T}\max_{t\in\omega_\tau}\|\psi\|_*. \tag{7.300}$$

Taking into account inequality (7.283) we obtain

$$\|z\|_A^2 + \frac{\tau}{3}\|Az\|_*^2 \leq c(h_1^2 + h_2^2 + \tau)^2.$$

Consequently the solution of the difference scheme (7.285), (7.278) converges to the solution of the differential problem (7.275), (7.276) in the norm $\|\cdot\|_1$ with the rate $O(h_1^2 + h_2^2 + \tau)$.

Chapter 8

DIFFERENCE SCHEMES OF DOMAIN DECOMPOSITION FOR NON-STATIONARY PROBLEMS

1. Introduction

In solving multi-dimensional problems of mathematical physics in complex computational domains one uses two main approaches of transition to problems in regular simpler domains. The first of them (the method of fictitious domains) is based on imbedding the original irregular domain into a regular domain with some continuations of the coefficients of the original equation into the adjoining domain and corresponding choice of boundary conditions on the boundary of the extended domain.

The second approach is applied when an irregular computational domain can be represented as a union of regular sub-domains. In domain decomposition (DD) methods the computational algorithms are thus based on solving problems in separate sub-domains. We can distinguish a variant of a DD method in which sub-domains do not overlap, whilst generally it is necessary to use a DD method with the overlapping of separate sub-domains.

Based on the domain decomposition method, computational algorithms can be constructed which are oriented to the use of computers with a parallel architecture. In this case of geometrical parallelism the solution of a problem in a sub-domain is related to a separate computing unit (processor, elementary computer of a network). The main problems here are generated by formulation of the interface (exchange) conditions on the boundaries of the involved sub-domains and by organization of the computational process.

Domain decomposition methods are widely discussed in the literature. Attention is mainly focused on elliptic boundary value problems and

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iterative methods of 'large block' grouping for solving large systems of linear algebraic equations. In this book we confine consideration to non-stationary problems of mathematical physics, which are different and distinguished in form from stationary problems. Their principal distinction is that the solution changes slightly when we pass from one time level to another. In particular, the exchange boundary conditions vary slightly as well. Taking this into account one can construct non-iterative domain decomposition schemes.

At first we give a general description of domain decomposition methods in solving problems of mathematical physics. We briefly discuss methods of solving non-stationary problems on parallel computers. As a basic object for research we choose a model boundary value problem for a parabolic equation in a rectangle for decomposition (splitting) with respect to separate variables.

The construction of domain decomposition schemes for non-stationary problems is based on using schemes of splitting (additive schemes). So these schemes are called regionally additive difference schemes. In the theory of additive schemes we distinguish the case of two-component splitting when the original space operator of the problem is decomposed into the sum of two operators. This case occurs in the domain decomposition method when the original domain is partitioned into two sub-domains, each of which is the union of disjoint sub-domains. For the model problem in a rectangle this corresponds to decomposition with respect to one variable.

We consider three main types of domain decomposition which is related to a certain organization of data exchange on the interface boundaries. In particular, we can assign the Dirichlet conditions, the Neumann exchange boundary conditions. After having decomposed the domain, we can use some kind of additive difference schemes when passing to a new time level. Attention is focused on the analysis of accuracy of regionally additive difference schemes.

The accuracy of decomposition schemes depends on the width of the overlapping region, and in the limiting case of nonoverlapping sub-domains (the most attractive variant from the viewpoint of parallelization) we have conditional convergence. It is shown that we obtain the most admissible estimates if we use regionally additive schemes with two-component splitting. The convergence is established in norms that are consistent with the choice of decomposition operators (in other words, with the choice of exchange boundary conditions).

In this chapter, we consider such regionally additive schemes as alternating direction schemes, two-component factorized schemes, additive schemes of multicomponent splitting. We also study regionally additive schemes based on vector additive schemes. Among most principal generalizations, we emphasize the construction of regionally additive schemes for second-order hyperbolic (evolutionary) equations.

2. Methods of Domain Decomposition

We consider general approaches to construction of parallel algorithms for numerical solution of multi-dimensional non-stationary problems of the mathematical physics on the basis of the decomposition of a domain into a set of sub-domains.

2.1 Introduction

Methods of decomposition of a domain into sub-domains are used for approximate solution of boundary value problems of the mathematical physics in complex irregular domains. These methods are also used for constructing efficient computational algorithms for modern computational systems with a parallel architecture. Nowadays the methods of domain decomposition for the second-order elliptic equations have been developed the most (see, e.g., [Agoshkov, 1991, Quarteroni, 1991, Quarteroni et al., 1994, Smith et al., 1996, Le Tallec, 1994]). We consider iterative methods of domain decomposition when separate sub-domains are overlapped or not. We study different types of boundary conditions on the boundaries of sub-domains, the so called exchange boundary conditions. Attention is specially paid to the development of asynchronous (parallel) algorithms which allow one to realize independent calculations in separate sub-domains on each iterative step.

To construct the *methods of domain decomposition* for non-stationary problems we suggest the following approaches.

- The first (see, e.g., [Kuznetsov, 1988, Le Tallec, 1994]) is based on the use of classical implicit schemes and methods of domain decomposition for numerical solution of a grid elliptic problem on a new time level.

- In the second approach the special features of non-stationary problems (the solution changes little in passing to a new time level) are taken into account more completely. The different types of the corresponding non-iterative schemes of domain decomposition (the so called regionally additive schemes) are studied in [Vabishchevich, 1989b, Laevskii, 1992, Dryja, 1990, Laevsky, 1990, Vabishchevich, 1994b]. Below, we shall note some later works in this field.

- Decomposition schemes with a special approximation of exchange boundary conditions, the so called inhomogeneous schemes, are considered (see [Dawson et al., 1991, Dawson and Dupont, 1992]).
- Parallel versions of standard splitting schemes (see, e.g., [Johnson et al., 1987]): schemes of alternating directions, factorized schemes, locally one-dimensional schemes (schemes of component-wise splitting) are constructed.

In the approximate solution of non-stationary problems of mathematical physics, attention should mainly be paid to non-iterative versions of the decomposition method, the so called regionally additive schemes. These algorithms most fully take into account the specific features of non-stationary problems when the passage to a new time level involves the solution of a set of the separate problems in the sub-domains. A theoretical analysis of the schemes of domain decomposition has been carried out using classical and new additive schemes (splitting schemes). The consideration is based on the modern theory of stability and convergence of operator-difference additive schemes of splitting [Samarskii, 1989, Samarskii and Goolin, 1973, Samarskii and Vabishchevich, 1995a].

We shall consider the construction of the schemes of domain decomposition on the example of the simplest two-dimensional second-order parabolic equation. Some fundamental aspects of the generalization of the results to more general problems will be mentioned separately.

2.2 Model Problem

A boundary value problem is considered in the rectangle Ω with the sides parallel to the axes. The solution of the following parabolic equation is sought:

$$\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha} \left(k_\alpha(x) \frac{\partial u}{\partial x_\alpha} \right) = 0, \quad x = (x_1, x_2) \in \Omega, \quad t > 0. \tag{8.1}$$

Equation (8.1) is supplemented with the homogeneous boundary conditions (the Dirichlet problem)

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{8.2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{8.3}$$

Let us introduce a grid uniform in each direction x_α with the steps h_α, $\alpha = 1, 2$ in the area Ω. Let ω be a set of interior nodes. More general problems with non-uniform rectangular grids are considered similarly. We define a set of the grid functions $v \in H$ so that $v(x) \equiv 0,\ x \notin \omega$, and the grid operator A is

$$A = \sum_{\alpha=1}^{2} \Lambda_\alpha, \tag{8.4}$$

$$\Lambda_\alpha v = -(a_\alpha(x) v_{\bar{x}})_x, \quad x \in \omega.$$

Here we use the standard notation of the theory of difference schemes. For the problems with sufficiently smooth coefficients we accept

$$a_1(x) = 0.5(k_1(x) + k_1(x_1 - h_1, x_2)),$$

$$a_2(x) = 0.5(k_2(x) + k_2(x_1, x_2 - h_2)).$$

In the Hilbert space H we introduce an inner product and a norm by the relations

$$(y, v) = \sum_{x \in \omega} y v h_1 h_2, \quad \|y\| = \sqrt{(y, y)}.$$

respectively. The operator A is self-adjoint and positive in the space H, i.e., $A = A^* > 0$.

If $v(x, 0),\ x \in \omega$ is given then from problem (8.1)–(8.3) we obtain the equation

$$\frac{dv}{dt} + Av = 0, \quad x \in \omega. \tag{8.5}$$

We shall use the method of domain decomposition for the approximate solution of the Cauchy problem for equation (8.5). In this case the computational algorithm is based on the solution of problems in separate sub-domains of the computational domain on each time step.

2.3 Domain Decomposition

Let the domain Ω consist of p separate sub-domains:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \Omega_p,$$

which can overlap one another. The method of working of separate computational nodes of a parallel computer involves the solution of separate problems in each sub-domain. We think that precisely the decomposition of the whole domain into separate sub-domains allows an efficient use of parallel computers for the approximate solution of multi-dimensional problems of mathematical physics. The method of domain decomposition makes it possible to adapt computational algorithm to a parallel

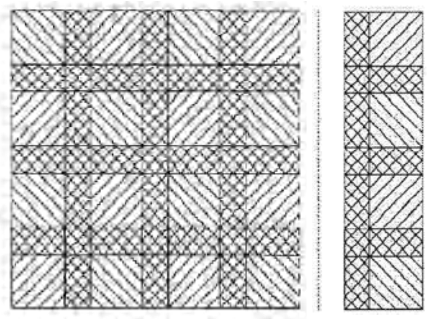

Figure 8.1. Decomposition with respect to one direction

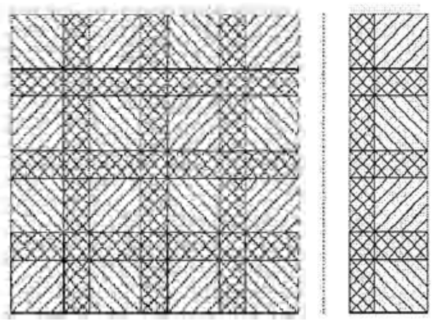

Figure 8.2. Links between separate processors

computational system with any organization of memory and arbitrary links between separate computational nodes.

Figure 8.1 schematically illustrates the decomposition of a computational domain (the rectangle Ω) into separate sub-domains overlapped by the straight lines $x_1 = \text{const}$. This decomposition corresponds to the simplest architecture of a parallel computational system like 'line' when separate processors (elementary computational machines) are connected consecutively with two nearest neighbors (Fig. 8.2).

Separate sub-domains can group for solving a separate subproblem on a separate processor. The grouping can be carried out only for two sub-domains Ω_1 and Ω_2. Each consisting of several disconnected sub-domains. Here we may talk about the two-coloring of the domain. Figure 8.3 illustrates a more complex decomposition which corresponds to four-coloring. In this case we must use the 'lattice' of processors (Fig. 8.4).

The idea of domain decomposition for approximate solution of non-stationary problems of the mathematical physics is most simply realized

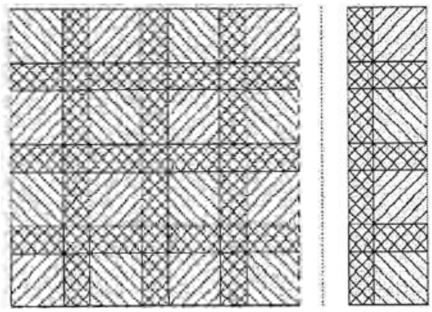

Figure 8.3. Decomposition in two directions

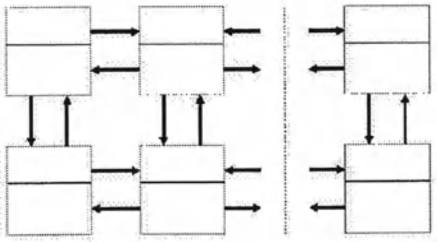

Figure 8.4. Architecture of parallel computer like 'lattice'

by the use of explicit difference schemes. Let y_n be a difference solution at the time moment $t_n = n\tau$, where $\tau > 0$ is a time step. For the model problem (8.5) we have

$$\frac{y_{n+1} - y_n}{\tau} + Ay_n = 0, \quad x \in \omega. \tag{8.6}$$

In this case the real width of the superposition area is minimal and is equal to h_1 or h_2. The exchanges between the elementary machines are also minimal. The fundamental difficulties of the use of explicit difference schemes for the approximate solution of non-stationary problems are also well known. First of all there are rigorous limitations on the relationships between grid steps. In the case of applied mathematical simulation, when the coefficients of equations change significantly, it is necessary to try to use implicit schemes. Let us consider the possibilities of constructing parallel difference schemes which are unconditionally stable like implicit schemes, but like the mentioned explicit schemes (8.6) as regards computing implementation .

2.4 Iterative Difference Schemes

The second simple approach to constructing difference schemes for solving non-stationary problems is the use of traditional (scalar) schemes. For instance, instead of the conditionally stable explicit difference scheme (8.6) we can use the unconditionally stable implicit scheme

$$\frac{y_{n+1} - y_n}{\tau} + Ay_{n+1} = 0, \quad x \in \omega. \tag{8.7}$$

It is implemented by solving on each time level the grid elliptic problem

$$(E + \tau A)y_{n+1} = y_n, \quad x \in \omega. \tag{8.8}$$

To solve equation (8.8) approximately use is made of the modern iterative methods [Samarskii and Nikolaev, 1978, Hackbush, 1993, Hageman and Young, 1981, Axelsson, 1994]. Computational algorithms are conveniently constructed on the basis of domain decomposition for parallel implementations. In this connection we additionally recall the reviews [Chan and Mathew, 1994, Demmel and van der Vorst, 1993, Xu, 1992].

Here we note the fundamental feature of the grid elliptic problem (8.8) associated with the presence of a small parameter τ which, generally can substantially influence the number of iterations. The classical versions of iterative methods are considered in [Samarskii and Vabishchevich, 1995a] and of the methods of domain decomposition are studied in [Kuznetsov, 1988, Le Tallec, 1994]. With an ample width of the superposition area it is sufficient to perform only one iteration for the consistency of the iterative method error with the accuracy of the implicit difference scheme. It allows the construction of non-iterative schemes of domain decomposition for the approximate solution of non-stationary problems.

2.5 Schemes of Splitting with Respect to Spatial Variables

Various classes of economical difference schemes have been developed for numerical solution of multi-dimensional non-stationary problems of the mathematical physics [Samarskii, 1989, Samarskii and Goolin, 1973, Marchuk, 1990, Samarskii and Vabishchevich, 1995a]. Realization of such schemes of splitting by spatial variables involves the solution of one-dimensional problems. We consider the following basic classes of economical schemes:

- schemes of alternating directions (the Peaceman–Rachford scheme, the Douglas–Rachford scheme);
- factorized schemes;

- schemes of summarized approximation (schemes of component-wise splitting).

To construct an additive difference scheme we rewrite equation (8.5) in the form

$$\frac{dv}{dt} + (A_1 + A_2)v = 0, \quad x \in \omega, \tag{8.9}$$

where, taking into account the form of the grid operator (8.4), we have

$$A_\alpha = \Lambda_\alpha, \quad \alpha = 1, 2.$$

For instance, we consider the factorized economical scheme for equation (8.9):

$$(E + \sigma\tau A_1)(E + \sigma\tau A_2)\frac{y_{n+1} - y_{n+1/2}}{\tau} + (A_1 + A_2)y_n = 0. \tag{8.10}$$

If $\sigma = 0.5$ then the factorized scheme (8.10) corresponds to the classical Peaceman–Rachford scheme. If $\sigma = 1$ then we have the Douglas–Rachford scheme (the scheme of stabilizing correction). Under the usual conditions of $\sigma \geq 0.5$ and $A_\alpha \geq 0$, $\alpha = 1, 2$, the factorized scheme (8.10) belongs to a class of unconditionally stable schemes.

Realization of economical schemes (e.g., scheme (8.10)) on parallel computers is provided by the use of parallel algorithms for solving systems of difference equations with the simplest three-diagonal matrices. This problem is widely considered in the linear algebra (see, e.g., [Ortega, 1988]). The most interesting versions of the sweep method (the Thomas algorithm) for systems of linear equations with a tape matrix are based on the ideas of domain decomposition. In this connection we note the pioneer paper [Yanenko et al., 1978] on parallelizing the sweep method.

To construct algorithms for three-dimensional non-stationary problems, we can apply the splitting with respect to a part of the variables. For instance, in equation (8.9) let the operator A_1 involve the difference derivatives by the variable x_1, and the operator A_2 by the remaining variables (x_2 and x_3). We may then talk about locally two-dimensional difference schemes (see, e.g., [Sukhinov, 1984]).

3. Regionally Additive Schemes of Two-Component Splitting

We begin investigation of the schemes of domain decomposition from consideration of regionally additive schemes in the case of two-component splitting, i.e., a computational domain is decomposed into two subdomains. In this case it is natural to use splitting schemes such as the classical schemes of alternating directions [Vabishchevich, 1994c]. Factorized schemes of domain decomposition [Samarskii and Vabishchevich, 1996a, Samarskii and Vabishchevich, 1997b] also belongs to a class of two-component splitting schemes.

3.1 Problem Statement

We consider in the rectangle Ω the boundary value problem

$$\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right) = 0, \quad x \in \Omega, \quad t > 0, \tag{8.11}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{8.12}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{8.13}$$

Carrying out discretization with respect to space we obtain the Cauchy problem

$$\frac{dv}{dt} + Av = 0, \quad x \in \omega, \tag{8.14}$$

$$v(x,0) = u_0(x), \quad x \in \omega. \tag{8.15}$$

The self-conjugate and positive operator A is defined by the following identity on a set of functions that vanish at the boundary nodes:

$$Ay = -\sum_{\alpha=1}^{2}(a_\alpha(x)y_{\overline{x}})_x, \quad x \in \omega. \tag{8.16}$$

The coefficients are given in a standard way.

3.2 Difference Operators of Domain Decomposition

Let the domain Ω consist of two separate sub-domains:

$$\Omega = \Omega_1 \cup \Omega_2.$$

Each of these sub-domains contain a set of non-intersecting sub-domains. This corresponds to two-color (red-black) splitting. This kind of splitting is implemented in decomposition of the original computational domain in one direction.

Construction of regionally additive schemes is based on the special representation of the operator of the original problem (8.16) and the use of these or other splitting schemes. The choice of a splitting operator and a splitting scheme corresponds to the choice of a certain computational scheme in separate sub-domains, in particular, to the choice of exchange conditions on the boundaries of the sub-domains.

