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# Blow-up in Quasilinear Parabolic Equations

by

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Translated from the Russian
by
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# Preface to the English edition

In the relatively brief time that has passed since the appearance of this book in Russian, a range of new results have been obtained in the theory of strongly non-stationary evolution equations, the main problems of this area have been more clearly delineated, specialist monographs and a large number of research papers were published, and the sphere of applications has expanded. It turns out, that as far as nonlinear heat equations with a source term are concerned, the present authors have, on the whole, correctly indicated the main directions of development of the theory of finite time blow-up processes in nonlinear media. We were gratified to see that the subject matter of the book had lost none of its topicality, in fact, its implications have widened. Therefore we thought it right to confine ourselves to relatively insignificant additions and corrections in the body of the work.

In preparing the English edition we have included additional material, provided an updated list of references and reworked the Comments sections wherever necessary.

It is well known that most phenomena were discovered by analyzing simple articular solutions of the equations and systems under consideration. This also applies to the theory of finite time blow-up. We included in the introductory Ch. I and II, and in Ch. IV, new examples of unusual special solutions, which illustrate unexpected properties of unbounded solutions and pose open problems concerning asymptotic behaviour. Some of these solutions are not self-similar (or invariant with respect to a group of transformations). Starting from one such solution and using the theory of intersection comparison of unbounded solutions having the same existence time, we were able to obtain new optimal estimates of evolution of fairly arbitrary solutions. This required changing the manner of presentation of the main comparison results and some subsequent material in Ch. IV.

We hope that this book will be of interest not only to specialists in the area of nonlinear equations of mathematical physics, but to everyone interested in the ideas and concepts of general rules of evolution of nonlinear systems. An important element of evolution of such systems is finite time blow-up behaviour, which represents a kind of stable intermediate asymptotics of the evolution. Without studying finite time blow-up, the picture of the nonlinear world would be incomplete. Of course, the degree to which a reader manages to extract such a picture from this somewhat specialized book, is entirely a matter for the authors' conscience; in writing this book they set themselves originally a much more limited goal: to present the mathematical basis of the theory of finite time blow-up in nonlinear heat equations.

The authors are grateful to the translator of the book, Dr. M. Grinfeld, who made a number of suggestions that led to improvements in the presentation of the material.

The authors would like to express their thanks to Professor J. L. Vazquez for numerous fruitful discussions in the course of preparation of the English edition.

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# Contents

# Introduction

Second order quasilinear parabolic equations and systems of parabolic quasilinear equations form the basis of mathematical models of diverse phenomena and processes in mechanics, physics, technology, biophysics, biology, ecology, and many other areas. For example, under certain conditions, the quasilinear heat equation describes processes of electron and ion heat conduction in plasma, adiabatic filtration of gases and liquids in porous media, diffusion of neutrons and alpha-particles; it arises in mathematical modelling of processes of chemical kinetics, of various biochemical reactions, of processes of growth and migration of populations, etc.

Such ubiquitous occurrence of quasilinear parabolic equations is to be explained, first of all, by the fact that they are derived from fundamental conservation laws (of energy, mass, particle numbers, etc). Therefore it could happen that two physical processes having at first sight nothing in common (for example, heat conduction in semiconductors and propagation of a magnetic field in a medium with finite conductivity), are described by the same nonlinear diffusion equation, differing only by values of a parameter.

In the general case the differences among quasilinear parabolic equations that form the basis of mathematical models of various phenomena lie in the character of the dependence of coefficients of the equation (thermal conductivity, diffusivity, strength of body heating sources and sinks) on the quantities that define the state of the medium, such as temperature, density, magnetic field, etc.

It is doubtful that one could list all the main results obtained in the theory of nonlinear parabolic equations. Let us remark only that for broad classes of equations the fundamental questions of solvability and uniqueness of solutions of various boundary value problems have been solved, and that differentiability properties of the solutions have been studied in detail. General results of the theory make it possible to study from these viewpoints whole classes of equations of a particular type.

There have also been notable successes in qualitative, or constructive, studies of quasilinear parabolic equations, concerned with the spatio-temporal structure of solutions (which is particularly important in practical applications). Research of this kind was pioneered by Soviet mathematicians and mechanicists. They studied properties of a large number of self-similar (invariant) solutions of various nonlinear parabolic equations used to describe important physical processes in nonlinear dissipative continua. Asymptotic stability of many of these solutions means that these particular solutions can be used to describe properties of a wide variety of solutions to nonlinear boundary value problems. This demonstrates the possibility of a "classification" of properties of families of solutions using a collection of stable particular solutions; this classification can, to a degree, serve as a "superposition principle" for nonlinear problems. Studies of this sort engendered a whole direction in the theory of nonlinear evolution equations, and this led to the creation of the qualitative (constructive) theory of nonlinear parabolic problems[1]. It turns out that, from the point of view of the constructive approach, each nonlinear parabolic problem has its own individuality and in general cannot be solved by a unified approach. As a rule, for such an analysis of certain (even very particular) properties of solutions, a whole spectrum of methods of qualitative study is required. This fact underlies the importance of the information contained even in the simplest model parabolic problems, which allow us to single out the main directions in the development of the constructive theory.

The main problems arising in the study of complicated real physical processes are related, primarily, to the nonlinearity of the equations that form the base of the mathematical model. The first consequence of nonlinearity is the absence of a superposition principle, which applies to linear homogeneous problems. This leads to an inexhaustible set of possible directions of evolution of a dissipative process, and also determines the appearance in a continuous medium of discrete spatio-temporal scales. These characterize the properties of the nonlinear medium, which are independent of external factors. Nonlinear dissipative media can exhibit a certain internal orderliness, characterized by spontaneous appearance in the medium of complex dissipative structures. In the course of evolution, the process of self-organization takes place.

These properties are shared by even the simplest nonlinear parabolic equations and systems thereof, so that a number of fundamental problems arise in the course of their constructive study. The principles of evolution and the spatio-temporal "architecture" of dissipative structures are best studied in detail using simple (and yet insightful) model equations obtained from complex mathematical models by singling out the mechanisms responsible for the phenomena being considered.

It is important to stress that the development of nonlinear differential equations of mathematical physics is inconceivable without the use of methods of mathematical modelling on computers and computational experimentation. It is always useful to verify numerically the conclusions and results of constructive theoretical investigation. In fact, this is an intrinsic requirement of constructive theory; this applies in particular to results directly related to applications.

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[1]Clearly, such a subdivision of the theory into general and constructive parts is arbitrary. The two directions of study are closely interlinked.

A well designed computational experiment (there are many examples of this) allows us not only to check the validity and sharpness of theoretical estimates, but also to uncover subtle effects and principles, which serve then to define new directions in the development of the theory. It is our opinion, that the level of understanding of physical processes, phenomena, and even of the properties of solutions of an abstract evolutionary problem, achieved through numerical experiments cannot be matched by a purely theoretical analysis.

A special place in the theory of nonlinear equations is occupied by the study of unbounded solutions, a phenomenon known also as blow-up behaviour (physical terminology). Nonlinear evolution problems that admit unbounded solutions are not solvable globally (in time): solutions grow without bound in finite time intervals. For a long time they were considered in the theory as exotic examples of a sort, good possibly only for establishing the degree of optimality of conditions for global solvability, which was taken to be a natural "physical" requirement. Nonetheless, we remark that the first successful attempts to derive unboundedness conditions for solutions of nonlinear parabolic equations were undertaken more than 30 years ago. The fact that such "singular" (in time) solutions have a physical meaning was known even earlier: these are problems of thermal runaway, processes of cumulation of shock waves, and so on.

A new impetus to the development of the theory of unbounded solutions was given by the ability to apply them in various contexts, for example, in self-focusing of light beams in nonlinear media, non-stationary structures in magnetohydrodynamics (the $T$-layer), shockless compression in problems of gas dynamics. The number of publications in which unbounded solutions are considered has risen sharply in the last decade.

It has to be said that in the mathematical study of unbounded solutions of nonlinear evolution problems, a substantial preference is given to questions of general theory: constructive studies in this area are not sufficiently well developed. This situation can be explained, on the one hand, by the fact that here traditional questions of general theory are very far from being answered completely, while, on the other hand, it is possible that a constructive description of unbounded solutions requires fundamentally new approaches, and an actual reappraisal of the theory. The important point here, in our understanding, is that so far there is no unified view of what constitutes the main questions in constructive study of blow-up phenomena, and the community of researchers in nonlinear differential equations does not know what to expect of unbounded solutions, in either theory or applications (that is, what properties of non-stationary dissipative processes these solutions describe).

These properties are very interesting; in some sense, they are paradoxical, if considered from the point of view of the usual interpretation of non-stationary dissipative processes.

In this book we present some mathematical aspects of the theory of blow-up phenomena in nonlinear continua. The principal models used to analyze the distinguishing properties of blow-up phenomena, are quasilinear heat equations and certain systems of quasilinear equations.

This book is based on the results of investigations carried out in the M. V. Keldysh Institute of Applied Mathematics of the Russian Academy of Sciences during the last 15 or so years. In this period, a number of extraordinary properties of unbounded solutions of many nonlinear boundary problems were discovered and studied. Using numerical experimentation, the spatio-temporal structure of blow-up phenomena was studied in detail; the common properties of their manifestations in various dissipative media were revealed. This series of studies defined the main range of questions and the direction of development of the theory of blow-up phenomena, indicated the main requirements for theoretical methods of study of unbounded solutions, and, finally, made it possible to determine the simplest nonlinear models of heat conduction and combustion, which exhibit the universal properties of blow-up phenomena.

The present book is devoted to the study of such model problems, but we emphasize again that most general properties are shared by unbounded solutions of nonlinear equations of different types. This holds, in particular, for the localization effect in blow-up phenomena in nonlinear continua: unbounded growth of temperature, for example, occurs only in a finite domain, and, despite heat conduction, the heat concentrated in the localization domain does not diffuse into the surrounding cold region throughout the whole period of the process.

The theory of blow-up phenomena in parabolic problems is by no means exhausted by the range of questions reflected in this book. It will not be an exaggeration to say that studies of blow-up phenomena in dissipative media made it possible to formulate a number of fundamentally new questions and problems in the theory of nonlinear partial differential equations. Many interesting results and conclusions, which do not have as yet a sufficient mathematical justification, have been left out of the present book.

One of the main ideas in the theory of dissipative structures and the theory of nonlinear evolution equations is the interpretation of the so-called eigenfunctions (e.f.) of the nonlinear dissipative medium as universal characteristics of processes that can develop in the medium in a stable fashion. The study of the architecture of the whole collection of e.f. of a nonlinear medium and, at the same time, of conditions of their resonant excitation, makes it possible to "control" nonlinear dissipative processes by a minimal input of energy.

Development of blow-up regimes is accompanied by the appearance in the medium of complex, as a rule discrete, collections of e.f. with diverse spatio-temporal structure. An intrinsic reason for such increase in the complexity of organization of a nonlinear medium is the localization of dissipative processes.

The problem of studying e.f. of a nonlinear dissipative medium, which is stated in a natural way in the framework of the differential equations of the corresponding mathematical model, is closely related to the fundamental problem of establishing the laws of thermodynamical evolution of non-equilibrium open systems.

Related questions are being intensively studied in the framework of synergetics. In open thermodynamical systems there are sources and sinks of energy, which, together with the mechanisms of dissipation, determine its evolution, which, in general, takes the system to a complex stable state different from the uniform equilibrium one. The latter is characteristic of closed isolated systems (the second law of thermodynamics).

The range of questions related to the analysis of fine structure of nonlinear dissipative media, represents the next, higher (and, it must be said, harder to investigate) level of the theory of blow-up phenomena.

The first two chapters of the book are introductory in nature. In Chapter I we present the necessary elementary material from the theory of second order quasilinear parabolic equations. Chapter II, the main part of which consists of results of analyses of a large number of concrete problems, should also be regarded as an introduction to the methods and approaches, which are systematically utilized in the sequel. These chapters contain the concepts necessary for a discussion of unbounded solutions and effects of localization of heat and combustion processes.

Chapters III, IV are devoted to the study of localization of blow-up in two specific problems for parabolic equations with power law nonlinearities. In subsequent chapters we develop methods of attacking unbounded solutions of quasilinear parabolic equations of general form; relevant applications are presented. At the end of each chapter we have placed comments containing bibliographical references and additional information on related results. There we also occasionally give lists of, in our opinion, the most interesting and important questions, which are as yet unsolved, and for the solution of which, furthermore, no approach has as yet been developed.

Chapter III deals, in the main, with the study of the boundary value problem in $(0, T) \times \mathbf{R}_+$ for the heat equation with a power law nonlinearity, $u_t = (u^\sigma u_x)_x$, $\sigma = \text{const} > 0$, with a fixed blow-up behaviour on the boundary $x = 0$: $u(t, 0) = u_1(t)$, $u_1(t) \to \infty$ as $t \to T_0 < \infty$.

For $\sigma > 0$ we mainly deal with the power law boundary condition, $u_1(t) = (T - t)^n$, where $n = \text{const} < 0$. In this class there exists the "limiting" localized S blow-up regime, $u_1(t) = (T - t)^{-1/\sigma}$; heat localization in this case is graphically illustrated by the simple separable self-similar solution[2]:

$$u_S(t, x) = (T - t)^{-1/\sigma} \left(1 - \frac{x}{x_0}\right)_+^{2/\sigma}, \quad x_0 = \left[\frac{2(\sigma + 2)}{\sigma}\right]^{1/2}. \tag{1}$$

[2]Here $(z)_+ = \max\{z, 0\}$.

By (1), heat from the localization region $\{0 < x < x_0\}$ never reaches the surrounding cold space, even though the temperature grows without bound in that region. In Ch. III we present a detailed study of localized ($n \geq -1/\sigma$) and non-localized ($n < -1/\sigma$) power law boundary conditions; corresponding self-similar solutions are constructed; analysis of the asymptotic behaviour of non-self-similar solutions of the boundary value problem is performed, and physical reasons for heat localization are discussed.

The case $\sigma = 0$ (the linear heat equation) has to be treated in a somewhat different manner. Here the localized S-regime is exponential, $u_1(t) = \exp\{(T - t)^{-1}\}$. In this case the heat coming from the boundary is effectively localized in the domain $\{0 < x < 2\}$: $u(t, x) \to \infty$ as $t \to T^-$, $0 < x \leq 2$, and $u(T^-, x) < \infty$ for all $x > 2$. The study of the asymptotic phase of the heating process uses approximate self-similar solutions, the general principles of construction of which are presented in Ch. VI.

Chapter IV contains the results of the study of the localization phenomenon in the Cauchy problem for the equation with power law nonlinearity: $u_t = \nabla \cdot (u^\sigma \nabla u) + u^\beta$, $t > 0$, $x \in \mathbf{R}^N$, where $\sigma \geq 0$, $\beta > 1$ are constants. A number of topics are investigated for $\sigma > 0$. We construct unbounded self-similar solutions, which describe the asymptotic phase of the development of the blow-up behaviour; conditions for global insolvability of the Cauchy problem are established, as well as conditions for global existence of solutions in the case $\beta > \sigma + 1 + 2/N$; we prove theorems on occurrence ($\beta \geq \sigma + 1$) and non-occurrence ($1 < \beta < \sigma + 1$) of localization of unbounded solutions.

Localization of the combustion process in the framework of this model is illustrated by the self-similar solution (S-regime) for $\beta = \sigma + 1$, $N = 1$, in the domain $(0, T_0) \times \mathbf{R}$:

$$u_S(t, x) = (T_0 - t)^{-1/\sigma} \begin{cases} \left( \frac{2(\sigma+1)}{\sigma(\sigma+2)} \cos^2 \frac{\pi x}{L_S} \right)^{1/\sigma}, & |x| < L_S/2, \\ 0, & |x| \geq L_S/2, \end{cases} \tag{2}$$

where $L_S = 2\pi(\sigma + 1)^{1/2}/\sigma$ is the fundamental length of the S-regime. The main characteristic of this solution is that the combustion process takes place entirely in the bounded region $\{|x| < L_S/2\}$; outside this region $u_S \equiv 0$ during all the time of existence of the solution which blows up ($t \in (0, T_0)$).

The study of the spatio-temporal structure of unbounded solutions is based on a particular "comparison" of the solution of the Cauchy problem with the corresponding self-similar solution (for example, with (2)). The main idea of this "comparison" consists of analyzing the number of intersections $N(t)$ of the spatial profiles of the two solutions, $u(t, x)$ and $u_S(t, x)$, having the same blow-up time. The fact that $N(t)$ does not exceed the number of intersections on the parabolic boundary of the domain under consideration (and in a number of cases is a non-decreasing function of $t$), is a natural consequence of the Strong Maximum

Principle for parabolic equations and goes back to the results by C. Sturm (1836). It turns out that in the comparison of unbounded solutions with equal intervals of existence, $N(t)$ cannot be strongly decreasing; in any case, if $N(0) > 0$ then $N(t) > 0$ for all $t \in (0, T_0)$. In Ch. IV we use comparison theorems of the form $N(t) \leq 1$ and $N(t) \equiv 2$.

Let us stress that to study particular properties of unbounded solutions the usual comparison theorem for initial conditions is not applicable. The reason is that majorization of one solution by another, for example, $u(t,x) \leq u_S(t,x)$ in $(0, T_0) \times \mathbf{R}$, usually means that the solutions $u \not\equiv u_S$ have different blow-up times, so that from a certain moment of time onwards such a comparison makes no sense.

In Chapter IV we also consider the case of a semilinear equation ($\sigma = 0$). Unbounded solutions of the equations with "logarithmic" nonlinearities, $u_t = \Delta u + (1+u)\ln^{\beta}(1+u)$, $t > 0$, $x \in \mathbf{R}^N$, have some very interesting properties for $\beta > 1$.

In Chapter V we prove comparison theorems for solutions of various nonlinear parabolic equations, based on special pointwise estimates of the highest order spatial derivative of one of the solutions; applications of this theory are given.

The idea of this comparison is the following. In the theory of nonlinear second order parabolic equations

$$u_t = \mathbf{A}(u), \quad (t,x) \in G = (0,T) \times \Omega, \tag{3}$$

where $\Omega$ is a smooth domain in $\mathbf{R}^N$, $\mathbf{A}(u)$ is a nonlinear second order elliptic operator with smooth coefficients, there is a well-known comparison principle for sub- and supersolutions. Let $u \geq 0$ and $v \geq 0$ be, respectively, a super- and a subsolution of equation (3), that is,

$$u_t \geq \mathbf{A}(u),\ v_t \leq \mathbf{A}(u) \ \text{ in } G. \tag{4}$$

and $u \geq v$ on $\partial G$, where $\partial G$ is the parabolic boundary of $G$. Then $u \geq v$ everywhere in $G$.

Propositions of this sort are often called Nagumo lemmas. A systematic constructive analysis of nonlinear parabolic equations started precisely from an understanding that a solution of the problem under consideration can be quite sharply bounded from above and below by solutions of the differential inequalities (4). Nagumo type lemmas are optimal in the sense that a further comparison of different functions $u$ and $v$ is impossible without using additional information concerning their properties.

The same operator $\mathbf{A}$ appears in both the inequalities of (4). Let us consider now the case when we have to determine conditions for the comparison of solutions $u^{(\nu)} \geq 0$ of parabolic equations

$$u_t^{(\nu)} = L^{(\nu)}(u^{(\nu)}, |\nabla u^{(\nu)}|, \Delta u^{(\nu)}), \quad (t,x) \in G, \quad \nu = 1, 2, \tag{5}$$

with different elliptic operators $L^{(1)} \not\equiv L^{(2)}$, where $L^{(\nu)}(p,q,r)$ are smooth functions of their arguments. Parabolicity of the equations means that

$$\frac{\partial}{\partial r} L^{(\nu)}(p,q,r) \geq 0, \quad p,q \in \mathbf{R}_+, \quad r \in \mathbf{R}. \tag{6}$$

From the usual comparison theorem of classical solutions it follows that the inequality $u^{(2)} \geq u^{(1)}$ will hold in $G$ if $u^{(2)} \geq u^{(1)}$ on $\partial G$ and for all $v \in C^{1,2}_{t,x}(G) \cap C(\overline{G})$

$$L^{(2)}(v, |\nabla v|, \Delta v) \geq L^{(1)}(v, |\nabla v|, \Delta v) \ \text{ in } G \tag{7}$$

(this claim is equivalent to the Nagumo lemma). The latter condition is frequently too cumbersome and does not allow us to compare solutions of equations (5) for significantly differing operators $L^{(\nu)}$.

Let us assume now, that, in addition, $u^{(2)}$ is a critical solution, that is

$$u^{(2)}_t \geq 0 \ \text{ in } G, \tag{8}$$

so that $L^{(2)}(u^{(2)}, |\nabla u^{(2)}|, \Delta u^{(2)}) \geq 0$ everywhere in $G$. Parabolicity of the equation for $\nu = 2$ allows us, in general, to solve the above inequality with respect to $\Delta u^{(2)}$, so that as a result we obtain the required pointwise estimate of the highest order derivative:

$$\Delta u^{(2)} \geq l^{(2)}_0(u^{(2)}, |\nabla u^{(2)}|) \ \text{ in } G. \tag{9}$$

Therefore for the comparison $u^{(2)} \geq u^{(1)}$ it suffices to verify that the inequality (7) holds not for all arbitrary $v$, but only for the functions that satisfy the estimate (9). This imposes the following conditions on the operators $L^{(\nu)}$ in (5):

$$\frac{\partial}{\partial r}(L^{(2)}(p,q,r) - L^{(1)}(p,q,r)) \geq 0, \ L^{(1)}(p,q,l^{(2)}_0(p,q)) \leq 0.$$

For quasilinear equations $L^{(\nu)} = K^{(\nu)}(p,q)r + N^{(\nu)}(p,q)$ these conditions have a particularly simple form: $K^{(2)} \geq K^{(1)}$, $K^{(1)}N^{(2)} \geq K^{(2)}N^{(1)}$ in $\mathbf{R}_+ \times \mathbf{R}_+$.

The criticality requirement (8) on the majorizing solution is entirely dependent on boundary conditions and frequently is easy to verify.

Vast possibilities are presented if we compare not the solutions themselves, but some nonlinear functions of these solutions; for example, $u^{(2)} \geqslant E(u^{(1)})$ in $G$, where $E: [0,\infty) \to [0,\infty)$ is a smooth monotone increasing function. The choice of this function is usually guided by the form of the elliptic operators $L^{(\nu)}$ in (5). In Ch. V we consider yet another direction of development of comparison theory; this is the derivation of more general pointwise estimates, which arise as a consequence of $\psi$-criticality of a solution: $u^{(2)}_t \geqslant \psi(u^{(2)})$ in $G$, where $\psi$ is a smooth function.

As applications, we obtain in Ch. V conditions for localization of boundary blow-up regimes and its absence in boundary value problems for the nonlinear heat equation of general type (by comparison with self-similar solutions of the equation $u_t = (u^\sigma u_x)_x$, $\sigma \geqslant 0$, which are studied in detail in Ch. III.) Using the concept of $\psi$-criticality, we derive conditions for non-existence of global solutions of quasilinear parabolic equations.

In Ch. VI we present a different approach to the study of asymptotic behaviour of solutions of quasilinear parabolic equations. There we also talk about comparing solutions of different equations.

As already mentioned above, an efficient method of analysis of non-stationary processes of nonlinear heat conduction, described, for example, by the boundary value problem

$$\begin{gathered} u_t = \mathbf{A}(u) \equiv (k(u)u_x)_x, \quad t \in (0, T), \ x > 0; \\ u(t, 0) = u_1(t) \to \infty, t \to T^- ; \ u(0, x) = u_0(x) \geq 0, x > 0, \end{gathered} \tag{10}$$

is the construction and analysis of the corresponding self-similar or invariant solutions. However, the appropriate particular solutions exist only in relatively rare cases, only for some thermal conductivities $k(u) \geqslant 0$ and boundary conditions $u(t, 0) = u_1(t) > 0$ in (10). Using the generalized comparison theory developed in Ch. V, it is not always possible to determine the precise asymptotics of the solutions by upper and lower bounds. On the whole this is related to the same cause, the paucity of invariant solutions of the problem (10). In Ch. VI we employ approximate self-similar solutions (a.s.s), the main feature of which is that they do not satisfy the equation, and yet nonetheless describe correctly the asymptotic behaviour of the problem under consideration.

In the general setting, a.s.s are constructed as follows. The elliptic operator $\mathbf{A}$ in equation (10), which by assumption, does not have an appropriate particular solution is decomposed into a sum of two operators,

$$\mathbf{A}(u) \equiv \mathbf{B}(t, u) + [\mathbf{A}(u) - \mathbf{B}(t, u)] \tag{11}$$

so that the equation

$$u_t = \mathbf{B}(t, u) \tag{12}$$

admits an invariant solution $u = u_s(t, x)$ generated by the given boundary condition: $u_s(t, 0) \equiv u_1(t)$. But the most important thing is that on this solution the operator $\mathbf{A} - \mathbf{B}$ in (11) is to be "much smaller" than the operator $\mathbf{B}$, that is, we want, in a certain sense, that

$$\|\mathbf{A}(u_s(t, \cdot)) - \mathbf{B}(t, u_s(t, \cdot))\| \ll \|\mathbf{B}(t, \ u_s(t, \cdot))\|$$

as $t \to T^-$. This can guarantee that the solution $u_s$ of (12) and the solution of the original problem are asymptotically close.

In Ch. VI, using several model problems, we solve two main questions: 1) a correct choice of the "defining" operator $\mathbf{B}$ with the above indicated properties; 2) justification of the passage to equation (12), that is, the proof of convergence, in a special norm of $u(t,\cdot) \to u_s(t,\cdot)$ as $t \to T^-$. It turns out that the defining operator $\mathbf{B}$ can be of a form at first glance completely unrelated to the operator $\mathbf{A}$ of the original equation. For example, we found a wide class of problems (10), the a.s.s. of which satisfy a Hamilton-Jacobi type equation:

$$(u_s)_t = \frac{k(u_s)}{u_s + 1}\,[(u_s)_x]^2 \equiv \mathbf{B}(u_s). \tag{13}$$

Thus at the asymptotic stage of the process we have "degeneration" of the original parabolic equation (10) into the first order equation (13).

Using the constructed families of a.s.s. we solve in Ch. VI the question of localization of boundary blow-up regimes in arbitrary nonlinear media.

A considerable amount of space is devoted in Ch. VII to the method of stationary states for nonlinear parabolic problems, which satisfy the Maximum Principle.

It is well known that if an evolution equation $u_t = \mathbf{A}(u)$ for $t > 0$, $u(0) = u_0$, has a stationary solution $u = u_*$ $(\mathbf{A}(u_*) = 0)$, there exists an attracting set $\mathcal{M}$ in the space of all initial functions, associated with that stationary state: if $u_0 \in \mathcal{M}$, then $u(t,\cdot) \to u_*$ as $t \to \infty$. This ensures that a large set of non-stationary solutions is close to $u_*$ for large $t$.

For strongly non-stationary solutions, for example, those exhibiting finite time blow-up $(\|u(t,\cdot)\| \to \infty$ as $t \to T_0^- < \infty)$, stabilization to $u_*$ is, of course, impossible. Nevertheless, as we show in Ch. VII, there still is a certain "closeness" of such solutions, now to a whole family of stationary states $\{U_\lambda\}$ (parametrized by $\lambda$). Using a number of examples we find that a family of stationary states $\{U_\lambda\}$ $(\mathbf{A}(U_\lambda) = 0)$, continuously depending on $\lambda$, contains in a "parametrized" way (in the sense of dependence $\lambda = \lambda(t)$) many important evolution properties of the solutions of the equation. Since to use the method we need only the most general information concerning the family $\{U_\lambda\}$, this fact allows us to describe quite subtle effects connected with the development of unbounded solutions.

In addition, in Ch. VII we analyse blow-up behaviour and global solutions of boundary value problems for quasilinear parabolic equations with a source. In the last section we consider difference schemes for quasilinear equations admitting unbounded solutions.

In the first two introductory chapters we use a consecutive enumeration (in each chapter) of theorems, propositions and auxiliary statements. In the following, more specialized chapters, theorems and lemmas are numbered anew in each section. In each section formulas are numbered consecutively as well. The number of references to formulas from other sections is reduced to a minimum; on the rare occasions when this is necessary, a double numeration scheme is used, with the first number being the section number.

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*Chapter I*

# Preliminary facts of the theory of second order quasilinear parabolic equations

In this introductory chapter we present well-known facts of the theory of second order quasilinear parabolic equations, which will be used below in our treatment of various more specialized topics.

The main goal of the present chapter is to show, using comparatively uncomplicated examples, the wide variety of properties of solutions of nonlinear equations of parabolic type and to give the reader an idea of methods of analysis to be used in subsequent chapters. In particular, we want to emphasize the part played by particular (self-similar or invariant) solutions of equations under consideration, which describe important characteristics of nonlinear dissipative processes and provide a "basis" for a description, in principle, of a wide class of arbitrary solutions. This type of representation is dealt with in detail in Ch. VI.

In this chapter we illustrate by examples the simplest propositions of the theory of quasilinear parabolic equations. A more detailed presentation of some of the questions mentioned here can be found in Ch. II; subsequent chapters develop other themes.

## § 1 Statement of the main problems. Comparison theorems

### 1 Formulation of boundary value and Cauchy problems

In the majority of cases we shall deal with quasilinear parabolic equations of the following type: *nonlinear heat equations*,

$$u_t = \mathbf{A}(u) = \nabla\cdot(k(u)\nabla u), \quad \nabla(\cdot) = \mathrm{grad}_x(\cdot), \ x \in \mathbf{R}^N, \tag{1}$$

or with *nonlinear heat equations with source (sink)*,

$$u_t = \mathbf{B}(u) = \nabla\cdot(k(u)\nabla u) + Q(u). \tag{2}$$

Here the function $k(u)$ has the meaning of nonlinear thermal conductivity, which depends on the temperature $u = u(t, x) \geq 0$. We shall take the coefficient $k$ to be a non-negative and sufficiently smooth function: $k(u) \in C^2((0, \infty)) \cap C([0, \infty))$.

If $u > 0$ is a sufficiently smooth solution, then (1) can be rewritten in the form

$$u_t = \mathbf{A}(u) \equiv k(u)\Delta u + k'(u)|\nabla u|^2, \tag{1'}$$

where $\Delta$ is the *Laplace operator*,

$$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}, \quad |\nabla u|^2 = \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}\right)^2.$$

Equation (1) is equivalent to the equation

$$u_t = \mathbf{A}(u) \equiv \Delta\phi(u), \tag{1''}$$

$$\phi(u) = \int_0^u k(\eta)\, d\eta, \quad u \geq 0.$$

The function $Q(u)$ in (2) describes the process of heat emission or combustion in a medium with nonlinear thermal conductivity if $Q(u) \geq 0$ for $u \geq 0$, or of heat absorption if $Q(u) \leq 0$. Unless explicitly stated otherwise, we shall consider the function $Q(u)$ to be sufficiently smooth: $Q(u) \in C^1([0, \infty))$. In most cases we assume that there is no heat emission (absorption) in a cold medium, $Q(0) = 0$.

In the following, we shall mainly deal with the first boundary value problem and with the Cauchy problem for the equations (1), (2). In the *first boundary value problem* we have to find a function $u(t, x)$, which satisfies in $(0, T) \times \Omega$, where $T > 0$ is a constant and $\Omega$ is a (possibly unbounded) domain in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$, the equation under consideration, together with the initial and boundary conditions

$$u(0, x) = u_0(x) \geq 0, \quad x \in \Omega; \quad u_0 \in C(\Omega), \ \sup u_0 < \infty; \tag{3}$$

$$\begin{gathered} u(t, x) = u_1(t, x) \geq 0, \quad t \in (0, T), \ x \in \partial\Omega; \\ u_1 \in C([0, T) \times \partial\Omega), \ \sup u_1 < \infty. \end{gathered} \tag{4}$$

The function $u_0(x)$ in (3) can be considered as the initial temperature perturbation. The condition (4) describes the exchange of heat with the surroundings on the boundary $\partial\Omega$ of the domain. The condition $\sup u_0 < \infty$ is of importance in the case of unbounded $\Omega$. The solution of problems (1), (3), (4) or of (2)–(4) is then also sought in the class of functions bounded uniformly in $x \in \overline{\Omega}$ for $t \in [0, T)$.

Apart from the first boundary value problem, we shall also consider the *Cauchy problem* in $(0, T) \times \mathbf{R}^N$ with the initial condition

$$u(0, x) = u_0(x) \geq 0, \quad x \in \mathbf{R}^N; \quad u_0 \in C(\mathbf{R}^N), \ \sup u_0 < \infty. \tag{5}$$

We are looking for a solution in the class of functions bounded uniformly in $x \in \mathbf{R}^N$ for $t \in [0, T)$.

In the above statement of the problems we omitted some details, which need to be clarified. First of all, it is not made clear in what sense the solution $u(t, x)$ is to satisfy the equation, and the boundary and initial conditions. This question is easily solved if we are looking for a *classical solution* $u \in C^{1,2}_{t,x}((0, T) \times \Omega) \cap C([0, T) \times \overline{\Omega})$, which has all the derivatives entering the equation, and which satisfies it in the usual sense. Naturally, for a classical solution to exist, we must have a compatibility condition between the initial and boundary conditions in the first boundary value problem:

$$u_0(x) = u_1(0, x), \quad x \in \partial\Omega.$$

In this case conditions (3), (4) or (5) are satisfied in the usual sense.

Secondly, the coefficients $k$, $Q$ were defined only for $u \geq 0$. Therefore the formulation of the problems assumes that the solution $u(t, x)$ is everywhere non-negative. This is ensured by the Maximum Principle, which plays a fundamental part in practically all aspects of the theory of nonlinear parabolic equations.

## 2 The Maximum Principle and comparison theorems

The Maximum Principle characterizes a kind of "monotonicity" property of solutions of parabolic equations with respect to initial and boundary conditions. We shall not present here the Maximum Principle for linear parabolic equations, which serves as the basis of proof of similar assertions for nonlinear problems. It is extensively dealt with in many textbooks and monographs (see, for example, [282, 101, 378, 338, 357, 320, 22, 361, 365, 42]). There the reader can also find the necessary restrictions on the smoothness and the structure of the boundary $\partial\Omega$ (they are especially important when the domain $\Omega$ is unbounded). Therefore we move on directly to assertions pertaining to the nonlinear problems discussed above.

Assertions of this kind are known under the heading of *Maximum Principle*, since they all share the same "physical" interpretation and are proved by broadly the same techniques, which are frequently used in the course of the book.

The comparison theorems we quote below are proved in detail, for example, in [101, 338, 356, 40]. We state the theorems in the case of boundary value problems, but they apply without changes also to the case of Cauchy problems.

**Theorem 1.** *Let $u^{(1)}$ and $u^{(2)}$ be non-negative classical solutions of equation* (2) *in $(0, T) \times \Omega$, such that, moreover,*

$$u^{(2)}(0, x) \geq u^{(1)}(0, x), \quad x \in \Omega, \tag{6}$$

$$u^{(2)}(t, x) \geq u^{(1)}(t, x), t \in [0, T), x \in \partial\Omega. \tag{7}$$

*Then*

$$u^{(2)}(t, x) \geq u^{(1)}(t, x) \text{ in } [0, T) \times \Omega; \tag{8}$$

The theorem can be easily explained in physical terms. Indeed, the bigger the initial temperature perturbation, and the more intensive the boundary heat supply, the higher will be the temperature in the medium. The proof of the theorem is based on the analysis of the "linear" parabolic equation for the difference $z = u^{(2)} - u^{(1)}$ and in essence uses the sign-definiteness of the derivative $\Delta z$ at an extremum point of the function $z$.

As a direct corollary of Theorem 1 we have the following

**Proposition 1.** *Let $Q(0) = 0$ and let $u(x, t)$ be a classical solution of the problem* (2)–(4). *Then $u \geq 0$ in $[0, T) \times \Omega$.*

Indeed, $u^{(1)} \equiv 0$ is a solution of equation (2). Then by setting $u^{(2)} = u$, we see that conditions (6), (7) hold, so that $u^{(2)} \geq u^{(1)} \equiv 0$ everywhere in $[0, T) \times \Omega$.

The comparison theorem makes it possible to compare different solutions of a parabolic equation and thus enables us, by using some fixed solution, to describe the properties of a wide class of other solutions. However, the fact that this theorem involves only exact solutions significantly restricts its applicability.

The following theorem has much wider applications in the analysis of nonlinear parabolic equations [101, 377, 338, 365].

**Theorem 2.** *Let be defined on $[0, T) \times \overline{\Omega}$ a classical solution $u(x, t) \geq 0$ of the problem* (2)–(4), *as well as the functions $u_\pm(t, x) \in C_{t,x}^{1,2}((0, T) \times \Omega) \cap C([0, T) \times \overline{\Omega})$, which satisfy the inequalities*

$$\partial u_+/\partial t \geq \mathbf{B}(u_+),\ \partial u_-/\partial t \leq \mathbf{B}(u_-) \text{ in } (0,\ T) \times \Omega, \tag{9}$$

*and furthermore*

$$u_-(0, x) \leq u_0(x) \leq u_+(0, x), \qquad x \in \Omega; \tag{10}$$

$$u_-(t, x) \leq u_1(t, x) \leq u_+(t, x), t \in [0, T), \quad x \in \partial\Omega. \tag{11}$$

*Then*

$$u_- \leq u \leq u_+ \text{ in } [0, T) \times \overline{\Omega}. \tag{12}$$

Let us emphasize that here we are talking about comparing a solution of the problem not with another solution of the same problem, as in Theorem 1, but with solutions of the corresponding differential inequalities (9). This extends the possibilities for analysis of properties of solutions of nonlinear parabolic equations, since it is much simpler to find a useful solution of a differential inequality than it is to find an exact solution of a parabolic equation.

The functions $u_+$ and $u_-$, which satisfy the inequalities (9)–(11) are called, respectively, a *supersolution* and a *subsolution* of the problem (2)–(4).

Statements similar to Theorems 1, 2 hold also for nonlinear parabolic equations of general form, in particular, for essentially nonlinear (not quasilinear) equations

$$u_t = F(u, \nabla u, \Delta u, t, x), \tag{13}$$

where $F(p, q, r, t, x)$ is a function which is continuously differentiable in $\mathbf{R}_+ \times \mathbf{R}^N \times \mathbf{R} \times [0, T) \times \overline{\Omega}$. The parabolicity condition here has the form

$$\partial F(p, q, r, t, x)/\partial r \geq 0. \tag{13'}$$

If we take for $F$ the operator in (1) or (2), then condition (13′) becomes the inequality $k(p) \geq 0$ for $p \geq 0$.

Under some additional restrictions on the domain $\Omega$ and its boundary, these assertions also hold for the *second boundary value problem*, in which instead of (4) we have on $\partial\Omega$, for example, the Neumann condition of the following type:

$$\partial u/\partial n = u_2(t, x), t \in (0, T), x \in \partial\Omega; \ u_2 \in C, \sup u_2 < \infty, \tag{14}$$

where $\partial/\partial n$ denotes the derivative in the direction of $n$, the outer normal to $\partial\Omega$. Condition (14) makes sense if the partial derivatives $u_{x_i}$ are continuous in $[0, T) \times \overline{\Omega}$. Then a new compatibility condition arises:

$$\partial u_0(x)/\partial n = u_2(0, x), \quad x \in \partial\Omega,$$

and then we can talk about a classical solution of the second boundary value problem.

In this case in Theorem 1 instead of the inequality (7) we must have the inequality

$$\partial u^{(2)}/\partial n \geq \partial u^{(1)}/\partial n, \quad t \in [0, T), \quad x \in \partial\Omega. \tag{14'}$$

Since the product $k(u)\partial u/\partial n$ equals the heat flux on the boundary, (14′) has a simple physical meaning. Correspondingly, in Theorem 2 the inequalities (11) are replaced by the inequalities

$$\partial u_- /\partial n \leq \partial u/\partial n \leq \partial u_+/\partial n, \ t \in [0, T), x \in \partial\Omega \tag{15}$$

(in this case additional smoothness conditions have to be imposed on super- and subsolutions $u_\pm$).

With the required changes, the theorems still hold if we have more general nonlinear boundary conditions of the third kind on $\partial\Omega$, such as

$$\partial u/\partial n = a(u, t, x), \ t \in [0, T), x \in \partial\Omega, \tag{16}$$

where $a(u, t, x)$ is a sufficiently smooth function [101, 338].

## § 2 Existence, uniqueness, and boundedness of the classical solution

Questions of existence and uniqueness of classical solutions of boundary value problems for nonlinear heat equations are studied in detail in the well-known monographs [282, 101, 361], where a wide spectrum of methods is used. Below we consider some important restrictions on coefficients, that are necessary for existence and uniqueness of a classical solution.

We shall be especially interested in questions of conditions for *global solvability* of boundary value problems, when the solution $u(t, x)$ is defined for all $t \geq 0$, and, conversely, in conditions for *global insolvability* or *insolvability in the large*. In other words, we want to know when a local solution $u(t, x)$, defined on some interval $(0, T)$, can be extended to arbitrary values $t > 0$, and when it cannot. Local solvability (solvability in the small) holds for a large class of quasilinear equations with sufficiently smooth coefficients without any essential restrictions on the nature of the nonlinearity of these coefficients. Such restrictions arise in the process of constructing a global solution.

For equations with a source,

$$u_t = \nabla \cdot (k(u)\nabla u) + Q(u), \tag{1}$$

the existence of a global solution is equivalent to its boundedness in $\overline{\Omega}$ on an arbitrary interval $(0, T)$. Namely: a global solution is defined and bounded in $\overline{\Omega}$ for all $t \geq 0$, while an unbounded solution is defined in $\overline{\Omega}$ on a finite interval $(0, T_0)$, such that moreover

$$\overline{\lim_{t \to T_0}} \sup_{x \in \Omega} u(t, x) = \infty, \tag{2}$$

which makes it impossible to continue the solution to values of $t > T_0$.

Questions related to the loss of requisite smoothness of a bounded solution are discussed in § 3.

### 1 Conditions for local existence of a classical solution

This question is now well understood [260, 282, 363, 101, 213]. Classical solutions of boundary value problems and of the Cauchy problem exist locally for smooth boundary data and under the necessary compatibility conditions for quite arbitrary quasilinear parabolic equations with smooth coefficients of the form

$$u_t = \sum_{i,j=1}^{N} a_{ij}(u, \nabla u, t, x) u_{x_i x_j} + a(u, \nabla u, t, x), \tag{3}$$

if they are *uniformly parabolic*. This means that

$$\nu(p)\|r\|^2 \le \sum_{i,j=1}^{N} a_{ij}(p,q,t,x)r_i r_j \le \mu(p)\|r\|^2 \tag{3'}$$

for arbitrary $t \in [0,T)$, $x \in \overline{\Omega}$, $p \ge 0$, $q$, $r \in \mathbf{R}^N$, where the continuous functions $\nu(p)$ and $\mu(p)$ are strictly positive. Condition (3′) means, in particular, that the second order elliptic operator in (3) is non-degenerate and that the matrix $\|a_{ij}\|$ is positive definite. Local solvability has been established also for a wide class of more general equations of the form (1.13)[1] (see [261, 69]). In this case the uniform parabolicity condition has the form

$$\nu(p) \le \partial F(p,q,r,t,x)/\partial r \le \mu(p).$$

For equations of the form (1) the uniform parabolicity condition has a particularly simple form.

**Proposition 2.** *Let the functions $k(u)$, $Q(u)$ be sufficiently smooth for $u \ge 0$, $Q(0) = 0$. If the condition*

$$k(u) \ge \epsilon_0 = \text{const} > 0 \text{ for } u > 0, \tag{4}$$

*holds, then there exists a local classical solution of the boundary value problem* (1.2)–(1.4)*; moreover, if $u_0 \not\equiv 0$ in $\overline{\Omega}$ or if $u_1(0,x) \not\equiv 0$ on $\partial\Omega$, then $u(t,x) > 0$ in $\Omega$ for all admissible $t > 0$.*

A non-negative solution of a uniformly parabolic equation (1) is strictly positive everywhere in its domain of definition. In other words, in heat transfer processes described by such equations, perturbations propagate with infinite speed. If, for example, in the Cauchy problem, the initial function $u_0 \not\equiv 0$ has compact support and possibly is non–differentiable, the local solution will still be a classical one for $t > 0$. Moreover, for all sufficiently small $t > 0$ the function $u(t,x)$ will be strictly positive in $\mathbf{R}^N$. Under appropriate restrictions on the coefficients of the equation in any admissible domain $(0,T) \times \Omega$ it will possess high order derivatives in $t$ and $x$.

If condition (4) does not hold, a solution of the Cauchy problem with an initial function $u_0$ of compact support, may also have compact support in $x$ for all $t > 0$, and as a result even its first derivatives in $t$ and $x$ can be not defined at a point where it vanishes. We shall treat generalized solutions in more detail in § 3, where we state a necessary and sufficient condition for existence of a strictly positive (and therefore classical) solution.

[1] In this way we refer to formulae from previous sections; in this case it is from § 1.

## 2 Condition for global boundedness of solutions

First of all let us observe that in the boundary value problem (1.1), (1.3), (1.4) without source, $Q \equiv 0$, the question of boundedness of solutions does not arise. This follows directly from Theorem 1 (§ 1). Setting in that theorem

$$u^{(2)}(t, x) \equiv M = \text{const} \geq \max\{\sup u_0, \sup u_1\}, \tag{5}$$

$$u^{(1)}(t, x) \equiv u(t, x),$$

we see that conditions (1.6) and (1.7) hold, so that $u(t, x) \leq M$, that is, $u$ is bounded in $\overline{\Omega}$ for all $t \in (0, T)$, where $T > 0$ is arbitrary. It is easy to verify that the same is true for equation (1) with a sink, when $Q(u) \leq 0$ for all $u \geq 0$. For equations with a source the situation is different.

**Proposition 3.** *In equation* (1), *let* $Q(u) > 0$ *for* $u > 0$. *Then the condition*

$$\int_1^\infty \frac{d\eta}{Q(\eta)} = \infty \tag{6}$$

*is a necessary and sufficient condition for global boundedness of any solution of the problem* (1.2)–(1.4).

*Proof.* Sufficiency. Let us use Theorem 1. As $u^{(2)}(t, x)$ let us take the spatially homogeneous solution $u^{(2)}(t)$ of (1):

$$\frac{du^{(2)}(t)}{dt} = Q(u^{(2)}(t)), \, t > 0; \; u^{(2)}(0) = M > 0, \tag{7}$$

where the constant $M$ satisfies (5). The function $u^{(2)}(t)$ is determined from the equation

$$\int_M^{u^{(2)}(t)} \frac{d\eta}{Q(\eta)} = t,$$

where, moreover, by (6) $u^{(2)}(t)$ is defined for all $t \in (0, \infty)$. Then from Theorem 1, by setting $u^{(1)} \equiv u$ we obtain that

$$u(t, x) \leq u^{(2)}(t), \quad t \in (0, T), \, x \in \Omega,$$

that is, $u$ is globally bounded.

Necessity. This follows from the following simple example.

**Example 1.** In the Cauchy problem for (1) let

$$u_0(x) \equiv m = \text{const} > 0, \quad x \in \mathbf{R}^N,$$

and let (6) be violated, that is,

$$\int_1^\infty \frac{d\eta}{Q(\eta)} < \infty, \tag{8}$$

where $Q(u) > 0$ for $u > 0$. The solution of the problem will then be spatially homogeneous: $u(t, x) \equiv u(t)$, where $u(t)$ satisfies (7) and the condition $u(0) = m$, that is,

$$\int_m^{u(t)} \frac{d\eta}{Q(\eta)} = t.$$

From this it can be seen that $u(t)$ is defined on a finite time interval $(0, T_0)$ where

$$T_0 = \int_m^\infty \frac{d\eta}{Q(\eta)} < \infty;$$

furthermore

$$u(t) \to \infty, t \to T_0.$$

□

Proposition 3 reflects one aspect of the problem of unboundedness of solutions. In a number of problems with specific boundary conditions, the existence of a global upper bound for the classical solution depends on the interplay of the coefficients $k$, $Q$, functions entering the statement of the boundary conditions, as well as the spatial structure of the domain $\Omega$. In the general setting the problem of unboundedness is quite a complicated one. For some classes of equations this problem will be analyzed in subsequent chapters (some examples are given below).

Let us observe that the necessary and sufficient condition (6) of global boundedness of all classical solutions arises in an analysis of an ordinary differential equation. In Example 1 we constructed an unbounded solution which grows to infinity as $t \to T_0^-$ on all of the space $\mathbf{R}^N$ at the same time.

What happens if we consider a boundary value problem in a bounded domain $\Omega$, such that, furthermore, on $\partial\Omega$ the solution is bounded from above uniformly in $t$? Can such spatially inhomogeneous solutions be unbounded in the sense of (2)? The following example gives a positive answer.

**Example 2.** Let us consider a boundary value problem for a semilinear equation,

$$u_t = \Delta u + Q(u), \quad t > 0, \quad x \in \Omega, \tag{9}$$

in a bounded domain $\Omega \in \mathbf{R}^N$ with a smooth boundary $\partial\Omega$. Let $u(0, x) = u_0(x) \geq 0$ in $\Omega$, $u_0 \in C(\overline{\Omega})$, $u_0 \not\equiv 0$, and

$$u(t, x) = 0, \quad t > 0, \quad x \in \partial\Omega. \tag{10}$$

Let us denote by $\lambda_1 = \lambda_1(\Omega) > 0$ the first (smallest) eigenvalue of the problem

$$\Delta\psi + \lambda\psi = 0, x \in \Omega; \ \psi(x) = 0, \ x \in \partial\Omega, \tag{11}$$

and by $\psi_1(x)$ the first eigenfunction, which is known [283, 362] to be of constant sign in $\Omega$. Let $\psi_1(x) > 0$ and

$$\|\psi_1\|_{L^1(\Omega)} \equiv \int_\Omega \psi_1(x)\, dx = 1. \tag{12}$$

Let $Q(u) - \lambda_1 u > 0$ for all $u \geq \delta_0 = \text{const} > 0$, and furthermore

$$\int_{\delta_0}^\infty \frac{d\eta}{Q(\eta) - \lambda_1\eta} < \infty \tag{13}$$

(let us note that if $Q(u) \gg u$ as $u \to \infty$ this condition is the same as (8)). Let us also assume that $Q \in C^2(\mathbf{R}_+)$ is a convex function:

$$Q''(u) \geq 0, u > 0. \tag{14}$$

Then for any initial functions $u_0(x) \geq 0$ such that

$$E_0 = \int_\Omega u_0(x)\psi_1(x)\, dx \geq \delta_0,$$

the solution of the prohlem is unbounded and exists till time

$$T_0 \leq T_* = \int_{E_0}^\infty \frac{d\eta}{Q(\eta) - \lambda_1\eta} < \infty.$$

To prove this, let us introduce the function

$$E(t) = \int_\Omega u(t, x)\psi_1(x)\, dx.$$

Then $E(0) = E_0$ and furthermore, as follows from (9), $E(t)$ satisfies the equality

$$\frac{dE(t)}{dt} = \int_\Omega \Delta u(t, x)\psi_1(x)\, dx + \int_\Omega Q(u(t, x))\psi_1(x)\, dx. \tag{15}$$

Integrating by parts and taking into account (10) and (11), we obtain

$$\int_\Omega \Delta u(t, x)\psi_1(x)\, dx =$$

$$= \int_\Omega u(t, x)\Delta\psi_1(x)\, dx = -\lambda_1 \int_\Omega u(t, x)\psi_1(x)\, dx = -\lambda_1 E(t).$$

Furthermore, from Jensen's inequality for convex functions [211] we obtain

$$\int_{\Omega} Q(u)\psi_1\,dx \geq Q\left(\int_{\Omega} u\psi_1\,dx\right) = Q(E)$$

(for this estimate to hold it is essential to have $\psi_1 > 0$ in $\Omega$ and for $\psi$ to be normalized by (12)), so that from (15) we have the inequality

$$\frac{dE(t)}{dt} \geq -\lambda_1 E + Q(E),\ t > 0;\ E(0) = E_0 \geq \delta_0.$$

Hence under our assumptions we have that $E(t) > E_0$ for all $t > 0$, and consequently

$$\int_{E_0}^{E(t)} \frac{d\eta}{Q(\eta) - \lambda_1\eta} \geq t,\quad t > 0.$$

Therefore by (13) $E(t) \to \infty$ as $t \to T_1^- \leq T_*$, and since $E(t) \leq \sup_x u(t, x)$, the solution $u(t, x)$ satisfies (2) for some $T_0 \leq T_1$ and is unbounded.

The interest of this example lies in the fact that for sufficiently "small" initial data $u_0(x)$ this boundary value problem has a global solution defined for all $t \gg 0$ (see Ch. VII, § 2). For "large" $u_0$ it grows unboundedly as $t \to T_0^-$, $T_0 < \infty$. One can then pose the question: in what portion of the domain $\Omega$ does it become unbounded as $t \to T_0^-$? This question, of localization of unbounded solutions, is considered in subsequent chapters.

We close the discussion of global boundedness conditions by an elementary example of a second boundary value problem.

**Example 3.** Let $\Omega$ be a smooth bounded domain, $\Omega \in \mathbf{R}^N$. Let $Q(u)$ be a function convex for $u > 0$, which satisfies (8). For (1), let us consider the second boundary value problem with no-flux Neumann boundary condition,

$$\partial u/\partial n = 0,\ t > 0,\ x \in \partial\Omega, \tag{16}$$

with initial perturbation $u(0, x) = u_0(x) \geq 0$ in $\Omega$. Let us show that any non-trivial ($u \not\equiv 0$) solution of the problem is unbounded.

Assuming sufficient smoothness of the solution, let us integrate equation (1) over the domain $\Omega$. Then, if we introduce the energy

$$H(t) = \int_{\Omega} u(t, x)\,dx,\ t \geq 0,$$

and integrate by parts, taking (16) into consideration, we have

$$\begin{gathered} \frac{dH(t)}{dt} = \int_{\Omega} Q(u(t, x))\,dx,\ t > 0; \\ H(0) = H_0 = \int_{\Omega} u_0(x)\,dx > 0. \end{gathered} \tag{17}$$

Using the Jensen inequality

$$\int_\Omega Q(u(t,x))\,dx \equiv (\operatorname{meas}\Omega)\int_\Omega \frac{1}{\operatorname{meas}\Omega} Q(u(t,x))\,dx \geq$$
$$\geq (\operatorname{meas}\Omega)\, Q\left(\int_\Omega \frac{1}{\operatorname{meas}\Omega} u(t,x)\,dx\right) \equiv (\operatorname{meas}\Omega)\, Q\left(\frac{H(t)}{\operatorname{meas}\Omega}\right),$$

we obtain from (17) the inequality

$$\frac{dH(t)}{dt} \geq (\operatorname{meas}\Omega)\, Q\left(\frac{H(t)}{\operatorname{meas}\Omega}\right), \quad t > 0.$$

Therefore by (8) it follows that the energy $H(t)$ (and therefore $u(t,x)$) is defined and bounded only on a bounded interval $(0, T_1)$, where

$$T_1 \leq T_* = \int_{H_0/\operatorname{meas}\Omega}^{\infty} \frac{d\eta}{Q(\eta)} < \infty,$$

and therefore $\overline{\lim}\sup_x u(t,x) = \infty$, $t \to T_0^- \leq T_1$.

## 3 Uniqueness conditions for the classical solution

Under the assumption of sufficient smoothness of the coefficient $Q$ in (1), the local classical solution is always unique. This follows directly from Theorem 1 of § 1. Indeed, if we assume that there exist two different solutions $u^*$ and $u_*$ of equation (1) corresponding to the same initial and boundary conditions, then it follows from Theorem 1, by first setting $u^{(1)} = u^*$, $u^{(2)} = u_*$ and then exchanging $u^*$ and $u_*$, that we have at the same time $u^* \leq u_*$ and $u^* \geq u_*$, that is, $u^* \equiv u_*$.

It remains to check how essential is the smoothness requirement on the coefficient $Q$, which is a non-negative function. In case of a heat sink ($Q(u) < 0$, $u > 0$), it is not hard to verify that uniqueness of the solution holds without any restrictions on the smoothness of $Q(u)$.

Thus, let a continuous function $Q(u)$, ($Q(0) = 0$; $Q(u) > 0$, $u > 0$) be non-differentiable for $u = 0$, $Q \in C^1((0, \infty))$. The following example shows what this can lead to.

**Example 4.** Let us consider the Cauchy problem for the equation

$$u_t = \Delta u + u^\alpha, \quad x \in \mathbf{R}^N, \tag{18}$$

where $\alpha \in (0, 1)$ is a constant. Here $Q(u) = u^\alpha$, $Q(0) = 0$, $Q'(0^+) = \infty$. Let

$$u(0, x) = u_0(x) \equiv 0, \quad x \in \mathbf{R}^N. \tag{19}$$

It is clear that the problem (18), (19) has the trivial solution $u(t, x) \equiv 0$. However, in addition it has an infinite number of other spatially homogeneous solutions $u(t, x) \equiv u(t)$, which satisfy the ordinary differential equation

$$u'(t) = u^{\alpha}(t), \ t > 0; \ u(0) = 0. \tag{20}$$

Solutions of this problem are the functions

$$u(t) = v_{\tau}(t) = \begin{cases} 0, 0 \le t < \tau, \\ (1 - \alpha)^{1/(1-\alpha)}(t - \tau)^{1/(1-\alpha)}, \ t \ge \tau, \end{cases} \tag{21}$$

where $\tau \geqslant 0$ is an arbitrary constant.

Therefore, due to non-differentiability of the source for $u = 0$, there appear from the zero initial condition (19) non-trivial solutions that grow at the same rate on the whole space. Let us note that for $\alpha \in (0, 1)$ all the functions $v_{\tau}(t)$ are classical solutions, since $v_{\tau} \in C^1([0, \infty))$.

It is not hard to see that similar non-trivial solutions of the Cauchy problem can be constructed in the case of arbitrary sources $Q(u) > 0$, $u > 0$, if

$$\int_0^1 \frac{d\eta}{Q(\eta)} < \infty. \tag{22}$$

Hence we obtain the condition

$$\int_0^1 \frac{d\eta}{Q(\eta)} = \infty, \tag{22'}$$

which is at least *necessary for the uniqueness of the solutions of the Cauchy problem.*

This example is entirely based on an analysis of spatially homogeneous solutions, which satisfy an ordinary differential equation. What if we consider a problem with boundary conditions that do not allow the solution to grow at the boundary? It turns out that in this case also lack of sufficient smoothness of the source for $u = 0$ may cause the solutions to be non-unique.

**Example 5.** Let $\Omega$ be a bounded domain, $\Omega \subset \mathbf{R}^N$, and let $\lambda_1 > 0$, $\psi_1(x) > 0$ in $\Omega$, be, respectively, the first eigenvalue and the corresponding eigenfunction of the problem (11). Let us consider in $\mathbf{R}_+ \times \Omega$ a boundary value problem for the equation

$$u_t = \Delta u + \lambda_1 u + \psi_1^{1-\alpha}(x)u^{\alpha}, \ t > 0, \ x \in \Omega, \tag{23}$$

with the conditions

$$u(0, x) \equiv 0, x \in \overline{\Omega}; \ u(t, x) = 0, \ t > 0, x \in \partial\Omega. \tag{24}$$

Let $\alpha \in (0, 1)$; then the source $\psi_1^{1-\alpha}(x)u^\alpha$, which depends not only on the solution $u$, but also on the spatial coordinate $x$, is non-differentiable in $u$ for $u = 0^+$ everywhere in $\Omega$. It is not hard to see that the problem (23), (24) has, in addition to the trivial solution $u \equiv 0$, the family of solutions

$$u(t, x) = v_T(t)\psi_1(x), \quad t > 0, \quad x \in \Omega,$$

where $v_T(t)$ are the functions defined in (21).

To conclude, let us observe that non-uniqueness is related to the particular formulation of a problem. If, for example, we take in the Cauchy problem for (18) an initial condition $u_0(x) \geq \delta_0 > 0$ in $\mathbf{R}^N$, then its solution will be classical and unique, since by Theorem 2, the solution will satisfy the condition $u(t, x) \geq \delta_0$ in $\mathbf{R}^N$. In the domain $u \geq \delta_0$ the coefficients of the equation are sufficiently smooth, which ensures uniqueness of the solution. Similarly, if in the problem (23), (24) $u_0 > 0$ in $\Omega$, then its solution will also be unique.

## § 3 Generalized solutions of quasilinear degenerate parabolic equations

In this section we consider equations (1.1), (1.2) which do not satisfy the uniform parabolicity condition. As above, we shall assume that the functions $k$ and $Q$ are sufficiently smooth: $k \in C^2((0, \infty)) \cap C([0, \infty))$, $Q \in C^1([0, \infty))$ (as was shown in § 2 this last condition is necessary for the uniqueness of the solution), $k(u) > 0$ for $u > 0$, and furthermore

$$k(0) = 0, \tag{1}$$

that is, the equation is degenerate. Formally this condition means that the second order equation (1.1′) that is equivalent to (1.1) degenerates for $u = 0$ into a first order equation (if $k'(0) \neq 0$ and $u(t, x)$ has two derivatives in $x$).

Before we move on to examples that elucidate certain properties of generalized (weak) solutions, we shall make a remark. When we dealt with classical solutions $u \in C_{t,x}^{1,2}$, there was no need to require continuity of the heat flux $W(t, x) = -k(u(t, x))\nabla u(t, x)$. This condition, as well as continuity of the solution itself (temperature), is a natural physical requirement on the formulation of the problem. In the present case we shall constantly have to monitor this property of generalized solutions.

## 1 Examples of generalized solutions (finite speed of propagation of perturbations, localization of boundary blow-up regimes and in media with sinks)

**Example 6.** (finite speed of propagation of perturbations) Let us consider equation (1.1) in the one-dimensional case:

$$u_t = (k(u)u_x)_x \tag{2}$$

and let us construct its particular self-similar solution of travelling wave type:

$$u_S(t, x) = f_S(\xi), \ \xi = x - \lambda t, \tag{3}$$

where $\lambda > 0$ is the speed of motion of the thermal wave. Substituting the expression (3) into (2), we obtain for $f_S(\xi) \geq 0$ the equation

$$\frac{d}{d\xi}\left(k(f_S)\frac{df_S}{d\xi}\right) + \lambda\frac{df_S}{d\xi} = 0,$$

or, which is the same,

$$k(f_S)\frac{df_S}{d\xi} + \lambda f_S = C. \tag{3'}$$

Setting $C = 0$ (what this corresponds to will be made clear in the following), we obtain the equality

$$\frac{k(f_S)}{f_S}\frac{df_S}{d\xi} = -\lambda. \tag{4}$$

Let us assume that

$$\int_0^1 \frac{k(\eta)}{\eta}\, d\eta < \infty, \tag{5}$$

so that the function

$$\Phi(u) = \int_0^u \frac{k(\eta)}{\eta}\, d\eta, \ u \geq 0; \ \Phi(0) = 0, \tag{5'}$$

makes sense. Then it follows from (4) that

$$\Phi(f_S(\xi)) = -\lambda(\xi - \xi_0), \ \xi \leq \xi_0 = \text{const}.$$

Let $\xi_0 = 0$, then

$$f_S(\xi) = \Phi^{-1}(-\lambda\xi), \ \xi \leq 0,$$

where $\Phi^{-1}$ is the function inverse to $\Phi$ (it exists by monotonicity of $\Phi$; see (5')). Let us extend $f_S$ into the domain $\{\xi > 0\}$ identically by zero; it follows from (3') that continuity of the heat flux $-k(f_S)f'_S$ will still hold at the point $\xi = 0$ for $C = 0$. As a result we obtain the following self-similar solution:

$$u_S(t,x) = \Phi^{-1}[\lambda(\lambda t - x)_+], \quad t > 0, \quad x \in \mathbf{R}, \tag{6}$$

where we have introduced the notation $(\kappa)_+ = [\kappa, \text{ if } \kappa \geq 0 \text{ and } 0, \text{ if } \kappa < 0\}$. Let us set $T_0 = \Phi(\infty)/\lambda^2 \leq \infty$. Then (6) can be considered as the solution in $(0, T_0) \times \mathbf{R}_+$ of the first boundary value problem for equation (2) with the conditions

$$u(0,x) = 0, \quad x > 0; \quad u(t,0) = \Phi^{-1}(\lambda^2 t), \quad 0 < t < T_0. \tag{7}$$

Thus if condition (5) holds, the problem (2), (7) has a solution with everywhere continuous heat flux, which has *compact support* in $x$ for each $t \in (0, T_0)$:

$$u_S(t,x) \equiv 0, \quad x \geq \lambda t, \quad t \in (0, T_0).$$

Therefore equation (2) describes *processes with finite speed of propagation of perturbations*. At the point where $u_S > 0$, the solution of the problem is a classical one and it is not necessarily sufficiently smooth at the front (the interface) of the thermal wave, $x_f(t) = \lambda t$, where it vanishes.

For a more detailed study of the behaviour of the solution at the points of degeneracy, let us consider the case

$$k(u) = u^\sigma, \quad \sigma = \text{const} > 0.$$

Then $\Phi(u) = u^\sigma/\sigma$, $\Phi^{-1}(u) = (\sigma u)^{1/\sigma}$, $T_0 = \infty$ and the travelling wave solution has an especially simple form

$$u_S(t,x) = [\sigma\lambda(\lambda t - x)_+]^{1/\sigma}, \quad t \geq 0, \quad x > 0. \tag{8}$$

Let us check again that the heat flux is continuous at the points $x_f(t) = \lambda t$. Indeed,

$$W(t,x) = -u_S^\sigma(u_S)_x = \sigma^{1/\sigma}\lambda^{(\sigma+1)/\sigma}[(\lambda t - x)_+]^{1/\sigma},$$

that is, $W(t, x_f^+(t)) = W(t, x_f^-(t)) = W(t, x_f(t)) = 0$ for all $t > 0$. At the same time, if $\sigma \geq 1$, at the degeneracy points $x = x_f(t)$ the derivatives $u_t, u_x, u_{xx}$ are not defined. In the case $\sigma \in [1/2, 1)$ the derivatives $u_t, u_x$ exist, but the derivative $u_{xx}(t, x_f(t))$ is not defined. If, on the other hand, $\sigma \in (0, 1/2)$, $u_t, u_x, u_{xx}$ are defined everywhere (that is, the compactly supported solution (8) is a classical one), however, higher order derivatives do not exist at the front points.

These are the main differentiability properties of the generalized solution we have constructed. The function (8) is schematically depicted, for different times, in Figure 1. This solution represents a thermal wave moving over the unperturbed (zero) temperature background with speed $\lambda \equiv dx_f(t)/dt$.

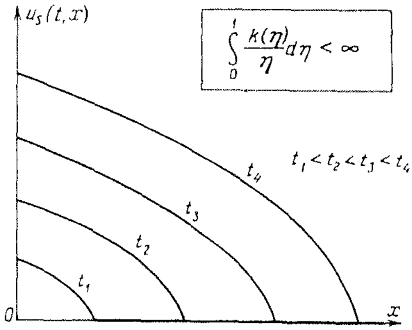


**Fig. 1.** Travelling wave in the case of finite speed of propagation of perturbations

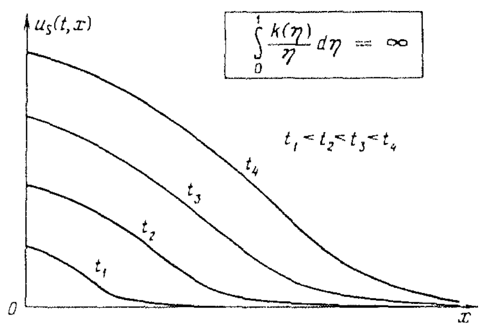


**Fig. 2.** Travelling wave in the case of infinite speed of propagation of perturbations

Condition (5) is necessary and sufficient for the existence of a compactly supported travelling wave solution. If it is violated, that is if

$$\int_0^1 \frac{k(\eta)}{\eta}\, d\eta = \infty, \tag{9}$$

then, as follows from (4), the function $f_S(\xi)$ is strictly positive for all admissible $\xi \in \mathbf{R}$, and therefore (3) represents a positive classical solution of the equation (2) (see Figure 2).

It is obvious that in the case $k(0) > 0$, that is, for uniform parabolicity of the equation (see Proposition 2, § 2), condition (9) holds. However, among coefficients $k(u)$, $k(0) = 0$, there are some for which (9) holds. This is true, for example, for the function $k(u) = |\ln u|^{-1}$, $u \in (0, 1/2)$, $k(u) > 0$ for $u \geq 1/2$. Then the travelling wave solution is strictly positive and therefore classical. Moreover, if $k(u) \in C^\infty(\mathbf{R}_+)$, then $u$ can be differentiated in $t$ and $x$ in the domain $(0, T_0) \times \mathbf{R}_+$ any number of times.

It is interesting that the condition (5), which was obtained without any difficulty, is not only sufficient, but also necessary for finite speed of propagation of perturbations in processes described by equation (1.1).

A travelling wave type solution has another exceptional quality: it demonstrates in a simple example localization in boundary heating regimes with blow-up. The study of this interesting phenomenon in various problems occupies a substantial part of the present book.

**Example 7.** (localization in a boundary blow-up regime) Let

$$k(u) = u\exp\{-u\},\ u \geq 0.$$

Then it is not hard to see that the solution (6) has the following form:

$$u_S(t,x) = \begin{cases} -\ln[1-\lambda(\lambda t - x)], & 0 \leq x \leq \lambda t, \\ 0, & x > \lambda t, \end{cases} \tag{10}$$

which is defined for a bounded time interval $[0, T_0)$, where $T_0 = 1/\lambda^2$. The boundary condition at $x = 0$ corresponding to (10) has the form

$$u_S(t,0) = u_1(t) = -\ln(1-\lambda^2 t),\ 0 < t < T_0, \tag{11}$$

and therefore $u_1(t) \to \infty$ as $t \to T_0^-$. However, though the temperature at the boundary blows up, heat penetrates only to a finite depth $L = 1/\lambda$, that is, $u_S(t,x) \equiv 0$ for all $x \geq L$ for all the times of existence of the solution, $t \in (0, T_0)$ (see Figure 3).

Here we have that everywhere apart from the boundary point $x = 0$, the solution is bounded from above uniformly in $t$:

$$u_S(t,x) \leq u_S(T_0^-, x) = \begin{cases} -\ln(\lambda x), & 0 < x < 1/\lambda, \\ 0, & x \geq 1/\lambda, \end{cases}$$

and it grows without bound due to the boundary blow-up regime at the single point $x = 0$. The limiting curve $u = u_S(T_0^-, x)$ is shown in Figure 3 by a thicker line. Let us note the striking difference between this halted thermal wave and the usual temperature waves shown in Figures 1, 2.

It is easy to see that in this case every boundary blow-up regime leads to localization. Indeed, for any boundary function $u_1(t) \to \infty$ as $t \to T_0$ (for simplicity we set $u_0(x) \equiv 0$), we can compare the solution $u(t, x)$ with the "shifted" self-similar solution $u_S(t, x - 1/\lambda)$, which is defined for $x > x_0(t) = \lambda t$. We have that $u \leq u_S = \infty$ for $x = x_0(t)$ for all $t \in (0, T_0)$. Therefore by the comparison theorem $u \leq u_S$ in $(0, T_0) \times \{x > x_0(t)\}$ and finally $u(T_0, x) \leq u_S(T_0, x - 1/\lambda) < \infty$ for $x > 1/\lambda$.

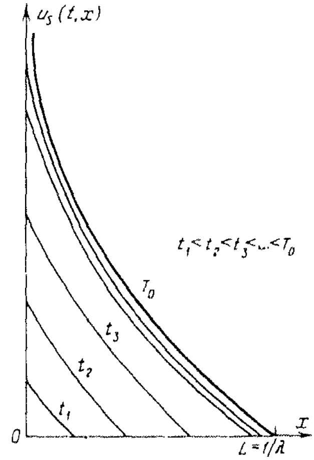


**Fig. 3.** Travelling wave self-similar solution (10) localized in the domain $(0, 1/\lambda)$

It is clear that the same comparison argument and thus the same result on localization of arbitrary boundary blow-up regimes, also holds in the case of coefficients $k$ that satisfy

$$\int_1^\infty \frac{k(\eta)}{\eta}\,d\eta < \infty.$$

This follows immediately from the representation (6) of the corresponding travelling wave solution generated by the blow-up regime.

Let us consider an example of a generalized solution of the heat equation in the multidimensional case.

**Example 8.** Let us find a solution of the Cauchy problem for an equation with a power law nonlinearity

$$u_t = \nabla\cdot(u^\sigma \nabla u),\ t > 0,\ x \in \mathbf{R}^N, \tag{12}$$

having constant energy

$$\int_{\mathbf{R}^N} u(t,x)\,dx = E_0 = \text{const} > 0 \tag{13}$$

(this is a solution of the instantaneous point source type).

We shall look for it in the self-similar ansatz

$$u(t,x) = t^\alpha \theta(\xi),\quad \xi = x/t^\beta \in \mathbf{R}^N, \tag{14}$$

where $\alpha$, $\beta$ are constants, and where $\theta(\xi) \geq 0$ is a continuous function. Substituting (14) into (12), we obtain the following equation:

$$\alpha t^{\alpha-1}\theta - \beta t^{\alpha-1} \sum_{i=1}^{N} \frac{\partial \theta}{\partial \xi_i} \xi_i = t^{\alpha(\sigma+1)-2\beta} \nabla_\xi \cdot (\theta^\sigma \nabla_\xi \theta). \tag{15}$$

From here we have the necessity of the equality $\alpha - 1 = \alpha(\sigma + 1) - 2\beta$; then the terms involving time can be cancelled. Furthermore, using the identity

$$\int_{\mathbf{R}^N} u(t, x)\, dx \equiv \int_{\mathbf{R}^N} t^\alpha \theta\left(\frac{x}{t^\beta}\right) dx \equiv t^{\alpha+N\beta} \int_{\mathbf{R}^N} \theta(\xi)\, d\xi$$

(it is assumed that $\theta \in L^1(\mathbf{R}^N)$), by (13) we have that $\alpha + N\beta = 0$. Hence we obtain a unique pair of parameters $\alpha = -N/(N\sigma + 2)$, $\beta = 1/(N\sigma + 2)$, that is, the desired solution has the form

$$u(t, x) = t^{-N/(N\sigma+2)}\theta(\xi), \quad \xi = x/t^{1/(N\sigma+2)}. \tag{16}$$

Then it follows from (15) that the function $\theta \geq 0$ satisfies the following quasilinear elliptic equation:

$$\nabla_\xi \cdot (\theta^\sigma \nabla_\xi \theta) + \frac{1}{N\sigma + 2} \sum_{i=1}^{N} \frac{\partial \theta}{\partial \xi_i} \xi_i + \frac{N}{N\sigma + 2} \theta = 0, \xi \in \mathbf{R}^N, \tag{17}$$

as well as the condition

$$\int_{\mathbf{R}^N} \theta(\xi)\, d\xi = E_0. \tag{18}$$

Let the function $\theta$ be *radially symmetric*, that is, let it depend only on one coordinate: $\theta = \theta(\eta)$, $\eta = |\xi| \geq 0$. Then equation (17) takes the form

$$\frac{1}{\eta^{N-1}} (\eta^{N-1} \theta^\sigma \theta')' + \frac{1}{N\sigma + 2} \theta' \eta + \frac{N}{N\sigma + 2} \theta = 0, \ \eta > 0; \tag{19}$$

moreover, by symmetry we have to require that the condition

$$\theta^\sigma \theta'(0) = 0$$

holds. Equation (19) is equivalent to the equation

$$(\eta^{N-1} \theta^\sigma \theta')' + \frac{1}{N\sigma + 2} (\theta \eta^N)' = 0, \ \eta > 0.$$

Integrating it, and setting the integration constant equal to zero (this, as is easily verified, is necessary for the existence of a solution with the required properties), we arrive at the first order equation

$$\eta^{N-1} \theta^\sigma \theta' + \frac{1}{N\sigma + 2} \theta \eta^N = 0, \ \eta > 0; \ \theta^\sigma \theta'(0) = 0.$$

Its solutions have the form

$$\theta(\eta) = \left[\frac{\sigma}{2(N\sigma+2)}(\eta_0^2 - \eta^2)_+\right]^{1/\sigma}, \quad \eta \geq 0. \tag{20}$$

Here $\eta_0$ is a constant, which is determined from the condition (18):

$$\eta_0(E_0) = \left\{\pi^{-N/2}\left[\frac{2(N\sigma+2)}{\sigma}\right]^{1/\sigma}\frac{\Gamma(N/2+1+1/\sigma)}{\Gamma(1/\sigma+1)}E_0\right\}^{\sigma/(N\sigma+2)}.$$

Thus the required self-similar solution with constant energy has the form

$$u_S(t,x) = t^{-N/(N\sigma+2)}\left[\frac{\sigma}{2(N\sigma+2)}\left(\eta_0^2 - \frac{|x|^2}{t^{2/(N\sigma+2)}}\right)_+\right]^{1/\sigma}. \tag{21}$$

For any $t > 0$ it has compact support in $x$, while as $t \to 0^+$, it goes to a $\delta$-function: $u_S(t,x) \to E_0\delta(x)$, $t \to 0^+$. Everywhere except on the degeneracy surface $\mathbf{R}_+ \times \{|x| = \eta_0 t^{1/(N\sigma+2)}\}$ it is classical (and infinitely differentiable), while on the surface of the front (on the interface) it has continuous heat flux. Differentiability properties of the solution (21) are the same as those of the particular solution of travelling wave type considered in Example 6. Since equation (12) is invariant under the change of $t$ to $T+t$, where $T = \text{const} > 0$, $u_S(t+T,x)$ will also be a solution with constant energy.

In the following example we use the solution constructed above to illustrate an intriguing property of a quasilinear degenerate parabolic equation with a sink.

**Example 9.** (localization of heat in media with absorption) Let us consider the equation

$$u_t = \nabla\cdot(u^\sigma\nabla u) - \gamma u, \quad t>0, \quad x\in\mathbf{R}^N, \tag{22}$$

where $\gamma > 0$ is a constant. Compared with (12), this equation has a linear sink of heat. Let us see how this is reflected in the properties of the generalized solution.

In equation (22) let us set

$$u(t,x) = \exp\{-\gamma t\}v(t,x),$$

where $v$ is a new unknown function. Then the equation for $v$ takes the form

$$\exp\{\gamma\sigma t\}v_t = \nabla\cdot(v^\sigma\nabla v), \quad t>0, \quad x\in\mathbf{R}^N.$$

Introducing the new independent (time) variable

$$\tau = \tau(t) \equiv \frac{1}{\gamma\sigma}[1-\exp\{-\gamma\sigma t\}], \quad \tau\in[0,\ 1/(\gamma\sigma)).$$

we obtain for $v = v(\tau, x)$ the equation

$$v_\tau = \nabla \cdot (v^\sigma \nabla v),$$

which we considered above; its particular solution we already know. Choosing as $v$, for example, the function $u_S(1+\tau, x)$ (see (21)), and inverting all the changes of variable, we obtain the following solution of equation (22):

$$u(t, x) =$$

$$= \exp\{-\gamma t\}[g(t)]^{-N/(N\sigma+2)} \left[ \frac{\sigma}{2(N\sigma+2)} \left( \eta_0^2 - \frac{|x|^2}{[g(t)]^{2/(N\sigma+2)}} \right)_+ \right]^{1/\sigma},$$

where $g(t) = 1 + \tau(t)$. This solution has the degeneracy surface

$$|x_f(t)| = \eta_0 \left[ 1 + \frac{1 - \exp\{-\gamma\sigma t\}}{\gamma\sigma} \right]^{1/(N\sigma+2)}, \quad t \geq 0, \tag{23}$$

on which the flux is continuous. But this is not its main distinguishing feature.

As in Example 8, the support of the generalized solution grows monotonically, however here we have

$$L = \lim_{t\to\infty} |x_f(t)| = \eta_0 \left( 1 + \frac{1}{\gamma\sigma} \right)^{1/(N\sigma+2)} < \infty,$$

that is, heat perturbations are localized due to the action of the sinks of energy in a bounded domain in the space, a ball with radius $L$.

## 2 Definition and main properties of generalized solutions

The examples we considered in subsection 1 allow us to demonstrate many of the properties of generalized solutions of quasilinear degenerate parabolic equations. Let us note again that a generalized solution does not necessarily have everywhere defined derivatives, but at points of degeneracy it possesses a certain regularity: the heat flux is continuous. At all other points where the equation is non-degenerate (and is, therefore, uniformly parabolic in a neighbourhood of these points), the solution is, as is to be expected, classical. Let us give a definition of a generalized solution, which takes into account all the indicated properties.

Let us consider in $(0, T) \times \Omega$ the first boundary value problem (1.2)–(1.4) for an equation with coefficients $k(u)$, $Q(u)$ sufficiently smooth for $u > 0$, such that, furthermore, $k$ does not satisfy the uniform parabolicity condition, that is $k(0) = 0$.

**Definition.** A non-negative continuous bounded function $u(t, x)$, which satisfies the boundary conditions (1.3), (1.4) will be called a *generalized (weak) solution* *of the problem* (1.2)–(1.4) if the generalized derivative $\nabla\phi(u) \equiv k(u)\nabla u$ exists, is square integrable in any bounded domain $\omega' \subset (0,T) \times \Omega$, and if for every continuously differentiable in $(0,T) \times \Omega$ function $f(t,x)$ with compact support, which is zero for $(t,x) \in [0,T) \times \partial\Omega$ and for $t = T$, we have the equality

$$\int_0^T \int_\Omega \left( u\frac{\partial f}{\partial t} - k(u)\nabla u \cdot \nabla f + Q(u)f \right) dx\,dt + \int_\Omega u_0(x)f(0,x)\,dx = 0. \quad (24)$$

Let us note that formally the equality (24) is obtained by multiplying equation (1.2) by $f$ and integrating over the domain $(0,T) \times \Omega$. Integration by parts (in the variable $x$) is then justified if the function $k(u)\nabla u$ is continuous in $\Omega$. This requirement is not contained in the definition, where weaker restrictions are imposed on the derivative $\nabla\phi(u)$ (existence in the sense of distributions and the condition $\nabla\phi(u) \in L^2_{\text{loc}}((0,T) \times \Omega)$, for which the integrals in (24) make sense). However for a wide range of degenerate equations the above restrictions are sufficient in order to prove continuity of $k(u)\nabla u$ (we deal with this in more detail below).

Naturally, it is necessary to define a solution in the generalized sense in the case when the solution $u(t,x)$ has degeneracy points in $(0,T)\times\Omega$, where $u(t,x) = 0$. In the opposite case, if, for example, $u_0(x) > 0$ in $\overline{\Omega}$ and $Q(u) \geq 0$, then $u(t,x) > 0$ in $(0,T) \times \Omega$ and the solution is a classical one, since the equation does not degenerate in the domain under consideration.

Generalized solutions of quasilinear degenerate parabolic equations were studied in detail in a large number of works (see, for example, [319, 341, 86, 377, 296]). Without entering into details, let us note one important point. As a rule, the generalized solution $u(t,x)$ of an equation with smooth coefficients is unique and can be obtained as the limit as $n \to \infty$ of a monotone sequence of smooth bounded positive solutions $u_n(t,x)$ of the same equation. As a result, in a neighbourhood of all the points $(t,x) \in ((0,T)\times\Omega$, where $u > 0$, the solution is classical, and it loses smoothness only on the degeneracy surface, which separates the domain $\{u > 0\}$ from the domain $\{u = 0\}$. To prove continuity of the heat flux $-k(u)\nabla u$, additional techniques must be mobilized (see, for example, [16]). Some additional information concerning differentiability and other properties of generalized solutions can be found in the Comments section of this chapter.

Below we shall treat in a more detailed manner the restrictions, under which it is necessary to consider solutions of a parabolic equation in the generalized sense. This will be done using the example of the nonlinear heat equation

$$u_t = \nabla \cdot (k(u)\nabla u), \quad t > 0,\ x \in \mathbf{R}^N, \quad (25)$$

for which we consider the Cauchy problem with an initial function of compact support

$$u(0,x) = u_0(x) \geq 0, \quad x \in \mathbf{R}^N; \quad \phi(u_0) \in C^1(\mathbf{R}^N), \quad (26)$$

so that

$$u_0(x) \equiv 0, \quad |x| > l = \text{const} > 0. \tag{27}$$

We return now to the condition we obtained in Example 6 concerning compact support of a travelling wave solution. It is quite general.

**Proposition 4.** *Convergence of the integral*

$$\int_0^1 \frac{k(\eta)}{\eta} < \infty \tag{28}$$

*is a necessary and sufficient condition for the solution of the Cauchy problem* (25)–(27) *to have compact support in* $x$.

In other words, if the integral in (28) diverges, then $u(t, x) > 0$ in $\mathbf{R}^N$ for all $t > 0$. The proof of this assertion is based on the comparison theorems for generalized solutions, which are essentially similar to the ones quoted in § 1. The second of these theorems is slightly different in the generalized setting.

## 3 Comparison theorems for generalized solutions

Theorem 1 extends to the generalized case word for word. In the general case its proof is based on the analysis of integral identities of the form (24) for solutions $u^{(1)}$, $u^{(2)}$ or by comparing a sequence of positive classical solutions $u_n^{(1)}$, $u_n^{(2)}$, which converge, respectively, to the generalized solutions $u^{(1)}$, $u^{(2)}$.

The statement of Theorem 2 has to be changed. In specific applications we shall use the following version.

**Theorem 3.** *Let there be defined in* $[0, T) \times \overline{\Omega}$ *a non-negative generalized solution of the boundary value problem* (1.2)–(1.4) *as well as the functions* $u_\pm \in C([0, T) \times \overline{\Omega})$, $u_\pm \in C^{1,2}$ *everywhere in* $(0, T) \times \overline{\Omega}$ *apart from a finite number of smooth non-intersecting surfaces* $(0, T) \times S_i(t)$ *on which the function* $\nabla\phi(u) \equiv k(u)\nabla u$ *is continuous. Let the inequalities* (1.9) *hold everywhere in* $(0, T) \times (\Omega \backslash \{x \in S_i(t)\})$, *while on the parabolic boundary of the domain* $(0, T) \times \Omega$ *we have the conditions* (1.10), (1.11). *Then*

$$u_- \le u \le u_+ \quad in\ (0, T) \times \Omega. \tag{29}$$

The new element in comparison with Theorem 2 is just the fact that the generalized supersolution $u_+$ and subsolution $u_-$ can have compact support, while on the degeneracy surfaces $(0, T) \times S_i(t)$ the corresponding heat fluxes must be continuous. Thus, roughly speaking, we are imposing the same requirements on the functions $u_\pm$ as on the generalized solution of the problem. If in (1.9) we replace the inequality signs by equality signs (in $(0, T) \times (\Omega \backslash \{x \in S_i(t)\})$), then the functions $u_\pm$ will be simply different generalized solutions of equation (1.2).

## 4 Proof of Proposition 4 (concerning finite speed of propagation of perturbations) and some of its corollaries

Sufficiency of the condition (28) follows directly from the analysis of self-similar solutions of travelling wave type, which was undertaken in Example 6. Let us place a bounded domain $\omega = \operatorname{supp} u_0$ in a parallepiped $P = \{|x_i| < l_0,\ i = 1, 2, \ldots, N\}$ with sides parallel to the axes $x_i$ so that $\overline{\omega} \subset P$. Let us show that the speed of propagation of perturbations along the $i$-th direction is finite. As in Example 6, let us construct a particular solution of travelling wave type, having the form

$$u_S^1(t, x) = \theta(x_i - l_0 - \lambda t),\quad u_S^1(0, x) \equiv \theta(x_i - l_0) = 0$$

for $x_i = l_0$. It is strictly positive in the left half-neighbourhood $\{l_0 - \epsilon < x_i < l_0\}$ of the plane $x_i = l_0$. However $\overline{\operatorname{supp}}\, u_0 \subset P$, so that there exists $\epsilon > 0$, such that $u_0(x) = 0$ for $x_i = l_0 - \epsilon$. By continuity of $u(t, x)$ for $x_i = l_0 - \epsilon$ for some time $t \in (0, \tau)$, we shall have the inequality $u(t, x) \le u_S^1(t, x)$, and by the comparison theorem, Theorem 1, $u(t, x) \le u_S^1(t, x)$ in the domain $\{t \in (0, \tau),\ x_i > l_0 - \epsilon\}$. Therefore $u(t, x)$ has compact support in $x$ along an arbitrary direction $x_i$. As $\operatorname{supp} u(t, x)$ grows, the parallepiped $P$ becomes larger, and the same argument applies.

To prove necessity we use a different self-similar solution of equation (25):

$$u_S(t, x) = f(\xi),\quad \xi = |x|/t^{1/2},\quad t > 0,\quad x \in \mathbf{R}^N, \tag{30}$$

where the function $f \ge 0$ satisfies the ordinary differential equation

$$\frac{1}{\xi^{N-1}} \left(\xi^{N-1} k(f) f'\right)' + \frac{1}{2} f' \xi = 0,\quad \xi > 0. \tag{31}$$

**Lemma.** *Condition* (28) *is a necessary and sufficient condition for existence of a non-negative generalized solution of equation* (31), *which vanishes at a point* $\xi = \xi_0 > 0$, *where the heat flux* $-\xi^{N-1} k(f) f'$ *is continuous.*

*Proof.* The existence of a solution $f = f(\xi)$, such that $f(\xi_0) = 0$, $(k(f)f')(\xi_0) = 0$, $f(\xi) > 0$ for all $\xi \in (0, \xi_0)$ is established by reducing (31) in a neighbourhood of the point $\xi = \xi_0$ to the equivalent integral equation with respect to the monotone decreasing function $\xi = \xi(f)$:

$$\xi(f) = \mathcal{M}(\xi)(f) = \xi_0 - \int_0^f \frac{2\xi^{N-1}(\eta) k(\eta)\, d\eta}{\int_0^\eta \xi^N(\zeta)\, d\zeta},\quad f \ge 0. \tag{32}$$

Local solvability follows from the Banach contraction mapping theorem. If the integral in (28) diverges, there is no solution with a finite front point $\xi = \xi_0$. Indeed, on the one hand

$$\lim_{f \to 0^+} \xi(f) = \xi_0 < \infty,$$

while on the other hand we obtain from (32) that

$$\lim_{f\to 0^+} \xi(f) = \lim_{f\to 0^+} \mathcal{M}(\xi)(f) = \xi_0 - 2\xi_0^{-1} \lim_{f\to 0^+} \int_0^f \frac{k(\eta)}{\eta}\, d\eta = -\infty.$$

In this case (31) has a monotone solution $f = f(\xi)$, strictly decreasing in $\mathbf{R}_+$, such that $f(\xi) \to 0$ as $\xi \to \infty$ (see e.g. [337, 24, 327]).

Necessity of condition (28) is also proved using Theorem 1. Let the integral in (28) diverge. Let us show that $u(t, x) > 0$ in $\mathbf{R}^N$ for all $t > 0$. Let us take the solution $f(\xi)$ of the lemma and set

$$u^{(1)}(t, x) = f(|x|/t^{1/2}) > 0,\ t > 0,\ x \in \mathbf{R}^N\setminus\{0\}. \tag{33}$$

The function $f(\xi)$ can be undefined for $\xi = 0$, but that is not essential. For us it is important that by (33) $u^{(1)}(t, x) \to 0$ as $t \to 0^+$ uniformly in any domain $\{|x| > \delta\}$, $\delta = \text{const} > 0$.

Without loss of generality let $0 \in \operatorname{supp} u_0$. Let us pick $\delta > 0$ small enough, so that $\{|x| \leq \delta\} \subset \operatorname{supp} u_0$. Then, obviously, there exists $\tau > 0$, such that $u^{(1)}(t, x) < u(t, x)$ for $|x| = \delta$, $t \in (0, \tau)$.