Let ω_α be the nodes of the grid ω lying in the sub-domain $\Omega_\alpha, \alpha = 1,2$. We shall construct schemes of domain decomposition on the basis of unit splitting of the domain Ω. Introduce the functions

$$\chi_\alpha(x) > 0, \quad x \in \Omega_\alpha, \quad \chi_\alpha(x) = 0, \quad x \notin \Omega_\alpha, \quad \alpha = 1,2, \tag{8.17}$$

where

$$\chi_1(x) + \chi_2(x) = 1, \quad x \in \Omega. \tag{8.18}$$

In considering the class of decomposition schemes when for the operator A there is the additive representation

$$A = A_1 + A_2 \tag{8.19}$$

the operators A_α, $\alpha = 1,2$ are coupled with separate sub-domains, which split (8.17), (8.18), and with the solution of separate subproblem in the sub-domains Ω_α, $\alpha = 1,2$. The simplest difference decomposition scheme is given by the following definition of the operators A_α, $\alpha = 1,2$:

$$A_\alpha = \chi_\alpha A, \quad \alpha = 1,2. \tag{8.20}$$

We can also suggest the following representation of *decomposition operators*:

$$A_\alpha = A\chi_\alpha, \quad \alpha = 1,2. \tag{8.21}$$

It is clear that for such splittings that $A_\alpha \neq A_\alpha^*, \alpha = 1,2$.

Traditionally the symmetrical splitting (8.18), (8.19) is used when

$$A_\alpha y = -\sum_{\beta=1}^{2} (a_\beta^\alpha(x) y_{\bar{x}_\beta})_{x_\beta}, \quad x \in \omega, \quad \alpha = 1,2. \tag{8.22}$$

The grid operators $A_\alpha, \alpha = 1,2$ approximate the generating differential elliptic operators

$$-\sum_{\beta=1}^{2} \frac{\partial}{\partial x_\beta} \left(k_\beta \chi_\alpha(x) \frac{\partial u}{\partial x_\beta} \right), \quad \alpha = 1,2,$$

where the coefficients a_β^α are defined by analogy with a_β. Under the choice of (8.18),(8.19), (8.22), we have $A_\alpha = A_\alpha^* \geq 0, \quad \alpha = 1,2$.

3.3 Difference Schemes

It is simple to construct unconditionally stable difference schemes for solving equation (8.14) with the initial condition (8.15) when we use the splitting (8.18), (8.19). In this case we use the decomposition operators (8.22), and the operators $A_\alpha, \alpha = 1,2$, are self-conjugate and non-negative. Among such schemes we note the corresponding analogies of difference schemes of alternating directions, factorized schemes, schemes of summarized approximation. In this connection the questions of accuracy of an approximate solution and its dependence on the superposition domain $\Omega_3 = \Omega_1 \cap \Omega_2$, on the functions $\chi_\alpha(x)$, $\alpha = 1,2$ are of great interest. Schemes with non-symmetrical decomposition operators require special investigation. Here we consider the analogue of the classical *Douglas–Rachford scheme* , the so called scheme of stabilizing correction) we denote by y_n the difference solution at the time instant $t_n = n\tau$, where $\tau > 0$ is a time step. Passage to a new time level involves the scheme

$$\frac{y_{n+1/2} - y_n}{\tau} + A_1 y_{n+1/2} + A_2 y_n = 0,$$

$$\frac{y_{n+1} - y_{n+1/2}}{\tau} + A_2(y_{n+1} - y_n) = 0. \tag{8.23}$$

It is well known that if $A_\alpha = A_\alpha^* \geq 0$, $\alpha = 1,2$, then the scheme (8.23) is unconditionally stable (see, e.g., [Samarskii, 1989]) and for the difference solution the following estimate holds:

$$\|(E + \tau A_2) y_{n+1}\| \leq \|(E + \tau A_2) y_n\|. \tag{8.24}$$

If we define decomposition operators by formulas (8.20), (8.21) then the difference scheme (8.23) belongs to the considered class of schemes with operator factors, to symmetrizable difference schemes.

At first we consider scheme (8.23) with splitting of the form (8.19), (8.20), i.e.,

$$\frac{y_{n+1/2} - y_n}{\tau} + \chi_1(x) A y_{n+1/2} + \chi_2(x) A y_n = 0,$$

$$\frac{y_{n+1} - y_{n+1/2}}{\tau} + \chi_2(x) A (y_{n+1} - y_n) = 0. \tag{8.25}$$

Letting $A^{1/2} y_{n+\alpha/2}$ by $v_{n+\alpha/2}$, $\alpha = 1,2$, and multiplying equation (8.25) by $A^{1/2}$, we obtain

$$\frac{v_{n+1/2} - v_n}{\tau} + \widetilde{A}_1 v_{n+1/2} + \widetilde{A}_2 v_n = 0,$$

$$\frac{v_{n+1} - v_{n+1/2}}{\tau} + \widetilde{A}_2 (v_{n+1} - v_n) = 0, \tag{8.26}$$

where

$$\widetilde{A}_\alpha = A^{1/2}\chi_\alpha(x)A^{1/2}, \quad \widetilde{A}_\alpha = \widetilde{A}_\alpha^* \geq 0, \quad \alpha = 1,2. \tag{8.27}$$

For the symmetrized difference schemes (8.26) as well as for the scheme (8.25) the following estimate like (8.24) is valid:

$$\|(E + \tau A_2)y_{n+1}\|_A \leq \|(E + \tau A_2)y_n\|_A. \tag{8.28}$$

The difference schemes (8.23) with the operators (8.19), (8.21) is considered similarly:

$$\begin{aligned} \frac{y_{n+1/2} - y_n}{\tau} + A\chi_1(x)y_{n+1/2} + A\chi_2(x)y_n = 0,\\ \frac{y_{n+1} - y_{n+1/2}}{\tau} + A\chi_2(x)(y_{n+1} - y_n) = 0. \end{aligned} \tag{8.29}$$

In order to reduce the difference scheme (8.29) to the form (8.26) we can accept $A^{-1/2}y_{n+\alpha/2} = v_{n+\alpha/2}$, $\alpha = 1,2$ and multiply one from the left by $A^{-1/2}$. Then we have

$$\widetilde{A}_\alpha = A^{-1/2}\chi_\alpha(x)A^{-1/2}, \quad \widetilde{A}_\alpha = \widetilde{A}_\alpha^* \geq 0, \quad \alpha = 1,2,$$

and the corresponding estimate of stability takes the form

$$\|(E + \tau A_2)y_{n+1}\|_{A^{-1}} \leq \|(E + \tau A_2)y_n\|_{A^{-1}}. \tag{8.30}$$

The estimates of stability obtained (8.24), (8.28), (8.30) for the splitting scheme (8.19), (8.23) with decomposition operators in the form (8.22), (8.28) and (8.21) allow us to formulate the following statement.

THEOREM 8.1 *For the difference decomposition scheme (8.19), (8.23) with the operators A_α, $\alpha = 1,2$ in the form (8.22), (8.28), (8.21) the following estimate of stability with respect to the initial data holds:*

$$\|(E + \tau A_2)y_{n+1}\|_D \leq \|(E + \tau A_2)y_n\|_D, \tag{8.31}$$

where $D = A$, A^{-1}, and E, respectively.

3.4 Accuracy of Difference Solution

Generally convergence of difference schemes is investigated by means of the corresponding estimates of a difference solution with respect to the right hand side. For simplicity we shall assume that an exact solution of the difference problem (8.11)–(8.13) is sufficiently smooth.

To study the accuracy of the decomposition scheme (8.19), (8.22), (8.23) we write the corresponding problem for the error $z_n = y_n -$

u_n, $u_n = u(x, t_n)$, $x \in \omega$. Assuming $z_{n+1/2} = y_{n+1/2} - u_{n+1}$, $z_0 = 0$, from scheme (8.23) we obviously obtain

$$\frac{z_{n+1/2} - z_n}{\tau} + A_1 z_{n+1/2} + A_2 z_n = \psi_n^{(1)},$$

$$\frac{z_{n+1} - z_{n+1/2}}{\tau} + A_2(z_{n+1} - z_n) = \psi_n^{(2)}. \tag{8.32}$$

For the truncation errors we have

$$\begin{aligned} \psi_n^{(1)} &= -\frac{u_{n+1/2} - u_n}{\tau} - A_1 u_{n+1/2} - A_2 u_n, \\ \psi_n^{(2)} &= -A_2(u_{n+1} - u_n). \end{aligned} \tag{8.33}$$

LEMMA 8.1 *For the scheme (8.32) with $A_\alpha \geq 0$, $\alpha = 1, 2$, the following estimate of stability with respect to the right hand side holds:*

$$\|(E + \tau A_2) z_{n+1}\| \leq \sum_{k=0}^{n} \tau(\|\psi_k^{(1)}\| + \|\psi_k^{(2)}\|). \tag{8.34}$$

Proof. To obtain estimate (8.34), we write scheme (8.32) in the form

$$(E + \tau A_1) z_{n+1/2} = (E - \tau A_2) z_n + \tau \psi_n^{(1)}, \tag{8.35}$$

$$(E + \tau A_2) z_{n+1} = z_{n+1/2} + \tau A_2 z_n + \tau \psi_n^{(2)}. \tag{8.36}$$

Taking into account equality (8.35) we can represent the right hand side of equation (8.36) in the following form:

$$\begin{aligned} z_{n+1/2} + \tau A_2 z_n &= z_{n+1/2} + \frac{1}{2}(E + \tau A_2) z_n - \frac{1}{2}(E - \tau A_2) z_n \\ &= z_{n+1/2} + \frac{1}{2}(E + \tau A_2) z_n \\ &\quad - \frac{1}{2}(E + \tau A_1) z_{n+1/2} - \frac{\tau}{2}\psi_n^{(1)} \\ &= \frac{1}{2}(E + \tau A_2) z_n + \frac{1}{2}(E - \tau A_1) z_{n+1/2} - \frac{\tau}{2}\psi_n^{(1)}. \end{aligned}$$

Substituting the previous expression in the formula (8.36), we obtain

$$\begin{aligned} &(E + \tau A_2) z_{n+1} \\ &\quad = \frac{1}{2}(E + \tau A_2) z_n + \frac{1}{2}(E - \tau A_1) z_{n+1/2} - \frac{\tau}{2}\psi_n^{(1)} + \tau \psi_n^{(2)}. \end{aligned} \tag{8.37}$$

For any operators $S \geq 0$ the following estimate is valid:

$$\|(E - S)y\| \leq \|(E + S)y\|.$$

From (8.35) we obtain

$$\|(E+\tau A_1)z_{n+1/2}\| \leq \|(E-\tau A_2)z_n\| + \tau\|\psi_n^{(1)}\|.$$

With (8.37) taken into account, it follows that

$$\begin{aligned}
&\|(E+\tau A_2)z_{n+1}\| \\
&\leq \frac{1}{2}\|(E+\tau A_2)z_n\| + \frac{1}{2}\|(E+\tau A_1)z_{n+1/2}\| + \frac{\tau}{2}\|\psi_n^{(1)}\| + \tau\|\psi_n^{(2)}\| \\
&\leq \|(E+\tau A_2)z_n\| + \tau\|\psi_n^{(1)}\| + \tau\|\psi_n^{(2)}\|.
\end{aligned}$$

Thus for scheme (8.32) the inequality

$$\|(E+\tau A_2)z_{n+1}\| \leq \|(E+\tau A_2)z_n\| + \tau(\|\psi_n^{(1)}\| + \|\psi_n^{(2)}\|).$$

holds, which yields the estimate of stability (8.34).

Consider a truncation error of the difference solution of the decomposition scheme (8.23). Let us write the truncation error (8.33) in a more convenient form. Taking into account the equality

$$A_1 u_{n+1} + A_2 u_n = A u_{n+1} - A_2(u_{n+1} - u_n),$$

for sufficiently smooth solutions we have

$$\begin{aligned}
\psi_n^{(1)} &= O(\tau + |h|^2) + \tau A_2\left(\frac{\partial u}{\partial t} + O(\tau)\right), \\
\psi_n^{(2)} &= -\tau A_2\left(\frac{\partial u}{\partial t} + O(\tau)\right),
\end{aligned} \tag{8.38}$$

where $|h|^2 = h_1^2 + h_2^2$.

Taking into account the representation (8.22) we have the inequality

$$\|A_2\frac{\partial u}{\partial t}\| \leq M_1\|\sum_{\alpha=1}^{2}|\chi_2(x)_{\bar{x}_\alpha}|\;\| \leq M_1\|\chi_2(x)\|_A.$$

Therefore estimate (8.34) for the error takes the form

$$\|(E+\tau A_2)z_{n+1}\| \leq M(\tau + |h|^2 + \tau\|\chi_2(x)\|_A). \tag{8.39}$$

Thus the convergence rate of the difference decomposition scheme depends on the width of the sub-domains Ω_1 and Ω_2 superposition (the term with $\|\chi_2(x)\|_A$ in estimate (8.39)). In the limiting case of the minimal sub-domains superposition, when the width of the superposition area is $O(|h|)$, the convergence rate is $O(|h|^{-1/2}\tau + |h|^2)$.

For the error of the decomposition scheme (8.19), (8.20), (8.23) we have the difference scheme

$$\begin{aligned} \frac{z_{n+1/2} - z_n}{\tau} + \chi_1 A z_{n+1/2} + \chi_2 A z_n = \psi_n^{(1)}, \\ \frac{z_{n+1} - z_{n+1/2}}{\tau} + \chi_2 A (z_{n+1} - z_n) = \psi_n^{(2)}, \end{aligned} \tag{8.40}$$

where from representation (8.20) and formula (8.33), we have

$$\begin{aligned} \psi_n^{(1)} &= -\frac{u_{n+1/2} - u_n}{\tau} - \chi_1 A u_{n+1/2} + \chi_2 A u_n, \\ \psi_n^{(2)} &= -\chi_2 A (u_{n+1} - u_n). \end{aligned} \tag{8.41}$$

By virtue of this representation we obtain for the error

$$\psi_n^{(1)} = O(\tau + |h|^2), \quad \psi_n^{(2)} = O(\tau).$$

Carrying out symmetrization (see (8.24), (8.25)), from the difference scheme (8.40) we obtain the estimate for the error:

$$\|(E + \tau A_2) z_{n+1}\|_A \le \sum_{k=0}^{n} \tau (\|\psi_k^{(1)}\|_A + \|\psi_k^{(2)}\|_A). \tag{8.42}$$

Taking into account formula (8.41), from inequality (8.42) we obtain the estimate which is similar to (8.39):

$$\|(E + \tau A_2) z_{n+1}\|_A \le M(\tau + |h|^2 + \tau \|\chi_2(x)\|_A).$$

From the study of the difference scheme (8.19), (8.21), (8.23) we obtain the estimate

$$\|(E + \tau A_2) z_{n+1}\|_{A^{-1}} \le M(\tau + |h|^2 + \tau \|\chi_2(x)\|_A).$$

Thus we have just proved the following statement:

THEOREM 8.2 *For the difference decomposition scheme (8.19), (8.23) with the operators A_α, $\alpha = 1, 2$ in the form (8.20), (8.21), (8.22) the estimate of the difference solution error is correct*

$$\|(E + \tau A_2) z_{n+1}\|_D \le M(\tau + |h|^2 + \tau \|\chi_2(x)\|_A), \tag{8.43}$$

where $D = A$, A^{-1}, and E, respectively.

3.5 Factorized Schemes of Domain Decomposition

The schemes of domain decomposition mentioned above involve classical schemes of alternating directions, or schemes of stabilizing correction. A more general class of schemes of alternating directions is a class of factorized schemes [Samarskii, 1989]. In this case in splitting the problem operator into two operators (two-component splitting) the *factorized scheme* has the form

$$(E+\sigma\tau A_1)(E+\sigma\tau A_2)\frac{y_{n+1}-y_n}{\tau}+Ay_n=0, \quad n=0,1,\dots . \tag{8.44}$$

If $\sigma = 1$ the scheme (8.44) corresponds to the scheme (8.23) mentioned above. If $\sigma = 0.5$ then we have the analogue of the classical *Peaceman–Rachford scheme* . Similarly to the case $\sigma = 1$ considered above we can show unconditional stability of the factorized scheme (8.44) under the standard restriction $\sigma \geq 0.5$ and conditional convergence in the sense of fulfilment of the estimate like (8.43).

It is possible to construct schemes of domain decomposition like (8.44) when the general theory of stability works directly. To this end we define the decomposition operators by the relations

$$A_1 = A\chi_1(\mathbf{x}), \quad A_2 = \chi_2(\mathbf{x})A. \tag{8.45}$$

We note specially that in this case equality (8.19) is not correct.

We shall carry out the study of the regionally additive scheme (8.44) with the operator (8.45) with the help of the general theory of stability of difference schemes with the use of the results mentioned above for the usual schemes with weights. We write scheme (8.44) with the operator (8.45) in the canonical form of two-level difference schemes:

$$B\frac{y_{n+1}-y_n}{\tau}+Ay_n=0, \quad n=0,1,\dots, \tag{8.46}$$

where the operator

$$B = E+\sigma\tau A+\sigma\tau(A\chi_1-\chi_1 A)+\sigma^2\tau^2 A\chi_1(1-\chi_1)A. \tag{8.47}$$

The operator $A\chi_1 - \chi_1 A$ is skew-symmetric. Taking into account the definition of the functions (8.17), (8.18), it is clear that the last term in relation (8.47) is non-negative. Hence $B \geq E + \sigma\tau A$. Consequently the following statement is valid:

THEOREM 8.3 *Assume*

$$\sigma \geq \frac{1}{2}-\frac{1}{\Delta\tau}, \tag{8.48}$$

where $A \leq \Delta E$. Then for the regionally additive factorized scheme (8.46) with the operator (8.47) the estimate of stability with respect to the initial data is correct

$$\|y_{n+1}\|_A \leq \|y_0\|_A. \tag{8.49}$$

In order to estimate the accuracy of the schemes of decomposition we need to engage estimates of stability with respect to the right hand side in addition to inequality (8.49). For the error of solution $z_n = y_n - u_n$, $x \in \omega$, we have

$$B\frac{z_{n+1} - z_n}{\tau} + Az_n = \psi_n, \quad n = 0, 1, \dots, \tag{8.50}$$

where the truncation error is

$$\psi_n = -B\frac{u_{n+1} - u_n}{\tau} - Au_n. \tag{8.51}$$

It is natural to assume that $z_0 = 0$, i.e., we can confine ourselves to the corresponding estimate of stability of the difference solution with respect to right hand side only.

If inequality (8.48) holds then for the difference scheme (8.50) the following estimate of stability with respect to the right hand side is valid (see theorem 2.16):

$$\|z_{n+1}\|_A \leq \|\psi_0\|_{A^{-1}} + \|\psi_n\|_{A^{-1}} + \sum_{k=1}^{n} \tau \|\psi_{\bar{t},k}\|_{A^{-1}}. \tag{8.52}$$

For the truncation error we obtain

$$\psi_n = O(\tau^2 + |h|^2) + (\sigma - \frac{1}{2})\tau A\frac{\partial u}{\partial t} + \sigma\tau A\chi_1\frac{\partial u}{\partial t} + \cdots .$$

Thus in general the choice of $\sigma = 0.5$ does not allow us to increase the accuracy. Substitution of ψ_n into estimate (8.52) is reduced to the following statement.

THEOREM 8.4 *For the error of the factorized scheme (8.46) with the operator (8.47) under condition (8.48) the following estimate is correct:*

$$\|z_{n+1}\|_A \leq M(\tau + |h|^2 + \tau\|\chi_2(x)\|_A). \tag{8.53}$$

Thus the accuracy of the factorized scheme of decomposition is the same as the accuracy of the regionally additive schemes considered above which were constructed by analogy with the classical schemes of alternating directions. It is necessary only to note the essential circumstance that the estimate (8.53) was obtained in the norm not associated with the splitting operators.

4. Regionally Additive Schemes of Summarized Approximation

We consider difference schemes of domain decomposition for the approximate solution of non-stationary problems of the decomposition of the initial domain into a large number of sub-domains. We investigate (see [Vabishchevich, 1997]) two classes of difference schemes: schemes of component wise splitting (operator analog of the ordinary locally one-dimensional schemes) and additive averaged schemes in which for the most part the specific features of parallel computers is considered.

4.1 Model Problem

Approaches which use non-iterative schemes (regionally additive difference schemes)are developed. We have considered above the regionally additive analogs of the classical schemes of alternating directions for the problems when splitting into two operators is possible. With the use of modern high-power multiprocessor systems it is necessary to use schemes with splitting into a larger number of operators (multicolor decomposition). This situation occurs in decomposition with respect to two and more variables.

For such problems it is necessary to use the analogs of unconditionally stable locally one-dimensional schemes (schemes of component wise splitting). Here we consider two basic classes of regionally additive difference schemes. The first of these [Samarskii, 1989] involves the use of two-level schemes of component wise splitting with variable weight factors. We study the dependence of accuracy on weight factors and on domain decomposition. We can obtain similar results for additive averaged difference schemes of domain decomposition [Gordeziani and Meladze, 1974, Samarskii and Vabishchevich, 1995a] which are actually parallel version of the schemes of component wise splitting. We can consider the schemes of component wise splitting as one-iteration implementation of usual implicit schemes by means of the classical synchronous (additive) version of the *Shwarz method* . In this sense the additive averaged schemes can be associated with nonsynchronous (multiplicative) version of the Shwarz method.

Let us consider a model two-dimensional problem for the parabolic equation

$$\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right) = 0, \quad x = (x_1, x_2) \in \Omega, \quad t > 0, \qquad (8.54)$$

supplemented with the simplest homogeneous boundary conditions of the first kind:

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0 \tag{8.55}$$

and the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{8.56}$$

After discretization with respect to space we obtain a differential-difference problem for the equation

$$\frac{dv}{dt} + Av = 0, \quad x \in \omega, \tag{8.57}$$

with the initial condition

$$v(x,0) = u_0(x), \quad x \in \Omega,$$

and the operator $A = A^* > 0$ in the space H.

4.2 Regionally Additive Schemes

Let the domain Ω contain p separate sub-domains:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_p.$$

Let ω_α be the nodes of the grid ω, lying in the sub-domain Ω_α, $\alpha = 1, 2, \ldots, p$. We define the functions

$$\chi_\alpha(x) > 0, \quad x \in \Omega_\alpha, \quad \chi_\alpha(x) = 0, \quad x \notin \Omega_\alpha, \quad \alpha = 1, 2, \ldots, p, \tag{8.58}$$

where

$$\sum_{\alpha=1}^{p} \chi_\alpha(x) = 1, \quad x \in \Omega. \tag{8.59}$$

We consider a class of decomposition schemes in multi-component splitting when

$$A = \sum_{\alpha=1}^{p} A_\alpha. \tag{8.60}$$

We shall again use the following three basic classes of decomposition operators:

$$A_\alpha = \chi_\alpha A, \quad \alpha = 1, 2, \ldots, p, \tag{8.61}$$

$$A_\alpha = A\chi_\alpha, \quad \alpha = 1, 2, \ldots, p, \tag{8.62}$$

$$A_\alpha y = -\sum_{\beta=1}^{2} (a_\beta^\alpha(x) y_{\bar{x}_\beta})_{x_\beta}, \quad x \in \omega, \quad \alpha = 1, 2, \ldots, p. \tag{8.63}$$

In the latter case the grid operators $A_\alpha, \alpha = 1, 2, \ldots, p$ approximate the differential degenerate elliptic operators

$$-\sum_{\beta=1}^{2} \frac{\partial}{\partial x_\beta} \left(k_\beta \chi_\alpha(x) \frac{\partial u}{\partial x_\beta} \right), \quad \alpha = 1, 2, \ldots, p.$$

In using of the scheme of component wise splitting for approximate solution of the Cauchy problem for equation (8.57), (8.58) the passage to a new time level is realized with the help of the scheme

$$\frac{y_{n+\alpha/p} - y_{n+(\alpha-1)/p}}{\tau} + A_\alpha(\sigma y_{n+\alpha/p} + (1-\sigma) y_{n+(\alpha-1)/p}) = 0, \quad \alpha = 1, 2, \ldots, p, \tag{8.64}$$

where $0 \leq \sigma \leq 1$ is a constant weight.

If we define the decomposing operators by (8.61), then scheme (8.64) takes the form

$$\frac{y_{n+\alpha/p} - y_{n+(\alpha-1)/p}}{\tau} + \chi_\alpha A(\sigma y_{n+\alpha/p} + (1-\sigma) y_{n+(\alpha-1)/p}) = 0, \quad \alpha = 1, 2, \ldots, p. \tag{8.65}$$

The difference scheme (8.65) can be interpreted as a scheme with a variable time step (the local step is equal to $\tau\chi_\alpha$).

The difference scheme (8.65) belongs to the class of symmetrizable difference schemes. Letting $A^{1/2} y_{n+\alpha/p} = v_{n+\alpha/p}$, $\alpha = 1, 2, ..., p$, and multiplying on the left (8.65) by $A^{1/2}$, we obtain the scheme

$$\frac{v_{n+\alpha/p} - v_{n+(\alpha-1)/p}}{\tau} + \tilde{A}_\alpha(\sigma y_{n+\alpha/p} + (1-\sigma) y_{n+(\alpha-1)/p}) = 0, \tag{8.66}$$

$$\alpha = 1, 2, \ldots, p,$$

where

$$\widetilde{A} = A^{1/2} \chi_\alpha(x) A^{1/2}, \quad \widetilde{A}_\alpha = \widetilde{A}_\alpha^* \geq 0, \quad \alpha = 1, 2, \ldots, p. \tag{8.67}$$

The general results of the theory of stability are used for investigation of stability of scheme (8.66) with symmetric difference operators of the form (8.67).

THEOREM 8.5 *The domain decomposition difference scheme of component wise splitting (8.64) is stable in the space H_A under the condition that $\sigma \geq 0.5$, and for the difference solution the following estimate holds:*

$$\|y_{n+1}\|_A \leq \|y_0\|_A. \tag{8.68}$$

P r o o f. After symmetrization of (8.65) we obtain the scheme (8.66). For each $\alpha = 1, 2, \ldots, p$ the following estimate (see Theorem 2.8) is valid:

$$\|v_{n+1}\| \le \|v_0\|, \tag{8.69}$$

From the latter inequality the statement of the theorem about stability of the regionally additive scheme (8.64) with respect to the initial data follows,

In addition to the scheme of component wise splitting (8.64) the additive averaged scheme is worthy of separate consideration. This scheme has great possibilities of parallelizing. Passage to a new time level is realized in the following way. At first we determine the auxiliary functions $\tilde{y}_{n+\alpha/p}$, $\alpha = 1, 2, \ldots, p$, from p problems:

$$\frac{\tilde{y}_{n+\alpha/p} - y_{n+(\alpha-1)/p}}{\tau} + A_\alpha(\sigma \tilde{y}_{n+\alpha/p} + (1-\sigma) y_{n+(\alpha-1)/p}) = 0, \tag{8.70}$$

$$\alpha = 1, 2, \ldots, p,$$

and then we determine the approximate solution at a new time level

$$y_{n+1} = \frac{1}{p} \sum_{\alpha=1}^{p} \tilde{y}_{n+\alpha/p}. \tag{8.71}$$

It is most essential that stage (8.70) is carried out in each separate sub-domains regardless of other sub-domains. This asynchronicity is fundamental for constructing parallel algorithms.

THEOREM 8.6 *The additive averaged difference scheme of domain decomposition (8.70), (8.71) is stable on the condition of* $\sigma \ge 0.5$ *in the space* H_A*, and for difference solution estimate (8.68) holds.*

P r o o f. Investigation is carried out completely similarly to the case of the scheme of component wise splitting (8.64).

4.3 Convergence of Schemes of Decomposition

In investigation of the regionally additive schemes we must pay main attention to the question of accuracy, in particular, to the dependence of an approximate solution error on decomposition, on the functions χ_α, $\alpha = 1, 2, \ldots, p$. For the schemes of two-component splitting we obtain above the optimal estimates of convergence rate of the decomposition schemes which have the form $O(\tau |h|^{-1/2})$ under the minimal superposition.

Assume that the exact solution of the differential problem (8.54)–(8.56) is sufficiently smooth. In order to investigate the accuracy of the difference scheme of component wise splitting (8.64), we write down the corresponding problem for the error $z_n = y_n - u_n$, $x \in \omega$. We can consider this scheme not as the scheme of summarized approximation, but as the scheme of total approximation. In addition we interpret it as the scheme with locally variable time steps.

We shall consider the grid function $y_{n+\alpha/p}$ as the approximate solution of the initial problem at the time moment

$$t_{n+\alpha/p}(x) = \sum_{\beta=1}^{\alpha} \chi_\beta(x) t_{n+1} + \left(1 - \sum_{\beta=1}^{\alpha} \chi_\beta(x)\right) t_n.$$

For error on the intermediate steps we denote $z_{n+\alpha/p} = y_{n+\alpha/p} - u_{n+\alpha/p}$, where

$$u_{n+\alpha/p} = \sum_{\beta=1}^{\alpha} \chi_\beta(x) u_{n+1} + \left(1 - \sum_{\beta=1}^{\alpha} \chi_\beta(x)\right) u_n. \tag{8.72}$$

From scheme (8.65) we obtain the following difference scheme for the error:

$$\frac{z_{n+\alpha/p} - z_{n+(\alpha-1)/p}}{\tau} + \chi_\alpha A(\sigma z_{n+\alpha/p} + (1-\sigma) z_{n+(\alpha-1)/p}) = \psi_{n+\alpha/p},$$
$$\alpha = 1, 2, \ldots, p, \tag{8.73}$$

where for the truncation error we have

$$\psi_{n+\alpha/p} = -\frac{u_{n+\alpha/p} - u_{n+(\alpha-1)/p}}{\tau} - \chi_\alpha A(\sigma u_{n+\alpha/p} + (1-\sigma) u_{n+(\alpha-1)/p}). \tag{8.74}$$

THEOREM 8.7 *Assume $\sigma \geq 0.5$. Then for the error of the difference scheme of decomposition (8.64) the following estimate is valid:*

$$\|z_{n+1}\|_A \leq M \left(\left(\sigma - \frac{1}{2}\right) \tau + \tau^2 + |h|^2 + \tau \sum_{\alpha=1}^{p} \|A\chi_\alpha(x)\| \right), \tag{8.75}$$

where the constant M does not depend on the grid.

Proof. Taking into account representation (8.72), we reduce expression (8.74) to the form

$$\psi_{n+\alpha/p} = \chi_\alpha \tilde{\psi}_{n+\alpha/p}, \tag{8.76}$$

$$\tilde{\psi}_{n+\alpha/p} = -\frac{u_{n+1} - u_n}{\tau} - A(\sigma u_{n+\alpha/p} + (1-\sigma) u_{n+(\alpha-1)/p}). \tag{8.77}$$

From (8.72) it follows that

$$\begin{aligned}
&\sigma u_{n+\alpha/p} + (1-\sigma)u_{n+(\alpha-1)/p} \\
&\qquad = \sigma\tau\chi_\alpha \frac{u_{n+1}-u_n}{\tau} + u_{n+(\alpha-1)/p} \\
&\qquad = \frac{u_{n+1}+u_n}{2} + \left(\xi_\alpha(x) + \left(\sigma - \frac{1}{2}\right)\right)\tau\frac{u_{n+1}-u_n}{\tau},
\end{aligned}$$

where

$$\xi_\alpha(x) = \sigma(\chi_\alpha(x)-1) + \sum_{\beta=1}^{\alpha-1}\chi_\beta(x).$$

Taking this into account in formula (8.77), we obtain the estimate

$$\tilde{\psi}_{n+\alpha/p} = O\left(\left(\sigma-\frac{1}{2}\right)\tau + \tau^2 + |h|^2\right) + \tau A\left(\xi_\alpha\frac{\partial u}{\partial t}\right), \qquad (8.78)$$

where the first term on the right hand side is truncation error in using the usual scheme with weights without domain decomposition. From estimate (8.78) we see that the value $\tilde{\psi}_{n+\alpha/p}$ has the first order by τ regardless of the choice of the weight σ.

Let us obtain the corresponding estimate for the error considered as the solution of equation (8.73). We shall use the special representation of the truncation error in the form of (8.76). We denote

$$\begin{aligned}
w_{n+\alpha/p} &= \sigma z_{n+\alpha/p} + (1-\sigma)z_{n+(\alpha-1)/p} \\
&= \left(\sigma-\frac{1}{2}\right)\tau z_{t,\alpha} + \frac{z_{n+\alpha/p} - z_{n+(\alpha-1)/p}}{2}, \\
z_{t,\alpha} &= \frac{z_{n+\alpha/p} - z_{n+(\alpha-1)/p}}{\tau}.
\end{aligned}$$

If $\sigma \geq 0.5$, then multiplying equation (8.73) by $Aw_{n+\alpha/p}$ scalarly we obtain

$$\frac{1}{2\tau}\left(\|z_{n+\alpha/p}\|_A^2 - \|z_{n+(\alpha-1)/p}\|_A^2\right) + \|\chi_\alpha^{1/2}Aw_{n+\alpha/p}\|^2 \leq$$

$$\leq (\chi_\alpha\tilde{\psi}_{n+\alpha/p}, Aw_{n+\alpha/p}).$$

For the right hand side of the latter inequality we have the estimate

$$(\chi_\alpha\tilde{\psi}_{n+\alpha/p}, Aw_{n+\alpha/p}) \leq \|\chi_\alpha^{1/2}Aw_{n+\alpha/p}\|^2 + \frac{1}{4}\|\chi_\alpha^{1/2}\tilde{\psi}_{n+\alpha/p}\|^2.$$

Therefore

$$\|z_{n+\alpha/p}\|_A^2 \leq \|z_{n+(\alpha-1)/p}\|_A^2 + \frac{\tau}{2}\|\chi_\alpha^{1/2}\tilde{\psi}_{n+\alpha/p}\|^2.$$

It follows that

$$\|z_{n+1}\|_A^2 \le \|z_n\|_A^2 + \frac{\tau}{2}\sum_{\alpha=1}^{p} \|\chi_\alpha^{1/2}\tilde{\psi}_{n+\alpha/p}\|^2.$$

Thus on the strength of the equality $\|z_0\|_A = 0$ we obtain

$$\|z_{n+1}\|_A^2 \le \sum_{k=1}^{n} \frac{\tau}{2}\sum_{\alpha=1}^{p} \|\chi_\alpha^{1/2}\tilde{\psi}_{k+\alpha/p}\|^2.$$

Taking into account representation (8.78) for the truncation error, we obtain the estimate of convergence (8.75) which is being proved.

In the limiting case of the minimal superposition of sub-domains (the width of superposition region is $O(|h|)$) the convergence rate is $O(|h|^{-3/2}\tau + |h|^2)$.

Similar statement holds for the convergence rate of the parallel version of regionally additive schemes.

THEOREM 8.8 *Let $\sigma \ge 0.5$. Then for the error of the additive averaged scheme of decomposition (8.70), (8.71) estimate (8.75) holds.*

Proof. For the error we have the problem

$$\frac{\tilde{z}_{n+\alpha/p} - z_{n+(\alpha-1)/p}}{\tau} + \chi_\alpha A(\sigma\tilde{z}_{n+\alpha/p} + (1-\sigma)z_{n+(\alpha-1)/p}) = \psi_{n+\alpha/p},$$
$$\alpha = 1, 2, \ldots, p, \tag{8.79}$$

$$z_{n+1} = \frac{1}{p}\sum_{\alpha=1}^{p} \tilde{z}_{n+\alpha/p}, \tag{8.80}$$

where

$$\psi_{n+\alpha/p} = -\frac{\tilde{u}_{n+\alpha/p} - u_{n+(\alpha-1)/p}}{\tau} - \chi_\alpha A(\sigma\tilde{u}_{n+\alpha/p} + (1-\sigma)z_{n+(\alpha-1)/p}). \tag{8.81}$$

We denote

$$\tilde{u}_{n+\alpha/p} = p\chi_\alpha(x)u_{n+1} + (1 - p\chi_\alpha(x))u_n. \tag{8.82}$$

Formula (8.82) conforms to splitting (8.59) and to definition of approximate solution on a new time level by formula (8.71).

Substituting expression (8.82) into formula (8.81), we obtain for the truncation error representation (8.76) with

$$\tilde{\psi}_{n+\alpha/p} = -\frac{u_{n+1} - u_n}{\tau}$$
$$- A\left(\frac{u_{n+1} + u_n}{2} + \left(\chi_\alpha(x) + \left(\sigma - \frac{1}{2}\right)\right)\tau\frac{u_{n+1} - u_n}{\tau}\right).$$

In what follows we reason by analogy with the proof of Theorem 8.7.

We have paid most attention to the decomposition schemes with the choice of splitting in the form of (8.59)–(8.61). The versions with decomposition operators defined by formulas (8.62), (8.63) are considered similarly. Convergence is established not in the space H_A but in H_D, where $D = A^{-1}$ and E, respectively. In addition, it is natural to interpret the corresponding schemes as the schemes of summarized approximation.