Let us use now the fact that the solution of the problem (25), (26) can be obtained in the form

$$u(t, x) = \lim_{\epsilon \to 0^+} u_\epsilon(t, x),\ t > 0,\ x \in \mathbf{R}^N,$$

where $u_\epsilon$ are classical solutions of equation (25), which correspond to the initial conditions $u_\epsilon(0, x) = \epsilon + u_0(x)$, $x \in \mathbf{R}^N$. But, as is easily seen, for every $\epsilon > 0$ we can always find $t_\epsilon \in (0, \tau)$ ($t_\epsilon \to 0$ as $\epsilon \to 0^+$), such that $u^{(1)}(t, x) < u_\epsilon(0, x)$ in $\{|x| > \delta\}$ for $t \in (0, t_\epsilon]$, while by construction of the family $\{u_\epsilon\}$ we have $u^{(1)}(t + t_\epsilon, x) \leq u_\epsilon(t, x)$ for $|x| = \delta$, $t \in (0, \tau - t_\epsilon)$.

Therefore from comparison Theorem 1 we obtain the inequality $u^{(1)}(t+t_\epsilon, x) \leq u_\epsilon(t, x)$ for $x \in \mathbf{R}^N\setminus\{|x| \leq \delta\}$, $t \in (0, \tau - t_\epsilon)$. Passing in this inequality to the limit $\epsilon \to 0^+$, we obtain that $u^{(1)}(t, x) \leq u(t, x)$ for $x \in \mathbf{R}^N\setminus\{|x| \leq \delta\}$, $t \in (0, \tau)$, which by (33) implies strict positivity in $\mathbf{R}^N$ of the solution of the problem (25), (26) for all arbitrarily small $t > 0$. This concludes the proof of Proposition 4. $\square$

Therefore if the condition

$$\int_0^1 \frac{k(\eta)}{\eta}\, d\eta = \infty \tag{34}$$

holds, there is no need to define the solution of the problem (25), (26) in generalized (weak) sense: any non-trivial solution is strictly positive and therefore a classical one. Naturally, this will also be correct for any equation (1.2) with a source.

**Proposition 5.** *Let $Q(u) \geq 0$, $Q \in C^1([0,\infty))$ and let the coefficient $k(u)$ satisfy condition* (34). *Then if $u_0(x) \not\equiv 0$, the solution of the Cauchy problem* (1.2), (1.5) *is strictly positive in $\mathbf{R}^N$ for all admissible $t > 0$.*

*Proof.* By comparison Theorem 3 the generalized solution of the problem under consideration (denoted by $u^{(2)}(t,x)$) is everywhere not smaller than the solution $u \equiv u^{(1)}$ of the Cauchy problem (25), (26) for the equation without a source: $u^{(2)} \geq u^{(1)}$. However, from Proposition 4 it follows that $u^{(1)} > 0$ in $\mathbf{R}^N$ for $t > 0$; therefore this also holds for $u^{(2)}$. □

For an equation with a sink the situation is more complicated. Here even if $k(0) > 0$, the solution $u(t,x)$ can have compact support. However, for that to happen the sink must be very powerful for low temperatures $u > 0$ and the function $Q(u)$ must be non-differentiable at zero. Otherwise, as shown in the example below, the solution will still be positive and a classical one.

**Example 10.** Let us consider the Cauchy problem for a semilinear equation with a sink:

$$u_t = \Delta u - Q(u), \ t > 0, x \in \mathbf{R}^N, \tag{35}$$

with an initial function $u_0(x) \not\equiv 0$ with compact support, $0 \leq u_0 \leq M$, $\operatorname{supp} u_0 \subset \{|x| < l_0\}$. Let $Q(u) > 0$ for $u > 0$, $Q(0) = 0$ and $Q \in C^1([0,\infty))$. Let us show that $u(t,x) > 0$ in $\mathbf{R}^N$ for $t > 0$.

First of all we immediately obtain from Theorem 1 that

$$0 \leq u(t,x) \leq M, \ t \geq 0, \ x \in \mathbf{R}^N.$$

Next, taking into account the restrictions on the coefficient $Q$ we deduce that

$$Q(u) \leq Cu, \ u \in [0, M]; \ C = \text{const} > 0.$$

Then, using Theorem 3 to compare the solutions of equation (35) and of the equation

$$v_t = \Delta v - Cv, \ t > 0, x \in \mathbf{R}^N,$$

which satisfy the same initial condition, we convince ourselves that

$$u(t,x) \geq v(t,x) \text{ in } \mathbf{R}_+ \times \mathbf{R}^N. \tag{36}$$

However, $v > 0$ for $t > 0$. Indeed, setting

$$v = \exp\{-Ct\} w \tag{37}$$

we obtain for $w$ the heat equation $w_t = \Delta w$, $w(0,x) = u_0(x) \geq 0$ in $\mathbf{R}^N$, $u_0 \not\equiv 0$, and therefore $w > 0$ in $\mathbf{R}^N$ for $t > 0$. The required result follows from (37), (36).

The next example shows that in a medium with a strong sink the thermal wave can be not only compactly supported, but also localized.

**Example 11.** Let us consider the first boundary value problem for a heat equation with a sink in the one-dimensional setting:

$$u_t = u_{xx} - u^\alpha, \quad t > 0, \; x > 0, \tag{38}$$

$$u(0, x) = u_0(x) \geq 0, \quad x > 0, \tag{39}$$

$$u(t, 0) = u_1(t) \geq 0, \quad t > 0, \tag{40}$$

where $\alpha \in (0, 1)$ is a constant, so that the function $-u^\alpha$ is non-differentiable for $u = 0$ (strong absorption). Let the initial perturbation $u_0$ have compact support: $u_0(x) = 0$ for all $x > l_0 > 0$, while the external heat supply is bounded: $u_1(t) \leq M < \infty$ for all $t \geq 0$. Let us show that under these conditions the solution always has compact support (even though $k(u) \equiv 1 > 0$) and is, moreover, localized in a bounded domain.

Both these assertions are proved by comparing the solution of the problem (38)–(40) with the stationary solution $v = v(x)$ of the same equation

$$v_{xx} - v^\alpha = 0, \tag{41}$$

which is determined in the following fashion.

Let us fix $l > 0$ and consider for (41) the Cauchy problem in the domain $\{0 < x < l\}$ with the conditions

$$v(l) = 0, \quad v'(l) = 0. \tag{42}$$

One solution of the problem (41), (42) is the trivial one. However, it is easily verified that there is another solution, which is positive on $(0, l)$.

Any solution of (41) satisfies the identity

$$\frac{1}{2}(v_x)^2 - \frac{1}{\alpha + 1} v^{\alpha+1} = C_1,$$

where the constant $C_1$ must be zero, which follows from (42). Then

$$v_x = -\sqrt{2/(\alpha + 1)}\, v^{(\alpha+1)/2}$$

and therefore

$$\frac{2}{1 - \alpha} v^{(1-\alpha)/2}(x) = -\sqrt{\frac{2}{\alpha + 1}}\, x + C_2.$$

Here the constant $C_2$, which is determined from the first of conditions (42), has the form $C_2 = l\sqrt{2/(\alpha+1)}$ and therefore

$$v(x) = \left[\frac{1-\alpha}{\sqrt{2(\alpha+1)}}(l-x)\right]^{2/(1-\alpha)}, \ 0 < x < l.$$

By construction the function

$$w_l(x) = \left[\frac{1-\alpha}{\sqrt{2(\alpha+1)}}(l-x)_+\right]^{2/(1-\alpha)}, \ x > 0, \tag{43}$$

is a classical stationary solution of equation (38) and has for $x > 0$ continuous derivatives $u_x$, $u_{xx}$ (let us note that at a front point $x = l$ higher derivatives do not necessarily exist). Let us choose now $l = l_* > 0$ large enough, so that $u_0 \le w_{l_*}(x)$ for $x > 0$ and furthermore

$$w_{l_*}(0) = \left[\frac{1-\alpha}{\sqrt{2(\alpha+1)}}l_*\right]^{2/(1-\alpha)} > M.$$

Then $u_1(t) < w_{l_*}(0)$ for all $t \ge 0$. Therefore by the comparison Theorem 1 we have the estimate

$$u(t,x) \le w_{l_*}(x), \ t \ge 0, \ x \in \mathbf{R}_+.$$

Thus, first, the function $u$ has compact support in $x$ for all $t > 0$ and, second, heat is localized in the domain $\{x \in (0, l_*)\}$ at all times $t \in (0, \infty)$.

Let us stress that these properties are possible only in the case $\alpha \in (0, 1)$; for $\alpha \ge 1$, as shown in Example 10, the solution is strictly positive for $t > 0$. Absence of non-trivial solutions of the stationary problem (41), (42) with finite $l > 0$ in the case $\alpha \ge 1$ also testifies to that.

## 5 Conditions of local and global existence of the generalized solution

On the whole, all the assertions stated in subsection 2 of § 2 concerning classical solutions, are valid here. Local existence of the generalized solution follows from the ability to construct it as a limit of a sequence of classical solutions defined on a finite interval $(0, T)$. Naturally, Proposition 3 is also valid, since the condition entering it has been obtained in an analysis of classical solutions. Analysis of unbounded classical solutions in Examples 2, 3 applies also to generalized solutions. Let us consider the following example (in a more general setting such problems are considered in § 2, Ch. VII).

**Example 12.** Let $\Omega$ be a bounded domain in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$; $\sigma > 0$ is a fixed constant. For a degenerate equation

$$u_t = \nabla \cdot (u^\sigma \nabla u) + u^{\sigma+1}, \ t > 0, \ x \in \Omega, \tag{44}$$

let us consider the boundary value problem with the conditions

$$\begin{aligned} u(0, x) &= u_0(x) \geq 0, \quad x \in \Omega, \quad u_0 \in C(\bar{\Omega}), \\ u(t, x) &= 0, \; t > 0, \quad x \in \partial\Omega. \end{aligned} \tag{45}$$

Let us denote by $\psi_1(x) > 0$ in $\Omega$, $\|\psi_1\|_{L^1(\Omega)} = 1$, the first eigenfunction of the problem $\Delta\psi + \lambda\psi = 0$ in $\Omega$, $\psi = 0$ on $\partial\Omega$, and by $\lambda_1 > 0$ the corresponding eigenvalue.

We shall show that for $\lambda_1 < \sigma + 1$ every non-trivial solution of the problem exists only for a finite time. We shall proceed as in Example 2. Let us form the scalar product in $L^2(\Omega)$ of the equation (44) with $\psi_1$. Introducing the notation $E(t) = (u(t, x), \psi_1(x))$, we obtain

$$\frac{dE}{dt}(t) = \int_\Omega \psi_1(x) \nabla \cdot (u^\sigma \nabla u)\, dx + \int_\Omega \psi_1(x) u^{\sigma+1}\, dx. \tag{46}$$

Here, as in the case of a classical solution, we can integrate by parts the first term in the right-hand side of (46).

Let us show that

$$\int_\Omega \psi_1(x) \nabla \cdot (u^\sigma \nabla u)\, dx = \frac{1}{\sigma+1} \int_\Omega u^{\sigma+1} \Delta\psi_1\, dx. \tag{47}$$

If $u_0 > 0$ in $\Omega$ then $u(t, x)$ is a classical solution. Let $\overline{\operatorname{supp}}\, u_0 \subset \Omega$. Let us denote by $\partial\omega(t)$ the degeneracy surface of equation (44) in this problem, that is, the boundary of the support of the solution $\omega(t) \equiv \operatorname{supp} u(t, x)$. Then $u(t, x) \equiv 0$ in $\Omega \backslash \omega(t)$, and by Green's formula

$$\begin{aligned} \int_\Omega \psi_1 \nabla \cdot (u^\sigma \nabla u) dx &\equiv \frac{1}{\sigma+1} \int_{\omega(t)} \psi_1 \Delta u^{\sigma+1}\, dx = \\ &= \frac{1}{\sigma+1} \left\{ \int_{\omega(t)} \Delta\psi_1 u^{\sigma+1}\, dx + \int_{\partial\omega(t)} \psi_1 \frac{\partial u^{\sigma+1}}{\partial n_t}\, ds - \int_{\partial\omega(t)} \frac{\partial\psi_1}{\partial n_t} u^{\sigma+1}\, dx \right\}. \end{aligned} \tag{48}$$

where we denoted by $\partial/\partial n_t$ the derivative in the direction of the outer normal to $\partial\omega(t)$. However, $u^{\sigma+1} = 0$ on $\partial\omega(t)$ and by continuity of the heat flux $\partial u^{\sigma+1}/\partial n_t \equiv \nabla u^{\sigma+1} \cdot n_t = 0$ for $x \in \partial\omega(t)$. Therefore the last two integrals in (48) are zero, which leads to the equality (47).

It must be said that in the analysis above we did not consider the question of regularity of the surface $\partial\omega(t)$ (in particular, the existence of the derivative $\partial u^{\sigma+1}/\partial n_t$ on $\partial\omega(t)$); for certain classes of equations this problem is quite well understood (see the Comments section). In this particular case this is not necessary; the definition of a generalized solution implies that integration by parts is justified and allows us to prove the equality (47).

Using (47) and taking into account the fact that $\Delta\psi_1 = -\lambda_1\psi_1$ in $\Omega$, we obtain from (46)

$$\frac{dE}{dt}(t) = \left(1 - \frac{\lambda_1}{\sigma+1}\right)(u^{\sigma+1}, \psi_1), \; t > 0. \tag{49}$$

If $\lambda_1 < \sigma+1$, then using Jensen's inequality ($u^{\sigma+1}$ is a convex function for $u \geq 0$), we arrive at the estimate

$$\frac{dE}{dt} \geq \left(1 - \frac{\lambda_1}{\sigma+1}\right)(u, \psi_1)^{\sigma+1} \equiv \left(1 - \frac{\lambda_1}{\sigma+1}\right)E^{\sigma+1}, \; t > 0.$$

Hence it follows that $E(t)$ (and thus $u(t, x)$) remains bounded for time not greater than

$$T_* = \frac{\sigma+1}{\sigma+1-\lambda_1}\frac{1}{\sigma}\left[\int_\Omega u_0(x)\psi_1(x)\,dx\right]^{-\sigma} < \infty,$$

that is, there exists $T_0 \in (0, T_*]$, such that $\overline{\lim}\sup_x u(t, x) = \infty$, $t \to T_0^-$.

Let us note that for $\lambda_1 \geq \sigma+1$ it follows from (49) that $E(t)$ is bounded for all $t \geq 0$. This can be considered as evidence of global boundedness of the solution (see § 2, Ch. VII).

To conclude, let us give some simple examples of unbounded generalized solutions which illustrate the property of heat localization in nonlinear media with volumetric energy sources.

**Example 13.** The equation

$$u_t = (u^\sigma u_x)_x + u^{\sigma+1}, \; \sigma = \text{const} > 0.$$

has in the domain $(-\infty, T_0) \times \mathbf{R}$ the following self-similar separable solution:

$$u(t, x) = (T_0 - t)^{-1/\sigma}\theta_S(x) \equiv$$

$$\equiv (T_0 - t)^{-1/\sigma}\begin{cases} \left(\frac{2(\sigma+1)}{\sigma(\sigma+2)}\cos^2\frac{\pi x}{L_S}\right)^{1/\sigma}, & |x| < L_S/2, \\ 0, \; |x| \geq L_S/2, & 0 < t < T_0, \end{cases} \tag{50}$$

where $L_S = 2\pi(\sigma+1)^{1/2}/\sigma$ and $T_0 > 0$ is an arbitrary constant.

Let us indicate the main features of this solution. First of all, it has compact support in $x$ and is a generalized solution; at the points of degeneracy $x = \pm L_S/2$ the heat flux is continuous.

Secondly, it exhibits finite time blow-up: $u(t, x) \to \infty$ as $t \to T_0^-$ for any $|x| < L_S/2$.

Thirdly, its support, $\operatorname{supp} u(t, x) = \{|x| < L_S/2\}$, is constant during the whole time of existence of the solution. It is localized; the heat from the localization domain $\{|x| < L_S/2\}$ does not penetrate into the surrounding cold space (see Figure 4), even though at all points of the localization domain the temperature

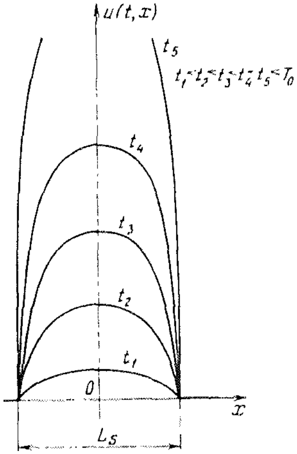


**Fig. 4.** Localization of a finite time blow-up combustion process in the S-regime (self-similar solution (50))

grows without bound as $t \to T_0^-$. The half-width $x_{ef}(t) > 0$ of this fast growing heat structure, that is, the coordinates of the point at which $u(t, x_{ef}(t)) = u(t,0)/2$ are also constant in time.

**Example 14.** The equation $u_t = (u^\sigma u_x)_x + u^{\sigma+1}$, $\sigma > 0$, also has quite an unusual exact non-self-similar solution of the following form:

$$u_*(t,x) = \{\phi(t)\,[\psi(t) + \cos(2\pi x/L_S)]_+\}^{1/\sigma} \geq 0$$

for $x \in (-L_S/2,\ L_S/2)$ and $u_*(t,x) = 0$ for $x \in \mathbf{R}\backslash(-L_S/2,\ L_S/2)$, where the function $\psi(t) \in (-1,1)$ satisfies for $t > 0$ the equation

$$\psi' = \sigma(\sigma+1)^{-1}C_0[1-\psi^2]^{\sigma/2},\quad t > 0;\ \psi(0) = -1,$$

and $\phi(t) = C_0[1-\psi^2(t)]^{-(\sigma+2)/2}$. If the constant $C_0$ is chosen in the form

$$C_0 = C_0(T_0) = (\sigma+1)\sigma^{-1}T_0^{-1}B(1+\sigma/2,\ 1/2),$$

then it is not hard to see by integrating the equation for $\psi(t)$ that the solution $u_*(t,x)$ will blow up at time $T_0$: $u_*(t,0) \to \infty$ as $t \to T_0^-$. It is easy to see that the fronts of the generalized solution $u_*$ are at the points

$$h_\pm^*(t) = \pm(L_S/2\pi)[\pi/2 + \arcsin\psi(t)] \to \pm L_S/2$$

as $t \to T_0^-$ and the solution grows without bound only in the localization domain $\{|x| < L_S/2\}$. It is interesting to note that since $\psi(0) = -1$, we have the equality

$h_{\pm}^{*}(0) = 0$ and the exact solution satisfies in the generalized sense the singular initial condition

$$u_*(0, x) = E_0\delta(x) \text{ in } \mathbf{R},$$

where $\delta(x)$ stands for the Dirac delta function, and the constant $E_0$ depends only on $T_0$ and $\sigma$. More precisely, for small $t > 0$ we have the representation

$$u_*(t, x) \simeq a_0 t^{-1/(\sigma+2)} \left[1 - \left(x/b_0^{1/2} t^{1/(\sigma+2)}\right)^2\right]_+^{1/\sigma},$$

where $a_0$ and $b_0$ are constants,

$$a_0 = 2^{-1/\sigma}[(\sigma+1)\sigma^{-1}]^{1/\sigma}(\sigma+2)^{-1/(\sigma+2)} T_0^{-2/\sigma(\sigma+2)} [B(1+\sigma/2, 1/2)]^{2/\sigma(\sigma+2)},$$

$$b_0 = (\sigma+1)\sigma^{-2}(\sigma+2)^{2/(\sigma+2)} T_0^{-2/(\sigma+2)} [B(1+\sigma/2, 1/2)]^{2/(\sigma+2)}.$$

Therefore it is easy to see that

$$\int_{-\infty}^{+\infty} u_*(t, x)\xi(x)\, dx \to E_0\xi(0) \text{ as } t \to 0$$

for any smooth compactly supported test function $\xi(x)$. From these asymptotics it follows immediately that $E_0 = a_0 b_0^{1/2} B(1 + 1/\sigma, 1/2)$.

As far as the behaviour of the solution $u_*(t, x)$ close to the blow-up time is concerned, it is not hard to check, by computing the asymptotics of the functions $\psi(t)$ and $\phi(t)$ as $t \to T_0^-$, that this exact solution converges asymptotically to the simpler self-similar solution (50), which we considered in Example 13. Below (see § 5, Ch. IV) we shall show that precisely this self-similar solution describes the asymptotics of a wide variety of unbounded solutions close to the blow-up time.

Finally, we observe that the above solution $u_*$, which is not self-similar, can be treated as follows. Setting $u^\sigma = v$ yields an equation with quadratic nonlinearities,

$$v_t = \mathbf{A}(v) \equiv v v_{xx} + \frac{1}{\sigma}(v_x)^2 + \sigma v^2.$$

The *nonlinear* operator $\mathbf{A}$ admits the following two-dimensional *linear invariant subspace*

$$W_2 = \mathcal{L}\{1, \cos(\lambda x)\} \equiv$$
$$\equiv \{w(x) \;:\; \exists C_0, C_1 \in \mathbf{R}, \text{ such that } w(x) = C_0 + C_1\cos(\lambda x)\},$$

where $\lambda = 2\pi/L_S$ ($\mathcal{L}\{\cdot\}$ denotes the linear span of given functions). This means that $\mathbf{A}(W_2) \subseteq W_2$. Therefore substituting $v(t, x) = C_0(t) + C_1(t)\cos(\lambda x) \in W_2$ into the equation gives us a dynamical system on the coefficients $\{C_0(t), C_1(t)\}$, which is precisely the parabolic equation on $W_2$.

## 6 Examples of non-uniqueness of the generalized solution

Obviously, the requirement of smoothness of the source $Q \geq 0$ in equation (1.2), which is necessary for unique solvability of the Cauchy problem (see subsection 3 of § 2) in the class of smooth functions, is still in force in the generalized setting. Example 4 applies in this case without changes. In addition, it is easy to give an example of a degenerate equation, constructed as in Example 5, which has in a bounded domain a non-unique spatially nonhomogeneous solution.

For example, the problem

$$u_t = \Delta u^{\sigma+1} + \lambda_1 u^{\sigma+1} + \psi_1^{(1-\alpha)/(\sigma+1)} u^\alpha,$$

$u = 0$ in $\Omega$ for $t = 0$ and in $\mathbf{R}_+ \times \partial\Omega$; $\alpha \in (0, 1)$, $\sigma > 0$ are constants; the rest of the notation is the same as in Example 5 of § 2, has the family of non-trivial solutions

$$u(t, x) = v_T(t)\psi_1^{1/(\sigma+1)}(x), \ t > 0, \ x \in \Omega.$$

Let us consider an example which demonstrates explicitly that if uniqueness conditions do not hold, the comparison theorems for generalized solutions are no longer valid.

**Example 15.** Let us fix an arbitrary $\sigma \in (0, 1)$ and let us consider for $t > 0$, $x > 0$, the equation

$$u_t = (u^\sigma u_x)_x + u^{1-\sigma}. \tag{51}$$

Here $Q(u) = u^{1-\sigma}$, so that $Q(0) = 0$, but $Q'(0^+) = \infty$. Let us find a travelling wave solution.

Setting $u(t, x) = f_S(\xi)$, $\xi = x - \lambda t$, $\lambda > 0$, we obtain for $f_S \geq 0$ the equation

$$-\lambda f_S' = (f_S^\sigma f_S')' + f_S^{1-\sigma},$$

which has for $\lambda > 2$ two different solutions

$$f_S^\pm(\xi) = C_\pm[(-\xi)_+]^{1/\sigma},$$

$$C_\pm = \left(\sigma\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}\right)^{1/\sigma} > 0.$$

Therefore the required self-similar solutions have the form

$$u^\pm(t, x) = C_\pm[(\lambda t - x)_+]^{1/\sigma}, \ t > 0, \ x > 0.$$

Let us compare these generalized solutions with the spatially homogeneous solution (Figure 5)

$$u^*(t, x) = (\sigma t)^{1/\sigma}, \ t \geq 0, \ x > 0.$$

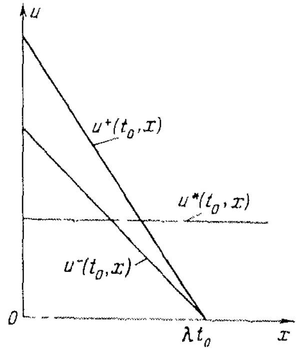


**Fig. 5.** Three different solutions of equation (51), which do not satisfy the Maximum Principle

First, all these solutions, as solutions to a boundary value problem in $\mathbf{R}_+ \times \mathbf{R}_+$, satisfy the same initial condition

$$u^{\pm}(0, x) = u^*(0, x) \equiv 0, \quad x \geq 0.$$

Secondly, for $\lambda > 2$ the boundary values satisfy the inequalities

$$u^*(t, 0) < u^-(t, 0) < u^+(t, 0), \quad t > 0.$$

Nonetheless, as seen from the position of these solutions relative to one another in Figure 5, they do not satisfy the comparison theorem. Let us note that already the existence of two solutions $u^{\pm}$ of travelling wave type with same speeds of motion and coinciding fronts, which correspond however to boundary regimes of different magnitudes, contradicts physical intuition.

## Remarks and comments on the literature

The necessary bibliographical references for most of the contents of § 1, 2 are contained in the text. Concerning Propositions 2, 3 in § 2, see [282, 320, 101, 338]; the restriction (6) in § 2 coincides with the Osgood criterion for global continuation of solutions of an ordinary differential equation [354]. The result stated in Example 2 was first obtained in [243]. Concerning Example 5, see [243, 116]. Non-uniqueness of solutions of boundary value problems in a bounded domain for a semilinear equation with source concave in $u$ was proved in [116] (see also [114]). The generalized self-similar solution of Example 8 was constructed in [385] ($N = 1$) and [28, 386] ($N \geq 1$ arbitrary). Asymptotic stability of the self-similar solution (21) of § 3 was established for $N = 1$ in [234] and by a different method in [187]. The proof of stability in the multi-dimensional case was done in [107] (qualitative formal results were obtained earlier, for example, in [5, 384, 386]); see also Ch. II.

The localized solution of Example 9 is taken from [302]. The definition of a generalized solution in subsection 2 of § 3 for a degenerate equation of general type without a source was formulated in [319, 341, 342]. These authors also prove existence and uniqueness theorems for generalized solutions for boundary value and Cauchy problems. For quasilinear parabolic equations with lower order terms such theorems are proved, for example, in [377, 231, 21, 43, 203, 294, 344, 345], where in a number of cases weaker generalized solutions are considered).

Differentiability properties of generalized solutions of the equation $u_t = (u^{\sigma+1})_{xx}$, $\sigma > 0$, were studied in [16, 17, 18]; in particular, continuity of the heat flux $-(u^{\sigma+1})_x$ was established, certain results concerning degeneracy curves were obtained, and Hölder continuity in $x$ with exponent $\nu = \min\{1, 1/\sigma\}$ was proved. This implies Hölder continuity in $t$ with exponent $\nu/2$ (see [202, 258]).

Under certain additional assumptions, it is shown in [75] that the Hölder continuity exponent in $t$ is also equal to $\nu$ (from the form of the solution in Example 8 it follows that this is an optimal result). Later some of these results were extended to the case of more general degenerate equations [230, 248, 203, 252, 253]. Properties of the degeneracy surface of the equation $u_t = \Delta u^{\sigma+1}$ were studied in [18, 58, 59, 252]; there it is shown that starting from some moment of time it is differentiable (many of these results are summarized in [103]; see also [328]).

We shall discuss in more detail the properties of generalized solutions of degenerate equations in Ch. II, III, and in Comments to these Chapters.

Sufficiency of condition (28) in § 3 in Proposition 4 (finite speed of propagation of perturbations) was established in [319] for the one-dimensional case; see also [33]. Necessity under some additional assumptions was proved in [229]. In the proof of Proposition 4 we use a method that was employed in [327] for $N = 1$. Concerning Theorem 3, see [231, 232, 248]. In the presentation of the result of Example 11, we used the approach of [231] (comparison with the stationary solution); in that paper conditions for localization in arbitrary media with volumetric absorption were obtained. For more details on localization in media with sinks see Ch. II and the surveys in [233, 162]. In the analysis of the parabolic equation in Example 12 we used a generalization of the method of [243] to the case of quasilinear problems [120, 121, 124] (see also [225], where the same method is used to study a quasilinear equation of a different type). The localized unbounded solution of Example 13 was first constructed in [391, 353] (see Ch. IV).

The localized solution of Example 14 was constructed and studied in [134, 176]. There one can also find a method of constructing similar exact solutions for a large class of evolution equations and systems with quadratic nonlinearities. Let us note that this solution is not invariant with respect to Lie groups or Lie-Bäcklund groups;

see [221, 322]. An example of this unusual kind of exact solution for a quasilinear equation with a sink was constructed in [49] (see also a similar solution in [313]. Some general ideas on construction of finite-dimensional linear subspaces that are invariant under a given nonlinear operator and of the corresponding explicit solutions via dynamical systems are presented in [136] and [139]. Example 15 is taken from [122]. In that paper were established conditions on the coefficients $k(u)$, $Q(u)$, under which a parabolic equation of general type admits at least two travelling wave type solutions. Existence of different travelling waves for an equation with power type nonlinearities, $u_t = \Delta u^m + u^p$, $p < 1 < m$, $m + p \geqslant 2$, was established in [323]; see also the general results of [324] on "almost" uniqueness (for $m + p < 2$) and nonuniqueness, and [6] for the case $m = 1$.

*Chapter II*

# Some quasilinear parabolic equations. Self-similar solutions and their asymptotic stability

In the present chapter, which, like the previous one, is of an introductory character, we briefly present results of analysis of specific quasilinear parabolic equations. As can be seen from the title, one of the principal methods of investigation consists of constructing and analyzing self-similar (or, in the general case, invariant) solutions of the problem being considered.

Using various examples, we shall try to show, what rôle these particular solutions play in the description of general properties of solutions of parabolic equations of most diverse types. Here we also introduce the concept of approximate self-similar solutions (a.s.s.) of nonlinear parabolic equations. Use of the construction of a.s.s as a tool in its own right will be considered in other chapters.

The examples presented below cover a sufficiently wide spectrum of nonlinear equations. Comparatively simple and frequently well known examples illustrate many ideas and methods of analysis, which will be developed in subsequent chapters in a more explicit and detailed fashion.

Many of the problems and questions considered below have been exhaustively researched; the corresponding references are given at the end of the chapter. From all the available results we choose only those that are, first, constructive, that is, ones that make it possible to show explicitly certain properties of the solutions of a problem, and second, which is particularly important for an introductory chapter, those that can be proved in a relatively simple and brief manner at least on the formal level. Wherever this cannot be done, we restrict ourselves to short remarks on the proof, or discuss only the "physical meaning" of the result, which contains the ideas of a rigorous proof. For that reason, we do not aim at a great generality in our presentation; frequently other proofs of well known facts are given; these, in our view, either make explicit the "physical basis" of a phenomenon, or illustrate mathematical methods to be used in the sequel. Let us note that this approach (frequently using similarity methods) makes it possible to obtain more optimal, and even new results.

We want to emphasize in particular the concept of asymptotic stability of self-similar solutions of nonlinear parabolic equations with respect to perturbations of the boundary data of the problem, as well as with respect to perturbations of the equation itself. Self-similar (invariant) solutions are not simply particular solutions appearing serendipitously. In many cases they serve as a sort of "centres of gravity" of a wide variety of solutions of the equation under consideration, as well as of solutions of other parabolic equations obtained as a result of a "nonlinear perturbation" of the original equation. The sense in which the expression "centre of gravity" is to be understood, will become clear below.

The specific form of self-similar solutions is to be determined from the conditions of invariance of an equation with respect to certain transformations. In the general case families of self-similar solutions are determined by a group classification of the equation. This allows us to find all classes of equations invariant with respect to a certain group of transformations (such as Lie groups of point transformations, or Lie-Bäcklund groups of contact transformations; see [221, 322]).

We start with an analysis of a simple linear problem; however, as we show below, this analysis will allow us to determine properties of a whole family of nonlinear problems.

## § 1 A boundary value problem in a half-space for the heat equation. The concept of asymptotic stability of self-similar solutions

For the linear equation

$$u_t = u_{xx}, \quad t > 0, \quad x > 0, \tag{1}$$

let us consider the boundary value problem with boundary data

$$u(0, x) = u_0(x) \geq 0, \quad x > 0; \quad \sup u_0 < \infty, \tag{2}$$

$$u(t, 0) = u_1(t) > 0, \quad t > 0. \tag{3}$$

It is assumed that the function $u_0(x)$ is Lipschitz continuous in $\mathbf{R}_+$. Here we analyse the "dimensionless" equation with thermal conductivity coefficient $k_0 = 1$. This does not restrict the generality of the results, since by scaling time $t \to k_0 t$ (or the spatial coordinate $x \to k_0^{1/2} x$) the linear equation $u_t = k_0 u_{xx}$ reduces to the original one. Thus in equation (1) the variables $t$, $x$ are also dimensionless quantities.

As we already mentioned, the problem (1)–(3) models the process of heat action on a medium with a constant thermal conductivity. Our goal is to describe explicitly the evolution of the heating process, establish the law governing the motion of the wave of heating, find how its depth of penetration (half-width) $x_{ef}(t)$ depends on time[1], and to determine the spatial profile of the wave.

Solution of the stated problem can be written down explicitly in terms of heat potentials [282]:

$$u(t,x) = \frac{x}{2\pi^{1/2}} \int_0^t \exp\left\{-\frac{x^2}{4(t-\tau)}\right\} \frac{u_1(\tau)}{(t-\tau)^{3/2}}\, d\tau + \\ + \frac{x}{2(\pi t)^{1/2}} \int_0^\infty \left[\exp\left\{-\frac{(x-\xi)^2}{4t}\right\} - \exp\left\{-\frac{(x+\xi)^2}{4t}\right\}\right] u_0(\xi)\, d\xi. \tag{4}$$

However it does not seem possible to glean directly from (4) the features of the process we are interested in. Therefore we proceed in a different way.

## 1 A self-similar solution

Let us consider a special form (power law) boundary regime:

$$u_1(t) = (1+t)^m, \quad t > 0. \tag{5}$$

where $m > 0$ is a constant. For such a boundary function equation (1) has a suitable self-similar solution:

$$u_S(t,x) = (1+t)^m \theta_S(\xi), \quad \xi = x/(1+t)^{1/2}. \tag{6}$$

Substituting the solution (6) into equation (1), we obtain for $\theta_S(\xi)$ the ordinary differential equation

$$\theta_S'' + \frac{1}{2}\theta_S'\xi - m\theta_S = 0, \quad \xi > 0. \tag{7}$$

Let

$$\theta_S(0) = 1. \tag{8}$$

Then the solution $u_S$ satisfies the boundary condition (3), (5). Taking into consideration the condition of boundedness of $u_S$ as $x \to \infty$ (see (2)), we shall require the inequality $\theta_S(\infty) < \infty$ to hold. From equation (7) it is easy to deduce that such a solution has to satisfy the condition

$$\theta_S(\infty) = 0. \tag{9}$$

[1]The quantity $x_{ef}(t)$ is determined for each time $t > 0$ by the equality $u(t, x_{ef}(t)) = u(t,0)/2$, that is, this is the point where the temperature is equal to half the temperature on the boundary.

Thus the problem of constructing a self-similar solution (6) of a partial differential equation has been reduced to the boundary value problem (7)–(9) for a considerably simpler ordinary differential equation.

The solution of the problem (7)–(9) exists, is unique, monotone, and strictly positive:

$$\theta_S(\xi) = 2^{2m+1}\frac{\Gamma(1+m)}{\pi^{1/2}}\exp\left\{-\frac{\xi^2}{4}\right\}H_{-(2m+1)}\left(\frac{\xi}{2}\right), \tag{10}$$

where $H_\nu(z)$ is the Hermite function:

$$H_\nu(z) = \frac{1}{\Gamma(-\nu)}\int_0^\infty \exp\{-t^2 - 2zt\}t^{-(\nu+1)}\,dt \tag{11}$$

(a special function of mathematical physics [35, 317]). The function $\theta_S(\xi)$ decays rapidly as $\xi \to \infty$:

$$\theta_S(\xi) \sim \exp\{-\xi^2/4\},\ \xi \to \infty. \tag{12}$$

The self-similar solution (6) constructed above has a simple spatio-temporal structure. From the form of the solution it is easy to determine the dependence of the depth of penetration (half-width) of the thermal wave on time:

$$x_{ef}^S(t) = \xi_{ef}(1+t)^{1/2}, \tag{13}$$

where the constant $\xi_{ef} = \xi_{ef}(m)$ is such that $\theta_S(\xi_{ef}) = \theta_S(0)/2 = 1/2$. The function $\theta_S(\xi)$ characterizes, for each $t > 0$, the spatial shape of the thermal wave.

## 2 Comparison with other (non-similarity) solutions

By the comparison theorem, $u_S$ majorizes a large set of solutions of the problem (1)–(3).

**Proposition 1.** *Let*

$$u_1(t) \le (1+t)^m,\ t > 0;\ u_0(x) \le \theta_S(x),\ x > 0. \tag{14}$$

*Then the solution of problem* (1)–(3) *satisfies the inequality*

$$u(t,x) \le (1+t)^m\theta_S(x/(1+t)^{1/2}),\ t > 0,\ x > 0. \tag{15}$$

Therefore if the inequalities (14) hold, we have an upper bound for the solution of the problem; this bound allows us to understand the form of the distribution in space of the heat coming in from the boundary. For example, let the boundary regime be of the self-similar form,

$$u_1(t) = (1+t)^m,\ t > 0, \tag{16}$$

while the initial perturbation satisfies $u_0(x) \le \theta_S(x)$ in $\mathbf{R}_+$. Then by (15) $x_{ef}(t) \le x_{ef}^S(t)$, that is,

$$x_{ef}(t) \le \xi_{ef}(m)(1+t)^{1/2}, \ t > 0.$$

Inequality (15) also gives us some information about the spatial profile of the non-self-similar thermal wave.

## 3 Asymptotic stability of the self-similar solution with respect to perturbation of the boundary data

Let us consider a different aspect of the problem. What would happen if the restrictions on the initial function $u_0(x)$ in (14) were not satisfied, for example, if $u_0(x) \equiv 1$ in $\mathbf{R}_+$ (then by the condition $\theta_S(x) \to 0$ as $x \to \infty$ the inequality $u_0(x) \le \theta_S(x)$ does not hold for all sufficiently large $x > 0$). In this case the self-similar solution allows us to obtain sharp bounds on the spatio-temporal structure of the heating wave, but, naturally, only for sufficiently large $t$. Below we shall deal with asymptotic stability of the solution (6) with respect to perturbations of the initial function.

Let equality (16) hold. Let us introduce the *similarity representation* (*similarity "transform"*) of the solution of problem (1)–(3), defined at each moment of time in accordance with the form of the self-similar solution (6):

$$\theta(t,\xi) = (1+t)^{-m} u(t, \xi(1+t)^{1/2}), \ t > 0, \xi > 0. \tag{17}$$

This expression is arranged in such a way that the similarity transform of $u_S(t,x)$ gives us exactly the function $\theta_S(\xi)$.

**Proposition 2.** *Let* $u_1(t) = (1+t)^m$, $t > 0$. *The self-similar solution* (6) *is asymptotically stable with respect to arbitrary (bounded) perturbations of the initial function; for any* $u_0(x)$

$$\begin{aligned} \|\theta(t,\cdot) - \theta_S(\cdot)\|_{C(\mathbf{R}_+)} &\equiv \sup_{\xi>0} |\theta(t,\xi) - \theta_S(\xi)| = \\ &= O((1+t)^{-m}) \to 0, \ t \to \infty. \end{aligned} \tag{18}$$

*Proof.* It follows from the Maximum Principle. Let us set $z = u - u_S$. Then $z$ satisfies the equation

$$z_t = z_{xx}, \ t > 0, x > 0,$$

and furthermore $z(t,0) = 0$, $t > 0$, and $\sup_{x>0} |z(0,x)| < \infty$. From the comparison theorem we obtain

$$|z(t,x)| \le M = \sup_{x>0} |z(0,x)|, \ t > 0.$$

Hence it follows immediately that

$$|\theta(t,\xi) - \theta_S(\xi)| \le M(1+t)^{-m} \to 0,\ t \to \infty,$$

for all $\xi \ge 0$. □

Thus for any initial function, the solution of the problem with a power law boundary regime after a certain time becomes quite close to the self-similar solution. From (18) it is not hard to derive, for example, the asymptotically exact expression for the depth of penetration of the thermal wave:

$$x_{ef}(t) = \xi_{ef}(m)\, t^{1/2} + o(t^{1/2}),\ t \to \infty, \tag{19}$$

which, for large $t$, is close to the self-similar one:

$$x_{ef}(t)/x^S_{ef}(t) \to 1,\ t \to \infty.$$

Here by (18) the similarity function correctly characterizes the profile of the heating wave at an advanced stage of the process.

This does not exhaust the properties of the constructed self-similar solution. It turns out that it is also stable with respect to small perturbations of the boundary regime. A general assertion concerning asymptotic stability of the self-similar solution (6) with respect to perturbations of the boundary data looks as follows (it is proved in exactly the same way as the previous one).

**Proposition 3.** *Let*

$$u_1(t)/(1+t)^m \to 1,\ t \to \infty. \tag{20}$$

*Then*

$$\|\theta(t,\cdot) - \theta_S(\cdot)\|_{C(\mathbf{R}_+)} = O[\max\{t^{-m},\ |1 - u_1(t)/t^m|\}] \to 0,\ t \to \infty. \tag{21}$$

If (20) holds, we have the same exact estimate (19) for the depth of penetration of the wave. This result gives us an explicit form of the evolution of the heating process for arbitrary initial perturbations and for boundary regimes asymptotically close in the sense of (20) to a power law dependence.

## 4 Asymptotic stability of the self-similar solution with respect to small perturbations of the equation

Let us show now that spatio-temporal structure of the self-similar solution is preserved for large $t > 0$ in the case of a "small nonlinear perturbation" of the original parabolic heat equation.

Suppose a sufficiently smooth thermal conductivity coefficient is not constant: $k = k(u) > 0$ for $u \geqslant 0$. However it is close to a constant for large temperatures:

$$k(u) \to 1, \quad u \to \infty. \tag{22}$$

Let us consider the same process of heating, but now in a nonlinear heat conducting medium:

$$u_t = (k(u)u_x)_x, \quad t > 0, \; x > 0, \tag{23}$$

where $u(t, x)$ satisfies the boundary conditions (2), (3). For convenience, let us introduce the function

$$G_k(u) = \int_0^u |1 - k^{1/2}(\eta)|^2 \, d\eta, \quad u \geq 0.$$

**Proposition 4.** *Let $u_1 = (1+t)^m$, $t > 0$, $u_0 \in L^2(\mathbf{R}_+)$; $u_0$ is non-increasing in $x$ and condition (22) holds. Then the self-similar solution (6) is stable with respect to the indicated perturbations of the thermal conductivity coefficient, and we have the estimate*

$$\begin{aligned} \|\theta(t,\cdot) - \theta_S(\cdot)\|^2_{L^2(\mathbf{R}_+)} &\equiv \int_0^\infty |\theta(t,\xi) - \theta_S(\xi)|^2 \, d\xi = \\ &= O\left[(1+t)^{-2m-1/2} \max\left\{1, \int_0^t (1+\tau)^{m-1/2} G_k|(1+\tau)^m| \, d\tau\right\}\right] \to 0 \end{aligned} \tag{24}$$

*as $t \to \infty$.*

If condition (22) holds, the right-hand side of the estimate (24) does indeed go to zero as $t \to \infty$, which is not hard to see by evaluating the indeterminates

$$\begin{aligned} \lim_{t\to\infty} \frac{\int_0^t (1+\tau)^{m-1/2} G_k[(1+\tau)^m] \, d\tau}{(1+t)^{2m+1/2}} &= \\ = \frac{1}{2m+1/2} \lim_{s\to\infty} \frac{G_k(s)}{s} &= \frac{1}{2m+1/2} \lim_{s\to\infty} (1 - k^{1/2}(s))^2 = 0. \end{aligned}$$

Let us note that convergence of $\theta(t,\cdot)$ to $\theta_S(\cdot)$ as $t \to \infty$ in the $L^2(\mathbf{R}_+)$ norm implies, in particular, pointwise convergence almost everywhere.

*Proof.* The function $w = u - u_S$ satisfies in $\mathbf{R}_+ \times \mathbf{R}_+$ the equation

$$w_t = [k(u)u_x - (u_S)_x]_x, \tag{25}$$

with $w(t, 0) \equiv 0$, $w(t, x) \to 0$ as $x \to \infty$ (this follows immediately from the Maximum Principle) and $w(0, \cdot) \in L^2(\mathbf{R}_+)$. The latter assertion is ensured by the requirement that $u_0 \in L^2(\mathbf{R}_+)$; by (12) the self-similar solution $u_S(t,\cdot)$ is in $L^2(\mathbf{R}_+)$ for all $m > 0$. We restrict ourselves to a formal analysis of equation (25).

Let us take the scalar product of equation (25) with $w$ in $L^2(\mathbf{R}_+)$, and, having convinced ourselves that this product makes sense, let us integrate the right-hand side by parts; this is allowed in view of uniform boundedness of the derivatives $u_x$ and $(u_S)_x$ in $\mathbf{R}_+ \times \mathbf{R}_+$ and the condition $w \to 0$ as $x \to \infty$. As a result we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\mathbf{R}_+)} = -(w_x, k(u)u_x - (u_S)_x).$$

It is not hard to verify the identity

$$\begin{aligned}
-(u_x - (u_S)_x)(k(u)u_x - (u_S)_x) &= \\
= -(k^{1/2}(u)u_x - (u_S)_x)^2 &+ (1 - k^{1/2}(u))^2 u_x (u_S)_x \equiv \\
&\equiv -(k^{1/2}(u)u_x - (u_S)_x)^2 + \frac{\partial}{\partial x}G_k(u)(u_S)_x,
\end{aligned}$$

using which the preceding equality takes the form

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\mathbf{R}_+)} = -\|k^{1/2}(u)u_x - (u_S)_x\|^2_{L^2(\mathbf{R}_+)} + \left(\frac{\partial}{\partial x}G_k(u),\ (u_S)_x\right).$$

Under our assumptions on $u_0(x)$, $u_x(t,x) \le 0$ in $\mathbf{R}_+$ for all $t > 0$ (this follows from the Maximum Principle; see § 1 of Ch. V). Taking into account in the last equality the fact that $(u_S)_x < 0$ in $\mathbf{R}_+ \times \mathbf{R}_+$, we arrive at the estimate

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\mathbf{R}_+)} &\le \left(\frac{\partial}{\partial x}G_k(u),\ (u_S)_x\right) \le \\
&\le -\sup_x |(u_S(t,x))_x| \int_0^\infty \frac{\partial}{\partial x}G_k(u(t,x))\,dx.
\end{aligned} \tag{26}$$

It is easily checked that

$$|(u_S(t,x))_x| \equiv (1+t)^{m-1/2}|\theta_S'(\xi)|,$$

while $\sup|\theta_S'(\xi)| = q_S < \infty$. Then from (26) we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{L^2(\mathbf{R}_+)} \le q_S(1+t)^{m-1/2}G_k[(1+t)^m]. \tag{27}$$

Since, as follows from (17),

$$\|w(t,\cdot)\|^2_{L^2(\mathbf{R}_+)} \equiv (1+t)^{2m+1/2}\|\theta(t,\cdot) - \theta_S(\cdot)\|^2_{L^2(\mathbf{R}_+)},$$

from (27) we immediately obtain the estimate

$$\|\theta(t,\cdot) - \theta_S(\cdot)\|^2_{L^2(\mathbf{R}_+)} \leq (1+t)^{-2m-1/2}\|u_0(\cdot) - \theta_S(\cdot)\|^2_{L^2(\mathbf{R}_+)} + \\ + 2q_S(1+t)^{-2m-1/2}\int_0^t (1+\tau)^{m-1/2} G_k|(1+\tau)^m|\,d\tau, \tag{28}$$

which is the same as (24). □

**Remark.** The estimate (24) holds for sufficiently arbitrary (non-monotone in $x$) initial functions $u_0 \in L^2(\mathbf{R}_+)$, such that $0 \leq u_0^- \leq u_0 \leq u_0^+$ in $\mathbf{R}_+$, where $u_0^\pm$ are monotone functions, $u_0^\pm(0) = u_1(0)$. Then the same method can be used to derive estimates of the form (24) for the similarity representations $\theta^\pm(t,\xi)$ of the solutions $u^\pm(t,x)$, which satisfy the initial conditions $u^\pm|_{t=0} = u_0^\pm$ in $\mathbf{R}_+$. Therefore the stabilization $\theta(t,\xi) \to \theta_S(\xi)$ as $t \to \infty$ will follow from the inequalities $u^- \leq u \leq u^+$ (or, equivalently, $\theta^- \leq \theta \leq \theta^+$) in $\mathbf{R}_+ \times \mathbf{R}_+$.

Thus, the self-similar solution (6) correctly describes for large $t$ properties of solutions of a large set of quasilinear parabolic equations. The estimate (19) of the depth of penetration of the thermal wave also holds here, while the function $\theta_S(\xi)$ determines its spatial form as $t \to \infty$. The function $u_S$ will be an approximate self-similar solution for the equation (23): $u_S$ does not satisfy that equation, but correctly describes asymptotic properties of solutions of this equation.

Therefore, using just the self-similar solutions (6) we can describe asymptotic behaviour of solutions of boundary value problems corresponding to different boundary data $u_0(x)$, $u_1(t)$ and different equations (in this case, equations with different heat conductivity coefficients $k(u)$). However, (1) admits also other self-similar solutions, such as, for example, a travelling wave type solution.

$$u_S(t,x) = \exp\{t - x\},\ t > 0,\ x > 0, \tag{29}$$

for which $u_1(t) = \exp\{t\}$. It is not hard to show that this solution is stable with respect to perturbations of initial and boundary functions, as well as to perturbations of the equation, which allows us to find asymptotically exact solutions for the class of boundary value problems with boundary regimes of exponential form.

Finally, let us observe that the properties of self-similar solutions of equation (1) (for example, of form (6) or (29)) are preserved also under perturbations of boundary regimes and the equation more drastic than those of (20) and (22) (see § 4, Ch. VI).

## § 2 Asymptotic stability of the fundamental solution of the Cauchy problem

In this section we consider the Cauchy problem for the heat equation,

$$u_t = u_{xx}, \quad t > 0, \quad x \in \mathbf{R}, \tag{1}$$

$$u(0, x) = u_0(x) \geq 0, \qquad x \in \mathbf{R};\ u_0 \in C(\mathbf{R}), \tag{2}$$

where the initial function has finite energy:

$$E_0 = \|u_0\|_{L^1(\mathbf{R})} < \infty, \quad E_0 > 0. \tag{3}$$

Then the solution of the problem (1), (2) will have the same property: its energy is constant in time:

$$\int_{-\infty}^{\infty} u(t, x)\,dx = E_0, \quad t \geq 0. \tag{4}$$

For simplicity we shall assume in the following that $u_0(x) = o(\exp\{-|x|^2\})$ as $|x| \to \infty$.

We set ourselves the same questions: how does the initial temperature profile spread, how do its amplitude and width change in time as $t \to \infty$?

We stress that this problem in the above setting is very different from that considered in § 1. Unlike the boundary value problem, here there is no "forgetting" of the properties of the initial condition, since the amount of energy $E_0$ in (4) (which is a characteristic of the function $u_0$) plays an important rôle at the asymptotic stage of the process. This fact imposes additional restrictions on the methods of studying asymptotic properties of solutions of the problem (1), (2).

Equation (1) has a well-known self-similar (fundamental) solution in $\mathbf{R}_+ \times \mathbf{R}$:

$$u_S(t, x) = (1 + t)^{-1/2} f_S(\xi), \quad \xi = x/(1 + t)^{1/2}, \tag{5}$$

where

$$f_S(\xi) = \frac{E_0}{2\pi^{1/2}} \exp\left\{-\frac{\xi^2}{4}\right\}, \quad \xi \in \mathbf{R}. \tag{6}$$

It satisfies the conservation law (4).

Solution (5) will solve the problem (1), (2) only if the initial function $u_0(x)$ is also of a self-similar form, that is, if

$$u_0(x) \equiv u_S(0, x) = \frac{E_0}{2\pi^{1/2}} \exp\left\{-\frac{x^2}{4}\right\}, \quad x \in \mathbf{R}. \tag{7}$$

## 1 Stability with respect to perturbations of the initial function

The analysis of this problem is not very complicated, since there is a representation of the solution of the problem (1), (2) in terms of a heat potential [282]:

$$u(t,x) = \frac{1}{2(\pi t)^{1/2}} \int_{-\infty}^{\infty} u_0(y) \exp\left\{-\frac{(x-y)^2}{4t}\right\} dy. \tag{8}$$

For convenience, let us introduce the similarity representation of the solution of the problem (1), (2) which corresponds to the spatio-temporal structure of the solution (5):

$$f(t,\xi) = (1+t)^{1/2} u(t, \xi(1+t)^{1/2}), \ t > 0, \ \xi \in \mathbf{R} \tag{9}$$

(substitution of the solution (5) into (9) gives us the function $f_S(\xi)$).

**Proposition 5.** *The self-similar solution* (5) *is stable with respect to arbitrary perturbations of the self-similar initial function* (7), *which preserve its energy: if* (3) *holds, we have pointwise convergence:*

$$f(t,\xi) \to f_S(\xi), \ t \to \infty; \ \xi \in \mathbf{R}. \tag{10}$$

*Proof.* Let us fix an arbitrary $\xi = x/(1+t)^{1/2}$. Then, using (8), after elementary transformations, we obtain

$$f(t,\xi) = \frac{1}{2}\left(\frac{1+t}{\pi t}\right)^{1/2} \exp\left\{-\frac{\xi^2}{4}\right\} \times$$
$$\times \int_{-\infty}^{\infty} u_0(y) \exp\left\{-\frac{\xi^2 + y^2 - 2\xi y(1+t)^{1/2}}{4t}\right\} dy.$$

Since $u_0 \in L^1(\mathbf{R})$ satisfies condition (3), the integral in the right-hand side converges to $E_0$ as $t \to \infty$. This means that (10) holds. □

What are the consequences of this result? First of all, it means that the amplitude of the thermal profile evolves for large times as

$$\sup_{x \in \mathbf{R}} u(t,x) \simeq \frac{E_0}{2\pi^{1/2}} t^{-1/2}, \ t \to \infty,$$

so that the width of the temperature inhomogeneity is

$$x_{ef}(t) \simeq 2(\ln 2)^{1/2} t^{1/2}, \ t \to \infty.$$

## 2 Stability with respect to nonlinear perturbations of the equation

Here we use the self-similar solution (5) to study the nonlinear heat equation

$$u_t = (k(u)u_x)_x, \quad t > 0, \quad x \in \mathbf{R}. \tag{11}$$

Since in the Cauchy problem (11), (2) the amplitude of the solution, $u_m(t) \equiv \sup_x u(t, x)$, goes to zero as $t \to \infty$, the asymptotic properties of the solution $u(t, x)$ depend on the character of behaviour of the coefficient $k(u)$ for small values of the temperature $u > 0$.

Below we shall demonstrate stability of the self-similar solution (5) of the heat equation with respect to the following perturbations of the constant coefficient: $k \in C^2((0, \infty)) \cap C([0, \infty))$, $k(u) > 0$, $k'(u) > 0$ for $u > 0$,

$$[k(u)/k'(u)]' \to \infty, u \to 0, \tag{12}$$

and furthermore,

$$\lim_{u \to 0} [k(\xi u)/k(u)] = 1, \quad \xi > 0. \tag{13}$$

These conditions are satisfied, for example by the coefficient

$$k(u) = |\ln u|^{-\alpha}, \quad \alpha = \text{const} > 0, \quad u \in (0, 1/2), \tag{14}$$

which differs significantly as $u \to 0$ from the coefficient $k \equiv 1$. Nonetheless, asymptotic properties of solutions of equation (11) can be described using the fundamental solution (5) by transforming it in a convenient manner. Therefore the problem of stability of the self-similar solution with respect to nonlinear perturbations of equation (1) is considered here in a new setting (compared to § 1). At the same time we shall prove stability of $u_S$ with respect to small perturbations of the thermal conductivity coefficient in the case $k(u) \to 1$ as $u \to 0^+$.

In addition to (14), all the conditions are satisfied by the coefficients $k(u) = [\ln|\ln u|]^{\alpha}$, $\alpha > 0$; $k(u) = \exp\{-|\ln u|^{\alpha}\}$, $\alpha \in (0, 1)$, and so forth.

Equation (11) with an arbitrary nonlinearity has no self-similar solution describing the "spread" of the initial profile in the Cauchy problem. Therefore we shall look for an *approximate self-similar solution* $u_s$ which does not satisfy equation (11):

$$u_s(t, x) = \frac{1}{\phi(t)} f_S(\zeta), \quad \zeta = \frac{x}{\phi(t)}, \tag{15}$$

where $\phi(t)$ is a monotone increasing positive function, $\phi(t) \to \infty$ as $t \to \infty$, while $f_S(\zeta)$ is the function (6) (it is precisely this function that connects the a.s.s. with the fundamental solution (5)). The function $u_s$ satisfies the energy conservation law (4).

The main problem is to determine the function $\phi(t)$ in (15), which depends on the behaviour of $k(u)$ for low temperatures. It provides both the rate of amplitude decay:

$$u_m(t) \simeq \frac{E_0}{\phi(t)} \frac{1}{2\pi^{1/2}}, \quad t \to \infty,$$

and the law governing the rate of change of the width of the temperature profile

$$x_{ef}(t) \simeq 2(\ln 2)^{1/2}\phi(t), \quad t \to \infty.$$

Let us introduce, as usual, the similarity representation of the solution of the problem (11), (2),

$$\theta(t, \zeta) = \phi(t)u(t, \zeta\phi(t)).$$

It is convenient to carry out the proof of convergence of $\theta(t, \zeta)$ to $f_S(\zeta)$ (which establishes similarity of asymptotic properties of the solution of the problem and a.s.s. (15)) by considering $u(t, \cdot)$ as an element of the Hilbert space $h^{-1}(\mathbf{R})$. To this space belong functions $w \in L^1(\mathbf{R})$, which satisfy the conditions

$$\int_{-\infty}^{\infty} w(x)\, dx = 0, \quad \int_{x}^{\infty} w(y)\, dy \in L^2(\mathbf{R}), \tag{16}$$

$$\left| \int_0^{\infty} dx \int_x^{\infty} w(y)\, dy \right| < \infty, \quad \left| \int_{-\infty}^{0} dx \int_{-\infty}^{x} w(y)\, dy \right| < \infty. \tag{17}$$

In the usual way we can introduce in this space the scalar product

$$(v, w)_{-1} = (v, (-d^2/dx^2)^{-1} w),$$

where the function $W = (-d^2/dx^2)^{-1}w$ is the solution of the problem $d^2W/dx^2 = -w$, $x \in \mathbf{R}$; $|W(\pm\infty)| < \infty$. It is not hard to verify that by (16) and (17) a solution of this problem exists. We shall denote by $\|\cdot\|_{h^{-1}(\mathbf{R})}$ the norm in $h^{-1}(\mathbf{R})$:

$$\|\cdot\|_{h^{-1}(\mathbf{R})} = (w, w)_{-1}^{1/2} = \left\{ \int_{-\infty}^{\infty} w(x) \left[ \left( -\frac{d^2}{dx^2} \right)^{-1} w \right] (x)\, dx \right\}^{1/2} \equiv$$

$$\equiv \left\| \left( \frac{d}{dx} \right)^{-1} w \right\|_{L^2(\mathbf{R})} = \left\| \int_x^{\infty} w(y)\, dy \right\|_{L^2(\mathbf{R})}.$$

**Proposition 6.** *Let conditions* (12), (13) *hold, and let* $u_0$ *satisfy* (3), *such that, moreover,* $u_0(\cdot) - f_S(\cdot) \in h^{-1}(\mathbf{R})$. *Then* $\phi(t) = [1 + \mu^{-1}(t)]^{1/2}$ *for all sufficiently large* $t$, *where* $\mu^{-1}$ *denotes the function inverse to the monotone increasing function*

$$\mu(t) = \int_0^t \frac{d\tau}{k[(1+\tau)^{-1/2}]} \simeq \frac{1+t}{k[(1+t)^{-1/2}]}, \quad t \to \infty.$$

*The solution of the problem* (11), (2) *converges to the a.s.s.* (15)*:*

$$\|\theta(t,\cdot) - f_S(\cdot)\|_{h^{-1}(\mathbf{R})} \to 0, \quad t \to \infty.$$

*Proof.* Let us make the change of variable $t \to \mu(t)$ in the problem (11), (2). Then[2]

$$\phi(\mu(t)) = (1+t)^{1/2},$$

and $u(\mu(t), x)$ satisfies the equation

$$u_t = \mu'(t)(k(u)u_x)_x.$$

Then the a.s.s. (15) becomes the function (5), that is,

$$u_S(\mu(t), \ x) \equiv u_S(t, x) \text{ as } t \to \infty.$$

Let us set $w(t,x) = u(\mu(t), x) - u_S(\mu(t), x)$. Then

$$\int_{-\infty}^{\infty} w(t,x)\,dx = 0, \quad t > 0$$

(since by assumption $u$ and $u_S$ have the same energy) and $w \in h^{-1}(\mathbf{R})$ for all $t \geq 0$. The function $w$ satisfies the equation

$$w_t = [\mu'(t)(k(u)u_x) - (u_S)_x]_x.$$

Taking the scalar product of this equation with $(-d^2/dx^2)^{-1}w$ and integrating by parts in the right-hand side, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{h^{-1}(\mathbf{R})} = \left(\mu'(t)(k(u)u_x) - (u_S)_x,\ \left(\frac{d}{dx}\right)^{-1} w\right). \tag{18}$$

It is easily verified that

$$\begin{aligned}(\mu'(t)(k(u)u_x) - (u_S)_x,\ (d/dx)^{-1}(u-u_S)) &\equiv \\ \equiv -\mu'(t)(F(u) - F(u_S),\ u - u_S) &+ ((\mu'(t)k(u_S) - 1)(u_S)_x,\ (d/dx)^{-1}w),\end{aligned}$$

where $F(u) = \int_0^u k(\eta)\,d\eta$. Since the first term in the right-hand side is non-positive, estimating the second one using the Cauchy-Schwarz inequality, we obtain from (18) that

$$\frac{d}{dt}\|w\|_{h^{-1}(\mathbf{R})} \leq \|[\mu'(t)k(u_S) - 1](u_S)_x\|_{L^2(\mathbf{R})}.$$

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[2]For the proof it is sufficient for this equality to hold for large $t > 0$.

Hence it follows (see the proof of Proposition 4) that

$$\|w\|_{h^{-1}(\mathbf{R})} \le \|w(0,\cdot)\|_{h^{-1}(\mathbf{R})} + \\ + (2q_S)^{1/2} \int_0^t (1+\tau)^{-1/2} H^{1/2}(f_S(0)(1+\tau)^{-1/2};\ \tau)\, d\tau, \tag{19}$$

$$q_S = \sup |f'_S(\zeta)| < \infty,$$

where the function $H$ has the form

$$H(s;t) = \int_0^s (\mu'(t)k(\eta) - 1)^2\, d\eta.$$

Since

$$\|w(t,\cdot)\|_{h^{-1}(\mathbf{R})} \equiv (1+t)^{1/4} \|\theta(\mu(t),\cdot) - f_S(\cdot)\|_{h^{-1}(\mathbf{R})},$$

we derive from (19) the estimate

$$\|\theta(\mu(t),\cdot) - f_S(\cdot)\|_{h^{-1}(\mathbf{R})} \le (1+t)^{1/4} \|w(0,\cdot)\|_{h^{-1}(\mathbf{R})} + \\ + (2q_S)^{1/2} (1+t)^{-1/4} \int_0^t (1+\tau)^{-1/2} H^{1/2}(f_S(0)(1+\tau)^{-1/2};\ \tau)\, d\tau. \tag{20}$$

Resolving consecutively all the indeterminacies that arise in the right-hand using the equality

$$\mu'(t) = 1/k[(1+t)^{-1/2}], \quad t \to \infty,$$

we obtain

$$\lim_{t\to\infty} \|\theta(\mu(t),\cdot) - f_S(\cdot)\|^2_{h^{-1}(\mathbf{R})} \le \\ \le 32 q_S \lim_{t\to\infty} (1+t)^{1/2} \int_0^{f_S(0)(1+t)^{-1/2}} [\mu'(t)k(\eta) - 1]^2\, d\eta = \\ = 32 q_S \lim_{t\to\infty} \int_0^{f_S(0)} \left\{ \frac{k[\zeta(1+t)^{-1/2}]}{k[(1+t)^{-1/2}]} - 1 \right\}^2 d\zeta = 0.$$

Convergence to a.s.s. now follows from condition (13). □

**Remark.** If in addition to (12), (13) we also impose the condition $k(u)/k(uk^{1/2}(u)) \to 1$ as $u \to 0$, then asymptotically we have

$$\phi(t) \simeq (1+t)^{1/2} k^{1/2}[(1+t)^{-1/2}], \quad t \to \infty.$$

This relation will hold, in particular, for the coefficient (14).

**Example 1.** Let $k(u) = |\ln u|^{-\alpha}$ for small $u > 0$, $\alpha = \text{const} > 0$. As we already mentioned, conditions (12), (13) are satisfied. From Proposition 6 we obtain in this case that

$$\phi(t) \simeq 2^{\alpha/2} t^{1/2} \ln^{-\alpha/2} t, \quad t \to \infty, \tag{21}$$

and therefore a.s.s. (15), to which the solution of the problem (11), (2) converges ($u_0$ satisfies (3)), has the form

$$u_s(t, x) \simeq \left(2^{\alpha+2}\pi \frac{t}{\ln^\alpha t}\right)^{-1/2} E_0 \exp\left\{-\frac{x^2}{2^{\alpha+2} t \ln^{-\alpha} t}\right\}.$$

where $E_0 = \|u_0\|_{L^1(\mathbf{R})} < \infty$. From here we obtain an estimate of the amplitude,

$$\sup_{x \in \mathbf{R}} u(t, x) \simeq u_s(t, 0) \cong (2^{\alpha+2}\pi)^{-1/2} E_0 (\ln^\alpha t/t)^{1/2}, \quad t \to \infty.$$

An estimate of the effective width of the inhomogeneous temperature profile for large times is given by (21).

In the next section we move on to analyze self-similar solutions of nonlinear heat equations.

## § 3 Asymptotic stability of self-similar solutions of nonlinear heat equations

Let us consider first the example of a self-similar solution already encountered in Ch. 1, which exists for arbitrary coefficients $k(u) \geq 0$.

### 1 A self-similar solution with constant temperature at the boundary

This example helps us to emphasize a fundamental property of self-similar solutions of nonlinear heat equations: their asymptotic stability with respect to perturbations of the initial function.

As in § 1, let us consider the boundary value problem in $\mathbf{R}_+ \times \mathbf{R}_+$ for the equation

$$u_t = (k(u)u_x)_x \tag{1}$$

($k(u) > 0$ for $u > 0$ is a sufficiently smooth function) with the initial and boundary conditions

$$u(0, x) = u_0(x) \geq 0, \ x > 0; \ u_0 \in C(\mathbf{R}_+), \tag{2}$$

$$u(t, 0) = 1, \ t > 0. \tag{3}$$

For arbitrary $k(u)$ equation (1) admits a self-similar solution which satisfies condition (3):

$$u_S(t,x) = g_S(\zeta), \quad \zeta = x/(1+t)^{1/2}, \tag{4}$$

where $g_S(\zeta)$ solves the problem

$$(k(g_S)g_S')' + \frac{1}{2}g_S'\zeta = 0, \quad \zeta > 0, \quad g_S(0) = 1, \quad g_S(\infty) = 0; \tag{5}$$

this solution will or will not have compact support depending on whether equation (1) admits finite speed of propagation of perturbations, or does not.

Below we restrict ourselves to the analysis of the case when the coefficient $k$ satisfies the condition for finite speed of propagation of perturbations:

$$\int_0^1 \frac{k(\eta)}{\eta}\,d\eta < \infty,$$

and we take for $g_S(\zeta)$ a solution of the problem (5) with compact support. We shall assume that $u_0$ in (2) also has compact support.

Existence of a self-similar solution of the form (4) is related to invariance of equation (1) for arbitrary $k(u)$ under the transformations $t \to t/\alpha$, $x \to x/\alpha^{1/2}$; $\alpha > 0$. Therefore, if $u(t, x)$ is a solution, so will be $u(t/\alpha, x/\alpha^{1/2})$. Let us try to find a solution which is invariant under these transformations, that is, such that $u(t,x) \equiv u(t/\alpha, x/\alpha^{1/2})$ for all $\alpha > 0$. Setting in that equality $\alpha = t$, we obtain $u(t,x) \equiv u(1, x/t^{1/2})$. Denoting $u(1, \zeta)$ by $g_S(\zeta)$ and using the change of variable $t \to 1+t$, which does not affect the form of the equation, we obtain (4).

Clearly, (4) is a solution of the original problem (1)–(3) only if $u_0 \equiv u_S(0,x) = g_S(x)$. Below we shall show that for any perturbations of initial function with compact support $u_0(x)$ the asymptotic behaviour of solutions $u(t, x)$ for large $t$ is described by the self-similar solution $u_S$. Therefore the law of motion of the half-width of the self-similar thermal wave, determined from (4):

$$x_{ef}^S(t) = \zeta_{ef}(1+t)^{1/2}, \ t > 0 \quad (g_S(\zeta_{ef}) = 1/2), \tag{6}$$

remains valid as $t \to \infty$ for other solutions of equation (1). Therefore the dependence of the wave speed on time is the same for equations (1) with a wide class of coefficients $k(u)$. Formulae (6) for different coefficients $k(u)$ differ only by the magnitude of the constant $\zeta_{ef}$, which, of course, depends on the form of $k(u)$.

Let us introduce the similarity representation of the problem,

$$g(t,\zeta) = u(t, \zeta(1+t)^{1/2})$$

and show that $g(t,\zeta) \to g_S(\zeta)$ as $t \to \infty$. This ensures that the main properties of the solutions $u(t, x)$ and $u_S(t, x)$ are similar for large $t$, so that, in particular, the estimate (6) holds for $u(t, x)$ as $t \to \infty$. In this case it is convenient to prove asymptotic stability of the self-similar solution in the norm of the space $h^{-1}(\mathbf{R}_+)$.

The Hilbert space $h^{-1}(\mathbf{R}_+)$ is the space of functions $v(x) \in L^1(\mathbf{R}_+)$, which satisfy the conditions

$$\int_x^\infty v(y)\,dy \in L^2(\mathbf{R}_+), \left|\int_0^\infty dx \int_x^\infty v(y)\,dy\right| < \infty. \tag{7}$$

The scalar product in $h^{-1}(\mathbf{R}_+)$ has the form

$$(v, w)_{-1} = \int_0^\infty v(x)\left[\left(-\frac{d^2}{dx^2}\right)^{-1} w\right](x)\,dx, \tag{8}$$

where we have denoted by $W = (-d^2/dx^2)^{-1}w$ the solution of the problem

$$d^2W/dx^2 = -w,\ x > 0;\ W(0) = 0,\ |W(\infty)| < \infty.$$

It is not hard to check that if (7) holds, a solution of this problem exists and is unique:

$$W(x) = \int_0^x dy \int_y^\infty w(z)\,dz,\ x \geq 0.$$

The norm in $h^{-1}(\mathbf{R}_+)$ is defined using (8):

$$\|w\|_{h^{-1}(\mathbf{R}_+)} = (w, w)_{-1}^{1/2},$$

that is,

$$\|w\|_{h^{-1}(\mathbf{R}_+)} \equiv \left\|\left(-\frac{d}{dx}\right)^{-1} w\right\|_{L^2(\mathbf{R}_+)} = \left\|\int_x^\infty w(y)\,dy\right\|_{L^2(\mathbf{R}_+)}.$$

In the norm of $h^{-1}(\mathbf{R}_+)$ convergence of $g(t, \cdot)$ to $g_S(\cdot)$ is especially easy to prove (naturally, it also holds in stronger norms; see the bibliographic comments). Convergence in $h^{-1}(\mathbf{R}_+)$ implies, in particular, pointwise convergence almost everywhere.

**Proposition 7.** *Let $u_0(x)$ be a function with compact support. Then*

$$\|g(t, \cdot) - g_S(\cdot)\|_{h^{-1}(\mathbf{R}_+)} = O\left((1+t)^{-3/4}\right) \to 0,\ t \to \infty.$$

*Proof.* The function $z = u - u_S$ satisfies the equation

$$z_t = [k(u)u_x - k(u_S)(u_S)_x]_x,\ t > 0,\ x > 0; \tag{9}$$

moreover, $z(t,0) = 0$, $z(t,x)$ has compact support in $x$ and $z(t,\cdot) \in h^{-1}(\mathbf{R}_+)$ for all $t \geq 0$. Taking the $h^{-1}(\mathbf{R}_+)$ scalar product of equation (9) with $z$ and integrating by parts, we obtain the equality

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{h^{-1}(\mathbf{R}_+)} = -(F(u) - F(u_S), u - u_S), \tag{10}$$

where

$$F(u) = \int_0^u k(\eta)\, d\eta$$

is a monotone increasing function. Therefore $(F(u) - F(u_S), u - u_S) \geq 0$ and then we have from (10) that

$$\|z(t,\cdot)\|_{h^{-1}(\mathbf{R}_+)} \leq \|z(0,\cdot)\|_{h^{-1}(\mathbf{R}_+)} \equiv \|u_0(\cdot) - g_S(\cdot)\|_{h^{-1}(\mathbf{R}_+)}$$

for all $t > 0$. Since in view of (4) and the way we defined the similarity representation $g(t, \zeta)$, we have the identity

$$\|z(t,\cdot)\|_{h^{-1}(\mathbf{R}_+)} \equiv (1+t)^{3/4}\|g(t,\cdot) - g_S(\cdot)\|_{h^{-1}(\mathbf{R}_+)},$$

we obtain the required estimate of the rate of convergence:

$$\|g(t,\cdot) - g_S(\cdot)\|_{h^{-1}(\mathbf{R}_+)} \leq \|z(0,\cdot)\|_{h^{-1}(\mathbf{R}_+)}(1+t)^{-3/4}.$$

□

Obviously, there is no need to discuss here asymptotic stability of the self-similar solution (4) with respect to perturbations of the coefficient $k$, as to each $k$ corresponds a different solution of the form (4).

## 2 The nonlinear heat equation with a power type nonlinearity

In this subsection we consider certain self-similar solutions of the boundary value problem for the quasilinear parabolic equation

$$u_t = (u^\sigma u_x)_x, t > 0, \ x > 0; \ \sigma = \text{const} > 0, \tag{11}$$

$$u(0,x) = u_0(x) \geq 0, \ x > 0; \ u_0^{\sigma+1} \in C^1(\mathbf{R}_+), \tag{12}$$

$$u(t,0) = u_1(t) > 0, \ t > 0, \tag{13}$$

where the boundary regime is strongly non-stationary: $u_1(t)$ grows without bound with $t$. Some examples of generalized self-similar solutions of this problem were considered in the previous chapter.

Let us make the prefatory remark that an equation with heat conductivity coefficient $k(u) = k_0 u^\sigma$, where $k_0 > 0$ is a constant of, in general, physical dimensions (in (11) it is assumed that $k_0 = 1$), can be non-dimensionalized by a change of variable of the form $t \to k_0 t$.

*I A power law boundary regime*

As in § 1, let

$$u_1(t) = (1 + t)^m, t > 0; \ m = \text{const} > 0.$$

Then equation (11) has a self-similar solution of the following form:

$$u_S(t, x) = (1 + t)^m \theta_S(\xi), \xi = x/(1 + t)^{(1+m\sigma)/2}, \tag{14}$$

which can be related to its invariance with respect to the transformations

$$t \to t/\alpha, x \to x/\alpha^{(1+m\sigma)/2}, u \to \alpha^m u; \ \alpha > 0 \tag{15}$$

(if $u$ is invariant, that is, if $u(t, x) \equiv \alpha^m u(t/\alpha, x/\alpha^{(1+m\sigma)/2})$, then, setting $\alpha = t$ and then by the change of variable $t \to 1 + t$, we obtain (14)).

The function $\theta_S(\xi)$ in (14) satisfies the following ordinary differential equation, obtained by substituting (14) into (11):

$$(\theta_S^\sigma \theta_S')' + \frac{1 + m\sigma}{2}\theta_S{}'\xi - m\theta_S = 0, \xi > 0, \tag{16}$$

where, as follows from the formulation of the problem and the spatio-temporal structure of the solution (14), the appropriate boundary conditions are

$$\theta_S(0) = 1, \theta_S(\infty) = 0. \tag{17}$$

A generalized solution of the problem (16), (17) exists, is unique and has compact support. This is not hard to see by transforming (16) into a first order equation (see Ch. III) or by first proving local solvability close to the point of degeneracy and then extending the obtained solution up to the point $\xi = 0$ ("shooting" to the first boundary condition in (17) is done by using the similarity transformation, which leaves equation (16) invariant). For $m = 1/\sigma$ the problem (16), (17) has the obvious generalized solution $\theta_S(\xi) = [(1 - \sigma^{1/2}\xi)_+]^{1/\sigma}$. In this case $u_S = (1 + t)^{1/\sigma}\theta_S(\xi)$, $\xi = x/(1 + t)$, and therefore the self-similar solution is just the travelling wave considered in Example 6 of Ch. I.

The depth of penetration of the thermal wave described by the self-similar solution (14) has the following dependence on time:

$$x_{ef}^S(t) = \xi_{ef}(1 + t)^{(1+m\sigma)/2}, t > 0; \ \theta_S(\xi_{ef}) = 1/2. \tag{18}$$

The wave moves at a higher speed than in a medium with constant heat conductivity and the same boundary regime (§ 1), since in (11) the thermal conductivity is an increasing function of temperature. This is also the speed of motion of the front of the thermal wave (the point at which $u_S$ vanishes) $x_f^S(t) = \xi_f(1 + t)^{(1+m\sigma)/2}$, where $\xi_f = \text{meas supp}\, \theta_S < \infty$. The evolution of this self-similar heating process

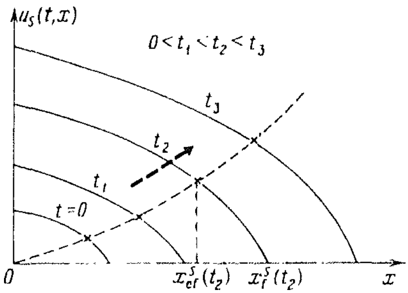


**Fig. 6.** Evolution of the self-similar solution (14) ($m > 0, \sigma > 1$)

is shown schematically in Figure 6. The trajectory of the half-width of the thermal wave is shown by the dashed line.

As in § 1, this self-similar solution is asymptotically stable with respect to small perturbations of the functions $u_0(x)$, $u_1(t)$, $k(u)$ entering the formulation of the problem (for the method of proof of such assertions see Ch. VI, § 3, 4). Therefore the expression (18) for the half-width is asymptotically true for a large class of quasilinear equations (1) with coefficients $k(u)$ not of power type, which are close to $u^\sigma$ as $u \to \infty$.

## 2 *Exponential boundary regime*

A different asymptotically stable self-similar solution of equation (11) exists in the case $u_1(t) = e^t$ for $t > 0$. Here $u_S$ has the form

$$u_S(t,x) = e^t f_S(\eta), \eta = x/\exp\{\sigma t/2\}. \tag{19}$$

The function $f_S \geq 0$ satisfies the boundary value problem

$$(f_S^\sigma f_S')' + \frac{\sigma}{2} f_S'\eta - f_S = 0, \eta > 0, f_S(0) = 1, f_S(\infty) = 0, \tag{19'}$$

solvability of which is proved as in the analogous problem for power law regimes. The nature of the motion of the thermal wave in this case is more or less the same as in Figure 6, the difference being that due to the more vigorous exponential boundary heating, the half-width of the wave grows with time faster than any power:

$$x_{ef}^S(t) = \eta_{ef}\exp\{\sigma t/2\}, t > 0 \quad (f_S(\eta_{ef}) = 1/2).$$

Due to asymptotic stability of the self-similar solution (19) this estimate holds for large $t$ for a large class of non-self-similar solutions. The same is true about the law of motion of the front point of the thermal wave:

$$x_f^S(t) = \eta_f\exp\{\sigma t/2\}, t > 0; \ \eta_f = \operatorname{meas}\operatorname{supp} f < \infty$$

(if the perturbed equation admits finite speed of propagation of perturbations and $u_0(x)$ is a function with compact support).

Analysis of self-similar solutions discloses the physically reasonable principle: the more vigorous the boundary regime, the higher will be the speed of the resulting thermal wave. If the regime is of power type, then so is the depth of penetration; if the regime is exponential (as $t \to \infty$ the heating is more intense than for any power type regime), then the motion of the half-width is given by an exponential function. The following question arises: do there exist boundary regimes to which correspond "slower" moving thermal waves? Such regimes exist, and to one of them corresponds a simple self-similar solution.

### *3 A power type boundary blow-up regime. Heat localization*

Let the dependence of the temperature on the boundary $x = 0$ exhibit finite time blow-up:

$$u_1(t) = (T_0 - t)^{-1/\sigma}, \quad 0 < t < T_0, \tag{20}$$

where $0 < T_0 < \infty$ is a constant (blow-up time). The boundary function in (20) becomes infinite in finite time: $u_1(t) \to \infty$ as $t \to T_0^-$. To this regime corresponds a self-similar solution of (11) of an unusual form, a *standing thermal wave*:

$$u_S(t, x) = (T_0 - t)^{-1/\sigma}[(1 - x/x_0)_+]^{2/\sigma}, \tag{21}$$

where $x_0 = [2(\sigma + 2)/\sigma]^{1/2}$. The position of the front point in (21), $x_f(t) \equiv x_0$, is constant during all the time of existence of the solution $t \in (0, T_0)$ and heat from the localization domain $x \in (0, x_0)$ does not penetrate into the surrounding cold space, even though everywhere in the domain $(0, x_0)$ the temperature grows without bound as $t \to T_0$.

A schematic drawing of such a heating process (heat localization in the *S-regime*) is to be seen in Figure 7, which shows the essential difference between the influence on a nonlinear medium of a boundary blow-up regime (20) and of ordinary regimes (see Figure 6). The depth of penetration of the localized wave is, just like the position of the front point, independent of time; from (21) it follows that $x_{ef}^S(t) \equiv x_0(1 - 2^{-\sigma/2})$, $0 < t < T_0$.

The self-similar solution (21) is asymptotically stable. In Ch. V we shall show that the heat localization of boundary heating regimes which exhibit finite time blow-up occurs also in arbitrary nonlinear media described by general heat equations of parabolic type.

It is important to note that not every boundary blow-up regime guarantees heat localization. For example, if we take a different power type regime:

$$u_1(t) = (T_0 - t)^n, \quad 0 < t < T_0, \tag{22}$$

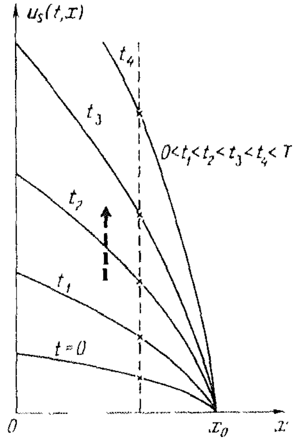


**Fig. 7.** Evolution as $t \to T_0^-$ of the localized self-similar solution (21) (S blow-up regime)

where $n < -1/\sigma$ (in (20) $n = -1/\sigma$), then there is no localization. To the regime (22) corresponds the self-similar solution

$$u_S(t, x) = (T_0 - t)^n \theta_S(\xi), \ \xi = x/(T_0 - t)^{(1+n\sigma)/2}, \tag{23}$$

where $\theta_S \geqslant 0$ satisfies an ordinary differential equation. For $n < -1/\sigma$ the function $\theta_S(\xi)$ has compact support, $\xi_f$ = meas supp $\theta_S < \infty$ (see Ch. III). Then it follows from (23) that the front point of the thermal wave moves according to

$$x_f^S(t) = \xi_f (T_0 - t)^{(1+n\sigma)/2},$$

and $x_f^S(t) \to \infty$ as $t \to T_0^-$.

Evolution of the thermal wave in this case is not substantially different from that of Figure 6; however, the heating of the whole space $\{x > 0\}$ to infinitely high temperature takes only a finite amount of time ($u_S(t, x) \to \infty$ as $\to T_0^-$ for all $x \geq 0$). For $n < -1/\sigma$ the boundary regime (22) is called the *HS blow-up regime*.

On the other hand, if $n \in (-1/\sigma, 0)$, then it is the *LS blow-up regime*, which leads to heating localization. Furthermore, from the spatio-temporal structure of the self-similar solution (23), unbounded growth of temperature as $t \to T_0^-$ occurs only at the point $x = 0$; everywhere in the space $\{x > 0\}$ it is bounded from above uniformly in $t \in (0, T_0)$. This is indicated, in particular, by the law of motion of the half-width of the thermal wave:

$$x_{ef}^S(t) = \xi_{ef} (T_0 - t)^{(1+n\sigma)/2}, \ t \in (0, T_0).$$

where the constant $\xi_{ef} \in \mathbf{R}_+$ is such that $\theta_S(\xi_{ef}) = 1/2$. For $n \in (-1/\sigma, 0)$ we have that $x^S_{ef}(t) \to 0$ as $t \to T_0^-$, so that the half-width (in a certain sense the depth of penetration) of the thermal wave decreases during the heating process down to zero. A detailed analysis of the localization phenomenon in boundary value problems for heat equations is presented in Ch. III (for equation (11) for $\sigma \geq 0$) and in Ch. V (for arbitrary nonlinear heat equations).

Equation (11) has a number of other interesting self-similar solutions (see Comments).

Let us present, for example, an interesting invariant solution, which especially clearly demonstrates localization of a thermal wave front under the action of the S boundary blow-up regime. It is not hard to check that equation (11) has the following exact generalized solution:

$$u_*(t, x) = (T_0 - t)^{-1/\sigma} \left[(1 - x/x_0)^2 - (1 - t/T_0)^{2/(\sigma+2)}\right]_+^{1/\sigma}, \tag{24}$$

where $x_0 = [2(\sigma + 2)/\sigma]^{1/2}$. It corresponds to the initial function $u_*(0, x) \equiv 0$ and a boundary regime which is close to a power type one:

$$u_*(t, 0) = (T_0 - t)^{-1/\sigma} \left[1 - (1 - t/T_0)^{2/(\sigma+2)}\right]^{1/\sigma}, \tag{25}$$

and obviously

$$u_*(t, 0) = (T_0 - t)^{-1/\sigma}(1 + o(1)) \text{ as } t \to T_0,$$

so that this is indeed a boundary blow-up S-regime and the solution (24) grows without bound in the localization domain $x \in [0, x_0)$. However, the front of the thermal wave, which corresponds to (24), is not (unlike (21)) immobile. It moves according to

$$x_f^*(t) = x_0[1 - (1 - t/T_0)^{1/(\sigma+2)}], \ t \in [0, T_0),$$

the wave is localized and $x_f^*(t) \to x_0$ as $t \to T_0$. By comparing (21) and (24) it is easy to see that close to the blow-up time $t = T_0$, the solution $u_*(t, x)$ is close to the self-similar solution (21). In Ch. IV we shall show that this self-similar solution is asymptotically stable not only with respect to small perturbations of the boundary function, as in (25), but also to perturbations of the nonlinear operator of the equation, that is, of the thermal conductivity coefficient.

## § 4 Quasilinear heat equation in a bounded domain

In this section we consider other problems for the nonlinear heat equation in the multi-dimensional case:

$$u_t = \Delta u^{\sigma+1}, \ \sigma = \text{const} > 0. \tag{1}$$

Let $\Omega$ be a bounded domain in $\mathbf{R}^N$ with a sufficiently smooth boundary $\partial\Omega$. Assume that in $\Omega$ an initial heat perturbation is given,

$$u(0, x) = u_0(x) \geq 0, \; x \in \Omega; \; u_0^{\sigma+1} \in C(\overline{\Omega}) \cap H^1(\Omega). \tag{2}$$

## 1 The boundary value problem with Dirichlet conditions

Let zero temperature be maintained on the boundary $\partial\Omega$ of the domain $\Omega$:

$$u(t, x) = 0, \; t > 0, \; x \in \partial\Omega, \tag{3}$$

which corresponds to outflow of heat from the boundary (processes with the adiabatic "isolation" condition on the boundary are considered in subsection 2.)

Clearly $u(t, x) \to 0$ as $t \to \infty$ everywhere in $\Omega$, as heat is taken away through the boundary. How does the evolution of the initial perturbation proceed? At what rate does the extinction process occur?

These questions can be answered by analyzing the self-similar solution admitted by equation (1):

$$u_S(t, x) = (T + t)^{-1/\sigma} f_S(x), \; t > 0, x \in \overline{\Omega}. \tag{4}$$

Here $T > 0$ is an arbitrary constant.

Substituting this expression into (1) and taking into consideration the boundary condition, we obtain for $f_S \geq 0$ the following elliptic problem:

$$\Delta f_S^{\sigma+1} + \frac{1}{\sigma} f_S = 0, \; x \in \Omega; \; f(x) = 0, \; x \in \partial\Omega. \tag{5}$$

For any $\sigma > 0$ it has a unique solution, strictly positive in $\Omega$ (existence of the solution can be established, for example, by constructing sub- and supersolutions of the problem; see [7, 21])

It turns out that (4) is stable with respect to arbitrary bounded perturbations of the initial function $u_0(x)$, that is, for $t \to \infty$ the expression (4) correctly describes the evolution of any heat perturbation. Without considering the details of this, let us restrict ourselves to proving a simple assertion.

To describe the asymptotics of the solution, let us introduce, as usual, a similarity representation of the solution of the problem (1)–(3) by the expression

$$f(t, x) = (1 + t)^{1/\sigma} u(t, x), \; t > 0, x \in \Omega. \tag{6}$$

Stability of the self-similar solution (4) will mean that $f(t, x) \to f_S(x)$ in $\Omega$ as $t \to \infty$.

**Proposition 8.** *Let the initial function $u_0(x)$ in (2) be such that*

$$T_2^{-1/\sigma} f_S(x) \le u_0(x) \le T_1^{-1/\sigma} f_S(x),\ x \in \Omega, \tag{7}$$

*where $0 < T_1 < T_2 < \infty$ are constants. Then*

$$\|f(t,\cdot) - f_S(\cdot)\|_{C(\Omega)} = O(t^{-1}) \to 0,\ t \to \infty. \tag{8}$$

*Proof.* Validity of (8) follows from the comparison theorem. Indeed, by (7)

$$(T_2+t)^{-1/\sigma} f_S(x) \le u(t,x) \le (T_1+t)^{-1/\sigma} f_S(x) \text{ in } \mathbf{R}_+ \times \Omega. \tag{9}$$

Hence (8) follows immediately. □

Therefore if conditions (7) hold, the amplitude of the heat perturbation decreases at the rate

$$\sup_{x\in\Omega} u(t,x) = \left(\sup_{x\in\Omega} f_S(x)\right) t^{-1/\sigma} + o(t^{-1/\sigma}),\ t \to \infty,$$

and furthermore the maximal value of the temperature is attained at an extremum point of $f_S(x)$. Thus in the framework of Proposition 8, the evolution of the heat conduction process for large times is entirely determined (in terms of the function $f_S(x)$) by the spatial structure of the domain $\Omega$ and by the exponent $\sigma$ in the thermal conductivity coefficient $k(u) = (\sigma+1)u^\sigma$.