5. Vector Additive Schemes of Domain Decomposition

At the present time various classes of additive difference schemes of total approximation under arbitrary multi-component splitting of a problem operator have been constructed. Here we consider the construction of schemes of domain decomposition on the basis of vector additive schemes [Samarskii and Vabishchevich, 1995b]. It is necessary to pay special attention to schemes of second-order approximation with respect to time.

5.1 Problem Statement

We have constructed above different types of schemes of domain decomposition like the classical difference schemes of alternating direction, locally one-dimensional schemes (schemes of component wise splitting). Having in mind parallel implementation of decomposition methods, we must consider regionally additive schemes (schemes of splitting into sub-domains) with an arbitrary number of sub-domains (groups of sub-domains). At the same time, most strong results are obtained in the convergence of approximate solution for schemes like that of alternating directions, i.e., under two-component splitting. Schemes like locally one-dimensional (schemes of component wise splitting) are constructed on the basis of the concept of summarized approximation. In general, these schemes have a low accuracy.

In the theory of splitting schemes (see, e.g., [Samarskii and Vabishchevich, 1997a]) significant attention is paid to new classes of multi-component schemes which in contrast to the schemes of component wise splitting are related to schemes of total (not summarized) approximation.

In the papers of V.N. Abrashin and A.N. Iakoubenia (see, e.g., [Abrashin, 1990, Abrashin and Mukha, 1992, Abrashin and Iakoubenia, 1988, Iakoubenia, 1988]) a class of unconditionally stable schemes of total approximation is suggested for arbitrary multi-component splitting. Such schemes are studied in the paper [Vabishchevich, 1996] with the help of the general theory of stability of difference schemes. Here we shall consider this class of schemes using the methods of domain decomposition. The accuracy of regionally additive difference schemes of the first- and second-order approximation with respect to time is investigated. We consider model boundary value problem for the second-order parabolic equation.

We study the Cauchy problem for the following differential-difference first-order equation:

$$\frac{dv}{dt} + Av = 0, \qquad x \in \omega, \tag{8.83}$$

$$v(x,0) = u_0(x), \quad x \in \omega. \tag{8.84}$$

The operator A is positive and self-adjoint, i.e., $A = A^* > 0$ in H. After decomposition of the domain into p subdomains one has the additive representation

$$A = \sum_{\alpha=1}^{p} A_\alpha. \tag{8.85}$$

The specific form of the decomposition operators is mentioned above when the schemes of domain decomposition were considered using two- and multi-component splitting. We shall use the symmetric splitting when $A_\alpha = A_\alpha^* \geq 0, \ \alpha = 1, 2, \dots, p$.

5.2 Vector Scheme

When we study *vector additive schemes* the problem for the vector $V = \{v^{(1)}, v^{(2)}, \dots, v^{(p)}\}$ is considered instead of the problem (8.83), (8.84). Each component of the vector V is defined by solving the one-type problems

$$\frac{dv^{(\alpha)}}{dt} + \sum_{\beta=1}^{p} A_\beta v^{(\beta)} = 0, \quad 0 < t \leq T, \tag{8.86}$$

$$v^{(\alpha)}(0) = u_0, \quad \alpha = 1, 2, \dots, p. \tag{8.87}$$

It is obvious that $v^{(\alpha)}(t) = v(t)$, and so any component of $V(t)$ is a solution of the original problem (8.83), (8.84).

We begin the consideration from the simplest difference scheme with weights for the system of equations (8.86), (8.87)

$$(E + \sigma\tau A_\alpha)\frac{y_{n+1}^{(\alpha)} - y_n^{(\alpha)}}{\tau} + \sum_{\beta=1}^{p} A_\beta y_n^{(\beta)} = 0, \quad \alpha = 1, 2, \dots, p. \tag{8.88}$$

Passage to a new time level in scheme (8.88) involves inversion of the operators $E+\sigma\tau A_\alpha$, $\alpha = 1,2,\dots,p$, on each time step like the standard (scalar) versions of additive difference schemes.

Let an approximate solution of problem (8.83), (8.84) be the function $\bar{y}$ defined by the following formula:

$$A\bar{y} = \sum_{\alpha=1}^{p} A_\alpha y^{(\alpha)}, \tag{8.89}$$

rather than the separate components of the vector $Y = \{y^{(1)}, y^{(2)}, ..., y^{(p)}\}$. Then scheme (8.88) is absolutely stable on the condition that $\sigma \geq p/2$, and for the approximate solution the following estimate holds:

$$\|A\bar{y}_{n+1}\| \leq \|A\bar{y}_n\|. \tag{8.90}$$

Thus stability is established in the norm of the Hilbert space H_D, $D = A^2$.

5.3 Convergence of the Scheme of Decomposition

The study of the convergence of difference schemes for non-stationary problems is based on stability estimates of difference solution with respect to the right hand side. The special features of the vector additive schemes of decomposition considered do not allow us to use the simplest estimates with respect only to the right hand side which follow immediately from solution stability with respect to initial data.

We determine via $z_n^{(\alpha)} = y_n^{(\alpha)} - u_n$, $\alpha = 1,2,\dots,p$,the error of the corresponding solution component at the time moment t_n, where $u_n = u(x,t_n)$, $x \in \omega$, is an exact sufficiently smooth solution of the original differential problem. Substituting $y_n^{(\alpha)}$ into (8.88), we obtain the system of equations for the errors

$$(E+\sigma\tau A_\alpha)\frac{z_{n+1}^{(\alpha)} - z_n^{(\alpha)}}{\tau} + \sum_{\beta=1}^{p} A_\beta z_n^{(\beta)} = \psi_n^{(\alpha)}, \quad \alpha = 1,2,\dots,p. \tag{8.91}$$

For the truncation errors we have

$$\psi_n^{(\alpha)} = -(E+\sigma\tau A_\alpha)\frac{u_{n+1}^{(\alpha)} - u_n^{(\alpha)}}{\tau} - \sum_{\beta=1}^{p} A_\beta u_n^{(\beta)}, \quad \alpha = 1,2,\dots,p.$$

Hence

$$\psi_n^{(\alpha)} = -\sigma\tau A_\alpha \frac{\partial u}{\partial t} + O(\tau + |h|^2), \quad \alpha = 1,2,\dots,p. \tag{8.92}$$

In order to obtain the corresponding estimate of convergence of the difference scheme (8.88), we rewrite the system of equations (8.91) in the form

$$(E+\sigma\tau A_\alpha)\frac{z_{n+1}^{(\alpha)}-z_n^{(\alpha)}}{\tau} + \frac{1}{2}\sum_{\beta=1}^{p} A_\beta(z_{n+1}^{(\beta)}+z_n^{(\beta)}) - \frac{\tau}{2}\sum_{\beta=1}^{p} A_\beta \frac{z_{n+1}^{(\beta)}-z_n^{(\beta)}}{\tau} = \psi_n^{(\alpha)},$$

$$\alpha = 1,2,\dots,p.$$

Taking the dot product of the each of the equations ($\alpha = 1,2,\dots,p$) with the term $2\tau A_\alpha z_t^{(\alpha)}$, $z_t^{(\alpha)} = (z_{n+1}^{(\alpha)} - z_n^{(\alpha)})/\tau$, and then summing them, we obtain the equality

$$\begin{aligned} &2\tau\sum_{\alpha=1}^{p}(A_\alpha z_t^{(\alpha)}, z_t^{(\alpha)}) \\ &+ 2\sigma\tau^2\sum_{\alpha=1}^{p}(A_\alpha z_t^{(\alpha)}, A_\alpha z_t^{(\alpha)}) - \tau^2\left(\sum_{\alpha=1}^{p}A_\alpha z_t^{(\alpha)}, \sum_{\alpha=1}^{p}A_\alpha z_t^{(\alpha)}\right) \\ &+ \left(\sum_{\alpha=1}^{p}A_\alpha(z_{n+1}^{(\alpha)}+z_n^{(\alpha)}), \sum_{\alpha=1}^{p}A_\alpha(z_{n+1}^{(\alpha)}-z_n^{(\alpha)})\right) = 2\tau\sum_{\alpha=1}^{p}(A_\alpha z_t^{(\alpha)}, \psi_n^{(\alpha)}). \end{aligned} \tag{8.93}$$

For the right hand side of expression (8.93) we use the estimate

$$\sum_{\alpha=1}^{p}(A_\alpha z_t^{(\alpha)}, \psi_n^{(\alpha)}) \le \sum_{\alpha=1}^{p}(A_\alpha z_t^{(\alpha)}, z_t^{(\alpha)}) + \frac{1}{4}\sum_{\alpha=1}^{p}(A_\alpha \psi_n^{(\alpha)}, \psi_n^{(\alpha)}). \tag{8.94}$$

Taking into account the inequality

$$\left(\sum_{\alpha=1}^{p}A_\alpha v^{(\alpha)}\right)^2 \le p\sum_{\alpha=1}^{p}(A_\alpha v^{(\alpha)})^2,$$

we have

$$\begin{aligned} &2\sigma\sum_{\alpha=1}^{p}(A_\alpha v^{(\alpha)}, A_\alpha v^{(\alpha)}) - \left(\sum_{\alpha=1}^{p}A_\alpha v^{(\alpha)}, \sum_{\alpha=1}^{p}A_\alpha v^{(\alpha)}\right) \\ &\qquad \ge (2\sigma - p)\sum_{\alpha=1}^{p}(A_\alpha v^{(\alpha)}, A_\alpha v^{(\alpha)}) \ge 0, \end{aligned} \tag{8.95}$$

under the above-mentioned constraint $\sigma \geq p/2$. In addition,

$$
\begin{aligned}
&\left(\sum_{\alpha=1}^{p} A_\alpha(z_{n+1}^{(\alpha)} + z_n^{(\alpha)}), \sum_{\alpha=1}^{p} A_\alpha(z_{n+1}^{(\alpha)} - z_n^{(\alpha)})\right) \\
&= \left(\sum_{\alpha=1}^{p} A_\alpha(z_{n+1}^{(\alpha)}, \sum_{\alpha=1}^{p} A_\alpha(z_{n+1}^{(\alpha)}\right) - \left(\sum_{\alpha=1}^{p} A_\alpha(z_n^{(\alpha)}, \sum_{\alpha=1}^{p} A_\alpha(z_n^{(\alpha)}\right) \\
&= \|A\bar{z}_{n+1}\|^2 - \|A\bar{z}_n\|^2,
\end{aligned}
$$

where, similarly to (8.89),

$$
A\bar{z}_n = \sum_{\alpha=1}^{p} A_\alpha z_n^{(\alpha)}, \tag{8.96}
$$

i.e., $\bar{z}_n = \bar{y}_n - u_n$. Taking into account (8.94), (8.95), we can pass from the equality (8.93) to the inequality

$$
\|A\bar{z}_{n+1}\|^2 \leq \|A\bar{z}_n\|^2 + \frac{\tau}{2}\sum_{\alpha=1}^{p}(A_\alpha\psi_n^{(\alpha)}, \psi_n^{(\alpha)}). \tag{8.97}
$$

Using (8.92), from (8.97) we obtain the desired estimate for the error.

THEOREM 8.9 *Assume that $\sigma \geq p/2$. Then the vector additive scheme (8.85), (8.88) of domain decomposition converges unconditionally, and for the error (8.96) the following estimate is valid:*

$$
\|A\bar{z}_{n+1}\| \leq M_1(\tau + |h|^2 + M_2\tau \max_\alpha \|A\chi_\alpha\|), \tag{8.98}
$$

where the constants M_1 and M_2 depend only on the exact solution of the problem.

Proof. To establish the correctness of inequality (8.98) it is necessary to substitute representation (8.92) for the truncation error in the estimate (8.97) and to take into account the choice of the decomposition operators.

The result mentioned represents the essential dependence of the convergence rate of the vector scheme of domain decomposition on splitting and on the width of the superposition sub-domains. It makes sense to single out the limiting case of non-overlapping sub-domains. Then from estimate (8.98) we obtain the estimate $O(\tau|h|^{-3/2} + +|h|^2)$ for the accuracy of the difference solution. Notice that in the case of $p = 2$ the schemes like that of alternating directions have substantially higher accuracy: $O(\tau|h|^{-1/2} + |h|^2)$.

5.4 Other Decomposition Operators

Consider the use of vector additive schemes like (8.88) in the case of non-self-adjoint decomposition operators. The decomposition operators A_α, $\alpha = 1, 2, \dots, p$, can be given in the form

$$A_\alpha = \chi_\alpha A, \quad \alpha = 1, 2, \dots, p. \tag{8.99}$$

We can also use the following representation for the decomposition operators:

$$A_\alpha = A\chi_\alpha, \quad \alpha = 1, 2, \dots, p. \tag{8.100}$$

It is clear that for splitting (8.99) and (8.100) we have $A_\alpha \neq A_\alpha^*$, $\alpha = 1, 2, \dots, p$. It should be recalled once again that the different types of definition of the decomposition operators conform to the different types of boundary conditions on the sub-domain borders.

Investigation of the stability and convergence of the vector additive schemes of domain decomposition (8.85), (8.88), (8.99) and (8.85), (8.88), (8.100) is carried out on the basis of symmetrization of these schemes. For instance, consider the scheme (8.85), (8.88), (8.99) which has the form

$$(E + \sigma\tau\chi_\alpha A)\frac{y_{n+1}^{(\alpha)} - y_n^{(\alpha)}}{\tau} + \sum_{\beta=1}^{p} \chi_\beta A y_n^{(\beta)} = 0, \quad \alpha = 1, 2, \dots, p. \tag{8.101}$$

Let us define the functions $v_n^{(\beta)} = A^{1/2} y_n^{(\beta)}$. Multiply each of equations (8.101) by $A^{1/2}$. Then we can rewrite system (8.101) in the form

$$(E + \sigma\tau\tilde{A}_\alpha)\frac{v_{n+1}^{(\alpha)} - v_n^{(\alpha)}}{\tau} + \sum_{\beta=1}^{p} \tilde{A}_\beta v_n^{(\beta)} = 0, \quad \alpha = 1, 2, \dots, p, \tag{8.102}$$

where $\tilde{A}_\alpha = A^{1/2}\chi_\alpha A^{1/2}$, and also $\tilde{A}_\alpha = \tilde{A}_\alpha^* \geq 0$. Scheme (8.102) belongs to the class of the schemes mentioned above. In particular, similarly to formula (8.89) we define $\tilde{v}$ from

$$\tilde{A}\bar{v} = \sum_{\alpha=1}^{p} \tilde{A}_\alpha v^{(\alpha)}.$$

Therefore we may say that the function $\tilde{y}$ defined by the formula

$$A\bar{y} = \sum_{\alpha=1}^{p} \chi_\alpha A y^{(\alpha)},$$

is an approximate solution of the original problem.

In the same way, from the decomposition scheme (8.85), (8.88), (8.100) we obtain the scheme (8.102), where $v_n^{(\beta)} = A^{-1/2} y_n^{(\beta)}$ and $\tilde{A}_\alpha = A^{-1/2} \chi_\alpha A^{-1/2}$, and the approximate solution is defined by the simplest formula

$$\bar{y} = \sum_{\alpha=1}^{p} \chi_\alpha y^{(\alpha)}.$$

The following statement is valid.

THEOREM 8.10 *Assume $\sigma \geq p/2$. Then the vector additive scheme of domain decomposition (8.85), (8.88), (8.99) and (8.85), (8.88), (8.100) converges unconditionally, and for the error the following estimate is correct:*

$$\|A\bar{z}_{n+1}\|_D \leq M_1(\tau + |h|^2 + M_2 \tau \max_\alpha \|A\chi_\alpha\|), \tag{8.103}$$

where $D = A$ for scheme (8.85), (8.88), (8.99), and $D = A^{-1}$ for scheme (8.85), (8.88), (8.100).

Comparing (8.98) and (8.103) we conclude that the use of different decomposition operators allows one to obtain schemes convergent in different norms.

5.5 Schemes of Second-Order Approximation with Respect to Time

The accuracy of the vector schemes of first-order approximation mentioned above depends strongly on the domain decomposition. We now consider schemes of second-order approximation with respect to time. We investigate only the simplest three-level schemes.

For problems (8.86), (8.87) with the self-adjoint operators of decomposition A_α, $\alpha = 1, 2, \ldots, p$ we shall use the scheme

$$\frac{y_{n+1}^{(\alpha)} - y_{n-1}^{(\alpha)}}{2\tau} + \sigma\tau^2 A_\alpha \frac{y_{n+1}^{(\alpha)} - 2y_n^{(\alpha)} + y_{n-1}^{(\alpha)}}{\tau^2} + \sum_{\beta=1}^{p} A_\beta y_n^{(\beta)} = 0, \quad \alpha = 1, 2, \ldots, p. \tag{8.104}$$

If $\sigma \geq 1/4$ then the scheme (8.85), (8.104) is unconditionally stable.

The corresponding problem for the error takes the form

$$\frac{z_{n+1}^{(\alpha)} - z_{n-1}^{(\alpha)}}{2\tau} + \sigma\tau^2 A_\alpha \frac{z_{n+1}^{(\alpha)} - 2z_n^{(\alpha)} + z_{n-1}^{(\alpha)}}{\tau^2} + \sum_{\beta=1}^{p} A_\beta z_n^{(\beta)} = \psi_n^{(\alpha)}, \quad \alpha = 1, 2, \ldots, p, \tag{8.105}$$

where

$$\psi_n^{(\alpha)} = -\frac{u_{n+1}^{(\alpha)} - u_{n-1}^{(\alpha)}}{2\tau} - \sigma\tau^2 A_\alpha \frac{u_{n+1}^{(\alpha)} - 2u_n^{(\alpha)} + u_{n-1}^{(\alpha)}}{\tau^2} - \sum_{\beta=1}^{p} A_\beta u_n^{(\beta)}.$$

Similarly to expression (8.92) we obtain

$$\psi_n^{(\alpha)} = -\sigma\tau^2 A_\alpha \frac{\partial^2 u}{\partial t^2} + O(\tau^2 + |h|^2), \quad \alpha = 1, 2, \ldots, p, \tag{8.106}$$

i.e., the original problem is approximated with the second order in time.

It is convenient to introduce the following notation:

$$v_{n+1}^{(\alpha)} = z_{n+1}^{(\alpha)} + z_n^{(\alpha)}, \quad w_{n+1}^{(\alpha)} = z_{n+1}^{(\alpha)} - z_n^{(\alpha)},$$

such that $w_{n+1}^{(\alpha)} + w_n^{(\alpha)} = v_{n+1}^{(\alpha)} - v_n^{(\alpha)}$:

$$\frac{w_{n+1}^{(\alpha)} + w_n^{(\alpha)}}{2\tau} + \sigma A_\alpha (w_{n+1}^{(\alpha)} - w_n^{(\alpha)})$$
$$+ \frac{1}{4}\sum_{\beta=1}^{p} A_\beta (v_{n+1}^{(\beta)} + v_n^{(\beta)} - (w_{n+1}^{(\beta)} - w_n^{(\beta)})) = \psi_n^{(\alpha)},$$

$$\alpha = 1, 2, \ldots, p.$$

Taking the dot product of each equation with the term $A(w_{n+1}^{(\alpha)} + w_n^{(\alpha)})$, and summing them up, we obtain

$$\frac{1}{2\tau}\sum_{\alpha=1}^{p} (A(w_{n+1}^{(\alpha)} + w_n^{(\alpha)}), w_{n+1}^{(\alpha)} + w_n^{(\alpha)})$$
$$+ \sigma \sum_{\alpha=1}^{p} (\|A(w_{n+1}^{(\alpha)}\|^2 - \|A(w_n^{(\alpha)}\|^2)$$
$$- \frac{1}{4}\left(\left\| \sum_{\alpha=1}^{p} A w_{n+1}^{(\alpha)} \right\|^2 - \left\| \sum_{\alpha=1}^{p} A w_n^{(\alpha)} \right\|^2 \right)$$
$$+ \frac{1}{4}\left(\left\| \sum_{\alpha=1}^{p} A v_{n+1}^{(\alpha)} \right\|^2 - \left\| \sum_{\alpha=1}^{p} A v_n^{(\alpha)} \right\|^2 \right)$$
$$= \sum_{\alpha=1}^{p} (\psi_n^{(\alpha)}, A(w_{n+1}^{(\alpha)} + w_n^{(\alpha)})).$$

Taking into account the relation

$$\sum_{\alpha=1}^{p}(\psi_n^{(\alpha)}, A(w_{n+1}^{(\alpha)} + w_n^{(\alpha)})) \leq \frac{1}{2\tau}\sum_{\alpha=1}^{p}(A(w_{n+1}^{(\alpha)} + w_n^{(\alpha)}), w_{n+1}^{(\alpha)} + w_n^{(\alpha)}) + \frac{\tau}{2}\sum_{\alpha=1}^{p}(A_\alpha\psi_n^{(\alpha)}, \psi_n^{(\alpha)}),$$

we obtain the inequality

$$\mathcal{E}_{n+1} \leq \mathcal{E}_n + \frac{\tau}{2}\sum_{\alpha=1}^{p}(A_\alpha\psi_n^{(\alpha)}, \psi_n^{(\alpha)}), \tag{8.107}$$

where

$$\mathcal{E}_n = \frac{1}{4}\|\sum_{\alpha=1}^{p} A_\alpha v_n^{(\alpha)}\|^2 + \sigma\sum_{\alpha=1}^{p}\|A_\alpha w_n^{(\alpha)}\|^2 - \frac{1}{4}\|\sum_{\alpha=1}^{p} A_\alpha w_n^{(\alpha)}\|^2.$$

Using the relation

$$\|\sum_{\alpha=1}^{p} A_\alpha w_n^{(\alpha)}\|^2 \leq p\sum_{\alpha=1}^{p}\|A_\alpha w_n^{(\alpha)}\|^2,$$

we obtain the estimate

$$\mathcal{E}_n \geq \frac{1}{4}\|\sum_{\alpha=1}^{p} A_\alpha v_n^{(\alpha)}\|^2, \tag{8.108}$$

provided that $\sigma \geq p/4$.

Using the inequalities (8.107), (8.108) and the expression for the truncation error (8.106) with the corresponding definition of $y_1^{(\alpha)}$, $\alpha = 1, 2, \ldots, p$, we obtain the desired estimate for the error. Thus we have just proved the following statement:

THEOREM 8.11 *The vector additive scheme of domain decomposition (8.85), (8.104) converges unconditionally for $\sigma \geq p/4$, and for the error the following estimate is valid:*

$$\|A(\bar{z}_{n+1} + \bar{z}_n)\| \leq M_1(\tau^2 + |h|^2 + M_2\tau^2 \max_\alpha \|A\chi_\alpha\|). \tag{8.109}$$

Thus the regionally additive scheme constructed has the second order of approximation with respect to time. If we use the decomposition without superposition of sub-domains, then we have the estimate like

$O(\tau^2|h|^{-3/2}+|h|^2)$. Such schemes of decomposition can be recommended for use in practice.

Likewise regionally additive schemes of second-order approximation in time with the decomposition operators (8.99), (8.100) are considered. Similarly to Theorem 8.10, their convergence is established in the space H_D, where $D = A$ and $D = A^{-1}$, respectively.

Note that the investigations developed can be continued if we consider problem (8.83), (8.84) with a non-self-adjoint operator A. Then (see [Samarskii and Vabishchevich, 1995b]) decomposition operators are constructed by considering the self-adjoint and skew-symmetric part of the operator A.

6. Schemes of Domain Decomposition for Second-Order Evolutionary Equations

The problems are studied which deal with the construction of domain decomposition schemes for second-order evolutionary equations. As an example, we consider the boundary value problem for the wave equation. Unconditionally stable vector additive schemes are constructed.

6.1 Introduction

Various classes of regionally additive difference schemes have been constructed above for approximate solution of boundary value problems for parabolic equations. The schemes of summarized approximation for first-order evolutionary equations are used. First of all, we note the schemes of component wise splitting (locally one-dimensional) of the first and the second order of accuracy. Note also the additive averaged schemes which can be considered as asynchronous (parallel) versions of the schemes of component wise splitting. The vector additive schemes (multi-component schemes of alternating directions) relate to the schemes of total approximation, i.e., each intermediate problem approximates an original one.

Certain difficulties occur when we try to construct the splitting schemes for the second-order evolutionary equations. In this connection mention should be made of the main paper [Samarskii, 1964]. The later researches have been summarized in the paper [Samarskii and Gordeziani, 1978]. In particular, the advantages of the vector additive schemes are evident from the observation that the problems of constructing splitting schemes for the second-order evolutionary equations (see, e.g., [Asmolik, 1996]) are not much more complex than for the first-order evolutionary equations.

Here we consider the Cauchy problem for the second-order evolutionary equation with its self-adjoint operator in the real grid Hilbert space. Investigation is performed with the help of the theory of stability of operator-difference schemes and the general methodological principle of constructing the difference schemes given capacity the principle of regularization [Samarskii, 1989, Samarskii, 1967b]. For the vector problem obtained three-level schemes of splitting are constructed. The corresponding estimates of stability with respect to initial data are proved and are on the right hand side. As a typical example, we take the multidimensional wave equation for which we construct regionally additive schemes (schemes of domain decomposition).

6.2 Problem Statement

We shall consider real grid functions y which act in the finite-dimensional real Hilbert space H with the usual notation of scalar product and norm. We seek the solution $u(t) \in H$ of the Cauchy problem for the second-order evolutionary equation

$$\frac{d^2u}{dt^2} + Au = f(t), \quad 0 < t \leq T, \tag{8.110}$$

with the initial conditions

$$u(0) = u_0, \tag{8.111}$$

$$\frac{du}{dt}(0) = u_1. \tag{8.112}$$

Let us study only the simplest case when the operator A is stationary, positive and self-adjoint, i.e., $A \neq A(t) = A^* > 0$.

In constructing difference schemes we shall use the following estimate of stability with respect to the initial data and right hand side for problem (8.110)–(8.112):

$$\|u(0)\|_* \leq \|u_0\|_A + \|u_1\| + \int_0^t \|f(s)\| ds, \tag{8.113}$$

where

$$\|u\|_*^2 \equiv \|u\|_A^2 + \left\|\frac{du}{dt}\right\|^2 .$$

In order to prove the estimate it is sufficient in the space H to take the dot product of equation (8.110) with du/dt.

The problem of constructing additive schemes for problem (8.110)–(8.112)is formulated. Assume that for the operator A the following additive representation is valid:

$$A = \sum_{\alpha=1}^{p} A^{(\alpha)}, \quad A^{(\alpha)} \neq A^{(\alpha)}(t) = (A^{(\alpha)})^* \geq 0, \quad \alpha = 1, 2, \dots, p. \tag{8.114}$$

Additive difference schemes are constructed on the basis of representation (8.114). In order to pass from one time level t_n to another time level $t_{n+1} = t_n + \tau$, where $\tau > 0$ is the time step, we solve the problems for the separate operators $A^{(\alpha)}, \alpha = 1, 2, \dots, p$, in the additive expansion (8.114). Therefore the original problem reduces to p more simple subproblems.

6.3 Vector Problem

Given the vector $\mathbf{u} = \{u^{(1)}, u^{(2)}, \dots, u^{(p)}\}$, each separate component is defined by problem solution of the same type problems

$$\frac{d^2 u^{(\alpha)}}{dt^2} + \sum_{\beta=1}^{p} A^{(\beta)} u^{(\beta)} = f(t), \quad 0 < t \leq T, \tag{8.115}$$

with the initial conditions

$$u^{(\alpha)}(0) = u_0, \tag{8.116}$$

$$\frac{du^{(\alpha)}}{dt}(0) = u_1, \quad \alpha = 1, 2, \dots, p. \tag{8.117}$$

It is clear that $u^{(\alpha)}(t) = u(t)$, and therefore as a solution of the original problem (8.110)–(8.112) we can take any component of the vector $\mathbf{u}(t)$.

Let us carry out a preliminary transformation of the system of equations (8.115). Multiplying each of equations (8.115) from the left by $A^{(\alpha)}$, $\alpha = 1, 2, \dots, p$, we obtain the system of equations

$$\mathbf{D} \frac{d^2 \mathbf{u}}{dt^2} + \mathbf{A}\mathbf{u} = \Phi(t), \quad 0 < t \leq T, \tag{8.118}$$

where for the operator matrix $\mathbf{A}$ we have the representation $\mathbf{A} \equiv [A^{(\alpha)} A^{(\beta)}]$, and for the vector on the right hand side we have $\Phi(t) = \{A^{(1)} f(t), A^{(2)}) f(t), \dots, A^{(p)} f(t)\}$, $\mathbf{D} \equiv [A^{(\alpha)} \delta_{\alpha\beta}]$ is the diagonal operator matrix, where $\delta_{\alpha\beta}$ is the Kronecker symbol.

Taking into account conditions (8.116), (8.117) we supplement the system of equations (8.118) with the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0 \equiv \{u_0, u_0, \dots, u_0\}, \tag{8.119}$$

$$\frac{d\mathbf{u}}{dt}(0) = \mathbf{u}_1 \equiv \{u_1, u_1, \ldots, u_1\}. \tag{8.120}$$

It is natural to consider problem (8.118)–(8.120) in the vector Hilbert space H^p, where the scalar product is defined by the expression

$$(\mathbf{u}, \mathbf{v}) = \sum_{\alpha=1}^{p} (w^{(\alpha)}, v^{(\alpha)}).$$

Since the operators $A^{(\alpha)}, \alpha = 1, 2, \ldots, p$ are self-adjoint the operator $\mathbf{A}$ in the space H^p is also self-adjoint. Furthermore we immediately see that

$$(\mathbf{A}\mathbf{u}, \mathbf{u}) = \left(\left(\sum_{\alpha=1}^{p} A^{(\alpha)} u^{(\alpha)} \right)^2, 1 \right). \tag{8.121}$$

From (8.121) it follows that the operator $\mathbf{A}$ is non-negative. With the splitting of (8.114) the operator $\mathbf{D}$ is also non-negative and self-adjoint. We arrive at the vector problem (8.118)–(8.120) with the operators satisfying the conditions

$$\mathbf{D} = \mathbf{D}^* \geq 0, \quad \mathbf{A} = \mathbf{A}^* \geq 0. \tag{8.122}$$

It is characteristics of the vector problem considered that the operators $\mathbf{D}$ and $\mathbf{A}$ are non-negative. This circumstance does not allow one to use directly the general results of the theory of stability of three-level difference schemes.

6.4 Difference Schemes with Weights

Let us give the estimate of stability with respect to the initial data and the right hand side of the usual scheme with weights for problem (8.110)–(8.112). For the approximate solution of this problem we shall use the second-order scheme

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} + A(\sigma y_{n+1} + (1 - 2\sigma) y_n + \sigma y_{n-1}) = f_n, \ n = 1, 2, \ldots, \tag{8.123}$$

where y_0, y_1 are defined.

For the difference scheme (8.123) the following *a priori* estimate holds:

$$\|y_{n+1}\|_* \leq \|y_n\|_* + \tau \|f_n\|, \tag{8.124}$$

where

$$\|y_{n+1}\|_*^2 \equiv \left\| \frac{y_{n+1} - y_n}{\tau} \right\|_{E + (\sigma - \frac{1}{4})\tau^2 A}^2 + \left\| \frac{y_{n+1} + y_n}{2} \right\|_A^2 .$$

Estimate (8.124) is consistent with estimate (8.113) for the solution of differential problem. If $\sigma \geq 0.25$ then it provides unconditional stability of the difference scheme with weights (8.123) with respect to the initial data and the right hand side.

For the proof it is convenient to introduce the new functions

$$v_n = \frac{y_n + y_{n-1}}{2}, \quad w_n = \frac{y_n - y_{n-1}}{\tau},$$

where

$$y_n = \frac{v_{n+1} + v_n}{2} - \frac{\tau^2}{4} \frac{w_{n+1} - w_n}{\tau}.$$

Using this notation we can rewrite scheme (8.123) in the form

$$B\frac{w_{n+1} - w_n}{\tau} + A\left(\frac{v_{n+1} + v_n}{2}\right) = f_n, \quad n = 1, 2, \ldots, \tag{8.125}$$

where the operator

$$B = E + (\sigma - \frac{1}{4})\tau^2 A.$$

Taking the dot product of equation (8.125) with

$$\tau(w_{n+1} + w_n) = 2(v_{n+1} - v_n),$$

we obtain the equality

$$\|w_{n+1}\|_B^2 - \|w_n\|_B^2 + \|v_{n+1}\|_A^2 - \|v_n\|_A^2 = \tau(f_n, w_{n+1} + w_n).$$

The left hand side of this equality can be represented in the form

$$\begin{aligned} &\|w_{n+1}\|_B^2 - \|w_n\|_B^2 + \|v_{n+1}\|_A^2 - \|v_n\|_A^2 \\ &\qquad = (\|y_{n+1}\|_* - \|y_n\|_*)(\|y_{n+1}\|_* + \|y_n\|_*), \end{aligned}$$

and for the scalar product on right hand side we obtain the estimate

$$(f_n, w_{n+1} + w_n) \leq \|f_n\|(\|w_{n+1}\| + \|w_n\|) \leq \|f_n\|(\|y_{n+1}\|_* + \|y_n\|_*).$$

From this follows the *a priori* estimate (8.124) being proved on the basis of which the convergence and accuracy of the difference schemes are investigated.

6.5 Additive Schemes

We shall use the general methodological principle, the so called principle of regularization, for constructing the difference schemes of the given capacity [Samarskii, 1967b, Samarskii, 1989]. In order to obtain an unconditionally stable schemes we:

- select some simplest (generating) difference scheme which is not unconditionally stable;
- write the scheme selected in a canonical form for which tests of stability are known from the general theory of stability of operator-difference schemes;
- construct an unconditionally stable scheme with the help of perturbation of operators of the generating difference scheme according to the tests of stability.

It is natural to use an explicit scheme for constructing vector additive schemes. For problem (8.118)–(8.120) we obtain the scheme

$$\mathbf{D}\frac{\mathbf{y}_{n+1} - 2\mathbf{y}_n + \mathbf{y}_{n-1}}{\tau^2} + \mathbf{A}\mathbf{y}_n = \Phi_n, \quad n = 1, 2, \ldots, \tag{8.126}$$

which is obtained from the explicit scheme

$$\frac{y_{n+1}^{(\alpha)} - 2y_n^{(\alpha)} + y_{n-1}^{(\alpha)}}{\tau^2} + \sum_{\beta=1}^{p} A^{(\beta)} y_n^{(\beta)} = f_n, \quad n = 1, 2, \ldots, \tag{8.127}$$

for problem (8.115)–(8.117).

Let us introduce

$$\mathbf{v}_n = \frac{\mathbf{y}_n + \mathbf{y}_{n-1}}{2}, \quad \mathbf{w}_n = \frac{\mathbf{y}_n - \mathbf{y}_{n-1}}{\tau}.$$

Then the vector difference scheme (8.126) can be written in the canonical form

$$\mathbf{B}\frac{\mathbf{w}_{n+1} - \mathbf{w}_n}{\tau} + \mathbf{A}(\frac{\mathbf{v}_{n+1} + \mathbf{v}_n}{2}) = \Phi_n, \quad n = 1, 2, \ldots, \tag{8.128}$$

where the operator B satisfies the relation

$$\mathbf{B} = \mathbf{D} - \frac{1}{4}\tau^2 \mathbf{A}.$$

If $\mathbf{B} \geq 0$, i.e.,

$$\mathbf{D} \geq \frac{1}{4}\tau^2 \mathbf{A}, \tag{8.129}$$

by analogy with the proof of Lemma 2 we obtain the estimate

$$\|\mathbf{y}_{n+1}\|_* \leq \|\mathbf{y}_n\|_* + \tau \|\Phi_n\|, \tag{8.130}$$

where

$$\|\mathbf{y}_{n+1}\|_*^2 \equiv \left\| \frac{\mathbf{y}_{n+1} - \mathbf{y}_n}{\tau} \right\|^2_{\mathbf{D} - \frac{1}{4}\tau^2 \mathbf{A}} + \left\| \frac{\mathbf{y}_{n+1} + \mathbf{y}_n}{2} \right\|^2_{\mathbf{A}}$$

In order to explain conditions (8.129) upper estimates of the operator terms $A^{(\alpha)}, \alpha = 1, 2, \dots, p$ are used. Let the following conditions hold

$$(A^{(\alpha)}u^{(\alpha)}, u^{(\alpha)}) \le \Delta^{(\alpha)} \|u^{(\alpha)}\|^2, \quad \alpha = 1, 2, \dots, p,$$

and $\Delta = \max\limits_{\alpha} \Delta^{(\alpha)}$. Then for the non-negative self-adjoint operators $A^{(\alpha)}, \alpha = 1, 2, \dots, p$ we have

$$(A^{(\alpha)}u^{(\alpha)}, u^{(\alpha)}) \ge \frac{1}{\Delta} \|A^{(\alpha)}u^{(\alpha)}\|^2, \quad \alpha = 1, 2, \dots, p.$$

Taking into account representation (8.121) we rewrite the inequality (8.129) in the form

$$\begin{aligned}(\mathbf{Du}, \mathbf{u}) &- \frac{1}{4}\tau^2 (\mathbf{Au}, \mathbf{u}) \\ &= \sum_{\alpha=1}^{p} (A^{(\alpha)}u^{(\alpha)}, u^{(\alpha)}) - \frac{1}{4}\tau^2 \left(\left(\sum_{\alpha=1}^{p} A^{(\alpha)}u^{(\alpha)} \right)^2, 1 \right) \ge 0.\end{aligned} \tag{8.131}$$

From the inequality

$$\left(\sum_{\alpha=1}^{p} A^{(\alpha)}u^{(\alpha)} \right)^2 \le p \sum_{\alpha=1}^{p} \left(A^{(\alpha)}u^{(\alpha)} \right)^2,$$

and relation (8.131), we obtain

$$\begin{aligned}\sum_{\alpha=1}^{p} (A^{(\alpha)}u^{(\alpha)}, u^{(\alpha)}) &- \left(\left(\sum_{\alpha=1}^{p} A^{(\alpha)}u^{(\alpha)} \right)^2, 1 \right) \\ &\ge \left(\frac{1}{\Delta} - \frac{p}{4}\tau^2 \right) \sum_{\alpha=1}^{p} \|A^{(\alpha)}u^{(\alpha)}\|^2 \ge 0.\end{aligned}$$

Therefore for the explicit scheme (8.127) we obtain the following restrictions on the time step:

$$\tau \le \sqrt{\frac{4}{p\Delta}}. \tag{8.132}$$

It is natural to compare conditions (8.132) with the usual restrictions on an admissible time step for the explicit scheme for problem (8.110)–(8.112).