The proof of convergence in the case of arbitrary $u_0 \not\equiv 0$ follows in essence along the same lines. We have to show that after a finite time $t_0 > 0$ the temperature distribution $u(t_0, x)$ will satisfy (7), whence the estimate (8) will follow. Let us clarify this assertion (the arguments below illustrate an application of criticality conditions for solutions of parabolic equations, which will be used systematically in Ch. V).

Let the initial function $u_0 \in C(\overline{\Omega})$, $u_0 \not\equiv 0$, be sufficiently small and have compact support in $\Omega$: $\overline{\operatorname{supp} u_0} \subset \Omega$. Then the lower bound of (7) does not hold for any $T_2 > 0$, since $f_S(x) > 0$ in $\Omega$. Let us show that we still have stability of the self-similar solution in the sense of (8). The equation for the similarity representation (6) has the form

$$\frac{\partial f}{\partial \tau} = \Delta f^{\sigma+1} + \frac{1}{\sigma} f,\ \tau > 0,\ x \in \Omega;\ f = 0,\ \tau > 0,\ x \in \partial\Omega, \tag{10}$$

where we have introduced the new "time" $\tau = \ln(1+t) : \mathbf{R}_+ \to \mathbf{R}_+$.

Since $f_S(x)$ satisfies the problem (5), the equality (8) has the interpretation that as $\tau \to \infty$, the solution of (10) stabilizes to its stationary solution, which, as we have mentioned already, is unique. For simplicity, let $0 \in \Omega$ and $0 \in \operatorname{supp} u_0$. Let us consider the family of stationary solutions $v = v(r)$, $r = |x|$, of equation (10):

$$\frac{1}{r^{N-1}}(r^{N-1}(v^{\sigma+1})')' + \frac{1}{\sigma} v = 0. \tag{11}$$

which satisfy for $r = 0$ the condition $v'(0) = 0$ (condition of symmetry with respect to the point $r = 0$) and $v(0) = v_0 = \text{const} > 0$.

The solution of this Cauchy problem for the ordinary differential equation (11) exists and is strictly positive in some ball $B_{r_0} = \{r < r_0\}$, where $r_0 = r_0(v_0) < \infty$, such that $v(r_0) = 0$. Here $r_0(v_0) \to 0$ as $v_0 \to 0$ (see § 3, Ch. IV). Let us choose $v_0$ so small that $B_{r_0} \subset \Omega$. Then we claim that the solution of equation (10) with the initial function

$$f(0, x) = v_0(|x|),\ x \in B_{r_0};\ f(0, x) = 0,\ x \in \Omega \backslash B_{r_0}, \tag{12}$$

is critical:

$$\partial f/\partial \tau \geq 0 \text{ in } \mathbf{R}_+ \times \Omega \cap \{x \in \Omega \mid f(\tau, x) > 0\}.$$

This is a direct consequence of the Maximum Principle (see Ch. V).

Therefore the function $f(\tau, x)$ does not decrease in $\tau$ everywhere in $\Omega$ and, if $v_0$ is small, is bounded from above by the stationary solution $f_S(x)$. Therefore at each point $x \in \Omega$ there exists the limit $f(\tau, x) \to f_*(x)$, $\tau \to \infty$. Then, by the usual Lyapunov arguments (see § 5, Ch. IV), we can prove that the limit function $f_*(x)$ has to coincide with the unique solution of the stationary problem (5).

As far as arbitrary, sufficiently small initial perturbations of $f(0, x)$ are concerned, note that under each of these we can "place" the indicated critical solution, which, by the comparison theorem, by stabilizing to the stationary solution, will force stabilization to it of any other solution lying between itself and the stationary solution.

Thus the self-similar solution provides us with information concerning the behaviour for large times of a wide variety of solutions of the problem for more or less arbitrary initial perturbations. Let us emphasize that the asymptotic spatio-temporal structure of solutions of the problem (1)–(3) depends in an essential way on the geometry of the domain $\Omega$. A slightly different situation arises in another boundary value problem for equation (1).

## 2 The boundary value problem with the Neumann condition

Let now the no heat flux condition

$$\partial u^{\sigma+1}/\partial n = 0,\ t > 0,\ x \in \partial\Omega, \tag{13}$$

be imposed on the boundary. Here $\partial/\partial n$ denotes the derivative in the direction of the outer normal to $\partial\Omega$. It is not hard to foretell the asymptotic properties of the solution, based on physical intuition concerning the behaviour of diffusion processes. By the adiabatic condition (13), the total heat energy in $\Omega$ is conserved:

$$\int_\Omega u(t, x)\, dx \equiv \int_\Omega u_0(x)\, dx = E_0. \tag{14}$$

Due to diffusion all inhomogeneities of the initial perturbation will be smoothed out with time, and as a result as $t \to \infty$ the temperature field must stabilize to a spatially homogeneous state. Its magnitude is uniquely determined from (14), and therefore we can expect that

$$u(t,x) \to \frac{1}{\operatorname{meas}\Omega}\int_\Omega u_0(x)\,dx = \overline{u}_{av},\ t \to \infty. \tag{15}$$

Without giving the detailed proof of (15), let us make some clarifications, using only two standard identities satisfied by the solution of the problem (1), (2), (13). The first of these is obtained by taking the scalar product in $L^2(\Omega)$ of equation (1) with $u^{\sigma+1}$ and integrating by parts:

$$\frac{1}{(\sigma+2)}\frac{d}{dt}\|u(t)\|^{\sigma+2}_{L^{\sigma+2}(\Omega)} = -\|\nabla u^{\sigma+1}(t)\|^2_{L^2(\Omega)},\ t \geq 0. \tag{16}$$

The second one is derived by multiplying the equation by $(u^{\sigma+1})_t$ and then integrating the resulting equality in $t$ (see § 2, Ch. VII). As a result we have

$$\begin{aligned}\frac{4(\sigma+1)}{(\sigma+2)^2}\int_0^t \|(u^{1+\sigma/2})_t(s)\|^2_{L^2(\Omega)}\,ds + \frac{1}{2}\|\nabla u^{\sigma+1}(t)\|^2_{L^2(\Omega)} &= \\ = \frac{1}{2}\|\nabla u_0^{\sigma+1}\|^2_{L^2(\Omega)}&.\end{aligned} \tag{17}$$

Passing in (17) to the limit as $t \to \infty$, we see that the first integral converges, so that the limit

$$\|\nabla u^{\sigma+1}(t)\|^2_{L^2(\Omega)} \to a_0 \geq 0,\ t \to \infty, \tag{18}$$

exists. Comparing (18) with the equality (16), we obtain $a_0 = 0$; otherwise the function $\|u\|^{\sigma+2}_{L^{\sigma+2}(\Omega)} \geq 0$ is negative for large $t$.

The condition $a_0 = 0$ in (18) means that $u^{\sigma+1}(t,x)$ converges to a spatially homogeneous state almost everywhere (in fact, by sufficient regularity of the generalized solution, everywhere in $\Omega$). Then the energy conservation law (14) guarantees stabilization (15).

It is not hard to derive an estimate of the rate of stabilization to the average value of the temperature.

**Proposition 9.** *We have the estimate*

$$\|u(t,\cdot) - \overline{u}_{av}\|^2_{L^2(\Omega)} \equiv \int_\Omega (u(t,x) - \overline{u}_{av})^2\,dx \leq Ke^{-\nu t} \to 0,\ t \to \infty, \tag{19}$$

*where $K > 0$, $\nu > 0$ are constants, and $\nu$ depends only on $\sigma$, $\overline{u}_{av}$ and the domain $\Omega$.*

*Proof.* Let us take the scalar product in $L^2(\Omega)$ of the equation (1) with $u$. Then, after integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} = -(\sigma+1)\int_\Omega u^\sigma(t,x)|\nabla u(t,x)|^2\,dx. \tag{20}$$

In view of stabilization of $u$ to $\overline{u}_{av} > 0$ as $t \to \infty$, there exists $t_* \geqslant 0$, such that for all $t > t_*$ we have the inequality $u(t,x) \geq \overline{u}_{av}/2$ in $\Omega$. Then, estimating from above the right-hand side of (20), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} \leq -(\sigma+1)\left(\frac{\overline{u}_{av}}{2}\right)^\sigma \int_\Omega |\nabla u(t,x)|^2\,dx,\ t > t_*. \tag{21}$$

Setting $u - \overline{u}_{av} = w$, we substitute in (21) $u = \overline{u}_{av} + w$. By (14)

$$\int_\Omega w(t,x)\,dx \equiv 0. \tag{22}$$

Then, since $\nabla u \equiv \nabla w$ and

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} &\equiv \frac{d}{dt}\int_\Omega (w^2 + \overline{u}^2_{av} + 2\overline{u}_{av}w)\,dx = \\ &= \frac{d}{dt}\left\{\int_\Omega w^2\,dx + \int_\Omega \overline{u}^2_{av}\,dx + 2\overline{u}_{av}\int_\Omega w\,dx\right\} \equiv \frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)},\end{aligned}$$

we derive from (21) the estimate

$$\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} \leq -2(\sigma+1)\left(\frac{\overline{u}_{av}}{2}\right)^\sigma \|\nabla w(t)\|^2_{L^2(\Omega)}. \tag{23}$$

Using the well-known inequality [362]

$$\|\nabla w\|^2_{L^2(\Omega)} \geq \lambda_1\|w\|^2_{L^2(\Omega)},$$

which holds for all functions $w \in H^1(\Omega)$, $\partial w/\partial n = 0$ on $\partial\Omega$, which satisfy the condition (22) (here $\lambda_1 = \lambda_1(\Omega) > 0$ is the first eigenvalue of the problem $\Delta\psi + \lambda\psi = 0$, $x \in \Omega$, $\partial\psi/\partial n = 0$ on $\partial\Omega$), we obtain from (23) that

$$\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} \leq -2(\sigma+1)\left(\frac{\overline{u}_{av}}{2}\right)^\sigma \lambda_1\|w(t)\|^2_{L^2(\Omega)},\ t > t_*.$$

Hence

$$\|w(t)\|^2_{L^2(\Omega)} \leq \|w(t_*)\|^2_{L^2(\Omega)}\exp\{-2(\sigma+1)\lambda_1(\overline{u}_{av}/2)^\sigma t\},\ t > t_*,$$

which coincides with the estimate (19), if we set $K = \|w(t_*)\|^2_{L^2(\Omega)} < \infty$ and $\nu = 2(\sigma+1)\lambda_1(\Omega)(\overline{u}_{av}/2)^\sigma > 0$. □

## § 5 The fast diffusion equation. Boundary value problems in a bounded domain

In this section we shall consider properties of solutions of quasilinear parabolic equations of nonlinear heat transfer with coefficient $k(u) > 0$ which grows unboundedly as $u \to 0$. These are the so-called *fast diffusion equations*. These include the equation with the power type nonlinearity

$$u_t = \Delta u^m, \tag{1}$$

where $0 < m < 1$ is a constant (if, as usual we set $m = \sigma + 1$, then in this case $\sigma = m - 1 \in (-1, 0)$). The heat conductivity $k(u) = mu^{m-1}$ grows without bound as $u \to 0$.

The name "fast diffusion" is related to the fact that since the heat conductivity is unbounded in the unperturbed (zero temperature) background, heat propagates from warm regions into cold ones much faster than, say in the case of constant ($m = 1$ in (1)) heat conductivity, and even more so than for $m > 1$, where we have finite speed of propagation of perturbations. This super-high speed of "dissolution" of heat implies a number of interesting properties of the process. We shall describe these in some detail, using mainly the technique of constructing various self-similar solutions of equation (1).

As we have not encountered such equations before, let us make the preliminary observations that for $m \in (0, 1)$ solutions of boundary value problems and of the Cauchy problem exist, are unique and satisfy the Maximum Principle; in particular, comparison theorems hold. Here, wherever this does not contradict the boundary conditions, the solution can be taken locally to be strictly positive and therefore classical (see the Comments).

Let us consider for (1) the boundary value problem in a bounded domain $\Omega$ ($\partial\Omega$ is its smooth boundary) with the conditions

$$u(0, x) = u_0(x) > 0, x \in \Omega; \; u_0 \in C(\overline{\Omega}), \tag{2}$$

$$u(t, x) = 0, \, t > 0, \quad x \in \partial\Omega. \tag{3}$$

In this problem we have *total extinction in finite time*. This is relatively simple to prove by constructing the self-similar solution

$$u_S(t, x) = [(T_0 - t)_+]^{1/(1-m)} \theta_S(x), \quad T_0 = \text{const} > 0. \tag{4}$$

The function (4) is such that $u_S \equiv 0$ for all $t \geq T_0$. Let us note that for $m \in (0, 1)$ the derivative $\partial u_S/\partial t$ has no jumps at $t = T_0$, so that $u_S$ is a classical solution. Substituting the expression (4) into the equation (1) and taking into account the boundary conditions, we obtain for the function $\theta_S > 0$ the elliptic problem:

$$\Delta \theta_S^m + \frac{1}{1-m} \theta_S = 0, \, x \in \Omega; \; \theta_S = 0, \, x \in \partial\Omega. \tag{5}$$

Setting $p_S^m = w_S$, we arrive at the equation

$$\Delta w_S + \frac{1}{1-m} w_S^\gamma = 0, \gamma = \frac{1}{m} > 1, \tag{5'}$$

with the same boundary condition $w_S = 0$ on $\partial\Omega$.

The function $w_S$ does not exist for all $\gamma = 1/m$; if $N \geq 3$ and $\gamma \geqslant (N+2)/(N-2)$, then the equation (5′) has solutions that are strictly positive in $\mathbf{R}^N$, while for $\Omega$ a ball of arbitrary radius, there is no solution with the condition $w_S|_{\partial\Omega} = 0$ (see § 3, Ch. IV). On the other hand, if $1 < \gamma < (N+2)/(N-2)_+$, the required similarity function can always be found.

However, for our ends it is not essential for the problem (5) to be solvable. We shall use the self-similar solution (4) only to find majorizing upper bounds for the solution of the problem (1)–(3).

**Proposition 10.** *Let $0 < m < 1$. Then for any initial function $u_0$ in the problem (1)–(3) there is complete extinction in finite time: there exists $T_0 > 0$ such that $u(t,x) \equiv 0$ in $\overline{\Omega}$ for all $t \geq T_0$.*

*Proof.* If $m \in ((N-2)/(N+2), 1)$, $N \geq 3$, or $m \in (0, 1)$, $N < 3$, let us take an arbitrary bounded domain $\Omega'$, such that $\overline{\Omega} \subset \Omega'$ and let us denote by $p_S(x)$ the solution of the equation (5), which is positive in $\Omega'$ and satisfies $p_S = 0$ on $\partial\Omega'$. Then, since $\overline{\Omega} \subset \Omega'$, we have $p_S > 0$ on $\partial\Omega$ and therefore we can always find $T_0 > 0$, such that $u_0(x) \leq T_0^{1/(1-m)} p_S(x)$, $x \in \overline{\Omega}$. By the comparison theorem we have

$$0 \leq u(t,x) \leq [(T_0 - t)_+]^{1/(1-m)} p_S(x), x \in \overline{\Omega},$$

and therefore $u \equiv 0$ in $\overline{\Omega}$ if $t \geq T_0$.

If on the other hand $m \in (0, (N-2)/(N+2)]$, $N \geq 3$ ($\gamma = 1/m \geq (N+2)/(N-2)$ and the boundary value problem (5′) can be insolvable), we take as $p_S(x)$ the solution of equation (5) which is strictly positive in $\mathbf{R}^N$. Then $p_S > 0$ on $\partial\Omega$ and the same argument applies. □

## § 6 The Cauchy problem for the fast diffusion equation

Let us see, whether it is possible to have *total extinction in finite time* in the Cauchy problem for the fast diffusion equation

$$u_t = \Delta u^m, t > 0, x \in \mathbf{R}^N; m \in (0, 1). \tag{1}$$

$$u(0,x) = u_0(x) > 0, x \in \mathbf{R}^N; \sup u_0 < \infty. \tag{2}$$

The situation here is more complex than for a boundary value problem in a bounded domain; however, it can also be analyzed using self-similar solutions.

It is assumed that $u_0(x) \to 0$ as $|x| \to \infty$. Naturally, if this condition is not met, and for example $u_0(x) \geq \delta > 0$ everywhere in $\mathbf{R}^N$, then by the comparison theorem $u(t, x) \geq \delta$ in $\mathbf{R}^N$ for all $t > 0$, that is, in principle there can be no total extinction.

## 1 Conditions for total extinction in finite time

Let us consider the self-similar solution, which describes the total extinction process in the Cauchy problem. We can derive a whole family of such solutions:

$$u_S(t, x) = [(T_0 - t)_+]^n p_S(\xi),\ \xi = x/[(T_0 - t)_+]^{[1+n(m-1)]/2}, \tag{3}$$

where $T_0 > 0$ and $n > 1$ are constants. Substitution of (3) into (1) gives the following elliptic equation for $w_S = p_S^m > 0$:

$$\Delta w_S - \frac{1 + n(m-1)}{2} \nabla w_S^{1/m} \cdot \xi + n w_S^{1/m} = 0,\ \ \xi \in \mathbf{R}^N. \tag{4}$$

For our ends, it suffices to consider radially symmetric solutions, which depend on one variable, $\eta = |\xi|$. All these satisfy a boundary value problem for an ordinary differential equation,

$$\frac{1}{\eta^{N-1}}(\eta^{N-1} w_S')' - \frac{1 + n(m-1)}{2}(w_S^{1/m})'\eta + n w_S^{1/m} = 0,\ \ \eta > 0, \tag{5}$$

$$w_S'(0) = 0,\ w_S(\infty) = 0. \tag{6}$$

This problem (in fact, just as (4)) is solvable not for all $m \in (0, 1)$, $n > 1$.

**Lemma 1.** *Let $N \geq 3$, $0 < m < (N-2)/N$. Then for any*[3]

$$n \geq [(N-2)/N - m]^{-1} \tag{7}$$

*the problem* (5), (6) *has an infinite number of strictly positive solutions.*

*Proof.* Let us consider the Cauchy problem for (5) in $\mathbf{R}_+$ with the conditions

$$w(0) = \mu,\ w'(0) = 0, \tag{8}$$

where $\mu > 0$ is an arbitrary constant. Let us prove that every solution of this problem defines, under the above assumptions, a required function $w_S$. Local solvability of the problem (5), (8) for small $\eta > 0$ is established by considering the equivalent integral equation.

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[3]Obviously, in this case $n > 1$, so that for $t = T_0$ $(u_S)_t$ is continuous and (3) is a classical solution in $\mathbf{R}_+ \times \mathbf{R}^N$.

Let us show that this local solution can be extended to the whole positive semi-axis $\eta \in \mathbf{R}_+$ and satisfies the second of conditions (6). First let us note that the solution is monotone decreasing in $\eta$, since assuming that at some point $\eta_m > 0$ the function $w$ has a minimum ($w(\eta_m) > 0$, $w'(\eta_m) = 0$) leads to a contradiction; this follows from the form of the equation.

Assume the contrary, that is, that the function $w$ vanishes at some point $\eta = \eta_* > 0$, so that $w(\eta) > 0$ on $(0, \eta_*)$ and $w(\eta_*) = 0$. Clearly, $w'(\eta_*) \leq 0$. Integrating equation (5) with the weight function $\eta^{N-1}$ over the interval $(0, \eta_*)$, we obtain the equality

$$\eta_*^{N-1} w'(\eta_*) + C(N, m, n) \int_0^{\eta_*} w^{1/m}(\eta)\eta^{N-1}\, d\eta = 0, \tag{9}$$

where we have introduced the notation

$$C(N, m, n) = -\frac{N}{2}\left[n\left(\frac{N-2}{N} - m\right) - 1\right];$$

$C(N, m, n) < 0$ if strict inequality in (7) holds. Therefore the equality in (9) is impossible, since its left-hand side is strictly negative.

Thus, $w(\eta)$ cannot vanish. From equation (5) it follows then that $w(\eta) \to 0$ as $\eta \to \infty$, that is, $w$ satisfies the boundary conditions (6).

If, on the other hand, we have in (7) the equality $n = [(N-2)/N - m[^{-1}$, then the problem (5), (6) has solutions of the form

$$w_S(\eta) = \left[\frac{1-m}{2m[(N-2) - mN[}(\eta_0^2 + \eta^2)\right]^{-m/(1-m)}, \eta \in \mathbf{R}_+, \tag{10}$$

where $\eta_0^2 > 0$ is an arbitrary constant. □

The family of self-similar solutions (3) makes it possible to obtain, using the comparison theorem, a condition on $u_0(x)$, which is sufficient for total extinction in finite time. For example, if $u_0(x) \leq u_S(0, x)$ in $\mathbf{R}^N$, then $u(t, x) \leq u_S(t, x)$ in $\mathbf{R}_+ \times \mathbf{R}^N$, and therefore $u(t, x) \equiv 0$ for all $t \geq T_0$. Self-similar solutions (3) provide us with the following law of motion of the half-width of the heat extinction wave:

$$|x_{ef}(t)| = \eta_{ef}[(T_0 - t)_+[^{[1+n(m-1)]/2}, \quad w_S^{1/m}(\eta_{ef}) = \frac{1}{2}w_S^{1/m}(0),$$

where, moreover, $1 + n(m-1) < 0$ for all $n$ satisfying (7). Therefore $|x_{ef}(t)| \to \infty$ as $t \to T_0^-$, which agrees well with the property of fast diffusion processes mentioned above: with ever increasing speed heat flows out of the region into infinitely distant regions, where the thermal conductivity coefficient is infinitely large.

It is convenient to formulate an optimal condition on the initial perturbation $u_0(x)$, which ensures total extinction, employing a solution of equation (4) of special form. It is not hard to check that for $0 < m < (N-2)_+/N$ there exists the solution

$$p_S^*(\xi) = w_S^{1/m}(\eta) = \left[\frac{2mN}{1-m}\left(\frac{N-2}{N} - m\right)\right]^{1/(1-m)} |\xi|^{-2/(1-m)}, \quad \xi \neq 0.$$

Here $p_S^*(\xi) \to \infty$ as $\xi \to 0$, which, as will be seen below, is not essential. This function corresponds to a solution, which becomes extinguished everywhere (apart from the point $x = 0$),

$$u_S^*(t, x) =$$

$$= [(T_0 - t)_+]^{1/(1-m)} \left[\frac{2mN}{1-m}\left(\frac{N-2}{N} - m\right)\right]^{1/(1-m)} |x|^{-2/(1-m)}, \quad x \in \mathbf{R}^N \backslash \{0\}. \tag{11}$$

Using (11) as the majorizing solution in the comparison theorem, we obtain

**Proposition 11.** *Let $N \geq 3$, $0 < m < (N-2)/N$, and let the initial function $u_0(x)$ be such that*

$$u_0(x) \leq K|x|^{-2/(1-m)}, \; x \in \mathbf{R}^N \backslash \{0\}; \; K = const > 0. \tag{12}$$

*Then there exists $T_0 > 0$ such that $u(t, x) \equiv 0$ in $\mathbf{R}^N$ for all $t \geq T_0$.*

**Corollary.** *For $0 < m < (N-2)_+/N$, in general, there is no conservation of energy: if $u_0 \in L^1(\mathbf{R}^N)$ and condition (12) holds, then*

$$\|u(t, \cdot)\|_{L^1(\mathbf{R}^N)} \to 0, \, t \to T_0^-, \tag{13}$$

*that is $\|u(t, \cdot)\|_{L^1(\mathbf{R}^N)} \not\equiv \|u_0(\cdot)\|_{L^1(\mathbf{R}^N)}$.*

*Proof* of Proposition 11. By condition (12) there exists $T_0 > 0$, such that $u_0(x) \leq u_S^*(0, x)$, $x \in \mathbf{R}^N \backslash \{0\}$. Therefore from the comparison theorem we obtain that $u(t, x) \leq u_S^*(t, x)$, $t > 0$, $x \in \mathbf{R}^N \backslash \{0\}$ and therefore $u(t, x) \equiv 0$ in $\mathbf{R}^N \backslash \{0\}$ for $t = T_0$. It remains to show that total extinction also occurs at the point $x = 0$. For that it suffices to notice that the function $u_S^*(t, x - x_0)$ where $x_0 \neq 0$ is an arbitrary point of $\mathbf{R}^N$, is also a solution of equation (1) in $\mathbf{R}_+ \times \{\mathbf{R}^N \backslash \{x = x_0\}\}$, and then compare $u(t, x)$ with this solution using similar arguments. □

**Example 2.** Let us set

$$\phi_k(u) = \min\{ku, u^m\}, \, u \geq 0; \; k = 1, 2, \ldots .$$

It is clear that the functions $\phi_k(u)$ are continuous for $u \geq 0$ and $\phi_k(u) \to u^m$ as $k \to \infty$ for any $u \geq 0$. Then the solution of the Cauchy problem for

$$(u_k)_t = \Delta\phi_k(u_k),\ t > 0,\ x \in \mathbf{R}^N,$$

with the initial condition (2) exists and is unique for any $k$. Since $\phi_k'(u)$ is not singular at $u = 0$, generalized solutions conserve energy:

$$\int_{\mathbf{R}^N} u_k(t,x)\,dx = \int_{\mathbf{R}^N} u_0(x)\,dx,\ t \geq 0;\ k = 1, 2, \ldots$$

(the fact that $\phi_k'(u)$ has a jump discontinuity at $u = k^{-1/(1-m)}$ is not important, since, for example, we could smooth $\phi_k$ in a neighbourhood of the point of discontinuity of the derivative). Therefore under the conditions of Proposition 11 (see the Corollary) the sequence $u_k(t,x)$ cannot converge in the norm of $L^1(\mathbf{R}^N)$ to $u(t,x)$, the solution of the original problem (1), (2), which corresponds to $k = \infty$.

## 2 Conditions for existence of a strictly positive solution

Let us show first that for $m < (N-2)/N$, $N \geq 3$, not every initial perturbation $u_0(x)$, such that $u_0(x) \to 0$ as $|x| \to \infty$, ensures total extinction in finite time. This is established by constructing other self-similar solutions of equation (1) in $\mathbf{R}_+ \times \mathbf{R}^N$, which do not have that property:

$$u_S(t,x) = \exp\{-\alpha(T+t)\}g_S(\xi), \tag{14}$$

$$\xi = |x|/\exp\{\alpha(1-m)(T+t)/2\},\ T = \text{const} \geq 0,$$

where $\alpha > 0$. Then $u_S \to 0$ in $\mathbf{R}^N$ as $t \to \infty$ and $u_S > 0$ everywhere. The function $g_S > 0$ satisfies the ordinary differential equation

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}(g_S^m)')' + \frac{\alpha(1-m)}{2}(g_S)'\xi + \alpha g_S = 0,\ \xi > 0, \tag{15}$$

$$g_S'(0) = 0,\ g_S(\infty) = 0. \tag{16}$$

Exactly as in Lemma 1 in subsection 1, we show that this problem has nontrivial solutions if $m \leq (N-2)_+/N$, that is, also in cases when total extinction in finite time is possible. However, (14) are strictly positive in $\mathbf{R}^N$ for all $t > 0$. In particular, if $m = (N-2)/N$ (the "critical" case), the problem (15), (16) can be integrated explicitly and the self-similar solutions have a simple form:

$$u_S(t,x) = \\ = e^{-\alpha t}\left[\frac{\alpha(1-m)^2}{4m}\left(\frac{|x|^2}{\exp\{\alpha(1-m)t\}} + \xi_0^2\right)\right]^{-1/(1-m)} > 0,\ t > 0,\ x \in \mathbf{R}^N, \tag{17}$$

$\xi_0^2 > 0$ is an arbitrary constant.

Therefore for $0 < m \le (N-2)_+/N$ there exists a solution of the Cauchy problem that becomes totally extinguished (Proposition 11) and a solution that does not. It is of interest to compare for which initial perturbations $u_0$ one or the other mode of evolution will occur. Determining the asymptotic behaviour of the problem (15), (16) as $\xi \to \infty$, we obtain the following

**Proposition 12.** *Let $N \ge 3$, $0 < m < (N-2)/N$ and let the initial function $u_0(x)$ be such that for all sufficiently large $|x|$*

$$u_0(x) \ge K|x|^{-2/(1-m)}[\ln|x|]^{1/(1-m)}, K = \text{const} > 0. \tag{18}$$

*Then $u(t,x) > 0$ in $\mathbf{R}^N$ for all $t > 0$.*

*Proof.* If (18) holds, we can always pick $\alpha > 0$ and $T \ge 0$ in (14), such that $u_0(x) \ge u_S(0,x)$ in $\mathbf{R}^N$, and then $u(t,x) \ge u_S(t,x)$ in $\mathbf{R}_+ \times \mathbf{R}^N$, which ensures strict positivity of the solution (so that there is no total extinction). This same inequality allows us to estimate the rate of decay of the amplitude of the temperature profile: it can be at most exponential. □

Let us note that the "boundaries" of the sets (12) and (18) in the space of initial functions (in the first set we have total extinction, which is absent in the second one) are very close and differ only by a slowly increasing logarithmic factor.

Let us show now that the restriction $m \in (0, (N-2)_+/N)$ is essential for total extinction in finite time to occur. Below we provide examples of positive self-similar solutions, which exist for $m > (N-2)_+/N$ and conserve energy. Let us seek these solutions in the form

$$u_S(t,x) = (T+t)^l p_S(\eta), \eta = |x|/(T+t)^{[1+l(m-1)]/2}, \tag{19}$$

where $l < 0$, $T > 0$ are constants. Here $u_S > 0$ in $\mathbf{R}^N$ for all $t > 0$.

Substituting (19) into equation (1), we obtain for the function $w_S = p_S^m(\eta) > 0$ the equation

$$\frac{1}{\eta^{N-1}}(\eta^{N-1}w_S')' + \frac{1+l(m-1)}{2}(w^{1/m})'\eta - lw_S^{1/m} = 0, \eta > 0, \tag{20}$$

$$w_S'(0) = 0, w_S(\infty) = 0. \tag{21}$$

It is not hard to show that if the condition $(N-2)_+/N < m < 1$ is satisfied for any $l \le [(N-2)/N - m]^{-1} < 0$, there exists an infinite number of functions $w_S(\eta) > 0$, which satisfy (20), (21) (see the proof of Lemma 1). In the particular case $l = -[m-(N-2)/N]^{-1}$ there exists a self-similar solution (19), which can be written down explicitly:

$$u_S(t,x) = (T+t)^{-N/[2+N(m-1)]} \left\{ \frac{(1-m)}{2m\,[mN-(N-2)]} \times \right.$$
$$\left. \times \left[ \eta_0^2 + \frac{|x|^2}{(T+t)^{2/[mN-(N-2)]}} \right] \right\}^{-1/(1-m)}, \quad t>0,\ x \in \mathbf{R}^N \tag{22}$$

($\eta_0^2 = \text{const} > 0$). It exists for all $(N-2)_+/N < m < 1$ and has finite energy, which is conserved:

$$\int_{\mathbf{R}^N} u_S(t,x)\,dx = \int_{\mathbf{R}^N} u_S(0,x)dx \equiv \|p_S\|_{L^1(\mathbf{R}^N)}.$$

The self-similar solution (22) is the analogue of a solution of instantaneous point energy source type, which was considered in Example 8 of Ch. I for the case $m > 1$ (that is, $\sigma = m-1 > 0$).

Thus, for $m > (N-2)_+/N$ there are solutions with conserved finite energy. In other words, in this case there is no absorption of heat in infinitely distant regions, which happens when $0 < m < (N-2)_+/N$ (Proposition 11). Furthermore, using the self-similar solutions (19) and the method of proof of Proposition 4 of Ch. I, it is not hard to show that in this case there is no finite time extinction and energy is conserved (see Comments).

## § 7 Conditions of equivalence of different quasilinear heat equations

Above, using a range of examples, we demonstrated asymptotic equivalence of solutions of nonlinear parabolic equations corresponding to different boundary data, as well as equivalence as $t \to \infty$ of solutions of different parabolic equations obtained by perturbing nonlinear operators. The idea of this asymptotic equivalence (asymptotic stability of approximate self-similar solutions) will be widely used in the sequel.

Here we consider the question of equivalence of equations, understood in a strict sense. Are there different quasilinear equations that can be reduced to each other by a certain transformation? In other words, is it possible to transform a nonlinear heat equation with a source or a sink into a simpler equation, one with better understood properties? In the general setting this problem is studied in the framework of the theory of transformation groups and is known as the Bäcklund problem (its precise formulation and constructive methods of solution are to be found in [221]).

The examples of really non-trivial and "non-obvious" strict equivalence are relatively few. For that reason their rôle in the systematic study of properties of solutions of quasilinear parabolic equations is, in general, not an important one. Nonetheless, this approach sometimes affords a considerable simplification of the problem.

We consider below certain simple transformations that establish equivalence of different equations. We will not analyse in detail the structure of such transformations or discuss their constructive aspect: how simple is it to reconstruct a solution of one equation using a solution of the other? From the practical point of view this last question is very important: frequently it is easier to solve numerically the equation itself than to implement numerically the equivalence transformation.

In most cases we shall deal with an equation, without posing a specific boundary value problem for it, and for that reason we will pay no attention to the behaviour of its coefficients. These have to be taken into account in the formulation of boundary value problems.

## 1 Simplest examples

We have already encountered an equation which can be reduced by an equivalence transformation into a simpler one. This is a quasilinear equation with a linear sink: $u_t = \Delta u^{\sigma+1} - u$, which can be transformed by a change of variables $u = e^{-t}v$, $e^{-\sigma t}dt = d\tau$ into an equation without a sink: $v_\tau = \Delta v^{\sigma+1}$. Using these elementary transformations we can establish localization of heat perturbations in nonlinear media with volumetric absorption. Let us consider another simple example.

**Example 3.** The semilinear parabolic equation

$$u_t = \Delta u + \frac{E''(u)}{E'(u)}|\nabla u|^2. \tag{1}$$

where $E : \mathbf{R}_+ \to \mathbf{R}_+$, $E \in C^2$, is an arbitrary monotone function, can be reduced by the change of variable $v = E(u)$ to the linear heat equation

$$v_t = \Delta v. \tag{2}$$

Let us consider more complicated transformations.

**Example 4.** Let $u = u(t, r)$, $r = |x|$, be a solution of a nonlinear heat equation with a source:

$$u_t = \frac{1}{r^{N-1}}(r^{N-1}u^\sigma u_r)_r + u^{\sigma+1}, \tag{3}$$

where $\sigma \neq -1$. Examples quoted earlier show that solutions of equations with a source have properties that are significantly different from those of solutions of nonlinear heat equations. Let us try to get rid of the source in the right-hand side of (3), so that only a diffusion operator remains there. To that end let us use the transformation

$$y = \phi(r), \quad u^{\sigma+1}(t, r) = \psi(r) v(t, y).$$

Then it is easily verified that the function $v \geq 0$ satisfies the equation

$$\begin{gathered}(\sigma+1)\psi^{1/(\sigma+1)}(v^{1/(\sigma+1)})_t = \psi\phi'^2 v_{yy} + \\ +\frac{1}{r^{N-1}}[r^{N-1}\psi'\phi' + (r^{N-1}\psi\phi')']v_y + \left[\frac{1}{r^{N-1}}(r^{N-1}\psi')' + (\sigma+1)\psi\right] v.\end{gathered} \tag{4}$$

To get rid of lower order terms in (4), we set

$$\frac{1}{r^{N-1}}(r^{N-1}\psi')' + (\sigma+1)\psi = 0, \tag{5}$$

$$r^{N-1}\psi'\phi' + (r^{N-1}\psi\phi')' = 0. \tag{6}$$

If the function $\psi$ satisfying (5) is known, then

$$\phi(r) = \int \frac{dr}{r^{N-1}\psi^2(r)}. \tag{7}$$

The system (5), (6) can be solved explicitly, for example, in the case $N = 1$ (concerning this see below), as well as for $N = 3$, when by the change of variable $\psi(r) = \kappa(r)/r$ equation (5) reduces to

$$\kappa'' + (\sigma+1)\kappa = 0.$$

In particular, if $\sigma + 1 < 0$, then for $N = 3$

$$\phi(r) = \frac{1}{r}\exp\{\pm|\sigma+1|^{1/2} r\},$$

$$\psi(r) = \mp\frac{1}{2|\sigma+1|^{1/2}}\exp\{\mp 2|\sigma+1|^{1/2} r\}.$$

If conditions (5), (6) hold, then the equation for the new function $v$ has the form

$$\left(v^{1/(\sigma+1)}\right)_t = \frac{\psi^{(3\sigma+4)/(\sigma+1)}(r)}{r^{2(N-1)}(\sigma+1)} v_{yy}, \quad y = \phi(r).$$

Setting $v^{1/(\sigma+1)} = U$, we obtain the one-dimensional equation without a source,

$$U_t = \frac{\psi^{(3\sigma+4)/(\sigma+1)}(r)}{r^{2(N-1)}}(U^\sigma U_y)_y. \tag{8}$$

It has a particularly simple form in the case $N = 1$, $\sigma = -4/3$. Then the system (5), (6) is easily solved. As a result we obtain the following

**Proposition 13.** *In the equation*

$$u_t = (u^{-4/3}u_x)_x + u^{-1/3} \tag{9}$$

*the transformation*

$$y = \frac{\sqrt{3}}{2}\exp\left\{\pm\frac{2x}{\sqrt{3}}\right\}, \quad u(t,x) = \exp\left\{\pm\frac{x}{\sqrt{3}}\right\} v(t,y) \tag{10}$$

*removes the source in the right-hand side, while the function $v$ satisfies the equation*

$$v_t = (v^{-4/3}v_y)_y. \tag{11}$$

For equations of general form

$$u_t = \Delta\phi(u) + Q(u) \tag{12}$$

there also exist transformations that remove the source term $Q(u)$; however, here the resulting equivalent equation is no longer autonomous.

**Example 5.** Let us set in (12) $u = E(t,v)$. Then for $v = v(t,x)$ we obtain the new equation

$$E'_t + E'_v v_t = \Delta\phi(E(t,v)) + Q(E(t,v)).$$

Let us choose the function $E$ by requiring that $\partial E/\partial t = Q(E)$, that is,

$$\int^{E(t,v)} \frac{d\eta}{Q(\eta)} = t + c(v),$$

where $c(v)$ is an arbitrary function. After this change of variables we obtain for $v$ a parabolic equation, whose coefficients depend on the variable $t$:

$$v_t = \frac{1}{E'_v(t,v)}\Delta\phi(E(t,v)). \tag{13}$$

For example, in the case of an equation with power law coefficients

$$u_t = \Delta u^{\sigma+1} + u^\beta, \quad \sigma \ge 0, \ \beta > 1, \tag{14}$$

which will be studied from different points of view in subsequent chapters, the transformation $E$ has the form

$$E(t,v) = [(\beta-1)(c(v)-t)]^{-1/(\beta-1)}.$$

It is convenient to choose the function $c(v)$ so that for $t = 0$ the transformation is the identity, $E(0,v) \equiv v$. This gives us $c(v) = v^{1-\beta}/(\beta-1)$, so that finally we have

$$E(t,v) = [v^{1-\beta} - (\beta-1)t]^{-1/(\beta-1)}.$$

Equation (13), which is equivalent to (12), then has the form

$$v_t = v^{\beta}[v^{1-\beta} - (\beta - 1)t]^{\beta/(\beta-1)}\Delta\left\{[v^{1-\beta} - (\beta - 1)t]^{-(\sigma+1)/(\beta-1)}\right\}.$$

Transformations of this kind turn out to be quite useful in the study of semilinear parabolic equations and will be employed in § 7, Ch. IV.

## 2 The "linear" equation $u_t = (u^{-2}u_x)_x$

We move on now to more complicated equivalence transformations. Let us show that the nonlinear heat equation with coefficient $k(u) = u^{-2}$ is equivalent to the linear equation.

Let $u(t, x)$ be a solution of the equation

$$u_t = (u^{-2}u_x)_x \tag{15}$$

such that $u(t, x)$ is a sufficiently smooth function which is not zero in the domain under consideration. Let us fix a point $(t_0, x_0)$. Integrating (15) in $x$ we obtain the equality

$$\frac{\partial}{\partial t}\int_{x_0}^{x} u(t, y)\,dy = u^{-2}(t, x)u_x(t, x) - u^{-2}(t, x_0)u_x(t, x_0),$$

or, equivalently,

$$\frac{\partial}{\partial t}\left\{\int_{x_0}^{x} u(t, y)\,dy + \int_{t_0}^{t} u^{-2}(\tau, x_0)u_x(\tau, x_0)\,d\tau\right\} = u^{-2}u_x.$$

Denoting the expression in braces by

$$\phi(t, x) = \int_{x_0}^{x} u(t, y)\,dy + \int_{t_0}^{t} u^{-2}u_x|_{(\tau, x_0)}\,d\tau$$

(from which it follows that $\phi_x \equiv u$), we obtain a new parabolic equation for the function $\phi$:

$$\frac{\partial \phi}{\partial t} = (\phi_x)^{-2}\phi_{xx}. \tag{16}$$

Let us introduce the new independent variables

$$\bar{x} = \phi(t, x), \quad \bar{t} = t. \tag{17}$$

Solving the first equality with respect to $x$, we obtain

$$x = \psi(\bar{t}, \bar{x}). \tag{18}$$

Let us derive from (16) an equation for the function $\psi$. It is easy to check that

$$\frac{\partial\phi}{\partial t} = -\frac{\partial\psi}{\partial\bar{t}}\left(\frac{\partial\psi}{\partial\bar{x}}\right)^{-1}, \quad \frac{\partial\phi}{\partial x} = \left(\frac{\partial\psi}{\partial\bar{x}}\right)^{-1}, \quad \frac{\partial^2\phi}{\partial x^2} = -\frac{\partial^2\psi}{\partial\bar{x}^2}\left(\frac{\partial\psi}{\partial\bar{x}}\right)^{-3},$$

and therefore

$$\left(\frac{\partial\psi}{\partial\bar{x}}\right)^{-1}\left\{\frac{\partial\psi}{\partial\bar{t}} - \frac{\partial^2\psi}{\partial\bar{x}^2}\right\} = 0.$$

Since $(\psi_{\bar{x}})^{-1} \neq 0$ (which is equivalent to the condition $u \neq 0$), $\psi(\bar{t}, \bar{x})$ satisfies the linear heat equation

$$\psi_{\bar{t}} = \psi_{\bar{x}\bar{x}}. \tag{19}$$

It is not hard to effect the inverse transformation and to show that a solution of equation (19) transforms into a solution of the original equation (15).

**Example 6.** Let us consider the fundamental solution of the heat equation (19):

$$\psi(\bar{t}, \bar{x}) = \frac{1}{\bar{t}^{1/2}} \exp\left\{-\frac{\bar{x}^2}{4\bar{t}}\right\}. \tag{20}$$

Equalities (17), (18) define the required function

$$\phi(t, x) = [-4t \ln(xt^{1/2})]^{1/2}.$$

However, $u = \phi_x$ is the solution of equation (15), that is, the fundamental solution (20) transforms into the following solution of the "linear" equation (15):

$$u(t, x) = -\frac{t^{1/2}}{x}\left[-\ln\left(xt^{1/2}\right)\right]^{-1/2}.$$

It makes sense for $xt^{1/2} \in (0, 1)$.

The equation with the coefficient $k(u) = u^{-2}$ has other interesting properties.

**Example 7.** Let us consider the "multi-dimensional" equation

$$u_t = \frac{1}{r^{N-1}}(r^{N-1}u^{-2}u_r)_r. \tag{21}$$

It is easy to check that the same transformations

$$u(t, r) = r^{1-N}\phi_r(t, r), \quad \bar{t} = t, \quad \bar{r} = \phi(t, r); \quad r = \psi(\bar{t}, \bar{r}),$$

reduce (21) to the form

$$\psi_{\bar{t}} = \psi^{2(N-1)}\psi_{\bar{r}\bar{r}} + (N-1)\psi^{2N-3}\psi_{\bar{r}}^2. \tag{22}$$

For $N = 1$ we obtain a linear equation; for $N = 2$ we have the equation

$$\psi_{\bar{t}} = \psi^2 \psi_{\bar{r}\bar{r}} + \psi \psi_{\bar{r}}^2 \equiv \psi(\psi \psi_{\bar{r}})_{\bar{r}}.$$

By a change of variable $\bar{u}(\bar{t}, \bar{r}) = \ln \psi$ this equation reduces to a one-dimensional equation with exponential nonlinearity:

$$\bar{u}_t = \left(e^{2\bar{u}} \bar{u}_r\right)_r .$$

If, on the other hand, $N \geq 3$, then, setting $\bar{u} = \psi^{2-N}$ we obtain from (22) the equation

$$\bar{u}_t = (\bar{u}^{\gamma_N} \bar{u}_r)_r ,$$

where $\gamma_N = 2(N-1)/(2-N) < 0$.

## 3 Equivalence conditions for equations of general form

Below, using the same transformations, we show that to each heat equation corresponds an equivalent heat equation with a different heat conductivity coefficient.

**Proposition 14.** *The transformation*

$$\bar{t} = t, \ \bar{x} = \int_{x_0}^{x} u(t, y)dy + \int_{t_0}^{t} k(u(\tau, x_0))u_x(\tau, x_0)d\tau, \tag{23}$$

$$\bar{u}(\bar{t}, \bar{x}) = 1/u(t, x), \tag{24}$$

*takes the solution $u(t, x) \neq 0$ of the equation*

$$u_t = (k(u)u_x)_x \tag{25}$$

*into a solution $\bar{u}(\bar{t}, \bar{x})$ of the equation*

$$\bar{u}_t = \left(\frac{1}{\bar{u}^2} k\left(\frac{1}{\bar{u}}\right) \bar{u}_x\right)_x . \tag{26}$$

*Proof.* Let us compute the derivatives that enter equation (26). From (24) it follows that

$$(\bar{u}(\bar{t}, \bar{x}))_t \equiv \left(\frac{1}{u(t, x)}\right)_t = -\frac{u_t + u_x x_t}{u^2} . \tag{27}$$

From the second equality in (23) it follows that

$$0 = u x_t + \int_{x_0}^{x} u_t(t, y)dy + k(u)u_x|_{(t, x_0)} . \tag{28}$$

However, from equation (25) we have that

$$\int_{x_0}^{x} u_t(t, y)\, dy = k(u)u_x - k(u)u_x|_{(t, x_0)},$$

and therefore (28) means that

$$x_t = -k(u)u_x/u.$$

Then from (27) we obtain

$$\tilde{u}_t = -\frac{u_t}{u^2} + \frac{k(u)}{u^3} u_x^2 . \tag{29}$$

Furthermore, since $x_x = 1/u$, the other derivatives are easily computed:

$$\tilde{u}_x = \left(\frac{1}{u}\right)_x = \left(\frac{1}{u}\right)_x x_x = -\frac{u_x}{u^3} . \tag{30}$$

$$\tilde{u}_{xx} = -\frac{u_{xx}u - 3u_x^2}{u^5} . \tag{31}$$

Finally, we obtain from (29)–(31)

$$\tilde{u}_t - \left(\frac{1}{\tilde{u}^2} k\left(\frac{1}{\tilde{u}}\right) \tilde{u}_x\right)_x = -\frac{1}{u^2}[u_t - (k(u)u_x)_x] = 0,$$

which completes the proof. □

Therefore *equations with coefficients*

$$k(u), \ K(u) = \frac{1}{u^2} k\left(\frac{1}{u}\right), \tag{32}$$

*are equivalent*. For a power coefficient $k(u) = u^\sigma$, the equivalent equation has the coefficient $K(u) = u^{-(\sigma+2)}$, which means that equations with coefficients $k_1(u) = u^{\sigma_1}$, $k_2(u) = u^{\sigma_2}$ are equivalent if

$$\sigma_1 + \sigma_2 = -2. \tag{32'}$$

If $\sigma_1 = 0$, then according to (32′) $\sigma_2 = -2$ and we obtain the known result on the equivalence of the equation with $k(u) = u^{-2}$ and the linear heat equation.

Proposition 14 opens new possibilities for constructing particular solutions of certain equations.

**Example 8.** The equation

$$u_t = (e^u u_x)_x \tag{33}$$

has a wide variety of symmetries. For example, it is invariant with respect to the transformations

$$t \to t/\alpha, \quad x \to x, \quad u \to -\ln\alpha + u,$$

that is, $-\ln\alpha + u(t/\alpha, x)$ is a solution of the equation if it is also satisfied by $u(t, x)$. Setting here $\alpha = -t$, $t < 0$, and then making the change of variables $t \to t - T_0$, $T_0 = \text{const} > 0$, we see that (33) has the self-similar solution

$$u_S(t, x) = -\ln(T_0 - t) + \theta_S(x), \quad 0 < t < T_0. \tag{34}$$

Substituting (34) into (33) provides us with the following equation for the function $\theta_S(x)$:

$$(e^{\theta_S}\theta_S')' = 1,$$

that is,

$$\theta_S(x) = \ln(x^2/2 + bx + c), \tag{35}$$

where $b$, $c$ are arbitrary constants.

The above equivalence of (33) to the equation (see (32))

$$\bar{u}_{\bar{t}} = \left(\frac{1}{\bar{u}^2} e^{1/\bar{u}} \bar{u}_{\bar{x}}\right)_{\bar{x}} \tag{36}$$

allows us to construct a particular solution of the latter equation. For example, let $b = c = 0$ in (35). Then, as follows from (23), (24), a solution of equation (36) will be a function $\bar{u}(\bar{t}, \bar{x})$, which is implicitly defined from the equalities

$$\bar{u}(\bar{t}, \bar{x}) = \left\{\ln\left[(T_0 - \bar{t})^{-1}\frac{\psi^2(\bar{t}, \bar{x})}{2}\right]\right\}^{-1},$$

where the function $\psi(\bar{t}, \bar{x})$ is such that

$$\bar{x} = \int_0^{\psi(t, x)} \ln\left[(T_0 - \bar{t})^{-1}\frac{y^2}{2}\right] dy \equiv \psi \ln\left[\frac{\psi^2 (T_0 - \bar{t})^{-1}}{2e^2}\right].$$

To conclude, let us state equivalence conditions for more general quasilinear parabolic equations.

**Proposition 15.** *The equations*

$$u_t = (k(u, u_x)u_x)_x, \quad \bar{u}_{\bar{t}} = \left(\frac{1}{\bar{u}^2} k\left(\frac{1}{\bar{u}}, -\frac{\bar{u}_{\bar{x}}}{\bar{u}^3}\right)\bar{u}_{\bar{x}}\right)_{\bar{x}},$$

*are equivalent. Transformation* (23), (24) *takes a solution $u \neq 0$ of the first equation into a solution of the second one.*

Let us note that the same method can be used to construct an unusual exact solution of the super-slow diffusion equation

$$u_t = (e^{-1/u})_{xx}. \tag{37}$$

Its name reflects the fact that the corresponding thermal conductivity coefficient $k(u) = u^{-2}e^{-1/u}$ changes for low temperatures $u > 0$ more slowly than any power. Therefore equation (37) can be formally considered as the limit as $\sigma \to \infty$ of the nonlinear heat equation $u_t = (u^\sigma u_x)_x$, some properties of whose solutions were described in § 4.

If we consider for (37) the Cauchy problem with a continuous non-negative compactly supported initial function $u(0, x) = u_0(x)$ in **R**, then the generalized solution $u(t, x)$ will satisfy the conservation law

$$\int_{-\infty}^{\infty} u(t, x)\, dx \equiv E_0 = \int_{-\infty}^{\infty} u_0(x)\, dx \quad \text{for all } t > 0$$

(see § 3 in Ch. I). The exact solution given below satisfies the conservation law. Formally it is implicitly given by $u_*(t, x) = -1/\ln(v_*(t, x; c))$,

$$v_*(t, x; c) = \frac{1}{2t}(c^2 - w^2)_+,$$

where $c > 0$ is an arbitrary constant and the function $w = w(t, x; c)$ is determined from the equation

$$|x| = (2 + \ln(2t))w + (c - w)\ln(c - w) - (c + w)\ln(c + w). \tag{38}$$

It is not hard to check that for $t \geq c^2/2$ equation (38) is uniquely solvable with respect to the function $w(t, x; c) \in [0, c)$ in terms of $x \in [0, x_*(t; c))$, where

$$x_*(t; c) = c\ln t + c\ln\left(\frac{e^2}{2c^2}\right).$$

Setting $v_*(t, x; c) = 0$ for $|x| \geq x_*(t; c)$, we obtain a generalized solution with compact support of equation (37), $u_*(t, x)$, which has continuous thermal flux at the front points of the solution $x = \pm x_*(t; c)$. It is easy to see that the conservation law

$$\int_{-\infty}^{\infty} u_*(t, x)\, dx \equiv E_0 = 2c \quad \text{for all } t \geq c^2/2$$

is satisfied. It is interesting that at time $t_0 = c^2/2$ the solution $u_*(x, t_0)$ behaves close to the point $x = 0$ as the unbounded singular function $|x|^{-2/3}$, which is integrable, but not a delta function.

## § 8 A heat equation with a gradient nonlinearity

In this section we consider the properties of generalized solutions of quasilinear parabolic equations, which describe diffusion of heat in a medium, heat conductivity of which depends not on the temperature, but rather on its spatial derivative (gradient). Typical examples of such equations are:

$$u_t = (|u_x|^\sigma u_x)_x, \tag{1}$$

in the one space dimension, while in the multi-dimensional case we have the equation

$$u_t = \nabla \cdot (|\nabla u|^\sigma \nabla u), \tag{2}$$

where $\sigma > 0$ is a constant. These equations are parabolic and degenerate; the thermal conductivity coefficient $k = k(|\nabla u|) = |\nabla u|^\sigma \geq 0$ vanishes wherever $\nabla u = 0$, in particular, at points of positive extremum of the function $u = u(t, x) \geq 0$, or, for example, at the points of the front of a thermal wave which propagates with a finite speed. Therefore, in general, solutions of the equations (1) and (2) are generalized ones.

**Example 9.** The Cauchy problem for (2) in $\mathbf{R}_+ \times \mathbf{R}^N$ has the solution

$$\begin{aligned} u_S(t, x) =& A_{\sigma,N}(T + t)^{-N/[\sigma(N+1)+2]} \times \\ & \times \left\{ \left[ a^{(\sigma+2)/(\sigma+1)} - \left( \frac{|x|}{(T+t)^{1/[\sigma(N+1)+2]}} \right)^{(\sigma+2)/(\sigma+1)} \right]_+ \right\}^{(\sigma+1)/\sigma} . \end{aligned} \tag{3}$$

where

$$A_{\sigma,N} = \left( \frac{\sigma}{\sigma+2} \right)^{(\sigma+1)/\sigma} \left\{ \frac{1}{\sigma(N+1)+2} \right\}^{1/\sigma},$$

$T \geq 0$, $a > 0$ are arbitrary constants. This is a self-similar solution of an instantaneous point energy source type. It is determined exactly as the analogous solution for the equation with $k(u) = u^\sigma$ which has constant energy:

$$\int_{\mathbf{R}^N} u_S(t, x)\,dx \equiv \int_{\mathbf{R}^N} u_S(0, x)\,dx,\ t > 0.$$

The solution (3) has compact support at each moment of time: $u_S(t, x) = 0$ for all $|x| \geqslant |x_f(t)| = a(T+t)^{1/[\sigma(N+1)+2]}$. Its degeneracy points are $x = 0$ (a positive maximum point) and the front surface $\{|x| = |x_f(t)|\}$. From (3) it follows that at $x = 0$ the second derivative $\Delta(u_S)$ does not exist, but that the product $|\nabla u_S|^\sigma \Delta u_S$ is finite, so that for $x = 0$ the derivative $u_t$ is defined, since (2) is equivalent to the equation

$$u_t = |\nabla u|^\sigma [\Delta u + \sigma \nabla |\nabla u| \cdot (\nabla u / |\nabla u|)].$$

At points of the front of the solution, $|\nabla u_S| = 0$, $\Delta u_S(t, x) \sim (|x_f| - |x|)^{(1-\sigma)/\sigma}$ as $|x| \to |x_f|^-$. Therefore if $\sigma < 1$, then $\Delta u_S(t, x_f) = 0$ ((3) is a classical solution for $x \neq 0$); if $\sigma = 1$, then $\Delta u_S(t, x_f) \neq 0$, while if $\sigma > 1$ then $\Delta u_S = \infty$ for $|x| = |x_f|^-$. In the two last cases $\Delta u_S$ has on the front surface a discontinuity of the first or second kind, respectively. Let us note in particular that at all points of degeneracy the heat flux

$$W(t, x) \equiv -k\nabla u = -|\nabla u|^\sigma \nabla u$$

is continuous. This is an important property of the generalized solution, which is taken into account when one introduces the integral identity which is equivalent to (2). The generalized solutions satisfy the Maximum Principle; comparison theorems with respect to boundary data hold for these solutions.

Equation (2) describes processes with a finite speed of propagation of heat perturbations over any constant temperature background. For example, the function

$$u(t, x) = 1 + u_S(t, x),\ t > 0,\ x \in \mathbf{R}^N,$$

is a solution with a finite front on a (temperature unity) background.

Equation (1) has a power nonlinearity. Therefore it is not difficult to construct self-similar solutions for it in the half-space $\{x > 0\}$ with a regime prescribed on the boundary $x = 0$ (see § 3). For example, if $u(t, 0) = (1 + t)^m$, $m > 0$, then the corresponding solution has the form

$$u_S(t, x) = (1 + t)^m f_S(\xi),\ \xi = x/(1 + t)^{(1+m\sigma)/(\sigma+2)}.$$

If on the other hand $u(t, 0) = e^t$, then

$$u_S(t, x) = e^t f_S(\xi),\ \xi = x/\exp\left\{\frac{\sigma}{\sigma + 2}t\right\}.$$

These self-similar solutions are asymptotically stable in the sense indicated above (see § 1, 2).

We shall consider more closely solutions evolving in a blow-up regime, which demonstrate the *heat localization* phenomenon.

**Example 10.** In a boundary value problem for equation (1) in the domain $(0, T_0) \times \mathbf{R}_+$, let

$$u(t, 0) = 1 + (T_0 - t)^n,\ 0 < t < T_0;\ n < 0. \qquad (4)$$

The corresponding self-similar solution has the form

$$u_S(t, x) = 1 + (T_0 - t)^n \theta_S(\xi),\ \xi = x/(T_0 - t)^{(1+n\sigma)/(\sigma+2)}. \qquad (5)$$

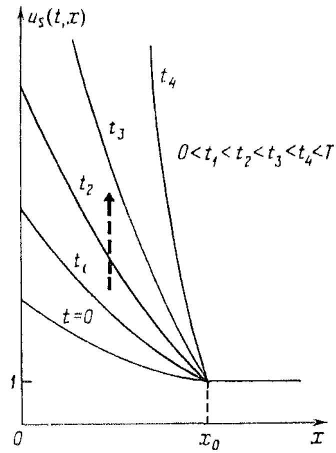


**Fig. 8.** Evolution as $t \to T_0^-$ of the localized S blow-up regime (5′)

where the function $\theta_S \geq 0$ is a generalized solution of the boundary value problem

$$(|\theta_S'|^\sigma \theta_S')' - \frac{1 + n\sigma}{\sigma + 2}\theta_S'\xi + n\theta_S = 0, \xi > 0,$$
$$\theta_S(0) = 1, \theta_S(\infty) = 0. \tag{6}$$

In the particular case $n = -1/\sigma$ (the S blow-up regime), equation (6) is easily integrated. The corresponding self-similar solution

$$u_S(t, x) = 1 + (T_0 - t)^{-1/\sigma}\left[(1 - x/x_0)_+\right]^{(\sigma+2)/\sigma},$$
$$x_0 = \frac{\sigma + 2}{\sigma}\left[\frac{2\sigma(\sigma + 1)}{(\sigma + 2)}\right]^{1/(\sigma+2)}. \tag{5'}$$

represents a thermal wave with a fixed front point, localized in the domain $0 < x < x_0$ during all the period of action of the boundary blow-up regime. Heat does not leave the localization domain, and for $x > x_0$ the homogeneous temperature background remains the same (Figure 8).

The spatio-temporal structure of the self-similar solution (5) indicates that if $n < -1/\sigma$, the influence of the blow-up regime will not be localized and $x_f(t) \sim (T_0-t)^{(1+n\sigma)/(\sigma+2)} \to \infty$ as $t \to T_0$ (HS-regime), while in the case $n \in (-1/\sigma, 0)$ we do have localization, such that, moreover, temperature grows without bound only at the point $x = 0$ (LS-regime). This classification coincides with the one given in § 3 for the thermal conductivity coefficient $k = u^\sigma$.

# § 9 The Kolmogorov-Petrovskii-Piskunov problem

From this section we begin to analyse specific self-similar solutions of quasilinear parabolic equations with an additional term $Q(u)$ (either a source or a sink) in the right-hand side. Some examples of such equations were given in Ch. I.

First we consider self-similar solutions of travelling wave type in active media with a source. This problem was studied first, and in an exhaustive manner, in the well-known paper [255]. It generated a whole range of papers (see Comments), which is the reason this problem is named after the authors of [255].

## 1 Statement of the problem

We consider the diffusion process

$$u_t = u_{xx} + Q(u), \quad t > 0, \; x \in \mathbf{R}, \tag{1}$$

in a medium with a source of a particular form:

$$\begin{aligned} Q(0) = Q(1) = 0;\; Q(u) > 0, &\quad u \in (0,1); \\ Q'(0) = \alpha > 0;\; Q'(u) < \alpha, &\quad u \in (0,1]. \end{aligned} \tag{2}$$

The behaviour of the function $Q(u)$ is shown in Figure 9. From the stated restrictions on $Q(u)$ it follows that

$$Q(u) \le \alpha u, \quad u \in [0,1] \tag{3}$$

(this is essential in the following). All the above conditions are satisfied, for example, by the source

$$Q(u) = \alpha u(1-u), \quad 0 \le u \le 1. \tag{4}$$

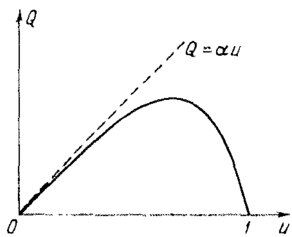


Fig. 9.

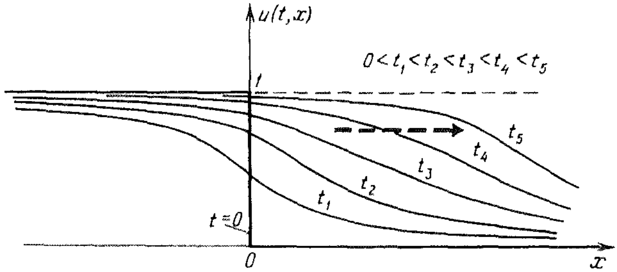


**Fig. 10.** Formation of a thermal wave in the problem (1), (6). The initial function is indicated by a thicker line

For equation (1) we consider the Cauchy problem with the initial condition

$$u(0, x) = u_0(x) \geq 0, \; u_0(x) \leq 1, x \in \mathbf{R}. \tag{5}$$

This problem is well-posed. Though we did not define the function $Q(u)$ for $u < 0$ and $u > 1$, this is not important, since from (5) and from the comparison theorem it follows that $0 \leq u(t, x) \leq 1$. Indeed, $u_+ \equiv 1$ and $u_- \equiv 0$ are solutions of equation (1), and by (5), $u_- \leq u_0(x) \leq u_+$; therefore $u_- \leq u(t, x) \leq u_+$ in $\mathbf{R}_+ \times \mathbf{R}$.

Let us consider now an initial perturbation of a simple form (see Figure 10):

$$u_0(x) \equiv 1, \; x \leq 0, u_0(x) \equiv 0, \; x > 0. \tag{6}$$

Then it is clear that the thermal wave will start to move to the right as shown in Figure 10. What is the law governing its motion? What is its spatial profile for large times?

In [255] it was shown that the asymptotic behaviour of the solution of the problem (1), (5) is determined by a self-similar solution of (1) of the travelling wave type:

$$u_S(t, x) = \theta_S(\xi), \; \xi = x - \lambda t, \tag{7}$$

where $\lambda > 0$ is a constant (the speed of motion of the wave). Substitution of (7) into (1) gives us the ordinary differential equation

$$\theta_S'' + \lambda\theta_S' + Q(\theta_S) = 0, \; \xi \in \mathbf{R}; \; \theta_S(-\infty) = 1, \; \theta_S(\infty) = 0. \tag{8}$$

The boundary conditions here were chosen based on the form (6) of the initial function.

The similarity equation (8) reduces to a first order equation. It is not hard to check [255] that this problem has a solution for any

$$\lambda \geq \lambda_0 = 2\sqrt{\alpha}. \tag{9}$$

The solution $\theta_S \geq 0$ corresponding to a given $\lambda \geqslant \lambda_0$ is unique up to a shift. This fact is important: if $\theta_S$ is a solution, so will be $\theta_S(\xi + \xi^*)$, $\xi^* = \text{const}$.

The natural question that arises is: what speed is selected for an initial perturbation of the "mesa"-like form (6)? In [255] the authors prove the following fundamentally important result: in the problem (1), (6), for large $t$ the wave moves at the speed $\lambda = \lambda_0$, that is, the minimal possible speed. For other non-compactly supported $u_0(x)$ the wave may move as $t \to \infty$ with a speed $\lambda > \lambda_0$. If we denote, as usual, by $x_{ef}(t)$ the depth of penetration of the thermal wave ($u(t, x_{ef}(t)) = 1/2$), then

$$dx_{ef}/dt = 2\sqrt{\alpha} + o(1), \quad t \to \infty. \tag{10}$$

In addition, at the asymptotic stage of the evolution the profile of the thermal wave coincides with the function $\theta_S^0(\xi)$, the solution of the problem (8) for $\lambda = \lambda_0$. This means that the similarity representation of the solution of the original non-self-similar problem, $\theta(t, \xi) = u(t, \xi + x_{ef}(t))$, converges as $t \to \infty$ to a shift of the function $\theta_S^0(\xi)$, that is,

$$\|\theta(t, \cdot) - \theta_S^0(\cdot)\|_{C(\mathbf{R})} \to 0, \quad t \to \infty. \tag{11}$$

Below we treat in detail several simple questions related to this problem, which at the same time illustrate methods of analysis to be used later.

## 2 Upper bound for the penetration depth of the wave

**Proposition 16.** *In the problem* (1), (5), (6) *we have the following estimate for the penetration depth of the wave:*

$$x_{ef}(t) \lesssim 2\sqrt{\alpha}\, t - \frac{1}{2\sqrt{\alpha}} \ln t + O(1), \quad t \to \infty. \tag{12}$$

*Proof.* By condition (3), the function $v(t, x)$, which satisfies the equation

$$v_t = v_{xx} + \alpha v, \quad t > 0, \quad x \in \mathbf{R}, \tag{13}$$

and the same initial condition (5), (6), is a supersolution of equation (1) (by the comparison theorem, Theorem 2 of Ch. 1). Therefore $u(t, x) \leq v(t, x)$ in $\mathbf{R}_+ \times \mathbf{R}$. The function $v$ can be easily computed (the change of variable $v = e^{\alpha t} w$ reduces equation (13) to the heat equation for $w$):

$$u(t, x) \leq v(t, x) = \frac{e^{\alpha t}}{2\pi^{1/2}} \int_{x/t^{1/2}}^{\infty} \exp\left\{-\frac{\eta^2}{4}\right\} d\eta.$$

By this inequality the required half-width $x_{ef}(t)$ will not exceed $s(t)$, "half-width" of the wave that corresponds to the function $v$, i.e., the solution of the equation

$$\frac{1}{2} = \frac{e^{\alpha t}}{2\pi^{1/2}} \int_{s(t)/t^{1/2}}^{\infty} \exp\left\{-\frac{\eta^2}{4}\right\} d\eta. \tag{14}$$

Hence we obtain the estimate (12). □

It is of interest to note that the exact value of $x_{ef}(t)$ does not differ significantly from the expression in the right-hand side of (12) (see [195]):

$$x_{ef}(t) = 2\sqrt{\alpha}\,t - \frac{3}{2\sqrt{\alpha}} \ln t + O(1), \quad t \to \infty. \tag{12'}$$

## 3 Asymptotic stability of the travelling wave

Let us show that the self-similar solution (7) is asymptotically stable; stability is not necessarily with respect to small perturbations of the initial function $u_0(x)$.

**Proposition 17.** *Let there exist a constant* $\delta \in (0, 1)$ *such that*

$$Q(\delta u) \geq \delta Q(u), \quad u \in (0, 1) \tag{15}$$

*(this condition is satisfied by the source* (4)*). Then the solution of the Cauchy problem for* (1) *with initial function*

$$u(0, x) = u_0(x) = \delta\theta_S^0(x), \quad x \in \mathbf{R}, \tag{16}$$

*converges asymptotically to the similarity function* $\theta_S^0(\xi)$ *in the following sense: there exists a constant* $\xi_0$, *such that*

$$u(t, \xi + \lambda_0 t) - \theta_S^0(\xi + \xi_0) \to 0, \quad t \to \infty, \tag{17}$$

*for all* $\xi \in \mathbf{R}$.

*Proof.* The proof is based on the lemma stated below (similar assertions in a more general setting are used in Ch. V). As a preliminary step, we pass from equation (1) to the equation satisfied by the function $\theta(t, \xi) = u(t, \xi + \lambda_0 t)$:

$$\theta_t = \theta_{\xi\xi} + \lambda_0\theta_\xi + Q(\theta), \quad t > 0, \quad \xi \in \mathbf{R}. \tag{18}$$

Under this transformation the initial function $\theta_0(\xi) = \theta(0, \xi)$ does not change. Comparison of (18) with the ordinary differential equation (8) shows that the problem of asymptotic stability of the travelling wave self-similar solution is reduced by the transformation to the analysis of stability of stationary solutions of the new equation (18).

**Lemma 2.** *Under these assumptions, the solution of equation* (18) *is critical:*

$$\theta_t(t,\xi) \ge 0,\ t > 0, \xi \in \mathbf{R}.$$

*Proof.* The function $z = \theta_t$ satisfies the linear parabolic equation

$$z_t = z_{\xi\xi} + \lambda_0 z_\xi + Q'(\theta)z,\ t > 0, \xi \in \mathbf{R},$$

which is derived from (18) by differentiation in $t$. In view of sufficient smoothness of $\theta$ and $Q$, this manipulation is justified; however, one can weaken the requirement $\theta \in C^{2,4}_{t\xi}$ (see Ch. V). Therefore by the Maximum Principle $z(t,\xi) \ge 0$ in $\mathbf{R}_+ \times \mathbf{R}$ if this inequality holds at the initial moment of time, that is, if

$$z(0,\xi) \equiv \theta_t(0,\xi) \ge 0,\ \xi \in \mathbf{R}. \tag{19}$$

Taking into consideration (18) and the form of the initial function $\theta_0(\xi) \equiv u_0(\xi) = \delta\theta^0_S(\xi)$, we obtain that it is necessary to verify the inequality

$$\theta_t(0,\xi) = (\theta_0)_{\xi\xi} + \lambda_0(\theta_0)_\xi + Q(\theta_0) \equiv \delta(\theta^{0''}_S + \delta_0\theta^{0'}_S) + Q(\delta\theta^0_S) \ge 0, \xi \in \mathbf{R}. \tag{20}$$

The function $\theta^0_S(\xi)$ satisfies equation (8) for $\lambda = \lambda_0$. Therefore (20) is equivalent to the inequality

$$-\delta Q(\theta^0_S(\xi)) + Q(\delta\theta^0_S(\xi)) \ge 0,\ \xi \in \mathbf{R},$$

which holds by assumption (15). □

To conclude the proof of Proposition 17, it suffices to observe that the function $\theta(t,\xi)$ is non-decreasing in $t$ for all $\xi \in \mathbf{R}$, and is, moreover, bounded from above:

$$\theta(t,\xi) \le \theta^0_S(\xi),\ \xi \in \mathbf{R}, \tag{21}$$

since this inequality holds for $t = 0$ (see (16), where $\delta \in (0,1)$, and $\theta^0_S$ is the solution of equation (18)).

Therefore for any $\xi \in \mathbf{R}$ there exists a limit $\theta(t,\xi) \to \theta^*(\xi)$, $t \to \infty$. Passing to the limit as $t \to \infty$ in the integral equation equivalent to (18), we see that $\theta^*(\xi)$ is a stationary solution of the equation (18), that is, a solution of the problem (8) for $\lambda = \lambda_0$. As was mentioned earlier, it is unique up to a shift. □

**Remark.** It is not hard to estimate just how different are the solution $u(t,x)$ and the corresponding limit function $\theta^0_S$, which depends on the magnitude of $\xi_0$ in (17) (that is, on the amount of shift). First of all, by (21) $\xi_0 \ge 0$. Second, let us compare the asymptotics of the function $\theta^0_S(\xi)$:

$$\theta^0_S(\xi) = C_0\exp\{-\sqrt{\alpha}\xi\} + \dots,\ \xi \to \infty,$$

and of the initial function

$$\theta_0(\xi) = \delta\theta_S^0(\xi) \equiv \delta C_0 \exp\{-\sqrt{\alpha}\xi\} + \dots, \quad \xi \to \infty$$

(here $C_0 > 0$ is a constant). Taking into account the fact that by Lemma 2 $\theta(t, \xi) \geq \theta_0(\xi)$ in $\mathbf{R}_+ \times \mathbf{R}$, we obtain from the inequality $\theta_S^0(\xi + \xi_0) \geq \theta_0(\xi)$, or, which is the same, from the condition

$$C_0 \exp\{-\sqrt{\alpha}(\xi + \xi_0)\} \geq \delta C_0 \exp\{-\sqrt{\alpha}\xi\} \text{ as } \xi \to \infty.$$

an upper bound on the magnitude of $\xi_0$: $C_0 \exp\{-\sqrt{\alpha}\xi_0\} \geq \delta C_0$, that is, $\xi_0 \leq -\alpha^{-1/2} \ln \delta$. If $\delta > 0$ in (16) is small and therefore $\theta_0(\xi)$ is very different from $\theta_S^0(\xi)$, then the difference between this function and the limit function, to which $\theta(t, \xi)$ converges as $t \to \infty$, can also be large.

Let us emphasize that the initial function $u_0(x)$ in (16), with which $u(t, x)$ stabilizes to the self-similar solution, is substantially different from (6): it does not have a finite front, and (the main difference) $u_0(x) \to \delta < 1$ as $x \to -\infty$. However, the law of motion of the half-width of the wave is in this case closer to that of the self-similar solution. It is not hard to deduce from (11) that $x_{ef}(t) = 2\sqrt{\alpha}t + O(1)$, $t \to \infty$ (compare with (12′), where there is another term, which grows logarithmically as $t \to \infty$).

To conclude, let us note that using the proof of Proposition 17 under the assumption of criticality of $u_0$, we can demonstrate stabilization of (17) to the minimal function $\theta_S^0$ without the restriction on the source term $Q'(u) \leq \alpha$, $u \in (0, 1)$. In this context, let us quote some examples of stable travelling waves $u_S(t, x) = \theta_S^0(\xi)$, $\xi = x - \lambda_0 t$, which can be written down explicitly.

If

$$Q(u) = u(1 - u^\nu)(1 + (\nu + 1)u^\nu), \quad u \in (0, 1); \ \nu = \text{const} > 0,$$

then for $\lambda = \lambda_0 = 2\sqrt{\alpha} = 2$ a solution of the problem (8) is

$$\theta_S^0(\xi) = \left\{e^{-\nu\xi} / \left(1 + e^{-\nu\xi}\right)\right\}^{1/\nu}, \quad \xi \in \mathbf{R}.$$

For a source of "trigonometric" type,

$$Q(u) = \pi^{-1} \sin(\pi u)(2 - \cos(\pi u)), \quad u \in (0, 1),$$

such a solution is

$$\theta_S^0(\xi) = \pi^{-1} \arccos\left\{(e^{2\xi} - 1)/(1 + e^{2\xi})\right\}, \quad \xi \in \mathbf{R}.$$

In both cases $\theta_S^0(\xi) \simeq e^{-\xi}$ as $\xi \to \infty$.

# § 10 Self-similar solutions of the semilinear parabolic equation $u_t = \Delta u + u \ln u$

In this section we consider the Cauchy problem for a semilinear equation of the particular form

$$u_t = \Delta u + u \ln u, \quad t > 0, x \in \mathbf{R}^N, \tag{1}$$

$$u(0, x) = u_0(x) \geq 0, x \in \mathbf{R}^N; \ \sup u_0 < \infty. \tag{2}$$

Here the function $Q(u) = u \ln u$ is a source ($Q > 0$) for $u > 1$ and a sink ($Q < 0$) for low temperatures $u \in (0, 1)$.

Equation (1) is interesting in that it admits a large two-parameter family of self-similar solutions, which allow us to give a detailed description of solutions of the Cauchy problem and, in particular, to determine conditions of asymptotic stability of the principal self-similar solution (it will be the first to be defined below).

The source $Q(u) = u \ln u > 0$ for $u > 1$ satisfies the condition

$$\int_2^\infty \frac{du}{Q(u)} = \int_{\ln 2}^\infty \frac{d\eta}{\eta} = \infty.$$

Therefore all solutions of the Cauchy problem are globally defined, that is, they exist for all $t > 0$. Since the function $Q(u)$ is differentiable everywhere apart from the point $u = 0$, in a neighbourhood of which it is a sink, the solution of the problem exists, is unique, and satisfies the Maximum Principle. Moreover, using the self-similar solutions constructed below, it is not hard to show (as in the proof of Proposition 4 of Ch. I) that every solution of the Cauchy problem with $u_0(x) \not\equiv 0$ is strictly positive in $\mathbf{R}_+ \times \mathbf{R}^N$ and is a classical one.

## 1 A one-parameter family of self-similar solutions

We shall seek the principal[4] separable self-similar solution in the form

$$u_S^*(t, x) = \psi_*(t)\theta_*(x), \ \theta_*(x) = \exp\{-|x|^2/4\}. \tag{3}$$

Then from (1) we obtain for $\psi_*(t) > 0$ the equation

$$\psi_*'(t) = -\frac{N}{2}\psi_*(t) + \psi_*(t) \ln \psi_*(t), \ t > 0,$$

from which $\psi_*(t) = \exp\{B_0 e^t + N/2\}$ and

$$u_S^*(t, x) = \exp\{B_0 e^t + N/2\} \exp\{-|x|^2/4\}. \tag{4}$$

[4] The sense in which "principal" is to be understood will become clear from the following.

Here $B_0$ is an arbitrary constant.

This solution corresponds to the initial perturbation

$$u_0(x) \equiv u_S^*(0, x) = \exp\{B_0 + N/2 - |x|^2/4\}, \ x \in \mathbf{R}^N. \tag{5}$$

It follows from (4) that if $B_0 > 0$, $u_S^*(t, x)$ grows without bound in $\mathbf{R}^N$ as $t \to \infty$; if $B_0 < 0$ then $u_S^*(t, x) \to 0$ in $\mathbf{R}^N$ as $t \to \infty$. The value $B_0 = 0$ corresponds to the stationary solution of equation (1), which is independent of time:

$$u_{SS}(x) = \exp\{N/2 - |x|^2/4\}, \ x \in \mathbf{R}^N. \tag{6}$$

Thus, there are three types of essentially different self-similar solutions (4):
1) a growing solution ($B_0 > 0$);
2) a decaying solution ($B_0 < 0$);
3) a stationary solution ($B_0 = 0$).

All these solutions can occur if we use quite a restricted set of initial functions (5). What is the domain of attraction of each of the three types of the principal self-similar solution; for what $u_0(x)$ will each type of evolution occur?

## 2 A two-parameter family of self-similar solutions

We can give partial answers to the questions posed above by constructing a larger family than (4) of self-similar solutions of equation (1). We shall look for these solutions in the self-similar form (now the variables do not separate as in (3)):

$$u_S(t, x) = \psi(t)\theta_*(\xi), \ \xi = |x|/\phi(t), \ \theta_*(\xi) = \exp\{-\xi^2/4\}. \tag{7}$$

Substituting the above expression into (1) leads to the following system of ordinary differential equations with respect to the functions $\psi(t)$, $\phi(t)$:

$$\psi'(t) = -\frac{N}{2}\frac{\psi(t)}{\phi^2(t)} + \psi(t)\ln\psi(t), \tag{8}$$

$$\frac{2\phi'(t)}{\phi(t)} = \frac{1}{\phi^2(t)} - 1, \ t > 0. \tag{9}$$

The second equation can be easily integrated:

$$\phi(t) = (1 - a_0 e^{-t})^{1/2}, \ t \geq 0, \tag{10}$$

where $a_0$ is a constant; here for (10) to make sense for all $t \geq 0$ we must have the inequality $a_0 < 1$. Then (8) gives us the following expression for the amplitude of the self-similar solution:

$$\psi(t) = \exp\left\{e^t\left[B_0 - \frac{N}{2a_0}\ln(1 - a_0 e^{-t})\right]\right\}. \tag{11}$$

where $B_0$ is a constant (the same as in (4)).

The constructed family of solutions (7) has the properties 1)–3), however, it is a larger family than (4), as it depends on two parameters $a_0$ and $B_0$. The corresponding initial functions have the form

$$u_0(x) \equiv u_S(0, x) = \exp\left\{B_0 - \frac{N}{2a_0}\ln(1 - a_0) - \frac{|x|^2}{4(1 - a_0)}\right\}, \quad x \in \mathbf{R}^N. \qquad (12)$$

For a fixed $B_0$ the one-parameter family of these initial functions ($a_0 < 1$ is a parameter) characterizes the domain of attraction of each of the three types of the principal self-similar solution (4).

### 3 Condition of stabilization to the stationary solution

Let us consider the case $B_0 = 0$ in (12). Then it easily follows from (7) that the corresponding self-similar solution converges as $t \to \infty$ to the principal self-similar solution (4), that is, in this case, to the stationary solution (6). From (10), (11) it is not hard to derive an estimate of the rate of stabilization to $u_{SS}(x)$.

For each fixed $x \in \mathbf{R}^N$ (for $B_0 = 0$)

$$\psi(t) = e^{N/2} + \frac{Na_0}{4}e^{N/2}e^{-t} + o(e^{-t}),\ \phi^2(t) = 1 - a_0e^{-t},\ t \to \infty.$$

Therefore for large $t$

$$u_S(t, x) = \exp\left\{\frac{N}{2} - \frac{|x|^2}{4}\right\}\left[1 + \frac{a_0}{4}(N - |x|^2)e^{-t}\right] + o(e^{-t}).$$

Hence it follows that on any compact set $K_L = \{|x| \leq L\}$ in $\mathbf{R}^N$

$$\|u_S(t, \cdot) - u_{SS}(\cdot)\|_{C(K_L)} = O(e^{-t}) \to 0,\ t \to \infty.$$

The process of stabilization to the stationary solution is schematically depicted in Figure 11.

Thus the stationary solution (6) is stable with repect to perturbations of the initial function, not to ones of arbitrary form, but to ones which make up the self-similar initial functions (12) for $B_0 = 0$. Here the perturbations can be arbitrarily large in amplitude; see Figure 11, where $u_S(t_1, 0)$ is several times larger than $u_{SS}(0)$.

This result is interesting, since with respect to arbitrary perturbations, no matter how small, the stationary solution $u_{SS}$ is *unstable*. This is demonstrated by the following simple claim.

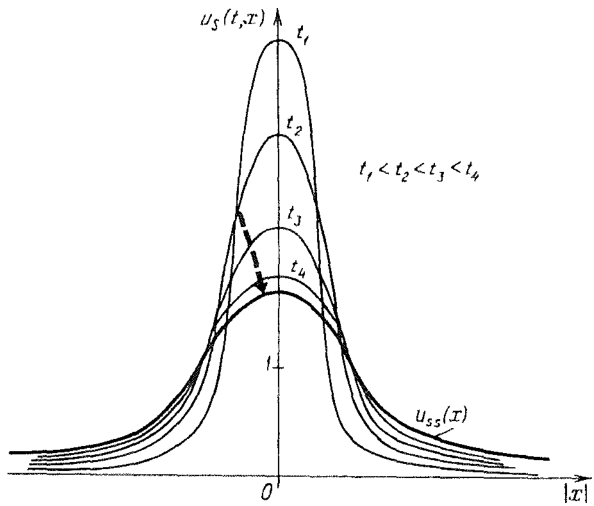


**Fig. 11.** Stabilization as $t \to \infty$ of a self-similar solution $u_S(t, x)$, $B_0 = 0$ to the unstable stationary solution $u_{SS}(x)$

**Proposition 18.** *Let*

$$u_0(x) = \delta u_{SS}(x), \quad x \in \mathbf{R}^N, \tag{13}$$

*where* $\delta > 1$ *is a constant (deviation from the stationary solution* $\|u_0(\cdot) - u_{SS}(\cdot)\|_{C(\mathbf{R}^N)} = (\delta - 1)e^{N/2}$ *can be arbitrarily small if* $\delta$ *is close to* 1*). Then*

$$\lim_{t \to \infty} u(t, x) = \infty, \quad x \in \mathbf{R}^N, \tag{14}$$

*that is, there is no stabilization to* $u_{SS}$.

*Proof.* Let us take in (5) an arbitrary $B_0 = B_0^+ \in (0, \ln \delta)$. It is not hard to check that in that case

$$u_0(x) \equiv \delta u_{SS}(x) \geq u_S^+(0, x), \quad x \in \mathbf{R}^N,$$

and therefore by the comparison theorem

$$u(t, x) \geq u_S^+(t, x) \equiv \exp\{B_0^+ e^t + N/2\} \exp\{-|x|^2/4\} \to \infty, \quad t \to \infty \text{ in } \mathbf{R}^N.$$

This concludes the proof of instability of the stationary solution (6) from above.

In a similar manner we can prove that it is also unstable in $C(\mathbf{R}^N)$ from below; to any initial function (13) with $\delta \in (0, 1)$ corresponds a decaying solution: $u(t, x) \to 0$ as $t \to \infty$ in $\mathbf{R}^N$, that is, again there is no stabilization to $u_{SS}$. It is proved exactly in the same way by comparing $u(t, x)$ with a self-similar solution (4), in which $B_0 = B_0^- \in (\ln \delta, 0)$. Then $u(t, x) \leq u_S^+(t, x) \to 0$, $t \to \infty$, in $\mathbf{R}^N$ since $B_0^- < 0$. □

Therefore the family of functions in (12) with $B_0 = 0$ is the attracting set of the unstable stationary solution in the space of initial functions. We note that it is unbounded in $C(\mathbf{R}^N)$.

## 4 Decaying solutions

These exist if $B_0 < 0$ in (12); then $u_S(t, x) \to 0$ as $t \to \infty$ for all $x \in \mathbf{R}^N$. Then it can be seen from (10), (11) that for large $t$ $u_S(t, x)$ has practically the same structure as the principal self-similar solution (4). To be more precise, introducing the similarity representation of the solution (7) (the similarity transform corresponding to the principal solution (4)),

$$\theta(t, x) = \frac{u_S(t, x)}{\psi(t)} \equiv \theta_* \left( \frac{|x|}{(1 - a_0 e^{-t})^{1/2}} \right), \tag{15}$$

we see that

$$\|\theta(t, \cdot) - \theta_*(\cdot)\|_{C(\mathbf{R}^N)} \to 0, \ t \to \infty. \tag{16}$$

Here $\theta_*(x) \equiv u_S^*(t, x)/\psi_*(t)$ has the meaning of the similarity representation of the principal self-similar solution (4).

This estimate implies asymptotic stability of the principal solution with respect to perturbations of the form (12) of the initial function (5).

## 5 Growing solutions

If $B_0 > 0$ in (12), it follows from (11) that $u_S(t, x) \to \infty$ in $\mathbf{R}^N$ as $t \to \infty$. Using the same formula (15) to introduce the similarity representation of the solutions $u_S(t, x)$, it is not hard to check that all these solutions (for any $B_0 > 0$, $a_0 < 1$) converge in the sense of (16) to the principal self-similar solution (4).

It is important to emphasize the following point. Let us determine the rate of change of the effective half-width of the growing heat structure $l_{ef}(t) = |x_{ef}(t)|$ defined by $u_S(t, x_{ef}(t)) = \psi(t)/2$. Using the explicit form of the function $u_S$ we obtain for the half-width the expression

$$l_{ef}^2(t) = 4 \ln 2 \cdot (1 - a_0 e^{-t}) \to 4 \ln 2, \ t \to \infty,$$

that is, for large $t$ it becomes practically constant. Nonetheless, the solution $u_S(t, x)$ grows without bound on the whole space (compare with example 13 of § 3, Ch. 1, where the half-width being constant went together with localization of the thermal structure in space).

To conclude, let us observe that it is very rarely that one is able to construct a large family of exact self-similar solutions which coincide asymptotically with the principal ("generating") solution. Frequently one can determine generating solutions in the framework of the theory of approximate self-similar solutions (see Ch. VI).

**Remark.** It is easy to construct a family of self-similar solutions of the form (7) for the equation

$$u_t = \Delta u - u \ln u, \quad t > 0, \; x \in \mathbf{R}^N. \tag{17}$$

Then, taking into account the fact that the function $Q(u) = -u \ln u > 0$ for $u \in (0, 1)$ is a source and $Q(0) = Q(1) = 0$, we obtain a problem similar to the one considered in § 9. However here $Q'(0^+) = \infty$; therefore the speed of the motion of the thermal wave will not be asymptotically constant.

Self-similar solutions of equation (17) have the form

$$u_S(t, x) = \exp\left\{ e^{-t} \left[ B_0 - \frac{N}{2a_0} \ln(a_0 e^t - 1) \right] - \frac{|x|^2}{4(a_0 e^t - 1)} \right\}, \quad t > 0, \; x \in \mathbf{R}^N, \tag{18}$$

where $B_0$, $a_0 > 1$ are constants. Let

$$B_0 - \frac{N}{2a_0} \ln(a_0 - 1) \leq 0.$$

Then, obviously, $0 < u_S(0, x) \leq 1$ in $\mathbf{R}^N$, and therefore, by the comparison theorem $u_S(t, x) \in (0, 1)$ in $\mathbf{R}_+ \times \mathbf{R}^N$ (this can be seen immediately from (18)). In this case (18) represents a thermal wave with nearly constant (as $t \to \infty$) amplitude that propagates in all directions (Figure 12). Its effective width has the form

$$|x_{ef}(t)| \simeq 2(a_0 \ln 2)^{1/2} e^{t/2}, \quad t \to \infty,$$

that is, the speed of motion grows exponentially as $t \to \infty$.

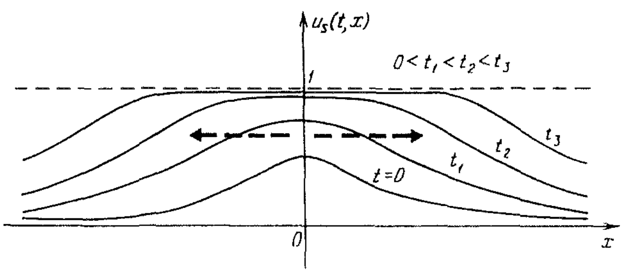


**Fig. 12.** A travelling wave in the Cauchy problem for equation (17)

As far as asymptotic stability of the family (18) is concerned, we have for all $B_0$ the estimate

$$\sup_{\xi \in \mathbf{R}^N} \left| u_S(t, \xi e^{t/2}) - \exp\left\{ -\frac{|\xi|^2}{4a_0} \right\} \right| = O(te^{-t}) \to 0, \; t \to \infty.$$

Therefore for large $t$

$$u_S(t, x) \simeq \exp\left\{ -\frac{|x|^2}{4a_0 e^t} \right\},$$

so that the principal self-similar solution here is a different one.

## § 11 A nonlinear heat equation with a source and a sink

Let us consider the quasilinear parabolic equation

$$u_t = (u^\sigma u_x)_x + u^{\sigma+1} - u, \; t > 0, \; x \in \mathbf{R}; \; \sigma > 0. \tag{1}$$

It differs from the one encountered before (Example 13, Ch. I) by the presence of the sink $-u$. This can significantly change the character of evolution of the combustion process.

We shall look for self-similar solutions of equation (1) in the separable form

$$u_S(t, x) = \psi(t)\theta(x), \; t > 0, \; x \in \mathbf{R}.$$

Substitution into (1) leads to the problem

$$\frac{\psi'(t) + \psi(t)}{\psi^{\sigma+1}(t)} = \frac{(\theta^\sigma \theta')' + \theta^{\sigma+1}}{\theta} = -\lambda = \text{const}. \tag{2}$$

For convenience let us set $\lambda = -1/\sigma$. Then we obtain for the function $\theta(x)$ exactly the same equation as for $\theta_S(x)$ in § 3, Ch. 1. Therefore we can take $\theta \equiv \theta_S$. The amplitude of the solution $\psi(t)$ is easily computed from (2), and as a result we obtain the family of self-similar solutions

$$u_S(t, x) = e^{-t} \left( \frac{1}{\sigma} e^{-\sigma t} + C_0 \right)^{-1/\sigma} \theta_S(x), \tag{3}$$

where $C_0$ is a constant. To each of those solutions corresponds an initial function

$$u_0(x) \equiv u_S(0, x) = (1/\sigma + C_0)^{-1/\sigma} \theta_S(x), \; x \in \mathbf{R}. \tag{4}$$

Hence we have that it is necessary to impose the restriction $C_0 > -1/\sigma$.

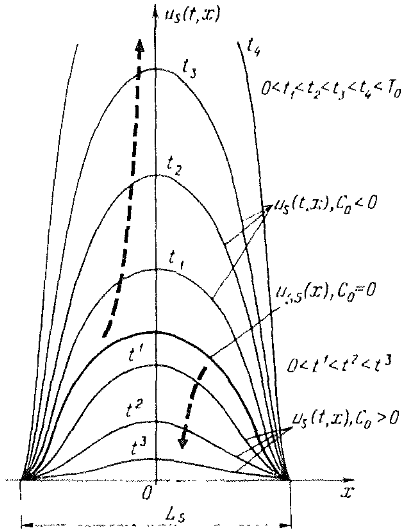


**Fig. 13.** Evolution of self-similar solutions (3) for $C_0 < 0$, $C_0 = 0$, $C_0 > 0$

For different $C_0$ in (3) there exist three types of self-similar solutions having different spatio-temporal evolution. If $C_0 = 0$, then (3) is a stationary solution (Figure 13):

$$u_{SS}(x) \equiv \sigma^{1/\sigma}\theta_S(x),\ x \in \mathbf{R}. \tag{5}$$

If $C_0 > 0$ then the solution $u_S$ decays (quenches):

$$u_S(t,x) = C_0^{-1/\sigma}e^{-t}\theta_S(x) + o(e^{-t}),\ t \to \infty. \tag{6}$$

These solutions are below the stationary one on Figure 13, and their existence has to do with the presence in (1) of a heat sink, which for small $u > 0$ is more powerful than the source.

On the other hand, if $C_0 \in (-1/\sigma, 0)$, that is, if the initial function lies above the stationary solution, then finite time blow-up occurs:

$$u_S(t,x) \to \infty,\ t \to T_0 = -\frac{1}{\sigma}\ln(-\sigma C_0) > 0$$

everywhere in the localization domain $|x| < (\text{meas supp}\,\theta_S)/2$. The perturbations do not leave this domain, even though the temperature grows without bound (see Figure 13). From (3) it follows that the solution $u_S$ grows according to the power law

$$u_S(t,x) \simeq (T_0 - t)^{-1/\sigma}\theta_S(x). \tag{7}$$

Therefore the stationary solution (5) is *unstable*: small negative perturbations lead to stabilization to a different stable stationary solution $u_{SS} \equiv 0$; positive perturbations lead to growing solutions which blow up in finite time.

Let us observe that the spatial dependence of heat transfer processes in this nonlinear medium (combustion or quenching) are determined by the same function $\theta_S(x)$; it is only the equations governing the change of the amplitude of the heat structure that depend on the type of the process. In this medium there is also a characteristic spatial scale, which is common to all the processes, the *fundamental length* $L_S = \text{meas supp}\,\theta_S = 2\pi(\sigma+1)^{1/2}/\sigma$.

## § 12 Localization and total extinction phenomena in media with a sink

In this section we consider in more detail certain properties of solutions of the nonlinear parabolic equation with a sink

$$u_t = (u^\sigma u_x)_x - u^\nu, \quad t > 0, \quad x \in \mathbf{R}, \tag{1}$$

where $\sigma > 0$, $\nu > 0$.

### 1 Localization of heat perturbations

We are already familiar with one of the important properties of solutions of this equation, the localization property (§ 3, Ch. 1): if the initial function $u_0(x)$ in the Cauchy problem is of compact support, then as $t_E \to \infty$ heat perturbations do not propagate beyond a certain finite length. As in example 11 of § 3, Ch. 1, we can prove a more general assertion concerning localization conditions of solutions of the Cauchy problem for (1).

**Proposition 19.** *Let $u_0(x)$ be a function with compact support and $\nu < \sigma + 1$. Then there exists a constant $L > 0$ such that $u(t,x) \equiv 0$ for all $|x| > L$ for any $t > 0$.*

This result can be extended to an arbitrary number of space variables. It is not hard to carry through the same kind of analysis for equations of the type (1) with arbitrary coefficients $k(u) \geq 0$, $Q(u) \leq 0$.

The localization condition $\nu < \sigma + 1$ for heat perturbations is obtained by a simple comparison of the solution of the Cauchy problem with a suitable stationary solution; moreover, this condition is both necessary and sufficient. This is indicated, in particular, by the fact that for $\nu \geq \sigma + 1$ there are no non-trivial stationary solutions that vanish together with the heat flux in some finite point. We shall return to consider the character of the motion of heat fronts a bit later, while now we consider another curious property of solutions of equation (1).

## 2 A condition for total extinction in finite time

In this case this phenomenon is related to the presence of heat sinks in the medium.

**Proposition 20.** *Let $\nu < 1$, $\sup u_0 = M < \infty$. Then there is $T_0 \leq T_* = M^{1-\nu}/(1-\nu)$, such that $u(t,x) \equiv 0$ in $\mathbf{R}$ for all $t \geq T_0$.*

*Proof.* Let us compare $u(t,x)$ with the spatially homogeneous solution $v(t)$ of equation (1):

$$v'(t) = -v^\nu(t), \ t > 0; \ v(0) = M.$$

By the comparison theorem, $u(t,x) \leq v(t)$ in $\mathbf{R}_+ \times \mathbf{R}$. However, it is not hard to see that $v(t) = 0$ for $t = T_*$, which completes the proof. □

From these arguments it follows that if we replace the term $-u^\nu$ in the equation by an arbitrary sink $-Q(u)$ ($Q(u) > 0$ for $u > 0$), total extinction occurs if

$$\int_0^1 \frac{d\eta}{Q(\eta)} < \infty.$$

By the comparison theorem, the same result holds in the multidimensional case.

Formally, we may asume that the asymptotic stage of the total extinction process ($t \to T_0$) is described by self-similar solutions

$$u_S(t,x) = (T_0 - t)^{1/(1-\nu)} f_S(\zeta), \ \zeta = x/(T_0 - t)^{(\sigma+1-\nu)/[2(1-\nu)]} \in \mathbf{R}, \tag{2}$$

where $T_0 > 0$ is a constant (the total extinction time) and $f_S(\zeta) \geq 0$ satisfies the equation

$$(f_S^\sigma f_S')' - \frac{(\sigma+1)-\nu}{2(1-\nu)} f_S'\zeta + \frac{1}{1-\nu} f_S - f_S^\nu = 0, \ \zeta \in \mathbf{R}. \tag{2'}$$

However, this fact depends strongly on the existence or nonexistence of non-trivial solutions of this ordinary differential equation which satisfy the condition $f_S \to 0$, $|\zeta| \to \infty$ (similar problems are treated in § 1, 3, Ch. IV). Then the expression (2) shows us the evolution of the extinction process. Asymptotic stability of these self-similar solutions can be studied by the methods used in § 4–6, Ch. IV in the analysis of self-similar solutions with blow-up (the difficulties that arise in the process are on the whole of the same nature, and have to do with "singularity" in time of the solutions under consideration). See also the Comments.

Both those properties, localization and total extinction, are illustrated by the following example, which demonstrates specificities of motion of heat fronts in media with volume (body) sinks.

**Example 11.** Let $\sigma \in (0, 1)$. Let us consider in $\mathbf{R}_+ \times \mathbf{R}^N$ the Cauchy problem for the equation

$$u_t = \nabla \cdot (u^\sigma \nabla u) - u^{1-\sigma}. \tag{3}$$

Let us assume that at the initial moment of time $t = 0$ all the heat energy is concentrated at the point $x = 0$, that is $u(0, x) \equiv 0$ in $\mathbf{R}^N \setminus \{0\}$ and that $u(0, 0) = \infty$. This is a typical "self-similar" statement, containing minimal information about initial data.

We shall look for a solution of the problem in the form

$$u_S(t, x) = \psi(t)\theta(\xi), \ \xi = |x|/\phi(t), \tag{4}$$

where $\psi(t) \geq 0$ and $\phi(t) \geq 0$ are, respectively, the amplitude and the width of the heat structure, while the compactly supported function $\theta(\xi) \geq 0$ has the form

$$\theta(\xi) = [(1 - \xi^2)_+]^{1/\sigma}, \ \xi \in \mathbf{R}.$$

Its regularity properties are satisfactory from our point of view: at the front points $\xi = \pm 1$ the heat flux is continuous.

Substituting the expression (4) into the original equation, we obtain for the functions $\psi$, $\phi$ a system of ordinary differential equations:

$$\begin{gathered} -\frac{2(2 + N\sigma)}{\sigma^2} + \frac{2\phi\phi'}{\sigma\psi^\sigma} - \frac{\phi^2\psi'}{\psi^{\sigma+1}} = 0, \\ \frac{4}{\sigma^2} - \frac{2\phi\phi'}{\sigma\psi^\sigma} - \frac{\phi^2}{\psi^{2\sigma}} = 0, \ t > 0; \ \psi(0) = \infty, \ \phi(0) = 0. \end{gathered} \tag{5}$$

Setting here $\phi^2 = Y(t)$, $\psi^{-\sigma} = Z(t)$, we obtain the system

$$(YZ)' = \frac{2(2 + N\sigma)}{\sigma}, \ \frac{1}{\sigma}Y'Z + YZ^2 = \frac{4}{\sigma^2},$$

which can be easily integrated.

Let us write down the expressions for the amplitude and the width of the thermal structure:

$$\begin{gathered} \psi^\sigma(t) = a_0 t^{-N\sigma/(2+N\sigma)} \left(A - b_0 t^{2(1+N\sigma)/(2+N\sigma)}\right)_+, \\ \phi^2(t) = c_0 t^{2/(2+N\sigma)} \left(A - b_0 t^{2(1+N\sigma)/(2+N\sigma)}\right)_+. \end{gathered} \tag{4'}$$

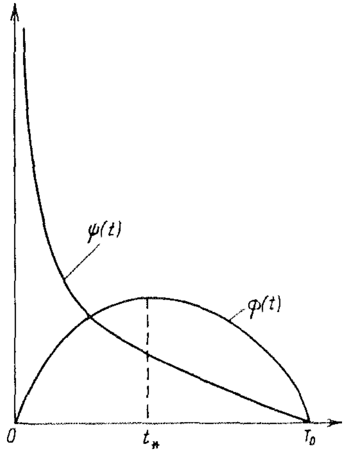


**Fig. 14.** The dependence of the amplitude $\psi(t)$ and half-width $\phi(t)$ of the localized solution (4), (4′) on time; $t = T_0$ is the total extinction time

Here

$$a_0 = \left[\frac{2(2+N\sigma)}{\sigma}\right]^{-N\sigma/(2+N\sigma)}, \quad c_0 = \left[\frac{2(2+N\sigma)}{\sigma}\right]^{2/(2+N\sigma)},$$

$$b_0 = \frac{\sigma^2}{4(1+N\sigma)}\left[\frac{2(2+N\sigma)}{\sigma}\right]^{2(1+N\sigma)/(2+N\sigma)}.$$

$A > 0$ is a constant. Graphs of the functions $\psi(t)$, $\phi(t)$ are sketched on Figure 14.

It is interesting to note that the size of the support of the generalized solution does not change monotonically with time. On the interval $(0, t_*)$, where

$$t_* = \left[\frac{A}{(2+N\sigma)b_0}\right]^{(2+N\sigma)/[2(1+N\sigma)]},$$

the width of the structure $\phi(t)$ grows; subsequently the surface of the heat front starts moving back towards the origin $x = 0$, and finally, at time

$$t = T_0 = (A/b_0)^{(2+N\sigma)/[2(1+N\sigma)]}$$

the functions $\psi(t)$ and $\phi(t)$ vanish simultaneously, that is, we have total extinction.

The self-similar solution (4) is localized: at every moment of time the diameter of the support does not exceed $2\phi(t_*)$. Observe that (4), (4′) imply that as $t \to T_0$ the solution has the following asymptotic extinction behaviour:

$$u_S(t,x) = \left[\sigma(T_0 - t)(1 - \xi^2)_+\right]^{1/\sigma}(1 + o(1))$$

with $\xi = |x|/d_0(T_0 - t)^{1/2}$ and $d_0^2 = 2(2 + N\sigma)T_0$. Thus it is not self-similar (cf (2) with $\nu = 1 - \sigma$) and is governed by the equation without diffusion, $u_t = -u^{1-\sigma}$

### 3 The motion of the thermal wave in the absence of localization

For $\nu \geq \sigma+1$ in (1) a compactly supported initial perturbation will not be localized. This is because at low temperatures the strength of heat absorption is not sufficient to halt the thermal wave. The nature of the motion of a front point in these cases is determined from the analysis of exact or approximate self-similar solutions of equation (1).

If $\nu = \sigma + 1$, then (1) admits a self-similar solution of a relatively unusual form:

$$u_S(t, x) = (T+t)^{-1/\sigma} f(\eta), \quad \eta = x - \lambda \ln(T+t), \tag{6}$$

where $T \geq 1$ and $\lambda > 0$ are constants. The function $f(\eta) \geqslant 0$ satisfies the equation

$$(f^{\sigma} f')' + \lambda f' + \frac{1}{\sigma} f - f^{\sigma+1} = 0, \quad \eta \in \mathbf{R}. \tag{7}$$

To formulate correctly the boundary conditions for this equation, let us consider the following analogy with the results of § 9. The quasilinear parabolic equation

$$v_\tau = (v^{\sigma} v_x)_x + \frac{1}{\sigma} v - v^{\sigma+1}, \quad \tau > 0, \ x \in \mathbf{R},$$

contains in its right-hand side the function $Q(v) = v/\sigma - v^{\sigma+1} > 0$ for $v \in (0, \sigma^{-1/\sigma})$ and $Q(0) = Q(\sigma^{-1/\sigma}) = 0$. Therefore we could formally consider a Kolmogorov-Petrovskii-Piskunov problem for that equation and try to find a travelling wave self-similar solution, $v(\tau, x) = f(\eta)$, $\eta = x - \lambda\tau$. Then the function $f \geq 0$ is a solution of equation (7), and therefore it is necessary to impose the boundary conditions

$$f(-\infty) = \sigma^{-1/\sigma}, \ f(\infty) = 0. \tag{7'}$$

We restrict ourselves to deriving estimates of the size of the support of the generalized solution of the Cauchy problem for $\nu = \sigma + 1$, when $u_0(x) = u(0, x)$ is a function with compact support.

First of all, it is easy to prove the following claim: *there exist constants $A > 0$ and $T \geq 1$, such that*

$$\operatorname{meas} \operatorname{supp} u(t, x) \leq A + \sigma^{-1} \ln(T+t), \quad t \geq 0. \tag{8}$$

To prove this, it suffices to check that the function

$$u_+ = (T+t)^{-1/\sigma} (1/2)^{1/\sigma} (-\eta)_+^{1/\sigma}, \ \eta = \eta_0 + x - \sigma^{-1} \ln(T+t),$$

is a supersolution of equation (1), and to choose the constants $T \geq 1$ and $\eta_0 \in \mathbf{R}$ so that $u_0(x) \leq u_+(0, x)$ in $\mathbf{R}$.

Secondly, we have a lower bound: *there are constants $B > 0$, $\lambda \in (0, 1)$, such that*

$$\text{meas supp}\, u(t, x) \geq B + \lambda \ln(1 + t),\ t \geq 0. \tag{8'}$$

The proof proceeds via construction of a subsolution of a different form:

$$u_- = (1 + t)^{-1/\sigma} H(1 - \eta^2/a^2)_+^{1/\sigma},\ \eta = \eta_0 + x - \lambda \ln(1 + t),$$

where $H > 0$, $a > 0$, $\lambda \in (0, 1)$ satisfy certain inequalities; if they do, they can be chosen to be arbitrarily small. Therefore there exists an $\eta_0 \in \mathbf{R}$, such that $u_0(x) \geq u_-(0, x)$ in $\mathbf{R}$.

In view of the last estimate (8'), compactly supported solutions of the Cauchy problem are not localized for $\nu = \sigma + 1$.

A sharp estimate of the support of any compactly supported solution can be derived by using a different particular solution. Namely, using the equation $v_t = \mathbf{A}(v) \equiv v v_{xx} + (1/\sigma)(v_x)^2 - \sigma v^2$, where $v = u^\sigma$, we observe that the quadratic operator $\mathbf{A}$ admits a linear invariant subspace $W_2 = \mathcal{L}\{1, \cosh(\lambda x)\}$, $\lambda = \sigma/(\sigma + 1)^{1/2}$. Therefore substiting $v(t, x) = C_0(t) + C_1(t) \cosh(\lambda x) \in W_2$, yields the dynamical system (cf. [49])

$$C_0' = -\sigma(C_0^2 + \frac{1}{\sigma + 1} C_1^2),\ C_1' = -\frac{\sigma(\sigma + 2)}{\sigma + 1} C_0 C_1,\ t > 0,$$

which can be easily studied. By using weak solutions of the form $(v)_+$ in comparison with $u(t, x)$ from above and from below, we derive the following estimate:

$$\text{meas supp}\, u(t, x) = \frac{2}{\sigma(\sigma + 1)^{1/2}} \ln t \left(1 + O\left(\frac{1}{\ln t}\right)\right) \quad \text{as } t \to \infty.$$

For $\nu > \sigma + 1$ asymptotic behaviour of the process is described by a self-similar solution of the usual form,

$$u_S = (T + t)^{-1/(\nu - 1)} g_S(\zeta),\ \zeta = |x|/(T + t)^{[\nu - (\sigma + 1)]/[2(\nu - 1)]}, \tag{9}$$

where the function $g_S \geq 0$ satisfies the problem

$$\begin{gathered}(g_S^\sigma g_S')' + \frac{\nu - (\sigma + 1)}{2(\nu - 1)} g_S' \zeta + \frac{1}{\nu - 1} g_S - g_S^\nu = 0,\ \zeta > 0, \\ g_S'(0) = 0,\ g_S(\infty) = 0.\end{gathered} \tag{10}$$

A non-trivial compactly supported generalized solution of this problem exists for $\sigma + 1 < \nu < \sigma + 3$ (similar problems are considered in Ch. IV). In this case using the comparison theorem, we obtain from (9) an estimate of the support of the generalized solution:

$$|x_f(t)| \sim t^{[\nu - (\sigma + 1)]/[2(\nu - 1)]} \to \infty,\ t \to \infty.$$

that is, unlike the case $\nu = \sigma + 1$, for $\nu > \sigma + 1$ the thermal wave moves according to a power type law (faster than in (8)). In this case (9) determines also, for example, the relation governing the change in time of the amplitude:

$$\sup_{x \in \mathbf{R}} u(t, x) \simeq g_S(0) t^{-1/(\nu-1)}, \quad t \to \infty.$$

The situation for $\nu > \sigma + 3$ is simpler. The problem (10) has no compactly supported solution. For $\nu > \sigma+3$ the heat absorption on the wave front is so small that it exerts practically no influence on the speed of its movement for large $t$. As a final result we have that as $t \to \infty$ the character of the motion of the front does not depend on absorption and is determined solely by the diffusion operator, that is $u(t, x)$ is in some sense close to the solution of the equation $v_t = (v^\sigma v_x)_x$. But for this equation we know a self-similar solution, which describes the asymptotic stage of spread of the heat perturbation (see Example 8, Ch. I). Therefore $|x_{ef}(t)| \sim t^{1/(\sigma+2)}$, $t \to \infty$, and furthermore $\sup_{x\in \mathbf{R}} u(t, x) \sim t^{-1/(\sigma+2)}$ as $t \to \infty$.

Heat perturbations penetrate arbitrarily far; there is no localization.

## § 13 The structure of attractor of the semilinear parabolic equation with absorption in $\mathbf{R}^N$

In this final section we study in more detail the asymptotic behaviour of solutions of the Cauchy problem for heat equation with absorption in the multi-dimensional case:

$$u_t = \Delta u - u^\beta, \ t > 0, x \in \mathbf{R}^N; \quad \beta = \text{const} > 1, \tag{1}$$

$$u(0, x) = u_0(x) \geq 0 \ (\not\equiv 0), \ x \in \mathbf{R}^N; \ \sup u_0 < \infty. \tag{2}$$

The initial function $u_0$ is uniformly Lipschitz continuous in $\mathbf{R}^N$. This is a semilinear equation ($\sigma = 0$); however, the same analysis can be carried out for the more general quasilinear equation considered in § 12.

Equation (1) is one of the few nonlinear parabolic equations in $\mathbf{R}^N$, whose asymptotic behaviour as $t \to \infty$ has been studied in sufficient detail. At present there exists a relatively complete, and for some parameter ranges, exhaustive, description of the attractor of the Cauchy problem for (1) as the manifold of asymptotically stable states (eigenfunctions of the nonlinear medium, e.f.), to each of which corresponds its attracting set $\mathcal{W}$ in the space of initial functions. A more detailed discussion of e.f. of a nonlinear medium can be found in Ch. IV (see [162, 268, 269]).

Below we present a simplified description of the structure of the attractor of equation (1), which determines the asymptotic behaviour of solutions of the Cauchy problem as $t \to \infty$.

We pursue two aims in concluding this introductory chapter with a detailed analysis of a particular problem. First, this is a result of a complex study of a rather complicated nonlinear problem. It turns out that the process of heat conduction with absorption in this case can evolve as $t \to \infty$ in many different ways. In particular, a measure of this variety is the fact that the attractor of the equation is infinite-dimensional.[5]

Secondly, we want to emphasize the essential difference in the structure of the attractor of the nonlinear equation with absorption (1) and of an equation with a source, which admits unbounded solutions. Analysis of the latter takes up a major part of this book. Without entering into details, let us indicate the main difference. If, roughly speaking, the equation with absorption has, for nearly all values of parameters, a "continuous" attractor, then in the case of an equation with a source term the attractor is "quantized" in a special way, and consists, apparently, of several collections of discrete states, combustion eigenfunctions of the nonlinear dissipative medium. Principles of constructing a discrete attractor are discussed in Ch. IV.

Let us return to the problem (1), (2). The first "candidates" to be elements of the attractor of the equation are, of course, its self-similar solutions.

## 1 Self-similar solutions and conditions for their asymptotic stability

Below we consider, for simplicity, radially symmetric self-similar solutions of equation (1) of the form

$$u_S(t,x) = (T+t)^{-1/(\beta-1)}\theta_S(\xi),\ \xi = |x|/(T+t)^{1/2}, \tag{3}$$

where $T \geq 0$ is a constant, while the function $\theta_S > 0$ satisfies the ordinary differential equation

$$\mathbf{A}_R(\theta_S) \equiv \xi^{1-N}(\xi^{N-1}\theta_S')' + \frac{1}{2}\theta_S'\xi + \frac{1}{\beta-1}\theta_S - \theta_S^\beta = 0,\ \xi > 0. \tag{4}$$

It has the obvious homogeneous solution

$$\theta_S(\xi) \equiv \theta_H = (\beta-1)^{-1/(\beta-1)},\ \xi \geq 0. \tag{5}$$

We shall be interested in its non-trivial solutions, satisfying the boundary conditions

$$\theta_S'(0) = 0,\ \theta_S(\infty) = 0. \tag{6}$$

A formal asymptotic analysis of equation (4) as $\xi \to \infty$ (that is, $\theta_S \to 0$) yields the possible asymptotics of the problem (4), (6): a power law one,

$$\theta_S(\xi) = C\xi^{-2/(\beta-1)} + \dots,\ \xi \to \infty;\ C > 0. \tag{7}$$

[5]For $N > 1$; for $N = 1$ it is at least two-dimensional.

or an "exponential" one:

$$\theta_S(\xi) = D\xi^{2/(\beta-1)-N}\exp\{-\xi^2/4\} + \dots, \xi \to \infty; \ D > 0. \tag{8}$$

Actually (8) is the limiting case of (7) for $C = 0$.

*1.1. The set of similarity functions $\{\theta_S\}$ in the cases $\beta \geq 1 + 2/N$ and $\beta < 1 + 2/N$.* The sets of the functions $\{\theta_S\}$ in these ranges of the parameter $\beta$ are significantly different, which eventually leads to differences in the asymptotic behaviour of solutions of the Cauchy problem (1), (2) for $\beta \geq 1 + 2/N$ and $\beta < 1 + 2/N$.

**Proposition 21.** *Let $\beta \geq 1+2/N$. Then there exists an infinite number of solutions of the problem (4), (6) with power law asymptotics (7) and there are no solutions with the exponential asymptotics (8).*

*In the case $\beta \in (1, 1 + 2/N)$ there is an infinite collection of functions $\theta_S(\xi)$ of the form (7) and at least one solution $\theta_S$ of exponential form (8).*

*Proof.* It is based on constructing super- and subsolutions, $\theta_+$ and $\theta_-$, of the problem (4), (6). We shall first seek them in the form

$$\theta_\pm(\xi) = A_\pm(a_\pm^2 + \xi^2)^{-1/(\beta-1)}, \ \xi \geq 0.$$

It is not hard to check that

$$\mathbf{A}_R(\theta_\pm) \equiv A_\pm(a_\pm^2 + \xi^2)^{-(2\beta-1)/(\beta-1)} \times$$
$$\times \left\{\left(\frac{a_\pm^2 - 2N}{\beta - 1} - A_\pm^{\beta-1}\right)a_\pm^2 + \left[\frac{a_\pm^2 - 2N}{\beta - 1} + \frac{4\beta}{(\beta-1)^2} - A_\pm^{\beta-1}\right]\xi^2\right\},$$

and therefore $\mathbf{A}_R(\theta_+) \leq 0$ in $\mathbf{R}_+$ (that is, $\theta_+$ is a supersolution), if

$$A_+^{\beta-1} \geq \frac{a_+^2 - 2N}{\beta - 1} + \frac{4\beta}{(\beta-1)^2}. \tag{9}$$

Similarly, $\mathbf{A}_R(\theta_-) \geq 0$ in $\mathbf{R}_+$ ($\theta_-$ is a subsolution) in the case

$$a_-^2 > 2N, \ A_-^{\beta-1} \leq \frac{a_-^2 - 2N}{\beta - 1}. \tag{10}$$

Varying the constants $a_\pm$, $A_\pm > 0$ which satisfy (9), (10), we can find an infinite number of distinct pairs of functions $\theta_+ \geq \theta_-$ in $\mathbf{R}_+$. Then, using a well-known principle in the theory of semilinear elliptic equations (see e.g. [356, 357, 378]), to each pair $\{\theta_+, \theta_-\}$ corresponds at least one positive solution $\theta_S(\xi)$ of the problem (4), (6), such that, moreover, $\theta_- \leq \theta_S(\xi) \leq \theta_+$ in $\mathbf{R}_+$.

Let us try now to find $\theta_+$ of exponential type:

$$\theta_\pm(\xi) = A_\pm \exp\{-\alpha_\pm \xi^2\}, \ \xi \geq 0. \tag{11}$$

Then

$$\mathbf{A}_R(\theta_\pm) \equiv A_\pm \exp\{-\alpha_\pm \xi^2\} \times$$
$$\times \left[\alpha_\pm(4\alpha_\pm - 1)\xi^2 + \frac{1}{\beta - 1} - 2N\alpha_\pm - A_\pm^\beta \exp\{-\alpha_\pm \xi^2(\beta - 1)\}\right],$$

which gives us the following restrictions on the values of the constants $\alpha_\pm$, $A_\pm > 0$:

$$\alpha_+ < 1/4,$$

$$A_+^{\beta-1} \geq \left(\frac{1}{\beta - 1} - 2N\alpha_+\right) \exp\left\{\frac{\beta - 1}{1 - 4\alpha_+}\left(\frac{1}{\beta - 1} - 2N\alpha_+\right)\right\}, \tag{12}$$

$$\alpha_- = 1/4, A_-^{\beta-1} \leq 1/(\beta - 1) - N/2. \tag{13}$$

From the last inequality it follows that the subsolution of the form (11) exists only if $\beta < 1 + 2/N$. In this case we can always pick, without violating (12), (13), the constants $\alpha_\pm$, $A_\pm$ so that $\theta_+ \geq \theta_-$ in $\mathbf{R}_+$, which proves the existence of a function $\theta_S(\xi)$ with exponential asymptotics for $\beta < 1 + 2/N$. □

Non-existence of such a solution for $\beta \geq 1+2/N$ is established by the following lemma, which will play an important part below. In preparation, let us formulate a family of Cauchy problems for equation (4):

$$\mathbf{A}_R(\theta) = 0, \ \xi > 0; \ \theta'(0) = 0, \theta(0) = \mu, \tag{14}$$

where $\mu \in (0, \theta_H)$ is an arbitrary parameter (clearly, $\theta = \theta(\xi; \mu) \to \infty$ as $\xi \to \infty$ in the case $\mu > \theta_H$). Naturally, if $\theta = \theta(\xi; \mu) > 0$ in $\mathbf{R}_+$ for some $\mu$ and $\theta(\infty; \mu) = 0$, then the solution $\theta(\xi; \mu)$ defines the required similarity function $\theta_S(\xi)$. We have the following

**Lemma 3.** *Let $\beta \geq 1 + 2/N$. Then $\theta(\xi; \mu) > 0$ in $\mathbf{R}_+$ for all $\mu \in (0, \theta_H)$, and $\theta$ cannot have exponential asymptotics.*

*Proof.* Let us multiply equation (14) by $\xi^{N-1}$ and integrate it over the interval $(0, \xi)$:

$$\xi^{N-1}\theta'(\xi) + \frac{1}{2}\theta(\xi)\xi^N = \int_0^\xi \eta^{N-1}\theta(\eta)\left[\frac{N}{2} - \frac{1}{\beta - 1} + \theta^{\beta-1}(\eta)\right] d\eta. \tag{15}$$

For $\beta \geq 1 + 2/N$ $(N/2 - 1/(\beta - 1) \geq 0)$ the right-hand side of this equality is strictly positive. If on the other hand $\theta(\xi_*) = 0$ ($\theta > 0$ on $(0, \xi_*)$, $\xi_* < \infty$), then,

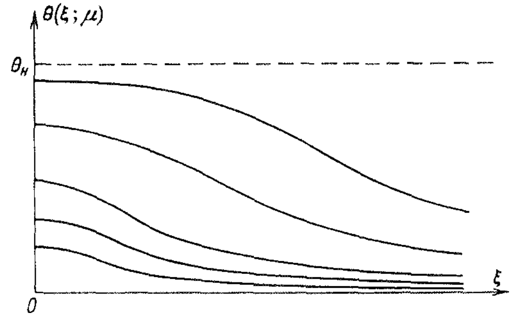


**Fig. 15.** The case $\beta \geq 1 + 2/N$

obviously, $\theta'(\xi_*) \leq 0$ and the left-hand side is non-positive for $\xi = \xi_*$, so that (15) cannot hold.

The second assertion of the lemma also follows from (15). If $\theta$ has exponential asymptotics (it is easily shown that in that case the derivative $\theta'$ will have the same property), then, setting $\xi = \infty$ in (15), we arrive at a contradiction in a similar way. □

Figure 15 shows schematically the functions $\theta = \theta(\xi;\mu)$ in the case $\beta \geq 1 + 2/N$ for different values of $\mu \in (0, \theta_H)$. From (15) it follows that for $\beta \geq 1+2/N$ the function $\theta(\xi;\mu)$ is monotone increasing in $\mu$ for $\xi \in \mathbf{R}_+$, so that different curves in Figure 15 cannot intersect.

In the case $\beta < 1 + 2/N$ the functions $\theta(\xi;\mu)$ have a more varied behaviour.

**Lemma 4.** *Let $\beta \in (1, 1 + 2/N)$. Then there exists a value $\mu_1 \in (0, \theta_H)$, such that for all $\mu \in (0, \mu_1)$ the solution of the problem (14) vanishes at some point $\xi = \xi_\mu < \infty$. For $\mu \in [\mu_1, \theta_H)$ there exists at least one positive solution $\theta$ with exponential asymptotics and an infinite number of solutions satisfying (7).*

*Proof.* The second assertion has already been proved. The first one is established by "linearizing" equation (14) around the trivial solution $\theta \equiv 0$. Setting $f_\mu(\xi) = \theta(\xi;\mu)/\mu$, we obtain for the new function $f_\mu$ the equation

$$\mathbf{F}_R(f_\mu) \equiv \xi^{1-N}(\xi^{N-1}f'_\mu)' + \frac{1}{2}f'_\mu\xi + \frac{1}{\beta-1}f_\mu = \mu^{\beta-1}f_\mu^\beta \tag{16}$$

(it is clear that $f_\mu(0) = 1$, $f'_\mu(0) = 0$) with a small parameter $\mu^{\beta-1}$ multiplying the nonlinear term. The corresponding linear problem for $\mu = 0$ has the form

$$\mathbf{F}_R(f_0) = 0,\ \xi > 0;\ f_0(0) = 1,\ f'_0(0) = 0.$$

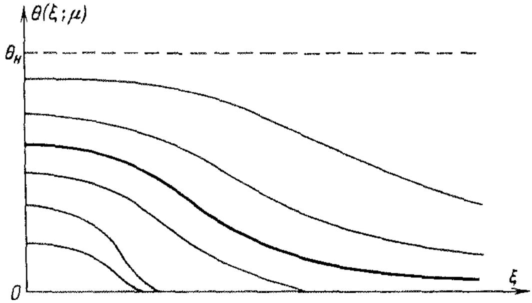


**Fig. 16.** The case $\beta \in (1, 1+2/N)$. The thick line denotes the function $\theta_S(\xi) = \theta(\xi;\mu_*)$ with "exponential" asymptotics

Using the change of variable $\xi = 2(-\eta)^{1/2}$, $\eta < 0$, it reduces to the boundary value problem for the degenerate hypergeometric equation:

$$\eta f_0'' + \left(\frac{N}{2} - \eta\right) f_0' - \frac{1}{\beta - 1} f_0 = 0,\ \ \eta < 0;\ \ f_0(0) = 1,$$

all solutions of which for $\beta < 1 + 2/N$ vanish at some point (see, for example [35]). By continuous dependence of solutions of equation (16) on $\mu^{\beta-1}$ this is also true for all sufficiently small $\mu \in (0, \mu_1)$. □

Figure 16 shows schematically the behaviour of solutions of (14) for different $\mu \in (0, \theta_H)$ in the case $\beta \in (1, 1+2/N)$.

To conclude, let us write down the solution of problem (4), (6), for the case $\beta = 2$, which has the explicit form

$$\theta_S(\xi) = \frac{A_N}{(a_N + \xi^2)^2} + \frac{B_N}{a_N + \xi^2} > 0,\ \xi \geq 0, \tag{17}$$

$$A_N = 48(-(N+14) - 10(1+N/2)^{1/2}),\ B_N = 24(2 + (1+N/2)^{1/2}),$$

$$a_N = 2(N + 14 + 10(1+N/2)^{1/2}).$$

It has power law asymptotics (7) as $\xi \to \infty$.

## 2 *Stability of self-similar solutions*

The main problem consists in determining in the space of initial functions the domains of attraction, corresponding to each eigenfunction of the nonlinear problem under consideration. In subsection 1.1 the similarity functions $\theta_S(\xi) > 0$ were ordered by introducing the parameter $\mu = \theta_S(0) \in (0, \theta_H)$. We shall denote the attracting set corresponding to a self-similar solution (3) by $\mathcal{W}_\mu$, and the solution

$u_S$ itself, by $u_S(t, x; T)$. By asymptotic stability we mean, as usual, convergence of the similarity representation of the solution of the original Cauchy problem (1), (2),

$$\theta_T(t, \xi) = (T + t)^{1/(\beta-1)} u(t, \xi(T + t)^{1/2}), \ t > 0, \xi \in \mathbf{R}^N. \tag{18}$$

to the corresponding function $\theta_S = \theta(|\xi|; \mu)$. The quantity $T \geq 0$ is conveniently determined from the form of the initial function $u_0 \in \mathcal{W}_\mu$. It is clear that representation (18) of the self-similar solution (3) gives us precisely the function $\theta_S(\xi)$.

The question of asymptotic stability of self-similar solutions is very easily solved in the case $\beta > 1 + 2/N$.

**Proposition 22.** *For $\beta > 1 + 2/N$ the attracting set $\mathcal{W}_\mu$ corresponding to a given self-similar solution* (3) *has the form*

$$\mathcal{W}_\mu = \{u_0 \geq 0 \mid \exists T > 0 : u_0(x) - T^{-1/(\beta-1)}\theta_S(|x|/T^{1/2}) = o(|x|^{-2/(\beta-1)}), |x| \to \infty \tag{19}$$

From Proposition 21 it follows that this attracting set is defined in an optimal fashion.

*Proof.* Let $u_0 \in \mathcal{W}_\mu$. Let us set $w_0^+(x) = \max\{u_0(x), u_S(0, x; T)\}$, $w_0^-(x) = \min\{u_0(x), u_S(0, x; T)\}$, and let us denote by $w^\pm(t, x)$ solutions of equation (1), $w^\pm(0, x) = w_0^\pm(x)$ in $\mathbf{R}^N$. Clearly, $w^+ \geq u_S$, $w^- \leq u_S$, $w^- \leq u \leq w^+$ in $\mathbf{R}_+ \times \mathbf{R}^N$. The function $z^+ = w^+ - u_S \geq 0$ is such that

$$z_t^+ = \Delta z^+ - (w^+)^\beta + u_S^\beta \leq \Delta z^+, \ t > 0, x \in \mathbf{R}^N. \tag{20}$$

and therefore

$$z^+(t, x) \leq \frac{1}{(4\pi t)^{N/2}} \int_{\mathbf{R}^N} \exp\left\{-\frac{|y|^2}{4t}\right\} z_0^+(x + y)\, dy. \tag{21}$$

By condition (19), $z_0^+(x) \leq \phi(|x|)$ in $\mathbf{R}^N$, where $\phi(|x|) = o(|x|^{-2/(\beta-1)})$ as $|x| \to \infty$. Then we obtain from (21)

$$\sup_{x \in \mathbf{R}^N} z^+(t, x) = O\left(t^{-N/2} \int_0^\infty y^{N-1} \exp\left\{-\frac{y^2}{4t}\right\} \phi(y)\, dy\right).$$

Deriving a similar estimate for $z^- = u_S - w^-$, we have

$$\|\theta_T - \theta_S\|_{C(\mathbf{R}^N)} = O\left(t^{1/(\beta-1)} \int_0^\infty \eta^{N-1} \exp\left\{-\frac{\eta^2}{4}\right\} \phi(t^{1/2}\eta)\, d\eta\right). \tag{22}$$

and it is not hard to see that the right-hand side goes to zero as $t \to \infty$. □

A simpler estimate of the rate of convergence can be obtained under different restrictions on $u_0$. For example, from inequalities of the form (21) it follows that in the case

$$u_0(x) - T^{-1/(\beta-1)}\theta_S(|x|/T^{1/2}) \in L^1(\mathbf{R}^N)$$

we have the estimate

$$\|\theta_T(t,\cdot) - \theta_S(|\cdot|)\|_{C(\mathbf{R}^N)} = O(t^{-N/2+1/(\beta-1)}) \to 0,\ t \to \infty. \tag{23}$$

It is interesting to note that from (19) we can deduce uniqueness of the similarity function $\theta_S$ with a fixed principal term in power type asymptotics (7).

As can be seen from the estimate (22), this method of proof does not work for $\beta \leq 1 + 2/N$. In this case (as, in fact, for any $\beta > 1$) the question of asymptotic stability can be partially resolved by using information about super- and subsolutions, $\theta_+$ and $\theta_-$, of equation (4).

Let us write down the equation for the similarity representation $\theta_T = \theta_T(\tau, \xi)$ in a new time variable $\tau = \ln(1 + t/T)$:

$$\frac{\partial \theta_T}{\partial \tau} = \mathbf{A}(\theta_T),\ \tau > 0,\ \xi \in \mathbf{R}^N, \tag{24}$$

$$\theta_T(0,\xi) = T^{1/(\beta-1)}u_0(T^{1/2}\xi),\ \xi \in \mathbf{R}^N. \tag{25}$$

Here $\mathbf{A}$ is the stationary operator

$$\mathbf{A}(\theta) \equiv \Delta_\xi \theta + \frac{1}{2}\sum_{i=1}^{N} \frac{\partial \theta}{\partial \xi_i}\xi_i + \frac{1}{\beta-1}\theta - \theta^\beta. \tag{26}$$

All the functions $\theta_S = \theta_S(|\xi|)$ satisfy the equation $\mathbf{A}(\theta_S) = 0$ in $\mathbf{R}^N$. Therefore it is necessary to study asymptotic stability of stationary solutions of equation (24). An important part is played by

**Lemma 5.** *Let $\theta_+$ ($\theta_-$) be some supersolution (subsolution) of equation* (4), *that is $\mathbf{A}_R(\theta_+) \leq 0$ ($\mathbf{A}_R(\theta_-) \geq 0$) in $\mathbf{R}^N$. Then the solution of equation* (24) *with initial function $\theta_T(0,\xi) = \theta_+(|\xi|)$ ($\theta_T(0,\xi) = \theta_-(|\xi|)$) is non-increasing (non-decreasing) in $\tau$:*

$$\partial\theta_T/\partial\tau \leq 0\ \ (\partial\theta_T/\partial\tau \geq 0),\ \tau > 0,\ \xi \in \mathbf{R}^N.$$

*Proof.* The proof is based on the Maximum Principle. Indeed, the function $z = (\theta_T)_\tau$ satisfies a linear parabolic equation, $z \leq 0$ in $\mathbf{R}^N$ for $\tau = 0$ and so on. □

Let us note that the subsolution $\theta_-$ in Lemma 5 does not have to be smooth; it is sufficient, for example to have $\theta_- \in C^2$ wherever it is positive. Therefore, if the radially symmetric function

$$\theta_T(0,\xi) = T^{1/(\beta-1)}u_0(T^{1/2}\xi),\ \xi \in \mathbf{R}^N,$$

is a super- or subsolution of equation (24), then $\theta_T(\tau, \xi)$ is monotone in $\tau$ and bounded; therefore by a standard monotone argument for semilinear parabolic equations (see [22, 357, 378]) there exists a solution $\theta_S = \theta_S(|\xi|)$ of the problem (4), (6), such that $\theta_T(\tau, \xi) \to \theta_S(|\xi|)$ in $\mathbf{R}^N$ as $\tau \to \infty$.

In the general case the problem of determining attracting sets is related to the problem of uniqueness classes of similarity functions $\theta_S = \theta_S(|\xi|)$. Namely, we have

**Proposition 23.** *Let $\theta_+, \theta_- \in C^2(\mathbf{R}^N)$ be, respectively, radially symmetric super- and subsolutions of equation* (14), *to which corresponds the same similarity function $\theta_S = \theta(|\xi|; \mu)$, $\theta_- \leq \theta_S \leq \theta_+$ in $\mathbf{R}^N$. Then the set*

$$\mathcal{H}_\mu = \{u_0 \geq 0 \mid \exists T > 0 : \ \theta_-(|\xi|) \leq T^{1/(\beta-1)} u_0(T^{1/2}\xi) \leq \theta_+(|\xi|) \ in \ \mathbf{R}^N\}$$

*is contained in $\mathcal{W}_\mu$.*

In the case $\beta < 1 + 2/N$ it is important to note the following asymptotic property of solutions.

**Lemma 6.** *Let $\beta \in (1, 1 + 2/N)$. If $u_0 \not\equiv 0$, then for some $T > 0$*

$$\varliminf_{\tau \to \infty} \theta_T(\tau, \xi) \not\equiv 0. \tag{27}$$

*Proof.* Without loss of generality we shall assume that $u_0(0) > 0$. Then it follows from Lemma 4 that there exist sufficiently small $T > 0$, $\mu > 0$, such that

$$u_0(x) \geq T^{-1/(\beta-1)} \theta(|x|/T^{1/2}; \mu), \ |x| < T^{1/2} \text{meas supp}\, \theta(|\xi|; \mu), \tag{28}$$

where $\theta(|\xi|; \mu)$ is a solution of (14). Therefore by the Maximum Principle (the function $\theta$ in (28) is a subsolution of equation (24))

$$u(t, x) \geq (T + t)^{-1/(\beta-1)} \theta(|x|/(T + t)^{1/2}; \mu),$$

$$t > 0, \ |x| < (T + t)^{1/2} \text{meas supp}\, \theta(|\xi|; \mu);$$

and consequently

$$\theta_T(\tau, \xi) \geq \theta(|\xi|; \mu) > 0, \ |\xi| < \text{meas supp}\, \theta(|\xi|; \mu),$$

which ensures that (27) holds. □

This is an important result. Practically, (27) shows that for $\beta \in (1, 1+2/N)$ the asymptotic behaviour of all small solutions is described precisely by self-similar solutions of the form (3). We shall show below that the situation for $\beta \geq 1 + 2/N$ is different. Let us indicate another interesting property which follows from the method of proof of Lemma 6.

**Proposition 24.** *For $\beta \in (1, 1+2/N)$ among solutions $\theta_S(|\xi|)$ of the problem* (4), (6) *there is a solution $\theta_S^*$ having the exponential asymptotics* (8), *which is minimal among all (including radially non-symmetric) solutions of the elliptic equation $\mathbf{A}(\theta) = 0$ in $\mathbf{R}^N$. The attracting set $\mathcal{W}_{\mu_*}$, $\mu_* = \theta_S^*(0)$, contains all sufficiently small initial functions $u_0$, and, in particular, the set*

$$\mathcal{H}_{\mu_*} = \{u_0 \geq 0, u_0 \not\equiv 0 \mid \exists T > 0 : u_0(x) \leq T^{-1/(\beta-1)}\theta_S^*(|x|/T^{1/2}) \text{ in } \mathbf{R}^N\}.$$

*Proof.* From Lemma 5 it follows that the solution of the Cauchy problem (24), (25) with the initial function $\theta_T(0, \xi) = \theta(|\xi|; \mu)$, where $\theta(|\xi|; \mu)$ is given in the proof of Lemma 6, is critical, that is $(\theta_T(\tau, \xi))_\tau \geq 0$ in $\mathbf{R}_+ \times \mathbf{R}^N$. Since $\theta_T(\tau, \xi)$ is bounded from above (for example, by a function $\theta_S(\xi)$ with power law asymptotics), the limit $\lim_{\tau \to \infty} \theta_T(\tau, \xi) = \theta_S^*(\xi)$ exists, and, obviously, $\theta_S^*$ has the exponential asymptotics (8). Monotone stabilization $\theta_T \to \theta_S^*(\xi)$ as $\tau \to \infty$ ensures that $\theta_S^*$ is minimal among all solutions of the equation $\mathbf{A}(\theta) = 0$ in $\mathbf{R}^N$, as well as the inclusion $\mathcal{H}_{\mu_*} \subset \mathcal{W}_{\mu_*}$. This follows from the fact that we can provide the estimate (28) for any $u_0(x) \not\equiv 0$. □

## 2 Non-self-similar eigenfunctions (approximate self-similar solutions)

Self-similar solutions with spatio-temporal behaviour (3) do not exhaust the set of elements of the attractor of equation (1). The remaining elements of the attractor are a.s.s., which do not satisfy equation (1), unlike the exact solutions (3).

### *1 Conditions of asymptotic degeneracy of the absorption process for $\beta > 1 + 2/N$*

Let us return to Proposition 22. Setting in (19) $\theta_S \equiv 0$, we obtain the attracting set

$$\mathcal{W}_0 = \{u_0 \geq 0 \mid u_0(x) = o(|x|^{-2/(\beta-1)}), |x| \to \infty\}. \tag{29}$$

If $u_0 \in \mathcal{W}_0$, then $\theta_T(t, \xi) \to 0$ uniformly in $\xi \in \mathbf{R}^N$ as $t \to \infty$. It is interesting to note that this simultaneously proves the known assertion (see Lemma 3) that for $\beta > 1 + 2/N$ there are no functions $\theta_S$ with exponential asymptotics.

Therefore in the set $\mathcal{W}_0$ the solution $u(t, x)$ does not evolves according to self-similar rules. The asymptotic behaviour of $u(t, x)$ for $u_0 \in \mathcal{W}_0$ is determined by self-similar solutions of the heat equation without a sink:

$$u_t = \Delta u, \quad t > 0, x \in \mathbf{R}^N. \tag{30}$$

If $u_0 \in \mathcal{W}_0$, then the sink $-u^\beta$ becomes negligible as $t \to \infty$ in comparison with the diffusion term.

We shall need the following self-similar solutions of equation (30):

$$u_s(t, x; T) = (T + t)^{-\gamma} f_s(\eta), \quad \eta = |x|/(T + t)^{1/2}. \tag{31}$$

Here $\gamma > 0$ is a parameter; the function $f_s > 0$ solves the problem

$$\begin{gathered} \eta^{1-N}(\eta^{N-1} f_s')' + \frac{1}{2} f_s' \eta + \gamma f_s = 0, \quad \eta > 0, \\ f_s'(0) = 0, \ f_s(\infty) = 0. \end{gathered} \tag{32}$$

It is well known [35, 317] that for $\gamma \in (0, N/2)$ this problem has a solution with power law asymptotics:

$$f_s(\eta) = M\eta^{-2\gamma} + \ldots, \ \eta \to \infty; \ M = \text{const} > 0. \tag{33}$$

If $\gamma = N/2$, then the only appropriate solution is

$$f_s(\eta) = f_s^*(\eta) = M \exp\{-\eta^2/4\}, \quad \eta \geq 0; \ M > 0. \tag{34}$$

For $\gamma > N/2$ (32) has no positive solutions.

Let $u_0 \in W_0$. It turns out that self-similar solutions of the form (33), (34) determine the asymptotic behaviour of almost all solutions $u(t, x)$ in the cases $u_0 \notin L^1(\mathbf{R}^N)$ and $u_0 \in L^1(\mathbf{R}^N)$, respectively.

Let us consider first the case $u_0 \notin L^1(\mathbf{R}^N)$, to which correspond the values $\gamma < N/2$. Let us introduce the similarity representation of the solution of the original problem (1), (2):

$$f_T(t, \eta) = (T + t)^{\gamma} u(t, \eta(T + t)^{1/2}), \ t > 0, \ \eta \in \mathbf{R}^N. \tag{35}$$

**Proposition 25.** *Let $\beta > 1 + 2/N$ and*

$$\beta(N - 2) \leq N, \tag{36}$$

*that is, $\beta \in (1 + 2/N, \infty)$ for $N = 1$ or $N = 2$ and $\beta \in (1 + 2/N, N/(N - 2))$ for $N \geq 3$. Let there exist $\gamma \in (1/(\beta - 1), N/2)$ and positive constants $T, M, A$, such that*

$$u_0(\cdot) - u_s(0, \cdot; T) \in L^1(\mathbf{R}^N), \tag{37}$$

$$u_0(x) \leq A u_s(0, x; T), \quad x \in \mathbf{R}^N. \tag{37'}$$

*Then $\|f_T(t, \cdot) - f_s(\cdot)\|_{L^1(\mathbf{R}^N)} \to 0$ as $t \to \infty$.*

*Proof.* Let us set $z = u_s - u$. Then

$$z_t = \Delta z + u^{\beta}, \ t > 0, \quad x \in \mathbf{R}^N; \ z(0, x) \in L^1(\mathbf{R}^N). \tag{38}$$

Let $z^+ = \max\{0, z\} \geq 0$, $z^- = -\min\{0, z\} \geq 0$ in $\mathbf{R}_+ \times \mathbf{R}^N$. Clearly

$$\|z(t)\|_{L^1(\mathbf{R}^N)} = \|z^+(t)\|_{L^1(\mathbf{R}^N)} + \|z^-(t)\|_{L^1(\mathbf{R}^N)},$$

and, furthermore, by (37) $z^\pm(0, \cdot) \in L^1(\mathbf{R}^N)$. From (38) it follows immediately that

$$\frac{d}{dt}\|z^+(t)\|_{L^1(\mathbf{R}^N)} \leq \int_{\mathbf{R}^N} u^\beta(t, x)\, dx, \quad \frac{d}{dt}\|z^-(t)\|_{L^1(\mathbf{R}^N)} \leq 0. \tag{39}$$

Since by (37′) $u \leq Au_s$ in $\mathbf{R}_+ \times \mathbf{R}^N$, from the first estimate (39) we deduce that

$$\frac{d}{dt}\|z^+(t)\|_{L^1(\mathbf{R}^N)} \leq A^\beta \int_{\mathbf{R}^N} u_s^\beta(t, x; T)\, dx =$$
$$= (T + t)^{-\gamma\beta + N/2} A^\beta \|f_s\|^\beta_{L^\beta(\mathbf{R}^N)}, \quad t > 0.$$

It is easy to check that for $\gamma > 1/(\beta - 1)$ and under condition (36), we have the inclusion $f_s \in L^\beta(\mathbf{R}^N)$ (see (33)), and therefore

$$\frac{d}{dt}\|z^+(t)\|_{L^1(\mathbf{R}^N)} \leq \text{const} \cdot (T + t)^{N/2 - \gamma\beta}, \quad t > 0.$$

The second estimate in (39) means that $\|z^-(t)\|_{L^1(\mathbf{R}^N)} \leq \text{const}$ for $t > 0$. Taking now into account the fact that

$$\|z(t)\|_{L^1(\mathbf{R}^N)} \equiv (T + t)^{-\gamma + N/2}\|f_T(t, \cdot) - f_S(\cdot)\|_{L^1(\mathbf{R}^N)},$$

we obtain now an estimate of the rate of convergence:

$$\|f_T(t, \cdot) - f_s(\cdot)\|_{L^1(\mathbf{R}^N)} = \begin{cases} O(t^{1-\gamma(\beta-1)}), & \gamma < (N + 2)/(2\beta), \\ O(t^{\gamma-N/2}\ln t), & \gamma = (N + 2)/(2\beta), \\ O(t^{\gamma-N/2}), & \gamma > (N + 2)/(2\beta). \end{cases}$$

Therefore if $\gamma \in (1/(\beta - 1), N/2)$, $f_t \to f_s$ as $t \to \infty$ in $L^1(\mathbf{R}^N)$. □

The restriction (36) in fact is not significant, and we can get rid of it by analyzing the stabilization $f_T \to f_s$ as $t \to \infty$ in the norm of $L^{p+1}(\mathbf{R}^N)$, where $p > 0$ is a constant (above we considered the case $p = 0$).

Next we shall touch on the case $u_0 \in \mathcal{W}_0 \cap L^1(\mathbf{R}^N)$ if $\gamma = N/2$. It is not hard to derive the equation for the similarity representation (35) for $\gamma = N/2$ in the new time $\tau = \ln(1 + t/T)$:

$$\frac{\partial f_T}{\partial \tau} = \mathbf{B}(f_T) - (Te^\tau)^{1-N(\beta-1)/2} f_T^\beta, \quad \tau > 0, \eta \in \mathbf{R}^N, \tag{40}$$

$$\mathbf{B}(f_T) \equiv \Delta_\eta f_T + \frac{1}{2}\sum_{i=1}^N \frac{\partial f_T}{\partial \eta_i}\eta_i + \frac{N}{2} f_T.$$

We shall assume that

$$f_T(0,\eta) \equiv T^{N/2}u_0(|\eta|T^{1/2}) \le C\exp\{-|\eta|^2/4\},\quad \eta\in\mathbf{R}^N,$$

where $C>0$ is a constant.

From (40), invoking the Maximum Principle, we obtain

$$f_T(\tau,\eta)\le C\exp\{-|\eta|^2/4\}\text{ in }\mathbf{R}_+\times\mathbf{R}^N,$$

so that $f_T$ is uniformly bounded. The differential operator in (40) is easily reduced to divergence form by multiplying both sides of the equation by $\exp\{|\eta|^2/4\}$:

$$e^{|\eta|^2/4}\frac{\partial f_T}{\partial\tau} = \nabla_\eta\cdot\left(e^{|\eta|^2/4}\nabla_\eta f_T\right)+$$

$$+\frac{N}{2}f_T e^{|\eta|^2/4}-(Te^\tau)^{1-(\beta-1)N/2}f_T^\beta e^{|\eta|^2/4},$$

from which, after taking the scalar product in $L^2(\mathbf{R}^N)$ with $(f_T)_\tau$, we easily obtain the important estimate

$$\int_1^\infty\left\|e^{|\cdot|^2/8}\frac{\partial}{\partial\tau}f_T(\tau,\cdot)\right\|^2_{L^2(\mathbf{R}^N)}d\tau<\infty.$$

It allows us to prove that every partial limit $f_T(\tau,\eta)\to f_*(\eta)$ as $\tau\to\infty$ in $L^2(\mathbf{R}^N)$ is a solution of the stationary equation $\mathbf{B}(f_*)=0$ in $\mathbf{R}^N$. As $f_T\le C\exp\{-|\eta|^2/4\}$ in $\mathbf{R}_+\times\mathbf{R}^N$, we have that $f_*=f_s=M\exp\{-|\eta|^2/4\}$ (see (34)).

The fact that the limit $f_T(\tau,\eta)\to f_*(\eta)$ for $\tau=\tau_i\to\infty$ does not depend on the choice of the sequence $\{\tau_i\}$ follows from a kind of monotonicity of the solution:

$$\frac{d}{d\tau}\|f_T(\tau,\cdot)\|_{L^1(\mathbf{R}^N)}\le 0,\quad \tau>0$$

(here we have used the fact that the functions $f_s\equiv M\exp\{-|\eta|^2/4\}$ are monotone in $M$ in $\mathbf{R}^N$).

It remains to show that $f_s\not\equiv 0$ (that is, $M>0$). This is easily done by constructing a special subsolution of equation (1):

$$u_-(t,x)=\psi(t)(T+t)^{-N/2}\exp\{-|x|^2/[4(T+t)]\}.$$

It is not hard to check that $(u_-)_t\le\Delta u_- - u_-^\beta$ in $\mathbf{R}_+\times\mathbf{R}^N$, if

$$\psi'(t)\le-\psi^\beta(t)(T+t)^{-N(\beta-1)/2},\quad t>0.$$

For $\beta>1+2/N$ we can take as $\psi(t)$ the function

$$\psi(t)=\psi_0+A(T+t)^{-\epsilon},\quad t>0;\ \psi_0>0,\ A>0,$$

where $\epsilon=N(\beta-1)/2-1$ and $\epsilon A>(\psi_0+AT^{-\epsilon})^\beta$. Therefore, under the appropriate restrictions on $u_0(x)$, we have $u\ge u_-$ in $\mathbf{R}_+\times\mathbf{R}^N$, that is, $f_T(\tau,\eta)>\psi_0\exp\{-|\eta|^2/4\}$ for any $\tau>0$, $\eta\in\mathbf{R}^N$.

*2 Conditions for asymptotic degeneration of the diffusion process*

Let us summarize briefly the above results. Let

$$u_0(x) \sim |x|^{-\alpha}, \ |x| \to \infty; \ \alpha = \text{const} > 0. \tag{41}$$

Then above we considered the case $\alpha \geq 2/(\beta - 1)$; $\alpha = 2/(\beta - 1)$ is the "resonance" case, while for $\alpha > 2/(\beta - 1)$, $\beta > 1 + 2/N$, the absorption process degenerates. Thus it remains to consider the case $\alpha < 2/(\beta - 1)$ in (41).

It turns out that in this case as $t \to \infty$ diffusion degenerates, and as a result the asymptotic behaviour of $u(t, x)$ is expected to be described by self-similar solutions of the first order equation

$$u_t = -u^{\beta}, \ t > 0, \ x \in \mathbf{R}^N, \tag{42}$$

which can be conveniently written in the form

$$\begin{aligned} u_s(t, x) &= (T + t)^{-1/(\beta-1)} f_s(\xi), \\ \xi &= x/(T + t)^{1/[\alpha(\beta-1)]} \in \mathbf{R}^N; \ T = \text{const} > 0. \end{aligned} \tag{43}$$

Substitution of (43) into (42) gives us the following equation for $f_s \geq 0$:

$$\begin{aligned} &\frac{1}{\alpha(\beta - 1)} \sum_{i=1}^{N} \frac{\partial f_s}{\partial \xi_i} \xi_i + \frac{1}{\beta - 1} f_s - f_s^{\beta} = 0, \ \xi \in \mathbf{R}^N, \\ &f_s(\xi) \to 0, \ |\xi| \to \infty. \end{aligned} \tag{44}$$

It is easily integrated, and the general solution of the problem (44) has the following form:

$$f_s(\xi) = \{(\beta - 1) + G^{1-\beta}(\xi/|\xi|)|\xi|^{\alpha(\beta-1)}\}^{-1/(\beta-1)}, \ \xi \in \mathbf{R}^N, \tag{45}$$

where $G(\omega) \geq 0$ is a sufficiently smooth function defined on the unit sphere $S = \{\omega \in \mathbf{R}^N \mid |\omega| = 1\}$. Let us note that in the general case the functions (45) are not radially symmetric.

Let us move on now to determine attracting sets corresponding to each of the a.s.s. (43). Below we denote by

$$f_T(t, \xi) = (T + t)^{1/(\beta-1)} u(t, \xi(T + t)^{1/[\alpha(\beta-1)]}) \tag{46}$$

the similarity representation of $u(t, x)$ defined using (43).

**Proposition 26.** *Let*

$$\frac{(2 - N)_+}{\beta - 1} < \alpha < \min\left\{\frac{2}{\beta - 1}, N\right\} \tag{47}$$

*and, moreover, assume that in* (45)

$$G(\omega) > 0,\ \omega \in S;\ G(\omega) \in C^2(S).$$

*Then there exists $p \geq 0$ such that for any initial function $u_0$, which satisfies for some $T > 0$ the condition $|u_0(\cdot) - u_s(0, \cdot)|^{(p+1)/2} \in H^1(\mathbf{R}^N)$, stabilization $f_T(\cdot) \to f_s(\cdot)$ in $L^{p+1}(\mathbf{R}^N)$ occurs as $t \to \infty$.*

*Proof.* We proceed with a formal analysis. The function $z = u - u_s$ satisfies in $\mathbf{R}_+ \times \mathbf{R}^N$ the equation

$$z_t = \Delta z - zg(t, x) + h(t, x), \tag{48}$$

$$g(t, x) = \beta \int_0^1 (\eta u(t, x) + (1 - \eta) u_s(t, x))^{\beta - 1}\, d\eta > 0,$$

$$h(t, x) = \Delta u_s(t, x).$$

By assumption $z(0, \cdot) \in L^{p+1}(\mathbf{R}^N)$, $\nabla |z(0, \cdot)|^{(p+1)/2} \in L^2(\mathbf{R}^N)$. Let us take the scalar products in $L^2(\mathbf{R}^N)$ of both sides of (48) with $|z|^{p-1}z$. Then using natural assumptions concerning the regularity of the generalized solution of the linear equation (48), we obtain

$$\begin{aligned} \frac{1}{p+1}\frac{d}{dt}\|z\|^{p+1}_{L^{p+1}(\mathbf{R}^N)} &= -\frac{4p}{(p+1)^2}\|\nabla |z|^{(p+1)/2}\|^2_{L^2(\mathbf{R}^N)} + \\ &+ (|z|^{p-1}z, \Delta u_s) - (|z|^{p+1}, g) \leq (|z|^{p-1}z, \Delta u_s). \end{aligned} \tag{49}$$

Let the condition

$$N - (\alpha + 2)(p + 1) < 0,\ N + |\alpha(\beta - 1) - 2|(p + 1) > 0 \tag{50}$$

be satisfied. Then it is easy to check that $\Delta_\xi f_s \in L^{p+1}(\mathbf{R}^N)$. Using the Hölder inequality

$$(|z|^{p-1}z, \Delta u_s) \leq \|z\|^p_{L^{p+1}(\mathbf{R}^N)} \|\Delta u_s\|_{L^{p+1}(\mathbf{R}^N)},$$

as well as the identity

$$\|\Delta u_s(t)\|_{L^{p+1}(\mathbf{R}^N)} \equiv (T + t)^\delta \|\Delta_\xi f_s\|_{L^{p+1}(\mathbf{R}^N)},$$

$$\delta = \frac{N - (\alpha + 2)(p + 1)}{\alpha(\beta - 1)(p + 1)} < 0,$$

which follows from (43), we obtain from (49) the following estimate

$$\frac{d}{dt}\|z(t)\|_{L^{p+1}(\mathbf{R}^N)} \leq (T + t)^\delta m_s,\ t > 0;$$

$$m_s = \|\Delta_\xi f_s\|_{L^{p+1}(\mathbf{R}^N)} < \infty.$$

Taking now into account the fact that

$$\|z(t)\|_{L^{p+1}(\mathbf{R}^N)} \equiv (T+t)^\epsilon \|f_T(t,\cdot) - f_s(\cdot)\|_{L^{p+1}(\mathbf{R}^N)},$$

where

$$\epsilon = \frac{1}{\beta - 1}\left[\frac{N}{\alpha(p+1)} - 1\right],$$

we obtain the final estimate of the rate of convergence:

$$\|f_T(t,\cdot) - f_s(\cdot)\|_{L^{p+1}(\mathbf{R}^N)} = \begin{cases} O(t^{\delta+1-\epsilon}), & \delta > -1, \\ O(t^{-\epsilon}\ln t), & \delta = -1, \\ O(t^{-\epsilon}), & \delta < -1, \end{cases} \tag{51}$$

as $t \to \infty$. One can see that $\delta + 1 - \epsilon < 0$ if $\delta > -1$. Let us require in addition to (50) that $\epsilon > 0$, that is,

$$N - \alpha(p+1) > 0. \tag{52}$$

Then it is not hard to see that (51) guarantees stabilization of $f_T$ to $f_s$ as $t \to \infty$, and that the system of inequalities (50), (52) is compatible under the condition (47). □

### *3 A.s.s. for the critical value of parameter $\beta = 1 + 2/N$, $u_0$ has exponential decay at infinity*

In this case there arises probably the most unusual a.s.s.:

$$u_s(t,x) = |(T+t)\ln(T+t)|^{-N/2} g_*\left(\frac{x}{(T+t)^{1/2}}\right), \; T > 1, \tag{53}$$

where the function $g_*(\xi) > 0$ will be defined below. This a.s.s. corresponds to the similarity representation

$$g_T(t,\xi) \equiv |(T+t)\ln(T+t)|^{N/2} u(t, \xi(T+t)^{1/2}). \tag{54}$$

The fact that for small $u_0(x) > 0$, $u(t,x)$ evolves as $t \to \infty$ according to the spatio-temporal structure (53), follows from the existence of a super- and subsolution of equation (1) of the following form:

$$u_-(t,x) = A|(T+t)\ln(T+t)|^{-N/2} \exp\left\{-\frac{|x|^2}{4(T+t)}\right\}, \tag{55}$$

$$A = \text{const} \in (0, (N/2)^{N/2}),$$

$$u_+(t,x) = H[(T+t)\ln(T+t)]^{-N/2}\exp\left\{-\frac{|x|^2}{4(T+t)[1+a\ln^{-1}(T+t)]^2}\right\}, \quad (56)$$

$a = \text{const} > 0$, $H = H(a) > 0$ can be arbitrarily large. This is easily verified by substituting these functions into (1).

It can be shown that for all sufficiently small $u_0(x)$ (for example, for $u_0 \sim \exp\{-|x|^2\}$, $|x| \to \infty$), after a finite time $t_1 > 0$, the condition $u_- \le u \le u_+$ will hold for $t = t_1$, $x \in \mathbf{R}^N$, and therefore $u_- \le u \le u_+$ for any $t \ge t_1$, $x \in \mathbf{R}^N$. Then from (54) we obtain the estimate

$$A\exp\left\{-\frac{|\xi|^2}{4}\right\} \le g_T(t,\xi) \le H\exp\left\{-\frac{|\xi|^2}{4[1+a\ln^{-1}(T+t)]^2}\right\}. \quad (57)$$

Therefore this is also true for any possible limiting function $g_*(\xi)$ and the estimates have the form

$$A\exp\left\{-|\xi|^2/4\right\} \le g_*(\xi) \le H\exp\left\{-|\xi|^2/4\right\}, \quad \xi \in \mathbf{R}^N. \quad (57')$$

The precise form of $g_*(\xi)$ becomes clear by using the equation satisfied by the similarity representation $g_T = g_T(\tau,\xi)$, with $\tau = \ln(1+t/T)$:

$$\frac{\partial g_T}{\partial \tau} = \Delta_\xi g_T + \frac{1}{2}\sum_{i=1}^{N}\frac{\partial g_T}{\partial \xi_i}\xi_i + \frac{N}{2}g_T + \frac{1}{\tau+\ln T}\left(\frac{N}{2}g_T - g_T^{1+2/N}\right). \quad (58)$$

By uniform boundedness of $g_T(\tau,\xi)$ (see (57)), from the last equation (which can be put in divergence form by multiplying by $\exp\{|\xi|^2/4\}$), it is not hard to deduce boundedness of $\exp\{|\xi|^2/8\}(g_T)_\tau$ in $L^2((1,\infty)\times\mathbf{R}^N)$. This estimate allows us to pass to the limit as $\tau \to \infty$ in (58). As a result we obtain for $g_*(\xi)$ the stationary equation

$$\mathbf{B}(g_*) \equiv \Delta_\xi g_* + \frac{1}{2}\sum_{i=1}^{N}\frac{\partial g_*}{\partial \xi_i}\xi_i + \frac{N}{2}g_* = 0, \quad \xi \in \mathbf{R}^N.$$

Therefore

$$g_*(\xi) = M\exp\{-|\xi|^2/4\}, \ M = \text{const}, \quad (59)$$

and by (57′) $M \in [A, H]$.

Below we only prove uniqueness of the limiting function $g_*(\xi)$.

**Proposition 27.** *Let $\beta = 1 + 2/N$, $u_0 = u_0(|x|) \sim \exp\{-|x|^2\}$ as $|x| \to \infty$. Let $g_T(\tau,\xi) \to g_*(\xi)$ as $\tau \to \infty$ uniformly on every compact set in $\mathbf{R}^N$. Then*

$$g_*(\xi) = (N/2)^{N/2}(1+2/N)^{N^2/4}\exp\{-|\xi|^2/4\}, \quad \xi \in \mathbf{R}^N. \quad (60)$$

*Proof.* By (59) we only have to show that

$$M = (N/2)^{N/2}(1 + 2/N)^{N^2/4}. \tag{61}$$

Integrating (58) over $\mathbf{R}^N$, which can be done by (57), we obtain

$$\begin{aligned} \frac{d}{d\tau}\|g_T(\tau,\cdot)\|_{L^1(\mathbf{R}^N)} &= \frac{1}{\tau + \ln T} G^*(g_T)(\tau) \equiv \\ &\equiv \frac{1}{\tau + \ln T}\left[\frac{N}{2}\|g_T(\tau,\cdot)\|_{L^1(\mathbf{R}^N)} - \|g_T(\tau,\cdot)\|^{1+2/N}_{L^{1+2/N}(\mathbf{R}^N)}\right], \tau > 1. \end{aligned} \tag{62}$$

From (57) we have that $\|g(\tau,\cdot)\|_{L^1(\mathbf{R}^N)}$ is bounded from above uniformly in $\tau$. Therefore it follows from (62) that the integral

$$\int_1^\infty \frac{G^*(g_T)(\tau)}{\tau + \ln T}\, d\tau$$

must converge.

Since under the conditions of the theorem $G^*(g_T)(\tau) \to G^*(g_*)$ as $\tau \to \infty$, we obtain the condition $\|g_*\|_{L^1(\mathbf{R}^N)} > 0$ (see (57′)) and

$$G^*(g_*) \equiv \frac{N}{2}\|g_*\|_{L^1(\mathbf{R}^N)} - \|g_*\|^{1+2/N}_{L^{1+2/N}(\mathbf{R}^N)} = 0,$$

from which follow (61) and (60). □

**Remark.** An elementary analysis of the behaviour of trajectories of the "ordinary differential equation" (62) as $\tau \to \infty$ also allows us to prove the stabilization $g_T(\tau,\xi) \to g_*(\xi)$ in $\mathbf{R}^N$ as $\tau \to \infty$, where $g_*$ is the function (60).

Under the stated conditions a.s.s. (53) satisfies the linear equation

$$\frac{\partial u_s}{\partial t} = \Delta u_s - \frac{N}{2}\frac{u_s}{(T+t)\ln(T+t)}, \quad t > 0,\ x \in \mathbf{R}^N, \tag{63}$$

which differs considerably from the original one.

To conclude, let us again remark on the curious transformations the semilinear parabolic equation (1) can undergo at the asymptotic stage. Depending on the magnitude of $\beta$ and the initial function $u_0(x)$, it is equivalent (in the sense of a.s.s.) to one of three types of equation: linear equation without sink (30), first order equation without diffusion (42), or, finally, equation (63) with a linear sink.

## Remarks and comments on the literature

**§ 1.** Basic material needed to derive the elementary results of subsections 1–3 is contained in the well-known monographs [282, 101, 338] (see also [378, 357, 361, 365]). Proposition 4 is proved in [119]; see also the more general statements of [187] and § 4, Ch. VI.

**§ 2.** Concerning Proposition 5, see [386, 384]. The presentation of subsection 2 follows [187]. Another method of proof of a statement similar to Proposition 6 is contained in [234] (let us note that the method of [234] cannot be used to derive an estimate of the convergence rate of $\theta \to f_S$ as $t \to \infty$, which is of importance in applications).

**§ 3.** Existence of solutions of the problem (5) under quite weak restrictions on the coefficient $k(u)$ has been proved by different methods in [23, 24, 68]. The proof of Proposition 7[6] uses the method of [187]; justification of the transformations used there can be found in [330].

The self-similar solutions (14) of subsection 2.1 were considered first in [32, 29, 30], where existence and uniqueness of solution of the problem (16), (17) were established (subsequently, they were established by a different method in [205]). Asymptotic stability of the solutions (14) with respect to perturbations of the initial function, boundary regime and the equation (the coefficient $u^\sigma \to k(u)$) is proved in [119, 184]; in this context, see Ch. VI, where similar solutions are used to construct families of a.s.s. of nonlinear heat equations with non-power type coefficients. Solvability of the problem (19′) and uniqueness of $f_S$ are established in [28, 29, 30, 32, 205]. Questions related to asymptotic stability of the solution (19) and construction of the corresponding family of a.s.s. of a large class of boundary value problems are considered in [184] (these questions are also briefly discussed in § 3, Ch. VI). The localized self-similar solution (21) is studied in [351, 393, 352]. Its asymptotic stability is proved in [119, 153] (see § 4, Ch. III); the corresponding family of a.s.s. is constructed in [119, 184, 187] (see § 3, 4, Ch. VI). Analysis of self-similar solutions (23) is the subject matter of much of Ch. III, where additional information and the relevant references can be found. The exact solution (24), which is invariant with respect to a Lie group of transformations (see [322]), is constructed from general considerations by the methods of [134], [176], though, of course, it is well known; see [44].

**§ 4.** Existence and uniqueness of the self-similar function $f_S$ in (5) has been established by different methods in [7, 21]. The estimate of the convergence rate (8) for arbitrary initial functions $u_0 \not\equiv 0$ has been obtained in [21]. In subsection 1 we present a different (and, in our opinion, simpler) method of proof of convergence. The boundary value problem with the condition (13) has been thoroughly considered in [10], where solutions of variable sign were also studied. It is of interest, that in that case, in distinction to Proposition 9, it is possible for a solution $u(t, x)$ to stabilize, in a special norm, as $t \to \infty$ to a spatially inhomogeneous function.

[6]Other estimates of the convergence rate are obtained in [360, 326].

**§ 5.** First existence and uniqueness theorems for fast diffusion equations of general form are proved in [343] ($N = 1$), where the total extinction effect was discovered. Similar studies of the multi-dimensional case for $m \in (0, 1)$ include, for example, [19, 43, 53, 328]. In [43] a method of proof of total extinction in the boundary problem (1)–(3) is presented.

The proof of Proposition 10 uses a different approach. Asymptotic stability of the self-similar solution (4) under some restrictions on $\Omega \subset \mathbf{R}^N$ was established in [47]. Let us note that in the course of the argument of [47], conditions of stabilization to an unstable stationary solution of a quasilinear parabolic equation are found (such problems arise in §§ 5, 7, Ch. IV).

**§ 6.** Here we mainly follow [168]. The first result concerning the total extinction phenomenon in the Cauchy problem (1), (2) was obtained in [43] in the case $0 < m < (N-2)/N$, $N \geq 3$, $u_0 \in L^1(\mathbf{R}^N) \cap L^p(\mathbf{R}^N)$, $p > (1-m)/(2N)$. Conditions imposed on $u_0$ in Proposition 11 are weaker. In case $0 < m < (N-2)$ equation (1) is shown [251] to admit a unique self-similar solution (3) with finite mass (this determines the exponent $n > 0$) which is asymptotically stable [166].

Example 2 is taken from [43]. Proposition 12 is proved in [168] by constructing a strictly positive in $\mathbf{R}_+ \times \mathbf{R}^N$ subsolution of the problem. Propositions 11, 12 provide a fairly precise description of the boundary between the sets of $\{u_0\}$ for which there is, or is no, total extinction in finite time. Non-occurrence of total extinction for all $m > (N-2)_+/N$ is proved in [19] (see also [328]).

**§ 7.** The elementary transformations $u \to E(u)$ (Example 3) will be used in § 2, Ch. V, to present the special comparison theory for solutions of two different nonlinear parabolic equations. The substantial simplification of equation (3) in Example 4 (the right-hand side of (8) is independent of $r$) obtains in the case $\sigma = -4/3$, $N = 1$, when equation (9) is invariant with respect to a five-parameter Lie group of point transformations [80, 81]. Concerning the transformation (10) and other properties of solutions of equation (9), see [278]. For an application of the transformation of Example 5 see § 7, Ch. IV, as well as [112, 114, 150]. The fact that it is possible to linearize equation (15) has been known for some time (for related results, see the references in [12]).

Group-theoretic aspects of the ability to linearize this equation were analyzed in [51] (note that (15) is the only nonlinear heat equation $u_t = (k(u)u_x)_x$ invariant with respect to a non-trivial Lie-Bäcklund group; see [221, 51, 262]). Example 6 is taken from [51]. Example 7 demonstrates new properties of the "multi-dimensional" equation with the coefficient $k(u) = u^{-2}$. Proposition 14 is proved in [57]; we remark that that paper uses the same transformations as in [262, 221]. Equivalence of equations with $k_i = u^{\sigma_i}$, $\sigma_1 + \sigma_2 + 2 = 0$, was established first in [310]; using group-theoretic methods the same result was proved in [221]. In Example 8 we present a new particular solution of equation (36), which cannot be obtained by known group-theoretic methods. Proposition 15 is a natural generalization of the result of [57]. An exact solution of the super-slow diffusion equation (37) is constructed in [250], using the approach of [57]. In [191] it is shown that precisely this solution describes the asymptotic behaviour of an arbitrary solution with the same initial mass. Estimates of solutions of the super-slow diffusion equation of general form are obtained in [99]. Asymptotics of solutions of the boundary value problem for equation (37) with the Dirichlet boundary conditions is contained in [145].

§ **8.** Existence and uniqueness theorems for solutions of degenerate equations of the form (1), (2) with lower-order terms are proved in [375, 296, 224, 371][7]. Continuity of the modulus of the gradient of the solution (that is, of the heat flux) of an equation of the form (2) has been established in [9]. The self-similar solution (3) in Example 9 is a particular instance of solutions of quasilinear equations of a more complex form, which were first constructed in [28]. By the change of variable $u_x = v$ equation (1) reduces to the previously considered equation $v_t = (|v|^\sigma v)_{xx}$; therefore all results concerning asymptotic stability and a.s.s. extend with minor modifications to the case of gradient nonlinearities (this refers also to the localized solution of Example 10).

§ **9.** For some generalizations and extensions of the results of [255] see [241, 242, 95, 195, 357, 358, 361]. The proof of Proposition 17 illustrates the technique of derivation of pointwise estimates of solutions of parabolic equations, as well as one of the simplest methods of comparison of different equations; more complicated examples are given in Ch. V.

§ **10.** The families of self-similar solutions (3), (7) were constructed in [80, 81]; presentation of the main conclusions uses the results of [82]. The result formulated here, concerning instability of the stationary solution (6) and the existence of the non-trivial attracting set corresponding to it, illustrates the analysis of §§ 5, 7, Ch. IV (where in principle we present a method of constructing a large attracting set of an unstable stationary solution of a quasilinear parabolic equation). The idea of using a two-parameter family of invariant solutions (18) to study the properties of travelling waves in the Cauchy problem for (17) is due to V. A. Dorodnitsyn.

§ **11.** The one-parameter family of solutions (3), feasibility of construction of which was discussed on group-theoretic grounds in [80, 81], is considered in [82].

§ **12.** The first example of a localized solution in a medium with a sink ($\nu = 1$ in (1)) is constructed in [302]; for details on that see the survey in [162]. A more general statement than Proposition 19, is proved in [231]. The first mention of total extinction in a medium with a absorption is contained in [343]; an analysis of this phenomenon for equations of general form was undertaken in [231]. To study asymptotic stability of self-similar solutions (2) and solvability of the problem (2′)

[7] See also the references contained therein.

we can apply the methods of §§ 1, 5, Ch. IV. See also [192]. A particular solution of equation (3) (Example 11) in the case $N = 1$ is contained in [248]; its multi-dimensional analogue is to be found in [301]. This explicit solution admits a natural $N$-dimensional generalization. Setting in (3) $u^{\sigma} = v$ yields the equation $v_t = v\Delta v + (1/\sigma)|\nabla v|^2 - \sigma \equiv \mathbf{A}(v)$, where the quadratic operator $\mathbf{A}$ has an $(N+1)$-dimensional invariant linear subspace given by the linear span $W_{N+1} = \mathcal{L}\{1, x_1^2, \ldots, x_N^2\}$, that is, $\mathbf{A}(W_{N+1}) \subseteq W_{N+1}$. For more general examples of invariant linear subspaces for nonlinear operators see [136]. Substituting into the equation $v = C_0(t) + C_1(t)x_1^2 + \cdots + C_N(t)x_N^2 \in W_{N+1}$, we arrive at a nonlinear dynamical system for the coeeficients. By analyzing its properties we derive, in particular, compactly supported solutions exhibiting *non-symmetric* extinction in finite time [167]. Asymptotic extinction behaviour of solutions of (3) is studied in [188]. Some of the results of subsection 3, which deal with estimating the size of the support of a generalized solution, are proved in [48], where references to earlier work can be found. More details concerning properties of solutions of nonlinear heat equations with a sink term can be found in [230, 231, 237, 248, 11, 50, 89, 208, 253] (see the survey of [162] and [233]).

**§ 13.** Presentation of all the results of subsections 1 and 2.1 follows [162]. Solution (17) was found in [34].

Let us note that in the case $T = 0$ every non-trivial self-similar solution (3) is generated by a singular initial function $u_0(x)$ of the following form: if $1 < \beta < 1 + 2/N$ and $\theta_S(\xi)$ is a solution of the form (8), then $u_0(x) \simeq D^l\delta^l(x)$, where $l = 2/N(\beta - 1) > 1$ (by a different method existence of such *very singular* self-similar solutions is proved in [55]; uniqueness is proved in [240]); if $\theta_S(\xi)$ has the power law asymptotics (7), then $u_0(x) = C|x|^{-2/(\beta-1)}$ in $\mathbf{R}^N\backslash\{0\}$, where $C > 0$ is the constant of (7). For $1 < \beta \leq 1 + 2/N$ all these functions are not in $L^1(\mathbf{R}^N)$. Therefore the self-similar solutions we construct seem to indicate the optimal degree of singularity necessary for the existence of a non-trivial solution of the Cauchy problem.

The above conclusions agree well with the results of [54], where it is shown that for $\beta \geq 1 + 2/N$ and $u_0(x) = \delta(x)$, a non-trivial solution does not exist, i.e., $u \equiv 0$ in $\mathbf{R}_+ \times \mathbf{R}^N$ (let us note that for $\beta \geq 1 + 2/N$ there is no self-similar solution (3) with $\theta_S \not\equiv 0$ of the form (8)).

An assertion, which is stronger than Proposition 22, is proved in [235], where for $\beta > 1 + 2/N$ the authors prove existence and asymptotic stability of an infinite-dimensional set of asymmetric self-similar solutions (3), $\xi = x/(T + t)^{1/2} \in \mathbf{R}^N$.

A generalization of Proposition 25 to the case of more general initial functions $u_0$ ($\beta > 1 + 2/N$) is contained in [208, 235]. In [235] the reader can find conditions of stabilization of $u(t, x)$ to stationary solutions of the form (31), where $\eta = x(T + t)^{-1/2} \in \mathbf{R}^N$ and $f_* \notin L^1(\mathbf{R}^N)$ ($u_0 \notin L^1(\mathbf{R}^N)$); the case $u_0 \in L^1(\mathbf{R}^N)$, is considered in [208].

Results of subsection 2.2 are contained in [163]. The stabilization $f_T \to f_*$, $t \to \infty$, of Proposition 26 means, in particular, that $t^{1/(\beta-1)}u(t, x) \to (\beta - 1)^{-1/(\beta-1)}$, $t \to \infty$ for any $x \in \mathbf{R}^N$. This result was proved first in [208] (obviously, it does not give any information on the spatial structure of the thermal perturbation). Let us note that since the function $G(\omega)$ in (45) is sufficiently arbitrary smooth, the set of asymptotically stable a.s.s. (43) is infinite-dimensional. A more complicated situation of extinction in finite time for the porous medium equation with absorption, when the limit profile satisfies the first order Hamilton-Jacobi equation is studied in [188]. Proposition 27 and all the auxiliary statements required in its proof are established in [162]. A lower bound for the amplitude was discussed previously in [208]. The same phenomenon of the appearance of unusual logarithmic perturbations of the asymptotics arises in the Cauchy problem for the quasilinear equation $u_t = \Delta u^{\sigma+1} - u^\beta$, $(\sigma > 0)$, $u_0 \in L^1(\mathbf{R}^N)$ has compact support, for the critical value of the parameter $\beta = \beta_* = \sigma + 1 + 2/N$. See the results of [178, 179] for the one-dimensional equation $(N = 1)$ and [190], where this equation is studied by a different method for arbitrary $N \geq 1$. In this connection, let us also mention the paper [239], where the occurrence of logarithmic perturbations of the asymptotics for $\beta > \beta_*$ is related to the behaviour of $u_0(x) \simeq |x|^{-N}$ as $|x| \to \infty$. A general classification of asymptotics of solutions depending on the parameters $\sigma > 0$ and $\beta > \sigma + 1$ is more or less contained in the papers [236, 237, 238, 239, 240, 178, 179, 188, 190, 74]. See also the references in the last papers and in the survey [233]. For the equation with absorption with gradient diffusivity $u_t = \nabla \cdot (|Du|^\sigma Du) - u^\beta$, $\sigma > 0$, the critical case $\beta = \sigma + 1 + (\sigma + 2)/N$ is considered in [190] (a uniqueness theorem for the asymptotics was proved earlier in [178, 179]). Therefore existence of a critical value of the parameter, for which the nonlinear interaction of various diffusion operators of diffusion equations with absorption generates an unusual non-self-similar asymptotics, is of a fairly general nature.

*Chapter III*

# Heat localization (inertia)

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This entire chapter is devoted to the study of unbounded solutions of parabolic equations. Here we consider the character of heat transfer in a medium, the temperature on the boundary of which follows a blow-up regime. We deal with the cases of power law dependence of the thermal conductivity coefficient on temperature, and of a constant coefficient. The study is based on an analysis of unbounded self-similar solutions and does not require any special mathematical methods.

The main effort is directed towards the study of physical properties of boundary blow-up regimes. We consider in detail different types of propagation of thermal waves, establish conditions for the appearance, and the physical meaning of the heat localization phenomenon, which reflects a kind of inertia of strongly non-stationary diffusion processes. Analysis of the heat transfer processes is conducted in dimensional form, which immediately allows us to use the obtained results to derive realistic physical estimates. These results are used in Ch. V, VI in the study of heat inertia in media with arbitrary thermophysical properties.

## § 1 The concept of heat localization

### 1 A boundary blow-up regime

We consider a one-dimensional process of heat propagation in a medium that occupies a half-space $\{x > 0\}$ with thermal conductivity coefficient which depends on the temperature: $k = k(u) > 0$ for $u > 0$, $k(0) \geq 0$.

On the boundary $x = 0$ the temperature follows a *blow-up regime*, that is it becomes infinite at a certain finite moment of time $t = T > 0$ ($T$ is the *blow-up time*).

The process is described by the first boundary value problem for the quasilinear parabolic equation

$$u_t = (\phi(u))_{xx} \equiv (k(u)u_x)_x, \ \ k \in C^2((0, \infty)) \cap C([0, \infty)), \tag{1}$$

with the initial condition

$$u(0, x) = u_0(x) \geq 0, \ x > 0; \ \sup u_0 < \infty, \ \sup |(\phi(u_0))_x| < \infty, \tag{2}$$

and the boundary condition

$$u(t, 0) = u_1(t) > 0, \ 0 < t < T; \ u_1 \in C^1([0, T)); \ u_1(t) \to \infty, \ t \to T^-. \tag{3}$$

The compatibility condition $u_1(0) = u_0(0)$ is taken to hold.

The main goal is to study the behaviour of the solution of the problem as $t \to T^-$. We shall be particularly interested in the conditions for heat localization, a paradoxical property of the heat conduction process, which shows itself at the asymptotic stage of a blow-up regime.

## 2 Examples of localization in boundary value problems

**Example 1.** A standing thermal wave (see § 3, Ch. I). Let us consider the problem (1)–(3), where

$$\begin{gathered}
k(u) = k_0 u^\sigma, \ \sigma = \text{const} > 0, \ k_0 = \text{const} > 0, \\
u_1(t) = A_S(T - t)^{-1/\sigma}, \ t < T, \ A_S = \text{const} > 0, \\
u_0(x) = \begin{cases} A_S T^{-1/\sigma}(1 - x/x_S)^{2/\sigma}, & 0 < x \leq x_S, \\ 0, & x > x_S. \end{cases}
\end{gathered} \tag{4}$$

This problem has a separable solution:

$$u_S(t, x) = \begin{cases} A_S(T - t)^{-1/\sigma}(1 - x/x_S)^{2/\sigma}, & 0 < x \leq x_S, \\ 0, & x > x_S, \end{cases} \tag{5}$$

where

$$x_S = [2k_0 A_S^\sigma (\sigma + 2)/\sigma]^{1/2}. \tag{6}$$

Let us indicate the main properties of the solution (5), (6):

a) for $0 \leq x < x_S$ the temperature $u_S(t, x)$ goes to infinity as $t \to T^-$;

b) $u_S(t, x) \equiv 0$ for all $t \in (0, T)$ for any $x \geq x_S$ (Figure 17); the point $x = x_S$, in which both the temperature and the heat flux are zero, is a fixed boundary, which separates heated matter from cold.