In accordance with the principle of regularization we perturb operators of the difference scheme to eliminate the limitations on the time step. In the scheme (8.126) we take the operator

$$\mathbf{D} = [(A^{(\alpha)} + \sigma\tau^2 (A^{(\alpha)})^2)\delta_{\alpha\beta}]. \tag{8.133}$$

In this case we keep a diagonal operator matrix and the second order of approximation. Such regularization corresponds to the use of the scheme with weights

$$\frac{y_{n+1}^{(\alpha)} - 2y_n^{(\alpha)} + y_{n-1}^{(\alpha)}}{\tau^2} + \sum_{\alpha \neq \beta = 1}^{p} A^{(\beta)} y_n^{(\beta)}$$
$$+ A^{(\alpha)}(\sigma y_{n+1}^{(\alpha)} + (1-2\sigma) y_n^{(\alpha)} + \sigma y_{n-1}^{(\alpha)}) = f_n, \quad n=1,2,\dots, \tag{8.134}$$

instead the explicit scheme (8.127). Implementation of the regularized scheme (8.134) is fully identifical to the implementation of the standard splitting schemes, and involves the inversion of the operators $(E+ + \sigma\tau^2 A^{(\alpha)})$, $\alpha = 1,2,\dots,p$.

Similarly to the explicit scheme (8.127), for scheme (8.126), (8.133) condition (8.129) leads to the requirement

$$(\mathbf{Du}, \mathbf{u}) - \frac{1}{4}\tau^2 (\mathbf{Au}, \mathbf{u})$$
$$\geq \left(\frac{1}{\Delta} + (\sigma - \frac{p}{4})\tau^2\right) \sum_{\alpha=1}^{p} \|A^{(\alpha)} u^{(\alpha)}\|^2 \geq 0.$$

Hence estimate (8.130) for scheme (8.126), (8.133) holds provided that

$$\sigma \geq \frac{p}{4} - \frac{1}{\Delta\tau^2}. \tag{8.135}$$

Under more tough restrictions $\sigma \geq 0.25p$, estimate (8.130) holds for any steps τ. Thus we have just proved the following statement.

THEOREM 8.12 *For the regularized vector difference scheme (8.134) the a priori estimate (8.130) holds provided (8.135).*

6.6 Stability of Additive Schemes

The *a priori* estimate (8.130) is an estimate of stability with respect to the initial data and the right hand side provided $\|\mathbf{u}\|_*$ is a norm. Under conditions (8.122) and restrictions (8.129) $\|\mathbf{u}\|_*$ is a semi-norm. To pass to a norm we can use the following restrictions that are tougher than (8.129):

$$\mathbf{D} > \frac{1}{4}\tau^2 \mathbf{A}.$$

We do not study various possibilities in this field (see [Samarskii and Goolin, 1973]). Confine ourselves to the case $\sigma \geq 0.25p$ when the following condition holds:

$$(\mathbf{Du}, \mathbf{u}) - \frac{1}{4}\tau^2 (\mathbf{Au}, \mathbf{u}) \geq \sum_{\alpha=1}^{p} (A^{(\alpha)} u^{(\alpha)}, u^{(\alpha)}). \tag{8.136}$$

Taking into account the limitations (8.136), from inequality (8.130) we obtain the estimate

$$\sqrt{\sum_{\alpha=1}^{p} \left\| \frac{y_{n+1}^{(\alpha)} - y_n^{(\alpha)}}{\tau} \right\|_{A^{(\alpha)}}^2} \leq \|\mathbf{y}_1\|_* + \sum_{k=1}^{n} \tau \|\Phi_k\|, \tag{8.137}$$

providing the stability of the separate component $y^{(\alpha)}$ of the vector $\mathbf{y}$ assuming the positivity of the operator term $A^{(\alpha)}, \alpha = 1, 2, \dots, p$. If $A^{(\alpha)} > 0, \alpha = 1, 2, \dots, p$, then from estimate (8.137) it follows that we can take any component $y^{(\alpha)}, \alpha = 1, 2, \dots, p$ in the capacity of the approximate solution of the scalar problem (8.110)–(8.112).

Similarly to the vector additive schemes of domain decomposition for the second-order evolutionary equations, we shall use the special definition of the approximate solution as a linear combination of separate components under more general conditions of nonnegativity of the operator terms $A^{(\alpha)}, \alpha = 1, 2, \dots, p$. Let the components $u^{(\alpha)}$, $\alpha = 1, 2, \dots, p$ be given. Denote the scalar function $\bar{u}$ by the relation

$$A\bar{u} = \sum_{\alpha=1}^{p} A^{(\alpha)} u^{(\alpha)}. \tag{8.138}$$

Taking into account formula (8.121) we obtain

$$(\mathbf{Au}, \mathbf{u}) = \|A\bar{u}\|^2. \tag{8.139}$$

Using relations (8.138), (8.139), from (8.130) it follows that

$$\|A\bar{u}_{n+1/2}\| \leq \|\mathbf{y}_1\|_* + \sum_{k=1}^{n} \tau \|\Phi_k\| \tag{8.140}$$

for the approximate solution at the half-integer node,

$$\bar{u}_{n+1/2} = \frac{1}{2}(\bar{u}_{n+1} + \bar{u}_n).$$

Estimate (8.140) can be considered as an analog of the corresponding *a priori* estimate for the differential problem (8.110)–(8.112) which we obtain by taking the dot product of the equation with Au.

6.7 The Wave Equation

Consider the simplest two-dimensional boundary value problem in the rectangle

$$\Omega = \{x \mid x = (x_1, x_2), \quad 0 < x_\alpha < l_\alpha, \quad \alpha = 1, 2\}.$$

We seek the solution $w(x,t)$ of the equation

$$\frac{\partial^2 w}{\partial t^2} - \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha}\left(k(x)\frac{\partial w}{\partial x_\alpha}\right) = f(x,t), \quad x \in \Omega, \quad 0 < t \leq T, \tag{8.141}$$

which satisfies the boundary and initial conditions

$$w(x,t) = 0, \quad x \in \partial\Omega, \quad 0 < t \leq T, \tag{8.142}$$

$$w(x,0) = u_0(x), \quad x \in \Omega, \tag{8.143}$$

$$\frac{\partial w}{\partial t}(x,0) = u_1(x), \quad x \in \Omega. \tag{8.144}$$

Let us introduce an uniform (for simplicity) grid with the steps h_α, $\alpha = 1,2$ in the rectangle Ω. We denote by ω a set of the interior grid nodes

$$\omega = \{x | x = (x_1, x_2), \quad x_1 = ih_1, \quad i = 1,2,...,N_1 - 1,$$

$$x_2 = jh_2, \quad j = 1,2,...,N_2 - 1, \quad N_\alpha h_\alpha = l_\alpha, \quad \alpha = 1,2\}.$$

On the set of the grid functions $u \in H$ vanishing for all $x \notin \omega$ we define the difference operator

$$Au = -\sum_{\alpha=1}^{2} (a^{(\alpha)} u_{x_\alpha})_{\bar{x}_\alpha} \quad x \in \omega. \tag{8.145}$$

Here the standard index-free notation has been used . If the coefficient of equation (8.141) is sufficiently smooth then we accept

$$a^{(1)}(x_1, x_2) = k(x_1 + 0.5h_1, x_2), \quad a^{(2)}(x_1, x_2) = k(x_1, x_2 + 0.5h_2).$$

It is well known that the grid operator A defined by formula (8.145) is self-adjoint and positive. Carrying out discretization with respect to space, we come from the continuous problem (8.141)–(8.144) to the Cauchy problem (8.110)–(8.112) for a differential-difference equation.

Now consider a non-iterative scheme of domain decomposition for the approximate solution of problem (8.141)–(8.144). In order to simplify the presentation we confine ourselves to the simplest version of decomposition in one direction with sub-domains overlapped. Define the function

$$\chi(x_1) = \begin{cases} 1, & x_1 < l_1 - \delta, \\ (l_1 + \delta - x_1)(2\delta)^{-1}, & l_1 - \delta \leq x_1 \leq l_1 + \delta, \\ 0, & x_1 > l_1 + \delta. \end{cases} \tag{8.146}$$

Here the parameter δ determines the half width of the domain of the sub-domains superposition:

$$\Omega_1 = \{x | x \in \Omega, x_1 < l_1 + \delta\}, \quad \Omega_2 = \{x | x \in \Omega, x_1 > l_1 - \delta\}.$$

Using the function $\chi(x_1)$ we construct the splitting

$$A = A_1 + A_2. \tag{8.147}$$

Taking into account the dependence of χ (see (8.146)) on only one variable, we have

$$A_1 u = -(\chi(x_1 + h_1/2) a^{(1)} u_{x_1})_{\bar{x}_1} - \chi(x_1)(a^{(2)} u_{x_2})_{\bar{x}_2}. \tag{8.148}$$

Similarly to (8.147), (8.148) other decomposition operators are also constructed. With the splitting (8.147), (8.148) considered we can guarantee only the nonnegativity of the operator terms, i.e., we have to use the special definition of the approximate solution and the estimates like (8.140).

The following statement is valid.

THEOREM 8.13 *The vector additive scheme of domain decomposition (8.134), (8.146)–(8.148) converges unconditionally under the condition that* $\sigma \geq 0.25p$, *and for the error the following estimate is correct:*

$$\|A(\bar{z}_{n+1} + \bar{z}_n)\| \leq M_1(\tau^2 + |h|^2 + M_2 \tau^2 \max_{\alpha} \|A\chi_\alpha\|).$$

Above we have obtained the similar estimates (see Theorem 8.11) when we considered three-level schemes of domain decomposition for parabolic equations.

References

[Abrashin, 1986] Abrashin, V. N. (1986). A class of difference schemes for nonlinear nonstationary problems of mathematical physics. I. *Differents. Uravneniya*, 22(7):1107–1119. (in Russian); transl. in *Differential Equations*, 22(7): 759–769.

[Abrashin, 1990] Abrashin, V. N. (1990). A variant of the method of variable directions for the solution of multidimensional problems in mathematical physics. I. *Differents. Uravneniya*, 26(2):314–323. (in Russian); transl. in *Differential Equations*, 26(2): 243–250.

[Abrashin and Iakoubenia, 1988] Abrashin, V. N. and Iakoubenia, A. N. (1988). Efficient schemes for the numerical solution of multidimensional problems with free boundaries. Preprint. AN of Belarus. Institute of Mathematics. No. 38 (348). Minsk, 1988. (in Russian).

[Abrashin and Lapko, 1993] Abrashin, V. N. and Lapko, S. L. (1993). A class of difference schemes for the solution of Navier-Stokes equations. II. *Differents. Uravneniya*, 29(4):673–688. (in Russian); transl. in *Differential Equations*, 29(4): 574–585.

[Abrashin and Mukha, 1992] Abrashin, V. N. and Mukha, V. A. (1992). A class of efficient difference schemes for solving multidimensional problems in mathematical physics. *Differents. Uravneniya*, 28(10):1786–1799. (in Russian); transl. in *Differential Equations*, 28(10): 1458–1469.

[Agoshkov, 1991] Agoshkov, V. I. (1991). Domain decomposition methods in problems of mathematical physics. In Marchuk, G. I., editor, *Computational Processes and Systems*, volume 8, pages 4–51. Nauka, Moscow. (in Russian).

[Akrivis, 1993] Akrivis, G. D. (1993). Finite difference discretization of the cubic Schrödinger equation. *IMA J. Numer. Anal.*, 13:115–124.

[Akrivis et al., 1997] Akrivis, G. D., Dougalis, V. A., and Karakashian, O. (1997). Solving the systems of equations arising in the discretization of some nonlinear p.d.e.'s by implicit Runge–Kutta methods. *Math. Model. Numer. Anal. (M^2AN)*, 31(2):251–287.

[Ananich, 1998] Ananich, S. E. (1998). Monotone conservative difference schemes of the second order of approximation on nonuniform grids for the Fokker-Planck equation. Preprint. AN of Belarus. Institute of Mathematics. No. 9 (521). Minsk, 1998.

[Anderson et al., 1984] Anderson, D., Tannenhill, J., and R., P. (1984). *Computational Fluid Mechanics and Heat Transfer.* Hemisphere, New York.

[Andreev, 1966] Andreev, V. B. (1966). On uniform convergence of certain difference schemes. *Zh. Vychisl. Mat. i Mat. Fiz.*, 6(2):519–524. (in Russian).

[Angrand et al., 1985] Angrand, F., Dervieux, A., Desideri, J. A., and Glowinski, R. (1985). *Numerical Methods for the Euler Equations of Fluid Dynamics*, volume 21 of *Proceedings in Applied Mathematics.* SIAM, Philadelphia.

[Asmolik, 1996] Asmolik, V. A. (1996). On a class of locally one-dimensional schemes for solving second-order multidimensional hyperbolic equations of arbitrary dimension. *Differents. Uravneniya*, 32(12):1678–1682. (in Russian); transl. in *Differential Equations*, 32(12): 1671–1676.

[Axelsson, 1994] Axelsson, O. (1994). *Iterative Solution Methods.* Cambridge University Press, New York.

[Axelsson and Barker, 1984] Axelsson, O. A. and Barker, V. A. (1984). *Finite Element Solution of Boundary Value Problems.* Academic Press, New York.

[Babuska et al., 1986] Babuska, I. P., Zienkiewicz, O. C., Gago, J. R., and Oliveira, E. R., editors (1986). *Accuracy Estimates and Adaptive Refinements in Finite Element Computations.* Wiley, London.

[Bayarunas and Čiegis, 1991] Bayarunas, E. and Čiegis, R. Y. (1991). Adaptive difference schemes for nonstationary one-dimensional problems. In *Differentsial'nye Uravneniya i Primenen*, volume 46, pages 25–41. Inst. Mat. i Kibernet., Vilnius. (in Russian; English, Lithuanian summary).

[Berezin, 1972] Berezin, Y. A. (1972). *Numerical study of nonlinear waves in a rarefied plasma.* Nauka, Novosibirsk. (in Russian).

[Berezin, 1982] Berezin, Y. A. (1982). *Modeling of Nonlinear Wave Processes.* Nauka, Novosibirsk. (in Russian); English transl.: Modelling nonlinear wave processes. VNU Science Press, b.v., Utrecht, 1987.

[Berkovsky and Polevikov, 1988] Berkovsky, B. M. and Polevikov, V. K. (1988). *Computational Experiment In Convection.* Universitetskoe, Minsk.

[Bitsadze, 1981] Bitsadze, A. V. (1981). *Some Classes of Partial Differential Equations.* Nauka, Moscow. (in Russian); English transl.: *Advanced Studies in Contemporary Mathematics*, 4. Gordon and Breach Science Publishers, New York, 1988.

[Brenner and Scott, 1994] Brenner, S. C. and Scott, L. R. (1994). *The Mathematical Theory of Finite Element Methods.* Texts in Applied Mathematics. Springer–Verlag, New York.

[Chan and Mathew, 1994] Chan, T. F. and Mathew, T. P. (1994). Domain decomposition algorithms. *Acta Numerica*, pages 61–143.

[Ciarlet, 1978] Ciarlet, P. G. (1978). *The Finite Element Method for Elliptic Problems.* North–Holland, Amsterdam, New York.

[Dar'in et al., 1988] Dar'in, N. A., Mazhukin, V. I., and Samarskii, A. A. (1988). A finite-difference method for solving gas dynamic equations using adaptive grids that are dynamically connected with the solution. *Zh. Vychisl. Mat. i Mat. Fiz.*, 28(8):1210–1225. (in Russian).

[Davis and Flaherty, 1982] Davis, S. and Flaherty, J. F. (1982). An adaptive finite element method for initial-boundary value problems for partial differential equations. *SIAM J. Sci. Statist. Comput.*, 3:6–27.

[Dawson et al., 1991] Dawson, C. N., Du, Q., and Dupont, T. F. (1991). A finite difference domain decomposition algorithm for the numerical solution of the heat equation. *Math. Comp.*, 57:63–67.

[Dawson and Dupont, 1992] Dawson, C. N. and Dupont, T. F. (1992). Explicit/implicit, conservative, Galerkin domain decomposition procedures for parabolic problems. *Math. Comp.*, 58:21–34.

[Degtyariov, 1994] Degtyariov, S. L. (1994). On the stability of difference schemes with variable weights for the one-dimensional heat equation. *Zh. Vychisl. Mat. i Mat. Fiz.*, 34(8–9):1316–1322. (in Russian); transl. in *Comput. Math. Math. Phys.*, 36(8–9): 1344–1186.

[Demmel and van der Vorst, 1993] Demmel, J. and van der Vorst, H. A. (1993). Parallel numerical linear algebra. *Acta Numerica*, pages 111–197.

[Diakonov, 1972] Diakonov, E. G. (1971 and 1972). *Difference Methods of Solving Boundary Value Problems*, volume 1 and 2. Moscow State University, Moscow.

[Diakonov, 1996] Diakonov, E. G. (1996). *Optimization in Solving Elliptic Problems.* CRC Press, Boca Raton.

[Dongarra et al., 1991] Dongarra, J. J., Duff, I. S., Sorensen, D., and van der Vorst, H. A. (1991). *Solving Linear Systems on Vector and Shared Memory Computers.* SIAM, Philadelfia.

[Drobyshevich, 1995] Drobyshevich, V. I. (1995). Difference schemes with different time steps in subdomains for the solution of multidimensional parabolic equations. *Sibirsk. Mat. Zh.*, 36:534–542. (in Russian); transl. in *Siberian Math. J.*, 36(3): 457–474.

[Dryja, 1990] Dryja, M. (1990). Substructuring methods for parabolic problems. In *IV Intern. Simp. on Domain Decomposition Methods for Partial Differential Equations*, pages 264–271, Philadelphia.

[Ewing and Lazarov, 1994] Ewing, R. E. and Lazarov, R. D. (1994). Approximation of parabolic problems on grids locally refined in time and space. *Appl. Numer. Math.*, 14:199–211.

[Ewing et al., 1989] Ewing, R. E., Lazarov, R. D., Pasciak, J. E., and Vassilevski, P. S. (1989). *Finite Element Methods for Parabolic Problems with Time Steps Variable in Space.* Enhanced Oil Recovery Institute, University of Wyoming, Laramie and Wyoming.

[Ewing et al., 1993] Ewing, R. E., Lazarov, R. D., Pasciak, J. E., and Vassilevski, P. S. (1993). Domain decomposition type iterative techniques for parabolic problems on locally refined grids. *SIAM J. Numer. Anal.*, 30:1537–1557.

[Ewing et al., 1992] Ewing, R. E., Lazarov, R. D., and Vassilev, A. T. (1992). Adaptive techniques for dependent problems. *Comput. Meth. Appl. Mech. Engrg.*, 101:113–126.

[Ewing et al., 1990] Ewing, R. E., Lazarov, R. D., and Vassilevski, P. S. (1990). Finite difference schemes on grids with local refinement in time and space for parabolic problems. I: derivation, stability, and error analysis. *Computing*, 45:193–215.

[Flaherty et al., 1989] Flaherty, J. F., Paslow, P. J., Shephard, M. S., and Vasilakis, J. D., editors (1989). *Adaptive Methods for Partial Differential Equations.* SIAM, Philadelphia.

[Fletcher, 1988] Fletcher, C. A. J. (1988). *Computational Techniques for Fluid Dynamics.* Springer, Berlin.

[Forsythe and Wasow, 1960] Forsythe, G. E. and Wasow, W. R. (1960). *Finite Difference Methods for Partial Differential Equations.* Wiley, New York.

[Fujita and Suzuki, 1991] Fujita, H. and Suzuki, T. (1991). Evolution problems. In Ciarlet, P. and Lions, J., editors, *Handbook of Numerical Analysis*, volume II of *Finite Element Methods (Part 1)*, pages 789–928. North-Holland, New York.

[Gardner, 1971] Gardner, C. S. (1971). Korteweg–de Vries equation and generalization. IV. the Korteweg–de Vries equation as a Hamiltonian system. *J. Math. Phis.*, 12:1548.

[Gardner et al., 1967] Gardner, C. S., Green, J. M., Kruskal, M. D., and Miura, R. M. (1967). Method for solving the Korteweg–de Vries equation. *Phys. Rev. Lett.*, 19:1095–1097.

[Godunov, 1971] Godunov, S. K. (1971). *Equations of Mathematical Physics.* Nauka, Moscow. (in Russian); French tranls.: *Equations de la Physique Mathematique.* Editions Mir, Moscow, 1973.

[Goolin, 1968] Goolin, A. V. (1968). Necessary and sufficient conditions for the stability of three-level difference schemes. *J. Vych. Mat. Mat. Fiz.*, 8(4):899–902. (in Russian).

[Goolin, 1979] Goolin, A. V. (1979). Theorems on the stability of nonselfadjoint difference schemes. *Mat. Sb. (N.S.)*, 110 (152)(2):297–303. (in Russian).

[Goolin and Degtyariov, 1994] Goolin, A. V. and Degtyariov, S. L. (1994). On the stability of difference schemes with variable weighted coefficients. *Vestnik Moskov. Univ., Ser. XV Vychisl. Mat. Kibernet.*, (3):23–29. (in Russian); transl. in *Moscow Univ. Comput. Math. Cybernet.*, 7974(3): 12–17.

[Goolin and Degtyariov, 1996] Goolin, A. V. and Degtyariov, S. L. (1996). A criterion for the stability of a two-dimensional difference scheme. *Differents. Uravneniya*, 32(7):1–8. (in Russian); transl. in *Differential Equations*, 32(7): 950–957.

[Goolin and Samarskii, 1976] Goolin, A. V. and Samarskii, A. A. (1976). Some results and problems in the theory of the stability of difference schemes. *Mat. Sb. (N.S.)*, 99 (141)(3):299–330. (in Russian).

[Goolin and Samarskii, 1993] Goolin, A. V. and Samarskii, A. A. (1993). On the stability of a class of difference schemes. *Differents. Uravneniya*, 29(7):1163–1173. (in Russian); transl. in *Differential Equations*, 29(7): 3210–6018.

[Goolin and Yukhno, 1996] Goolin, A. V. and Yukhno, L. F. (1996). Numerical investigation of the stability of two-layer difference schemes for the two-dimensional heat equation. *Zh. Vychisl. Mat. i Mat. Fiz.*, 36(8):118–126. (in Russian); transl. in *Comput. Math. Math. Phys.*, 36(8): 1079–1085.

[Goolin and Yukhno, 1998] Goolin, A. V. and Yukhno, L. F. (1998). Stability boundaries for two-dimensional difference schemes. *Mat. Model.*, 10(1):44–50. (in Russian).

[Gordeziani and Meladze, 1974] Gordeziani, D. G. and Meladze, G. V. (1974). The simulation of the third boundary value problem for multidimensional parabolic equations in an arbitrary domain by one-dimensional equations. *J. Vych. Mat. Mat. Fiz.*, 14(14):246–250. (in Russian); transl. in *USSR Comput. Math. Math. Phys.*, 14(1): 249–253.

[Greig and Morris, 1976] Greig, I. G. and Morris, J. L. (1976). A hopscotch method for the Korteweg–de Vries equation. *J. Comput. Phys.*, 20(64–80).

[Gropp, 1987] Gropp, W. D. (1987). Local uniform mesh refinement with moving grids. *SIAM J. Sci. Statist. Comput.*, 8(3):292–304.

[Hackbush, 1993] Hackbush, W. (1993). *Iterative Solution of Large Sparse Systems of Equations.* Berlin. Springer-Verlag.

[Hageman and Young, 1981] Hageman, L. A. and Young, D. M. (1981). *Applied Iterative Methods.* New York. Academic Press.

[Hedstrom and Rodrique, 1960] Hedstrom, G. W. and Rodrique, G. H. (1960). Adaptive-grid method for time-dependent partial differential equations. *Lect. Notes Math.*, (960):474–484.

[Hockney and Eastwood, 1981] Hockney, R. W. and Eastwood, J. W. (1981). *Computer Simulation Using Particles.* McGraw Hill, New York.

[Iakoubenia, 1988] Iakoubenia, A. N. (1988). A class of difference schemes on adaptive nets for the numerical solution of two-dimensional Stefan problems. In *Differentsial'nye Uravneniya i Primenen*, volume 43, pages 107–114. Inst. Mat. i Kibernet., Vilnius. (in Russian).

[Johnson, 1987] Johnson, G. M. (1987). *Numerical Solution of Partial Differential Equations by the Finite Element Methods.* Cambridge Univ. press, Cambridge.

[Johnson et al., 1987] Johnson, S. L., Saad, V., and Schultz, M. H. (1987). Alternating direction methods on multiprocessors. *SIAM J. Sci. Stat. Comput.*, 8:686–700.

[Korteweg and de Vries, 1895] Korteweg, D. and de Vries, G. (1895). On the change of form of long waves advancing in a rectangular channel and on a new type of long stationary waves. *Phyl. May*, 39(5):422–443.

[Korzjuk, 1968] Korzjuk, V. I. (1968). A conjugacy problem for equations of hyperbolic and parabolic types. *Differents. Uravneniya*, 4(10):1855–1866. (in Russian).

[Korzjuk, 1970] Korzjuk, V. I. (1970). A mixed problem for certain nonstationary equations with discontinuous coefficients. *Differents. Uravneniya*, 6(2):343–357. (in Russian).

[Kreiss and Lorenz, 1989] Kreiss, H. O. and Lorenz, J. (1989). *Initial-boundary value problems and the Navier-Stokes equations.* Acad. press, San Diego.

[Krizer and Neittaanmaki, 1987] Krizer, M. and Neittaanmaki, P. (1987). *Acta Applicandae Mathematicae*, 9:175–198.

[Kruskal et al., 1970] Kruskal, M. D., Miura, R. M., and Gardner, C. S. (1970). Korteweg–de Vries equation and generalization. V. Uniqueness and nonexistence of polynomial conservation laws. *J. Math. Phis.*, 11:952–960.

[Kuznetsov, 1988] Kuznetsov, Y. A. (1988). New algorithms for approximate realization of implicit difference schemes. *Sov. J. Numer. Anal. Math. Model.*, 3:99–114.

[Ladyzhenskaya, 1973] Ladyzhenskaya, O. A. (1973). *Boundary Value Problems of Mathematical Physics.* Nauka, Moscow. (in Russian); English tranls.: *The Boundary Value Problems of Mathematical Physics.* Applied Mathematical Sciences, 49. Springer-Verlag, New York, 1985.

[Laevskii, 1992] Laevskii, Y. M. (1992). A domain decomposition algorithm without overlapping subdomains for the solution of parabolic equations. *Zh. Vychisl. Mat. i Mat. Fiz.*, 32:1744–1755. (in Russian); transl. in *Comput. Math. Math. Phys.*, 32(11): 1563–1580.

[Laevsky, 1990] Laevsky, Y. M. (1990). Quadratic elements in splitting methods. *Sov. J. Numer. Anal. Math. Model.*, 5:244–249.

[Lax and Richtmyer, 1956] Lax, P. D. and Richtmyer, R. D. (1956). Survey of the stability of linear finite difference equations. *Comm. Pure Appl. Math.*, 9:267–293.

[Le Tallec, 1994] Le Tallec, P. (1994). Domain decomposition methods in computational mechanics. *Computational Mechanics Advances*, 1(121–220).

[Marchuk, 1990] Marchuk, G. I. (1990). Splitting and alternating direction methods. In *Handbook of Numerical Analysis*, volume 1, pages 197–464. North-Holland, Amsterdam.

[Matus, 1990] Matus, P. P. (1990). A class of difference schemes on composite grids for non-steady-state problems in mathematical physics. *Differents. Uravneniya*, 26(7):1241–1254. (in Russian); transl. in *Differential Equations*, 26(7): 911–992.

[Matus, 1991] Matus, P. P. (1991). On the construction of difference schemes for multidimensional parabolic equations on adaptive-time grids. *Differents. Uravneniya*, 27(11):1961–1971. (in Russian); transl. in *Differential Equations*, 27(11): 1402–1414.

[Matus, 1993a] Matus, P. P. (1993a). Conservative difference schemes for quasilinear parabolic equations in subdomains. *Differents. Uravneniya*, 29(7):1222–1231. (in Russian); transl. in *Differential Equations*, 29(7): 1060–1068.

[Matus, 1993b] Matus, P. P. (1993b). Conservative difference schemes for second-order parabolic and hyperbolic equations in subdomains. *Differents. Uravneniya*, 29(4):700–711. (in Russian); transl. in *Differential Equations*, 29(4): 595–604.

[Matus, 1993c] Matus, P. P. (1993c). Difference analogues of embedding theorems and their application to the study of the convergence of difference schemes for differential equations with generalized solutions. *Vestsi Akad. Navuk Belarusi, Ser. Fiz. Mat. Navuk*, (4):15–21. (in Russian).

[Matus, 1994] Matus, P. P. (1994). Difference schemes on composite grids for hyperbolic equations. *Zh. Vychisl. Mat. i Mat. Fiz.*, 34(6):870–885. (in Russian); transl. in Comput. Math. Math. Phys. 34 (1994), no. 6, 749–767.

[Matus, 1997] Matus, P. P. (1997). Stability of difference schemes with variable weights. In Vulkov, L., editor, *Proceedings of WNAA'96, Rousse, Bulgaria, June 24-26, 1996*, volume 1196 of *Lecture Notes of Computer Science*, pages 306–318, Berlin. Springler.

[Matus and Mikhiliouk, 1993] Matus, P. P. and Mikhiliouk, I. A. (1993). Difference schemes with variable weights for systems of hyperbolic equations. *Mat. Model.*, 5(12):35–60. (in Russian).

[Matus and Vabishchevich, 1994] Matus, P. P. and Vabishchevich, P. N. (1994). Difference schemes on the grids locally refining in space as well as in time. In *Advances in Numerical Methods and Applications*, pages 146–153. World Shcientific, Singapore.

[McCormick, 1989] McCormick, S. F. (1989). *Multilevel Adaptive Methods for Partial Differential Equations.* SIAM, Philadelphia.

[Mikhailov, 1983] Mikhailov, V. P. (1983). *Partial Differential Equations.* Nauka, Moscow, 2 edition. (in Russian); English transl.: Mir, Moscow; distributed by Imported Publications, Inc., Chicago, Ill., 1977.

[Mikhlin, 1988] Mikhlin, R. G. (1988). *Some Problems in Error Theory.* Leningrad. Univ., Leningrad. (in Russian).

[Mitchell and Griffiths, 1980] Mitchell, A. R. and Griffiths, D. F. (1980). *The Finite Difference Methods in Partial Differential Equations.* Wiley, Chichester.

[Miura, 1968] Miura, R. M. (1968). Korteweg–de Vries equation and generalizations. I. A remarkable explicit non-linear transformation. *J. Math. Phis.*, 9:1202–1204.

[Miura, 1976] Miura, R. M. (1976). The Korteweg–de Vries equation: a survey of results. *SIAM Review*, 18:412–459.

[Miura et al., 1968] Miura, R. M., Gardner, C. S., and Kruskal, M. D. (1968). Korteweg–de Vries equation and generalization. II. Existence of conservation laws and constants of the motion. *J. Math. Phis.*, 9:1204–1209.

[Morton, 1980] Morton, K. W. (1980). Stability of finite difference approximations to a diffusion-convection equation. *Int. J. Numer. Meths. Engrg.*, 15:677–683.

[Moskal'kov, 1974] Moskal'kov, M. N. (1974). The accuracy of difference schemes approximating the wave equation that have piecewise smooth solutions. *Zh. Vychisl. Mat. i Mat. Fiz.*, 14(2):390–401. (in Russian).

[Newell, 1997] Newell, A. C. (1997). Finite amplitude instabilities of partial differential equations. *SIAM J. Appl. Math.*, 33:133–160.

[Oganesian and Rukhovets, 1979] Oganesian, L. A. and Rukhovets, L. A. (1979). *Variational difference methods for solving equations.* Akad. Nauk Arm. SSR, Erevan.

[Ortega, 1988] Ortega, J. M. (1988). *Introduction to Parallel and Vector Solution of Linear Systems.* Plenum Press, New York.

[Osher and Sanders, 1983] Osher, S. and Sanders, R. (1983). Numerical approximations to nonlinear conservation laws with locally varying time and space grids. *Math. Comput.*, 41:321–386.

[Patankar, 1980] Patankar, S. V. (1980). *Numerical Heat Transfer and Fluid Flow.* Hemisphere, Washington, DC.

[Patel and Marcatos, 1986] Patel, M. K. and Marcatos, N. C. (1986). An evaluation of eight discretization schemes for two-dimensional convection-diffusion equations. *Int. J. Numer. Methods Fluids*, 6:129–154.

[Peyret and Taylor, 1983] Peyret, R. and Taylor, T. D. (1983). *Computational Methods for Fluid Flow.* Springer, Berlin.

[Quarteroni, 1991] Quarteroni, A. (1991). Domain decomposition and parallel processing for the numerical solution of partial differential equations. *Surveys on Mathematics for Industry*, 1:75–118.

[Quarteroni et al., 1994] Quarteroni, A., Periaux, J., Kuznetsov, Y., and Widlund, O. B. (1994). *Domain Decomposition Methods in Science and Engineering.* AMS, Providence, RI.

[Quarteroni and Valli, 1994] Quarteroni, A. and Valli, A. (1994). *Numerical Approximation of Partial Differential Equations.* Springer, Berlin.

[Quarteroni and Valli, 1999] Quarteroni, A. and Valli, A. (1999). *Domain Decomposition Methods for Partial Differential Equations.* Oxford University Press, Oxford.

[Revilla, 1986] Revilla, M. A. (1986). Simple time and space adaptation in one-dimensional evolutionary partial differential equations. *Int. J. Numer. Meth. Eng.*, 23:2263–2270.

[Richtmyer and Morton, 1967] Richtmyer, R. D. and Morton, K. W. (1967). *Difference Methods for Initial-Value Problems.* Interscience Publisher, John Wiley & Sons, New York.

[Roache, 1982] Roache, P. J. (1982). *Computational Fluid Dynamics.* Hermosa, Albuquerque (N. M.).

[Samarskii, 1964] Samarskii, A. A. (1964). Locally one-dimensional difference schemes for higher-dimensional equations of hyperbolic type in an arbitrary region. *Zh. Vychisl. Mat. i Mat. Fiz.*, 4(4):638–648. (in Russian).

[Samarskii, 1967a] Samarskii, A. A. (1967a). Classes of stable schemes. *Zh. Vychisl. Mat. i Mat. Fiz.*, 7(5):1096–1133. (in Russian).

[Samarskii, 1967b] Samarskii, A. A. (1967b). Regularization of difference schemes. *Zh. Vychisl. Mat. i Mat. Fiz.*, 7(1):62–93. (in Russian).

[Samarskii, 1968] Samarskii, A. A. (1968). Necessary and sufficient conditions for the stability of double layer difference schemes. *Dokl. Akad. Nauk SSSR*, 181(4):808–811. (in Russian).

[Samarskii, 1970] Samarskii, A. A. (1970). The stability of three-level difference schemes. *Dokl. Akad. Nauk SSSR*, 192(5):998–1001. (in Russian).

[Samarskii, 1971] Samarskii, A. A. (1971). *Introduction to the Theory of Difference Schemes.* Nauka, Moscow. (in Russian).

[Samarskii, 1984] Samarskii, A. A. (1984). Investigation of the accuracy of difference schemes for problems with generalized solutions. In *Current problems in Mathematical Physics and Numerical Mathematics*, pages 174–183. Nauka, Moscow. (in Russian).