The process of heat transfer is localized in the finite domain $0 < x < x_S$, even though in that domain the temperature grows without bound as $t \to T^-$.

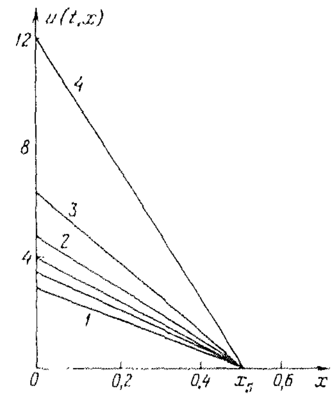


**Fig. 17.** A standing thermal wave. The parameters are: $\sigma = 2$, $n = -0.5$, $k_0 = 0.5$, $A_S = 0.354$, $x_S = 0.5$, $T = 1$; 1: $T - t = 1.25 \cdot 10^{-2}$, 2: $T - t = 5.3 \cdot 10^{-3}$, 3: $T - t = 2.9 \cdot 10^{-3}$, 4: $T - t = 9 \cdot 10^{-4}$

**Example 2.** Effective heat localization in a medium with constant thermal conductivity. The problem (1)–(3), where

$$k(u) = k_0 > 0;\ u_1(t) = A_S \exp\{R_0(T-t)^{-1}\},\ R_0 = \mathrm{const} > 0;\ u_0(x) \equiv 0,$$

has the solution

$$u(t,x) = \frac{x}{2\sqrt{k_0\pi}} \int_0^t \exp\left\{-\frac{x^2}{4k_0(t-\tau)}\right\} (t-\tau)^{-3/2} A_S \exp\{R_0(T-\tau)^{-1}\}\, d\tau, \quad (7)$$

satisfying the following properties:

a) for

$$0 \le x \le x_S = 2(k_0 R_0)^{1/2} \quad (8)$$

the temperature goes to infinity as $t \to T^-$;

b) for $x > x_S$ the temperature is non-zero and is bounded for all $0 \le t < T$ by the function

$$u(T^-, x) \equiv \lim_{t \to T^-} u(t,x) = \frac{A_S}{\sqrt{\pi}} \left[1 - \left(\frac{x_S}{x}\right)^2\right]^{-1/2} \int_{(x^2 - x_S^2)/(4k_0T)}^{\infty} e^{-v} v^{-1/2}\, dv < \infty; \quad (9)$$

c) for any $x_1 > x_S$ the energy

$$E(t, x_1) = \int_{x_1}^{\infty} u(t,\xi)\, d\xi < \mathrm{const},\ t \in (0, T).$$

contained in the domain $\{x_1 \le x < \infty\}$, is bounded.

As in Example 1, practically all the energy is localized in the finite domain $\{x < x_S = x_S(k_0, R_0)\}$. The difference is that the temperature to the right of the point $x_S$ is nonzero (Figure 18), but is uniformly bounded during the entire course of the process.

## 3 Definition of localization and its physical meaning

**Definition 1.** The problem (1)–(3) exhibits *strict heat localization* if there exists a constant $l > 0$ such that $u(t, x) = 0$ everywhere in $(0, T) \times (l, \infty)$.

We shall call the smallest such number $l$ the *localization depth* $l^*$, and the set $\{0 < x < l^*\}$ will be called the *localization domain*.

This definition has content if the following two conditions are satisfied:

a) $$\int_0^1 \left[\frac{k(\xi)}{\xi}\right] d\xi < \infty,$$

which is a necessary and sufficient condition for finite speed propagation of disturbances in processes described by equation (1) (see § 3, Ch. I). Therefore Definition 1 makes sense, for example, for $k(u) = k_0 u^\sigma$, $\sigma > 0$ (but is not applicable to a medium with constant thermal conductivity $k \equiv k_0 > 0$).

b) $u_0(x) = 0$ for $x > l_0$, $l_0 < \infty$, that is, the function $u_0(x)$ must be of compact support (there must exist a region of "cold background" initial temperature in the medium).

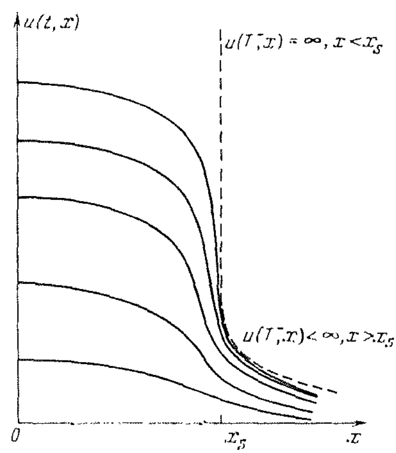


**Fig. 18.** The concept of effective heat localization

Example 1 demonstrates strict heat localization, and in this case the depth of localization is $l^* = x_S$ (see (6)).

**Definition 2.** Problem (1)–(3) exhibits *effective localization* if the set

$$\omega_L = \left\{ x > 0 \mid \overline{\lim}_{t \to T} \; u(t, x) = \infty \right\}$$

is bounded.

We shall call the quantity $L^* = \text{meas}\, \omega_L$ the *effective localization depth*, while the set $\{0 < x < L^*\}$ will be called the *effective localization domain*.

In Example 2 we have heat localization in the sense of Definition 2 with effective depth $L^* = x_S$ (see (8)).

For most physical problems the function $u(t, x)$ (temperature) is never zero; therefore Definition 2 is the more natural one.

Heat localization (inertia of heat) makes it possible to attain any temperature and concentrate any amount of energy in a bounded portion of space, and to contain them for a finite time practically without loss from the localization domain. This unusual property of the heat transfer process can be used in many applications.

The concept of localization for media without heat absorption makes sense only for boundary blow-up regimes. If, on the other hand, instead of (3), we have in the problem (1), (2), a boundary regime without blqw-up, that is $u(t, 0) = u_1(t) \to \infty$ as $t \to \infty$, then, as is easily shown, $u(t, x) \to \infty$ for all $0 < x < \infty$, that is, there is no localization. The proof of this fact proceeds by comparing $u(t, x)$ with self-similar solutions of equation (1) of the form $u_S(t, x) = \theta(x^2/t) \geq 0$, which exist for arbitrary functions $k(u)$ (see § 3, Ch. I).

There are two possible regimes (modes) of heat propagation with localization. We shall say that an S-*regime* obtains if $L^* > 0$. Then the temperature and the energy grow unboundedly in the localization domain as $t \to T$. If $L^* = 0$, then we have the LS-*regime*: $u(t, x) \to \infty$ as $t \to T$ only on the boundary $x = 0$. The amount of heat that enters the domain $(0, x)$,

$$E(t, x) = \int_0^x [u(t, \xi) - u_0(\xi)]\, d\xi, \quad 0 < x < \infty, \quad t \in (0, T),$$

can in this case be either bounded or unbounded as $t \to T$.

If, on the other hand, we have

$$\underline{\lim}_{t \to T} \; u(t, x) = \infty$$

for any $x > 0$, then *there is no localization* and we say that the HS-*regime* obtains.

In the present chapter we consider the case $k(u) = k_0 u^\sigma$, $\sigma \geq 0$, $k_0 = \text{const} > 0$. We study the heat localization effect both in the strict sense (§§ 2, 3) and in the effective sense (§ 4). There is a close relation between the two definitions; it is established in § 4.

# § 2 Blowing-up self-similar solutions

## 1 Formulation of the problem

In this section we construct self-similar solutions of the problem (1.1)–(1.3) in the case $k(u) = k_0u^\sigma$, $\sigma > 0$. Together with comparison theorems, they provide an efficient apparatus for studying the localization property.

For $k(u) = k_0u^\sigma$, equation (1.1) has power law self-similar solutions corresponding to boundary regimes with blow-up:

$$u(t,0) = u_1(t) = A_0(T-t)^n, \quad A_0 = \text{const} > 0, \quad n < 0.$$

The required self-similar solution $u_S$ satisfies the following problem:

$$u_t = (k_0u^\sigma u_x)_x, \quad -\infty < t < T, \quad x > 0, \tag{1}$$

$$u(-\infty, x) = 0, \quad x > 0, \tag{2}$$

$$u(t,0) = A_0(T-t)^n, \quad -\infty < t < T. \tag{3}$$

Its solution has the form

$$u_S(t,x) = A_0(T-t)^n f(\xi), \tag{4}$$

where

$$\xi = \frac{x}{k_0^{1/2}A_0^{\sigma/2}(T-t)^{(1+n\sigma)/2}} \geq 0 \tag{5}$$

is the similarity variable. The heat flux $W = -k(u)u_x$ has the representation

$$W(t,x) = A_0^{(\sigma+2)/2}k_0^{1/2}(T-t)^{[-1+n(\sigma+2)]/2}\omega(\xi). \tag{6}$$

Here $f(\xi)$ and $\omega(\xi) = -f^\sigma(\xi)f'(\xi)$ ($f' = df/d\xi$) are dimensionless functions of temperature and flux, respectively.

The function $f(\xi) \geq 0$ is determined from the equation

$$(f^\sigma f')' - [(1+n\sigma)/2]f'\xi + nf = 0, \quad 0 < \xi < \infty, \tag{7}$$

with the boundary conditions

$$f(0) = 1, \quad f(\infty) = 0. \tag{8}$$

We remind the reader that in order for the solution to make physical sense, we are seeking a non-negative, continuous solution of the problem (7), (8), while the function $\omega(\xi)$ has to be continuous and bounded.

## 2 Construction of self-similar solutions

Equation (7) admits the similarity transformation

$$\tilde{\xi} = \alpha\xi, \ \tilde{f}(\tilde{\xi}) = \alpha^{2/\sigma} f(\xi), \tag{9}$$

and, using the change of variables

$$\eta = \ln \xi, \ f(\xi) = \xi^{2/\sigma}\phi(\eta) \geq 0, \ \psi = \frac{d\phi}{d\eta} = -\frac{2}{\sigma}\phi + \xi^{-2/\sigma+1} f', \tag{10}$$

reduces to the first order equation

$$\frac{d\psi}{d\phi} = -\frac{1}{\phi^{\sigma}\psi}\left[\frac{2}{\sigma}\left(\frac{2}{\sigma}+1\right)\phi^{\sigma+1} + \left(\frac{4}{\sigma}+3\right)\phi^{\sigma}\psi + \sigma\phi^{\sigma-1}\psi^2 - \frac{\phi}{\sigma} - \frac{1+n\sigma}{2}\psi\right] \tag{11}$$

We shall say that a point $\xi_f \in (0, \infty)$ is a front point if $f(\xi) = 0$, $\xi \geq \xi_f$, $f(\xi) > 0$ for $\xi < \xi_f$. In the $(\phi, \psi)$ plane a front corresponds to $\phi = 0$, the value of $\psi$ is not known a priori. The behaviour of integral curves of the equation (11) is different in the cases $1 + n\sigma < 0$ and $1 + n\sigma > 0$.

In a neighbourhood of the line $\phi = 0$ the integral curves of equation (11) have the form

$$\psi = A\phi^{-\sigma} + \frac{1+n\sigma}{2}\phi^{1-\sigma} + O(\phi^{2-\sigma}), \ A = \text{const}, \tag{12}$$

$$\psi = -\frac{2}{1+n\sigma}\phi + O(\phi^2). \tag{13}$$

Only the curve (12) with $A = 0$ satisfies the second of conditions (8) for $1+n\sigma < 0$ (the heat flux be continuous at a front point only for this choice of the constant $A$). For $1 + n\sigma > 0$ the condition on the front is satisfied by the curve (13).

Therefore if the solution exists, it is unique.

From (10), (12), (13) we obtain the asymptotics of the solution in a neighbourhood of the front: for $1 + n\sigma < 0$

$$\begin{aligned} f(\xi) = &\left(-\frac{1+n\sigma}{2}\sigma\xi_f\right)^{1/\sigma} (\xi_f - \xi)^{1/\sigma} + \\ &+ \frac{1-n\sigma}{4(\sigma+1)}\left(-\frac{1+n\sigma}{2}\sigma\xi_f\right)^{1/\sigma-1} (\xi_f - \xi)^{1/\sigma+1} + \ldots, \xi \to \xi_f; \end{aligned} \tag{14}$$

for $1 + n\sigma > 0$

$$f(\xi) = C\xi^{2n/(1+n\sigma)} + C_1\xi^{(2n-2)/(1+n\sigma)} + \ldots, \ \xi \to \infty, \tag{15}$$

where $\xi_f = \xi_f(n, \sigma) < \infty$ is the similarity coordinate of the front, $C = C(n, \sigma) > 0$, $C_1 = -C^{n+1}[2n(2n + n\sigma - 1)]/(1 + n\sigma)^2 < 0$. In general case the values of $\xi_f$ and $C$ are to be determined numerically.

On the straight line $\psi = -2\phi/\sigma$ we have the inequality

$$\frac{d\psi}{d\phi} = -\frac{2}{\sigma} + \frac{n\sigma}{2\phi^{\sigma}} < -\frac{2}{\sigma},$$

that is, as $\phi$ is increased the integral curves intersect the straight line $\psi = -2\phi/\sigma$ with a slope larger than that of the straight line itself. Since for $\phi > 0$, $\psi < -2\phi/\sigma$ there are no isoclines of infinity, the integral curves do not leave that region. The desired trajectory (see (12)) for $A = 0$ lies below the straight line $\psi = -2\phi/\sigma$, and therefore we may restrict ourselves to the analysis of the equation (11) in the region $\phi > 0$, $\psi \leq -2\phi/\sigma$.

Then it follows from (10) that $f'_{\xi} < 0$, $\xi < \xi_f$, that is, the required solution is a monotone decreasing function (monotonicity of the solution can be easily established directly from equation (7)).

In the $\{\phi, \psi\}$-plane, to the boundary point $\xi = 0$ correspond $\phi = \infty$, $\psi = -\infty$. In the domain $\phi > 0$, $\psi < -2\phi/\sigma$ there is a unique direction, $\psi = -2\phi/\sigma$, along which there is a bundle of integral curves

$$\psi = -\frac{2\phi}{\sigma} + B\phi^{-\sigma/2+1} + o(\phi^{1-\sigma/2}), \tag{16}$$

where $B < 0$ parametrizes the bundle, which enter the point $\phi = \infty$, $\psi = -\infty$. The required solution corresponds to some value $B^* = C_0(n, \sigma) < 0$.

In the plane $\{\xi, f\}$, to any curve in the plane of $\phi$, $\psi$, there corresponds a family of similar curves, obtained by the transformation (9); the solution $f(\xi)$ is chosen using the first of conditions (8).

Integrating (16), taking into account (10) and (8), we obtain the first terms of the asymptotic expansion of the solution in a neighbourhood of $\xi = 0$:

$$f(\xi) = 1 + C_0(n, \sigma)\xi + \ldots,$$

where $C_0(n, \sigma) = f'(0) < 0$ is computed numerically.

The expression in square brackets in (11) is a quadratic polynomial in $\psi$ with a non-negative discriminant. Therefore (11) can be rewritten in the form

$$\frac{d\psi}{d\phi} = -\frac{1}{\phi^{\sigma}\psi}\left[\psi - \psi^{+}(\phi)\right]\left[\psi - \psi^{-}(\phi)\right],$$

where $\psi = \psi^{\pm}(\phi)$ are isoclines of zero of the equation. The continuous curve $\psi = \psi^{-}(\phi)$ is wholly contained in the domain $\phi \geq 0$, $\psi \leq -2\phi/\sigma$. The isocline $\psi = \psi^{+}(\phi)$ lies above the line $\psi = -2\phi/\sigma$.

Once the critical points corresponding to the front and the boundary, are analyzed (the other critical points are of no interest), it is not hard to construct the whole field of integral curves, using which we can prove existence of the solution, a trajectory that connects the front point with the boundary point.

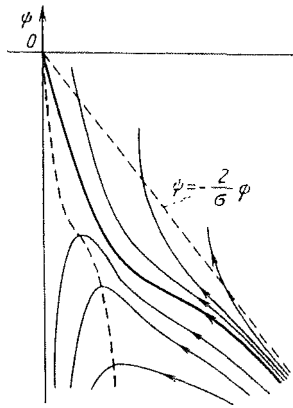


**Fig. 19.** The phase portrait of (11) in the case $1 + n\sigma > 0$

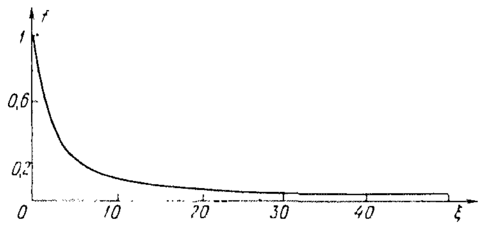


**Fig. 20.** Numerical solution of the problem (7), (8) for $\sigma = 2$, $n = -0.25$

Figure 19 shows the "phase portrait" of equation (11) in the case $1 + n\sigma > 0$. The thick line denotes the required solution, while the dotted one shows the null-isocline $\psi = \psi\ (\phi)$. In Figures 20, 21 we present results of numerical solution of the problem (7), (8) for $1 + n\sigma > 0$ and $1 + n\sigma < 0$, respectively.

Thus, the solution of the problem (7), (8) exists, is unique and monotone. For $1 + n\sigma < 0$ the front of the solution is at a finite point. If $1 + n\sigma > 0$, then $f(\xi) > 0$ for all $0 < \xi < \infty$.

**Remark.** In the case $n = -1/(\sigma + 2)$, equation (7) has a first integral $E = f^{\prime\prime} f^{\prime} - f\xi/(\sigma + 2)$, $E = \text{const}$. It is easy to establish existence, uniqueness and monotonicity of the solution for some $E = E(\sigma) \in (-\infty, 0)$.

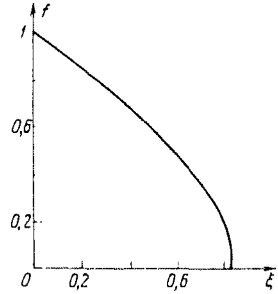


**Fig. 21.** Numerical solution of the problem (7), (8) for $\sigma = 2$, $n = -1$

## 3 Physical properties of solutions

Solutions of equation (1) and of similar equations describing diffusion of heat from the boundary $x = 0$ in a half-space are usually called *thermal waves*.

Let us review some concepts related to thermal waves (see Ch. I, II). The *front* of a wave is the point with the coordinate $x_f(t)$, such that

$$u(t, x) = 0,\ x > x_f(t);\quad u(t, x) > 0,\ x < x_f(t).$$

The quantity $x_f(t)$ determines the depth of penetration of heat into the medium.

The point with the coordinate $x_{ef}(t)$, such that $u(t, x_{ef}(t)) = u(t, 0)/2$, is called the *half-width* point of the thermal wave.

For the self-similar solutions, from (5) we have

$$x_f(t) = \xi_f k_0^{1/2} A_0^{\sigma/2} (T - t)^{(1+n\sigma)/2}, \tag{17}$$

$$x_{ef}(t) = \xi_{ef} k_0^{1/2} A_0^{\sigma/2} (T - t)^{(1+n\sigma)/2}, \tag{18}$$

where the constant $\xi_{ef} > 0$ is such that $f(\xi_{ef}) = 1/2$.

The amount of heat contained in wave at time $t$ is

$$E(t) = \int_0^\infty u(t, x)\, dx.$$

For the self-similar solution we obtain

$$E(t) = \frac{2\omega(0) A_0^{(\sigma+2)/2} k_0^{1/2}}{1 + n(\sigma + 2)} (T - t)^{[1+n(\sigma+2)]/2} \Big|_{-\infty}^{t},\quad n \neq -\frac{1}{\sigma + 2}, \tag{19}$$

where $\omega(0) = -f'(0) > 0$ (the heat flux at the boundary is positive).

Below we shall make use of the following fact. For a fixed coordinate $0 < x_0 < \infty$ the similarity coordinate $\xi = \xi(t, x_0)$ changes in time according to (see (5))

$$\xi(t, x_0) = x_0 / \left[k_0^{1/2} A_0^{\sigma/2} (T-t)^{(1+n\sigma)/2}\right], \tag{20}$$

that is, $\xi(t, x_0) \to \infty$, $t \to T^-$ if $n > -1/\sigma$ and $\xi(t, x_0) \to 0$ as $t \to T^-$ if $n < -1/\sigma$.

Let us now analyze the physical properties of the self-similar solutions.

In the case $n = -1/\sigma$, the solution of the problem (1)–(3) (the S-regime) is given in Example 1. The front of the thermal wave and the half-width are constant, $E(t) < \infty$, $t \in (-\infty, T)$ and $E(t) \to \infty$ as $t \to T^-$.

For $n < -1/\sigma$ the solution has the following characteristics:

1) The front is at a finite point $\xi_f < \infty$ and in a neighbourhood of the front we have the asymptotics

$$u_S(t, x) = A_0 (T-t)^n f(\xi) = A_0 (T-t)^n \left[-\frac{1+n\sigma}{2} \sigma \xi_f (\xi - \xi_f)\right]_+^{1/\sigma} + \dots .$$

2) The width $x_f(t)$ and the effective depth of heat penetration $x_{ef}(t)$ grow without bound as time approaches the blow-up time. In the limit the thermal wave covers all the space.

3) $E(t) < \infty$, $t \in (-\infty, T)$ and $E(t) \to \infty$ as $t \to T^-$.

4) $u_S(t, x_0)/u_S(t, 0) \to 1$ as $t \to T^-$, that is, in time, at every point of the space the temperature behaves essentially as on the boundary $x = 0$.

Thus there is no localization for $n < -1/\sigma$, and the HS-regime obtains.

In some sense the HS-regime is similar to regimes without blow-up: with time, the influence of the boundary condition is felt in more and more distant regions of the medium (compare this, for example, with the self-similar solutions of subsection 2, § 3, Ch. II and of § 3, Ch. I, which correspond to boundary regimes without blow-up: $u(t, 0) = A_0 t^n$, $A_0 > 0$, $n > 0$, $t > 0$). However, infinite values of temperature are reached not for $t = \infty$, but at the finite blow-up time. The HS-regime (HS comes from "higher" than S) is a "superfast" way of heating the medium; the boundary heating for $t \to T^-$ is "faster" than in the S-regime; see Figure 22.

Let us enumerate the physical properties of self-similar solutions for $n > -1/\sigma$.

1) It follows from (15) that the front of the wave is at an infinitely far point[1]: $x_f(t) = \infty$, $t \in (-\infty, T)$. This result is a priori obvious from physical considerations. Indeed, from (17) it can be seen that under the assumption $\xi_f < \infty$, $x_f(t)$ would go to zero as $t \to T^-$. This would mean shrinking of the domain encompassed by the thermal wave, which would be impossible.

[1]This result does not contradict the finite speed of propagation of heat in a medium with $k(u) = k_0 u^\sigma$, $\sigma > 0$: infinite time elapses from the start of the process till time $t \in (-\infty, T)$.

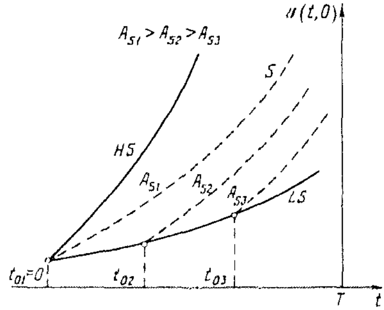


**Fig. 22.** The field of boundary blow-up regimes

2) The half-width decreases as $t \to T^-$ at the rate given by (18). Energy which enters the medium is concentrated in a part of the space, which becomes smaller with time. We shall call solutions of this kind *thermal waves of decreasing effective dimension*.

3) Taking into account (20) and the asymptotics (15), for all $x > 0$ we obtain

$$
\begin{aligned}
u_S(t,x)_{t \to T} &= C\frac{A_0}{(k_0 A_0^{\sigma})^{n/(1+n\sigma)}} x^{2n/(1+n\sigma)} + C_1 \frac{A_0 x^{(2n-2)/(1+n\sigma)}}{(k_0 A_0)^{(n-1)/(1+n\sigma)}}(T-t) + \dots, \\
W(t,x)_{t \to T} &= -C^{\sigma+1}\frac{2n}{1+n\sigma} A_0^{\sigma+1} k_0^{(1-n)/(1+n\sigma)} x^{(2n+n\sigma-1)/(1+n\sigma)} + \dots,
\end{aligned}
\tag{21}
$$

that is, the self-similar solution converges from below to a limiting curve. The presence of this limiting curve, the "trace" of the boundary regime for $t = T^-$, which restricts the growth of heat-related quantities at each point of the material, is equivalent to the definition of the LS-regime, and is an important property of that regime (from that we also have shrinking of the half-width in the case of LS-regime).

4) From (19) we obtain the following. For $-1/\sigma < n < -1/(\sigma+2) = n_*$, the amount of energy is finite: $E(t) < \infty$, $t \in (-\infty, T)$, and $E(t) \to \infty$, $t \to T^-$, that is, the medium is imparted infinite energy, which is being concentrated in a neighbourhood of the boundary. For $n > n^*$ we have $E(t) = \infty$, $t \in (-\infty, T)$; neighbourhoods of the front contain an infinite amount of energy. However, $E(T^-) - E(t) < \infty$ for all $t \in (-\infty, T]$, which means that only a finite amount of energy enters the medium as the blow-up time approaches. Finally,

$$E(t) = \infty,\ E(T^-) - E(t) = \infty,\ t \in (-\infty, T),$$

when $n = n^*$.

Thus, for $n > -1/\sigma$ self-similar solutions belong to the class of LS-regimes (LS comes from "lower," the boundary regime as $t \to T^-$ is "slower" than the S-regime; see Figure 22) and we have effective heat localization.

**Remark.** Suppose it is not the temperature, but the heat flux on the boundary, that blows up in finite time:

$$W(t,0) \equiv -k_0 u^\sigma(t,0)\frac{\partial u}{\partial x}|_{x=0} = W_0(T-t)^{n_1},\ n_1 < -1/2, t < T. \tag{22}$$

The problem (1), (2) with condition (22) also has a self-similar solution:

$$u_S(t,x) = k_0^{\,1/(\sigma+2)} W_0^{2/(\sigma+2)}(T-t)^{(2n_1+1)/(\sigma+2)}F(J),$$

$$J = k_0^{-1/(\sigma+2)} W_0^{-\sigma/(\sigma+2)} x(T-t)^{(-\sigma n_1+\sigma+1)/(\sigma+2)}.$$

The function $F(J) \geq 0$ satisfies an equation which reduces to (7) by the change of variable $n = (2n_1+1)/(\sigma+2)$, and the conditions $F(\infty) = 0$, $-F^\sigma(0)F'(0) = 1$.

Taking into account (9), we have for $F(J)$

$$F(J) = C_0^{-2/(\sigma+2)}f(\xi),\ J = C_0^{-\sigma/(\sigma+2)}\xi,\ C_0(n,\sigma) = -f'(0).$$

Therefore self-similar solutions of the second boundary value problem (1), (2), (22) can be expressed in terms of already analyzed solutions and have the same properties. For $n_1 < -(\sigma+1)/\sigma$, $n_1 = -(\sigma+1)/\sigma$ and $n_1 > -(\sigma+1)/\sigma$, HS-, S-, and LS-regimes obtain, respectively.

Analysis of self-similar solutions that blow up in finite time is the first important step in the study of the localization phenomenon. The ideas of three types of thermal waves, of "fast" and "slow" solutions, will be frequently used in the sequel. Comparison theorems, which express continuous dependence of the heat conduction process on boundary data, together with self-similar solutions, allow us to map out the classes of blow-up regimes with differing physical properties.

## § 3 Heat "inertia" in media with nonlinear thermal conductivity

In this section, using the self-similar solutions of § 2 and comparison theorems, we study the influence of boundary blow-up regimes on a medium with a power law dependence of the heat conductivity coefficient on the temperature: $k(u) = k_0 u^\sigma$, $\sigma > 0$. Physical grounds for heat "inertia" are discussed in the framework of studying the evolution of an initial temperature distribution in the Cauchy problem. Localization of heat in multi-dimensional problems of nonlinear heat conduction is also considered.

## 1 A class of boundary regimes leading to heat localization

**Theorem 1.** *Let the boundary conditions in the problem* (1.1)–(1.3) *satisfy the inequalities*

$$u_0(x) \le \begin{cases} A_S T^{-1/\sigma}(1 - x/x_S)^{2/\sigma}, & x \le x_S, \\ 0, & x > x_S = (2k_0 A_S^\sigma(\sigma+2)/\sigma)^{1/2}, \end{cases} \tag{1}$$

$$u_1(t) \le A_S(T-t)^{-1/\sigma}, \; 0 < t < T, \tag{2}$$

*where $A_S > 0$ is a constant. Then we have heat localization. In particular, the following estimates hold:*

$$l^* \le x_S, \; u(t,x) \le u_S(t,x) = \begin{cases} A_S(T-t)^{-1/\sigma}(1 - x/x_S)^{2/\sigma}, & x \le x_S, \\ 0, & x > x_S = (2k_0 A_S^\sigma(\sigma+2)/\sigma)^{1/2}. \end{cases} \tag{3}$$

By comparison theorems, validity of the estimates (3) follows immediately from the properties of the self-similar solution $u_S$.

The self-similar S-regime defines a class of "slow" boundary regimes, which ensure heat localization. It is interesting to note that the estimate (3) of localization depth is independent of the period of action of the boundary regime.

Figure 23 shows the dynamics of the thermal wave in the case $u_0(x) \equiv 0$, $u_1(t) = A_0(T-t)^{-1/\sigma}$. The half-width of the wave (crosses) increases initially, and then stabilizes. The front of the wave does not penetrate beyond the localization depth $l^* = x_S = 0.5$. The dashed line in that Figure shows solution (1.5).

Another assertion concerning localization is established using the self-similar solutions of the LS-regime constructed in § 2. Let us consider first the case $u_0(x) = 0$ for $x > 0$.

**Theorem 2.** *If in the problem* (1.1)–(1.3)

$$u_1(t) \le A_0(T-t)^n, \; 0 < t < T; \; n = \text{const} \in (-1/\sigma, 0), \tag{4}$$

*then we have heat localization in the LS-regime, and the following estimates hold:*

$$l^* \le x_S = (2k_0 A_0^\sigma(\sigma+2)/\sigma)^{1/2} T^{(1+n\sigma)/2}, \tag{5}$$

$$\overline{\lim}_{t \to T} u(t,x) \le C(n,\sigma)(A_0 k_0^{-n})^{1/(1+n\sigma)} x^{2n/(1+n\sigma)}. \tag{6}$$

*Proof.* Since $n > -1/\sigma$, (2) follows from (4) for some constant $A_S$, that is, the boundary function is majorized by the self-similar S-regime. To determine the smallest constant $A_S$ in (2), we insist that at $t = 0$ the temperature on the boundary does not exceed the boundary value (Figure 22) of the solution (1.5):

$$u_1(0) \le A_0 T^n = A_S T^{-1/\sigma}. \tag{7}$$

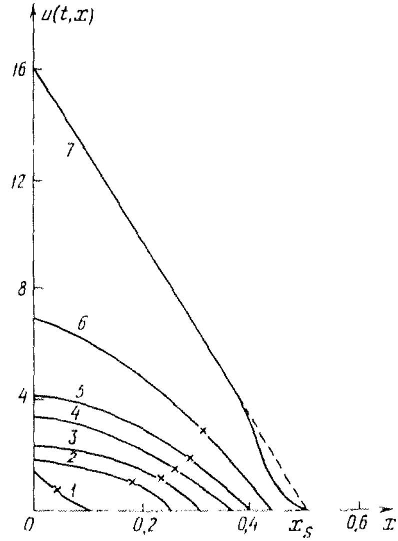


**Fig. 23.** Heat localization in the case of a boundary S-regime. The parameters are: $\sigma = 2$, $n = -0.5$, $k_0 = 0.5$, $A_0 = 0.354$, $T = 0.1125$; 1: $T-t = 1.105 \cdot 10^{-1}$, 2: $T-t = 4.04 \cdot 10^{-2}$, 3: $T-t = 3.05 \cdot 10^{-2}$, 4: $T-t = 1.05 \cdot 10^{-2}$, 5: $T-t = 6.5 \cdot 10^{-3}$, 6: $T-t = 2.5 \cdot 10^{-3}$, 7: $T-t = 5 \cdot 10^{-4}$

Taking into account the fact that $u_0(x) \equiv 0$, we obtain localization from Theorem 1, while from (7) follows the estimate (5) of the localization depth.

For $x \geq 0$, $0 \leq t < T$ the solution $u(t,x)$ is majorized by the self-similar solution $u_S(t,x)$ for the LS-regime, which corresponds to the same values of the parameters $\sigma$, $n$, $A_0$, $k_0$ (this follows from (4) and the condition $u_S(0,x) > u(0,x) \equiv 0$). Then from the inequality $u \leq u_S$ in $(0,T) \times \mathbf{R}_+$ we obtain the estimate $u(t,x) \leq u_S(T^-, x)$, which is the same as (6); see subsection 3, § 2. □

The theorem is true for any initial function $u_0(x)$ with compact support, since we can always find a constant $A_S > 0$, such that $u_0(x) \leq u_S(0,x)$ in $\mathbf{R}_+$. Then the estimates (5), (6) would depend not only on the parameters of the boundary regime, but on the initial data as well.

Thus, if condition (4) holds, we have localization in the LS-regime, there exists a limiting curve, and the half-width of the thermal wave decreases. Since the boundary regime acts for a finite time, unlike the case of self-similar LS-regime (see subsection 3, § 2) the thermal wave has a finite front. The estimate of the magnitude of $l^*$ in terms of the localization depth of the majorizing S-regime depends on the length of time of the heating process.

Figure 24 shows the results of numerical computation of solution of the problem (1.1)–(1.3). Here $u_0(x) \equiv 0$ and the boundary regime $u_1(t) = A_0(T-t)^n$ corresponds to the self-similar LS-regime. The half-width of the thermal wave first increases and then begins to shrink. The solution is bounded by the limiting curve (6) (dashed line); the front of the wave does not penetrate beyond $l^* \le x_S = 0.87$.

## 2 Conditions for the absence of localization

**Theorem 3.** *If in the problem* (1.1)–(1.3) *the boundary regime satisfies the inequality*

$$u_1(t) \ge A_0(T-t)^n, \quad 0 \le t^* \le t < T; \ n < -1/\sigma, \tag{8}$$

*then there is no localization (HS-regime) and as $t \to T^-$ we have the estimates*

$$\begin{gathered} \varlimsup_{t \to T^-} u(t,x) = \infty \ \textit{everywhere in } \mathbf{R}_+, \\ x_f(t) \ge \xi_f k_0^{1/2}(A_0^*)^{\sigma/2}(T-t)^{(1+n\sigma)/2} \to \infty, \ t \to T^-, \end{gathered} \tag{9}$$

*where $A_0^* > 0$ is some constant.*

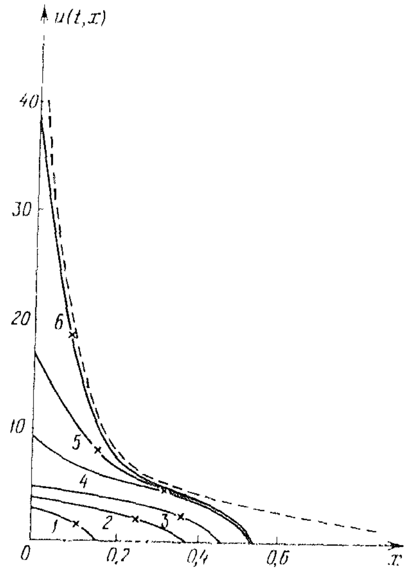


**Fig. 24.** Heat localization in the LS-regime. The parameters are: $\sigma = 2$, $n = -0.25$, $k_0 = 0.5$, $A_0 = 1.06$, $T = 0.1125$; 1: $T-t = 1.02 \cdot 10^{-1}$, 2: $T-t = 3.1 \cdot 10^{-2}$, 3: $T-t = 1.05 \cdot 10^{-2}$, 4: $T-t = 3 \cdot 10^{-4}$, 5: $T-t = 2.4 \cdot 10^{-5}$, 6: $T-t = 1 \cdot 10^{-6}$

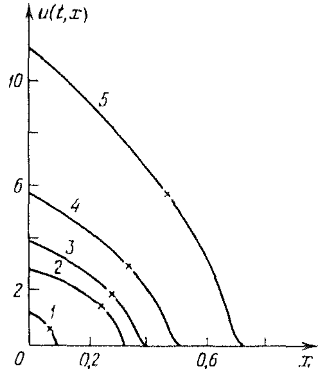


**Fig. 25.** Heat propagation in HS-regime. The parameters are: $\sigma = 2$, $n = -1$, $k_0 = 0.5$, $A_0 = 0.12$, $T = 0.1125$, $u_0(x) = 0$, $x > 0$ ; 1: $T - t = 1.023 \cdot 10^{-1}$, 2: $T - t = 4.11 \cdot 10^{-2}$, 3: $T - t = 3.09 \cdot 10^{-2}$, 4: $T - t = 2.07 \cdot 10^{-2}$, 5: $T - t = 1.05 \cdot 10^{-2}$

Here $\xi_f = \xi_f(n, \sigma) > 0$ is the dimensionless coordinate of the front of the corresponding self-similar HS-regime.

*Proof.* Let us show that under the assumptions made $u \geq u_S$ in $(t^*, T) \times \mathbf{R}_+$, where $u_S = u_S(t, x; A_0^*)$ is some non-localized self-similar solution of the HS-regime (2.4), (2.5) for $A_0 = A_0^*$.

Without loss of generality let $u(t^*, x) > 0$ on an interval $(0, x^*)$, $x^* > 0$. Let us choose the value of $A_0 \equiv A_0^* > 0$ in the self-similar solution (2.4) for $n < -1/\sigma$ so small that

$$u_S(t^*, x; A_0^*) \leq u(t^*, x), \quad x \geq 0. \tag{10}$$

Existence of such a constant $A_0^* > 0$ follows from the obvious conditions (see (2.4), (2.5)): $u_S(t^*, x; A_0^*) \to 0$, $\operatorname{supp} u_S(t^*, x; A_0^*) \to \{0\}$ as $A_0^* \to 0^+$.

Then by (8), (10) and the comparison theorem we have that $u(t, x) \geq u_S(t, x; A_0^*)$ in $(t^*, T) \times \mathbf{R}_+$, which proves (9). □

Numerical solution of the problem (1.1)–(1.3) in the case $u_1(t) = A_0(T - t)^n$, $n < -1/\sigma$ (HS-regime), is shown in Figure 25.

Theorems 1-3 allow us to classify boundary conditions that lead to blow-up, and establish important properties of regimes of heat propagation. The "boundary" between different regimes is the boundary condition corresponding to the self-similar S-regime.

Let us note that if on the boundary we are given not the temperature, but time-dependent heat flux that blows up in finite time: $W(t, 0) \to \infty$, $t \to T^-$, then if the inequalities

$$W(t,0) \le W_0(T-t)^{-(\sigma+1)/\sigma};$$

$$W(t,0) \le W_0(T-t)^{n_1}, \ -(\sigma+1)/\sigma < n_1 < -1/2;$$

$$W(t,0) \ge W_0(T-t)^{n_1}, \ n_1 < -(\sigma+1)/\sigma,$$

are satisfied, respective analogues of Theorems 1-3 hold. These results follow from properties of self-similar solutions of the second boundary value problem, which blow up in finite time.

## 3 Physical basis for heat localization. A class of temperature profiles with inertia

Results of subsections 1, 2 show that localization is conditioned not only by the speed of the process, but also by "internal" properties of the heat conducting medium.

Let us consider the evolution of a thermal perturbation in a medium, which is not acted upon by any boundary regime, or the Cauchy problem for equation (1.1) with the initial condition

$$u(0,x) = u_0(x) \ge 0, \ -\infty < x < \infty, \tag{11}$$

The function $u_0(x)$ has compact support and $\operatorname{supp} u_0 = (-x_0, x_0)$, $x_0 > 0$ is a constant.

**Definition.** There is *heat localization (inertia)* in the problem (1.1), (11), if there exists $t_l$, such that $\operatorname{supp} u(t,x) = \operatorname{supp} u_0$ for all $0 < t < t_l$.

In other words, heat contained initially in the domain $|x| < x_0$ does not propagate out of the domain during the finite localization time $t_l = t_l(\sigma, k_0; u_0)$. In the following theorem we characterize a class of "inertial" temperature profiles.

**Theorem 4.** *If the initial heat profile satisfies the condition*

$$0 < u_0(x) \le u_m(1-|x|/x_0)^{2/\sigma}, \ |x| < x_0, \tag{12}$$

*then there is heat localization in the Cauchy problem* (1.1), (11) *and the localization time satisfies the estimate*

$$t_l \ge t^* = x_0^2\sigma / \left[2k_0(\sigma+2)u_m^\sigma\right]. \tag{13}$$

*Proof.* By the Maximum Principle,

$$u(t,0) \le u_m, \ t > 0. \tag{14}$$

The function in the right-hand side of (12) has for $x > 0$ the same initial data as (1.5), the self-similar solution $u_S(t, x)$ of the S-regime, if we set there $T = x_0^2\sigma/[2k_0(\sigma+2)u_m^\sigma]$, $A_S = \{x_0^2\sigma/[2k_0(\sigma+2)]\}^{1/\sigma}$ (here $x_S = x_0$, $u_S(0, 0) = u_m$).

Let us compare the solutions $u(t, x)$ and $u_S(t, x)$ in $(0, T) \times \mathbf{R}_+$. From (12), (14), we have that $u(0, x) = u_0(x) \le u_S(0, x)$ for $x \ge 0$ and $u(t, 0) \le u_m \le u_S(t, 0)$ for all $0 < t < T$. Therefore by the comparison theorem

$$u(t, x) \le u_S(t, x), \quad x \ge 0, \quad 0 < t < T.$$

Hence, using the properties of the solution $u_S$ (see (1.5)), we obtain

$$u(t, x) = 0, \quad x \ge x_0, \quad 0 < t < T = t^*. \tag{15}$$

Similarly, it is proved that

$$u(t, x) = 0, \quad x < -x_0, \quad 0 < t < t^*. \tag{16}$$

Combining (15) and (16), we have that $u(t, x) = 0$ for $|x| > x_0$, $0 < t < t^*$, which concludes the proof. □

Thus, there always exists a class of initial temperature profiles which have the localization property. The estimate (13) of localization time depends on the parameters of the medium ($k_0$, $\sigma$) and the initial temperature profile (parameters $x_0$, $u_m$).

The heat inertia phenomenon has a simple physical interpretation. The rate of temperature growth at any point of the medium is determined by its spatial profile in a neighbourhood of that point. If the temperature profile is sufficiently "convex" (in the case $\sigma = 2$ convexity has the usual meaning), the temperature does not change, or changes only slightly in a neighbourhood of the front points $x = \pm x_0$.

Therefore immobility of the thermal front depends on the behaviour of $u_0(x)$ in a small neighbourhood of the front, though, of course, the length of localization time is determined by the global spatial structure of the initial perturbation (in other words, by the "degree of convexity" of its profile everywhere in $\operatorname{supp} u_0$; this is reflected in the estimate (12)).

It is clear that in a medium without absorption a convex temperature profile can exist only for a finite time[2]. Thermal energy enters colder regions from the hotter ones, a "concave" temperature profile is formed, and the wave starts to move.

This is easily seen by comparing $u_0(x) \not\equiv 0$ with the function

$$\bar{u}(0, x) = \bar{u}_0(x) = \bar{u}_m(0)(1 - x^2/x_f^2(0))_+^{1/\sigma}, \tag{17}$$

[2]Localization of thermal perturbations coming from the presence in the medium of heat sinks, which was briefly considered in Ch. I, II, is of a different "physical" nature. In particular, in the presence of absorption, localization for any length of time is possible.

that is, the self-similar solution

$$\bar{u}(t,x) = \bar{u}_m(t)(1 - x^2/x_f^2(t))_+^{1/\sigma} \tag{18}$$

of equation (2.1) at time $t = 0$; here

$$x_f(t) = C_1(\sigma)k_0^{1/(\sigma+2)}Q_0^{\sigma/(\sigma+2)}(t+t_0)^{1/(\sigma+2)},$$

$$u_m(t) = C_2(\sigma)k_0^{-1/(\sigma+2)}Q_0^{2/(\sigma+2)}(t+t_0)^{-1/(\sigma+2)},$$

and $C_1$, $C_2$ are some positive constants. This solution of "instantaneous point source" type describes the evolution of a thermal perturbation of energy $Q_0 > 0$ concentrated at the point $x = 0$ at time $t = -t_0 < 0$ (see § 3, Ch. I).

Choosing the magnitudes of $t_0$ and $Q_0$ so that $u_0(x) \geq \bar{u}_0(x)$ in $\mathbf{R}$, by the comparison theorem we have that $u(t,x) \geq \bar{u}(t,x)$ in $\mathbf{R}_+ \times \mathbf{R}$, and therefore meas supp $u(t,x) \geq$ meas supp $\bar{u}(t,x) \sim t^{1/(\sigma+2)} \to \infty$ as $t \to \infty$. Localization in the Cauchy problem is possible only for a finite period of time.

The initial function (17) is an example of a "concave" temperature profile, which does not have the inertia property. Here the thermal wave is in motion for all $t > 0$. By the comparison theorem the same is true for all initial perturbations $u_0(x)$, which majorize (17), when the fronts of the perturbations $u_0(x)$ and $\bar{u}_0(x)$ are the same.

Figure 26a shows the results of a numerical computation, which illustrates Theorem 4 in the case $u_0(x) = u_m(1 - |x|/x_0)_+^{2/\sigma}$. Till time $t = t^*$ heat is localized in the domain $(-x_0, x_0)$. In the course of time, the temperature profile rearranges itself into a concave shape, and the wave starts to move. Evolution of an initially concave profile is shown in Figure 26b, where the initial function has the form (17) (the size of both perturbations and the amount of energy they contain is the same in both cases).

Figure 26c shows the dynamics of a "combined" profile: for $x > 0$ we have taken $u_0(x) = u_m(1 - x/x_0)_+^{2/\sigma}$, while for $x < 0$ we have taken the function (17). As a result the right side is localized, while on the left the wave starts to move immediately, that is, for a finite time we have directional heat conductance.

The above properties of temperature profiles allow us to explain the physical nature of the localization phenomenon under the action of slow blow-up regimes on the medium (subsection 1). Boundary S- and LS-regimes expose the inertia of the heat conductance process by creating and supporting localized temperature profiles. In the S-regime the rate of energy supply into the material is so adjusted to the properties of the medium that the heat is distributed over the whole profile (see (1.5) and Figure 17). With a slower energy "provision," heat is mainly concentrated near the boundary, the profile is more "convex" (compare Figure 24 with Figures 23, 17), and the LS-regime is brought about.

Formation of inertial profiles takes place at the localization depth, which is determined by the parameters of the problem.

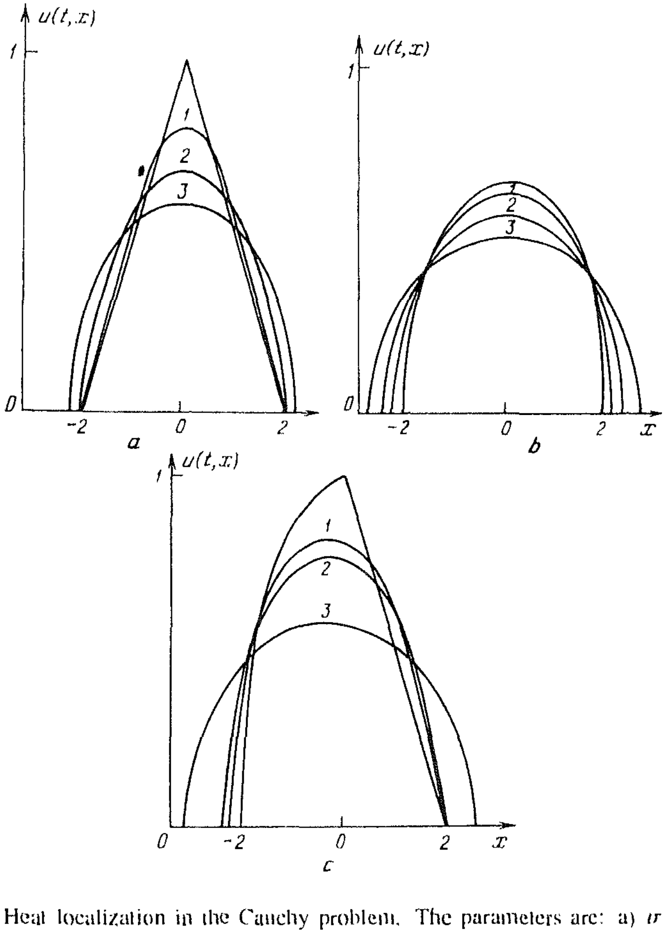


**Fig. 26.** Heat localization in the Cauchy problem. The parameters are: a) $\sigma = x_0 = 2$, $k_0 = u_m = t^* = 1$; 1: $t = 0.25$, 2: $t = 1$, 3: $t = 2.5$. b) $\sigma = x_f(0) = 2$, $k_0 = 1$, $\overline{u}_m(0) = 0.62$; 1: $t = 0.5$, 2: $t = 2$, 3: $t = 5$. c) $\sigma = x_0 = x_f(0) = 2$, $u_m = \overline{u}_m(0) = k_0 = 1$; 1: $t = 0.5$, 2: $t = 1$, 3: $t = 2$

As the thermal wave approaches the boundary of the localization domain, the concave propagating profile rearranges itself into a convex form, which is easily seen in Figures 23, 24. From that moment the localization effect becomes manifest: the size of the heated domain does not change significantly, the half-width is either constant or decreases, heat does not penetrate beyond the localization depth. If, after the formation of the inertial profile, energy is no longer supplied (the heat flux at the boundary $x = 0$ vanishes), then during the period leading to blow-up of the original boundary regime, the thermal wave hardly propagates (see Figure 26, $a$).

During heating which is faster than that of the S-regime, a concave temperature profile is formed (compare Figure 25 with Figures 24, 23); the domain occupied by the thermal wave expands, there is no localization, and we have the HS-regime.

Let us note that the action of boundary regimes that do not blow up always creates concave profiles and there is no localization (see the Remark in subsection 3, § 1).

Therefore the heat localization phenomenon is related not only to the speed of the process. Interaction between the rate of heating of the medium and its properties determines the nature of the temperature profiles being formed, inertial or otherwise. This conclusion is also true in the case of arbitrary media (see Ch. V), including media with volumetric energy sources (see Ch. IV).

## 4 Heat localization in multi-dimensional problems. The "thermal crystal"

The main properties of blow-up regimes in heat conducting media established in subsections 1-3 for one-dimensional media, also characterize the case of many spatial variables. The method of analysis of multi-dimensional equations is also based on the construction of certain particular solutions and the use of comparison theorems.

The new element, in comparison with one-dimensional geometry, is the shape of the heat localization domain, which can be quite contrary to intuition about diffusional dissipative processes.

Let us illustrate this remark using easy examples. First of all let us find a particular solution of the equation

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(k(u)\frac{\partial u}{\partial x_i}\right), \tag{19}$$

which is the multi-dimensional analogue of the self-similar S-regime. In equation (19) $x = (x_1, \ldots, x_N) \in \mathbf{R}^N$ are the spatial coordinates, $u = u(t, x) \geq 0$ is the temperature; $k(u) = k_0 u^\sigma$, $\sigma = \text{const} > 0$, is the thermal conductivity coefficient.

As in (1.5), we shall seek a separable solution of (19):

$$u_S(t, x) = V(t)\theta(x). \tag{20}$$

Substituting (20) into (19), we obtain the following equation for the functions $V(t)$, $\theta(x)$:

$$\frac{1}{V^{\sigma+1}}\frac{dV}{dt} = \frac{1}{\theta}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(k_0\theta^\sigma\frac{\partial\theta}{\partial x_i}\right) = C > 0. \tag{21}$$

Hence

$$V(t) = A_S(T-t)^{-1/\sigma},\ A_S = (C\sigma)^{-1/\sigma},\ 0 < t < T < \infty, \tag{22}$$

that is, the required solutions will blow up as $t \to T^-$ at the same rate as in the one-dimensional case.

The elliptic equation (21) satisfied by the function $\theta$ has a solution of the following kind:

$$\theta = \theta(\eta),\ \eta = \sum_{i=1}^{N} \alpha_i x_i;\ \alpha_i = \text{const} > 0;\ \sum_{i=1}^{N} \alpha_i^2 = 1.$$

Then $\theta(\eta) \geq 0$ satisfies the one-dimensional equation

$$\frac{1}{\theta}\frac{d}{d\eta}\left(k_0\theta^\sigma \frac{d\theta}{d\eta}\right) = \frac{A_S^{-\sigma}}{\sigma},$$

and, for example,

$$\theta(\eta) = \left(1 - \frac{\eta}{x_S}\right)_+^{2/\sigma},\ x_S = \left(2k_0 A_S^\sigma \frac{\sigma+2}{\sigma}\right)^{1/2}. \tag{23}$$

Therefore the desired solution has the form

$$u_S(t,x) = A_S(T-t)^{-1/\sigma}\left(1 - \sum_{i=1}^{N}\frac{\alpha_i x_i}{x_S}\right)_+^{2/\sigma}. \tag{24}$$

The spatio-temporal structure of this solution is the same as that of the one-dimensional one. It can be considered as the solution of the boundary value problems in $(0,T) \times \{x \in \mathbf{R}^N \mid x_i > 0,\ i = 1,\dots,N\}$ with the corresponding initial and boundary conditions.

Let us indicate the main properties of the solution, first of all in two-dimensional ($N = 2$) geometry. The boundary temperature is prescribed on the $x_1$, $x_2$ axes, is equal to zero for $x_1 \geq x_S/\alpha_1$, $x_2 \geq x_S/\alpha_2$ and blows up in finite time for $0 \leq x_1 < x_S/\alpha_1$, $0 \leq x_2 < x_S/\alpha_2$. Nonetheless, unbounded growth of the temperature as $t \to T^-$ takes place only in a finite localization domain, the *triangle* with vertices at $(0,0)$, $(x_S/\alpha_1, 0)$, $(0, x_S/\alpha_2)$. In the rest of the medium (for $\eta > x_S$) the temperature is zero for all $0 \leq t < T$. The localization domain is separated by the stationary front, which is the piece of the straight line connecting the points $(x_S/\alpha_1, 0)$, $(0, x_S/\alpha_2)$.

In the three-dimensional case the localization domain is the *pyramid* with the apex at $(0, 0, 0)$, triagonal base and vertices at $(x_S/\alpha_1, 0, 0)$, $(0, x_S/\alpha_2, 0)$, $(0, 0, x_S/\alpha_3)$. On the lateral boundaries of the pyramid the temperature blows up in finite time in accordance with (24), while outside the boundaries it is equal to zero. Inside the pyramid $u_S(t,x) \to \infty$ as $t \to T^-$. The temperature is maximal at the apex of the pyramid and decreases to zero as we approach its base ($\eta \to x_S$), which is the stationary boundary of the localization domain.

By analogy with (1.5), the solution (24) can be called a *multi-dimensional standing thermal wave*.

Let us clarify the relation between the solution constructed above with the one-dimensional one (Figure 27):

$$u_S(t, x_1) = \begin{cases} A_S(T-t)^{-1/\sigma}(1 - x_1/x_S)^{2/\sigma}, & x_1 \leq x_S, \\ 0, & x_1 > x_S, \end{cases} \tag{25}$$

which does not depend on other spatial coordinates in the planes $x_1 = \text{const}$. In Figure 27 the coordinate axes are oriented so that the $x_3$ axis is perpendicular to its plane.

Let us rotate the $x_1, x_2$ axes by an angle $\beta \in (0, \pi/2)$ with respect to the $x_3$ axis and let us consider solution (25) in a triangle with vertices at the points $(0, 0)$, $(x_S/\alpha_1, 0)$, $(0, x_S/\alpha_2)$; $\alpha_1 = \cos\beta$, $\alpha_2 = \sin\beta$. Inside it the temperature depends on $(t, x_1, x_2)$, therefore $u_S(t, x)$ can be considered as a solution of the two-dimensional equation (19). The boundary conditions and the solution itself are easily computed from (25), with rotation of the $x_1, x_2$ axes taken into account, and coincide with (24). The three-dimensional solution is obtained in a similar way after an additional rotation of the $x_3$ axis. This method can be used to construct multi-dimensional standing thermal waves in more complicated domains (dashed line in Figure 27). Boundary conditions are determined from (25) if the boundary of the domain is prescribed.

If the boundary regime in the multi-dimensional problem is slow (majorized by the S-regime boundary dependence), then we have heat localization, and upper bounds both for the solution and the localization domain can be obtained using the function (24). Figure 28 shows the results of numerical solution of equation (19) for $N = 2$ in the domain $\{x_1 > 0,\ x_2 > 0\}$ with zero initial conditions and boundary conditions

$$u(t, x_1, 0) = A_0(1-t)^n(1 - \alpha_1 x_1/x_S)_+^{2/\sigma},$$

$$u(t, 0, x_2) = A_0(1-t)^n(1 - \alpha_2 x_2/x_S)_+^{2/\sigma},$$

which are majorized by boundary values of the solution (24) for $A_S = 0.5$, $\alpha_1 = \alpha_2 = 1/\sqrt{2}$, $T = 1$. The localization domain is the triangle with vertices at $(0, 0)$, $(\sqrt{2}, 0)$, $(0, \sqrt{2})$. The results illustrate heat localization in the LS-regime, when $n > -1/\sigma$. The thermal wave for all $0 < t < 1$ is inside the domain of localization of the majorizing S-regime. From some moment of time onwards, a concave temperature profile is formed, and the effective dimensions of the hot domain shrink.

As in one-dimensional geometry, under the action of fast blow-up regimes there is no localization. Figure 29 shows the evolution of the multi-dimensional HS

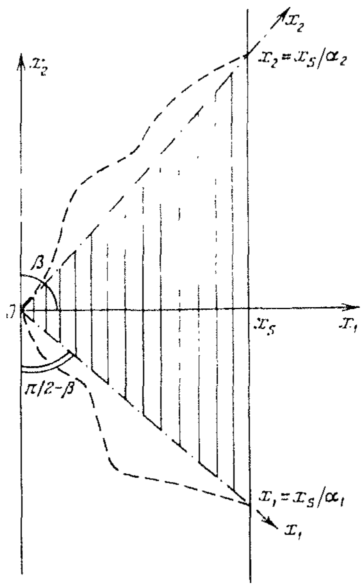


**Fig. 27.** Geometric interpretation of the solution (24)

blow-up regime under the same initial and boundary conditions as in the previous computation, but for $n < -1/\sigma$. Here a convex temperature profile is formed and heat propagates into infinitely far regions during the finite time of existence of the solution.

Therefore the main results and intuition concerning the influence of boundary blow-up regimes in a heat-conducting medium extend to the multi-dimensional case. Let us note that the rate of temperature growth in the S-regime, which separates slow and fast blow-up regimes, does not depend on the dimension of the space and is determined only by the properties of the medium.

A new feature of multi-dimensional geometry is the shape of the heat localization domain, which can vary considerably. Let us give some appropriate examples.

Let us consider a solution $u(t, x_1, x_2)$ of the problem for equation (19), $N = 2$, in the domain $(0, T) \times \{x_1 > 0, x_2 > 0\}$ with the initial function (24) for $t = 0$. Let us prescribe zero heat fluxes on the axes $x_1, x_2$: $k_0 u^\sigma u_{x_1} = 0$ for $x_2 = 0$, $k_0 u^\sigma u_{x_2} = 0$ for $x_1 = 0$, $t \in (0, T)$, so that no energy enters the medium.

Since heat fluxes on the boundary, corresponding to the self-similar solution $u_S(t, x)$, are positive, by the comparison theorem we have $u(t, x) \le u_S(t, x)$ in

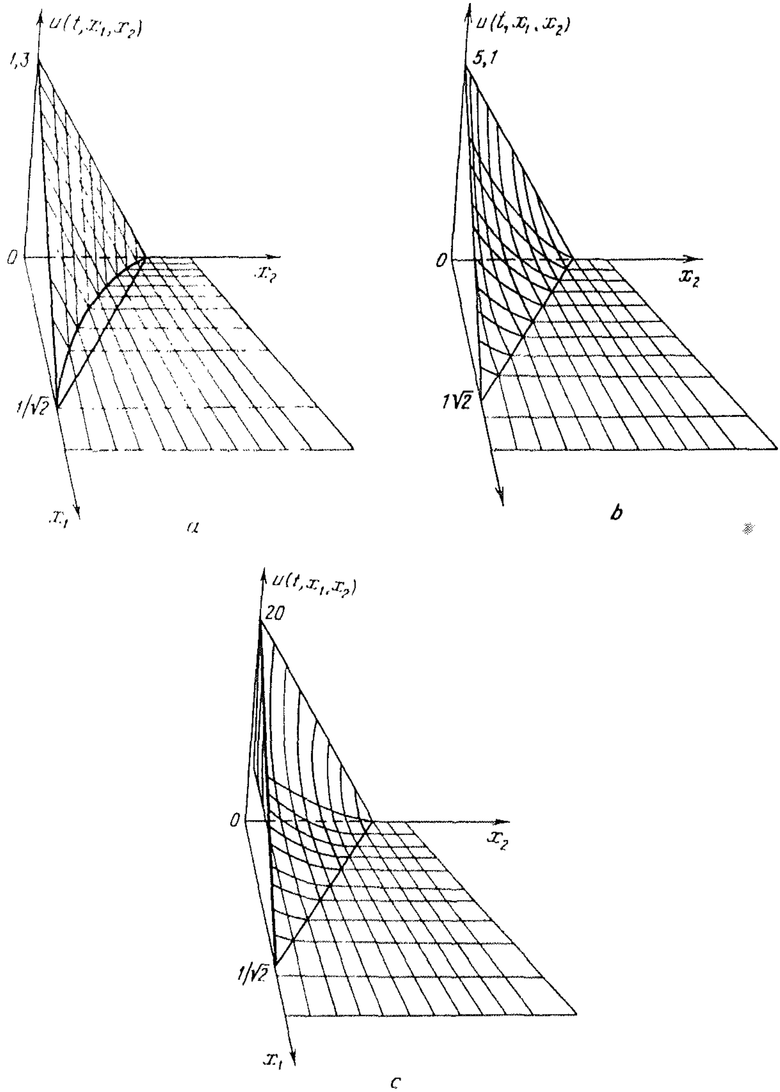


**Fig. 28.** LS-regime in two-dimensional geometry. The parameters are: $\sigma = 2$, $n = -0.25$, $k_0 = 1$, $A_0 = 0.5$, $T = 1$, $\alpha_1 = \alpha_2 = 1/\sqrt{2}$, $x_S = 1$; a) $T - t = 0.22$, b) $T - t = 9.1 \cdot 10^{-5}$, c) $T - t = 4 \cdot 10^{-7}$

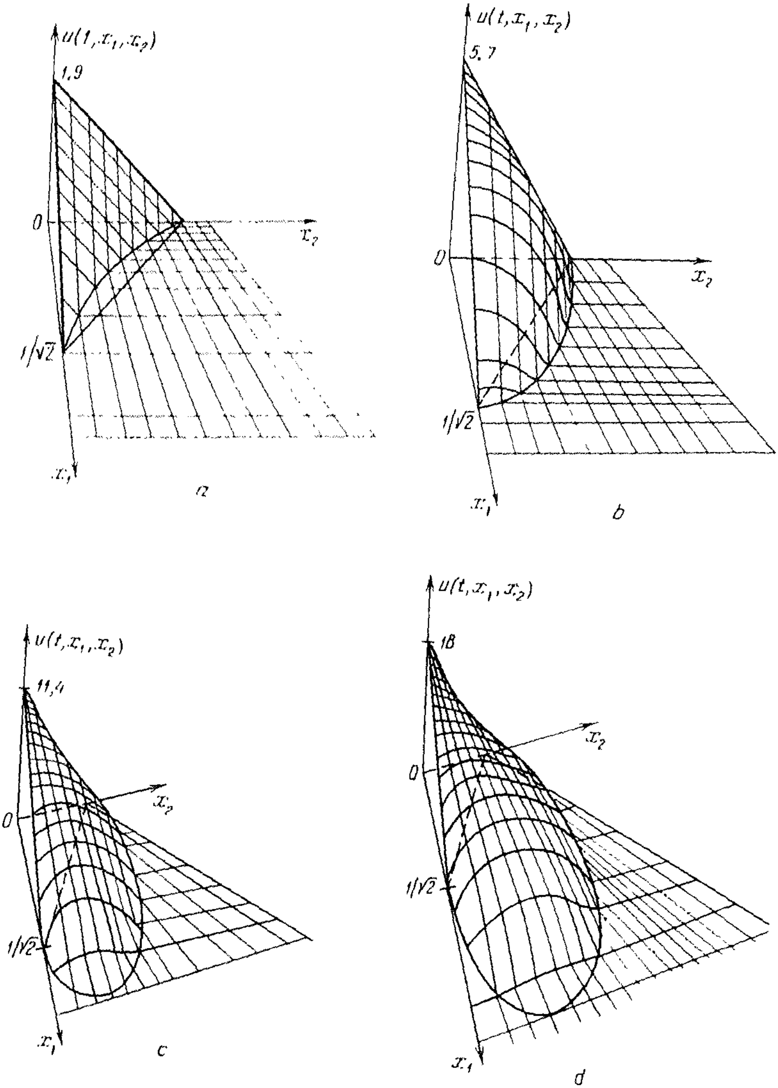


**Fig. 29.** HS-regime of heat propagation in two-dimensional geometry. The parameters are: $\sigma = 2$, $n = -1$, $k_0 = 1$, $A_0 = 0.5$, $T = 1$, $\alpha_1 = \alpha_2 = 1/\sqrt{2}$; a) $T - t = 0.27$, b) $T - t = 8.7 \cdot 10^{-2}$, c) $T - t = 4.3 \cdot 10^{-2}$, d) $T - t = 2.7 \cdot 10^{-2}$

$(0, T) \times \{x_1 \geq 0, x_2 \geq 0\}$, and therefore

$$u(t, x) = 0, \ t \in (0, T), \alpha_1 x_1 + \alpha_2 x_2 \geq x_S. \tag{26}$$

Moreover, by construction $u(t, x) > 0$ for $t \in (0, T)$ and $\alpha_1 x_1 + \alpha_2 x_2 < x_S$, $x_1 \geq 0$, $x_2 \geq 0$.

Solutions of equation (19) in the spatial domains $\{x_1 > 0, x_2 < 0\}$, $\{x_1 < 0, x_2 > 0\}$ and $\{x_1 < 0, x_2 < 0\}$ with the same boundary conditions have the same properties. Therefore $u(t, x) = 0$ for $t \in (0, T)$ for all $\alpha_1|x_1| + \alpha_2|x_2| \geq x_S$. However, by symmetry all these solutions coincide, in their respective quadrants, with the solution of the Cauchy problem for equation (19) in $\mathbf{R}_+ \times \mathbf{R}^2$ with initial function

$$u(0, x_1, x_2) = u_m[1 - (\alpha_1|x_1| + \alpha_2|x_2|)/x_0]_+^{2/\sigma}, \tag{27}$$

where $u_m = A_S T^{-1/\sigma}$, $x_0 = x_S = (2k_0 A_S^\sigma(\sigma + 2)/\sigma)^{1/2}$.

Thus, the initial temperature distribution (27) is localized in the *rhombus* (diamond) $\alpha_1|x_1| + \alpha_2|x_2| \leq x_0$ for time not less than $t_l \geq t^* = \sigma x_0^2/(2k_0(\sigma + 2)u_m^\sigma)$. The estimate of localization time is identical to formula (13) for one-dimensional geometry.

Performing the same construction for the three-dimensional case, we see that the initial perturbation

$$u_0(x_1, x_2, x_3) = \begin{cases} u_m\left(1 - \sum_{i=1}^3 \alpha_i \dfrac{|x_i|}{x_0}\right)^{2/\sigma}, & \sum_{i=1}^3 \alpha_i|x_i| \leq x_0, \\ 0, & \sum_{i=1}^3 \alpha_i|x_i| > x_0, \end{cases} \tag{28}$$

is localized for a finite time ($t_l$ is estimated using a similar formula) in the *octahedron* $\sum_{i=1}^3 \alpha_i|x_i| \leq x_0$.

For a finite time a *thermal crystal*, an octahedron that preserves its shape, exists in the medium. Inside it, the temperature is different from zero, while on its boundaries and outside, it is zero for all $t \in (0, t_l)$.

A numerical simulation of a two-dimensional analogue of the thermal crystal is shown in Figure 30. Initial data is the function (27). For a finite length of time thermal energy is localized inside the square. With time, the temperature profile inside the localization domain rearranges itself into a convex shape, and heat starts to spread (compare with Figure 26, *a*).

By the comparison theorem, the functions (27), (28) define a class of inertial temperature profiles in multi-dimensional geometries. The temperature distribution is localized if for each of its front points we can find a function of the form (27) or (28), which majorizes this distribution and has the same fronts. By this method it is not hard to construct localization domains of various shapes (ball, ellipsoid, etc.) and to determine the corresponding initial temperature profiles.

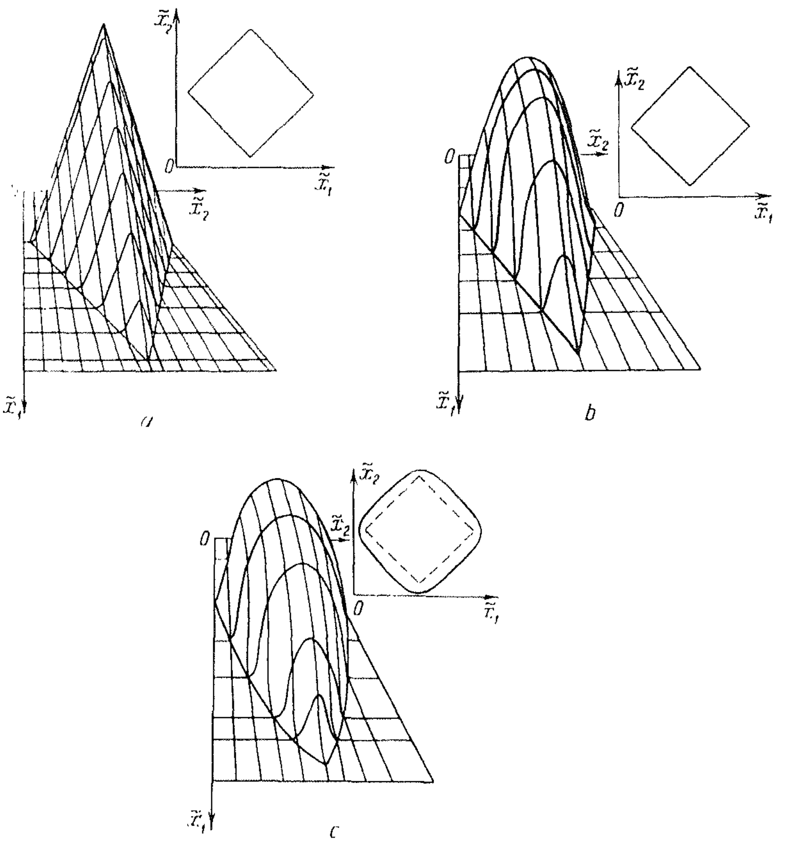


**Fig. 30.** Thermal crystal in two-dimensional geometry. The parameters are: $\sigma = 2$, $k_0 = 1$, $u_m = 0.5$, $x_0 = 1$, $\alpha_1 = \alpha_2 = 1/\sqrt{2}$, $t^* = 1$; a) $t = 0$, $\max u(0, x_1, x_2) = 0.5$, b) $t = 0.8$, $\max u(t, x_1, x_2) = 0.24$, c) $t = 4$, $\max u(t, x_1, x_2) = 0.18$

# § 4 Effective heat localization

## 1 Independence of effective localization on the initial state

In the study of the thermal inertia phenomenon in § 2, 3, we approximated the "zero background," by taking the initial function to have compact support.

Let us consider the original problem (1.1)–(1.3), in which the bounded continuous function $u_0(x)$ is arbitrary. In this case heat localization is to be understood in the sense of effective localization.

First of all let us establish the connection between two solutions $u^{(\nu)}$ ($\nu = 1, 2$) of problem (1.1)–(1.3) corresponding to different constant initial functions, $u_0^{(\nu)}(x) \equiv C^{(\nu)} = \text{const} \geq 0$ and the same boundary regimes as $t \to T^-$, $u^{(\nu)}(t, 0) = u_1^{(\nu)}(t) > 0$, $t \in (0, T)$.

**Lemma 1.** *Let $C^{(1)} \leq C^{(2)}$, and assume that the functions $u_1^{(\nu)}$ do not decrease in $t \in (0, T)$, and that there exists $\tau \in (0, T)$ such that*

$$u_1^{(1)}(t) = u_1^{(2)}(t) \equiv u_1(t), \ t \in [\tau, T);$$
$$u_1^{(1)} \leq u_1^{(2)}, \ t \in (0, \tau). \tag{1}$$

*Then*

$$\Delta E(t) \equiv \Delta E^{(2)}(t) - \Delta E^{(1)}(t) \leq \Delta E(\tau), \ t \in [\tau, T), \tag{2}$$

*where*

$$\Delta E^{(\nu)}(t) = \int_0^\infty \{u^{(\nu)}(t, x) - C^{(\nu)}\}\, dx \in [0, \infty), \ t \in (0, T); \ \nu = 1, 2.$$

The functions $\Delta E^{(\nu)}(t)$ have the meaning of energy supplied to the medium up to the moment $t \in (0, T)$.

*Proof.* First of all let us note that by the Maximum Principle $u^{(\nu)} \geq C^{(\nu)}$ in $(0, T) \times \mathbf{R}_+$, so that $\Delta E^{(\nu)}(t) \geq 0$. Under the assumptions of the lemma, equation (1.1) can be integrated over $(\tau, t) \times \mathbf{R}_+$, $t \in (\tau, T)$. This follows from an integral identity satisfied by the generalized solution for a particular choice of a sequence of test functions with compact support $f = f(x/\alpha) \to 1$ as $\alpha \to \infty$ everywhere in $\mathbf{R}_+$, $|f'(\xi)| \leq 1$, and known regularity of the generalized solution (see § 3, Ch. 1). The result of formal integration of equation (1.1) over $(\tau, t) \times \mathbf{R}_+$ is:

$$\Delta E^{(\nu)}(t) - \Delta E^{(\nu)}(\tau) = -\int_\tau^t k\left[u^{(\nu)}(s, 0)\right] \frac{\partial u^{(\nu)}}{\partial x}(s, 0)\, ds \geq 0, \ \nu = 1, 2$$

(modulo sign, the integrands are heat fluxes at the boundary). By (1), we have for all $t \in [\tau, T)$

$$\Delta E(t) - \Delta E(\tau) = \int_\tau^t k[u_1(s)] \left[\frac{\partial u^{(1)}}{\partial x}(s, 0) - \frac{\partial u^{(2)}}{\partial x}(s, 0)\right] ds. \tag{3}$$

By the comparison theorem $u^{(1)} \leq u^{(2)}$ in $(0, T) \times \mathbf{R}_+$. Since $u^{(1)}(t, 0) = u^{(2)}(t, 0)$ for $t \in [\tau, T)$, we obtain

$$\frac{\partial u^{(1)}}{\partial x}(t, 0) \leq \frac{\partial u^{(2)}}{\partial x}(t, 0).$$

Therefore the integrand in (3) is non-positive, which is equivalent to (2). □

Therefore, for the same heating regime, the amount of heat entering a colder medium is not less than that supplied to a warmer one (if $\Delta E(\tau) \le 0$).

Of course this lemma can be extended to cover a wider class of initial perturbations. If $u_0^{(\nu)}$ are non-constant, but, for example,

$$u_0^{(1)} \le u_0^{(2)},\ x > 0;\ u_0^{(\nu)} \in L^1(\mathbf{R}_+),\ \left[\phi(u_0^{(\nu)})\right]_x \to 0,\ x \to \infty, \tag{4}$$

then if conditions (1) hold, instead of (2) we derive the estimate

$$0 \le \int_0^\infty [u^{(2)} - u^{(1)}](t,x)\,dx \le \int_0^\infty [u^{(2)} - u^{(1)}](\tau,x)\,dx, \tag{5}$$

or, in other words, $\|u^{(2)}(t,\cdot) - u^{(1)}(t,\cdot)\|_{L^1(\mathbf{R}_+)}$ is non-increasing in $t \in [\tau, T)$.

The lemma we proved above expresses a kind of stability in $L^1(\mathbf{R}_+)$ of the heat diffusion process to perturbations of the initial function. Using it, we can establish the following assertion concerning independence of effective localization depth from the initial function.

**Theorem 1.** *Let $u^{(\nu)}$ ($\nu = 1, 2$) be solutions of problem* (1.1)–(1.3) *with boundary conditions $u_0^{(\nu)}$, $u_1^{(\nu)}$, respectively, such that conditions* (1) *hold. Furthermore, let*

$$u_1(t) \to \infty,\ t \to T^-,$$
$$\left[u_1^{(\nu)}(t)\right]' \ge 0,\ t \in [0,T); \tag{6}$$

*$u^{(\nu)}(x)$ are non-increasing in $x > 0$; $\nu = 1, 2$. Let $u^{(1)}$ be effectively localized and let the depth of localization be $L^{(1)*}$. Then $u^{(2)}$ is also localized, and $L^{(2)*} = L^{(1)*}$.*

*Proof.* Let us consider $\overline{u}^{(1)}, \overline{u}^{(2)}$, solutions of problem (1.1)–(1.3) satisfying the conditions $\overline{u}^{(1)}(0,x) \equiv 0$,

$$\overline{u}^{(2)}(0,x) \equiv C = \max\{\sup u_0^{(1)}, \sup u_0^{(2)}\} \equiv \max\{u_0^{(1)}(0), u_0^{(2)}(0)\},$$

$$\overline{u}^{(1)}(t,0) \equiv \overline{u}^{(2)}(t,0) \equiv u_1(t) \text{ for } t \in [\tau, T).$$

Let us extend the functions $\overline{u}^{(\nu)}(t,0)$ in $[0,\tau)$ so that Lemma 1 could be applied to the solutions $\overline{u}^{(\nu)}$ and $\overline{u}^{(1)}(t,0) \le u_1^{(1)}(t),\ u_1^{(2)}(t) \le \overline{u}^{(2)}(t,0)$.

From the comparison theorem it follows that

$$\overline{u}^{(1)} \le u^{(1)},\ u^{(2)} \le \overline{u}^{(2)} \text{ in } (0,T) \times \mathbf{R}_+. \tag{7}$$

Applying the lemma to the functions $\overline{u}^{(\nu)}$, we rewrite inequality (2) for $C^{(1)} = 0$, $C^{(2)} = C$ in the form

$$\int_0^\infty \{\overline{u}^{(2)}(t,x) - C - \overline{u}^{(1)}(t,x)\}\,dx \le \text{const},\ t \in [\tau, T).$$

Decomposing this integral into the sum of integrals over $(0, x_0)$ and $(x_0, \infty)$, where $x_0 > L^{(1)*}$ is an arbitrary constant, we obtain

$$\int_0^{x_0} \{\overline{u}^{(2)}(t,x) - \overline{u}^{(1)}(t,x)\}\,dx - \int_0^{x_0} C\,dx + \int_{x_0}^{\infty} \{\overline{u}^{(2)}(t,x) - C\}\,dx -$$

$$- \int_{x_0}^{\infty} \overline{u}^{(1)}(t,x)\,dx \equiv I_1 - I_2 + I_3 - I_4 \le \text{const}, \quad t \in [\tau, T).$$

Let us prove uniform boundedness in $t \in [\tau, T)$ of the integral $I_3$. Since by the Maximum Principle $\overline{u}^{(2)} \ge \overline{u}^{(1)}$ in $(0,T) \times \mathbf{R}_+$, we have that $I_1 \ge 0$. Furthermore, $I_2 = Cx_0$ and therefore $I_3 \le \text{const} + I_4$ for all $t \in [\tau, T)$.

Let us consider the integral $I_4$. Since the solution $u^{(1)}$ is localized and $x_0 > L^{(1)*}$, there exists a constant $M > 0$, such that $u^{(1)} \le M$ in $(0,T) \times [x_0, \infty)$. By (7) this means that $\overline{u}^{(1)} \le M$ in $(0,T) \times [x_0, \infty)$.

Then by the comparison theorem $\overline{u}^{(1)} \le u_S$ in $(0,T) \times (x_0, \infty)$, where $u_S(t,x) \equiv \theta((x-x_0)/t^{1/2})$ is the self-similar solution of equation (1.1), which satisfies the conditions $u_S(0,x) = 0$, $x > x_0$; $u_S(t, x_0) = M$, $t \in (0,T)$. Concerning the existence and uniqueness of the solution $u_S \ge 0$ see subsection 4, § 3, Ch. I, as well as the Comments to Ch. I. Here $\theta \in L^1(\mathbf{R}_+)$.

Thus

$$I_4 \equiv \int_{x_0}^{\infty} \overline{u}^{(1)}(t,x)\,dx \le \int_{x_0}^{\infty} u_S(t,x)\,dx \equiv$$

$$\equiv \int_{x_0}^{\infty} \theta\left(\frac{x-x_0}{t^{1/2}}\right)dx \equiv t^{1/2}\int_0^{\infty} \theta(\xi)\,d\xi \le T^{1/2}\|\theta\|_{L^1(\mathbf{R}_+)} < \infty$$

for all $t \in [\tau, T)$. Therefore

$$I_3 \equiv \int_{x_0}^{\infty} \{\overline{u}^{(2)}(t,x) - C\}\,dx \le \text{const} \tag{8}$$

for any $t \in [\tau, T)$. From this we immediately deduce uniform boundedness of $\overline{u}^{(2)}$ in $[\tau, T) \times (x_0, \infty)$. Indeed, by monotonicity in $t \in (0, T)$ of the boundary function $\overline{u}_1^{(2)}(t)$, first, $\overline{u}^{(2)} \ge C$ (that is, $I_3 \ge 0$), and, second, $\overline{u}^{(2)}(t, x)$ is non-increasing in $x$ for any $t \in (0, T)$ (see §§ 1, 2, Ch. V). Therefore for any $x_1 > x_0$

$$I_3 \ge (x_1 - x_0)[\overline{u}^{(2)}(t, x_1) - C],$$

and therefore the assumption

$$\overline{\lim}_{t \to T}\ \overline{u}^{(2)}(t, x_1) = \infty$$

leads to a contradiction with (8).

By the second of inequalities (7) we have that for any $x_0 > L^{(1)*}$ there exists a constant $M > 0$, such that $u^{(2)} < M$ in $(0, T) \times (x_0, \infty)$. Therefore the solution $u^{(2)}$ is localized and $L^{(2)*} \leq L^{(1)*}$.

Exchanging $u^{(1)}$ and $u^{(2)}$, and using the same argument, we obtain the opposite estimate $L^{(1)*} \leq L^{(2)*}$, so that $L^{(1)*} = L^{(2)*}$, which concludes the proof. □

Let us now consider a case of absence of localization.

**Theorem 2.** *Under the conditions of Theorem 1 let the solution $u^{(1)}$ be not localized (HS-regime). Then $u^{(2)}$ is not localized either.*

*Proof.* If we assume the contrary, viz., that $u^{(2)}$ is localized, then by Theorem 1 $u^{(1)}$ is also localized, which contradicts the assumption. □

**Remark.** The requirements (6) on the boundary data can be substantially weakened. Actually, for Theorems 1, 2 to hold, it is sufficient to satisfy the first of conditions (6), and to have the continuous initial functions $u_0^{(r)}$ uniformly bounded.

Thus, the properties of regimes that blow up as $t \to T^-$ do not depend on the initial temperature profile. If a boundary dependence ensures heat localization (in either strict or effective sense) for any initial condition, then localization will occur for any other bounded initial perturbation. Depth of localization and class of regime (S- or LS-) are also preserved.

In particular, many asymptotic properties of heat diffusion processes in the problem (1.1)–(1.3) remain the same. These were studied in §§ 2, 3 for $k(u) = k_0 u^\sigma$ and a function $u_0(x)$ with compact support. For example, we have

**Theorem 3.** *Assume that in the problem* (1.1)–(1.3)

$$k(u) = k_0 u^\sigma, \sigma > 0, \textit{ and } u_1(t) = A_S(T-t)^{-1/\sigma}, t \in (0, T).$$

*Then the solution is effectively localized and*

$$L^* = (2k_0 A_S^\sigma(\sigma+2)/\sigma)^{1/2}.$$

Figure 31 shows the results of a numerical computation, which illustrates Theorem 3.

Asymptotic stability of self-similar solutions of the HS-, S-, and LS-regimes is proved in Ch. VI, where we also derive convergence rate estimates. For example, Figure 32 shows the "arrival" of a non-self-similar solution at the spatio-temporal structure of the self-similar HS-regime (thick line).

Without introducing a precise notion of closeness, let us observe that the influence of initial data in a domain covered by a thermal wave becomes negligible if the medium is supplied with an amount of energy, which is at least an order of

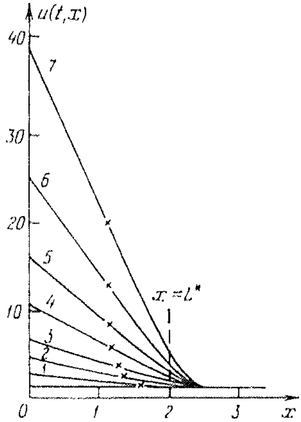


**Fig. 31.** Effective heat localization in a medium with nonlinear heat conductivity. The parameters are: $\sigma = 2$, $k_0 = 1$, $u_0(x) \equiv 1$, $A_S = T = 1$, $L^* = 2$; 1: $T - t = 1.7 \cdot 10^{-1}$, 2: $T - t = 5.48 \cdot 10^{-2}$, 3: $T - t = 2.37 \cdot 10^{-2}$, 4: $T - t = 8.14 \cdot 10^{-3}$, 5: $T - t = 3.9 \cdot 10^{-3}$, 6: $T - t = 1.58 \cdot 10^{-3}$, 7: $T - t = 6.37 \cdot 10^{-4}$

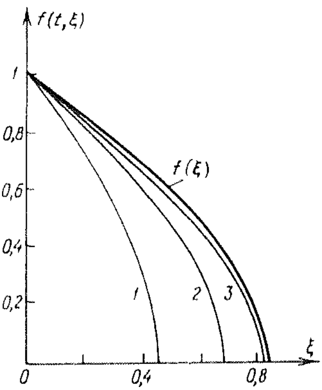


**Fig. 32.** "Arrival" of a solution at a self-similar HS-regime. The parameters are: $\sigma = 2$, $n = -1$, $k_0 = 0.5$, $A_0 = 0.12$, $T = 1.12 \cdot 10^{-1}$; 1: $T - t = 9.2 \cdot 10^{-3}$, 2: $T - t = 6.15 \cdot 10^{-3}$, 3: $T - t = 1.05 \cdot 10^{-3}$; $f(t, \xi) = \frac{u(t,\ \xi(k_0 A_0^\sigma (T-t)^{1+n\sigma})^{1/2})}{A_0 (T-t)^n}$

magnitude larger than the initial amount. For example, for the S-regime, when the characteristic size of the resulting thermal wave is constant, convergence to self-similar structures occurs when the temperature at the boundary is approximately 10 times larger than the characteristic initial temperature. This fact is reflected in Figure 31.

Therefore the self-similar solutions that blow up in finite time, which were constructed in § 2, are stable asymptotic states of thermal processes.

Heat localization in a medium, or the lack thereof, is determined only by the form of the boundary regime, unlike some other phenomena of nonlinear heat conductance (for example, finite speed of propagation of perturbations), for the existence of which special initial data are required.

These results extend the sphere of applicability of the phenomena we are considering in various physical situations. However, the following question arises: is heat localization a property of a medium with precisely the nonlinear thermal conductivity $k(u) = k_0 u^\sigma$, $\sigma > 0$, or is the localization phenomenon present in arbitrary media? In particular, is it possible to obtain the different heat diffusion regimes in a medium described by the classical heat equation?

## 2 Influence of boundary blow-up regimes on a medium with constant thermo-physical properties

Let us consider the problem of heating a medium with constant thermal conductivity in a boundary blow-up regime,

$$\frac{\partial u}{\partial t} = k_0 \frac{\partial^2 u}{\partial x^2}, \quad 0 < t < T, \; x > 0, \tag{9}$$

$$u(0, x) = 0, \quad x > 0, \tag{10}$$

$$\begin{gathered} u(t, 0) = u_1(t) \geq 0, \quad t \in [0, T); \\ u_1 \in C([0, T)); \; u_1 \to \infty, \; t \to T^- . \end{gathered} \tag{11}$$

which is a particular case of the problem (1.1)–(1.3). For simplicity, we have taken the initial temperature of the material equal to zero, which is not essential, due to the superposition principle (see also subsection 1).

In processes described by equation (9), perturbations propagate with infinite speed, so that localization must be understood in the effective sense.

Solution of the problem (9)–(11) is expressed in terms of the double layer potential

$$u(t, x) = \frac{x}{2\sqrt{k_0 \pi}} \int_0^t \exp\left\{ -\frac{x^2}{4k_0(t-\tau)} \right\} \frac{u_1(\tau)}{(t-\tau)^{3/2}} \, d\tau. \tag{12}$$

To determine conditions for localization in the problem (9)–(11), let us pass in (12) to the limit as $t \to T^-$:

$$\lim_{t \to T^-} u(t, x) \equiv u(T^-, x) = \frac{x}{2\sqrt{k_0 \pi}} \int_0^T \exp\left\{ -\frac{x^2}{4k_0(T-\tau)} \right\} \frac{u_1(\tau)}{(T-\tau)^{3/2}} \, d\tau. \tag{13}$$

From (13) it can be seen that the most interesting class is of "exponential" blow-up regimes. Indeed, if

$$u_1(t) = A_S(T-t)^{\nu}\exp\{R_0(T-t)^{-1}\},\ R_0, A_S > 0, \tag{14}$$

then $u(t, x) \to \infty$ as $t \to T^-$, where $x \in (0, x_S)$, where

$$x_S = 2\sqrt{k_0 R_0}. \tag{15}$$

For any $x > x_S$ the temperature at $t = T^-$ is bounded:

$$u(T^-, x) = \frac{A_S x^{2\nu}}{2^{2\nu}\sqrt{\pi}} k_0^{-\nu}\left[1-\left(\frac{x_S}{x}\right)^2\right]^{-\nu-1/2}\int_{(x^2-x_S^2)/(4k_0T)}^{\infty} e^{-u}u^{-\nu-1/2}\,du < \infty. \tag{16}$$

For $t = T^-$ the heat flux and the amount of contained energy in the domain $x > x_S$ are bounded. The parameter $\nu$ determines the nature of the change in temperature and heat flux at the point $x = x_S$: for $\nu > 1/2$ ($\nu > 3/2$) the temperature (heat flux) is bounded at $t = T^-$, while for $\nu \leq 1/2$ ($\nu \leq 3/2$) it is not.

Solution (12), (14) is an example of the S blow-up regime. It is the analogue of the standing thermal wave (1.5) for the case of constant thermal conductivity.

From the comparison theorem we obtain, that for boundary regimes majorized by (14),

$$u_1(t) \leq A_S(T-t)^{\nu}\exp\{R_0(T-t)^{-1}\},\ t \in (0, T), \tag{17}$$

localization of depth $L^* \leq x_S$ occurs, and for $x > x_S$ the solution is bounded for all $0 < t < T$ by the limiting curve (16). Condition (17) distinguishes the class of slow boundary blow-up regimes in this problem. If

$$\begin{gathered} u_1(t) \leq A_0(T-t)^{\nu}\exp\{R_0(T-t)^{n}\},\ 0 < t < T; \\ -1 < n \leq 0,\ A_0 > 0 \end{gathered} \tag{18}$$

(for $n = 0$ we assume $\nu < 0$), the integral (13) converges for all $x > 0$ and the function $u(t, x)$ is infinite only at the point $x = 0$. In the rest of the space it is bounded:

$$u(T^-, x) \leq \frac{A_0 x^{2\nu}}{2^{2\nu}\sqrt{\pi}} k_0^{-\nu}\int_{x^2/(4k_0T)}^{\infty}\exp\left\{-u+\frac{R_0 x^{2n}}{(4k_0)^n}u^{-n}\right\}u^{-\nu-1/2}\,du. \tag{19}$$

Therefore if condition (18) holds, we have the LS blow-up regime.

Finally, in the case of fast regimes

$$u_1(t) \geq A_0(T-t)^{\nu}\exp\{R_0(T-t)^{n}\},\ 0 < t < T,\ n < -1, \tag{20}$$

the integral (13) diverges for all $x > 0$, $u(t, x) \to \infty$ as $t \to T^-$ everywhere in $\mathbf{R}_+$, and the HS-regime obtains.

Thus, in a medium with constant thermo-physical properties, exactly as in the case $k(u) = k_0 u^\sigma$ ($\sigma > 0$), there are three regimes of heat diffusion. Heat inertia, and the appearance of a finite thermal process localization domain also occurs in a homogeneous medium with infinite speed of propagation of perturbations.

Let us consider the question of localization in the multi-dimensional case. Here a great variety of localization domain shapes can be constructed, in particular, domains with a non-smooth boundary.

Solution of the heat equation ($k_0 = 1$)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \quad 0 < t < T, \ x_1 > 0, \ -\infty < x_2 < \infty,$$

with the conditions

$$u(0, x_1, x_2) \equiv 0; \ u(t, 0, x_2) = \Phi(t, x_2) \geq 0, \ t \in (0, T), \ x_2 \in \mathbf{R},$$

where $\Phi(t, x_2) \to \infty$ as $t \to T^-$ for all $x_2 \in E_0 \subseteq \mathbf{R}$, $E_0 \neq \emptyset$, has a representation in terms of the two-dimensional heat potential:

$$u(t, x_1, x_2) = \frac{x_1}{4\pi} \int_0^t \exp\left\{-\frac{x_1^2 + x_2^2}{4(t-\tau)}\right\} \times \\ \times \frac{d\tau}{(t-\tau)^2} \int_{-\infty}^{\infty} \exp\left\{-\frac{y^2 - 2x_2 y}{4(t-\tau)}\right\} \Phi(\tau, y)\, dy.$$

Taking $t$ to $T^-$, we obtain the limiting temperature distribution:

$$u(T^-, x_1, x_2) = \frac{x_1}{4\pi} \int_0^T \exp\left\{-\frac{x_1^2 + x_2^2}{4(T-\tau)}\right\} \times \\ \times \frac{d\tau}{(T-\tau)^2} \int_{-\infty}^{\infty} \exp\left\{-\frac{y^2 - 2x_2 y}{4(T-\tau)}\right\} \Phi(\tau, y)\, dy. \tag{21}$$

For example, let the boundary regime have the form

$$\Phi(t, x_2) = \exp\left\{\frac{x_2^2}{4(T-t)}\right\} \mu(t, x_2),$$

where

$$\mu(t, x_2) = \begin{cases} \exp\left\{\dfrac{2b x_2}{4(T-t)}\right\}, 0 \leq x_2 \leq d = \text{const} > 0, \\ 0, \ x_2 > d, \ x_2 < 0. \end{cases}$$

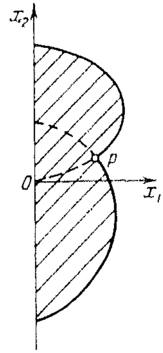


**Fig. 33.** Heat localization domain (dashed) with a non-smooth boundary ($P$ is a corner point)

where $b > 0$ is a constant The inner integral in (21) converges for all $x_2 \in \mathbf{R}$ and equals

$$I(x_2, \tau) = \begin{cases} \dfrac{2(T-\tau)}{x_2+b}\left[\exp\left\{\dfrac{(x_2+b)d}{2(T-\tau)}\right\} - 1\right], & x_2 \neq -b, \\ d, & x_2 = -b. \end{cases}$$

Then we have from (21) that the localization domain is the set

$$x_1^2 + (x_2 - d)^2 < d^2 + 2db, \; x_1 > 0,$$

the boundary of which is composed of a segment of a straight line and a half-circle. Inside the domain the temperature goes to infinity as $t \to T^-$, while outside it is bounded uniformly in time.

Using (21) it is not hard to devise localization domains with boundaries given by any second order curve (parabola, ellipse, hyperbola). The principle of superposition allows us to combine domains corresponding to different boundary regimes and to obtain as a result localization domains with non-smooth boundaries. In Figure 33 the boundary consists of segments of a circle and an ellipse.

## 3 Asymptotic stage of development of blow-up regimes in a medium with constant thermal conductivity

Using the integral representation (12) of the solution of problem (9)–(11), it is hard to characterize in detail the asymptotic stage of the process. To do that, we shall construct self-similar and approximate self-similar solutions of the problem. Let us consider two important examples.

1. Boundary regimes of power type.

$$u_1(t) = A_0(T-t)^n,\ 0 < t < T,\ n < 0, \tag{22}$$

lead to the occurrence of LS-regime. We shall analyse the problem (9)–(11) using the self-similar solutions

$$u_S(t,x) = A_0(T-t)^n f_S(\xi),\ \xi = x[k_0(T-t)]^{-1/2},\quad -\infty < t < T,\ x > 0.$$

For the function $f_S(\xi)$ we then obtain the problem

$$f_S'' - \frac{1}{2}f_S'\xi + nf_S = 0,\ 0 < \xi < \infty;\ f_S(0) = 1,\ f_S(\infty) = 0,$$

which has a unique positive monotone solution

$$f_S(\xi) = \frac{\Gamma(1/2-n)}{\sqrt{\pi}\Gamma(-n)}\int_0^\infty \exp\left\{-\frac{\xi^2 s}{4}\right\} s^{-n-1}(1+s)^{n-1/2}\,ds,\ 0 < \xi < \infty.$$

From the Maximum Principle it follows that the difference between a self-similar and a non-self-similar solution satisfies the estimate

$$0 \le u_S(t,x) - u(t,x) \le u_S(0,0) = A_0T^n. \tag{23}$$

Introducing the "similarity representation" of the solution,

$$f(t,\xi) = A_0^{-1}(T-t)^{-n}u(t,\xi[k_0(T-t)]^{1/2}),$$

we obtain from (23) for all $0 < t < T$ the estimate

$$\|f(t,\cdot) - f_S(\cdot)\|_{C(\mathbf{R}_+)} \le T^n(T-t)^{-n} \to 0,\ t \to T^-,$$

that is, asymptotic stability of the self-similar solution (for results of a numerical computation see Figure 34.) Stability ensures that all the main properties of the solutions $u(t,\ x)$ and $u_S(t,\ x)$ are the same at the asymptotic stage of evolution. For example, the half-width of the thermal wave $x_{ef}(t)$, determined from the equation

$$u(t, x_{ef}(t)) = \frac{1}{2}u(t,0) \equiv \frac{1}{2}A_0(T-t)^n,\ t \to T^-,$$

satisfies by (23) the inequalities

$$1/2 \le f(t, x_{ef}(t)/[k_0(T-t)]^{1/2}) \le 1/2 + T^n(T-t)^{-n}. \tag{24}$$

Let $f_S^{-1}(1/2) = \xi_{ef} < \infty$, where $f_S^{-1}$ is the function inverse to $f_S(\xi)$ (it exists by monotonicity of $f_S(\xi)$). Then we obtain from (24) an expression for the half-width,

$$x_{ef}(t) = \xi_{ef}(T-t)^{1/2} + O[(T-t)^{-n+1/2}],\ t \to T^-.$$

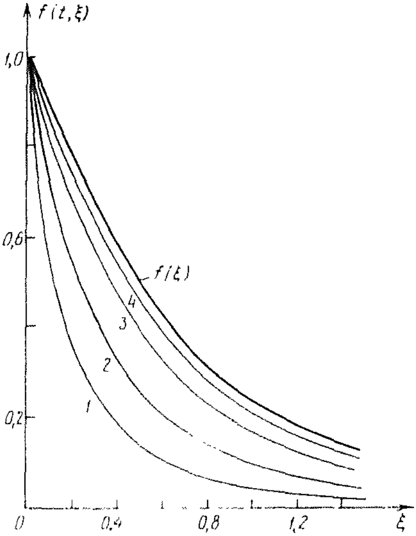


**Fig. 34.** Convergence of a solution to the power law self-similar LS-regime in a medium with constant thermal conductivity. The parameters are: $k_0 = 1.0$, $n = -0.5$, $A_0 = T = 1$; 1: $T - t = 0.77$, 2: $T - t = 0.70$, 3: $T - t = 0.55$, 4: $T - t = 0.41$

A similar expression is true for the quantity $x_{en}(t)$, defined by the equation

$$W(t, x_{en}(t)) = \frac{1}{2} W(t, 0), \quad 0 < t < T,$$

where $W(t, x)$ is the heat flux, so that $x_{en}(t)$ is the coordinate of the point on each side of which the amounts of energy entering the medium are equal.

Figure 35 shows the results of a numerical solution of problem (9), (10), (22) for $n = -1$. Dashed and dash-dotted lines show, respectively, the trajectories of $x = x_{ef}(t)$ and $x = x_{en}(t)$. Self-similar behaviour is established once the temperature on the boundary becomes 5-10 times larger than the initial one (which is close to the criterion obtained for media with $k(u) = k_0 u^{\sigma}$, $\sigma > 0$).

Nonetheless, there are certain differences between the solutions $u(t, x)$ and $u_S(t, x)$. The asymptotic behaviour of the limiting distribution of $u(t, x)$ as $x \to \infty$,

$$u(T^-, x) = \frac{A_0 x^{2n}}{2^{2n}\sqrt{\pi}} k_0^{-n} \int_{x^2/(4k_0 T)}^{\infty} e^{-s} s^{-n-1/2} \, ds \tag{25}$$

is exponential, unlike the power law asymptotics of the self-similar solution,

$$u_S(T^-, x) = \frac{A_0 x^{2n}}{2^{2n}\sqrt{\pi}} k_0^{-n} \Gamma\left(\frac{1}{2} - n\right) + \ldots, \quad x \to \infty.$$

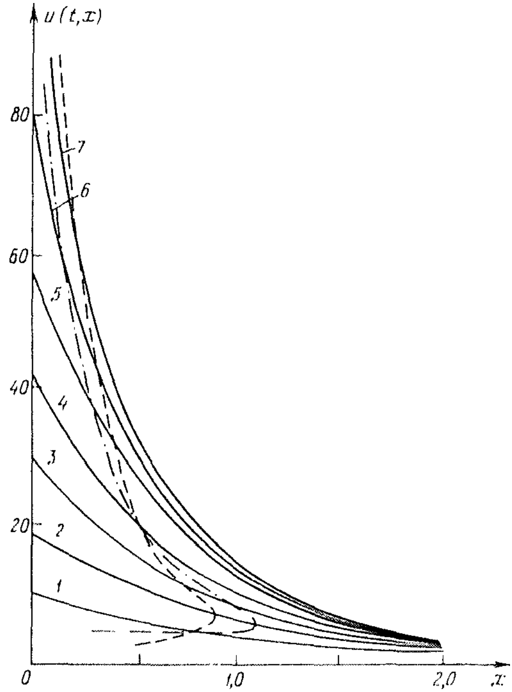


**Fig. 35.** Dynamics of the power law LS-regime in a medium with constant thermal conductivity. The parameters are: $k_0 = 1.0$, $A_0 = T = 1$; 1: $T - t = 0.113$, 2: $T - t = 5.8 \cdot 10^{-2}$, 3: $T - t = 3.5 \cdot 10^{-2}$, 4: $T - t = 2.3 \cdot 10^{-2}$, 5: $T - t = 1.8 \cdot 10^{-2}$, 6: $T - t = 1.2 \cdot 10^{-2}$, 7: $T - t = 7 \cdot 10^{-3}$

This, naturally, has to do with the fact that the self-similar profile $u_S(T^-, x)$ takes the infinite amount of time $t \in (-\infty, T)$ to form.

2. Let us consider now the asymptotic stage of exponential boundary blow-up regimes:

$$u_1(t) = A_0[\exp\{R_0(T - t)^n\} - 1], \; 0 < t < T; \; n < 0. \tag{26}$$

For $\nu = 0$, (26) differs from (14) by a constant, which is not essential.

The problem (9), (10), (26) contains at least two parameters with the dimension of length, $|k_0(T - t)|^{1/2}$ and $|k_0 R_0^{\;1/n}|^{1/2}$ and, therefore, has no self-similar solutions. We shall show that the asymptotics of the solution of the problem as $t \to T^-$ is described by self-similar solutions of a "degenerate" equation.

It is constructed as follows. The change of variable

$$V(t, x) = A_0 \ln|1 + u(t, x)/A_0| \tag{27}$$

takes the original problem into the form

$$\frac{\partial V}{\partial t} = k_0\frac{\partial^2 V}{\partial x^2} + \frac{k_0}{A_0}\left(\frac{\partial V}{\partial x}\right)^2, \quad 0 < t < T, \ x > 0, \tag{28}$$

$$V(t, 0) = A_0 R_0 (T-t)^n, \quad 0 < t < T, \tag{29}$$

$$V(0, x) = 0, \quad x > 0. \tag{30}$$

If we neglect the highest order derivative term in (28), we arrive at the degenerate problem

$$\frac{\partial V_s}{\partial t} = \frac{k_0}{A_0}\left(\frac{\partial V_s}{\partial x}\right)^2, \quad 0 < t < T, x > 0,$$

$$V_s(t, 0) = A_0 R_0 (T-t)^n, \quad 0 < t < T,$$

which has the self-similar solution

$$V_s(t, x) = A_0 R_0 (T-t)^n \theta_s(\xi), \quad \xi = \frac{x}{(k_0 R_0)^{1/2}}(T-t)^{-(1+n)/2}. \tag{31}$$

The function $\theta_s(\xi) \geq 0$ satisfies the equation

$$(\theta_s')^2 - \frac{1+n}{2}\theta_s'\xi + n\theta_s = 0, \quad \xi > 0; \quad \theta_s(0) = 1, \tag{32}$$

and wherever it is positive, is defined implicitly from the equality

$$\left[\sqrt{\left(\frac{1+n}{4}\right)^2 - n\theta_s\xi^{-2}} - \frac{1+n}{4}\right]^{(1+n)/2} \times \\ \times \left[\sqrt{\left(\frac{1+n}{4}\right)^2 - n\theta_s\xi^{-2}} + \frac{1-n}{4}\right]^{(1-n)/2} = \frac{(-n)^{1/2}}{\xi}. \tag{33}$$

At all the other points we set $\theta_s(\xi) = 0$.

The properties of the monotone function $\theta_s$ depend on the parameter $n$.

1) If $-1 < n < 0$ (LS-regime), then $\theta_s(\xi) > 0$ for all $\xi > 0$ and

$$\theta_s(\xi) = C(n)\xi^{2n/(1+n)} + \ldots, \xi \to \infty;$$

$$C(n) = -\frac{1+n}{2n}2^{-2n/(1+n)}(-n)^{1/(1+n)}.$$

2) For $n = -1$ (S-regime) the solution has the form

$$\theta_s(\xi) = \begin{cases} (1-\xi/2)^2, & 0 < \xi < 2, \\ 0, & \xi \geq 2. \end{cases}$$

3) In the case $n < -1$ (HS-regime) $\theta_s$ is a function with compact support: $\theta_s(\xi) > 0$ for $0 \le \xi < \xi_f = 2(-n)^{n/2}(-1-n)^{-(1+n)/2}$, $\theta_s(\xi) = 0$ for all $\xi \ge \xi_f$, such that moreover

$$\theta_S(\xi) = -|(1+n)/2|\xi_f(\xi_f - \xi) + o((\xi_f - \xi)) \text{ as } \xi \to \xi_f^- .$$

In all the cases $\theta_s'(0) = -(-n)^{1/2}$, $\theta_s''(\xi) > 0$ wherever $\theta_s > 0$ and $\theta_s''(\xi) \le \theta_s''(0) = (1-n)/4$ for $\xi \in (0, \xi_f)$.

From the properties of the self-similar solutions and the Maximum Principle, we obtain the estimates (see § 2, Ch. VI)

$$-A_0 R_0 T^n \le V(t,x) - V_s(t,x) \le A_0 \|\theta_s''(\xi)\|_{C(0,\xi_f)} \ln \frac{T}{T-t}, \quad t \in (0,T),\ x > 0.$$

Then for the solution of the original problem we obtain from (27)

$$\begin{aligned} &A_0\left[\exp\left\{\frac{V_s(t,x)}{A_0} - R_0 T^n\right\} - 1\right] \le \\ &\le u(t,x) \le A_0 T^{-(n-1)/4}(T-t)^{(n-1)/4}\exp\left\{\frac{V_s(t,x)}{A_0}\right\}. \end{aligned} \tag{34}$$

For the similarity representation of the solution $V$,

$$\theta(t,\xi) = (A_0 R_0)^{-1}(T-t)^{-n} V(t, \xi(k_0 R_0)^{1/2}(T-t)^{(1+n)/2}),$$

we obtain from the preceding inequalities the following estimate of the rate of convergence to the approximate self-similar solution:

$$\|\theta(t,\cdot) - \theta_s(\cdot)\|_{C(\mathbf{R}_+)} = O\left((T-t)^{-n}|\ln(T-t)|\right) \to 0,\ t \to T^- . \tag{35}$$

In the case of the S-regime, convergence (35) is illustrated by the results of numerical solution of the problem (28)–(30), shown in Figure 36. Significant growth of the temperature in the main part of the localization domain, as compared with the temperature for $x > x_S = 2(k_0 R_0)^{1/2}$, occurs when the temperature on the boundary grows by a factor of 10-20.

The estimate (34) makes it possible to analyze in detail the fully developed stage of the process. For example, in the S-regime we obtain, as $t \to T^-$, from (34) for all $0 < x < x_S = 2(k_0 R_0)^{1/2}$ the estimates

$$\begin{aligned} A_0|\exp\{R_0(T-t)^{-1}(1-x/x_s)^2 - R_0 T^{-1}\} - 1| \le u(t,x) \le \\ \le A_0 T^{1/2}(T-t)^{-1/2}\exp\{R_0(T-t)^{-1}(1-x/x_S)^2\}, \end{aligned}$$

whence it follows that inside the localization domain the temperature changes according to

$$u(t,x) \sim \exp\{R_0(T-t)^{-1}(1-x/x_S)^2\},\ 0 < x < x_S;\ t \to T^- .$$

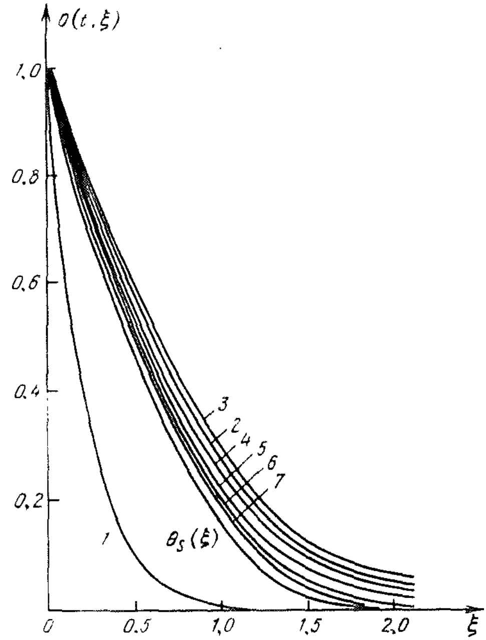


**Fig. 36.** The parameters are: $n = -1$, $k_0 = A_0 = R_0 = T = 1$, $x_S = 2$; 1: $T - t = 0.95$, 2: $T - t = 0.47$, 3: $T - t = 0.2$, 4: $T - t = 0.1$, 5: $T - t = 5 \cdot 10^{-2}$, 6: $T - t = 2.5 \cdot 10^{-2}$, 7: $T - t = 1.2 \cdot 10^{-2}$

In the case of the HS-regime, we have for all $x > 0$

$$u(t, x) \sim \exp\{R_0(T - t)^n - x(-R_0 n/k_0)^{1/2}(T - t)^{(n-1)/2}\}, \quad t \to T^-,$$

which means that at each point in space the temperature grows to infinity, but more slowly than the temperature at the boundary.

Using (34), it is also not hard to determine the dynamics of the evolution of the process at the asymptotic stage:

$$\begin{aligned} x_{ef}(t) &= \ln 2 \left[\frac{k_0}{R_0(-n)}\right]^{1/2} (T - t)^{(1-n)/2} + O\left[(T - t)^{(3-n)/2}\right], \\ x_{en}(t) &= \left[\frac{k_0}{R_0(-n)}\right]^{1/2} \frac{1-n}{4} (T - t)^{(1-n)/2} + O\left[(T - t)^{(3-n)/2}\right]. \end{aligned} \tag{36}$$

Unlike a medium with a power law nonlinearity, in all the cases here $x_{ef} \to 0$, $x_{en} \to 0$ as $t \to T^-$, including the HS-regime, when the temperature grows without bound in all of $\mathbf{R}_+$. This shows, in particular, that shrinking of the half-width is not a sign of localization in a blow-up regime. Let us note that the frequently used dimensional estimate of the half-width, $x_{ef}(t) \sim |k_0(T-t)|^{1/2}$, in this case does not describe the process correctly at the fully developed stage.

Thus, under the influence of exponential boundary blow-up regimes there is a kind of degeneration of the parabolic equation (9) into the first order equation

$$\frac{\partial u}{\partial t} = k_0 \frac{(\partial u/\partial x)^2}{A_0 + u},$$

self-similar solutions of which provide us with the principal term of the asymptotics as $t \to T^-$.

The function (31) is an approximate self-similar solution of the problem (28)–(30). The general theory of a.s.s. of parabolic equations and its applications are presented in Ch. VI.

Results of this chapter testify to the generality of the heat inertia phenomenon and show that conditions for its occurrence are not hard to satisfy. These results will be used in Ch. V, VI in the study of boundary blow-up regimes in media with quite general thermo-physical properties.

## Remarks and comments on the literature

§ **1.** The self-similar solution of the S-regime (5) was constructed in [351] (existence of separable solutions for equation (1) was known before; see, for example [33], where for the first time the standing thermal wave was studied; that paper also verified numerically its asymptotic stability. The paper [351] led to detailed studies of the heat localization phenomenon in media with nonlinear thermal conductivity [390, 264, 265, 266], where all the main concepts and definitions are developed. A detailed analysis of the localized solution of Example 2 is presented in [347, 348, 149].

§ **2.** The three types of self-similar blow-up regimes (S-, LS-, and HS-regimes) were studied in [352, 393, 267, 165]. In the presentation of subsection 2 we follow [165]. By a different method the existence and uniqueness of self-similar solutions are proved in [205, 206].

§ **3.** Theorems on presence or absence of localization (subsections 1 and 2) are proved in [304] (an interesting criterion of localization depending on the form of the boundary function which blows up in finite time has been obtained by [204]). A more detailed discussion of the physical basis of localization can be found in [393, 267, 268]; these papers also discuss the possibilities of its experimental study. Localization in the Cauchy problem has been studied in [352, 393, 267].

An interesting example of a localized initial function was constructed earlier in [17], where the exact value of the localization time was calculated; it agrees with the calculations of subsection 3. Results of subsection 4 are contained in the main in [277, 331, 267].

A list of papers dealing with the analysis of local properties of the degeneracy surface in problems for quasilinear parabolic equations can be found in the Comments sections of Ch. I and II.

§ 4. Results of subsection 1 appear partially in [153]. The study of subsections 2, 3 was published in [348, 347, 149].

An elementary presentation of some of the questions relating to the localization phenomenon can be found in [394].

Possible applications of the discussed phenomena were considered in [392, 350]. Blow-up regimes in compressible media with various physical processes are studied in [15, 70, 71, 72, 73, 382, 387, 388, 389, 366, 318]. A more complete bibliography can be found in [267, 268, 269].

# *Chapter IV*

# Nonlinear equation with a source. Blow-up regimes. Localization. Asymptotic behaviour of solutions.

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The present chapter deals with the study of spatio-temporal structure and conditions for the appearance of unbounded solutions of the Cauchy problem for quasilinear equations with power law nonlinearities:

$$u_t = \nabla \cdot (u^\sigma \nabla u) + u^\beta, \quad t > 0, \; x \in \mathbf{R}^N, \tag{0.1}$$

$$u(0, x) = u_0(x) \geq 0, \quad x \in \mathbf{R}^N, \; u_0^{\sigma+1} \in C^1(\mathbf{R}^N), \tag{0.2}$$

where $\sigma > 0$, $\beta > 1$ are constants.

Equation (0.1) describes processes with a finite speed of propagation of perturbations (see § 3, Ch. I). Therefore, if $u_0$ is a function with compact support, $u(t, x)$ will also have compact support in $x$ for all $0 < t < T_0$, where $T_0 \leq \infty$ is the time of existence of the solution. The main question, considered in §§ 1, 2, 4 is to define conditions of localization of unbounded solutions.

**Definition 1.** An unbounded solution of the problem (0.1), (0.2) is called *(strictly) localized* if the set

$$\Omega_L = \left\{ x \in \mathbf{R}^N \mid u(T_0, x) \equiv \overline{\lim}_{t \to T_0} u(t, x) > 0 \right\} \tag{0.3}$$

is bounded.

The set $\Omega_L$ is called the *localization domain*. Boundedness of $\Omega_L$ means, in particular, that $u(t, x) \equiv 0$ in $\mathbf{R}^N \setminus \Omega_L$ for all $0 \leq t < T_0$. This follows from general properties of solutions of parabolic equations with a source. A strictly localized solution grows unboundedly as $t \to T_0$ in a domain

$$\omega_L = \left\{ x \in \mathbf{R}^N \mid u(T_0, x) = \infty \right\}$$

of finite size, which, in general, is different from $\Omega_L$. As in the case of boundary value problems (Ch. III), localized solutions can be conveniently divided into two classes: S-*regime* solutions, for which $0 < \operatorname{meas} \omega_L < \infty$, and LS-*regime* solutions, for which meas $\omega_L = 0$. In the latter case the solution $u(t, x)$ grows to infinity, for example, in one point, while at all the other points it is bounded from above uniformly in $t \in (0, T_0)$. In the most general case the classification of blow-up regimes should be based on the measure of the blow-up set having the form $B_L = \{x \in \mathbf{R}^N \mid \exists$ sequences $t_n \to T_0$ and $x_n \to x$, such that $u(t_n, x_n) \to \infty$ as $n \to \infty\}$. Obviously, by definition of an unbounded solution $B_L \neq \emptyset$ for bell-shaped data.

**Definition 2.** There is *no localization* in the problem (0.1), (0.2) if the domain $\Omega_L$ in (0.3) is unbounded.

We put non-localized unbounded solutions in the class of HS *(blow-up) regimes*. A combustion process is not localized if as $t \to T_0^-$ heat propagates into arbitrarily distant regions. In a number of cases the condition of Definition 2 is equivalent to the requirement

$$\overline{\lim_{t \to T_0}}\, u(t, x) = \infty, \ x \in \mathbf{R}^N,$$

that is, the non-localized solution grows to infinity as $t \to T_0^-$ in the whole space.

In §§ 1, 2, 4 it is shown that for $\beta \geq \sigma + 1$ the problem exhibits localization; the case $\beta = \sigma + 1$ corresponds to the S-regime of combustion, while the case $\beta > \sigma + 1$ corresponds to the LS-regime; for $1 < \beta < \sigma + 1$ there is no localization (HS-regime). The study is conducted by constructing unbounded similarity solutions (§ 1), as well as by the qualitative method of averaging (§ 2), which establishes their asymptotic stability in certain parameter ranges.

In § 3 we prove various assertions concerning conditions of existence of unbounded solutions of the problem (0.1), (0.2), and we show that for $\beta > \sigma + 1 + 2/N$ it can be globally solvable (for "small" data $u_0$), which confirms the qualitative conclusion of § 2.

Rigorous results on the existence ($\beta \geq \sigma + 1$) and non-existence ($1 < \beta < \sigma + 1$) of localization of unbounded solutions for $N = 1$ are given in § 4.

The next section, § 5, is wholly devoted to the study of asymptotic stability of similarity solutions.

In § 6 we show that for some $u_0(x)$ in the case $\sigma + 1 < \beta \leq (\sigma + 1)N/(N-2)_+$ the problem (0.1), (0.2) evolves in LS-regime of blow-up, in which meas $\omega_L = 0$. There we also obtain bounds from above and below for $u(T_0^-, x)$ in a neighbourhood of the singular point where $u(T_0^-, x) = \infty$.

In § 7 we use the above approach to study the semilinear equation (0.1) for $\sigma = 0$ with a reasonably general form of source. There we obtain, in particular *effective localization* conditions for blow-up regimes, that is, conditions for boundedness of the set $\omega_L$. We consider in detail the phenomenon of degeneration of the equation $u_t = \Delta u + (1 + u)\ln^\beta(1 + u)$, $\beta > 1$, at the asymptotic stage of blow-up. Asymptotics of the combustion process is described by invariant solutions of a Hamilton-Jacobi first order equation. This degeneration phenomenon has already been considered in the context of boundary value problems (see § 4, Ch. III).

## § 1 Three types of self-similar blow-up regimes in combustion

It is convenient to start the study of the relatively complicated problem (0.1), (0.2) by an analysis of particular self-similar solutions of the equation (0.1). Here we construct unbounded self-similar solutions, the spatio-temporal structure of which is substantially different in three cases: $1 < \beta < \sigma + 1$ (HS blow-up regime), $\beta = \sigma + 1$ (S-regime; a solution of this type in the one-dimensional case was considered in Ch. 1, Example 13 of § 3), $\beta > \sigma + 1$ (LS-regime). Though these particular solutions arise only for a special choice of the initial function $u_0(x)$, the analysis of their spatio-temporal structure allows us to make assertions concerning the character of evolution of combustion processes with finite time blow-up in the general case (see § 5). Moreover, they can be used to establish conditions for existence of blow-up, that is, conditions for global insolvability of the Cauchy problem (see §§ 3, 4). They are also used to prove localization of unbounded solutions in the case $\beta \geq \sigma + 1$.

The spatio-temporal structure of unbounded self-similar solutions contains important and nearly exhaustive information about general properties of evolution of unbounded solutions of the equation (0.1). Therefore it is not an exaggeration to call the particular solutions we construct *eigenfunctions* (e.f.) of combustion of the nonlinear dissipative medium corresponding to the equation (0.1).

### 1 Formulation of self-similar problems

For any[1] $\sigma > 0$ and $\beta > 1$, equation (0.1) has unbounded self-similar solutions of the following form:

$$u_S(t, x) = (T_0 - t)^{-1/(\beta - 1)}\theta_S(\xi), \quad \xi = x/(T_0 - t)^m, \tag{1}$$

where

$$m = |\beta - (\sigma + 1)|/|2(\beta - 1)|.$$

Here the constant $T_0 > 0$ is the time of existence of the solution $u_S$, for $t \geq T_0$ the solution (1) is, in general, not defined and the amplitude of the solution grows without bound as $t \to T_0^-$. The function $\theta_S(\xi) \geq 0$ satisfies in $\mathbf{R}^N$ an elliptic equation obtained by substituting the expression (1) into (0.1):

$$\nabla_\xi \cdot (\theta_S^\sigma \nabla_\xi \theta_S) - m\nabla_\xi \theta_S \cdot \xi - \frac{1}{\beta - 1}\theta_S + \theta_S^\beta = 0, \quad \xi \in \mathbf{R}^N. \tag{2}$$

This equation has the trivial solution $\theta_S(\xi) \equiv 0$, as well as the spatially homogeneous solution

$$\theta_S(\xi) \equiv \theta_H = (\beta - 1)^{-1/(\beta - 1)}. \tag{3}$$

[1]The case $\sigma = 0$ is considered in § 7.

According to (1), this solution corresponds to the process of spatially homogeneous (homothermic) combustion with blow-up.

Below we restrict ourselves to an analysis of radially symmetric self-similar solutions:

$$\xi = r/(T_0 - t)^m, \quad r = |x|. \tag{4}$$

Then (2) becomes the ordinary differential equation

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}\theta_S^\sigma \theta_S')' - m\theta_S'\xi - \frac{1}{\beta - 1}\theta_S + \theta_S^\beta = 0, \quad \xi > 0. \tag{5}$$

The first operator can be written in the form

$$(\theta_S^\sigma \theta_S')' + \frac{N-1}{\xi}\theta_S^\sigma \theta_S',$$

and therefore, if we want the solution $\theta_S$ to be defined in $\mathbf{R}^N$, we have to impose the symmetry condition

$$\theta_S'(0) = 0 \quad (\theta_S(0) > 0). \tag{6}$$

Moreover, we shall require the following condition to be satisfied:

$$\theta_S(\infty) = 0. \tag{7}$$

In this section we mainly deal with a study of the problem (5)–(7), and with an analysis of the properties of the corresponding radially symmetric solutions of (1).

The equation (7) is degenerate for $\theta_S = 0$; therefore in general (5)–(7) admits a generalized solution, not having the requisite smoothness at the points of degeneracy. However, in all cases the self-similar heat flux, $-\xi^{N-1}\theta_S^\sigma\theta_S'$, must be continuous (similarly, in the case of equation (2) the derivative $\nabla\theta_S^{\sigma+1}$ must be continuous in $\mathbf{R}^N$). This means, in particular, that $\theta_S^\sigma\theta_S' = 0$ wherever $\theta_S = 0$.

Any solution of the equation (5) can be considered in its domain of non-monotonicity as some kind of oscillation around the homothermic solution $\theta \equiv \theta_H$. This analogy has to do with the fact that the maximum of the function $\theta_S$ can be attained only at a point where $-\theta_S/(\beta - 1) + \theta_S^\beta > 0$, that is, for $\theta_S > \theta_H$, and the minimum at a point where $0 \le \theta_S < \theta_H$.

## 2 Localization of combustion in the self-similar S-regime, $\beta = \sigma + 1$

In this case equation (2) assumes the simpler form

$$\frac{1}{\sigma + 1}\Delta_\xi \theta_S^{\sigma+1} - \frac{1}{\sigma}\theta_S + \theta_S^{\sigma+1} = 0, \quad \xi \in \mathbf{R}^N, \tag{8}$$

while the corresponding radially symmetric problem (5)–(7) can be written as

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}\theta_S^\sigma\theta_S')' - \frac{1}{\sigma}\theta_S + \theta_S^{\sigma+1} = 0,\ \xi > 0, \tag{9}$$

$$\theta_S'(0) = 0\ (\theta_S(0) > 0),\ \theta_S(\infty) = 0. \tag{10}$$

*1 The case $N = 1$*

In one-dimensional geometry the equation (9) becomes an autonomous one, and can be integrated. In particular, it is not hard to obtain the following solution of the equation (9):

$$\theta_S(\xi) = \left(\frac{2(\sigma+1)}{\sigma(\sigma+2)}\cos^2\frac{\pi\xi}{L_S}\right)^{1/\sigma},\ \xi \geq 0, \tag{11}$$

where

$$L_S = \frac{2\pi}{\sigma}(\sigma+1)^{1/2}. \tag{12}$$

As follows from (1), for $\beta = \sigma + 1$, $\xi \equiv x$; therefore (1) is a separable solution:

$$u_S(t,x) = (T_0 - t)^{-1/\sigma}\theta_S(x),\ 0 < t < T_0,\ x \in \mathbf{R} \tag{13}$$

(the function $\theta_S$ is here evenly extended into the domain of negative values of $x$).

The solution (13) looks unusual from the point of view of traditional ideas about propagation of heat in diffusional media. The point is that in (11) $\theta_S(x)$ is a periodic function; it vanishes at the points $x_k = (1/2 \pm k)L_S$ ($k = 0, 1, \ldots$); furthermore the heat flux $-\theta_S^\sigma\theta_S' = 0$ is continuous: $\theta_S^\sigma\theta_S' \to 0$ as $x \to x_k$. Therefore a generalized solution of the problem (9), (10) will be obtained if take a function $\theta_S$ consisting from only one "wave" of the general solution (11), while at all other points we can set $\theta_S \equiv 0$.

Hence it follows that, in particular, the following function is also a self-similar solution:

$$u_S(t,x) = \begin{cases} (T_0 - t)^{-1/\sigma}\left(\frac{2(\sigma+1)}{\sigma(\sigma+2)}\cos^2\frac{\pi x}{L_S}\right)^{1/\sigma}, & |x| < L_S/2, \\ 0, & |x| \geq L_S/2;\ 0 < t < T_0. \end{cases} \tag{14}$$

This is the elementary temperature profile of the self-similar S blow-up regime, which is *localized* in the domain $\{|x| < L_S/2\}$ during all the course of its existence. Despite unbounded growth of the solution as $t \to T_0^-$ at all points of localization $\{|x| < L_S/2\}$, heat does not penetrate the surrounding cold space.

The quantity $L_S$ is called the *fundamental length* of the S-regime of combustion in the nonlinear medium. It is shown by numerical computations that for practically arbitrary non-monotone initial perturbations, for $\beta = \sigma + 1$, the unbounded

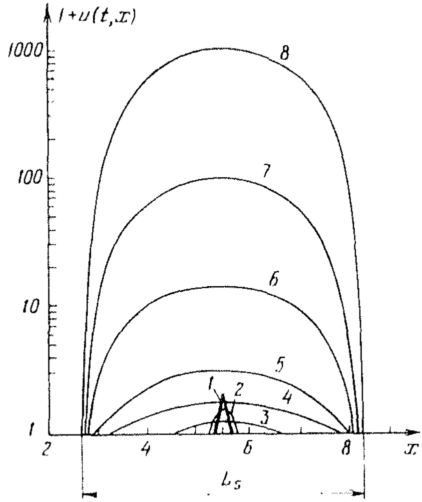


**Fig. 37.** Numerical manifestation of the S-regime. The parameters are: $\sigma = 2$, $\beta = 3$, $N = 1$, $L_S = 2\pi(\sigma+1)^{1/2}/\sigma \simeq 5.44$; 1: $t_1 = 0$, 2: $t_2 = 7.92 \cdot 10^{-2}$, 3: $t_3 = 19.6$, 4: $t_4 = 73.0$, 5: $t_5 = 74.9$, 6: $t_6 = 74.951$, 7: $t_7 = 74.9548$, 8: $t_8 = 74.9551$

solution goes to infinity on a set of length $L_S$. If, on the other hand, we have that $u_0(x) > 0$ on a small interval, meas supp $u_0 < L_S$, then we have strict localization on an interval of length $L_S$. Furthermore, $L_S$ characterizes the maximal length of propagation of heat perturbations with compact support during the course of existence of the unbounded solution (see § 4).

We present here the results of two numerical computations. In Figure 37 we show the evolution of an initial perturbation $u_0(x)$ of small energy, distributed over a small region (smaller than $L_S$). It can be clearly seen that first the heat profile spreads to a certain *resonance* (*critical*) *length*: only after that, starting with time $t_4$, does the combustion process become intensive, and as $t \to T_0^-$ it evolves according to the self-similar solution (14).

In Figure 38 the initial energy is large and occupies an extensive domain, inside which $u_0(x)$ is close to a spatially homogeneous profile $u_0 \equiv 1$. Actually Figure 38 shows instability of the spatially homogeneous (homothermic) solution in the S-regime.

In both cases an unbounded heat profile is formed, which follows, as $t \to T_0^-$ ($T_0$ is the interval of time for which the profile exists: it is different for different profiles), the course of evolution of the self-similar solution (14). In the first case, as can be seen from Figure 37, the solution is strictly localized in an interval of length $L_S$. In the second case (see Figure 38) there is no strict localization,

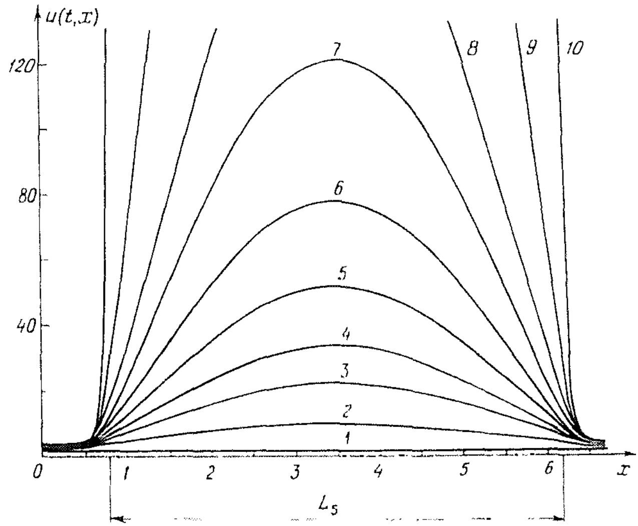


**Fig. 38.** Numerical manifestation of the S-regime. The parameters are: $\sigma = 2$, $\beta = 3$, $N = 1$, $L_S \simeq 5.44$; 1: $t_1 = 0$, 2: $t_2 = 0.43$, 3: $t_3 = 0.4464$, 4: $t_4 = 0.4484$, 5: $t_5 = 0.4491$, 6: $t_6 = 0.4495$, 7: $t_7 = 0.44963$, 8: $t_8 = 0.449699$, 9: $t_9 = 0.449738$, 10: $t_{10} = 0.449747$

however unbounded growth takes place with the fundamental length scale $L_S$. Similarity transformation of any non-stationary solution of the problem shows that its solution for all initial data is in a certain sense close to the corresponding self-similar solution (14), the spatio-temporal structure of which is a fundamental property of the S-regime. The proof of this fact is given in § 5.

For $N = 1$ there exists a countable set of different self-similar solutions, composed of an arbitrary number of the elementary solutions (14), which by the thermal isolation condition, burn independently of each other. Any elementary structure can be removed, without any consequences for the evolution of the neighbouring ones. It turns out that a finite spectrum of similarity structures is also possible for $\beta > \sigma + 1$. However, in this case the principle of superposition, of combining elementary structures to give more complex ones, is not as simple (see Remarks).

## 2 *The multi-dimensional case $N > 1$*

A self-similar solution of the S-regime exists in spaces of arbitrary dimension. However, unlike the one-dimensional case here there are no non-monotone solutions.

**Theorem 1.** *For any $N > 1$ there exists a solution $\theta_S(\xi)$ of problem* (9), (10) *with compact support. The function $\theta_S$ is monotone decreasing wherever it is positive. The problem has no non-monotone solutions.*

First of all let us note that the fact that any possible solution $\theta_S$ has compact support follows from an analysis of the equation for small $\theta_S > 0$ using fixed point theorems for continuous mappings (first (9) is reduced to the equivalent integral equation). This simple analysis provides us with the only possible asymptotics of the function $\theta_S$: it has compact support and if meas supp $\theta_S = \xi_0 > 0$, then

$$\theta_S(\xi) = \left\{ \frac{\sigma}{2(\sigma+2)} (\xi_0 - \xi)^2 \right\}^{1/\sigma} (1 + \epsilon(\xi)), \ \xi < \xi_0 \tag{15}$$

($\theta_S(\xi) \equiv 0$ for all $\xi \geq \xi_0$), where $\epsilon(\xi) \to 0$ as $\xi \to \xi_0$.

For the proof of existence, it is convenient to consider, side by side with (9), (10), the family of Cauchy problems for equation (9):

$$\frac{1}{\xi^{N-1}} (\xi^{N-1} |\theta|^\sigma \theta')' - \frac{1}{\sigma}\theta + |\theta|^\sigma \theta = 0, \ \xi > 0, \tag{16}$$

$$\theta(0) = \mu, \ \theta'(0) = 0, \tag{17}$$

where $\mu > 0$ is a constant. At the points where $\theta \geq 0$, equation (16) coincides with (9). We must find a value $\mu = \theta_0 > 0$, such that the solution of the Cauchy problem (16), (17), $\theta = \theta(\xi;\mu)$, is non-negative for all $\xi \geq 0$ and satisfies the second condition of (10), that is, $\theta(\infty;\mu) = 0$. Local existence and uniqueness of solutions of the problem (16), (17) for all sufficiently small $\xi > 0$ is established by analyzing the equivalent integral equation using the Banach contraction mapping theorem.

The main interest lies in the global analysis of properties of solutions $\theta(\xi;\mu)$, which is presented below. First of all let us note that every local solution $\theta(\xi;\mu)$ can be extended to the whole semi-axis $\xi \in \mathbf{R}_+$; $\theta(\xi;\mu)$ can go only to the values $0$ or $\pm\sigma^{-1/\sigma}$ as $\xi \to \infty$.

Proof of Theorem 1 is based on the following lemmas.

**Lemma 1.** *Let*

$$0 < \mu < \mu_* = \left[\frac{2(\sigma+1)}{\sigma(\sigma+2)}\right]^{1/\sigma}. \tag{18}$$

*Then $\theta(\xi;\mu) > 0$ for all $\xi > 0$. Furthermore, for any $\mu > \theta_H = \sigma^{-1/\sigma}$ the solution is bounded in $\mathbf{R}_+$:*

$$|\theta(\xi;\mu)| < \mu, \ \xi > 0. \tag{19}$$

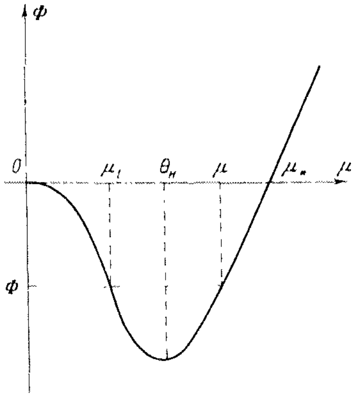


**Fig. 39.** The function $\Phi(\mu)$ (see (21)) for $\beta = \sigma + 1$

*Proof.* The proof relies on an identity, to derive which we multiply (16) by $|\theta|^\sigma\theta'$ and integrate the resulting equality over the interval $(0, \xi)$, taking into account conditions (17). As a result we have

$$\frac{1}{2}(|\theta|^\sigma\theta')^2(\xi) + (N-1)\int_0^\xi (|\theta|^\sigma\theta')^2(\eta)\,\frac{d\eta}{\eta} + \Phi(|\theta(\xi)|) = \Phi(\mu). \tag{20}$$

where

$$\Phi(\mu) = \frac{1}{2(\sigma+1)}\mu^{2\sigma+2} - \frac{\mu^{\sigma+2}}{\sigma(\sigma+2)}, \quad \mu \geq 0 \tag{21}$$

(the graph of this function is sketched in Figure 39).

From (20) it follows that $\Phi(|\theta(\xi)|) \leq \Phi(\mu)$ for all $\xi > 0$ (the equality is attained only in the case $\theta \equiv \theta_H$, $\mu = \theta_H$). Therefore if $\mu < \mu_*$, the solution of the problem satisfies the estimate

$$\mu_1 < \theta(\xi;\ \mu) < \mu,\ \xi > 0,\ \mu \neq \theta_H,$$

where $\mu_1 > 0$ is the second (different from $\mu$) root of the equation $\Phi(\mu_1) = \Phi(\mu)$. The estimate (19) follows immediately from the inequality $\Phi(|\theta(\xi)|) < \Phi(\mu)$ for $\mu > \theta_H$. □

**Remark.** By (20) the possible oscillations of $\theta$ around the homothermic solution $\theta \equiv \theta_H$ are damped, that is, if $\xi_1 < \xi_2$ are maxima (minima) of the function $\theta \geq 0$, then

$$\theta(\xi_1;\mu) > \theta(\xi_2;\mu) \quad (\theta(\xi_1;\mu) < \theta(\xi_2;\mu)). \tag{22}$$

Let us show now that for some sufficiently large $\mu$ the solution $\theta$ is not strictly positive.

**Lemma 2.** *There exists $\mu = \mu^* > \theta_H$, for which the solution of the Cauchy problem* (16), (17) *becomes zero at a point $\xi > 0$.*

*Proof.* Let us assume the contrary. Let $\theta(\xi; \mu) > 0$ in $\mathbf{R}_+$ for all $\mu > \theta_H$. The problem (16), (17) is equivalent to the integral equation

$$\phi'(\xi) = (\sigma + 1)\xi^{1-N} \int_0^\xi \eta^{N-1} \left[\frac{1}{\sigma}|\phi|^{-\sigma/(\sigma+1)}\phi(\eta) - \phi(\eta)\right] d\eta, \ \xi > 0, \qquad (23)$$

where we have introduced the notation $\phi(\xi) = |\theta|^\sigma \theta(\xi; \mu)$, $\phi(0) = \mu^{\sigma+1}$. Let us set $\psi_\mu(\xi) = \phi(\xi)/\phi(0) \equiv \phi(\xi)/\mu^{\sigma+1}$. Then the equation for $\psi_\mu$ takes the form

$$\psi'_\mu(\xi) = (\sigma + 1)\xi^{1-N} \int_0^\xi \eta^{N-1} \left[\frac{1}{\sigma}\mu^{-\sigma}|\psi_\mu|^{-\sigma/(\sigma+1)}\psi_\mu - \psi_\mu\right] d\eta, \ \xi > 0, \qquad (24)$$

and by (19)

$$|\psi_\mu(\xi)| \le 1, \ \xi \ge 0; \ \mu > \theta_H. \qquad (25)$$

Moreover, from (24) we derive the estimate

$$\begin{aligned} |\psi'_\mu(\xi)| &\le \frac{\sigma + 1}{\xi^{N-1}} \int_0^\xi \eta^{N-1} \left[\frac{1}{\sigma}\mu^{-\sigma} + 1\right] d\eta \equiv \\ &\equiv \frac{\sigma+1}{N}\xi\left[1 + \frac{1}{\sigma}\mu^{-\sigma}\right], \ \xi > 0. \end{aligned} \qquad (26)$$

From (25) and (26) it follows that for any $\mu > \theta_H$ the functions $\psi_\mu$ and $\psi'_\mu$ are uniformly bounded on any compact interval $[0, \xi_m]$. Then from the Arzela-Ascoli compactness theorem it follows that there exists a sequence $\mu_k \to \infty$, $k \to \infty$, such that the corresponding sequence $\psi_{\mu_k}(\xi)$ converges uniformly on $[0, \xi_m]$ to some function $w(\xi)$. The equation for $w$ is obtained from (24) by passing to the limit $\mu = \mu_k \to \infty$ (convergence of $\psi_{\mu_k}{}'$ to $w'$ is established by passing from (24) to the corresponding integral equation). It has the form

$$w'(\xi) = -(\sigma + 1)\xi^{1-N} \int_0^\xi \eta^{N-1} w(\eta)\, d\eta, \ \xi > 0; \ w(0) = 1. \qquad (27)$$

and $w \in C([0, \infty)) \cap C^1(\mathbf{R}_+)$.

Taking now into account the assumption that $\psi_\mu > 0$ in $\mathbf{R}_+$ for any $\mu > \theta_H$, we obtain that $w(\xi) \ge 0$ in $\mathbf{R}_+$. However, by (27) $w(\xi)$ is a monotone strictly decreasing function, so that $w > 0$ in $\mathbf{R}_+$. This immediately leads to a contradiction, because (27) is equivalent to the boundary value problem

$$w'' + \frac{N-1}{\xi}w' + (\sigma + 1)w = 0, \ \xi > 0; \ w'(0) = 0, \ w(0) = 1.$$

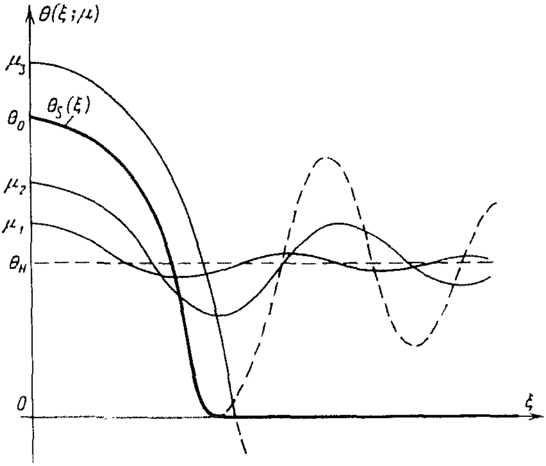


**Fig. 40.** A sketch of the curves $\theta = \theta(\xi;\ \mu)$, solutions of the problem (16), (17) for different $\mu > \theta_H$, $\beta = \sigma + 1$ (S-regime)

whose solution $w = C_1\xi^{(2-N)/2}J_{(N-2)/2}((\sigma+1)^{1/2}\xi)$ ($C_1 > 0$ is a constant, $J_{(N-2)/2}$ is a Bessel function) vanishes at the point $\xi = \xi_1 \equiv z_N^{(1)}/(\sigma+1)^{1/2}$, where $z_N^{(1)} > 0$ is the first root of the function $J_{(N-2)/2}$. □

Properties of solutions of the problem (16), (17) are shown in Figure 40. To values $\mu_1 > \theta_H$, $\mu_2 > \mu_1$ there correspond solutions $\theta(\xi;\mu)$ that are strictly positive in $\mathbf{R}_+$ (Lemma 1), while to a value $\mu_3 > \mu_2$ there corresponds a solution $\theta(\xi;\mu_3)$, which vanishes at a point (Lemma 2). Therefore there can exist a value $\mu = \theta_0 \in (\mu_2, \mu_3]$, for which the function $\theta^{\sigma+1}(\xi;\theta_0)$ is "tangent" to the $\xi$ axis at some point $\xi = \xi_0$, and this "tangency" allows us to extend $\theta(\xi;\theta_0)$ into the domain $\xi > \xi_0$ identically by zero. As a result we have a generalized solution of the original problem (9), (10) with a continuous heat flux $-\xi^{N-1}|\theta_S|^\sigma\theta_S'$; it is marked by a thick line in Figure 40.

But to be able to use the above properties of the solutions $\theta(\xi;\mu)$, we shall need a condition of continuous dependence of $\theta(\xi;\mu)$ on the parameter $\mu$. Observe that in general there is no continuous dependence. This is clearly seen in Figure 40.

**Lemma 3.** *Let the solution $\theta = \theta(\xi;\mu_1)$, $\mu_1 > 0$ be such, that on the compact set $K = [0, \xi_m]$ there are no points for which $|\theta|^\sigma\theta = (|\theta|^\sigma\theta)' = 0$. Then $\theta(\xi;\mu)$ and $(|\theta|^\sigma\theta')(\xi;\mu)$ depend continuously on the parameter $\mu$ in a neighbourhood of $\mu = \mu_1$ on $K$.*

*Proof.* Let us consider equation (23), which is equivalent to the problem (16), (17), for $\mu = \mu_1$. The integrand contain the function $|\phi|^{-\sigma/(\sigma+1)}\phi$, which is not differentiable at $\phi = 0$. Obviously, if $\theta(\xi;\mu_1) > 0$ on $K$, then we have continuous dependence on $\mu$. Let $\xi = \xi_1$ be the first point where $\theta(\xi;\mu_1) = 0$. By assumption, $(|\theta|^\sigma\theta)'(\xi_1;\mu_1) \neq 0$. Then we have continuous dependence on $\mu$ on any interval $]0, \xi_1 - \epsilon[$, where $\epsilon > 0$ is a small number. In a neighbourhood of $\phi \equiv 0$ the operator in the right-hand side of (23) is not a contraction, but the term $|\phi|^{-\sigma/(\sigma+1)}\phi$ is small on $(\xi_1 - \epsilon, \xi_1 + \epsilon)$. Therefore, as we extend $\theta(\xi;\mu)$, with $|\mu - \mu_1|$ small, into this neighbourhood, we shall preserve continuity of the derivative $\phi'(\xi)$ in $\xi$ and $\mu$, and, of course, of the solution $\phi(\xi) \equiv (|\theta|^\sigma\theta)(\xi;\mu)$, which has a unique extension. In a similar way, we can extend $\theta(\xi;\mu)$ to the whole compact set $K$, preserving in the process continuous dependence of $\phi$ and $\phi'$ on $\mu$ in a neighbourhood of $\mu = \mu_1$. □

*Proof of Theorem 1.* It is based entirely on Lemmas 1-3. Let us introduce the set $\mathcal{M} = \{\mu^0 > 0 \,|\, \theta(\xi;\mu) > 0 \text{ in } \mathbf{R}_+ \text{ for all } 0 < \mu < \mu^0\}$. From Lemma 1 it follows that $\mathcal{M} \neq \{\mu^0 \leq \theta_H\}$. By Lemma 2 $\mathcal{M}$ is bounded from above. Therefore there exists $\theta_0 = \sup \mathcal{M} < \infty$. From Lemma 3 it follows then that the solution of problem (16), (17) is the required function $\theta_S$, satisfying (10), with the asymptotic behaviour given by (15). Monotonicity of any non-negative solution of the problem (9), (10) follows immediately from the Remark to Lemma 1. □

## 3 Non-localized self-similar solutions of the HS-regime, $\beta < \sigma + 1$

Here we use the same method to prove the theorem concerning solvability of the self-similar problem (5)–(7) for $\beta \in (1, \sigma + 1)$. Direct inspection of the equation shows that a solution $\theta_S(\xi)$ can only be a function with compact support, having in a neighbourhood of the degeneracy point $\xi_0 = \text{meas supp}\,\theta_S$ asymptotic behaviour different from that of (15):

$$\theta_S(\xi) = \left[\frac{(\sigma + 1 - \beta)\sigma}{2(\beta - 1)}\xi_0(\xi_0 - \xi)\right]^{1/\sigma}(1 + \omega(\xi)). \tag{28}$$

where $\omega(\xi) \to 0$ as $\xi \to \xi_0$.

**Theorem 2.** *For any $1 < \beta < \sigma + 1$ there exists a compactly supported solution $\theta_S$ of the problem (5), (7), which is strictly decreasing wherever $\theta_S > 0$. The problem has no non-monotone solutions. For $N = 1$ the compactly supported solution $\theta_S$ is unique.*

Let us briefly describe the main steps of the proof. The counterpart of identity (20) has here the form

$$
\begin{aligned}
\frac{1}{2}(|\theta|^{\sigma}\theta')^2(\xi)+(N-1)\int_0^{\xi}(|\theta|^{\sigma}\theta')^2(\eta)\frac{d\eta}{\eta}-\\
-m\int_0^{\xi}\eta(|\theta|^{\sigma}\theta'^2)(\eta)\,d\eta+\Phi(|\theta(\xi)|)=\Phi(\mu).
\end{aligned}
\tag{29}
$$

where the function

$$
\Phi(\mu)=\frac{1}{\beta+\sigma+1}\mu^{\beta+\sigma+1}-\frac{1}{(\beta-1)(\sigma+2)}\mu^{\sigma+2},\quad \mu\geq 0,
$$

has the same form as in Figure 39. Therefore, taking into account that $m=[\beta-(\sigma+1)]/[2(\beta-1)]<0$, we have that $\Phi(|\theta(\xi)|)\leq\Phi(\mu)$; in particular, $\theta(\xi)>0$ in $\mathbf{R}_+$ for all

$$
0<\mu<\mu_*=\left[\frac{\beta+\sigma+1}{(\beta-1)(\sigma+2)}\right]^{1/(\beta-1)}.
$$

Hence it also follows that for any $\mu>\theta_H=(\beta-1)^{-1/(\beta-1)}$ the solution is uniformly bounded: $|\theta(\xi)|\leq\mu$ in $\mathbf{R}_+$. Thus we have proved for the case $\beta<\sigma+1$ the counterpart of Lemma 1.

To prove the counterpart of Lemma 2, problem (5), (7) (first equation (5) is extended into the domain of negative values of $\theta$) is reduced, after the change of variable $\phi=|\theta|^{\sigma}\theta$, to the integral equation

$$
\begin{aligned}
\phi'(\xi)=&m(\sigma+1)\xi|\phi|^{-\sigma/(\sigma+1)}\phi+\\
+(\sigma+1)\xi^{1-N}&\int_0^{\xi}\eta^{N-1}\left[\left(\frac{1}{\beta-1}-mN\right)|\phi|^{-\sigma/(\sigma+1)}-|\phi|^{[\beta-(\sigma+1)]/(\sigma+1)}\right]\phi\,d\eta,
\end{aligned}
$$

which after the transformation

$$
\psi_{\mu}(\xi)=\mu^{-(\sigma+1)}\phi\left(\frac{\xi}{\mu^{[\beta-(\sigma+1)]/2}}\right)
\tag{30}
$$

assumes the form

$$
\begin{aligned}
\psi_{\mu}'(\xi)=m(\sigma+1)\mu^{1-\beta}\xi|\psi_{\mu}|^{-\sigma/(\sigma+1)}\psi_{\mu}+(\sigma+1)\xi^{1-N}\times\\
\times\int_0^{\xi}\eta^{N-1}\left[\left(\frac{1}{\beta-1}-mN\right)\mu^{1-\beta}|\psi_{\mu}|^{-\sigma/(\sigma+1)}-|\psi_{\mu}|^{[\beta-(\sigma+1)]/(\sigma+1)}\right]\psi_{\mu}\,d\eta.
\end{aligned}
\tag{30'}
$$

As in the proof of Lemma 2, we have from here that the assumption $\psi_{\mu}>0$ in $\mathbf{R}_+$ for any $\mu>\theta_H$ leads, by the compactness theorem, to the existence of a sequence $\{\mu_k\}$, such that $\psi_\mu \to w > 0$ for $\mu = \mu_k \to \infty$, where $w(\xi)$ satisfies the problem

$$w' = -(\sigma + 1)\xi^{1-N} \int_0^\xi \eta^{N-1} w^{\beta/(\sigma+1)}(\eta)\, d\eta, \ \xi > 0; \ w(0) = 1.$$

It is equivalent to the problem

$$w'' + \frac{N-1}{\xi} w' + (\sigma + 1) w^{\beta/(\sigma+1)} = 0, \ \xi > 0; \ w'(0) = 0, w(0) = 1, \qquad (31)$$

whose solution has a zero. This is proved in subsection 4.1 of § 3, where the case of arbitrary $\beta > 1$, $\sigma \geq 0$, is considered. Proof of Theorem 2 is concluded as in subsection 2, using an assertion analogous to Lemma 3.

Uniqueness of the compactly supported self-similar function $\theta_S$, $N = 1$, will be proved in § 5, by analyzing a quasilinear partial differential equation. Dependence of $\theta(\xi; \mu)$ on $\mu$ for $\beta < \sigma + 1$ is in principle the same as in Figure 40.

Having convinced ourselves of the existence of a suitable function $\theta_S$, let us now indicate the main properties of the self-similar solution (1) for $1 < \beta < \sigma + 1$. This is the HS blow-up regime, and the unbounded solution is not localized. This last property follows directly from the change with time of the radius of the support of the unbounded solution $u_S$ in (4). Indeed, from (4) we obtain the following expression for $|x_f(t)|$, the radius of the spherical front of the propagating thermal wave:

$$|x_f(t)| = \xi_0 (T_0 - t)^{[\beta - (\sigma+1)]/[2(\beta - 1)]}.$$

In view of the condition $\beta < \sigma + 1$, we have that $|x_f(t)| \to \infty$ as $t \to T_0^-$, that is, the thermal wave engulfs the whole space in finite time. Furthermore, it is not hard to deduce from (1) that in the HS-regime

$$u_S(t, x) \to \infty \text{ in } \mathbf{R}^N, \ t \to T_0^-. \qquad (32)$$

From (4) we can also derive the analogous expression for the half-width of the spatial profile of the wave:

$$|x_{ef}(t)| = \xi_* (T_0 - t)^{[\beta - (\sigma+1)]/[2(\beta - 1)]}, \ 0 < t < T_0,$$

where $\xi_* > 0$ is a root of the equation $\theta_S(\xi) = \theta_S(0)/2$. By monotonicity of $\theta_S$, $\xi_*$ is unique.

It turns out that sufficiently general initial perturbations $u_0(x)$ in the problem (0.1), (0.2) behave according to self-similar rules. For $\beta < \sigma + 1$ all unbounded solutions $u(t, x)$ satisfy (32) and are not localized. The theorem concerning absence of localization for $\beta < \sigma + 1$ is proved in § 4. As an illustration, we show in Figure 41 the results of a numerical computation of the Cauchy problem (0.1), (0.2) for $N = 1$. It is clearly seen that here, in distinction to the S-regime (see Figure 37), as $t \to T_0^-$, the thermal wave accelerates, engulfing and heating to infinite temperature all the space $\{-\infty < x < \infty\}$.

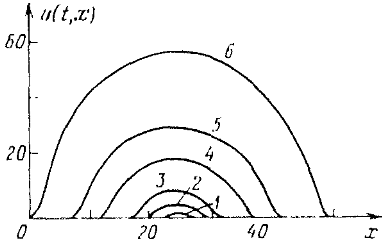


**Fig. 41.** Numerical manifestation of the HS-regime. The parameters are: $\sigma = 2$, $\beta = 5/3$, $N = 1$; 1: $t_1 = 0$, 2: $t_2 = 1.63$, 3: $t_3 = 2.030$, 4: $t_4 = 2.320$, 5: $t_5 = 2.410$, 6: $t_6 = 2.505$

## 4 Localization in the self-similar LS blow-up regime, $\beta > \sigma + 1$

In this subsection we consider self-similar solutions for $\beta > \sigma + 1$, which illustrate even more clearly than in the S-regime the property of localization of processes of heat diffusion and combustion.

Let us consider the boundary value problem (5)–(7) for $\beta > \sigma + 1$. It is not hard to show that unlike the cases $\beta = \sigma + 1$ (subsection 2) and $\beta < \sigma + 1$ (subsection 3), for $\beta > \sigma + 1$ there are no generalized solutions with compact support, and $\theta_S$ has the following asymptotics:

$$\theta_S(\xi) = C_S \xi^{-2/[\beta-(\sigma+1)]}(1 + \nu(\xi)),\quad \nu(\xi) \to 0,\quad \xi \to \infty, \tag{33}$$

where $C_S = C_S(\sigma, \beta, N) > 0$ is a constant (see Remarks). The fact that there does not exist a point $\xi = \xi_0 > 0$ such that $\theta_S(\xi_0) = 0$, $(\theta_S^\sigma \theta_S')(\xi_0) = 0$ and $\theta_S(\xi) > 0$ for $0 < \xi < \xi_0$ follows directly from a local analysis of equation (5) in a left half-neighbourhood of $\xi = \xi_0$.

Below we shall prove existence of the simplest monotone solution of the problem (5)–(7). More complicated non-monotone solutions (so-called *combustion eigenfunctions of the nonlinear medium*) were studied in detail in [349, 391, 267, 268, 274, 90, 1, 2].

As usual, side by side with the boundary value problem (5)–(7), we consider the family of Cauchy problems for the same equation:

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}|\theta|^\sigma \theta')' - m\theta'\xi - \frac{1}{\beta - 1}\theta + |\theta|^{\beta-1}\theta = 0,\quad \xi > 0, \tag{34}$$

$$\theta(0;\mu) = \mu,\ \theta'_\xi(0;\mu) = 0;\quad m = \frac{\beta - (\sigma + 1)}{2(\beta - 1)} > 0, \tag{35}$$

where $\mu > 0$ is a constant. Let us show that for some $\mu$ the solution $\theta = \theta(\xi;\mu) \geq 0$ satisfies condition (7) at infinity, and thus defines the required function $\theta_S$. Observe the following property of solutions of the problem (34), (35). Earlier, in the cases $\beta = \sigma + 1$ and $\beta < \sigma + 1$ it was shown that in the class of all solutions $\theta(\xi; \mu)$ of the Cauchy problem for different $\mu > 0$, there was always a family of strictly positive functions $\theta$, oscillating around the spatially homogeneous (homothermic) solution $\theta \equiv \theta_H$ (see Figure 40). The oscillations there were damped, and their amplitude decreased with $\xi$, which ensured strict positivity of the solutions. For $\beta > \sigma + 1$, when $m > 0$, this, in general, is not the case.

For example, for $N = 1$ the identity (29) ensures exactly the opposite, i.e., if for some $\mu > 0$ there exists a solution $\theta = \theta(\xi; \mu) > 0$, which oscillates about $\theta \equiv \theta_H$, and $\xi_1, \xi_2$ ($\xi_1 < \xi_2$) are any two maximum (minimum) points, then

$$\theta(\xi_1; \mu) < \theta(\xi_2; \mu) \quad (\theta(\xi_1; \mu) > \theta(\xi_2; \mu)). \tag{36}$$

We shall take into account the following easily established fact: if $\theta(\xi; \mu) \to s$ as $\xi \to \infty$ ($s \geq 0$), then $s = 0$ (to prove this, it suffices to analyse the equation locally in a neighbourhood of $\xi = \infty$ (see [1, 2])).

*1 Linearization around $\theta \equiv \theta_H$*

A fairly precise picture of undamped oscillations for $\mu$ sufficiently close to $\theta_H$ is provided by solutions $v(\xi)$ of the problem obtained by linearizing the original problem around the homogeneous solution $\theta \equiv \theta_H$.

Let us set

$$\theta(\xi; \mu) = \theta_H + \epsilon v(\xi), \; \xi \geq 0; \; \theta_H = (\beta - 1)^{-1/(\beta-1)}, \tag{37}$$

where $\epsilon > 0$ is a constant, which plays the role of a small parameter in the sequel. Then, after substitution of (37) into (34), (35) we obtain the following problem for $v(\xi)$:

$$\theta_H^\sigma \frac{1}{\xi^{N-1}} \left(\xi^{N-1} v'\right)' - m v' \xi + v = \epsilon \Phi_\epsilon(v), \; \xi > 0, \tag{38}$$

$$v(0) = \nu, \; v'(0) = 0. \tag{39}$$

Here $\Phi_\epsilon(v) : C^2 \to C$ is a bounded quasilinear second order operator. Boundary values $\nu$ in (39) and $\mu$ in (35) are related by

$$\mu = \theta_H + \epsilon \nu. \tag{37'}$$

From (38) it follows that by continuous dependence of the solution of the equation on a parameter, for sufficiently small $\epsilon > 0$ the solution $v(\xi)$ of the problem (38), (39) is close to the solution of the corresponding linear problem:

$$\theta_H^\sigma \frac{1}{\xi^{N-1}} \left(\xi^{N-1} y'\right)' - m y' \xi + y = 0, \; \xi > 0. \tag{40}$$

$$y(0) = \nu \neq 0, \; y'(0) = 0. \tag{41}$$

Because of that, let us consider the problem (40), (41) in more detail. The change of the independent variable

$$\xi = \left(\frac{2\theta_{II}''}{m}\right)^{1/2} \eta^{1/2} \equiv \left[\frac{4(\beta-1)^{1-\sigma/(\beta-1)}}{\beta-(\sigma+1)}\right]^{1/2} \eta^{1/2} \tag{42}$$

reduces (40) to the degenerate hypergeometric equation

$$\eta y''_{\eta\eta} + y'_\eta(c-\eta) - ay = 0, \quad \eta > 0; \; y(0) = \nu, \tag{43}$$

where $c = N/2$, $a = -1/(2m) = -(\beta-1)/[\beta-(\sigma+1)]$. Then the second boundary condition assumes the form

$$\eta^{1/2} y'_\eta(\eta)|_{\eta=0} = 0.$$

Therefore a suitable solution of (43) is one with a bounded derivative $y'_\eta(0)$. It can be written down in a convergent Kummer series [35, 317]:

$$y(\eta) = \nu \left(1 + \frac{a}{c}\frac{\eta}{1!} + \frac{a(a+1)}{c(c+1)}\frac{\eta^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{\eta^3}{3!} + \dots\right), \tag{44}$$

which defines the degenerate hypergeometric function

$$y(\eta) = \nu M(a, c, \eta) = \nu e^\eta \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} \int_0^1 e^{-\eta s} s^{c-a-1}(1-s)^{a-1}\, ds.$$

In general, the function (44), and thus the solution $y(\xi)$ of the problem (40), (41), is non-monotone. In the cases when

$$-a = (\beta-1)/[\beta-(\sigma+1)] = K, \tag{45}$$

where $K > 1$ is an integer, the function $y(\eta)$ is a polynomial of degree $K$, since the series (44) terminates at the $(K+1)$-st term. Furthermore, it is known [35] that it has for $\eta > 0$ exactly $K$ "zeros". Equality (45) holds if

$$\beta = \beta_K = -\frac{1}{K-1} + \frac{K}{K-1}(\sigma+1), \quad K = 2, 3, \dots \tag{46}$$

(for convenience we set $\beta_1 = \infty$). For example, if $\beta = \beta_2 = 2\sigma + 1$, then

$$y(\eta) = \nu\left(1 - \frac{4}{N}\eta + \frac{4}{N(N+2)}\eta^2\right), \quad \eta > 0,$$

and therefore the equation $y(\eta) = 0$ has two positive roots:

$$\eta_0^\pm = \left[N + 2 \pm (2(N+2))^{1/2}\right]/2,$$

to which there correspond the following zeros of the solution $y(\xi)$ of the original problem (40), (41):

$$\xi_0^{\pm} = 2\sqrt{2}(2\sigma)^{-1/4}\left(\eta_0^{\pm}\right)^{1/2} > 0.$$

The equality (45) determines the number of zeros (and thus the nature of non-monotonicity, or, we might say, the degree of complexity) of the function $y$ for all values $\beta > \sigma + 1$. Namely, for any $\beta_{K+1} \le \beta < \beta_K$ ($K = 1, 2, \dots$) the function $y(\xi)$ has exactly $K+1$ zeros in $\xi > 0$ (see [35], where approximate formulae for computation of zeros and positions of extremum points of the function $y(\xi)$ can be found).

Combining all the cases considered above, we obtain a general formula for the number of zeros of the solution of the problem (40), (41):

$$K = -[a], \quad a = -(\beta - 1)/|\beta - (\sigma + 1)| < 0, \tag{47}$$

which is valid for any $\beta > \sigma + 1$. By (47) $K \ge 2$ for all $\beta > \sigma + 1$, that is, the solution $y(\xi)$ will always be non-monotone[2]. Let us note that the oscillations around zero are undamped (this follows directly from the form of equation (40)).

Returning to the original linearized problem (38), (39), we see that by continuous dependence on the parameter $\epsilon > 0$ in a neighbourhood of $\epsilon = 0$ of the solution $\nu$ on any compact set, there exists an $\epsilon > 0$ small enough, such that for all $|\nu| \le 1$ the function $\nu(\xi)$ has for $\xi > 0$ at least $K$ ($K \ge 2$) zeros. For the original problem (34), (35) it means that for any $0 < \theta_H - \epsilon < \mu < \theta_H + \epsilon$ the solution $\theta(\xi;\ \mu)$ has for $\xi \ge 0$ at least $K$ extrema. In particular, if $\theta_H < \mu < \theta_H + \epsilon$, then there exist at least $[K/2]$ minima and $K - [K/2]$ maxima, for which (36) holds.

## 2 *Global properties of the solutions $\theta(\xi; \mu)$*

Thus, we have determined the behaviour of $\theta(\xi; \mu)$ for all $\mu$ close to $\theta_H$. Let us show now that for large enough $\mu$ the function $\theta(\xi; \mu)$ vanishes at some point. For that we could use the same method as in the case $\beta \le \sigma + 1$ (though due to (36) some additional difficulties appear). However, for $N = 1$ this result can be obtained in a much simpler way. In the following we confine ourselves to a fairly brief analysis of the case $N = 1$, and at the same will present some results pertaining to the multi-dimensional case.

**Lemma 4.** *Let $N = 1$, $\beta > \sigma + 1$. Then for all*

$$\mu \ge \mu^* = \left[\frac{\beta + \sigma + 1}{(\beta - 1)(\sigma + 2)}\right]^{1/(\beta - 1)} \tag{48}$$

[2] Let us note that in the case of a semilinear equation ($\sigma = 0$), $K \equiv 1$ for all $\beta$. This conclusion plays an important part in § 7.

*the solution* $\theta = \theta(\xi;\mu)$ *of the problem* (34), (35) *vanishes at some point and has no extrema in* $\{\bar{\xi} > 0 \mid \theta(\xi;\mu) > 0, 0 < \xi < \bar{\xi}\}$ *(that is, it is strictly decreasing).*

*Proof.* Let us consider the identity (29), which for $N = 1$ has the form

$$\frac{1}{2}(|\theta|^{\sigma}\theta')^2(\xi) - m\int_0^{\xi} \eta(|\theta|^{\sigma}(\theta')^2)(\eta)\,d\eta + \Phi(|\theta(\xi)|) = \Phi(\mu). \tag{49}$$

Here the function

$$\Phi(\mu) = \frac{1}{\beta+\sigma+1}\mu^{\beta+\sigma+1} - \frac{1}{(\beta-1)(\sigma+2)}\mu^{\sigma+2}, \quad \mu \geq 0,$$

has the same form as in the case $\beta \leq \sigma + 1$ (see Figure 39).

Let us assume the opposite. For example, assuming that (48) is satisfied, let the solution $\theta(\xi;\mu)$ have a point of minimum at $\xi = \xi_* < \infty$, where, naturally, $\theta < \theta_H$. Then, setting in (49) $\xi = \xi_*$, in view of the condition $m > 0$, we obtain the inequality $\Phi(\theta(\xi_*)) > \Phi(\mu)$, and therefore we must have that $\theta(\xi_*;\mu) > \mu > \theta_H$, which is impossible. In the same way it is proved that for $\mu \geq \mu^*$ the function $\theta(\xi;\mu)$ cannot be a positive solution in $\mathbf{R}_+$ (i.e. the case $\xi_* = \infty$ is also impossible). □

In a similar fashion, we derive from (49)

**Corollary.** *Any solution of the problem* (5)–(7) *satisfies for* $\beta > \sigma + 1$, $N = 1$, *the estimate*

$$\theta_S(\xi) < \mu^* = \left[\frac{\beta+\sigma+1}{(\beta-1)(\sigma+2)}\right]^{1/(\beta-1)}. \tag{50}$$

*Proof.* Let $\xi = \xi_*$ be a point of absolute extremum of the function $\theta_S(\xi)$. Setting in (49) first $\xi = \infty$, and then $\xi = \xi_*$, and subtracting the second equality from the first, we obtain $\Phi(\theta_S(\xi_*)) < 0$, which guarantees (50). □

Let us move on now to prove solvability of the problem (5)–(7) for $\beta > \sigma + 1$, $N = 1$. Let us set $\mathcal{N} = \{\mu > \theta_H \mid$ there exists a compact set $K = [0, \xi_K]$, such that $\theta(\xi;\mu) > 0$ on $K$ and has at least one minimum point on $K\}$. Then $\mathcal{N} \neq \emptyset$ (by the analysis of the linearized equation) and $\mathcal{N}$ is bounded from above (Lemma 4). Therefore there exists

$$\sup \mathcal{N} = \theta_0 \in (\theta_H, \infty). \tag{51}$$

It is not hard to see that by the choice of $\theta_0$ the function $\theta(\xi;\theta_0)$, first, has no minima for $\xi > 0$ (this follows from continuous dependence of $\theta(\xi;\mu)$ on $\mu$ on any compact set, where $\theta > 0$), and, second, cannot vanish (see Lemma 3). Therefore $\theta(\xi;\theta_0)$ is a positive, strictly monotone solution $\theta_S(\xi)$ of the problem (5)–(7) for $\beta > \sigma + 1$, $N = 1$.

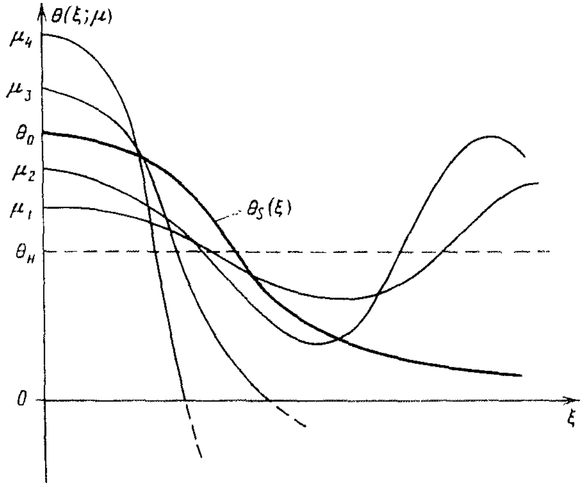


**Fig. 42.** The function $\theta = \theta(\xi;\ \mu)$ for different $\mu$. $\beta > \sigma + 1$

In Figure 42 we sketch the behaviour of the functions $\theta(\xi;\mu)$ for different values of $\mu \equiv \theta(0;\mu)$; the thick line shows the solution $\theta_S(\xi)$, which corresponds to $\mu = \theta_0 = \sup \mathcal{N}$.

Thus, we have proved the following

**Theorem 3.** *Let* $\beta > \sigma + 1$, $N = 1$. *Then the problem* (5)–(7) *has a strictly monotone positive solution.*

Some additional properties of the function $\theta_S(\xi)$ will be mentioned in subsection 4.3, as well as in § 6.

**Remark.** As we already mentioned above, in Theorem 3 we determine the simplest self-similar solution; in fact the problem (5)–(7) has at least $K' = K - 1$ different solutions $\theta_S^1, \theta_S^2, \ldots, \theta_S^{K-1}$. Each one of these has one more extremum point than the preceding one. Among these there are at least $[K'/2]$ solutions for which $\theta_S(0) < \theta_H$ ($\xi = 0$ is a point of minimum) and at least $K' - [K'/2]$ solutions such that $\theta_S(0) > \theta_H$ ($\xi = 0$ is a point of maximum). Proof of existence of these solutions follows the same lines; the set $\mathcal{N} \neq \emptyset$ now including values of $\mu$ for which the corresponding function $\theta(\xi;\mu)$ has a more complicated spatial profile than the desired function $\theta_S(\xi)$ (see [1, 2]).

*3 The multi-dimensional case, $N > 1$*

The method of proof of Lemma 2 can be used to show quite easily that whether or not a statement analogous to Lemma 4 holds for $N > 1$ depends on the properties of the solution of the stationary equation

$$\frac{1}{\xi^{N-1}}\left(\xi^{N-1}|f|^{\sigma}f'\right)' + |f|^{\beta-1}f = 0,\ \xi > 0; \qquad f(0) = 1,\ f'(0) = 0. \tag{52}$$

Indeed, let us rewrite (30′) in the form

$$\psi'_\mu(\xi) = -(\sigma+1)\xi^{1-N}\int_0^\xi \eta^{N-1}|\psi_\mu|^{[\beta-(\sigma+1)]/(\sigma+1)}\psi_\mu\,d\eta + \mu^{1-\beta}G(\psi_\mu), \tag{53}$$

where $\psi_\mu(0) = 1$, $\psi'_\mu(0) = 0$ and $G(\psi_\mu)$ is the following integral operator, which is bounded in $C$:

$$G(\psi_\mu) = m(\sigma+1)\xi|\psi_\mu|^{-\sigma/(\sigma+1)}\psi_\mu + \\ + (\sigma+1)\xi^{1-N}\int_0^\xi \eta^{N-1}\left[\frac{1}{\beta-1} - mN\right]|\psi_\mu|^{-\sigma/(\sigma+1)}\psi_\mu\,d\eta,$$

which is not a contraction in a neighbourhood of $\psi_\mu \equiv 0$.

Unlike the case $\beta \le \sigma+1$, for $\beta > \sigma+1$, a priori nothing can be said about boundedness of $\psi_\mu$ and $\psi'_\mu$ on compact sets, since possible oscillations of $\theta(\xi;\ \mu)$ around $\theta \equiv \theta_H$ are undamped (see (36)). Therefore we shall use a method based on continuous dependence of $\psi_\mu$ on $\mu$ in a neighbourhood of $\mu = \infty$.

For $\mu = \infty$, (53) becomes formally the equation

$$\psi'_\infty(\xi) = -(\sigma+1)\xi^{1-N}\int_0^\xi \eta^{N-1}\psi_\infty^{[\beta-(\sigma+1)]/(\sigma+1)}\psi_\infty\,d\eta,\ \xi > 0;$$

$$\psi_\infty(0) = 1,\ \psi'_\infty(0) = 0,$$

and its solution coincides with the function $|f|^\sigma f(\xi)$, where $f(\xi)$ is the solution of the problem (52). In § 3 we shall show that for any $\beta < (\sigma+1)(N+2)/(N-2)_+$ the function $f(\xi)$ of (52) vanishes at some point $\xi = \xi_* > 0$, and, moreover, $\psi'_\infty(\xi_*) < 0$. Therefore on every compact set $K_\epsilon = [0, \xi_* - \epsilon]$, $\epsilon > 0$, on which $\psi_\infty > 0$ there is continuous dependence of $\psi_\mu(\xi)$ on $\mu$ in a neighbourhood of $\mu = \infty$, that is, $\psi_\mu(\xi)$ is close to $\psi_\infty(\xi)$ on $K_\epsilon$ for all sufficiently large $\mu > 0$. Let us fix a sufficiently small $\epsilon > 0$. Then

$$\psi_\mu(\xi_* - \epsilon) \to f^{\sigma+1}(\xi_* - \epsilon),\ \psi'_\mu(\xi_* - \epsilon) \to (f^{\sigma+1}(\xi_* - \epsilon))' < 0 \text{ as } \mu \to \infty.$$

But $G(\psi_\mu) = O(\|\psi_\mu\|_C^{1/(\sigma+1)}) \to 0$ as $\|\psi_\mu\|_C \to 0$. Therefore in order to extend $\psi_\mu(\xi)$ from a point $\xi = \xi_* - \epsilon$ into a neighbourhood of $\xi = \xi_*$ we can use the

Schauder fixed point theorem. Then by "smallness" of $G(\psi_\mu)$ the derivative $\psi'_\mu(\xi)$ does not change significantly and as a result $\psi_\mu(\xi)$ will be zero if $\mu$ is sufficiently large ($\mu^{1-\beta}$ is a small number). This fact allows us to prove the following result.

**Theorem 4.** *Let $\sigma+1<\beta<\infty$ for $N=1$ or $N=2$ and $\sigma+1<\beta<(\sigma+1)(N+2)/(N-2)$ for $N\geq 3$. Then the problem (5)–(7) has a strictly positive monotone solution.*

**Remark.** In the case $\beta\geq(\sigma+1)(N+2)/(N-2)_+$ the solution $f(\xi)$ of (52) is strictly positive in $\mathbf{R}_+$ (see Lemma 1 in § 3) and the question of existence of $\theta_S(\xi)$ remains open.

In the sequel we shall need the following surprising property of the self-similar function $\theta_S$.

**Theorem 5.** *Let $\sigma+1<\beta\leq(\sigma+1)N/(N-2)_+$ and let $\theta_S(\xi)$ be an arbitrary solution of the problem (5)–(7). Then*

$$F(\xi)\equiv\frac{\beta-(\sigma+1)}{2(\beta-1)}\theta'_S(\xi)\xi+\frac{1}{\beta-1}\theta_S(\xi)>0,\ \ \xi\geq 0. \tag{54}$$

Inequality (54) displays some important properties of the unbounded self-similar solution (1), (4). For example, from it immediately follows

**Corollary 1.** *For $\sigma+1<\beta\leq(\sigma+1)N/(N-2)_+$ the solution $u_S(t,x)$ is critical, that is,*

$$\frac{\partial}{\partial t}u_S(t,x)\equiv(T_0-t)^{-1/(\beta-1)-1}\left[m\theta'_S(\xi)\xi+\frac{1}{\beta-1}\theta_S(\xi)\right]>0, \tag{55}$$

$$t\in(0,T_0),\ x\in\mathbf{R}^N,$$

*and therefore for any $t\in(0,T_0)$*

$$u_S(t,x)<u_S(T_0^-,x)\equiv C_S|x|^{-2/[\beta-(\sigma+1)]},\ \ x\in\mathbf{R}^N\setminus\{0\}, \tag{56}$$

*where $C_S=C_S(\sigma,\beta,N)>0$ is the constant of the asymptotic expansion (33).*

Integrating the inequality (54), we obtain the following estimate, which again demonstrates strict positivity of $\theta_S(\xi)$ (it correctly reflects the asymptotic behaviour of the function $\theta_S$ as $\xi\to\infty$).

**Corollary 2.** *Let $\sigma+1<\beta\leq(\sigma+1)N/(N-2)_+$. Then for all $\xi\geq\xi_0>0$*

$$\theta_S(\xi)\geq\theta_S(\xi_0)(\xi/\xi_0)^{-2/[\beta-(\sigma+1)]}.$$

*Proof of Theorem* 5. Let us rewrite equation (5) in the form

$$\theta_S^\sigma\left(\theta_S'' + \frac{N-1}{\xi}\theta_S' + \sigma\frac{\theta_S'}{\theta_S}\theta_S'\right) + \theta_S^\beta = F(\xi). \tag{57}$$

Let $F(\xi_1) \le 0$, $\theta_S(\xi_1) > 0$, $\xi_1 \ge 0$. Then $\theta_S'(\xi_1) < 0$ and therefore

$$F'(\xi_1) = m\xi_1\left[\theta_S''(\xi_1) + \frac{\beta-\sigma+1}{\beta-(\sigma+1)}\frac{\theta_S'(\xi_1)}{\xi_1}\right] \le$$
$$\le m\xi_1\left[\theta_S''(\xi_1) + \frac{1}{\xi_1}\left(N-1-\frac{2\sigma}{\beta-(\sigma+1)}\right)\theta_S'(\xi_1)\right],$$

as $(\beta-\sigma+1)/[\beta-(\sigma+1)] \ge N-1-2\sigma/[\beta-(\sigma+1)]$ for $\sigma+1<\beta\le(\sigma+1)N/(N-2)_+$.

Then from (57) we have

$$\frac{\theta_S^\sigma(\xi_1)}{m\xi_1}F'(\xi_1) < F(\xi_1) \le 0.$$

since

$$\frac{\theta_S'}{\theta_S} \le -\frac{1}{m(\beta-1)}\frac{1}{\xi_1}.$$

From here it follows that $F(\xi_1) < 0$, so that the function $F(\xi)$ is decreasing on an interval $(\xi_1, \xi_1+\delta)$, $\delta > 0$, where

$$\frac{\theta_S^\sigma(\xi)}{m\xi}F'(\xi) < F(\xi) < 0.$$

Therefore $F(\xi) < 0$ for any $\xi > \xi_1$. But by (57) it also means that

$$\frac{1}{\xi^{N-1}}\left(\xi^{N-1}\theta_S^\sigma\theta_S'\right)' + \theta_S^\beta < 0,\ \ \xi > \xi_1;\ \theta_S'(\xi_1) < 0.$$

The last inequality, under the assumptions we have made, ensures that the function $\theta_S(\xi)$ vanishes at some point and, consequently, is not a solution of the problem (5)–(7). This is especially simple to prove when $N=1,2$, $\beta>\sigma+1$. Analysis of the case $N\ge 3$ uses the same method as in the proof of Lemma 1 in subsection 4.1 in § 3. □

## *4 Properties of the self-similar LS-regime*

Let us again write down the expression for the time dependence of the half-width of the self-similar thermal structure:

$$|x_{ef}(t)| = \xi_*(T_0-t)^{[\beta-(\sigma+1)]/[2(\beta-1)]},\ \ 0<t<T_0.$$

Hence we have that in the LS-regime ($\beta > \sigma + 1$) the half-width decreases with time and $|x_{ef}(T_0^-)| = 0$. Thus intensive combustion takes place in an ever shrinking central region of the structure. As a result blow-up occurs only at one point; in the rest of the space the temperature is bounded from above uniformly in $t$ by the limiting profile $u_S(T_0^-, x)$ (see (56)).