[Samarskii, 1989] Samarskii, A. A. (1989). *Theory of Difference Schemes.* Nauka, Moscow. (in Russian); English transl.: *The Theory of Difference Schemes.* Monographs and Textbooks in Pure and Applied Mathematics, 240. Marcel Dekker, Inc., New York, 2001.

[Samarskii and Andreev, 1976] Samarskii, A. A. and Andreev, V. B. (1976). *Difference Methods for Elliptic Equations.* Nauka, Moscow. (in Russian); French transl.: *Methodes aux Differences pour Equations Elliptiques.* Editions Mir, Moscow, 1978.

[Samarskii and Fryazinov, 1961] Samarskii, A. A. and Fryazinov, I. V. (1961). On the convergence of difference schemes for a heat-conduction equation with discontinuous coefficients. *Zh. Vychisl. Mat. i Mat. Fiz.*, 1(5):806–824. (in Russian); transl. in *U.S.S.R. Comput. Math. and Math. Phys.*, 2(4): 982–982.

[Samarskii and Goolin, 1973] Samarskii, A. A. and Goolin, A. V. (1973). *Stability of Difference Schemes.* Nauka, Moscow. (in Russian).

[Samarskii and Goolin, 1993] Samarskii, A. A. and Goolin, A. V. (1993). Criteria for the stability of a family of difference schemes. *Dokl. Akad. Nauk*, 330(6):694–695. (in Russian); trancl. in *Russian Acad. Sci. Dokl. Math.*, 47 (3): 629–611.

[Samarskii and Gordeziani, 1978] Samarskii, A. A. and Gordeziani, D. G. (1978). Some problems of plate and envelopes thermoelasticity and summary approximation methods. In *Complex Analysis and Its Applications*, pages 173–186. Nauka, Moscow.

[Samarskii et al., 1997a] Samarskii, A. A., Jovanovich, B. S., Matus, P. P., and Shchehlik, V. S. (1997a). Difference schemes on time-adaptive grids for parabolic equations with generalized solutions. *Differents. Uravneniya*, 33(7):975–984. (in Russian); transl. in *Differential Equations*, 33(7): 981–990.

[Samarskii et al., 1987] Samarskii, A. A., Lazarov, R. D., and Makarov, V. L. (1987). *Difference Schemes for Differential Equations with Generalized Solutions.* Nauka, Moscow. (in Russian).

[Samarskii et al., 1997b] Samarskii, A. A., Mazhukin, V. I., Matus, P. P., and Mikhiliouk, I. A. (1997b). L_2–conservative schemes for the Korteweg-de Vries equation. *Dokl. Akad. Nauk*, 357(4):458–561. (in Russian).

[Samarskii and Nikolaev, 1978] Samarskii, A. A. and Nikolaev, E. S. (1978). *Methods of Solving Grid Equations*, volume Moscow. Nauka. (in Russian); English tranls.: *Numerical Methods for Grid Equations. Vol. I. Direct Methods*. Birkhauser Verlag, Basel, 1983; *Numerical Methods for Grid Equations. Vol. II. Iterative Methods.* Birkhauser Verlag, Basel, 1989.

[Samarskii and Popov, 1980] Samarskii, A. A. and Popov, Y. P. (1980). *Difference Methods for the Solution of Problems of Gas Dynamics.* Nauka, Moscow. (in Russian).

[Samarskii and Vabishchevich, 1990] Samarskii, A. A. and Vabishchevich, P. N. (1990). Difference schemes for unstable problems. *Mat. Model.*, 2(11):89–98. (in Russian).

[Samarskii and Vabishchevich, 1992] Samarskii, A. A. and Vabishchevich, P. N. (1992). Regularized difference schemes for evolutionary second order equations. *Mathematical Models and Methods in Applied Sciences*, 2:295–315.

[Samarskii and Vabishchevich, 1995a] Samarskii, A. A. and Vabishchevich, P. N. (1995a). *Computational Heat Transfer.* Wiley, Chichester.

[Samarskii and Vabishchevich, 1995b] Samarskii, A. A. and Vabishchevich, P. N. (1995b). Vector additive schemes of domain decomposition for parabolic problems. *Differents. Uravneniya*, 31:742–745. (in Russian); transl. in *Differential Equations*, 31 (9): 1525–1528.

[Samarskii and Vabishchevich, 1996a] Samarskii, A. A. and Vabishchevich, P. N. (1996a). Factorized region-additive schemes for convection/diffusion problems. *Dokl. Akad. Nauk.*, 346:742–745. (in Russian).

[Samarskii and Vabishchevich, 1996b] Samarskii, A. A. and Vabishchevich, P. N. (1996b). Stability of three-layer projection-difference schemes. *Mat. Model.*, 8(9):74–84. (in Russian).

[Samarskii and Vabishchevich, 1997a] Samarskii, A. A. and Vabishchevich, P. N. (1997a). Additive difference schemes for problems of mathematical physics. In

Sydow, A., editor, *Numerical Mathematics*, volume 2 of *15th IMACS World Congress on Scientific Computation, Modelling and Applied Mathematics: Berlin, August 1997*, pages 93–98, Berlin. Wissenschaft und Technik Verl.

[Samarskii and Vabishchevich, 1997b] Samarskii, A. A. and Vabishchevich, P. N. (1997b). Factorized domain decomposition difference schemes for convection-diffusion problems. *Differents. Uravneniya*, 33(7):967–974. (in Russian); transl. in *Differential Equations*, 33 (7): 972–970.

[Samarskii et al., 1998a] Samarskii, A. A., Vabishchevich, P. N., Lemeshevskii, S. V., and Matus, P. P. (1998a). Difference schemes for the problem of the conjugation of equations of hyperbolic and parabolic types. *Sibirsk. Mat. Zh.*, 39(4):954–962. (in Russian); transl. in *Siberian Math. J.*, 39(4): 805–853.

[Samarskii et al., 1996a] Samarskii, A. A., Vabishchevich, P. N., and Matus, P. P. (1996a). Difference schemes of increased order of accuracy on nonuniform grids. *Differents. Uravneniya*, 32(2):265–274. (in Russian); transl. in *Differential Equations*, 32(2): 269–280.

[Samarskii et al., 1996b] Samarskii, A. A., Vabishchevich, P. N., and Matus, P. P. (1996b). Difference schemes of increased order of approximation on nonuniform grids for elliptic equations. *Dokl. Akad. Nauk Belarusi*, 40(5):9–14. (in Russian).

[Samarskii et al., 1997c] Samarskii, A. A., Vabishchevich, P. N., and Matus, P. P. (1997c). Stability of difference schemes in norms that are integral with respect to time. *Dokl. Akad. Nauk*, 354(6):745–747. (in Russian).

[Samarskii et al., 1997d] Samarskii, A. A., Vabishchevich, P. N., and Matus, P. P. (1997d). Strong stability of operator-differential and operator-difference schemes. *Dokl. Akad. Nauk*, 356(4):455–457. (in Russian).

[Samarskii et al., 1998b] Samarskii, A. A., Vabishchevich, P. N., and Matus, P. P. (1998b). Difference schemes of second-order accuracy on nonuniform grids. *Zh. Vychisl. Mat. i Mat. Fiz.*, 38(3):413–424. (in Russian); transl. in *Comput. Math. Math. Phys.*, 38(3): 399–110.

[Samarskii et al., 1998c] Samarskii, A. A., Vabishchevich, P. N., Zyl, A. N., and Matus, P. P. (1998c). A difference scheme of increased order of approximation for the Dirichlet problem for an arbitrary domain. *Dokl. Nats. Akad. Nauk Belarusi*, 42(1):13–17. (in Russian).

[Saul'ev, 1960] Saul'ev, V. K. (1960). *Integration of Parabolic Equations by the Grid Method.* Izdat. Fiz.-Mat. Lit., Moscow. (in Russian); English tranls.: *Integration of Equations of Parabolic Type by the Method of Nets.* in: *International Series of Monographs in Pure and Applied Mathematics*, Vol. 54. Pergamon Press, London-Edinburgh-New York, 1920.

[Shyy, 1985] Shyy, W. (1985). A study of finite difference approximations to steady-state, convection-dominated flow problems. *J. Comp. Phys.*, 57:415–438.

[Siemieniuch and Gladwell, 1978] Siemieniuch, J. E. and Gladwell, I. (1978). Analysis of explicit difference methods for a diffusion-convection equation. *Int. J. Numer. Meths. Engrg.*, 12:899–916.

[Smith et al., 1996] Smith, B. F., Bjørstad, P. E., and Gropp, W. D. (1996). *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations.* Cambridge Univ. Press, New York.

[Smith, 1992] Smith, G. D. (1992). *Numerical Solution of Partial Differential Equations: Finite Difference Method.* Clarendon press, Oxford.

[Sod, 1985] Sod, G. A. (1985). *Numerical Methods in Fluid Dynamics.* Cambridge Univ. Press, Cambridge.

[Strang and Fix, 1973] Strang, G. and Fix, G. J. (1973). *An Analysis of the Finite Element Method.* Prentice–Hall, Engelwood Cliffs.

[Streit, 1986] Streit, U. (1986). Difference schemes for a parabolic equation of general form with generalized solutions. *Differents. Uravneniya*, 22(7):1246–1259. (in Russian); transl. in *Differential Equations*, 22(7): 880–890.

[Strikwerda, 1989] Strikwerda, J. C. (1989). *Finite Difference Schemes and Partial Differential Equations.* Wadsworth & Brooks Cole, Pacific Grove.

[Su and Gardner, 1969] Su, G. and Gardner, C. S. (1969). Korteweg–de Vries equation and generalization. III. Derivation of the Korteweg–de Vries equation and Burger's equation. *J. Math. Phis.*, 10:536–539.

[Sukhinov, 1984] Sukhinov, A. I. (1984). Locally two-dimensional schemes for solving multidimensional parabolic equations on computational systems of matrix type. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (11):45–53. (in Russian); transl. in *Soviet Math. (Izv. VUZ)*, 28(11): 56–65.

[Tadmor, 1987] Tadmor, E. (1987). Stability analysis of finite-difference, pseudospecrtal and Fourier–Galerkin approximations for time-dependent problems. *SIAM Review*, 29(4):525–555.

[Temam, 1979] Temam, R. (1979). *Navier-Stokes equations. Theory and Numerical Analysis.* North-Holland, Amsterdam.

[Thomas, 1995] Thomas, J. W. (1995). *Numerical Partial Differential Equations: Finite Difference Methods*, volume 22 of *Texts in Applied Mathematics.* Springer, Berlin.

[Thomée, 1969] Thomée, V. (1969). Stability theory for partial difference operators. *SIAM Review*, 11(2):152–195.

[Thomée, 1984] Thomée, V. (1984). *Galerkin Finite Element Methods for Parabolic Problems*, volume 1054 of *Lecture Notes in Math.* Springer, Berlin and New York.

[Thomée, 1990] Thomée, V. (1990). Finite difference methods for linear parabolic equations. In Ciarlet, P. G. and Lions, J. L., editors, *Handbook of Numerical Analysis*, volume I of *Finite Difference Methods 1*, pages 8–193. North-Holland, Amsterdam.

[Thomée, 1997] Thomée, V. (1997). *Galerkin Finite Element Methods for Parabolic Problems*, volume 25 of *Springer Series in Computational Mathematics.* Springer, Berlin and Heidelberg.

[Tikhonov and Samarskii, 1959] Tikhonov, A. N. and Samarskii, A. A. (1959). Convergence of difference schemes in the class of discontinuous coefficients. *Dokl. Akad. Nauk SSSR*, 124(3):529–532. (in Russian).

[Tikhonov and Samarskii, 1961] Tikhonov, A. N. and Samarskii, A. A. (1961). Homogeneous difference schemes. *Zh. Vychisl. Mat. i Mat. Fiz.*, 1(1):5–63. (in Russian).

[Tikhonov and Samarskii, 1972] Tikhonov, A. N. and Samarskii, A. A. (1972). *The Equations of Mathematical Physics.* Nauka, Moscow, 4 edition. (in Russian); Enlgish transl.: *Partial Differential Equations of Mathematical Physics.* Vol. I. Holden-Day, Inc., San Francisco–London–Amsterdam, 1964; Vol. II. Holden-Day, Inc., San Francisco–London–Amsterdam, 1967.

[Turetaev, 1992] Turetaev, I. D. (1992). Investigation of finite-difference schemes for the Korteweg-de Vries equation. *Zh. Vychisl. Mat. i Mat. Fiz.*, 32(8):1275–1290. (in Russian); transl. in *Comput. Math. Math. Phys.*, 32(8): 1135–1149.

[Vabishchevich, 1989a] Vabishchevich, P. N. (1989a). Adaptive grids of composite type in problems of mathematical physics. *Zh. Vychisl. Mat. i Mat. Fiz.*, 29(6):902–914. (in Russian); transl. in *U.S.S.R. Comput. Math. and Math. Phys.*, 29(3): 180–187.

[Vabishchevich, 1989b] Vabishchevich, P. N. (1989b). Domain splitting for solving nonstationary PDE problems. *Zh. Vychisl. Mat. i Mat. Fiz.*, 29(12):1822–1829. (in Russian); transl. in *U.S.S.R. Comput. Math. and Math. Phys.*, 29(6): 155–160.

[Vabishchevich, 1994a] Vabishchevich, P. N. (1994a). Monotone difference schemes for convection/diffusion problems. *Differents. Uravneniya*, 30(3):503–513. (in Russian); transl. in *Differential Equations*, 30(3): 466–474.

[Vabishchevich, 1994b] Vabishchevich, P. N. (1994b). Parallel domain decomposition algorithms for time-dependent problems of mathematical physics. In *Advances in Numerical Methods and Applications*, pages 293–299. World Scientific, Singapore.

[Vabishchevich, 1994c] Vabishchevich, P. N. (1994c). Regionally additive difference schemes of stabilizing correction for parabolic problems. *Zh. Vychisl. Mat. i Mat. Fiz.*, 34(12):1832–1842. (in Russian); transl. in *Comput. Math. Math. Phys.*, 34(12): 1573–1581.

[Vabishchevich, 1995] Vabishchevich, P. N. (1995). Difference schemes on grids that are locally refining in time. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (4):22–28. (in Russian); transl. in *Russian Math. (Iz. VUZ)*, 39(4): 19–25.

[Vabishchevich, 1996] Vabishchevich, P. N. (1996). Vector additive difference schemes for first-order evolution equations. *Zh. Vychisl. Mat. i Mat. Fiz.*, 36(3):44–51. (in Russian); transl. in *Comput. Math. Math. Phys.*, 36(3): 317–322.

[Vabishchevich, 1997] Vabishchevich, P. N. (1997). Parallel domain decomposition algorithms for parabolic problems. *Mat. Model.*, 9(5):77–86. (in Russian).

[Vabishchevich and Matus, 1993a] Vabishchevich, P. N. and Matus, P. P. (1993a). Operator-difference schemes with variable weights. Preprint No 31. Moscow: IMM RAS, 1993. (in Russian).

[Vabishchevich and Matus, 1993b] Vabishchevich, P. N. and Matus, P. P. (1993b). Two-layer difference schemes with variable weights. *Dokl. Akad. Nauk Belarusi*, 37(6):15–17. (in Russian).

[Vabishchevich and Matus, 1994] Vabishchevich, P. N. and Matus, P. P. (1994). Difference schemes with variable time steps. *Vestsi Akad. Navuk Belarusi, Ser. Fiz. Mat. Navuk*, (4):5–11. (in Russian).

[Vabishchevich and Matus, 1995] Vabishchevich, P. N. and Matus, P. P. (1995). Difference schemes for problems with variable coefficients on dynamic locally condensing grids. *Dokl. Akad. Nauk Belarusi*, 39(5):11–16. (in Russian).

[Vabishchevich et al., 1995] Vabishchevich, P. N., Matus, P. P., and Rychagov, V. G. (1995). On a class of difference schemes on dynamic locally condensing grids. *Differents. Uravneniya*, 31(5):849–857. (in Russian); transl. in *Differential Equations*, 31(5): 787–795.

[Vabishchevich et al., 1993] Vabishchevich, P. N., Matus, P. P., and Shchehlik, V. S. (1993). Three-level operator-difference schemes with variable weights. Preprint No. 30. Moscow: IMM RAS, 1993. (in Russian).

[Vabishchevich et al., 1994a] Vabishchevich, P. N., Matus, P. P., and Shchehlik, V. S. (1994a). Difference schemes with variable weights for second-order evolution equations. *Dokl. Akad. Nauk Belarusi*, 38(3):13–15. (in Russian).

[Vabishchevich et al., 1994b] Vabishchevich, P. N., Matus, P. P., and Shchehlik, V. S. (1994b). Operator-difference equations of divergence type. *Differents. Uravneniya*, 30(7):1175–1186. (in Russian); transl. in *Differential Equations*, 30(7): 1088–1099.

[Vabishchevich and Samarskii, 1995] Vabishchevich, P. N. and Samarskii, A. A. (1995). Stability of projection-difference schemes for nonstationary problems in mathematical physics. *Zh. Vychisl. Mat. i Mat. Fiz.*, 35(7):1011–1021. (in Russian); transl. in *Comput. Math. Math. Phys.*, 35(7): 805–813.

[Vabishchevich and Samarskii, 1997] Vabishchevich, P. N. and Samarskii, A. A. (1997). On the stability of difference schemes for convection-diffusion problems. *Zh. Vychisl. Mat. i Mat. Fiz.*, 37(2):182–186. (in Russian); transl. in *Comput. Math. Math. Phys.*, 37(2): 184–188.

[Vabishchevich and Samarskii, 1998] Vabishchevich, P. N. and Samarskii, A. A. (1998). Difference schemes for nonstationary convection-diffusion problems. *Zh. Vychisl. Mat. i Mat. Fiz.*, 38(2):207–219. (in Russian); transl. in *Comput. Math. Math. Phys.*, 38(2): 199–210.

[Wesseling, 1992] Wesseling, P. (1992). *An Introduction to Multigrid Methods.* John Wiley & Sons, Chichester.

[Xu, 1992] Xu, J. (1992). Iterative methods by space decomposition and subspace correction. *SIAM Review*, (34):581–613.

[Yanenko et al., 1978] Yanenko, N. N., Konovalov, A. N., Bugrov, A. N., and Shustov, G. V. (1978). On the organization of parallel computations and "parallelization" of the sweep method. *Chisl. Metody Mekh. Sploshn. Sredy*, (7):139–146.

[Zabusky and Kruskal, 1965] Zabusky, N. J. and Kruskal, M. D. (1965). Interaction of "Solutions" in a collisionless plasma and the reccurence of initial states. *Phys. Rev. Lett.*, 15(6):240–243.

[Zienkiewicz, 1977] Zienkiewicz, O. C. (1977). *The Finite Element Method.* McGraw Hill, London.

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