Thus the self-similar solution is effectively localized. Strict localization, however, cannot obtain here, since $u_S(t, x)$ is strictly positive in $(0, T_0) \times \mathbf{R}^N$. Numerical computations show that self-similar estimates hold, and, moreover, testify to the occurrence of strict localization in the LS-regime (for a proof see § 4).

An example of such a computation is presented in Figure 43. Here $u_0(x)$ is a compactly supported (not self-similar) initial function. Up to the time $t = t_1$ the initial perturbation spreads out, then reaches its *resonance length* ($t = t_2$), after which fast growth of the solution starts. It is clearly seen that as $t \to T_0^-$ combustion occurs in an ever narrowing central region of the structure. During the process the front points of the solution $u(t, x)$ hardly move at all and heat is localized in the fundamental length $L_{LS}$ of LS-regime. We emphasize that here, unlike the situation in the S-regime, $L_{LS}$ depends on the initial perturbation $u_0(x)$.

Figure 44 shows the LS-regime, which is close to the self-similar one as $t \to T_0^-$, which develops from a spatially homogeneous initial perturbation $u_0(x) \equiv 1$, due to instability of homothermic combustion with finite time blow-up.

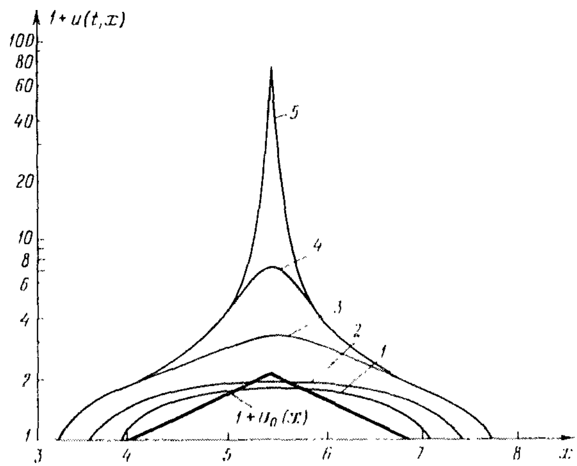


**Fig. 43.** Numerical manifestation of the LS-regime. The parameters are: $\sigma = 2$, $\beta = 5$, $N = 1$; 1: $t_1 = 1.14$, 2: $t_2 = 2.34$, 3: $t_3 = 3.559$, 4: $t_4 = 3.5712$, 5: $t_5 = 3.5714$

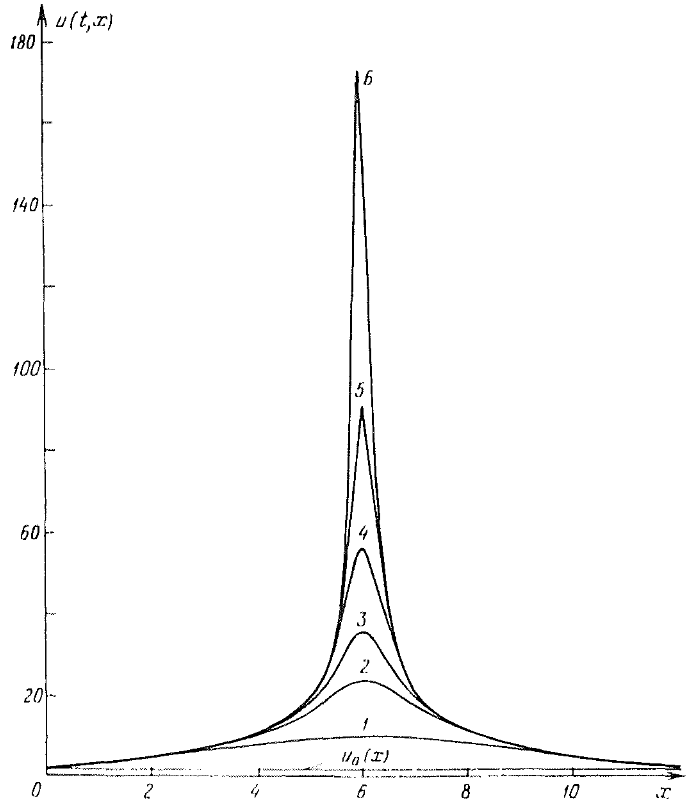


**Fig. 44.** Numerical manifestation of the LS-regime. The parameters are: $\sigma = 2$, $\beta = 4$, $N = 1$; 1: $t_1 = 0.2824$, 2: $t_2 = 0.28358$, 3: $t_3 = 0.28365$, 4: $t_4 = 0.283674$, 5: $t_5 = 0.283679$, 6: $t_6 = 0.283681$

## § 2 Asymptotic behaviour of unbounded solutions. Qualitative theory of non-stationary averaging

In this section we consider questions connected with asymptotic stability of self-similar solutions of the problem

$$\mathbf{A}(u) \equiv u_t - \nabla \cdot (u^\sigma \nabla u) - u^\beta = 0, \quad t > 0, \, x \in \mathbf{R}^N, \tag{1}$$

$$u(0, x) = u_0(x) \geq 0, \quad x \in \mathbf{R}^N; \; u_0 \in C(\mathbf{R}^N), \, u_0^{\sigma+1} \in H^1(\mathbf{R}^N). \tag{2}$$

We shall be interested in the following question: under what conditions does the unbounded solution of the problem, $u(t, x)$, acquire in the domain of intense heating the spatio-temporal structure characteristic of the self-similar solution

$$\begin{aligned} u_S(t, x) &= (T_0 - t)^{-1/(\beta-1)}\theta_S(\xi),\ \xi = x/(T_0 - t)^m, \\ m &= |\beta - (\sigma + 1)|/|2(\beta - 1)|. \end{aligned} \tag{3}$$

Difficulties of the analysis of asymptotic behaviour of unbounded solutions are related to the speed of evolution of the blow-up regime, which is not stable with respect to arbitrary, even infinitesimally small, perturbations of the initial function $u_0(x)$.

Here we present the results of a qualitative analysis. Qualitative theory allows us to obtain a number of quite subtle results; for example, for $\beta > \sigma + 1 + 2/N$ we can find a family of global solutions of the problem (1), (2), which correspond to sufficiently small initial functions $u_0(x)$. At the same time we show that for $\beta \leq 1 + \sigma + 2/N$ there are no global solutions $u \not\equiv 0$. These results are justified in § 3.

The idea of the averaging method consists of reducing the problem (1), (2) for a partial differential equation to a system of two ordinary differential equations with respect to certain parameters that characterize the evolution in time of the spatial profile of the thermal structure. As such parameters we can choose, for example, the amplitude and the half-width of the structure, or the amplitude and the position of the front of a radially symmetric structure which has compact support in $x$. The latter averaging, "amplitude-front position" allows us, in particular, to describe the localization of unbounded solutions in the S- and LS-regimes, and absence of localization in the HS-regime.

## 1 The non-stationary averaging "amplitude-half-width"

Let the initial function $u_0 \in L^1(\mathbf{R}^N)$ in (2) have compact support and be elementary in the sense that $u_0(x)$ has a unique maximum at $x = 0$. We shall take $u_0(x)$ to be close to a radially symmetric function. Then we should expect that the solution $u(t, x)$ will also be almost radially symmetric and that the half-width of the evolving thermal structure will be approximately the same in all directions. Taking this as our departure point, let us seek an approximate solution of the problem (1), (2) in the form

$$u(t, x) = \psi(t)\theta(\xi),\ \xi = (|x_1|/\phi(t), \ldots, |x_N|/\phi(t)), \tag{4}$$

where $\psi(t)$ and $\phi(t)$ are, respectively, the amplitude and the half-width of the structure, which depend on time, and $\theta(\xi)$ is some fixed function of compact support, which is monotone decreasing in all its arguments and such that $\theta(0) = 1$, $\theta^{\sigma+1} \in H^1(\mathbf{R}^N)$.

In its form (4) is the same as the self-similar solution (3), which was studied in § 1, where the specific form of the functions $\psi(t)$, $\phi(t)$ is determined by substituting (4) into the original equation (1). Therefore the self-similar solution (3) satisfies the problem (1), (2) for some specially chosen initial functions $u_0(x)$.

In our case $u_0$ is, in general, an arbitrary function; therefore we do not require that the approximate solution satisfy equation (1) in strict sense. Instead, we shall demand that (4) satisfy the two following equalities (conservation laws)[3]:

$$\int_{\mathbf{R}^N} \mathbf{A}(u(t,x))\,dx = 0,\quad \int_{\mathbf{R}^N} \mathbf{A}(u(t,x))u(t,x)\,dx = 0,\quad t > 0. \tag{5}$$

After integration by parts, these equalities have the form

$$\frac{d}{dt}\|u(t)\|_{L^1(\mathbf{R}^N)} = \|u(t)\|^{\beta}_{L^{\beta}(\mathbf{R}^N)}, \tag{6}$$

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbf{R}^N)} = -\int_{\mathbf{R}^N} u^{\sigma}|\nabla u|^2\,dx + \|u(t)\|^{\beta+1}_{L^{\beta+1}(\mathbf{R}^N)}. \tag{7}$$

In the first equality, which is the energy equation, there is no contribution from the diffusion operator, while in the second there are contributions of both the source term and the diffusion operator.

Substitution of (4) into (6), (7) gives us the following system of ordinary differential equations for the functions $\psi(t)$, $\phi(t)$:

$$\frac{d}{dt}[\psi(t)\phi^N(t)] = \nu_1\psi^{\beta}(t)\phi^N(t), \tag{8}$$

$$\frac{d}{dt}[\psi^2(t)\phi^N(t)] = -\nu_2\psi^{\sigma+2}(t)\phi^{N-2}(t) + \nu_3\psi^{\beta+1}(t)\phi(t), \tag{9}$$

where $\nu_1$, $\nu_2$, $\nu_3$ are positive constants:

$$\begin{aligned} \nu_1 &= \frac{\int_{\mathbf{R}^N}\theta^{\beta}(\xi)\,d\xi}{\int_{\mathbf{R}^N}\theta(\xi)\,d\xi},\\ \nu_2 &= 2\frac{\int_{\mathbf{R}^N}\theta^{\sigma}|\nabla\theta|^2\,d\xi}{\int_{\mathbf{R}^N}\theta^2\,d\xi},\\ \nu_3 &= 2\frac{\int_{\mathbf{R}^N}\theta^{\beta+1}\,d\xi}{\int_{\mathbf{R}^N}\theta^2\,d\xi}. \end{aligned} \tag{10}$$

Here we are assuming that the function $\theta$ in (4) is such that all these expressions make sense.

[3] Instead of these two laws we could take others; for example, instead of the second one we could take the identity $(\mathbf{A}(u(t,x)), x) = 0$ and retain the first one as it is the simplest possible. This does not affect the results of the analysis below.

It is not hard to resolve the system (8), (9) with respect to derivatives:

$$\psi' = \frac{\psi^{\sigma+1}}{\phi^2}\left[(\nu_3 - \nu_1)\psi^{\beta-(\sigma+1)}\phi^2 - \nu_2\right], \tag{11}$$

$$\phi' = \frac{\psi^{\sigma}}{N\phi}\left[(2\nu_1 - \nu_3)\psi^{\beta-(\sigma+1)}\phi^2 + \nu_2\right], \ t > 0, \tag{12}$$

while from (11), (12) we easily pass to the single equation

$$\frac{d\psi}{d\phi} = -N\frac{\psi}{\phi}\frac{a\psi^{\beta-(\sigma+1)}\phi^2 - 1}{b\psi^{\beta-(\sigma+1)}\phi^2 - 1}, \ \psi > 0, \ \phi > 0, \tag{13}$$

where

$$a = (\nu_3 - \nu_1)/\nu_2, b = (\nu_3 - 2\nu_1)/\nu_2. \tag{14}$$

We shall take the condition $\nu_3 > 2\nu_1$ to hold, so that

$$a > 0, \ b > 0. \tag{15}$$

The inequalities (15) are, generally speaking, necessary for (11), (12) to admit blow-up regimes.

Let us move on now to analyse the equation (13), which describes the dependence of the amplitude of the thermal structure on its half-width.

*I S-regime*, $\beta = \sigma + 1$

In this case equation (13) has an especially simple form:

$$\frac{d\psi}{d\phi} = -N\frac{\psi}{\phi}\frac{a\phi^2 - 1}{b\phi^2 - 1}, \ \psi > 0, \ \phi > 0. \tag{16}$$

It is easily integrated, and its general solution has the form

$$C_0 = \psi^{-1}\phi^{-N}\left|1 - \frac{\nu_3 - 2\nu_1}{\nu_2}\phi^2\right|^{-N\nu_1/[2(\nu_3-2\nu_1)]}, \tag{17}$$

where $C_0 \geq 0$ is a constant determined by the initial conditions: if $\psi(0) = \psi_0 > 0$ and $\phi(0) = \phi_0 > 0$, then

$$C_0 = \psi_0^{-1}\phi_0^{-N}\left|1 - \frac{\nu_3 - 2\nu_1}{\nu_2}\phi_0^2\right|^{-N\nu_1/[2(\nu_3-2\nu_1)]},$$

where $\phi_0^2 \neq \nu_2/(\nu_3 - 2\nu_1)$. For (17) to correspond to a blow-up regime, we have to demand that the inequality $\nu_3 > 2\nu_1$ holds, that is, that both conditions (15) are satisfied.

We are in a position to check how exact the averaging is, using the self-similar solution of § 1 for the case $\beta = \sigma + 1$, $N = 1$:

$$u_S(t,x) = \begin{cases} (T_0 - t)^{-1/\sigma} \left( \dfrac{2(\sigma+1)}{\sigma(\sigma+2)} \cos^2 \dfrac{\pi x}{L_S} \right)^{1/\sigma}, |x| \le L_S/2, \\ 0, |x| > L_S/2; \; 0 < t < T_0, \end{cases} \tag{18}$$

where $L_S = 2\pi(\sigma+1)^{1/2}/\sigma$ is the fundamental length.

Taking the precise structure of (18) into account, let us set

$$\theta(\xi) = \begin{cases} \cos^{2/\sigma}(\pi\xi/2), & |\xi| < 1, \\ 0, & |\xi| \ge 1, \end{cases} \tag{19}$$

to which corresponds the amplitude

$$\psi(t) = \left[ \frac{2(\sigma+1)}{\sigma(\sigma+2)} \right]^{1/\sigma} (T_0 - t)^{-1/\sigma}, \tag{20}$$

as well as the half-width

$$\phi(t) = L_S/2. \tag{20'}$$

For the function (19) the coefficients $\nu_1, \nu_2, \nu_3$ have the following form:

$$\nu_1 = \frac{\sigma+2}{2(\sigma+1)}, \quad \nu_2 = \frac{\pi^2}{\sigma(\sigma+2)}, \quad \nu_3 = \frac{\sigma+4}{\sigma+2} \tag{21}$$

(let us stress that here $\nu_3 > 2\nu_1$, so that the inequalities (15) hold). Substitution of the functions (20), (20') into (16) leads to an identity. Therefore, the averaging method gives us an absolutely exact description of the self-similar solution (18).

A clear understanding of the evolution of the thermal structure (the dependence of the amplitude on the half-width for different initial data) can be obtained from considering the behaviour of phase trajectories of equation (16), which are depicted schematically in Figure 45. The thick line denotes the trajectory

$$\phi \equiv \phi_S = \left( \frac{\nu_2}{\nu_3 - 2\nu_1} \right)^{1/2},$$

which corresponds to the self-similar solution of the S-regime (it is also the isocline of infinity of equation (16)); the dashed line denotes the nullcline $\phi = a^{-1/2} < \phi_S$.

That figure shows, in particular, that for $\beta = \sigma + 1$ all solutions become in fact infinite in finite time, and as the amplitude $\psi(t)$ grows, all the trajectories converge to the self-similar one: $\phi(t) \to \phi_S$, $t \to T_0$. Using this conclusion, we immediately obtain from (11) for $\beta = \sigma + 1$

$$\psi'(t) \simeq \psi^{\sigma+1}(t)\nu_1, \; t \to T_0.$$

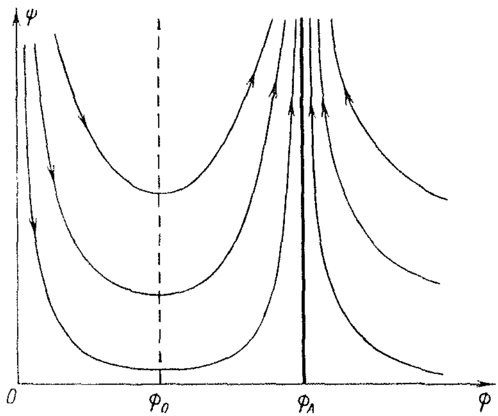


**Fig. 45.** Phase plane of equation (16) ($\beta = \sigma + 1$, S-regime)

that is,

$$\psi'(t) \simeq (\sigma\nu_1)^{-1/\sigma}(T_0 - t)^{-1/\sigma}, \quad t \to T_0^-,$$

which is the same as the dependence on time of the amplitude of the self-similar structure (see § 1). If for $N = 1$ we take the value of $\nu_1$ from (21), then we obtain precisely the expression (20) for the self-similar solution.

In conclusion, let us remark that the results of numerical computations agree well with the phase plane picture of Figure 45.

*2 HS-regime, $\beta < \sigma + 1$*

Let us first determine the general solution of equation (13) for $\beta \neq \sigma + 1$. Let us set $\psi^{\beta-(\sigma+1)}\phi^2 = z(\phi)$. Then from (13) we obtain a separable equation:

$$\frac{dz}{d\phi}\phi = z\left[2 - N(\beta - \sigma - 1)\frac{az - 1}{bz - 1}\right], \tag{22}$$

the general solution of which for $\beta \neq \sigma + 1 + 2/N$ has the form

$$(\psi\phi^N)^{\frac{\beta-(\sigma+1)}{N|\beta-(\sigma+1+2/N)|}}\left|\psi^{\beta-(\sigma+1)}\phi^2 - d\right|^{\alpha} = C_0, \tag{23}$$

where

$$\begin{aligned} d &= \frac{\nu_2}{\nu_1}\left[\frac{\nu_3}{\nu_1} - \frac{\beta - (\sigma + 1 + 4/N)}{\beta - (\sigma + 1 + 2/N)}\right]^{-1}, \\ \alpha &= \frac{\beta - (\sigma + 1)}{N|\beta - (\sigma + 1 + 2/N)|^2}\left[\frac{\nu_3}{\nu_1} - \frac{\beta - (\sigma + 1 + 4/N)}{\beta - (\sigma + 1 + 2/N)}\right]^{-1}. \end{aligned} \tag{23'}$$

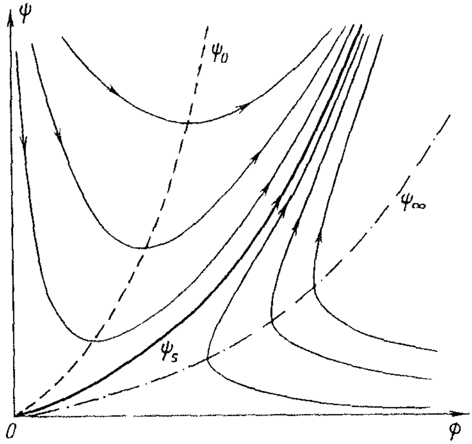


**Fig. 46.** Evolution of trajectories of equation (13) in the HS-regime ($\beta < \sigma + 1$)

Thus, let $\beta < \sigma + 1$. Using (23), it is not hard to draw the phase plane of equation (13). For convenience, we have shown in Figure 46 the nullcline

$$\psi = \psi_0(\phi) = a^{-1/[\beta-(\sigma+1)]}\phi^{-2/[\beta-(\sigma+1)]} \tag{24}$$

and the isocline of infinity

$$\psi = \psi_\infty(\phi) = b^{-1/[\beta-(\sigma+1)]}\phi^{-2/[\beta-(\sigma+1)]}. \tag{25}$$

The thick line is used to show the separatrix

$$\psi = \psi_S(\phi) = d^{1/[\beta-(\sigma+1)]}\phi^{-2/[\beta-(\sigma+1)]}, \tag{26}$$

which is an exact solution of the equation. In (23) to this trajectory there corresponds the value $C_0 = 0$. For $\beta < \sigma + 1$ we have the inequalities $\psi_\infty(\phi) < \psi_S(\phi) < \psi_0(\phi)$, which define the nature of the evolution of trajectories in Figure 46.

Thus, as can be seen from that figure, in the HS-regime all the trajectories converge as $\psi \to \infty$ to the separatrix $\psi = \psi_S(\phi)$, that is,

$$\psi(t) \simeq d^{1/[\beta-(\sigma+1)]}\phi^{-2/[\beta-(\sigma+1)]}(t),\quad t \to T_0^-. \tag{27}$$

Substituting this estimate into equations (11), (12), we deduce that at a fully developed stage of evolution

$$\psi(t) \sim (T_0 - t)^{-1/(\beta-1)},\quad \phi(t) \sim (T_0 - t)^{[\beta-(\sigma+1)]/[2(\beta-1)]}, \tag{28}$$

that is, as $t \to T_0^-$ unbounded solutions develop a spatio-temporal structure which is closed to the self-similar one.

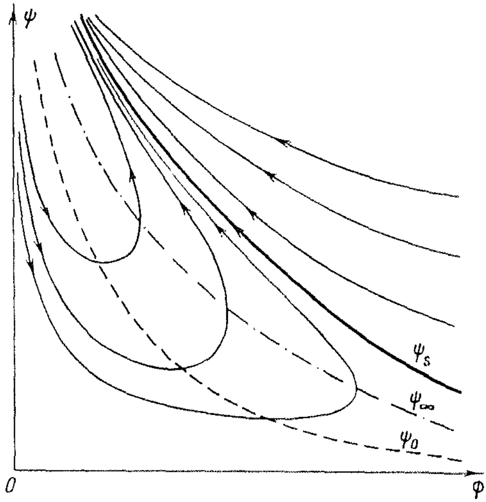


**Fig. 47.** Evolution of trajectories for $\beta \in (\sigma + 1, \sigma + 1 + 2/N)$ (LS-regime)

### 3 *LS-regime, $\beta > \sigma + 1$*

As can be seen from (23), for $\beta > \sigma + 1$ the phase planes are different in the cases $\sigma + 1 < \beta < \sigma + 1 + 2/N$, $\beta = \sigma + 1 + 2/N$, and $\beta > \sigma + 1 + 2/N$.

*The case $\sigma + 1 < \beta < \sigma + 1 + 2/N$: unbounded solutions.* Figure 47 shows the phase portrait for $\beta \in (\sigma + 1, \sigma + 1 + 2/N)$. Here the separatrix $\psi = \psi_S(\phi)$ (see (26)) lies above the isoclines (24), (25). With time, all trajectories converge to the separatrix, that is the asymptotic equality (27) holds, so that as $t \to T_0^-$ the estimates (28) are satisfied. Thus for $\beta < \sigma + 1 + 2/N$ all solutions of the problem are unbounded and as $t \to T_0^-$ their evolution follows that of the self-similar solution.

*The case $\beta = \sigma + 1 + 2/N$: unbounded solutions.* For $\beta = \sigma + 1 + 2/N$ equation (22) assumes the form

$$\frac{dz}{d\phi}\phi = 2(b - a)\frac{z^2}{bz - 1}.$$

Its general solution is determined from the algebraic relation

$$\psi^{\frac{1}{N}\left(2 - \frac{\nu_2}{\nu_1}\right)}\phi^{1-\frac{\nu_2}{\nu_1}} \exp\left\{-\frac{\nu_2}{2\nu_1}\phi^{-2}\psi^{-\frac{2}{N}}\right\} = C_0 > 0. \tag{29}$$

For $\beta = \sigma + 1 + 2/N$ the phase plane of equation (13) has no separatrix; this follows from (29). As $a > b$, $\psi_0(\phi) < \psi_\infty(\phi)$, and the trajectories behave more or less as in Figure 47, except that in this case there is no special trajectory $\psi_S$.

Thus, for $\beta = \sigma + 1 + 2/N$, as before, all the trajectories correspond to unbounded solutions of the problem. Moreover, as the amplitude $\psi(t)$ grows, the half-width $\phi(t)$ decreases.

Let us find out what is the asymptotic behaviour of $\psi$, $\phi$ as $t \to T_0^-$. From (29) or directly from equation (13) it is easy to deduce that as the solution grows without bound, the relation between $\psi$ and $\phi$ can be determined from the approximate equality

$$\frac{d\psi}{d\phi} \simeq -N\frac{\psi}{\phi}\frac{a}{b}, \tag{30}$$

that is,

$$\psi(\phi) \simeq B_0\phi^{-Na/b}, \quad \phi \to 0, \tag{30$'$}$$

where $B_0 > 0$ is a constant that depends on the initial conditions. The dependence (30′) is not a self-similar one, which corresponds to the asymptotic equality (27), since here

$$N\frac{a}{b} > \frac{2}{\beta - (\sigma + 1)} = N \tag{31}$$

(we remind the reader that $a > b$ by (14)).

The estimate (30′) gives us the following asymptotic expressions for the amplitude and the half-width of the thermal structure as $t \to T_0$:

$$\psi(t) \sim (T_0 - t)^{-1/(\beta-1)}, \quad \phi(t) \sim (T_0 - t)^{\alpha}, \tag{32}$$

where

$$\alpha = \left[\frac{Na}{b}(\beta - 1) - 1\right]^{-1} \equiv \left[\frac{a}{b}(N\sigma + 2) - 1\right]^{-1}.$$

Let us compare (32) with the self-similar expressions (28). The amplitude $\psi(t)$ is the self-similar one (there is a rigorous justification for that), while the half-width $\phi(t)$ behaves as $t \to T_0^-$ in a non-self-similar way, since in general $\alpha$ is different from the exponent $|\beta - (\sigma + 1)|/|2(\beta - 1)| \equiv 1/(N\sigma + 2)$, which appears in (28). Thus there arises the question: what invariant or approximate self-similar solution describes the asymptotic ($t \to T_0^-$) stage of the process?

Therefore for $\beta = \sigma + 1 + 2/N$ we can in principle expect unbounded solutions which evolve not according to self-similar laws at the asymptotic stage. Let us note that in (30′) and (32), apart from the non-self-similar exponent, we also have significant dependence on the initial conditions (through the constant $B_0$ in (30′), which differs from one trajectory to another); we recall that for $\beta < \sigma + 1 + 2/N$ all the trajectories converged to the self-similar separatrix (26) with a fixed constant $d$ determined from (23′).

At this point it has to be stressed that the averaging theory considers solutions with compact support, $u(t, \cdot) \in L^1(\mathbf{R}^N)$ for any $t < T_0$. It is well known (see § 1) that for $\beta > \sigma + 1$ self-similar solutions $u_S$ do not have compact support; however,

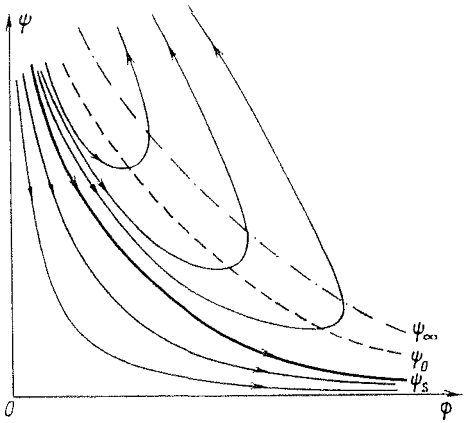


**Fig. 48.** LS-regime. $\beta > \sigma + 1 + 2/N$

if $\sigma+1 < \beta < \sigma+1+2/N$, the inclusion $u_S \in L^1(\mathbf{R}^N)$ still obtains, that is, $u_S$ have finite energy. This, apparently, can guarantee self-similar asymptotics of compactly supported solutions. For $\beta = \sigma + 1 + 2/N$ (and *a fortiori* for $\beta > \sigma + 1 + 2/N$) the energy of $u_S$ is infinite[4] and therefore $u_S$ does not necessarily describe the asymptotics of a solution with compact support and finite energy.

*The case $\beta > \sigma + 1 + 2/N$: unbounded solutions.* In this case there exists a separatrix (26), and in the phase plane it is placed so that $\psi_S(\phi) < \psi_0(\phi) < \psi_\infty(\phi)$. This determines the behaviour of trajectories in Figure 48. The asymptotics of unbounded solutions as $t \to T_0^-$ here is the same as in the case $\beta = \sigma + 1 + 2/N$, that is, it is non-self-similar (see (30′), (32)).

*The case $\beta > \sigma+1+2/N$: global solutions.* From Figure 48 it can be seen that for $\beta > \sigma + 1 + 2/N$ there are initial conditions to which there correspond global solutions (that do not blow up in finite time). The corresponding global trajectories lie below the separatrix $\psi = \psi_S(\phi)$. As $t \to \infty$, the amplitude of the global solutions goes to zero, while the half-width grows without bound (extinction). A rigorous construction of the family of global solutions is presented in § 3.

Let us determine the dynamics of this extinction process. First of all let us note that to the separatrix $\psi = \psi_S(\phi)$ there corresponds a thermal perturbation for which

$$\psi(t) \sim t^{-1/(\beta-1)}, \quad \phi(t) \sim t^{[\beta-(\sigma+1)]/[2(\beta-1)]}, \quad t \to \infty. \tag{33}$$

[4]This follows from the nature of asymptotics of $\theta_S(\xi)$ as $\xi \to \infty$ (see § 1): $\theta_S(\xi) \simeq C_S \xi^{-2/[\beta-(\sigma+1)]}$.

In § 3 we shall construct a family of global self-similar solutions of equation (1) with the spatio-temporal structure of (33), so that the separatrix $\psi = \psi_S(\phi)$ is the image of some self-similar solution.

What is the behaviour as $t \to \infty$ of the remaining global trajectories? From (23) it is not hard to deduce that for them

$$\psi(\phi) \simeq D_0\phi^{-N}, \ \phi \to \infty, \tag{34}$$

where the constant $D_0$, which depends on initial conditions, has the form

$$D_0 = |C_0 d^{-\sigma}|^{-\frac{N[\beta-(\sigma+1+2/N)]}{\beta-(\sigma+1)}}. \tag{35}$$

Substitution of (34) into the original system (11), (12) gives us the following asymptotics of global solutions:

$$\psi(t) \sim t^{-N/(N\sigma+2)}, \ \phi(t) \sim t^{1/(N\sigma+2)}, \ t \to \infty. \tag{36}$$

It is clear that the dependence in (34) corresponds to the energy conservation law $(d/dt)(\psi(t)\phi^N(t)) \simeq 0$ (see (8)), i.e., as $t \to \infty$ the self-similar solutions are close in a certain sense to the self-similar solutions of the nonlinear heat equation without a source term,

$$u_t = \nabla \cdot (u^\sigma \nabla u), \ t > 0, x \in \mathbf{R}^N \tag{37}$$

(for a proof of this fact see § 3). Another indication of this is given by the asymptotics (36) (a self-similar solution of equation (37), which satisfies (36), is given in § 3, Ch. I).

Therefore for $\beta > \sigma + 1 + 2/N$, the separatrix, to which there correspond self-similar solutions, is unstable in the class of both unbounded and of global solutions. Therefore it has to be expected that at the asymptotic stage the combustion process evolves according to different, non-self-similar rules, and, as shown by averaging, the form of the limiting thermal structure depends on initial data.

These are the qualitative properties of the evolution of thermal structures initiated by an elementary perturbation with finite energy. As shown by numerical experiments, this method of "amplitude-half-width" averaging affords us quite a precise description of the behaviour of unbounded solutions on a large time interval. The question of the nature of the front motion, and thus the question of combustion localization, remain unanswered. To that end, we present below another method of averaging.

## 2 Non-stationary averaging "amplitude-wave front position"

Let an elementary initial perturbation $u_0 \in L^1(\mathbf{R}^N)$ be radially symmetric and have compact support. Then $u = u(t, r)$, $r = |x|$, for all $t \in (0, T_0)$. We shall look for an approximate solution of the same form:

$$u(t, x) = \psi(t)\theta(\xi), \quad \xi = |x|/g(t), \tag{38}$$

where $\psi(t) > 0$ is the amplitude of the solution, and now $g(t) > 0$ is not the half-width, but the position of the front of the solution (in the symmetric case the front is the surface $|x| = g(t)$). The function $\theta(\xi) \geq 0$ is such that $\theta(\xi) > 0$ for $\xi \in [0, 1)$, $\theta(\xi) = 0$ for all $\xi \geq 1$, $\theta(0) = 1$, $\theta'(0) = 0$, $\theta^{\sigma+1} \in H^1((0, 1))$.

As the first equation for the functions $\psi$, $g$, we choose the energy equation

$$\frac{d}{dt}\left[\psi(t)g^N(t)\right] = \nu_1\psi^\beta(t)g^N(t), \quad t > 0. \tag{39}$$

The second equation is obtained from the well-known expression for the motion of the front of a thermal wave (see Remarks):

$$\frac{dg(t)}{dt} = -\lim_{r \to g(t)} \frac{\partial u(t, r)/\partial t}{\partial u(t, r)/\partial r}. \tag{40}$$

Hence, using the original equation (1), we obtain

$$\frac{dg(t)}{dt} = -\lim_{u \to 0} \frac{r^{-(N-1)}(r^{N-1}u^\sigma u_r')_r' + u^\beta}{u_r'},$$

and, finally, resolving the indeterminacy in the right-hand side, using the known differentiability properties of the solution $u(t, x)$, we arrive at the equality

$$\frac{dg}{dt} = -\lim_{u \to 0} u^{\sigma-1}u_r'. \tag{41}$$

In the derivation of this equality we assumed that the singularity of the solution close to the front has the algebraic form $u(t, x) \sim (g(t) - |x|)_+^{1/\sigma}$. Therefore the presence of the source term $u^\beta$ has no bearing on the final form of (41).

Substituting the approximate equality (38) into (41), we obtain the second equation:

$$\frac{dg(t)}{dt} = \nu_4\frac{\psi^\sigma(t)}{g(t)}, \quad t > 0, \tag{42}$$

where $\nu_4 = -(\theta^\sigma)'(1)/\sigma > 0$.

The required system of equations for $\psi$, $g$ has been obtained. Solving (39) for $\psi'(t)$, we rewrite it in the form

$$\psi' = \nu_1\psi^\beta - N\nu_4\psi^{\sigma+1}g^{-2}, \tag{43}$$

$$g' = \nu_4\psi^\sigma g^{-1}, \tag{44}$$

and then pass to the single equation

$$\frac{d\psi}{dg} = N\frac{\psi}{g}\left[\mu\psi^{\beta-(\sigma+1)}g^2 - 1\right], \quad g > 0, \tag{45}$$

which describes the evolution of the amplitude of the thermal structure as a function its front position. Here $\mu = \nu_1/(N\nu_4)$ is a constant.

Equation (45) is much simpler than the one obtained using the "amplitude-half-width" averaging. Let us write down its general solution and discuss its main properties. Let us note that (45) is not applicable for the S-regime, when $\beta = \sigma + 1$. From the analysis of self-similar solutions (see § 1) it is known that the asymptotics close to the front in this case must have the form $u \sim (g(t) - |x|)_+^{2/\sigma}$, that is, in this case we should put $(\theta^\sigma)'(1) = 0$, $\nu_4 = 0$. As a result, we obtain from (42) $g = \text{const}$, while for $\beta = \sigma + 1$ (43) becomes the equation

$$\psi'(t) = \nu_1 \psi^{\sigma+1}(t), \quad t > 0.$$

From that we derive the self-similar dependence of the amplitude of the localized unbounded solution on time:

$$\psi(t) \sim (T_0 - t)^{-1/\sigma}, \quad t \to T_0^-. \tag{46}$$

For the HS- ($\beta < \sigma + 1$) and LS- ($\beta > \sigma + 1$) regimes equation (45) makes sense. In the case $\beta \neq \sigma + 1 + 2/N$ its general solution has the form

$$|\psi^{\beta-(\sigma+1)} g^2 - l_0|^{1/(\sigma+1-\beta)} \psi g^N = C_0, \tag{47}$$

where

$$l_0 = \frac{1}{\mu} \frac{\beta - (\sigma + 1 + 2/N)}{\beta - (\sigma + 1)}, \tag{48}$$

$C_0 \geqslant 0$ is a constant determined by the initial values $g_0$, $\psi_0$. In the case $\beta = \sigma + 1 + 2/N$ we have, instead, the expression

$$\psi^{2/N} = |g^2(C_0 - 2\mu \ln g)|^{-1}, \quad C_0 = \text{const} > 0. \tag{49}$$

Schematic behaviour of trajectories of equation (45) in the cases $\beta < \sigma + 1$, $\sigma + 1 < \beta < \sigma + 1 + 2/N$, $\beta > \sigma + 1 + 2/N$ is shown in Figures 49-51, respectively. The dashed lines in all these figures denote the non-trivial nulleline:

$$\psi_0(\phi) = \mu^{-1/[\beta-(\sigma+1)]} g^{-2/[\beta-(\sigma+1)]}.$$

For $\beta < \sigma + 1$, $\beta > \sigma + 1 + 2/N$ equation (45) has a special trajectory, the separatrix

$$\psi = \psi_S(\phi) = l_0^{1/[\beta-(\sigma+1)]} g^{-2[\beta-(\sigma+1)]}, \tag{50}$$

which corresponds to $C_0 = 0$ in (47). For $\sigma + 1 < \beta \leq \sigma + 1 + 2/N$ we have from (48) $l_0 \leq 0$, and therefore there is no separatrix.

Let us indicate the main properties of the solutions. For $\beta < \sigma + 1$ (see Figure 49) all the trajectories evolve to the separatrix (50), which determines the asymptotic self-similar regime

$$\psi(t) \sim (T_0 - t)^{-1/(\beta-1)}, \quad g(t) \sim (T_0 - t)^{|\beta-(\sigma+1)|/[2(\beta-1)]}, \quad t \to T_0^-.$$

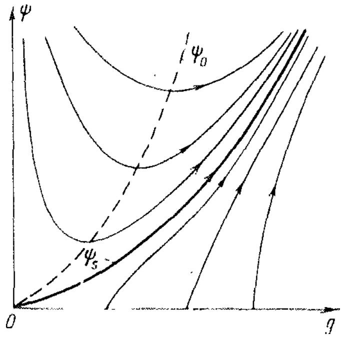


**Fig. 49.** Evolution of trajectories of equation (45) in the HS-regime ($\beta < \sigma + 1$)

and here $g(t) \to \infty$ as $t \to T_0$, that is, there is no heat localization in if $\beta < \sigma+1$.

For $\sigma + 1 < \beta < \sigma + 1 + 2/N$ (see Figure 50) each trajectory has its own vertical asymptote with coordinate

$$g_* = C_0^{\frac{1}{N}\frac{\beta-(\sigma+1)}{\beta-(\sigma+1+2/N)}},$$

that is, $g(t) \to g_*$ as $t \to T_0$. This implies heat localization in the domain $|x| < g_*$. In this case the amplitude grows according to the self-similar rule

$$\psi(t) \sim (T_0 - t)^{-1/(\beta-1)}, \quad t \to T_0. \tag{51}$$

The same conclusions are true also for $\beta = \sigma + 1 + 2/N$ and $\beta > \sigma + 1 + 2/N$, except that in the first case the expression for the wave penetration depth has the form $g_* = \exp\{C_0/(2\mu)\}$, which follows from (49).

In the case $\beta > \sigma + 1 + 2/N$ (see Figure 51) there exists the separatrix (50) in the phase plane. It separates unbounded trajectories from the family of global solutions. It follows from (47) that the latter evolve according to

$$\psi(\phi) \simeq F_0 g^{-N}, \quad F_0 = C_0 I_0^{1/(\sigma+1-\beta)} > 0, \quad g \to \infty.$$

From (43), (44) we then obtain the following asymptotic bounds for the global solutions:

$$\psi(t) \sim t^{-N/(N\sigma+2)}, \quad g(t) \sim t^{1/(N\sigma+2)}, \quad t \to \infty,$$

which are typical of self-similar solutions of equation (37) without a source term, which have finite energy: $\|u(t,\cdot)\|_{L^1(\mathbf{R}^N)} \equiv \text{const}$.

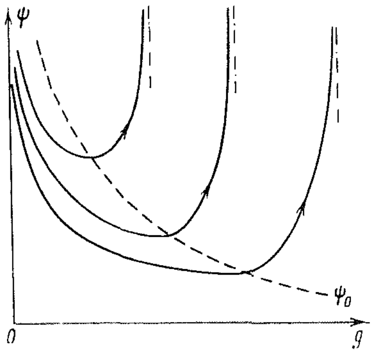


**Fig. 50.** LS-regime, $\beta \in (\sigma + 1, \sigma + 1 + 2/N)$

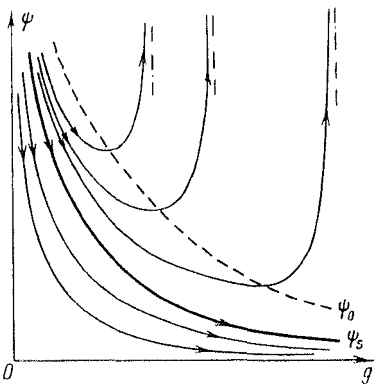


**Fig. 51.** LS-regime, $\beta > \sigma + 1 + 2/N$

## § 3 Conditions for finite time blow-up. Globally existing solutions for $\beta > \sigma + 1 + 2/N$

In this section we justify some of the qualitative derivations, obtained in § 2 and deal with conditions of global solvability and insolvability of the Cauchy problem for equations with power type nonlinearities,

$$\mathbf{A}(u) \equiv u_t - \nabla \cdot (u^\sigma \nabla u) - u^\beta = 0, \; t > 0, \; x \in \mathbf{R}^N, \tag{1}$$

$$u(0, x) = u_0(x) \geq 0, \; x \in \mathbf{R}^N; \; u_0 \in C(\mathbf{R}^N), \; u_0^{\sigma+1} \in H^1(\mathbf{R}^N), \tag{2}$$

The main approach is to construct and analyze suitable sub- and supersolutions of equation (1).

## 1 Construction of unbounded subsolutions

Let us consider the function[5]

$$u_-(t,x) = (T-t)^{-1/(\beta-1)}\theta_-(\xi), \ \xi = |x|/\zeta(t), \tag{3}$$

$$\theta_-(\xi) = A(1-\xi^2/a^2)_+^{1/\sigma}, \ 0 < t < T, \ \xi > 0, \tag{4}$$

where $\zeta(t) = (T-t)^{[\beta-(\sigma+1)]/[2(\beta-1)]}$, $A, a, T$ are positive constants. The function $u_-$ blows up as $t \to T^-$ in a self-similar fashion.

Let us find under what conditions this function will be an unbounded subsolution of equation (1). From Theorem 3, Ch. 1, it follows that for this it is sufficient for $u_-$ to satisfy everywhere in $(0,T)\times \mathbf{R}_+$, apart from on the degeneracy surface $(0,T)\times\{|x| = a\zeta(t)\}$, the inequality

$$\mathbf{A}(u_-) \equiv (u_-)_t - \frac{1}{r^{N-1}}\left(r^{N-1}(u_-)^\sigma(u_-)_r\right)_r - (u_-)^\beta \le 0,$$

which reduces after simplifications to

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}\theta_-^\sigma\theta_-')' - \frac{\beta-(\sigma+1)}{2(\beta-1)}\theta_-'\xi - \frac{1}{\beta-1}\theta_- + \theta_-^\beta \ge 0, \ \xi\in(0,a), \tag{5}$$

where $(\cdot)' = (d/d\xi)(\cdot)$. The left-hand side of the resulting inequality contains the operator of the self-similar equation (1.5), which is not surprising, since $u_-$ and $u_S$ have the same spatio-temporal structure. Substituting here the function $\theta_-(\xi)$ of (4), we obtain after relatively simple computations the inequality

$$\Phi_{\sigma\beta}(\Delta) \equiv m - n\Delta + A^{\beta-1}\Delta^{(\beta-1)/\sigma+1} \ge 0, \tag{6}$$

which is equivalent to (5). Here

$$\Delta = (1-\xi^2/a^2)_+,$$

$$m = \frac{4A^\sigma}{\sigma^2 a^2} + \frac{\beta-(\sigma+1)}{(\beta-1)\sigma}, \quad n = \frac{1}{\sigma}\left[1 + 2\left(\frac{A^\sigma}{a^2}\right)\left(N+\frac{2}{\sigma}\right)\right].$$

Inequality (6) must be satisfied for all $\Delta\in(0,1]$. Let us determine the restrictions on $A$, $a$.

First of all, we must have the inequality

$$\Phi_{\sigma\beta}(0) > 0,$$

from which we obtain the restriction $m > 0$, that is

$$\frac{4}{\sigma}\frac{A^\sigma}{a^2} > \frac{\sigma+1-\beta}{\beta-1}. \tag{7}$$

[5]Here, as usual, $(f)_+ = \max\{0, f\}$.

Secondly, it is easy to see that for $m > 0$ inequality (6) holds for all $\Delta \in (0, \Delta^*)$, where $\Delta^* = m/n \in (0, 1)$. Hence it follows that (6) will be satisfied for all $\Delta \in (0, 1]$ if

$$m - n\Delta + A^{\beta-1}\Delta_*^{(\beta+\sigma-1)/\sigma} \geq 0, \quad \Delta \in (\Delta^*, 1),$$

which is equivalent to the condition

$$A^{\beta-1} \geq (n-m)\Delta_*^{-(\beta+\sigma-1)/\sigma}.$$

Therefore the second inequality, which, together with (7), guarantees that (5) holds, has the form

$$A^{\beta-1} \geq \left(\frac{1}{\beta-1} + \frac{2N}{\sigma}\frac{A^\sigma}{a^2}\right)\left\{\frac{1 + 2\left(N + \dfrac{2}{\sigma}\right)\dfrac{A^\sigma}{a^2}}{\dfrac{\beta-(\sigma+1)}{\beta-1} + \dfrac{4}{\sigma}\dfrac{A^\sigma}{a^2}}\right\}^{(\beta+\sigma-1)/\sigma}. \tag{8}$$

The system of inequalities (7), (8) has a solution $(a, A)$ for all $\sigma > 0$, $\beta > 1$. Indeed, for $\beta < \sigma + 1$ condition (7) imposes a restriction on the ratio $A^\sigma/a^2$. Then, by increasing $A$ and $a$ in such a way that the ratio $A^\sigma/a^2$ does not decrease, we can always cause (8) to be satisfied. If $\beta \geq \sigma + 1$, everything is much easier, since then (7) is not taken into account. Therefore we have established the following assertion.

**Theorem 1.** *Let*

$$u_0(x) \geq u_-(0, x) = T^{-1/(\beta-1)}\theta_-\left(\frac{|x|}{T^{[\beta-(\sigma+1)]/[2(\beta-1)]}}\right), \quad x \in \mathbf{R}^N, \tag{9}$$

*where* $\theta_-(\xi) = A(1 - \xi^2/a^2)_+^{1/\sigma}$ *and* $T, a, A$ *are positive constants, the two last ones being related by (7), (8). Then the solution of the Cauchy problem (1), (2) is unbounded and exists at most for time $T$.*

An elementary analysis of the subsolution (3) for $\beta < \sigma + 1$ leads to the following result.

**Corollary.** *Let* $1 < \beta < \sigma + 1$, $u_0(x) \not\equiv 0$. *Then the solution of the problem is unbounded.*

*Proof.* Since $u_0 \not\equiv 0$, there exists a ball $\{x \in \mathbf{R}^N \mid |x - x_0| < \rho\}$, $\rho > 0$, in which $u_0(x) \geq \epsilon > 0$. Then, choosing in (3), (4), $T$ so large that the inequalities $AT^{-1/(\beta-1)} < \epsilon$, $aT^{[\beta-(\sigma+1)]/[2(\beta-1)]} < \rho$ hold, we have that $u_0(x) \geq u_-(0, x - x_0)$. Therefore by Theorem 1 the solution is unbounded and blows up at $T_0 \leq T_*$, where

$$T_* = \max\left\{(A/\epsilon)^{\beta-1}, (a/\rho)^{2(\beta-1)/(\sigma+1-\beta)}\right\}.$$

□

A stronger result will be obtained below.

## 2 Non-existence of global solutions for $1 < \beta < \sigma + 1 + 2/N$

**Theorem 2.** *Let $\beta \in (1, \sigma + 1 + 2/N)$, $u_0(x) \not\equiv 0$. Then the solution of the Cauchy problem* (1), (2) *is unbounded.*

*Proof.* It proceeds by comparing the solution $u(t, x)$ with a known self-similar solution of the Cauchy problem for the equation without a source,

$$v_t = \nabla \cdot (v^\sigma \nabla v), \quad t > 0, \quad x \in \mathbf{R}^N. \tag{10}$$

In the $N$-dimensional case this solution has the form (see § 3, Ch. I)

$$v_S(t, x) = (T_1 + t)^{-N/(N\sigma+2)} f(\eta), \quad \eta = \frac{|x|}{(T_1 + t)^{1/(N\sigma+2)}}, \tag{11}$$

where

$$f(\eta) = B(\eta_0^2 - \eta^2)_+^{1/\sigma}, B = \left[\frac{\sigma}{2(N\sigma + 2)}\right]^{1/\sigma}. \tag{12}$$

Here $T_1$, $\eta_0$ are arbitrary positive constants. Let us show that in the case $\beta < \sigma + 1 + 2/N$ for any $u_0 \not\equiv 0$, after a finite time the solution $u(t, x)$ of problem (1), (2) would have to satisfy condition (9) of Theorem 1, and thus is unbounded. The stage at which the initial perturbation spreads, when the amplitude of the spatial profile decreases, will be described using the solution (11) of equation (10), in which production of energy due to combustion is not taken into account (for most of the spreading stage it is negligible).

Without loss of generality, let $u_0(0) > 0$ and $u_0(x) \geq \epsilon > 0$ in a ball $\{x \in \mathbf{R}^N \mid |x| < \delta\}$. Let us choose the number $\eta_0 = \eta_0(T_1)$, such that $u_0(x) \geq v_S(0, x)$ in $\mathbf{R}^N$. For this it is sufficient that

$$B\eta_0^{2/\sigma} T_1^{-N/(N\sigma+2)} \leq \epsilon, \quad \eta_0 T_1^{1/(N\sigma+2)} \leq \delta \tag{13}$$

(here $T_1$ can be arbitrary).

Then by the comparison theorem (see Ch. I)

$$u(t, x) \geq v_S(t, x), \quad t > 0, \quad x \in \mathbf{R}^N. \tag{14}$$

Let us show that for $1 < \beta < \sigma + 1 + 2/N$ there exists $t_1$, such that for some $T_1$ the function $v_S(t_1, x)$ satisfies condition (9). Then by (14) this condition will also hold for the solution $u(t_1, x)$. The inequality $v_S(t_1, x) \geq u(0, x)$ in $\mathbf{R}^N$ will hold if

$$(T_1 + t_1)^{-N/(N\sigma+2)} B\eta_0^{2/\sigma} \geq T^{-1/(\beta-1)} A, \tag{15}$$

$$\eta_0 (T_1 + t_1)^{1/(N\sigma+2)} \geq a T^{[\beta-(\sigma+1)]/[2(\beta-1)]} \tag{16}$$

(here $A, a$ is an arbitrary solution of the inequalities (7), (8)).

Let us show that the system (15), (16) is always solvable with respect to $t_1$, $T$ if $\beta < \sigma + 1 + 2/N$.

Suppose that equality is attained in (15), that is

$$T_1 + t_1 = (B\eta_0^{2/\sigma}/A)^{(N\sigma+2)/N} T^{(N\sigma+2)/[N(\beta-1)]}. \tag{17}$$

Here $T_1$ is fixed and $T$ is sufficiently large. It remains to check, whether for sufficiently large $T$ inequality (16) is satisfied. It then has the form

$$\eta_0 (B\eta_0^{2/\sigma}/A)^{1/N} T^{1/[N(\beta-1)]} \geq a T^{[\beta-(\sigma+1)]/[2(\beta-1)]},$$

or, which amounts to the same,

$$T^{\frac{1}{2(\beta-1)}[\beta-(\sigma+1+2/N)]} \leq \frac{1}{a}\left(\frac{B}{A}\right)^{1/N} \eta_0^{2/N\sigma+1}. \tag{18}$$

It is clear that in the case $\beta \in (1, \sigma+1+2/N)$ it holds for large $T$, which concludes the proof. □

**Remark.** In the course of the proof we have in fact showed that for $1 < \beta < \sigma + 1 + 2/N$ the blow-up time is composed of two parts: $T_0 \leq t_1 + T$, where $t_1$ is the time of spreading out of the initial perturbation practically with almost no energy production, $T$ is the time of rapid growth of the resonant solution towards finite time blow-up, which was determined in Theorem 1.

For $\beta > \sigma + 1 + 2/N$ the spreading out stage can take infinitely long time, so that non-trivial global solutions are possible. Justification of this conclusion is the subject matter of subsection 3.

Let us use inequalities (13), (18) to analyze the case of the "critical" value $\beta = \sigma + 1 + 2/N$.

**Corollary.** *Let $\beta = \sigma + 1 + 2/N$ and let the initial function be such that $u_0(x) \geq \epsilon$ in $\{|x| < \delta\}$, $\epsilon > 0$, $\delta > 0$, where*

$$\epsilon\delta^N \geq A a^N, \tag{19}$$

*where $a$, $A$ satisfy the inequality (8). Then the solution of problem (1), (2) is unbounded.*

Since the product $\epsilon\delta^N$ characterizes the amount of energy of the initial perturbation, condition (19) means that for $\beta = \sigma + 1 + 2/N$ unbounded solutions are all solutions with sufficiently large energy. Let us recall that the qualitative results of § 2 indicate that in this case all non-trivial solutions are unbounded. This can be proved; see Remarks. This is also attested to by the analogy with the results of § 7 concerning semilinear ($\sigma = 0$) equations.

To conclude this subsection, let us note that in the case $\beta > \sigma+1+2/N$, analysis of inequalities (15), (16) allows us to enlarge substantially the *unstable set* $\mathcal{V}$ of

Theorem 1 (if $u_0 \in \mathcal{V}$, then $u(t, x)$ is unbounded). The set $\mathcal{V}$ contains not only resonant initial perturbations, solutions through which start growing immediately and blow up in finite time (these are displayed in Theorem 1), but also $u_0(x)$, to which there correspond unbounded solutions with initially decreasing amplitude.

## 3 Conditions of global solvability of the Cauchy problem for $\beta > \sigma + 1 + 2/N$

This will be obtained by constructing bounded supersolutions $u_+$, which, as in subsection 2, are sought in the self-similar form

$$u_+(t, x) = (T + t)^{-1/(\beta-1)}\theta_+(\xi), \quad \xi = |x|/(T + t)^{[\beta-(\sigma+1)]/[2(\beta-1)]}, \tag{20}$$

where $\theta_+(\xi) = A(1 - \xi^2/a^2)_+^{1/\sigma}$; $A, T, a > 0$ are constants.

The choice of the function (20) is suggested by the form of the global self-similar solution of equation (1), which is considered in subsection 4. Taking into consideration the fact that $u_+(t, x)$ has a continuous derivative $\nabla u_+^{\sigma+1}$, we conclude that (20) will be a supersolution if $\mathbf{A}(u_+) \geq 0$ in $\mathbf{R}_+ \times \mathbf{R}^N \backslash \{\xi = a\}$, which gives us the inequality

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}\theta_+^\sigma\theta_+')' + \frac{\beta - (\sigma + 1)}{2(\beta - 1)}\theta_+'\xi + \frac{1}{\beta - 1}\theta_+ + \theta_+^\beta \leq 0, \quad \xi \neq a \tag{21}$$

(compare with (5)), which is equivalent to the inequality

$$F_{\sigma\beta}(\Delta) = m_* + n_*\Delta + A^{\beta-1}\Delta^{(\beta+\sigma-1)/\sigma} \leq 0, \quad \Delta \in (0, 1], \tag{22}$$

where

$$\Delta = (1 - \xi^2/a^2)_+,$$

$$m_* = \frac{4A^\sigma}{\sigma^2 a^2} - \frac{\beta - (\sigma + 1)}{\sigma(\beta - 1)}, \quad n_* = \frac{1}{\sigma}\left[1 - \frac{2A^\sigma}{a^2}\left(N + \frac{2}{\sigma}\right)\right].$$

Since the function $F_{\sigma\beta}$ is convex ($F''_{\sigma\beta} \geq 0$), to satisfy (22) it is sufficient to have $F_{\sigma\beta}(0) \leq 0$ and $F_{\sigma\beta}(1) \leq 0$. From that we obtain the required restrictions on the numbers $A, a$: $m_* \leq 0$, $m_* + n_* + A^{\beta-1} \leq 0$, or

$$\frac{4}{\sigma^2}\frac{A^\sigma}{a^2} \leq \frac{\beta - (\sigma + 1)}{\sigma(\beta - 1)}, \tag{23}$$

$$A^{\beta-1} \leq \frac{2N}{\sigma}\frac{A^\sigma}{a^2} - \frac{1}{\beta - 1}. \tag{24}$$

Let us show that the system of inequalities (23), (24) has a solution only in the case $\beta > \sigma + 1 + 2/N$ (for $\beta < \sigma + 1 + 2/N$ by Theorem 2 there cannot be any solutions). From (24) it follows the necessity of the restriction $2NA^{\sigma}/(\sigma a^2) > 1/(\beta - 1)$, which, combined with (23), gives us

$$\frac{2}{N\sigma(\beta-1)} < \frac{4}{\sigma^2}\frac{A^{\sigma}}{a^2} \leq \frac{\beta-(\sigma+1)}{\sigma(\beta-1)}. \tag{25}$$

The quantity $A^{\sigma}/a^2$, which satisfies (25), exists if $2/|N\sigma(\beta-1)| < |\beta - (\sigma + 1)|/|\sigma(\beta-1)|$, which is equivalent to the inequality $[\beta - (\sigma + 1 + 2/N)]/[\sigma(\beta - 1)] > 0$. Hence arises the restriction $\beta > \sigma + 1 + 2/N$. Then, varying the numbers $A$ and $a$ so that the ratio $A^{\sigma}/a^2$ remains constant and within the bounds of (25), we can ensure that (24) always holds by decreasing $A$. Thus we have proved

**Theorem 3.** *Let $\beta > \sigma + 1 + 2/N$, and let the function $u_0(x)$ satisfy for some $T > 0$ the inequality*

$$u_0(x) \leq u_+(0,x) = T^{-1/(\beta-1)}\theta_+\left(\frac{|x|}{T^{[\beta-(\sigma+1)]/[2(\beta-1)]}}\right), \quad x \in \mathbf{R}^N, \tag{26}$$

*where $\theta_+(\xi) = A(1 - \xi^2/a^2)_+^{1/\sigma}$ and the constants $A, a > 0$ satisfy inequalities (23), (24). Then the Cauchy problem (1), (2) has a global solution and*

$$u(t,x) \leq (T+t)^{-1/(\beta-1)}\theta_+\left(\frac{|x|}{(T+t)^{[\beta-(\sigma+1)]/[2(\beta-1)]}}\right) \text{ in } \mathbf{R}_+ \times \mathbf{R}^N. \tag{27}$$

**Remark 1.** From this follows, in particular, the estimate

$$\sup_x u(t,x) \leq A(T+t)^{-1/(\beta-1)}, \quad t > 0.$$

Furthermore, using (27) we can estimate the diameter $d(t)$ of the support of a generalized solution: $d(t) \leq 2a(T+t)^{[\beta-(\sigma+1)]/[2(\beta-1)]}$. Naturally, these estimates coincide with the self-similar ones.

Combining the results of Theorems 1, 3, we arrive at the following statement: *for $\beta > \sigma + 1 + 2/N$ for all large initial functions the problem (1), (2) is globally insolvable, while for sufficiently small $u_0$ there exists a global solution.*

**Remark 2.** For $\beta > \sigma + 1 + 2/N$, we distinguished in Theorem 3 the stable set $\mathcal{W}$ of the Cauchy problem (1), (2), such that the inclusion $u_0 \in \mathcal{W}$ entails global solvability of the problem. The set $\mathcal{W} = \{u_0 \geq 0 \mid \exists T > 0 : u_0(x) \leq u_+(0,x) \text{ in } \mathbf{R}^N\}$ contains only functions with compact support, and its "boundary" consists of a one-parameter family (with parameter $T > 0$) of also compactly supported functions. This does not mean that only compactly supported solutions can be global. In subsection 4 we construct a non-compactly supported stable set in the case $\beta > \sigma + 1 + 2/N$, the boundary of which consists of non-compactly supported global self-similar solutions of equation (1).

## 4 Global self-similar solutions for $\beta > \sigma + 1 + 2/N$. A lemma concerning stationary solutions

This subsection is wholly devoted to the study of one particular case of global solutions of equation (1) of the form

$$u_S = u_S(t, x; T) = (T + t)^{-1/(\beta-1)} f_S(\xi), \tag{28}$$

$$\xi = x/(T + t)^m, \; m = [\beta - (\sigma + 1)]/[2(\beta - 1)],$$

where $T > 0$ is an arbitrary constant.

After substitution of (28) into (1), we obtain for $f_S \geq 0$ the following elliptic equation:

$$\nabla_\xi \cdot (f_S^\sigma \nabla_\xi f_S) + m \nabla_\xi f_S \cdot \xi + \frac{1}{\beta - 1} f_S + f_S^\beta = 0, \; \xi \in \mathbf{R}^N, \tag{29}$$

$$f_S(\xi) \to 0, \; |\xi| \to \infty,$$

which differs from the equation which corresponds to self-similar blow-up regimes only in the signs of the second and the third terms. This, however, significantly alters the properties of the solution $f_S$ as compared with $\theta_S$ of § 1.

At this stage we confine ourselves to the study of radially symmetric solutions

$$f_S = f_S(\xi) \geq 0, \; \xi = |x|/(T + t)^m \in \mathbf{R}_+, \tag{30}$$

which, as follows from (29), satisfy the equation

$$\frac{1}{\xi^{N-1}} (\xi^{N-1} f_S^\sigma f_S')' + m f_S' \xi + \frac{1}{\beta - 1} f_S + f_S^\beta = 0, \; \xi > 0, \tag{31}$$

and the boundary conditions

$$f_S'(0) = 0, \; f_S(\infty) = 0 \quad (f_S(0) > 0). \tag{32}$$

The generalized solution $f_S$ must have continuous heat flux, that is, if $f_S$ is a function with compact support, then $\left(f_S^{\sigma+1}\right)'(\xi_0) = 0$ at the point of degeneracy $\xi_0 = \text{meas supp} f_S$.

Let us consider a family of Cauchy problems for the same equation:

$$\frac{1}{\xi^{N-1}} \left(\xi^{N-1} f^\sigma f'\right)' + m f' \xi + \frac{1}{\beta - 1} f + f^\beta = 0, \; \xi > 0, \tag{33}$$

$$f'(0) = 0, \; f(0) = \mu > 0, \tag{34}$$

and choose $\mu$ so that $f = f(\xi; \mu)$ satisfies (32). Before stating the main theorem, let us note two properties of the solution $f_S$ which follow directly from the form of equation (31).

First of all, $f_S(\xi)$ is monotone, since (31) does not admit points of minimum $\xi = \xi_m$, such that $f_S(\xi_m) > 0$, $f'_S(\xi_m) = 0$, $f''_S(\xi_m) \geq 0$. Therefore for all $\mu > 0$ any monotone classical solution of the problem (33), (34), defined for small $\xi \geq 0$, can be extended either onto the whole axis $\xi \in \mathbf{R}_+$ (then it is the required solution $f_S$), or till it becomes zero. Local solvability of (33), (34) is demonstrated by analyzing the equivalent integral equation using the Banach contraction mapping theorem.

Secondly, analysis of (31) for small $f_S$ reveals the possible forms of asymptotic behaviour of the solution as $f_S \to 0$. First is the asymptotics of a non-compactly supported solution:

$$f_S(\xi) = C\xi^{-2/[\beta-(\sigma+1)]}(1+\epsilon(\xi));\ \epsilon(\xi) \to 0,\ \xi \to \infty, \tag{35}$$

where $C > 0$ is a constant. Second is the asymptotics of a solution with compact support

$$f_S(\xi) = \left[\frac{\beta-(\sigma+1)}{2(\beta-1)}\sigma\xi_0(\xi_0-\xi)\right]^{1/\sigma}(1+\omega(\xi)), \tag{36}$$
$$\xi \to \xi_0^- = \text{meas supp}\, f_S < \infty,$$

where $\omega(\xi) \to 0$ as $\xi \to \xi_0^-$. The asymptotics (35), (36) make sense for $\beta > \sigma+1$. Let us note that a solution with compact support (36) formally corresponds to the value $C = 0$ in (35), that is, (35) becomes (36) for $C = 0$.

Properties of various solutions of the problem (31), (32) depend on the relation among the parameters $\beta$, $\sigma$ and the dimension of the space $N$.

**Theorem 4.** *Let $\beta > 1$, $\sigma > 0$. Then:*

(a) *if $\beta < \sigma + 1 + 2/N$, then the problem (31), (32) has no positive solutions (i.e., for any $\mu > 0$ the function $f(\xi;\mu)$ becomes zero at some point $\xi = \xi_*$ and $(f^{\sigma+1})'_\xi(\xi_*;\mu) \neq 0$);*

(b) *for all $\beta > \sigma + 1 + 2/N$, if $N = 1, 2$, or for $\sigma + 1 + 2/N < \beta < (\sigma+1)(N+2)/(N-2)$, if $N \geq 3$, the problem (31), (32) has at least one solution $f_S$ with compact support and an infinite number of strictly positive solutions;*

(c) *if $\beta \geq (\sigma + 1)(N + 2)/(N - 2)$, $N \geq 3$, then the problem (31), (32) has no solutions with compact support. For any $\mu > 0$ the solution of the Cauchy problem (33), (34) is strictly positive and satisfies condition (32) at infinity.*

By analogy with results obtained for tbe case $\sigma = 0$ (see § 7), we can expect tbat for $\beta = \sigma + 1 + 2/N$ all non-trivial solutions of the problem (1), (2) are unbounded, and therefore a function $f_S \geq 0$ does not exist in this case. Tbis is true for $\sigma > 0$, see Remarks.

*Proof.* Assertion (a) follows immediately from Theorem 2 concerning non-existence of non-trivial global solutions of the problem (1), (2) for $1 < \beta <$

$\sigma + 1 + 2/N$. Indeed, there were the function $f_S \geq 0$ to exist, (28) would have been the global solution $u_S \not\equiv 0$, which, as we showed above, does not exist.

Let us note a peculiarity of this argument. Here, in order to study an ordinary differential equation, we use results from an analysis of much more complex partial differential equations. Advantages of this approach in this case are not significant, since (a) admits another, simple proof. However, in the sequel (in the proof of (c)) this approach leads to a noticeable simplification.

The same result can be obtained by a different method. By (35) for $\beta < \sigma + 1 + 2/N$ equation (31) can be integrated over $(0, \infty)$ with the weight function $\xi^{N-1}$. As a result, after integration by parts we obtain the equality

$$\int_0^\infty f_S^\beta(\eta)\eta^{N-1}\,d\eta = -\frac{N}{2(\beta - 1)}\left(\sigma + 1 + \frac{2}{N} - \beta\right)\int_0^\infty f_S(\eta)\eta^{N-1}\,d\eta, \quad (37)$$

which for $\beta < \sigma + 1 + 2/N$ cannot be satisfied, as in the right hand-side we have a negative quantity (to derive (37) we need an estimate of $f_S'(\xi)$ as $\xi \to \infty$, which is easily obtained from the equation).

(b), (c). Proofs of these assertions are based on the properties of a family of stationary solutions of the original equation (1), which are established below.

*1 A lemma concerning stationary solutions*

Let us consider the stationary equation

$$\nabla \cdot (U^\sigma \nabla U) + U^\beta = 0 \quad (38)$$

for arbitrary values of parameters $\sigma \geq 0$, $\beta > 0$. For our purposes it suffices to analyze the family $\{U \geq 0\}$ of radially symmetric solutions, which satisfy the equation

$$\frac{1}{r^{N-1}}(r^{N-1}U^\sigma U')' + U^\beta = 0, \quad r = |x| > 0. \quad (38')$$

Let us set $U^{\sigma+1} = V$ and make the change of variable $r \to r(\sigma + 1)^{1/2}$. Let us denote by $V_\lambda$ the solution of the following problem:

$$\frac{1}{r^{N-1}}(r^{N-1}V_\lambda')' + V_\lambda^\alpha = 0, \quad r = |x| > 0, \quad (39)$$

$$V_\lambda(0) = \lambda, \quad V_\lambda'(0) = 0, \quad (40)$$

where $\lambda > 0$ is a constant (which parametrizes the family $\{V_\lambda\}$), $\alpha = \beta/(\sigma + 1) > 0$.

**Lemma 1.** *Let $\alpha > 0$. Then:*

1) *For any $\alpha > 0$, if $N = 1, 2$, or $0 < \alpha < (N + 2)/(N - 2)$, if $N \geq 3$, the problem (39), (40) has no solutions $V_\lambda \geq 0$ in $\mathbf{R}_+$ (that is, for any $\lambda > 0$ the function $V_\lambda$ becomes zero at some finite point, where $V_\lambda' \neq 0$);*

2) *if* $\alpha \geq (N+2)/(N-2)$ *for* $N \geq 3$, *then for any* $\lambda > 0$ *the solution of the problem is defined and strictly positive in* $\mathbf{R}_+$, $V_\lambda(r) \to 0$ *as* $r \to \infty$.

*Proof.* Local solvability of the problem is proved by reducing (39), (40) to an equivalent integral equation.

1) Let us first consider the case $N \leq 2$. From (39) we have

$$r^{N-1}V'_\lambda(r) = -\int_0^r \eta^{N-1}V_\lambda^\alpha(\eta)\,d\eta, \tag{41}$$

and therefore (if we assume that $V_\lambda > 0$ everywhere), for each $r \geq 1$

$$V'_\lambda(r) \leq -c_1 r^{1-N}, \quad c_1 = \int_0^1 \eta^{N-1}V_\lambda^\alpha(\eta)\,d\eta > 0.$$

Hence

$$V_\lambda(r) \leq V_\lambda(1) - c_1 \int_1^r \eta^{1-N}\,d\eta, \quad r > 1,$$

that is,

$$\begin{aligned} V_\lambda(r) &\leq V_\lambda(1) - c_1(r-1) \quad (N=1), \\ V_\lambda(r) &\leq V_\lambda(1) - c_1 \ln r \quad (N=2);\ r > 1. \end{aligned}$$

This means that for $N \leq 2$ $V_\lambda$ becomes zero at a point; furthermore, by (41) at that point $V'_\lambda(r) \neq 0$ (that is, the heat flux cannot be continuous).

Now let $N \geq 3$. Then by monotonicity of $V_\lambda$ we have from (41)

$$r^{N-1}V'_\lambda(r) < -V_\lambda^\alpha(r)\int_0^r \eta^{N-1}\,d\eta = -V_\lambda^\alpha(r)\frac{r^N}{N}. \tag{42}$$

Integrating this inequality we derive the following estimate; for $\alpha < 1$

$$V_\lambda(r) < \left[\lambda^{1-\alpha} - \frac{r^2}{2N}(1-\alpha)\right]^{1/(1-\alpha)},$$

and therefore $V_\lambda(r)$ is defined on an interval of length not exceeding $r_\lambda = [2N/(1-\alpha)]^{1/2}\lambda^{(1-\alpha)/2}$.

Let us note that for $\alpha < 1$ the function $V_\lambda$ depends in a monotone way on the boundary value $\lambda = V_\lambda(0)$ (Figure 52).

For $\alpha = 1$ the function $V_\lambda$ is positive on the interval $(0, z_N^{(1)})$, where $z_N^{(1)} > 0$ is the first root of the Bessel function $J_{(N-2)/2}$.

If, on the other hand, $\alpha > 1$, then from (42) follows the "non-compact" estimate

$$V_\lambda(r) < \left[\lambda^{1-\alpha} + \frac{r^2}{2N}(\alpha-1)\right]^{-1/(\alpha-1)}, \quad r > 0, \tag{43}$$

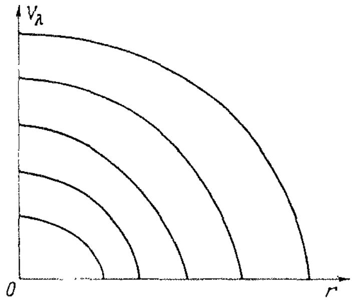


**Fig. 52.** Solutions of the problem (39), (40) for different $\lambda > 0$ in the case $\alpha < 1$.

from which we cannot draw a conclusion concerning extension of $V_\lambda(r) > 0$ into the domain of large $r$. From here it follows that

$$V_\lambda(r) < \left(\frac{2N}{\alpha - 1}\right)^{1/(\alpha-1)} r^{-2/(\alpha-1)}, \quad r > 0. \tag{43'}$$

Let us assume that 1) does not hold, and that $V_\lambda > 0$ in $\mathbf{R}_+$ for some $\alpha < (N+2)/(N-2)$, $N \geq 3$. Then, using the identities we derive below, we shall arrive at a contradiction. To derive the first of these, we multiply (39) by $r^{N-1}V_\lambda$ and integrate the resulting equality over the interval $(0, r)$. As a result we have

$$-r^{N-1}V'_\lambda(r)V_\lambda(r) + \int_0^r \eta^{N-1}V'^2_\lambda(\eta)\,d\eta = \int_0^r \eta^{N-1}V_\lambda^{\alpha+1}(\eta)\,d\eta, \quad r > 0.$$

Since $V'_\lambda(r) < 0$ and (by assumption) $V_\lambda > 0$, we have from here

$$\int_0^\infty \eta^{N-1}V'^2_\lambda(\eta)d\eta \leq \int_0^\infty \eta^{N-1}V_\lambda^{\alpha+1}(\eta)d\eta < \infty, \tag{44}$$

where convergence of the integral in the right-hand side is ensured by the estimate (43).

Now let us multiply (39) by $r^N V'_\lambda(r)$ and integrate over $(0, r)$. This results in a different equality:

$$\begin{aligned}&\frac{r^N}{2}V'^2_\lambda(r) + \frac{r^N}{\alpha+1}V_\lambda^{\alpha+1}(r) = \\ &= \frac{N}{\alpha+1}\int_0^r \eta^{N-1}V_\lambda^{\alpha+1}(\eta)\,d\eta - \frac{N-2}{2}\int_0^r \eta^{N-1}V'^2_\lambda(\eta)\,d\eta, \quad r > 0.\end{aligned} \tag{45}$$

It is not hard to see that $p(r) \equiv r^N V'^2_\lambda(r)/2 + r^N V_\lambda^{\alpha+1}(r)/(\alpha+1) \to 0$ as $r \to \infty$.

Indeed, from (45), since the integrals converge, it follows that $p(r) \to p_0$ as $r \to \infty$, while from convergence of the integral $\int_0^\infty |p(\eta)/\eta|\,d\eta$, which follows from (44), we deduce that $p_0 = 0$. Passing in (45) to the limit as $r \to \infty$, we have

$$\frac{N-2}{2}\int_0^\infty \eta^{N-1} V_\lambda'^2\, d\eta = \frac{N}{\alpha+1}\int_0^\infty \eta^{N-1} V_\lambda^{\alpha+1}\, d\eta,$$

which, combined with (44), gives us the inequality $(N-2)/2 \geq N/(\alpha+1)$, i.e., $\alpha \geq (N+2)/(N-2)_+$, which leads to a contradiction.

2) Assume the opposite, that is, that there exists $\alpha \geq (N+2)/(N-2)_+$ (the case of the equality will be considered separately), such that for some $\lambda > 0$ the solution $V_\lambda$ vanishes at some point $r_\lambda > 0$. Then $V_\lambda \not\equiv 0$ is a solution of the boundary value problem (39) on the interval $(0, r_\lambda)$, satisfying the boundary conditions

$$V_\lambda'(0) = 0, V_\lambda(r_\lambda) = 0. \tag{46}$$

However, as we shall show, the problem (39), (46) has no solutions. For that, as in the proof of 1), we first take the scalar product of (39) with $N\kappa_N r^{N-1} V_\lambda(r)$ and then with $N\kappa_N r^N V_\lambda'(r)$, where $\kappa_N$ is the volume of the unit sphere in $\mathbf{R}^N$. As a result, after integrating by parts and taking into account the conditions (46), we obtain

$$\|V_\lambda'\|_{L^2}^2 = \|V_\lambda\|_{L^{\alpha+1}}^{\alpha+1} \quad (L^p = L^p(\{|x| < r_\lambda\})), \tag{47}$$

$$\frac{N\kappa_N}{2} r_\lambda^N V_\lambda'^2(r_\lambda) + \frac{N-2}{2}\|V_\lambda'\|_{L^2}^2 - \frac{N}{\alpha+1}\|V_\lambda\|_{L^{\alpha+1}}^{\alpha+1} = 0. \tag{48}$$

Substituting into (48) the expression for $\|V_\lambda'\|_{L^2}^2$ from (47), we obtain the equality

$$N\frac{\kappa_N}{2} r_\lambda^N V_\lambda'^2(r_\lambda) = \frac{N-2}{2(\alpha+1)}\left[\frac{(N+2)}{(N-2)} - \alpha\right]\|V_\lambda\|_{L^{\alpha+1}}^{\alpha+1}, \tag{49}$$

which, of course, cannot be satisfied for $\alpha > (N+2)/(N-2)_+$.

Finally, for the critical value $\alpha = (N+2)/(N-2)_+$ the problem (39), (40) has for all $\lambda > 0$ the positive solution

$$V_\lambda(r) = \left[\frac{N(N-2)\lambda^{2/(N-2)}}{N(N-2) + \lambda^{4/(N-2)} r^2}\right]^{(N-2)/2}, r \geq 0; \; V_\lambda(0) = \lambda. \tag{50}$$

Insolvability of the boundary value problem in this case also follows from (49), since here $V_\lambda'(r_\lambda) \neq 0$ by (41). □

**Remark.** Returning to equation (38′) we obtain that for all $\beta > 0$, $N = 1, 2$ or for $0 < \beta < (\sigma+1)(N+2)/(N-2)$, $N \geq 3$, there are no stationary solutions in $\mathbf{R}_+$. On the other hand, for $\beta \geq (\sigma+1)(N+2)/(N-2)_+$ all its solutions are strictly positive.

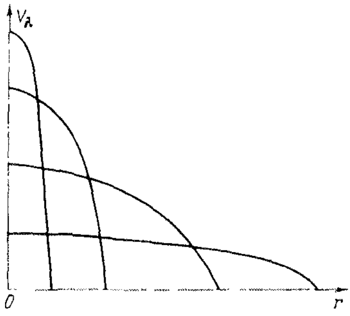


**Fig. 53.** Functions $V_\lambda(r)$ for $1 < \alpha < (N+2)/(N-2)_+$

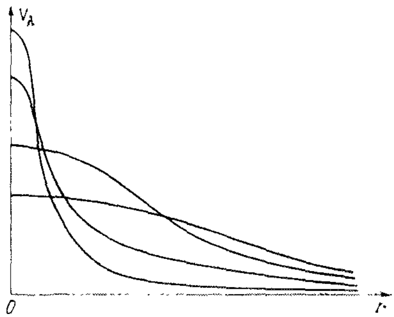


**Fig. 54.** Functions $V_\lambda(r)$ for $\alpha \geq (N+2)/(N-2)_+$

In Figures 53, 54 we sketch the behaviour of the functions $V_\lambda(r)$ for different $\lambda > 0$ in the compactly supported (Figure 53) and non-compactly supported (Figure 54) cases. For $\alpha > 1$ (unlike $\alpha < 1$) there is no monotone dependence of $V_\lambda(r)$ on $\lambda$ if $\alpha < 1 + 4/(N - 4 - 2\sqrt{N-1})$, $N \geq 11$ [227, 378].

In conclusion, let us note that a statement similar to the one proved above, is valid in the case of equation (39) with a fairly general nonlinear term $q(V)$ in the place of $V^\alpha$ (see § 1, Ch. VII).

*2 Proof of assertion (b) of Theorem 4*

Let us first establish a simple claim, which is relevant to assertions (b), (c).

**Lemma 2.** *Let $\beta > \sigma + 1 + 2/N$. Then for all*

$$0 < \mu \leq \left\{ \frac{N}{2(\beta - 1)} \left[ \beta - \left( \sigma + 1 + \frac{2}{N} \right) \right] \right\}^{1/(\beta - 1)} = \mu_1 \tag{51}$$

*the solution of problem* (33), (34) *is strictly positive in* $\mathbf{R}_+$ *(and, consequently, has the asymptotic behaviour* (35)).

*Proof.* Let us rewrite the equality obtained by integrating equation (33) multiplied by $\xi^{N-1}$ over the interval $(0, \xi)$, as follows:

$$\begin{aligned} &\xi^{N-1} f^\sigma f' + m f \xi^N = \\ &= \int_0^\xi \eta^{N-1} f(\eta) \left\{ \frac{N}{2(\beta-1)} \left[ \beta - \left( \sigma + 1 + \frac{2}{N} \right) \right] - f^{\beta-1}(\eta) \right\} d\eta. \end{aligned} \tag{52}$$

By (51) and monotonicity of $f$ the right-hand side is strictly positive for $\xi > 0$. Let us assume that $f(\eta)$ vanishes at $\xi = \xi_* \in \mathbf{R}_+$. Then $f^\sigma f'(\xi_*) \leq 0$, $f(\xi_*) = 0$, so that the left-hand side of (52) is non-positive for $\xi = \xi_*$, which leads to a contradiction. □

Thus we have established the second part of assertion (b). Its proof is completed by appealing to the following lemma.

**Lemma 3.** *Let* $\beta > \sigma + 1 + 2/N$ *if* $N = 1, 2$, *or* $\sigma + 1 + 2/N < \beta < (\sigma + 1)(N+2)/(N-2)$ *if* $N \geq 3$. *Then there exists* $\mu > 0$, *such that the solution of the problem* (33), (34) *becomes zero.*

Proof of the lemma follows the lines of the proof of Theorem 4 in § 1 (see also the analysis of the case $\beta < \sigma + 1$ in subsection 3, § 1). In the final count, the assumption contrary to Lemma 3, that is, that for all $\mu > 0$ the solution of problem (33), (34) is strictly positive in $\mathbf{R}_+$, leads, after passing to the limit, to a conclusion that for $\lambda = 1$ there exists a strictly positive solution of the stationary problem (39), (40), which is impossible by Lemma 1.

Next, denoting by $\mathcal{M}$ the set of all $\mu^0 > 0$, such that $f(\xi; \mu) > 0$ in $\mathbf{R}_+$ for all $0 < \mu < \mu^0$, we have that $\mathcal{M} \neq \emptyset$ (see Lemma 2) and that $\mathcal{M}$ is bounded from above (see Lemma 3). Therefore there exists $\mu_* = \sup \mathcal{M} < \infty$, and, using standard methods, we can show that the function $f_S = f(\xi; \mu_*)$, which corresponds to $\mu = \mu_*$ is a solution of the problem (31), (32) and has the asymptotics (36).

In Figure 55 we show curves $f(\xi; \mu)$ for different values of $\mu > 0$. The thick line shows the compactly supported solution.

### 3 *Proof of assertion (c) of Theorem 4*

Let us first note that strict positivity of all radially symmetric solutions of the stationary equation (38) for $\beta \geq (\sigma+1)(N+2)/(N-2)$, $N \geq 3$ (Lemma 4), indicates, in principle, that assertion (c) is true. A proof which proceeds by analyzing the ordinary differential equation (31) encounters various difficulties. Therefore our

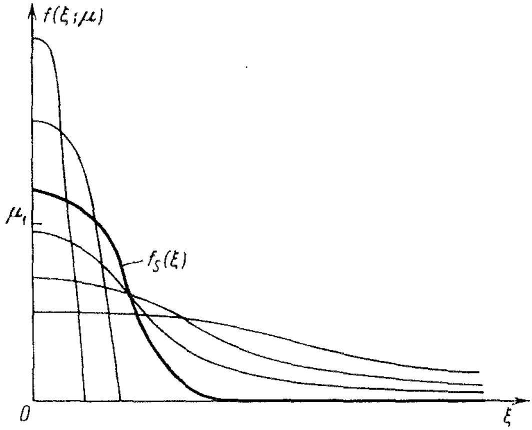


**Fig. 55.** Solutions of the problem (33), (34) for different $\mu = f(0)$, the case $\sigma + 1 + 2/N < \beta < (\sigma + 1)(N + 2)/(N - 2)_+$

proof will be based on a curious property of solutions of the corresponding partial differential equation (1).

Thus, let us assume the opposite: that in the conditions of (c) there exists $\mu > 0$, such that $f(\xi;\mu)$ vanishes at some point. Observe that by Lemma 2

$$\mu > \left\{ \frac{N}{2(\beta - 1)} \left[ \beta - \left( \sigma + 1 + \frac{2}{N} \right) \right] \right\}^{1/(\beta - 1)}.$$

Then, as in the proof of (b), we conclude that there exists a non-trivial solution $f_S(\xi) \geq 0$ of the problem (31), (32), with compact support and having the asymptotic behaviour (36). As we show below, this conclusion leads to a contradiction.

We established that for $\beta \geq (\sigma + 1)(N + 2)/(N - 2)_+$ the Cauchy problem for equation (1) has the self-similar solution

$$u \equiv u_S(t, x) = (T + t)^{-1/(\beta - 1)} \theta_S \left( \frac{|x|}{(T + t)^{|\beta - (\sigma + 1)|/[2(\beta - 1)]}} \right), \tag{53}$$

which has finite energy

$$E(t) = \frac{1}{2(\sigma + 1)} \|\nabla u^{\sigma + 1}\|^2_{L^2(\mathbf{R}^N)} - \frac{\sigma + 1}{\beta + \sigma + 1} \|u\|^{\beta + \sigma + 1}_{L^{\beta + \sigma + 1}(\mathbf{R}^N)}. \tag{54}$$

Indeed, $f_S$ has compact support, and $f_S^\sigma f_S' \in C(\mathbf{R}_+)$; it is not hard to check that $\nabla u^{\sigma + 1}(t) \in L^2(\mathbf{R}^N)$, $u(t) \in L^{\beta + \sigma + 1}(\mathbf{R}^N)$, so that $|E(t)| < \infty$ for all $t > 0$. Energy functionals of the form (54) are an important attribute of solutions of the problem, and we shall frequently use them in the following (see § 2, Ch. VII).

It is easy to see that $E(t)$ is non-increasing. Indeed, using equation (1) we obtain

$$
\begin{aligned}
E'(t) &\equiv \left[\frac{1}{2(\sigma+1)}\int_{\mathbf{R}^N} |\nabla u^{\sigma+1}|^2\,dx - \frac{\sigma+1}{\beta+\sigma+1}\int_{\mathbf{R}^N} u^{\beta+\sigma+1}\,dx\right]_t' = \\
&= -\int_{\mathbf{R}^N}\left(\frac{1}{\sigma+1}\Delta u^{\sigma+1} + u^{\beta}\right)(u^{\sigma+1})_t\,dx = -\int_{\mathbf{R}^N} u_t(u^{\sigma+1})_t\,dx = \qquad (55)\\
&= -\frac{4(\sigma+1)}{(\sigma+2)^2}\|(u^{1+\sigma/2})_t\|^2_{L^2(\mathbf{R}^N)} \le 0.
\end{aligned}
$$

For convenience, let us introduce the functional

$$
G(t) = \|u^{1+\sigma/2}\|^2_{L^2(\mathbf{R}^N)} \equiv \int_{\mathbf{R}^N} u^{\sigma+2}\,dx.
$$

Then

$$
\begin{aligned}
G'(t) &= (\sigma+2)\int_{\mathbf{R}^N} u^{\sigma+1}u_t\,dx = (\sigma+2)\int_{\mathbf{R}^N} u^{\sigma+1}\left(\frac{1}{\sigma+1}\Delta u^{\sigma+1} + u^{\beta}\right)dx = \\
&= -(\sigma+2)\frac{\beta+\sigma+1}{\sigma+1}\left[\frac{1}{\beta+\sigma+1}\|\nabla u^{\sigma+1}\|^2_{L^2(\mathbf{R}^N)} - \frac{\sigma+1}{\beta+\sigma+1}\|u\|^{\beta+\sigma+1}_{L^{\beta+\sigma+1}(\mathbf{R}^N)}\right]. \qquad (56)
\end{aligned}
$$

Since $\beta > \sigma + 1$, we have hence the estimate

$$
G'(t) \ge -\frac{(\sigma+2)(\beta+\sigma+1)}{\sigma+1}E(t),\; t > 0. \qquad (57)
$$

All the above transformations are justified for the function (53).

Let us see now what energy corresponds to a global solution (53). It is easily computed that for it

$$
\begin{aligned}
E(t) =& (T+t)^{\frac{N-2}{2(\beta-\sigma-1)}\left[\beta-(\sigma+1)\frac{N+2}{N-2}\right]} \times \\
&\times\left\{\frac{1}{2(\sigma+1)}\|\nabla_\xi f_S^{\sigma+1}(\xi)\|^2_{L^2} - \frac{\sigma+1}{\beta+\sigma+1}\|f_S(\xi)\|^{\beta+\sigma+1}_{L^{\beta+\sigma+1}}\right\}. \qquad (58)
\end{aligned}
$$

Let us show that for $\beta > (\sigma+1)(N+2)/(N-2)$, $N \ge 3$, a global solution cannot have such energy. For the critical case $\beta = (\sigma+1)(N+2)/(N-2)$ this is obvious. From (58) it follows that $E(t) = \text{const}$, that is, $E'(t) \equiv 0$, which contradicts (55), as $u_t \not\equiv 0$.

Assume now that $\beta > (\sigma+1)(N+2)/(N-2)_+$. Then from (58) we have that for the energy $E$ to be strictly decreasing (see (55)), we must have $E(0) \le 0$. Then $E(t) < 0$ for all $t > 0$ (if $E(0) > 0$, then by (58) $E'(t) > 0$, which contradicts (55)).

It turns out that the resulting condition

$$E(t) < 0, \ t > 0, \tag{59}$$

is incompatible with global existence of a solution.

**Lemma 4.** *Let $u(t,x) \not\equiv 0$ be a solution of the Cauchy problem* (1), (2) *having compact support, which satisfies condition* (59). *Then $u(t,x)$ exists for a finite time.*

*Proof.* Under the above assumptions we can take $G(t) > 0$, $\|u^{\sigma+1}(t)\|_{L^2} \neq 0$, and $\|\nabla u^{\sigma+1}(t)\|_{L^2} \neq 0$ for all $t > 0$. Then by (55), (56) we have $G'(t) > 0$, $E'(t) < 0$, and, furthermore, the inequality (57) is strict:

$$G'(t) > -\frac{(\sigma+2)(\beta+\sigma+1)}{\sigma+1}E(t), t > 0. \tag{60}$$

Using the Cauchy-Schwarz inequality, we derive from (55), (56), (60) the following estimate:

$$-G(t)E'(t) = \frac{4(\sigma+1)}{(\sigma+2)^2}\|u^{1+\sigma/2}\|_{L^2}^2\|(u^{1+\sigma/2})_t\|_{L^2}^2 \geq$$

$$\geq \frac{4(\sigma+1)}{(\sigma+2)^2}(u^{1+\sigma/2}, (u^{1+\sigma/2})_t)^2 = \frac{4(\sigma+1)}{(\sigma+2)^2}\frac{G'(t)G'(t)}{4} \geq -\frac{\beta+\sigma+1}{\sigma+2}G'(t)E(t),$$

that is,

$$GE' - \frac{\beta+\sigma+1}{\sigma+2}G'E \leq 0, \ t > 0,$$

or, which is the same,

$$(G^{(\beta+\sigma+1)/(\sigma+2)}/E)'(t) \geq 0, \ t > 0.$$

Hence, taking into account the fact that $E(t) < 0$, we obtain for all $t \geq t^* > 0$ the estimate

$$G^{(\beta+\sigma+1)/(\sigma+2)}(t) \leq c_* E(t), \tag{61}$$

where $c_* = G^{(\beta+\sigma+1)/(\sigma+2)}(t_*)/E(t_*) < 0$. From (60), (61) it follows the inequality

$$G^{(\beta+\sigma+1)/(\sigma+2)}(t) \leq |c_*||E(t)| \leq \frac{\sigma+1}{\sigma+2}\frac{|c_*|}{\beta+\sigma+1}G'(t), \ t > t_*.$$

From the inequality

$$G'(t) \geq \frac{\sigma+2}{\sigma+1}\frac{\beta+\sigma+1}{|c_*|}G^{(\beta+\sigma+1)/(\sigma+2)}(t), \ t > t_*.$$

it follows that the function $G(t) \equiv \|u^{1+\sigma/2}(t)\|^2_{L^2}$ cannot be bounded for all $t > 0$, and that there exists

$$T_0 \leq t_* + \frac{\sigma + 1}{(\beta - 1)(\beta + \sigma + 1)} |E(t_*)|^{-1} G(t_*) < \infty,$$

such that $G(t) \to \infty$ as $t \to T_0^-$, i.e., the solution $u(t, x)$ is unbounded. □

Therefore for $\beta \geq (\sigma + 1)(N + 2)/(N - 2)_+$ the compactly supported solution $u_S$ in (53) cannot be a global one, so that in this case all solutions (33), (34) are strictly positive, which concludes the proof of assertion (c) of Theorem 4. □

This analysis establishes a certain similarity of solutions of equation (31) and of the stationary equation (38′), which is quite obvious for sufficiently large values of $\mu = f(0)$. There are also significant differences between them: for small $\mu > 0$ lower order (linear) terms in equation (31) are of importance. A consequence of this is existence for a range of $\beta$ of the solution $f_S(\xi)$ with compact support. In the case of the stationary equation (38′) (see subsection 4.1) a solution with the required asymptotic behaviour exists only for the one value $\beta = (\sigma + 1)(N + 2)/(N - 2)_+$.

## 5 A non-compactly supported stable set

Using the results of subsection 4, it is not hard to determine the stable set $\mathcal{W}$ of the problem (1), (2) for $\beta > \sigma + 1 + 2/N$, which consists of non-compactly supported functions. The boundary of $\mathcal{W}$ consists of non-compactly supported global self-similar solutions (28). Here we shall assume that the solution $u(t, x)$ obeys the Maximum Principle and depends monotonically on the initial function (see § 3, Ch. I).

Let us denote by $\mathcal{F}$ the set of non-compactly supported solutions $f_S(\xi)$ of the problem (31), (32); to $\mathcal{F}$ belong, for example all $f(\xi; \mu)$ of Lemma 2 ($\mathcal{F} \neq \emptyset$ for $\beta > \sigma + 1 + 2/N$). The set $\mathcal{W}$ is determined in the following statement.

**Theorem 5.** *Let $\beta > \sigma + 1 + 2/N$. Then there exists a non-compactly supported stable set $\mathcal{W}$ of the problem* (1), (2)*:*

$$\begin{aligned} \mathcal{W} = \{u_0(x) \geq 0 \mid \exists\, f_S \in \mathcal{F},\ T > 0 = \text{const} :& \\ u_0(x) \leq T^{-1/(\beta-1)} f_S(|x|/T^{[\beta-(\sigma+1)]/[2(\beta-1)]})\}.& \end{aligned} \tag{62}$$

*Proof.* By the Maximum Principle the restriction (62) on the initial function provides us with the bound for the solution:

$$u(t, x) \leq (T + t)^{-1/(\beta-1)} f_S(|x|/(T + t)^m), \quad t > 0, \ x \in \mathbf{R}^N.$$

Therefore the solution $u(t, x)$ is globally bounded:

$$\sup_{x \in \mathbf{R}^N} u(t, x) \leq f_S(0)(T + t)^{-1/(\beta - 1)} \to 0, \ t \to \infty.$$

□

## 6 Asymptotic behaviour of global solutions for $\beta > \sigma + 1 + 2/N$

Let us find out the relation of the self-similar solutions constructed in subsection 4 to the asymptotic behaviour of arbitrary global solutions of the Cauchy problem. Is it true that the particular solutions (28) describe for large $t$ the amplitude and spatial profile of decaying thermal structures, which exist for $\beta > \sigma + 1 + 2/N$?

Below we present an analysis of asymptotic stability of symmetric in $x$ self-similar solutions (28).

First we shall show that the question posed above can be answered in the affirmative if $f_S(|\xi|)$ has the power law asymptotics (35). In particular, if we denote by $f_F(t, \xi)$ the similarity representation of the solution of the Cauchy problem (1), (2),

$$f_F(t, \xi) = (T + t)^{1/(\beta - 1)} u(t, \xi(T + t)^m), \ t > 0, \xi \in \mathbf{R}^N,$$

the following statement is true:

**Theorem 6.** *Let $\beta > \sigma + 1 + 2/N$, and let the self-similar function $f_S$ in (30) be such that*

$$-\delta \equiv \frac{N}{2(\beta - 1)} \left( \sigma + 1 + \frac{2}{N} - \beta \right) + \beta (f_S(0))^{\beta - 1} < 0. \tag{63}$$

*Then the self-similar solution (28) is asymptotically stable in $L^1(\mathbf{R}^N)$ in the following sense: if for some $T > 0$*

$$u_0(x) \leq u_S(0, x; T), \ x \in \mathbf{R}^N, \tag{64}$$

$$u_S(0, \cdot; T) - u_0(\cdot) \in L^1(\mathbf{R}^N), \tag{64'}$$

*then*

$$\|f_F(t, \cdot) - f_S(\cdot)\|_{L^1(\mathbf{R}^N)} = O(t^{-\delta}) \to 0, \ t \to \infty. \tag{65}$$

**Remark.** The inequality (68) provides the following restriction on the size of $f_S(0) = \sup f_S(|\xi|)$:

$$f_S(0) \leq \left\{ \frac{N}{2(\beta - 1)\beta} \left[ \beta - \left( \sigma + 1 + \frac{2}{N} \right) \right] \right\}^{1/(\beta - 1)}.$$

From Lemma 2 (see subsection 4.2) it follows that then $f_S(|\xi|) > 0$ in $\mathbf{R}^N$, and therefore the theorem deals with asymptotic stability of a non-compactly supported self-similar solution $u_S$. Let us note that by (35) $u_S(t, \cdot; T) \notin L^1(\mathbf{R}^N)$ for any $t > 0$.

*Proof.* Formally, the argument is the same as in § 13, Ch. II. Let $z = u_S - u \in L^1(\mathbf{R}^N)$ for each $t \geq 0$. By (64) $u \leq u_S$, i.e., $z \geq 0$ in $\mathbf{R}_+ \times \mathbf{R}^N$. From the parabolic equation for the function $z$ it follows that

$$\frac{d}{dt}\|z(t)\|_{L^1(\mathbf{R}^N)} \leq (z(t), a(u, u_S)), \quad t > 0,$$

where

$$a(u, u_S) = \beta \int_0^1 (\eta u_S + (1-\eta)u)^{\beta-1}\, d\eta \leq \beta (f_S(0))^{\beta-1}(T+t)^{-1}.$$

Therefore

$$\|z(t)\|_{L^1(\mathbf{R}^N)} = O\left(t^{\beta(f_S(0))^{\beta-1}}\right), \quad t \to \infty. \tag{66}$$

However

$$\|z(t)\|_{L^1(\mathbf{R}^N)} \equiv (T+t)^{\frac{N}{2(\beta-1)}\left[\beta-\left(\sigma+1+\frac{2}{N}\right)\right]}\|f_T - f_S\|_{L^1(\mathbf{R}^N)}. \tag{67}$$

and the estimate (65) follows from (66), (67); by (63) it implies the stabilization $f_T \to f_S$ as $t \to \infty$ in the norm of $L^1(\mathbf{R}^N)$. □

Theorem 6 demonstrates asymptotic stability of non-compactly supported self-similar solutions in the class of initial functions (see (64′))

$$u_0(x) \sim C|x|^{-2/[\beta-(\sigma+1)]}, \quad |x| \to \infty. \tag{68}$$

Thus if $u_0(x)$ satisfies (68) (then $u_0 \notin L^1(\mathbf{R}^N)$), and to a given initial function there corresponds a global solution of the Cauchy problem, then the amplitude and the half-width are estimated asymptotically exactly as $t \to \infty$ by

$$\sup_{x \in \mathbf{R}^N} u(t, x) \simeq t^{-1/(\beta-1)}, \quad |x_{ef}(t)| \sim t^{[\beta-(\sigma+1)]/[2(\beta-1)]}. \tag{69}$$

What will happen if (68) is not satisfied, for example, in the case of an initial perturbation with compact support $u_0 \in L^1(\mathbf{R}^N)$? It follows from Theorem 4 that for $\sigma+1+2/N < \beta < (\sigma+1)(N+2)/(N-2)_+$ there exists a compactly supported self-similar solution $u_S$ of the form (28), which, it would seem, should describe the asymptotic behaviour of such solutions. However, this is not the case. Unlike the non-compactly supported solutions in Theorem 6, this self-similar solution is unstable as $t \to \infty$. In this case the asymptotic stage of combustion with extinction is described by self-similar solutions of the equation without source:

$$v_t = \nabla \cdot (v^\sigma \nabla v), \ t > 0, \ x \in \mathbf{R}^N, \tag{70}$$

i.e., for large times combustion is negligible compared with diffusion.

A similar situation was already encountered in § 13, Ch. II. Therefore we shall not attempt an exhaustive analysis, and will study only the most interesting (and hardest where proofs are concerned) case of a compactly supported initial function $u_0 \in L^1(\mathbf{R}^N)$. It will be shown that in this case the global solution of the Cauchy problem evolves as $t \to \infty$ according to the rules determined by the spatio-temporal structure of the self-similar solution of equation (70) (see § 3, Ch. I):

$$\begin{gathered} v_S(t, x; T, a) = (T+t)^{-N/(N\sigma+2)} g_S(\eta; a), \\ \eta = x/(T+t)^{1/(N\sigma+2)}, \end{gathered} \tag{71}$$

where $T > 0$ is an arbitrary constant, $g_S(\eta; a) \geq 0$ satisfies in $\mathbf{R}^N$ the equation

$$\mathbf{B}_\sim(g_S) \equiv \nabla_\eta \cdot (g_S^\sigma \nabla_\eta g_S) + \frac{1}{N\sigma+2} \nabla_\eta g_S \cdot \eta + \frac{N g_S}{N\sigma+2} = 0 \tag{72}$$

and has the form

$$g_S(\eta; a) = A_0(a^2 - |\eta|^2)_+^{1/\sigma}, \ A_0 = \left[\frac{\sigma}{2(N\sigma+2)}\right]^{1/\sigma}. \tag{73}$$

Here $a > 0$ is a constant.

To prove the above-mentioned fact, let us introduce, corresponding to (71), the similarity representation of the solution $u(t, x)$ of the problem (1), (2):

$$g(t, \eta) = (T+t)^{N/(N\sigma+2)} u(t, \eta (T+t)^{1/(N\sigma+2)}), \tag{74}$$

and write down the equation it satisfies with the new time $\tau = \ln(T+t)$:

$$\begin{gathered} \frac{\partial g}{\partial \tau} = \mathbf{B}_\tau(g) \equiv \nabla_\eta \cdot (g^\sigma \nabla_\eta g) + \\ + \frac{1}{N\sigma+2} \nabla_\eta g \cdot \eta + \frac{N g}{N\sigma+2} + e^{-\nu\tau} g^\beta, \tau > \tau_0 = \ln T, \ \eta \in \mathbf{R}^N, \end{gathered} \tag{75}$$

$$g(\tau_0, \eta) = T^{N/(N\sigma+2)} u_0(\eta T^{1/(N\sigma+2)}) \equiv g_0(\eta), \ \eta \in \mathbf{R}^N. \tag{76}$$

In (75) we denoted by $\nu$ the constant $\nu = N[\beta - (\sigma + 1 + 2/N)]/(N\sigma+2)$. For $\beta > \sigma + 1 + 2/N$ it is positive, which is important.

Let us prove first that in the case of a function $u_0(x)$ with compact support the behaviour of global solutions of the problem (1), (2) as $t \to \infty$ "obeys" (71). First of all, it is clear that the function (73) is a subsolution of equation (75), since

$$\frac{\partial g_S}{\partial \tau} \equiv 0 \leq \mathbf{B}_\tau(g_S) \equiv e^{-\nu\tau} g_S^\beta.$$

Thus if $u_0(0) > 0$, there exist constants $T > 0$, $a_- > 0$, such that

$$u_0(x) \geq T^{-N/(N\sigma+2)} g_S(xT^{-1/(N\sigma+2)}; a_-) \text{ in } \mathbf{R}^N. \tag{77}$$

Therefore $g(\tau, \eta) \geq g_S(\eta; a_-)$ in $\mathbf{R}^N$ for all admissible $\tau > \tau_0$.

It remains to construct a similar supersolution of equation (75). Naturally, the functions $g_S$ cannot be used to that end. However, they can be easily modified to give a supersolution.

**Lemma 5.** *Let* $\beta > \sigma + 1 + 2/N$ *and*

$$1 - \frac{\sigma}{\nu} A_0^{\beta-1} (a^2)^{(\beta-1)/\sigma} T^{-\nu} > 0. \tag{78}$$

*Then the function*

$$g_+(\tau, \eta) = |1 - be^{-\nu\tau}|^{1/\sigma} g_S(\eta/|1 - be^{-\nu\tau}|^{1/2}; a), \tag{79}$$

*where* $b = \sigma A_0^{\beta-1}(a^2)^{(\beta-1)/\sigma}/\nu$, *is a supersolution of equation* (75).

*Proof.* We shall seek a supersolution in the form $g_+ = \psi(\tau) g_S(\eta/\phi(\tau); a)$. Then we have $\partial g_+/\partial\tau \geq \mathbf{B}_\tau(g_+)$ in $\mathbf{R}_+ \times \mathbf{R}^N \setminus \{|\eta| = a\phi(\tau)\}$, if

$$\begin{aligned} \frac{\psi'}{\psi} \geq \frac{2}{\sigma}\left(1 - \frac{a^2}{d}\right) \frac{1}{(N\sigma + 2)} \left[1 - \frac{\psi^\sigma}{\phi^2} + (N\sigma + 2)\frac{\phi'}{\phi}\right] + \\ + \frac{N}{N\sigma + 2}\left(1 - \frac{\psi^\sigma}{\phi^2}\right) + e^{-\nu\tau}\psi^{\beta-1} A_0^{\beta-1} d^{(\beta-1)/\sigma}, \end{aligned} \tag{80}$$

where $d = (a^2 - \xi^2)_+ \in (0, a^2]$.

Let $\phi' \geq 0$, $\psi^\sigma = \phi^2$ (this holds in (79)). Then validity of (80) follows from the following inequality:

$$\frac{\psi'(\tau)}{\psi(\tau)} \geq e^{-\nu\tau}\psi^{\beta-1}(\tau) A_0^{\beta-1} (a^2)^{(\beta-1)/\sigma}, \quad \tau \geq \tau_0.$$

Setting here $\psi^\sigma(\tau) = 1 - be^{-\nu\tau}$ (by (78) $\psi > 0$ in $\mathbf{R}_+$), we see that $g_+$ is a supersolution if $b \geq \sigma A_0^{\beta-1}(a^2)^{(\beta-1)/\sigma}/\nu$.

Thus, if $u_0(x)$ is a function with compact support and there exist constants $T > 0$, $a_+ > 0$, such that

$$g_0(\eta) \le (1 - bT^{-\nu})^{1/\sigma} g_S(\eta/(1 - bT^{-\nu})^{1/2}; a_+), \eta \in \mathbf{R}^N, \tag{81}$$

where $b = \sigma A_0^{\beta-1}(a_+^2)^{(\beta-1)/\sigma}/\nu$, then we have in $(\tau_0, \infty) \times \mathbf{R}^N$ the estimate

$$g(\tau, \eta) \le g_+(\tau, \eta) \tag{82}$$

(and, actually, problem (75), (76) will have a global solution). □

Combining the above results, we arrive at the following conclusion.

**Theorem 7.** *Let $\beta > \sigma + 1 + 2/N$, and let the function $u_0(x)$ satisfy the conditions* (77), (81). *Then, if $g(\infty, \eta)$ exists,*

$$g_S(\eta; a_-) \le g(\infty, \eta) \le g_S(\eta; a_+), \quad \eta \in \mathbf{R}^N. \tag{83}$$

This result shows that for $\beta > \sigma + 1 + 2/N$ the evolution of global solutions of the Cauchy problem (1), (2) is described by the self-similar solutions (71) of equation (70). The estimates (83) mean, in particular, that

$$\sup_{x \in \mathbf{R}^N} u(t, x) \sim t^{-N/(N\sigma+2)}, \quad |x_{ef}(t)| \sim t^{1/(N\sigma+2)}, \quad t \to \infty.$$

Recall that the same conclusions were obtained earlier using the qualitative non-stationary averaging theory (see § 2).

As far as asymptotic stability of the self-similar solutions (71) is concerned, we shall confine ourselves here to proving one simple assertion.

**Theorem 8.** *Let $N = 1$, $\beta > \sigma + 1 + 2/N \equiv \sigma + 3$ and let $u_0(x)$ have compact support. Then under the conditions of Theorem 7 we can find $a \in [a_-, a_+]$, such that*

$$g(t, \eta) \to g_S(\eta; a), \; t \to \infty,$$

*almost everywhere in* $\mathbf{R}$.

*Proof.* First of all, by (83) the Cauchy problem (75), (76) is equivalent to a boundary value problem in some bounded domain $\Omega \supset [a_-, a_+]$, $g = 0$ on $\mathbf{R}_+ \times \partial\Omega$. In addition, we can derive the estimates

$$(g^{1+\sigma/2})_\tau \in L^2((\tau_1, \infty) \times \Omega), \; g^{\sigma+1} \in L^\infty((\tau_1, \infty); \; H_0^1(\Omega)), \quad \tau_1 = \tau_0 + 1, \tag{84}$$

by taking scalar products in $L^2((\tau_1, \infty) \times \Omega)$ of both sides of equation (75) with

$$\frac{\partial}{\partial \tau}\left(\frac{1}{\sigma+1} g^{\sigma+1} + \frac{1}{\sigma+2}\int_0^\eta g(\tau, \zeta)\zeta d\zeta\right).$$

These estimates show that the $\omega$-limit set, $\omega(g_0) = \{g^*(\eta) \mid \exists \tau_n \to \infty : g^{\sigma+1}(\tau_n, \cdot) \to [g^*(\cdot)]^{\sigma+1}$ in $L^2(\Omega)\}$, consists of "stationary" solutions of equation (75) for $\tau = \infty$, that is, $\mathbf{B}_\infty(g^*) = 0$. Indeed, using the estimate (83) and (84) in passing to the limit in equation (75), we have that, given a monotone sequence $\tau_n \to \infty$, $g(\tau_n + s, \cdot) \to h(s, \cdot)$ in $L^2_{loc}((\tau_1, \infty) \times \Omega)$, where $h(s, \cdot)$ is a weak solution of the limit equation $h_s = \mathbf{B}_\infty(h)$ for $s > 0$, $h(0) \in \omega(g_0)$. It follows from (84) that uniformly in $s \in [0, 1]$ ($\alpha = 1 + \sigma/2$)

$$\|g^\alpha(\tau_n + s) - g^\alpha(\tau_n)\|^2_{L^2(\Omega)} \leq \left\| \int_{\tau_n}^{\tau_n+1} \left| \frac{\partial}{\partial \tau} g^\alpha(\tau) \right| d\tau \right\|^2_{L^2(\Omega)} \leq$$

$$\leq \int_{\tau_n}^{\tau_n+1} \left\| \frac{\partial}{\partial \tau} g^\alpha(\tau) \right\|^2_{L^2(\Omega)} d\tau \to 0$$

as $\tau_n \to \infty$, that is, $h$ does not depend on $s$ and is a weak stationary solution, $\mathbf{B}_\infty(h) = 0$. By (83) $g^*$ are functions with compact support, and therefore $\omega(g_0) \subseteq \{g_S(\eta; a), a \in [a_-, a_+]\}$. Finally, independence of the limit function $g^*(\eta)$ of the choice of the sequence $\tau_n \to \infty$ follows from "monotonicity" of the solution of the problem (75), (76):

$$\frac{d}{d\tau} \|g(\tau)\|_{L^1(\Omega)} = e^{-\nu\tau} \|g(\tau)\|^\beta_{L^\beta(\Omega)} > 0, \ \tau > 0$$

($-\|g(\tau)\|_{L^1(\Omega)}$ is a Liapunov function), and also strict monotonicity in $a > 0$ of the expression $\|g_S(\cdot; a)\|_{L^1(\Omega)}$. □

**Remark.** It is not hard to show by the Bernstein technique that the derivative $(g^{\sigma+1})_\eta$ is uniformly bounded in $(\tau_0 + 1, \infty) \times \mathbf{R}$, so that stabilization $g^{\sigma+1}(\tau, \cdot) \to g_S^{\sigma+1}(\cdot; a)$ as $\tau \to \infty$ is uniform in $\mathbf{R}$.

## § 4 Proof of localization of unbounded solutions for $\beta \geq \sigma + 1$; absence of localization in the case $1 < \beta < \sigma + 1$

Results of preceding sections give us quite a good overall picture of the main properties of unbounded solutions of the Cauchy problem we are considering. The aim of the present section is to prove localization of unbounded compactly supported solutions of the S- ($\beta = \sigma + 1$) and LS- ($\beta > \sigma + 1$) regimes, as well as absence of localization in the HS-regime ($1 < \beta < \sigma + 1$). At the same time we shall obtain a number of important estimates of the size of the support of unbounded solutions. The method of proof presented here will be used in § 5.

where, applying this method, we shall solve the question of describing asymptotic spatio-temporal structure of unbounded solutions for times close to the blow-up time.

We shall consider the Cauchy problem in the one-dimensional case,

$$u_t = (u^\sigma u_x)_x + u^\beta, \ t > 0, \ x \in \mathbf{R}; \ \sigma > 0, \ \beta > 1, \tag{1}$$

$$u(0, x) = u_0(x) \geq 0, \ x \in \mathbf{R}; \ u^{\sigma+1} \in C^1(\mathbf{R}), \tag{2}$$

where the initial perturbation $u_0(x) \not\equiv 0$ is a compactly supported function with a connected support:

$$\omega(0) = \operatorname{supp} u_0 \equiv \{x \in \mathbf{R} \mid u_0(x) > 0\} = (h_-(0), h_+(0)), \tag{3}$$

$$-\infty < h_-(0) < h_+(0) < \infty.$$

Then, for all $t > 0$ for which the solution exists, the support of the generalized solution $u(t, x)$ will also be bounded and connected,

$$\omega(t) = \operatorname{supp} u(t, x) = (h_-(t), h_+(t)), \tag{4}$$

$$-\infty < h_-(t) < h_+(t) < \infty.$$

The functions $h_-(t)$ and $h_+(t)$, which determine at each moment of time the position of the (left and right, respectively) front points of the generalized solution, are, respectively, non-increasing and non-decreasing, so that the length of the support $\operatorname{meas} \omega(t) \equiv h_+(t) - h_-(t)$ is non-decreasing with time. It is not hard to show by comparison with travelling wave solutions that $h_\pm \in C([0, T_0))$.

Let $t = T_0(u_0) < \infty$ be the blow-up time of the problem (1), (2). First of all we shall be interested in the behaviour of the functions $h_\pm(t)$ as $t \to T_0^-$. It turns out that for $\beta \geq \sigma + 1$ the functions $h_\pm(t)$ are bounded on $(0, T_0)$ and $|h_\pm(T_0^-)| < \infty$, which, as can be easily seen, is equivalent to localization of the unbounded solution. Conversely, it will be shown that in the case $\beta \in (1, \sigma + 1)$ the functions $h_\pm(t)$ are unbounded, and $h_\pm(t) \to \pm\infty$ as $t \to T_0^-$ (there is no localization).

In § 1 we studied in detail self-similar unbounded solutions, which explicitly illustrated various interesting properties of blow-up regimes. To prove that these properties are shared by a wide class of solutions of equation (1), we shall use the method of intersection comparison of the solutions being considered with exact self-similar solutions having the same blow-up time. We start by presenting the main ideas of this comparison theory, which is especially suited to analyze the spatial structure of unbounded solution close to the finite blow-up time.

## 1 The number of intersections of different unbounded solutions having the same blow-up time (main comparison theorems)

Let us observe at the outset that Proposition 1 below, concerning the non-increase of the number of spatial intersections of two different solutions $u(t, x)$ and $v(t, x)$, is an immediate corollary of the Maximum Principle for linear parabolic equations. Certain technical difficulties, which crop up also in the definition of the intersection used here, have to do with the fact that (1) is a degenerate equation, which admits generalized solutions. Therefore we shall not present this result in its maximal generality, or in all the possible detail. We shall mainly emphasize the parts of comparison theorems that make essential use of unboundedness of solutions under consideration.

Thus, let $v(t, x)$ be a generalized solution of equation (1) with a bounded non-negative initial function

$$v(0, x) = v_0(x) \geq 0, \quad x \in \mathbf{R}; \; v_0^{\sigma+1} \in C^1(\mathbf{R}). \tag{5}$$

We shall assume that the function $v(t, x)$ is defined for $[0, T_0) \times \mathbf{R}$.

**Definition.** For a fixed $t_0 \in [0, T_0)$ the interval $[a_1, a_2] \subset \mathbf{R}$ is called an *intersection interval* (or an *intersection point* if $a_1 = a_2$) of functions $u(t_0, x)$ and $v(t_0, x)$ if the difference $w(t_0, x) \equiv u(t_0, x) - v(t_0, x)$ is such that $w(t_0, x) = 0$ for all $x \in [a_1, a_2]$; for any sufficiently small $\delta > 0$ the function $w(t_0, x)$ does not have the same sign in $[a_1 - \delta, a_2 + \delta]$ and $w(t_0, x) \not\equiv 0$ for all $x \in (a_1 - \delta, a_1)$ and $x \in (a_2, a_2 + \delta)$. If, on the other hand, $w(t_0, x) = 0$ on $[a_1, a_2]$, $w(t_0, x)$ has constant sign on $[a_1 - \delta, a_2 + \delta]$ for any sufficiently small $\delta > 0$ and $w(t_0, x) \not\equiv 0$ for all $x \in (a_1 - \delta, a_1)$ and $x \in (a_2, a_2 + \delta)$, then we call $[a_1, a_2]$ a *tangency interval* (or *point* if $a_1 = a_2$) of the functions $u(t_0, x)$ and $v(t_0, x)$.

We shall denote the number of spatial intersections in $\mathbf{R}$ (of intervals or points of intersection) of the solutions $u(t_0, x)$ and $v(t_0, x)$ by $N(t_0)$, and we shall always assume that $N(0) < \infty$. It is clear that $N(t_0)$ is precisely the number of sign changes in $\mathbf{R}$ of the difference $w(t_0, x)$.

**Proposition 1.** *The function $N(t)$ is non-increasing with time, and in particular*

$$N(t) \leq N(0) \quad \textit{for all} \quad t \in (0, T_0). \tag{6}$$

*Proof.* As we already mentioned above, technical difficulties arise in the analysis of generalized solutions $u(t, x)$ and $v(t, x)$ having compact support. In this case, in accordance with the method of construction of generalized solutions (see § 3, Ch. I) we shall first consider the functions $u_\epsilon(t, x)$ and $v_\epsilon(t, x)$, classical strictly positive solutions of equation (1) with initial functions

$$u_\epsilon(0, x) = u_0(x) + \epsilon, \; v_\epsilon(0, x) = v_0(x) + \epsilon, \quad x \in \mathbf{R}, \tag{7}$$

where $\epsilon > 0$ is a fixed sufficiently small positive constant. By the comparison theorem $u_\epsilon \geq \epsilon$, $v_\epsilon \geq \epsilon$ everywhere in the domain of existence of the solutions, and therefore equation (1) is uniformly parabolic on these solutions. Let us denote the number of spatial intersections of the functions $u_\epsilon(t, x)$ and $v_\epsilon(t, x)$ by $N_\epsilon(t)$. It is easily seen that by construction $N_\epsilon(0) \equiv N(0)$ for every $\epsilon > 0$.

The difference $w(t, x) = u_\epsilon(t, x) - v_\epsilon(t, x)$ satisfies the linear parabolic equation

$$w_t = (a(t, x)w_x)_x + b(t, x)w, \tag{8}$$

where the coefficients of the equations

$$a(t, x) = \int_0^1 |\eta u_\epsilon + (1 - \eta)v_\epsilon|^\sigma \, d\eta, \; b(t, x) = \beta \int_0^1 |\eta u_\epsilon + (1 - \eta)v_\epsilon|^{\beta - 1} \, d\eta,$$

are sufficiently smooth, and $a(t, x) \geq \epsilon^\sigma$. Equation (8) is uniformly parabolic in the domain under consideration. Therefore Proposition 1 is valid for the function $N_\epsilon(t)$, which is a consequence of the Maximum Principle. A simple short proof of this fact is presented, for example, in [355]; for similar statements see [13, 316, 303, 315, 171, 175, 263]. The original general ideas of such a comparison go back to C. Sturm, 1836 [368].

Thus, $N_\epsilon(t)$ is non-increasing and $N_\epsilon(t) \leq N_\epsilon(0)$. Using now the fact that $u_\epsilon(t, x) \to u(t, x)$, $v_\epsilon(t, x) \to v(t, x)$ as $\epsilon \to 0$ uniformly on every compact set in $[0, T_0) \times \mathbf{R}$, after the necessary elementary consideration of the possible configurations of the intersections of the generalized solutions $u(t, x)$ and $v(t, x)$ and the corresponding regularized solutions $u_\epsilon(t, x)$ and $v_\epsilon(t, x)$, we arrive at the desired result for the function $N(t)$; see [140, 175]. □

Let us note that for $u \geq \epsilon$ the coefficients of equation (1) are analytic. Therefore its solutions $u_\epsilon(t, x)$ and $v_\epsilon(t, x)$ for $t > 0$ are analytic in $x$ functions (see, e.g. [100, 249, 256]). Therefore each of their intersections for $t > 0$ is an isolated point. However, as we pass to the limit $\epsilon \to 0$, an intersection point of $u_\epsilon(t, x)$ and $v_\epsilon(t, x)$ can be transformed into an interval of intersection of the generalized solutions $u(t, x)$ and $v(t, x)$. Nonetheless, any intersection in the domain of strict positivity of the generalized solutions $u(t, x)$ and $v(t, x)$, where the equation (8) for their difference $w = u - v$ is uniformly parabolic, and the solutions are classical and as smooth as is allowed by the coefficients (therefore we can also claim that the solutions are analytic in $x$ in the domain of their strict positivity), is an isolated point. Hence an intersection interval can only arise when it contains end-points of the supports of the functions under consideration.

Thus the number of spatial intersections is non-increasing in time. As we already observed, this fact is true for a wide range of parabolic equations, and we always have the upper bound (6) for the number of spatial intersections. However, to be able to use intersection comparison of solutions we also need, roughly speaking, a lower bound of the number of intersections. Indeed, for example, the fact that the number of intersections cannot decrease to zero during the evolution of solutions would mean that there exists a certain relation between spatial profiles of the solutions on the whole interval of their existence. Unfortunately, it does not appear possible to obtain such a lower bound for the number of intersections in the general case; for that, solutions must share some common properties. In this case such a shared property will be equality of blow-up times.

Thus we shall assume that $u(t, x)$ and $v(t, x)$ have the same blow-up time:

$$\overline{\lim_{t \to T_0}} \sup_{x \in \mathbf{R}} u(t, x) = \overline{\lim_{t \to T_0}} \sup_{x \in \mathbf{R}} v(t, x) = \infty. \tag{9}$$

Let us now state the main intersection comparison theorem.

**Proposition 2.** *Let $u(t, x)$ and $v(t, x)$ have the same blow-up time $t = T_0$. Then*

$$\begin{gathered} \mathcal{W}^* \equiv \{t \in [0, T_0) \mid u(t, x) \geq v(t, x) \text{ in } \mathbf{R} \text{ and} \\ \overline{\operatorname{supp}}\, v(t, x) \subset \operatorname{supp} u(t, x)\} = \emptyset. \end{gathered} \tag{10}$$

*Proof.* Let us assume the contrary: let $\mathcal{W}^* \neq \emptyset$ and there exists $t_* \in [0, T_0)$, such that

$$u(t_*, x) \geq v(t_*, x) \text{ in } \mathbf{R},$$

$$\overline{\operatorname{supp}}\, v(t_*, x) \subset \operatorname{supp} u(t_*, x).$$

Then, first of all, by the Strong Maximum Principle, applied to equation (8) in any subdomain where it is uniformly parabolic, in which $u(t, x)$ and $v(t, x)$ are separated uniformly from zero, and also using continuity of solutions and of the boundaries of their support, there exists a sufficiently small time $\tau_1 > 0$, such that

$$\begin{aligned} &u(t_* + \tau_1, x) > v(t_* + \tau_1, x) \text{ in } \overline{\operatorname{supp}}\, v(t_* + \tau_1, x), \\ &\overline{\operatorname{supp}}\, v(t_* + \tau_1, x) \subset \operatorname{supp} u(t_* + \tau_1, x). \end{aligned}$$

Using again continuity of the solution $v(t, x)$ and of the boundaries of its support, we conclude that there exists a sufficiently small $\tau_2 > 0$, such that

$$u(t_* + \tau_1, x) \geq v(t_* + \tau_1 + \tau_2, x) \text{ in } \mathbf{R}.$$

Therefore by the usual comparison theorem

$$u(t, x) \geq v(t + \tau_2, x) \text{ in } (t_* + \tau_1, T_0) \times \mathbf{R}.$$

Setting here $t = T_0 - \tau_2$, we arrive at the estimate

$$v(T_0, x) \leq u(T_0 - \tau_2, x) \text{ in } \mathbf{R},$$

which, obviously, contradicts (9), since the function in the right-hand side is uniformly bounded in **R**. □

Proposition 2 can be described (with certain caveats) as providing a lower bound for the number of intersections: under the above assumptions $N(t) > 0$ for all $t \in [0, T_0)$. Without any caveats, one such result is presented below.

**Corollary.** *Let $u(t, x)$ and $v(t, x)$ have the same blow-up time $t = T_0$; assume that $u_0(x)$ has compact support and $v_0(x) > 0$ in* **R**. *Then the solutions $u(t, x)$ and $v(t, x)$ intersect for all $t \in [0, T_0)$. Furthermore,*

$$N(t) \geq 2 \text{ in } [0, T_0). \tag{11}$$

*Proof.* Since under these assumptions the solution $u(t, x)$ has compact support, and $v(t, x) > 0$ in $[0, T_0) \times \mathbf{R}$, the estimate (11) follows immediately from (10) if we replace $u(t, x)$ by $v(t, x)$. □

In the sequel we shall have to compare a solution of the problem (1), (2) with exact solutions, which are not defined everywhere in $[0, T_0) \times \mathbf{R}$. Below we formulate an intersection comparison theorem for a solution $u(t, x)$ of the Cauchy problem and a solution $v(t, x)$ of a boundary value problem for equation (1).

Thus, now we shall assume that a generalized solution $v(t, x)$ is defined in some domain of the form $[0, T_0) \times (\eta_1(t), \eta_2(t))$, where $\eta_1(t) < \eta_2(t)$, $v(t, \eta_i(t))$ are continuous functions in $[0, T_0)$, and it is unbounded in the sense that

$$\overline{\lim_{t \to T_0}} \sup_{x \in (\eta_1(t), \eta_2(t))} v(t, x) = \infty.$$

In this context we shall denote by $N(t_0)$ the number of spatial intersections of the solutions $u(t_0, x)$ and $v(t_0, x)$ in the domain $(\eta_1(t_0), \eta_2(t_0))$. We shall take $N(0) < \infty$.

**Proposition 3.** 1) *For any $t_0 \in [0, T_0)$ the number $N(t_0)$ does not exceed the number of changes of sign of the difference $w(t, x) \equiv u(t, x) - v(t, x)$ on the parabolic boundary of the domain $[0, T_0) \times (\eta_1(t), \eta_2(t))$.*

2) *Let the solutions $v(t, x)$ and $u(t, x)$ exist for the same time $T_0$. Then*

$$\begin{aligned} \mathcal{V}^* = \Big\{ t_0 \in (0, T_0) \mid u(t_0, x) \geq v(t_0, x) \text{ in } (\eta_1(t_0), \eta_2(t_0)), \\ \sup_{t \in (t_0, T_0)} v(t, \eta_i(t)) \leq \inf_{t \in (t_0, T_0)} u(t, \eta_i(t)) \text{ for } i = 1, 2 \Big\} = \emptyset. \end{aligned} \tag{12}$$

*Proof.* The first statement is a corollary of the Maximum Principle and, as Proposition 1, is proved by first regularizing.

The proof of statement 2) is also similar to Proposition 2. Indeed, if there exists $t_0 \in \mathcal{V}^+$, $t_0 < T_0$, then, after "small translations in time" of size $\tau_1$ and $\tau_2$, justified by the Strong Maximum Principle and boundary data comparison theorem, we conclude that the solutions $u(t, x)$ and $v(t, x)$ have different blow-up times (that of $v(t, x)$ is somewhat larger). Observe that Proposition 2 is a direct corollary of this more general assertion. □

**Remark.** Since proof of statement 2) is entirely based on the Strong Maximum Principle and boundary data comparison theorem, it will still hold if $v(t, x)$ is a generalized subsolution of equation (1) in the domain $[0, T_0) \times (\eta_1(t), \eta_2(t))$ with blow-up time $T_0$.

We shall start our applications of the intersection comparison theory by analyzing the S blow-up regime.

## 2 Localization for $\beta = \sigma + 1$ (S-regime)

The main localization result is the following claim.

**Theorem 1** (localization in the S-regime). *For $\beta = \sigma + 1$ unbounded solution of the Cauchy problem* (1), (2) *is localized, and if* (3) *holds, we have for all $t \in (0, T_0)$ the estimates*

$$h_+(t) \le h_+(0) + L_S/2, h_-(t) \ge h_-(0) - L_S/2 \tag{13}$$

*and, in particular,*

$$\operatorname{meas} \omega(T_0^-) \le \operatorname{meas} \omega(0) + L_S. \tag{13'}$$

*where $L_S = 2\pi(\sigma + 1)^{1/2}/\sigma$ is the fundamental length of the S-regime.*

Below we shall also prove other theorems, which describe more precisely the penetration depth of the thermal wave for specific initial perturbations.

Actually, the word "unbounded" in the statement of the theorem is superfluous, since, as we showed in § 3, for all $\beta \in (1, \sigma + 3)$, $N = 1$, to any initial function $u_0 \not\equiv 0$ of the Cauchy problem there corresponds a solution that exists only for a finite time.

The estimates (13) mean that in the S-regime the front of a thermal wave can advance a distance which does not exceed half of the fundamental length $L_S$. We stress that the estimates (13) and (13′) are independent of the spatial structure and amplitude of the initial perturbation $u_0(x)$; therefore the length $L_S$ is indeed a fundamental (independent of $u_0$) characteristic of the nonlinear medium.

Proof of Theorem 1 is based on intersection comparison of the solution $u(t, x)$ with a family of exact non-self-similar solutions $u_*(t, x)$ presented in Example 14 in § 3, Ch. 1. Since during the time of existence thermal fronts of such an exact solution travel a distance exactly equal to $L_S/2$, for arbitrary compactly supported solutions the estimates (13) are optimal.

*1 Comparison with an exact self-similar solution*

Proof of Theorem 1 proceeds in two stages. First we shall prove a weaker result, which will serve as a simple and illustrative example of the power of the intersection comparison method applied to an exact self-similar solution.

**Theorem 1′.** *In the conditions of Theorem* 1, *for all* $t \in (0, T_0)$

$$h_+(t) \le h_+(0) + L_S,\; h_-(t) \ge h_-(0) - L_S. \tag{13''}$$

*Proof of Theorem* 1′. In § 1 we presented an example of a simple localized self-similar unbounded solution for $\beta = \sigma + 1$:

$$u_A(t,x) = (T_0 - t)^{-1/\sigma}\theta_S(x),\; 0 < t < T_0,\; x \in \mathbf{R}, \tag{14}$$

where

$$\theta_S(x) = \begin{cases} \left[\frac{2(\sigma+1)}{\sigma(\sigma+2)}\cos^2\left(\frac{\pi x}{L_S}\right)\right]^{1/\sigma}, & |x| < L_S/2, \\ 0, & |x| \ge L_S/2. \end{cases} \tag{15}$$

The support of this solution is constant in time, and its length is $L_S \equiv \text{meas supp}\,\theta_S$. We shall prove the estimate (13″) by intersection comparison with the above self-similar solution existing for the same length of time. Since (14) is a solution in separated variables and has a very simple spatio-temporal structure, this allows us to give an exhaustive graphical illustration of the proof.

Let us denote by $u_S(t, x; x_0, T_0)$ the self-similar solution (14), symmetric with respect to the point $x = x_0$ ((14) is symmetric with respect to $x = 0$). Let us prove the first inequality (13″); the second one is established in a similar manner. Let us set $x_0 = h_+(0) + L_S/2$, and in addition to $u(t, x)$ let us consider a different solution $v(t,x) = u_S(t, x; x_0, T_0)$, localized in the domain $\{|x - x_0| < L_S/2\}$, having the same blow-up time $t = T_0$. The interrelation of graphs of the corresponding initial functions $u_0(x)$ and $v(0,x) \equiv u_S(0, x; x_0, T_0) \equiv T_0^{-1/\sigma}\theta_S(x - x_0)$ is shown in Figure 56. It is clear that they intersect only at the point $x = h_+(0)$, so that $N(0) = 1$. Then by Proposition 1

$$N(t) \le N(0) = 1 \text{ for all } t \in (0, T_0). \tag{16}$$

Let us prove that $h_+(t) \le h_+(0) + L_S$, i.e., that the thermal wave cannot advance beyond the right front point of the solution $v(t,x)$. Let us assume the contrary. Let $t^* = \sup\{t > 0 \mid h_+(t) \le h_+(0) + L_S\} < T_0$, that is, $u(t, x_*) > 0$ for all $t \in (t^*, T_0)$

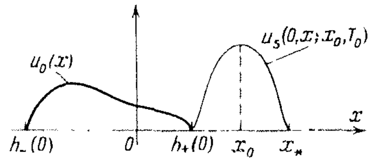


**Fig. 56.** $N(0) = 1$; initial functions have a unique intersection at the point $x = h_+(0)$

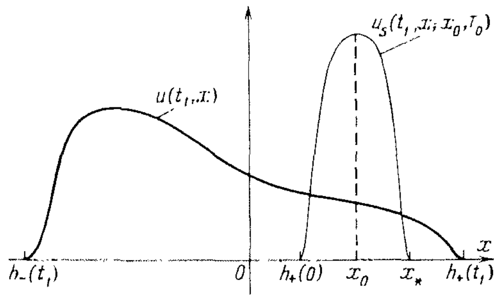


**Fig. 57.** $N(t_1) = 2$; such a situation contradicts Proposition 1

at the point $x_* = h_+(0) + L_S$. Then there exist two ways the front point $x = h_+(t)$ can move past the right front point $x \equiv x_*$ of the self-similar solution $u_S$.

The first of these corresponds to the case $N(t^*) = 1$. Then, since, as is well known, the moving front of the solution $u(t,x)$ cannot stop, for any arbitrarily small $\tau > 0$ there exists $t_1 \in (t^*, t^* + \tau)$ such that $N(t_1) \geq 2$. This situation is shown in Figure 57, where $N(t_1) = 2$. Indeed, by continuity of the solutions, under a small shift in time the intersection that existed for $t = t^*$ persists, and at least one "new" one is created to the left of the point $x = x_*$ due to the motion of the right front of the solution $u(t,x)$. Hence $N(t_1) \geq 2$. In other words, during the evolution the number of intersections of $u(t,x)$ and $v(t,x)$ increases, which is forbidden by (16) (and contradicts the Maximum Principle).

In accordance with (16), the only other way of violating the first bound in (13″) is by having $N(t^*) = 0$. Then by Proposition 1 we have $N(t) \leq N(t^*) = 0$ for all $t \in (t^*, T_0)$, and therefore by the usual comparison theorem with respect to initial functions, at any moment of time $t = t_2 \in (t^*, T_0)$ we must have the situation as in Figure 58. (The case of $u_S(t_2, x; x_0, T_0)$ being tangent "from inside" to the spatial profile of $u(t_2, x)$ is ruled out by the Strong Maximum Principle applied to equation (8) for the difference of these solutions in the domain of uniform parabolicity, where the solutions are uniformly separated from zero.) Clearly, such a configuration of the graphs of the functions $u(t_2, x)$ and $u_S(t_2, x; x_0, T_0)$ contradicts the condition of them having the same blow-up time.

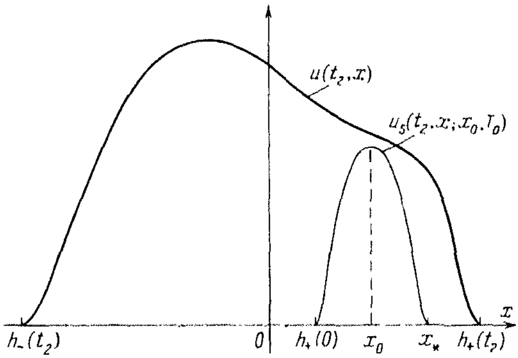


**Fig. 58.** $N(t_2) = 0$; such a situation is prohibited by Proposition 2

Thus, we have considered the two cases: $N(t^*) = 1$ (which leads to Figure 57) and $N(t^*) = 0$ (see Figure 58). By (16) there are no other possibilities. Therefore $t^* = T_0$, which concludes the proof of Theorem 1′. □

Next we are going to exploit similar ideas of the intersection comparison method applied to the self-similar solution (14), (15) to study in more detail the dependence of the character of the motion of the front of a thermal wave in the S-regime on the spatial structure of the initial perturbation $u_0(x)$.

*2 Condition of time-independence of the support of an unbounded solution*

The support (localization domain) of the self-similar solution (14), (15) does not change during the time of existence of the solution $t \in (0, T_0)$. Let us show that in addition there are many other (non-self-similar) solutions, that are localized in the domain $\operatorname{supp} u_0(x)$ of their positivity at the initial moment of time.

**Theorem 2.** *Assume that* $\beta = \sigma + 1$, *conditions* (3) *are satisfied and* $\operatorname{meas}(\operatorname{supp} u_0) > L_S$. *Let the initial function* $u_0(x)$ *satisfy the following condition:*

*there exists a constant* $\lambda_0 > 0$, *such that*
$u_S(0, x; x_0, \lambda_0) \le u_0(x)$ *in* $\mathbf{R}$, *where* $x_0 = h_+(0) - L_S/2$,
*while the functions* $u_0(x)$ *and* $u_S(0, x; x_0, \lambda)$ *have exactly*
*one intersection point for all* $\lambda \in (0, \lambda_0)$

*(this situation is shown in Figure 59). Then*

$$h_+(t) = h_+(0) \text{ for all } t \in (0, T_0).$$

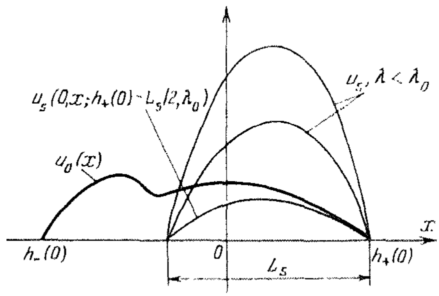


**Fig. 59.**

*Proof.* Let us denote by $N(t;\lambda)$ the number of spatial intersections of the solutions $u(t,x)$ and $u_S(t,x;x_0,\lambda)$. From the condition of the theorem it follows that $T_0 \le \lambda_0$. If $T_0 = \lambda_0$, that is, if $N(0;\lambda_0) = 0$, then the hypothesis concerning the motion of the right front leads to a contradiction with Proposition 2 (here obtains the situation of Figure 58). Therefore we have to consider the case $T_0 < \lambda_0$. Let us fix $\lambda = T_0$ and consider the solution $u_S(t,x;x_0,\lambda)$, which has by construction blow-up time $T_0$. Then for $t = 0$ there is a unique intersection point of the initial functions $u_0(x)$ and $u_S(0,x;x_0,\lambda) \equiv \lambda^{-1/\sigma}\theta_S(x-x_0)$, i.e., $N(0;\lambda) = 1$. Arguing now as in the proof of Theorem 1′, we conclude that there are two possible scenarios for the motion of the right front of the solution $u(t,x)$. According to the first of these ($N(t^*) = 1$), we arrive at the configuration of spatial profiles as in Figure 57 (which contradicts Proposition 1). The second scenario ($N(t^*) = 0$)leads to Figure 58, that is, to a contradiction with Proposition 2. Hence the front of the solution $u(t,x)$ must be perfectly immobile, which concludes the proof. □

**Remark.** It is of interest that for the right front point of the solution to be immobile throughout all the time of its existence, we need a non-local condition on the behaviour of the initial function $u_0(x)$ in an $L_S$-neighbourhood $(h_+(0)-L_S, h_+(0))$ of the front point $x = h_+(0)$. If the condition of Theorem 2 holds, the behaviour of $u_0(x)$ in the rest of the space, $\{x \le h_+(0) - L_S\}$, has no influence on the immobility of the right front of the solution. Formally, the initial function can go to any large value as $x \to -\infty$ (as long as a local (in time) solution exists). This again emphasizes the universality of the fundamental length $L_S$ characteristic of a nonlinear medium, which here plays the part of a kind of effective radius of influence of thermal perturbations.

Imposing similar conditions on the behaviour of $u_0(x)$ in a neighbourhood of the left front point, we obtain a set of initial perturbations $u_0(x)$, which generate unbounded solutions with a constant support. It is easy to show that this set $\{u_0\}$

is quite large and contains functions other than the initial functions corresponding to the self-similar solutions (14).

*3 Condition for localization on the fundamental length $L_S$*

Let us show that under certain conditions an initial perturbation with a small support (meas(supp $u_0$) $< L_S$) cannot propagate beyond the domain $\{|x| < L_S/2\}$ in finite time of existence of the solution. Here we shall assume that in addition to (3) we also have the conditions

$$u_0(-x) = u_0(x),\ x \in \mathbf{R};\ u_0(x) \text{ is non-increasing for } x > 0. \tag{17}$$

Under these conditions, due to uniqueness of the solution and the Maximum Principle we have that $u = u(t, |x|)$, $u_x(t, x) \leq 0$ for $x \in ]0, h_+(t))$ and $\sup_x u(t, x) \equiv u(t, 0)$.

**Theorem 3.** *Assume that* $\beta = \sigma + 1$, *conditions* (3), (17) *hold and* meas(supp $u_0$) $< L_S$. *Let* $u_0(x)$ *also satisfy the condition*

*there exists* $\lambda_0 > 0$, *such that* $u_S(0, x; 0, \lambda_0) \geqslant u_0(x)$ *in* $\mathbf{R}$,
*while the functions* $u_0(x)$ *and* $u_S(0, x; 0, \lambda)$
*intersect precisely at two points for all* $\lambda > \lambda_0$

*(see Figure 60). Then*

$$|h_+(t)| \leq L_S/2 \text{ for all } t \in (0, T_0)$$

*and in particular*

$$\text{meas}\ \omega(T_0) \leq L_S. \tag{18}$$

*Proof.* By Proposition 2 $T_0 > \lambda_0$. Setting $\lambda = T_0$ and denoting by $N_+(t; \lambda)$ the number of spatial intersections of the solutions $u(t, x)$ and $u_S(t, x; 0, \lambda)$ in the domain $\{x > 0\}$, we obtain $N_+(0; \lambda) = 1$. Then by Proposition 1, in view of the condition $u = u(t, |x|)$, we have that $N_+(t; \lambda) \leq 1$ for all $t \in (0, T_0)$. Therefore we can now use the method of proof of previous theorems. □

Thus in the conditions of Theorem 3 the thermal wave can move in any direction, but the total distance covered by the thermal perturbations up to the blow-up time cannot exceed

$$L_S - \text{meas}\ \omega(0) < L_S. \tag{19}$$

*4 Comparison with a family of exact non-self-similar solutions. Proof of Theorem 1*

We move on now to prove the optimal bounds (13). As we already mentioned, the proof is based on intersection comparison with the more complex solution $u_*(t, x)$

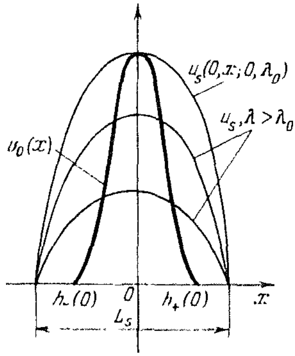


**Fig. 60.**

of equation (1), presented in § 3, Ch. I. It has the form

$$u_*(t,x) = \{\phi(t)\,[\psi(t) + \cos(2\pi x/L_S)]_+\}^{1/\sigma} \geq 0 \tag{20}$$

for $x \in (-L_S/2, L_S/2)$ and $u_*(t,x) = 0$ for $x \in \mathbf{R}\setminus(-L_S/2, L_S/2)$. The function $\psi(t)$ is determined from the equation

$$\psi' = \sigma(\sigma+1)^{-1}C_0\left[1-\psi^2\right]^{-\sigma/2},\ t > 0;\ \psi(0) = -1, \tag{21}$$

and $\phi(t) = C_0[1-\psi^2(t)]^{-(\sigma+2)/2}$, where

$$C_0 = C_0(T_0) = (\sigma+1)\sigma^{-1}T_0^{-1}B(1+\sigma/2, 1/2). \tag{22}$$

Then $\psi(t)$ is defined on $(0, T_0]$, and $\psi(T_0) = 1$ (hence $\phi(T_0) = +\infty$), so that (20) is an unbounded solution with blow-up time $T_0$. The generalized solution $u_*(t,x)$, which is symmetric with respect to $x = 0$, has compact support, and its right front is located at the point

$$h_+^*(t) = g(t) \equiv (L_S/2\pi)[\pi/2 + \arcsin\psi(t)]. \tag{23}$$

Clearly, $h_+^*(t)$ is a strictly increasing function and $h_+^*(t) < L_S/2$ for all $t \in (0, T_0)$, so that the unbounded solution (20) is localized in the domain $\{|x| < L_S/2\}$.

In § 3, Ch. I we showed that $u_*(t,x)$ satisfies the singular initial condition $u_*(0,x) = E_0\delta(x)$, $E_0 > 0$ is a constant, and $\operatorname{supp} u_*(t,x) \to \{0\}$ as $t \to 0$. Therefore for intersection comparison we shall in the following take functions of the form $u_*(t+\epsilon, x)$ (the constant $\epsilon > 0$ is taken to be sufficiently small), to which correspond regular bounded initial functions $u_*(\epsilon, x)$.

In view of the fact that the support of this exact solution (unlike the self-similar one) varies with time, we shall modify somewhat the proof. We shall exhibit new facets of the intersection comparison method by comparing not with a single fixed solution, but in fact with a continuously parametrized family of exact solutions, all having the same blow-up time.

For a fixed $\epsilon > 0$, $\delta \in \mathbf{R}$, let us denote by $v(t, x) \equiv v(t, x; \epsilon, \delta)$ the function $u_*(t + \epsilon, x - (h_+(0) + \delta))$, where $C_0 = C_0(T_0 + \epsilon)$. Then $v(t, x)$ is an unbounded solution of the Cauchy problem (1), (2) with initial function $u_0 = u_*(\epsilon, x - (h_+(0) + \delta))$ and support $\operatorname{supp} v(t, x) \equiv \{|x - (h_+(0) + \delta)| < g(t + \epsilon)\}$. Let us note that the function $u_*(t + \epsilon, x - (h_+(0) + \delta))$ is continuous in $\epsilon, \delta$ in $[0, T_0) \times \mathbf{R}$. By construction, for all $\delta \in \mathbf{R}$ the solutions $u(t, x)$ and $v(t, x; \epsilon, \delta)$ have the same blow-up time $T_0$. For any $t \in [0, T_0)$ let us denote by $N(t; \epsilon, \delta)$ the number of spatial intersections in $\mathbf{R}$ of the functions $u(t, x)$ and $v(t, x)$.

It is not hard to check that for any $\epsilon > 0$ and $\delta \geq \delta_\epsilon = g(\epsilon)$ supports of the initial functions $u_0(x)$ and $v(0, x)$ do not intersect, so that $N(0; \epsilon, \delta) = 1$. Then by Proposition 1 we have that

$$N(t; \epsilon, \delta) \leq 1 \text{ for all } t \in (0, T_0). \tag{24}$$

Thus, let us fix sufficiently small $\epsilon > 0$ and $\delta = 2\delta_\epsilon$; then $\operatorname{supp} v(t, x; \epsilon, 2\delta_\epsilon) = \{|x - l_\epsilon| < g(t + \epsilon)\}$, where $l_\epsilon = h_+(0) + 2\delta_\epsilon$. Obviously, $l_\epsilon \to h_+(0)$ as $\epsilon \to 0$. Let us show that $h_+(t) \leq l_\epsilon + g(t + \epsilon)$ in $[0, T_0)$. Assume that this is not the case and

$$\begin{gathered} t_* = \sup\{t \in [0, T_0) \mid h_+(t') \leq l_\epsilon + g(t' + \epsilon) \\ \text{for all } t' \in [0, t]\} < T_0. \end{gathered} \tag{25}$$

Clearly, $h_+(t_*) = l_\epsilon + g(t_* + \epsilon) \equiv x_*$. By Proposition 1, two cases are possible.

*Case 1:* $N(t_*; \epsilon, 2\delta_\epsilon) = 1$. Here we shall arrive at a contradiction similar to the one in Figure 57, where $u_S(\cdot)$ should be replaced by the solution $u_*(\cdot)$. In this case the difference $w(t_*, x) \equiv u(t_*, x) - v(t_*, x)$ changes sign in $\mathbf{R}$ exactly once. Then we can find $-\infty < x_2 < x_1 < x_* = h_+(t_*)$, such that either

$$u(t_*, x_1) < v(t_*, x_1; \epsilon, 2\delta_\epsilon), \; u(t_*, x_2) > v(t_*, x_2; \epsilon, 2\delta_\epsilon), \tag{26}$$

or, on the contrary,

$$u(t_*, x_1) > v(t_*, x_1; \epsilon, 2\delta_\epsilon), \; u(t_*, x_2) < v(t_*, x_2; \epsilon, 2\delta_\epsilon). \tag{27}$$

If (26) is satisfied, then choosing $\delta_1 \in (\delta_\epsilon, 2\delta_\epsilon)$, $\delta_1 \sim 2\delta_2$, we have that by continuity of the function $v(t, x; \epsilon, \delta)$ in $\delta$, inequalities (26) would still hold if the function $v(t_*, x; \epsilon, 2\delta_\epsilon)$ is replaced by $v(t_*, x; \epsilon, \delta_1)$ and, furthermore $v(t_*, x'; \epsilon, \delta_1) = 0 < u(t_*, x')$ at the point $x' = |x_* - (2\delta_\epsilon - \delta_1)| \in (x_1, x_*)$. Therefore $N(t_*; \epsilon, \delta_1) \geq 2$, which contradicts Proposition 1. If, on the other hand, we have (27), then the same contradiction is obtained by comparing the functions $v(t_*, x; \epsilon, \delta_1)$ and $u_*(t, x)$ for $\delta_1 > 2\delta_\epsilon$, with $\delta_1 - 2\delta_\epsilon$ sufficiently small.

*Case 2:* $N(t_*;\epsilon, 2\delta_\epsilon) = 0$. We shall show that this case leads to the situation of Figure 58 ($u_S(\cdot)$ is replaced by $u_*(\cdot)$). Then either $u(t_*, x) \le v(t_*, x)$ in $\mathbf{R}$ (but then by the usual comparison theorem $u(t,x) \le v(t,x)$, $\operatorname{supp} u(t,x) \subseteq \operatorname{supp} v(t,x)$ in $[t_*, T_0) \times \mathbf{R}$, which contradicts (25)), or $u(t_*, x) \ge v(t_*, x)$ in $\mathbf{R}$. In the latter case by (25) there exists $t_2 \in (t_*, T_0)$, such that $h_+(t_2) > l_\epsilon + g(t_2+\epsilon)$ and in addition obviously $u(t_2, x) \ge v(t_2, x)$ in $\mathbf{R}$. But then we can find $\delta_1 \in ]0, h_+(t_2) - (l_\epsilon + g(t_2+\epsilon)))$, such that $\overline{\operatorname{supp}}\, v(t_2, x;\epsilon, \delta_1) \subseteq \operatorname{supp} u(t_2, x)$ and $v(t_2, x;\epsilon,\delta_1) \le u(t_2, x)$ for all $x \in \mathbf{R}$. This contradicts Proposition 2.

Thus, $h_+(t) \le l_\epsilon + g(t+\epsilon)$ in $(0, T_0)$ for any arbitrarily small $\epsilon > 0$. Passing in this inequality to the limit $\epsilon \to 0$ (then $l_\epsilon \to h_+(0)$, $g(t+\epsilon) \to g(t)$), we obtain the first of the bounds (13), which completes the proof of Theorem 1. □

Let us note that the above argument proves a sharper optimal time-dependent upper bound for the motion of the front.

**Corollary.** *In the conditions of Theorem 1*

$$h_+(t) \le h_+(0) + (L_S/2\pi)[\pi/2 + \arcsin \psi(t)], \quad t \in [0, T_0). \tag{28}$$

Clearly, by (23), for the solution $u_*(t,x)$ instead of the inequality we have in (28) an exact equality. Since the initial function for the solution $u_*(t,x)$ is singular, the estimate (28) describes, in particular, the maximal speed of motion of the thermal front for small $t > 0$. It is not hard to compute from (28) that

$$h_+(t) \le h_+(0) + b_0^{1/2} t^{1/(\sigma+2)}(1 + o(1)) \text{ as } t \to 0,$$

where

$$b_0 = (\sigma+1)\sigma^{-2}(\sigma+2)^{2/(\sigma+2)} T_0^{-2/(\sigma+2)} [B(1+\sigma/2, 1/2)]^{2/(\sigma+2)}.$$

## 3 Localization for $\beta > \sigma + 1$ (LS-regime)

The main assertion concerning localization in the case of the LS blow-up regimes has the following form:

**Theorem 4** (localization in the LS-regime). *Let $\beta > \sigma + 1$. Then an unbounded solution of the problem* (1), (2) *having blow-up time $T_0 = T_0(u_0) < \infty$ is localized and*

$$h_+(T_0^-) \le h_+(0) + \xi^* T_0^m, \; h_-(T_0^-) \ge h_-(0) - \xi^* T_0^m, \tag{29}$$

*i.e.,*

$$\operatorname{meas} \omega(T_0^-) \le \operatorname{meas} \omega(0) + 2\xi^* T_0^m < \infty, \tag{30}$$

*where $m = [\beta - (\sigma+1)]/[2(\beta-1)] > 0$ and $\xi^* > 0$ is a constant that depends only on $\sigma, \beta$.*

**Remark.** For $\beta = \sigma + 1$ we have $m = 0$, and as will be seen in the following, $\xi^* = L_S$, i.e., in the case $\beta = \sigma + 1$ this theorem becomes Theorem 1′.

Let us stress that unlike the S-regime (Theorem 1) the "fundamental" length of the LS-regime $L_{LS} = \operatorname{meas} \omega(T_0^-)$ depends, via its dependence on $T_0$, on the initial function. An upper bound on the blow-up time $T_0 = T_0(u_0)$ in this problem was obtained in § 3.

*1 Construction of a self-similar subsolution*

In § 1 it was shown that for $\beta > \sigma + 1$ equation (1) has no localized self-similar solutions. All the self-similar solutions constructed there are strictly positive and are only effectively localized ($u_S(t, x)$ grow without bound as $t \to T_0^-$ only at the point $x = 0$ and remain uniformly in $t$ bounded in $\mathbf{R}\setminus\{0\}$).

However, it is easy to verify that Proposition 2 still holds, if as the second solution $v(t, x)$ we take some unbounded subsolution of equation (1). We shall construct such a subsolution for the LS blow-up regime, and, since it is not defined in $(0, T_0) \times \mathbf{R}$, we will in fact be using as our main intersection comparison theorem Proposition 3, which is specifically suited to deal with this case. We shall seek the self-similar localized subsolution in the usual form:

$$u_S^-(t, x) = (T_0 - t)^{-1/(\beta-1)}\theta_-(\xi), \quad \xi = x/(T_0 - t)^m, \tag{31}$$

where the function $\theta_-(\xi) \geq 0$ satisfies almost everywhere in $\mathbf{R}$ the equation

$$(\theta_-^\sigma \theta_-')' - m\theta_-'\xi - \frac{1}{\beta - 1}\theta_- + \theta_-^\beta = 0. \tag{32}$$

**Lemma 1.** *For any $\beta > \sigma + 1$ there exists a non-trivial solution $\theta_-(\xi)$ satisfying* (32) *on an interval*[6] $(-\xi^*, 0)$, $\xi^* > 0$, *as well as the conditions*

$$\theta_-(0) = 0, \ (\theta_-^\sigma \theta_-')(0) = 0, \tag{33}$$

$\theta_-(\xi) > 0$ *on* $(-\xi^*, 0)$ *and* $\theta_-(\xi^*) = 0$.

From (33) it follows that the function $\theta_-(\xi)$, which is the same as the function of Lemma 1 for $\xi \in (-\xi^*, 0]$ and zero for $\xi > 0$, is a generalized solution of equation (32) on $(-\xi^*, \infty)$. Then (31) is a generalized unbounded solution of equation (1) in the domain $(0, T_0) \times (x_*(t), \infty)$ with a moving left boundary $x_*(t) = -\xi^*(T_0 - t)^m$, on which $u_S^-(t, x_*(t)) = 0$. Hence we have that (31) is a localized solution: even though $u_S^-(t, x)$ grows without bound in any left neighbourhood of the point $x = 0$ as $t \to T_0^-$, the front of the solution $x_f(t) \equiv 0$ is immobile, and perturbations do not penetrate into the domain $x > 0$ (see Figure 61).

[6] Here the constant $\xi^* = \xi^*(\sigma, \beta)$ is the same as in the statement of Theorem 4.

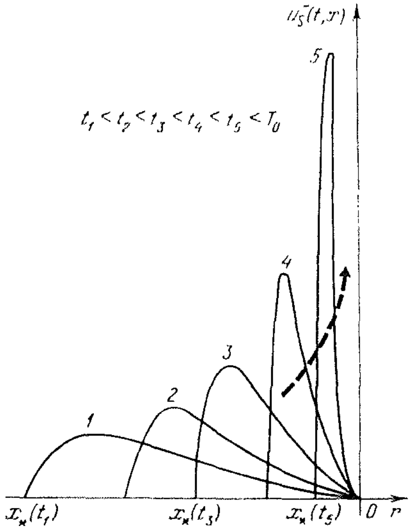


**Fig. 61.** Localized subsolution (31) for $\beta > \sigma + 1$ at different times

If now we set $\theta_-(\xi) \equiv 0$ for $\xi \leq -\xi^*$, then $u_S^-(t, x)$ will be an unbounded subsolution of equation (1) in $(0, T_0) \times \mathbf{R}$.

*Proof of Lemma* 1. It is not hard to demonstrate local solvability of the problem (32), (33) for small $|\xi|$ by reducing it to an equivalent integral equation and using the Schauder fixed point theorem. The property of the solution extended into the domain of $\xi < 0$, alluded to in the lemma, follows immediately from the results of § 1 (see (29)) for $\beta > \sigma + 1$, $N = 1$. □

## 2 *Proof of Theorem 4*

Let us denote by $u_{LS}^-(t, x; x_0, T_0)$ the function which coincides for $(0, T_0) \times \{-\xi^*(T_0 - t)^m < x - x_0 < 0\}$ with (31) ($\theta_-(\xi) \geq 0$ is as in Lemma 1 and $u_{LS}^- \equiv 0$ outside that domain). As we already mentioned $u_{LS}^-$ is an unbounded subsolution of the Cauchy problem[7] in $(0, T_0) \times \mathbf{R}$, i.e., if $u_0(x) \geq u_{LS}^-(0, x; x_0, T_0)$ in $\mathbf{R}$ then $u(t, x) \geq u_{LS}^-(t, x; x_0, T_0)$ in $\mathbf{R}$ for all admissible $t > 0$. Therefore Proposition 3 remains valid if as the function $v(t, x)$ we take $u_{LS}^-(t, x; x_0, T_0)$ (or any other subsolution of a similar form). Let us note that at the same time the function $u_{LS}^-$ satisfies the equation in a generalized sense in the domain

[7]Note that this fact is useful in deriving conditions for global insolvability of boundary value problems for (1).

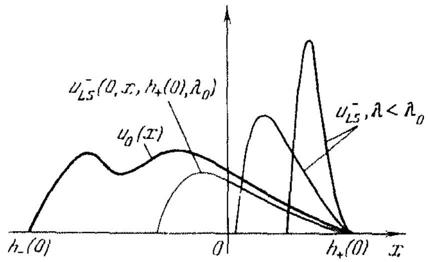


**Fig. 62.**

$(0, T_0) \times \{\mathbf{R}\backslash\{x = x_0 - \xi^*(T_0 - t)^m\}\}$. All this allows us to compare the solution $u(t, x)$ with $u^-_{LS}$ as was done in the previous subsection. In brief, it goes like this.

Let us prove localization on the right. Set $x_0 = h_+(0) + \xi^* T_0^m$, $x_*(t) = x_0 - \xi^*(T_0 - t)^m$ ($T_0$ is the blow-up time of $u(t, x)$). Then $u_0(x)$ and $u^-_{LS}(0, x; x_0, T_0)$ do not intersect in $(x_*(0), \infty)$, and, clearly, in the course of the evolution there can be only one intersection of $u(t, x)$ and $u_{LS}(t, x; x_0, T_0)$ in $(x_*(t), \infty)$ (if $u \gg u^-_{LS}$ for $x = x_*(t)$). Therefore $u(t, x)$ will become larger than $v \equiv u^-_{LS}(t, x; x_0, T_0)$ at the point $x = x_0$ (there $v \equiv 0$) only after it becomes at least as large as $v(t, x)$ for all $x < x_0$. However, by Proposition 3 it contradicts the fact that the solution $u$ and the subsolution $v$ have the same blow-up time. Therefore $h_+(t) \leq x_0$, which is the same as (29). □

In conclusion, we present a result for the LS-regime, which is similar to Theorem 2 in statement and method of proof.

*3 Condition of immobility of front points of an unbounded solution*

**Theorem 5.** *Assume that $\beta > \sigma + 1$ and let $u_0(x)$ also satisfy the condition*

*there exists $\lambda_0 > 0$, such that $u_0(x) \geq u^-_{LS}(0, x; h_+(0), \lambda_0)$ in* $\mathbf{R}$,
*while the functions $u_0(x)$ and $u^-_{LS}(0, x; h_+(0), \lambda)$*
*intersect precisely at one point for all $0 < \lambda < \lambda_0$.*

*Then*

$$h_+(t) \equiv h_+(0) \text{ for all } t \in (0, T_0),$$

*where $T_0 < \infty$ is the blow-up time of the solution $u(t, x)$.*

A graphical interpretation of the condition of the theorem is presented in Figure 62. In the LS-regime the length of the part of the support $\operatorname{supp} u_0(x)$, which, in accordance with the condition of the theorem, influences immobility of the front point $x = h_+(t) \equiv h_+(0)$ is $\xi^* T_0^m$. Unlike the S-regime case, this length depends on the behaviour of the initial perturbation $u_0(x)$ (via $T_0(u_0)$) on practically the whole space.

## 4 Non-localized unbounded solutions of the HS-regime, $1 < \beta < \sigma + 1$

Absence of localization of solutions of the Cauchy problem in this case (all nontrivial solutions are unbounded; see § 3) is easily proved by the method of stationary states, which is presented in Ch. VII. It is used there to study the localization phenomenon in arbitrary nonlinear media.

However, for equation (1) sharper results can be obtained by comparing the solution $u(t, x)$ with a self-similar solution of the HS-regime, which was constructed in § 1.

**Theorem 6** (absence of localization in the HS-regime). *Let $1 < \beta < \sigma + 1$, and let condition (3) be satisfied. Then the unbounded solution of the Cauchy problem (1), (2) is not localized, and if $t = T_0$ is the blow-up time, we have the estimates*

$$\begin{aligned} h_+(t) &\geq h_-(0) + \xi_0[(T_0 - t)^m - T_0^m], \\ h_-(t) &\leq h_+(0) - \xi_0[(T_0 - t)^m - T_0^m], \quad t \in (0, T_0). \end{aligned} \tag{34}$$

*where $m = [\beta - (\sigma + 1)]/[2(\beta - 1)] < 0$ and therefore $|h_\pm(t)| \to \infty$ as $t \to T_0^-$. The constant $\xi_0 > 0$ in (34) depends only on $\sigma, \beta$.*

Inequalities (34) mean that as $t \to T_0^-$

$$\operatorname{meas} \omega(t) \equiv h_+(t) - h_-(t) \geq \operatorname{meas} \omega(0) + 2\xi_0[(T_0 - t)^m - T_0^m] \to \infty. \tag{35}$$

*Proof.* It is similar to the proofs of previous theorems; there is a direct connection, for example, with the proof of Theorem 1. Let us write down the unbounded self-similar solution of equation (1) for $\beta < \sigma + 1$ (see § 1):

$$u_S(t, x) = (T_0 - t)^{-1/(\beta-1)}\theta_S(\xi), \quad \xi = x/(T_0 - t)^m \in \mathbf{R}. \tag{36}$$

The function $\theta_S(\xi)$ has compact support: $\operatorname{meas} \operatorname{supp} \theta_S(\xi) = 2\xi_0 < \infty$, where the constant $\xi_0$ is as in the right-hand sides of (34). As before, let us denote by $u_{HS}(t, x; x_0, T_0)$ the self-similar solution (36) symmetric in $x$ with respect to the point $x = x_0$. Let us, for example, sketch the proof of the second bound in (34). Let us place $u_{HS}(0, x; x_0, T_0)$ relative to the initial function $u_0(x)$ as in Figure 56 (replace $u_S$ in that figure by $u_{HS}$). For that we have to set $x_0 = h_+(0) + \xi_0 T_0^m$.

The relative position of these two functions is, in principle, the same as in the proof of Theorem 1′. However, the general structure of the proof is somewhat different due to the nature of the bounds in (34) (they are lower, not upper bounds as in all the other theorems).

With these positions of the supports of $u_0$ and $u_{HS}(0, x; x_0, T_0)$ ($\operatorname{supp} u_0 \cap \operatorname{supp} u_{HS}(0, x; x_0, T_0) = \emptyset$), the number of their intersections in $\mathbf{R}$ is equal to one. As the solutions evolve, this number cannot increase. Therefore the number of intersections of different solutions $u(t, x)$ and $u_{HS}(t, x; x_0, T_0)$ having the same blow-up time $t = T_0(u_0) < \infty$ does not exceed one for all $t \in [0, T_0)$. This means that the left front of the self-similar solution $u_{HS}$, which for all $t \in [0, T_0)$ is at the point

$$x_f^-(t) = x_0 - \xi_0(T_0 - t)^m \equiv h_+(0) - \xi_0[(T_0 - t)^m - T_0^m]. \tag{37}$$

cannot overtake the left front point $x = h_-(t)$ of the solution $u(t, x)$. Were that to happen, then either at some moment of time $t = t_1$ we would have two intersections of $u(t, x)$ and $u_{HS}(t, x; x_0, T_0)$, which contradicts the Maximum Principle, or for some $t_2 \in (0, T_0)$ we would have a situation precluded by Proposition 2. Therefore $h_-(t) \leq x_f^-(t)$. Taking into account (37), we obtain the second bound of (34), which concludes the proof. □

Therefore for any compactly supported initial perturbation, the fronts of the thermal wave in the HS-regime move as $t \to T_0^-(u_0)$ not slower than at the self-similar rate

$$|h_\pm(t)| \gtrsim \xi_0(T_0 - t)^{[\beta-(\sigma+1)]/[2(\beta-1)]}, \quad t \to T_0^-. \tag{38}$$

In the next section we shall show that as $t \to T_0^-$ the motion of the front points $x = h_\pm(t)$ approaches asymptotically the self-similar one, that is, in addition to the inequality (38) we also have the reverse one.

As far as equivalence of conditions $\{h_\pm(t) \to \infty, t \to T_0\}$ and $\{u(t, x) \to \infty \text{ in } \mathbf{R}, t \to T_0^-\}$ for $1 < \beta < \sigma + 1$ is concerned, we shall prove assertions of that sort in § 1, Ch. VII. For example, it is especially easy to prove

**Theorem 7.** *Let $1 < \beta < \sigma + 1$ and assume that conditions (3), (17) hold. Then $u(t, x) \to \infty$ in $\mathbf{R}$ as $t \to T_0^-(u_0)$.*

## § 5 Asymptotic stability of unbounded self-similar solutions

We have already discussed above certain essential difficulties in the analysis of the spatio-temporal structure of unbounded (singular in time) solutions, which always arise when solutions are unstable with respect to small perturbation of the initial function. Therefore in this section we shall not strive for maximal generality of presentation, which would entail exerting a great amount of effort in overcoming complications that are not of any particular importance. We shall use the example of the Cauchy problem in $N = 1$ to present the crucial stages of the proof.

First, to simplify the presentation below, we state in a compact form the methods of comparison of unbounded solutions we used in § 4 in the study of the Cauchy problem

$$u_t = (u^\sigma u_x)_x + u^\beta, \ t > 0, x \in \mathbf{R}; \ \sigma > 0, \beta > 1, \tag{1}$$

$$u(0, x) = u_0(x) \geq 0, \ x \in \mathbf{R}; \ u_0 \in C(\mathbf{R}), \tag{2}$$

where $\operatorname{supp} u_0 = (h_-(0), h_+(0))$, $u_0^\sigma$ is Lipschitz continuous in $\mathbf{R}$; $\operatorname{supp} u(t, x) = (h_-(t), h_+(t))$.

## 1 A lower bound for the amplitude of the unbounded solution

This is the simplest corollary of the comparison theorem.

**Theorem 1.** *Let $\sigma \geq 0$, $\beta > 1$, and let $u(t, x)$ be an unbounded solution of the Cauchy problem* (1), (2). *Then*

$$\sup_x u(t, x) > \theta_H (T_0 - t)^{-1/(\beta-1)}, \ t \in [0, T_0); \ \theta_H = (\beta - 1)^{-1/(\beta-1)}, \tag{3}$$

*where $T_0 = T_0(u_0)$ is the blow-up time.*

*Proof.* The estimate (3) follows from the corollary to Proposition 2 in § 4, if we take as $v(t, x)$ a spatially homogeneous solution of equation (1) with the same blow-up time:

$$v(t) = \theta_H (T_0 - t)^{-1/(\beta-1)}, \ t \in [0, T_0). \tag{4}$$

Then we have that $u(t, x)$ and $v(t)$ must intersect for each $t \in [0, T_0)$. Moreover, each intersection point in the domain of strict positivity of both solutions is an isolated point. For the number of intersections we have

$$N(t) \geq 2 \text{ for all } t \in [0, T_0). \tag{5}$$

Obviously, this means that

$$\sup_{x \in \mathbf{R}} u(t, x) > v(t), \ t \in [0, T_0), \tag{6}$$

from which (3) follows. □

Let us note that this estimate can be obtained directly from equation (1).

## 2 Similarity transformation. Restrictions on the form of initial functions

Self-similar solutions of equation (1) have the form

$$u_S(t,x) = (T_0 - t)^{-1/(\beta-1)}\theta_S(\xi),\ \xi = x/(T_0 - t)^m \in \mathbf{R}, \tag{7}$$

where the function $\theta_S(\xi) \geq 0$ satisfies the ordinary differential equation

$$(\theta_S^\sigma \theta_S')' - m\theta_S'\xi - \frac{1}{\beta - 1}\theta_S + \theta_S^\beta = 0,\ \xi \in \mathbf{R}. \tag{8}$$

In § 1 we found that for all $\sigma > 0$, $\beta > 1$ it has an even solution $\theta_S(\xi)$, which is non-increasing for $\xi \geq 0$. We shall study asymptotic stability of precisely these solutions (for $\beta \geq \sigma + 1$ there are other, non-monotone solutions $\theta_S(\xi)$).

Consequently, we shall introduce the following restrictions on the compactly supported initial perturbation:

$$u_0(-x) = u_0(x),\ x \in \mathbf{R};\ \text{meas supp}\, u_0 = 2l_0 < \infty, \tag{9}$$

$$u_0(x) \text{ is non-increasing for } x > 0. \tag{10}$$

Then $u(t,x)$, an unbounded solution of problem (1), (2) is even in $x$, non-increasing in $x$ for $x \geq 0$, and $\sup_x u(t,x) = u(t,0)$ for any $t \in (0, T_0(u_0))$.

Corresponding to (7), let us introduce the similarity representation $\theta(t,\xi)$ of the solution of problem (1), (2):

$$\theta(t,\xi) = (T_0 - t)^{1/(\beta-1)}u(t, \xi(T_0 - t)^m),\ t \in (0, T_0), \xi \in \mathbf{R}, \tag{11}$$

where $m = [\beta - (\sigma + 1)]/[2(\beta - 1)]$. Similarity transformation of the solution (7) gives us exactly the function $\theta_S(\xi)$.

We shall be interested in the behaviour of $\theta(t,\xi)$ as $t \to T_0^-$. Asymptotic stability of the self-similar solution (7) means that

$$\theta(t,\xi) \to \theta_S(\xi), t \to T_0^-(u_0), \tag{12}$$

for a sufficiently large set of initial functions $u_0$.

Let us note that under the assumptions (9), (10) the limiting function is necessarily even and non-increasing for $\xi > 0$. Therefore we are analyzing asymptotic stability of the most elementary (in its spatial "architecture") self-similar solution. Many of the results stated below extend to the multi-dimensional case (see § 6).

## 3 Asymptotic stability of the self-similar solution for $\beta = \sigma + 1$ (S-regime)

If (9), (10) are satisfied, the only "candidate" for a stable self-similar solution is the following one (§ 1):

$$u_S(t,x) = (T_0 - t)^{-1/\sigma}\theta_S(x),\ 0 < t < T_0, x \in \mathbf{R}, \tag{13}$$

where the function

$$\theta_S = \begin{cases} \left(\dfrac{2(\sigma+1)}{\sigma(\sigma+2)}\cos^2\dfrac{\pi x}{L_S}\right)^{1/\sigma}, & |x| < L_S/2, \\ 0, \ |x| \geq L_S/2 = \pi(\sigma+1)^{1/2}/\sigma, \end{cases} \tag{14}$$

satisfies the ordinary differential equation

$$(\theta_S^\sigma \theta_S')' - \frac{1}{\sigma}\theta_S + \theta_S^{\sigma+1} = 0, \ x \in \mathbf{R}. \tag{15}$$

In the S-regime the similarity transformation has an especially simple form:

$$\theta(t,x) = (T_0 - t)^{1/\sigma} u(t,x); \ T_0 = T_0(u_0) < \infty. \tag{16}$$

**Theorem 2.** *Let $\beta = \sigma + 1$, assume that the conditions (9), (10) are satisfied, and let $T_0 < \infty$ be the blow-up time for the unbounded solution of the problem (1), (2). Then*

$$\theta(t,x) \to \theta_S(x), \ t \to T_0^-, \tag{17}$$

*uniformly in $\mathbf{R}$, where $\theta_S(x)$ is the function (14).*

The main obstacle that arises in the proof of (17) is the derivation of bounds in $L^\infty(\mathbf{R})$ for the similarity representation, which are uniform in $t \in (0, T_0)$. Upper bounds guarantee global boundedness of $\theta(t,x)$, while a lower bound is needed in order that the limiting function $\theta(T_0^-, x)$ in (17) be non-trivial. These are the two most crucial stages of the proof. The point is that the function $\theta = \theta_S(x)$ is an unstable stationary solution of the parabolic equation satisfied by the similarity representation $\theta(t,x)$ (for a similar example see § 11, Ch. II). Therefore in the course of derivation of (17), we single out in the space of initial functions $\{\theta(0,x)\}$ the attracting set of an unstable stationary solution. We emphasize that similar problems of asymptotic stability of stationary solutions arise precisely in the analysis of singular solutions of evolution problems, which have a singularity in time.

### *1 Auxiliary results*

**Lemma 1.** *In the conditions of Theorem 2, for all $t \in (0, T_0)$ we have the estimates*

$$\operatorname{supp} u(t,x) \subset \,]-l_0 - L_S, l_0 + L_S[; \tag{18}$$

$$\sup_{x \in \mathbf{R}} u(t,x) > \sigma^{-1/\sigma}(T_0 - t)^{-1/\sigma}; \tag{19}$$

*there exists $\theta_* > \sigma^{-1/\sigma}$, such that*

$$\sup_{x \in \mathbf{R}} u(t,x) < \theta_*(T_0 - t)^{-1/\sigma}. \tag{20}$$

A sharper estimate than (19), which is "distributed" over $\mathbf{R}$, can be derived using the method of stationary states (see § 1, Ch. VII).

*Proof.* The estimate (18) was obtained in the course of proving Theorem 1 in § 4. Inequality (19) is none other than (3) for $\beta = \sigma + 1$.

Inequality (20) follows from Proposition 3 in § 4. Let us pick the value $\theta_* > \theta_H = \sigma^{-1/\sigma}$ large enough, so that, first, $\theta_* T_0^{-1/\sigma} > u_0(0)$, and, second, that the Cauchy problem for equation (15) for $x > 0$ with the conditions

$$\theta(0) = \theta_*, \ \theta'(0) = 0,$$

has a solution $\theta = \theta(x)$, which vanishes at a point $x = x_*(\theta_*) > 0$. This is always possible, as can be immediately seen from equation (15), which can be integrated in quadratures (see Lemma 2 in § 1). Let us set $\theta(-x) = \theta(x)$ in $(-x_*, 0)$. Then $v(t, x) = (T_0 - t)^{-1/\sigma}\theta(x)$ is an unbounded solution in $(0, T_0) \times (-x_*, x_*)$.

It is easily checked that $x_*(\theta_*) \to \pi/[2(\sigma + 1)^{1/2}]$ and $(\theta^\sigma)_x(x_*^-) \to -\infty$ as $\theta_* \to \infty$. Since $u_0^\sigma$ is uniformly Lipschitz continuous, we can pick $\theta_* > 0$ so large, that if (9), (10) hold, the initial function $u_0$ either does not intersect $v(0, x)$ at all in $(-x_*, x_*)$ (i.e., $N(0) = 0$), or intersects it exactly at two points, which are symmetric with respect to $x = 0$ ($N(0) = 2$). Then by the comparison theorem $N(t) \le 2$ for all $t \in (0, T_0)$.

Let us show that $u(t, 0) \equiv \sup_x u \le \sup_x v \equiv v(t, 0)$ (this immediately results in the estimate (20)). If $u(t_1, 0) > v(t_1, 0)$ for some $t_1 \in (0, T_0)$, then $u(t_1, x) > v(t_1, x)$ in $(-x_*, x_*) \equiv \operatorname{supp} v$. Indeed, if this is not the case, then $u(t_1, x) = 0$ for $x = \pm x_*$ (since $N(t_1) \le 2$), and this equality holds for $t \in [0, t_1[$. Therefore $N(0) = 0$, and by assertion 1) of Proposition 3 $N(t_1) = 0$, which is impossible.

Thus, $u(t_1, x) > v(t_1, x)$ in $(-x_*, x_*)$. If $\overline{\operatorname{supp}}\, v(t_1, x) \subset \operatorname{supp} u(t_1, x)$, we obtain a contradiction to Proposition 2, § 4. If, on the other hand, $\operatorname{supp} u(t_1, x) = (-x_*, x_*)$, then, by slightly increasing the value of $\theta_*$, and thus decreasing $2x_* = \operatorname{meas} \operatorname{supp} \theta$, we are back at the previous case. □

### 2 Proof of Theorem 2. a) Equation for the function $\theta(t, x)$

It is not hard to check that the similarity representation (16) satisfies the Cauchy problem

$$(T_0 - t)\theta_t = (\theta^\sigma \theta_x)_x - \frac{1}{\sigma}\theta + \theta^{\sigma+1}, 0 < t < T_0, x \in \mathbf{R}, \tag{21}$$

$$\theta(0, x) = \theta_0(x) \equiv T_0^{1/\sigma} u_0(x), \ \ x \in \mathbf{R}. \tag{22}$$

Setting in (21) $\tau = -\ln(1 - t/T_0) : [0, T_0) \to [0, \infty)$, we obtain the equivalent equation

$$\theta_\tau = (\theta^\sigma \theta_x)_x - \frac{1}{\sigma}\theta + \theta^{\sigma+1}, \ \ \tau > 0, \ \ x \in \mathbf{R}. \tag{23}$$

The initial condition (22) remains the same.

Comparing (23) with the "self-similar" ordinary differential equation (15), we see that in the new notation the study of asymptotic stability of the unbounded self-similar solution of the S-regime is equivalent to the analysis of asymptotic stability of the non-trivial stationary solution (14) of equation (23). It is important to note that for arbitrary initial perturbations $\theta_0(x)$, when the quantity $T_0$ in (22) has been chosen "incorrectly", and does not equal the blow-up time of the solution $u(t,x)$, the problem (22), (23) can have both unbounded ($\sup_x \theta(\tau,x) \to \infty$ as $\tau \to \tau_0 < \infty$) and global solutions, which stabilize to the trivial stationary solution $\theta \equiv 0$ as $\tau \to \infty$. In other words, the stationary solution (14) is unstable with respect to arbitrarily small perturbations. A proof of this fact is presented in § 11, Ch. II.

*b) Estimates of the function $\theta(t,x)$.* In the conditions of the theorem (with a "correct" choice of $T_0 = T_0(u_0)$ in (22)), the Cauchy problem (23), (22) always has a global solution, which stabilizes to the function (14). The proof is based on the estimates (18)–(20), which assume the following form in the new notation:

**Corollary of Lemma 1.** *In the conditions of Theorem* 2

$$\operatorname{supp}\theta(\tau,x) \subset [-l_0 - L_S, l_0 + L_S],\ L_S = 2\pi(\sigma+1)^{1/2}/\sigma; \tag{24}$$

$$\sup_{x\in\mathbf{R}}\theta(\tau,x) > \sigma^{-1/\sigma}; \tag{25}$$

$$\theta(\tau,x) < \theta_+ \tag{26}$$

*for all* $\tau \geq 0$, $x \in \mathbf{R}$.

From that we immediately have

**Lemma 2.** *Assume that conditions* (9), (10) *hold, and that* $T_0 = T_0(u_0)$. *Let* $\Omega$ *be a domain in* $\mathbf{R}$, *such that* $(-l_0 - L_S, l_0 + L_S) \subset \Omega$. *Then*

$$\theta^{1+\sigma/2} \in L^\infty(\mathbf{R}_+;\ L^2(\Omega)), \tag{27}$$

$$(\theta^{1+\sigma/2})_\tau \in L^2(\mathbf{R}_+;\ L^2(\Omega)), \tag{28}$$

$$\theta^{\sigma+1} \in L^\infty(\mathbf{R}_+;\ H_0^1(\Omega)). \tag{29}$$

*Proof.* By (25) $\theta = 0$ on $\partial\Omega$ for any $\tau \geq 0$. Taking the scalar product in $L^2(\Omega)$ of equation (23) with $(\theta^{\sigma+1})_\tau$ and integrating over $\tau$, we obtain the equality

$$\frac{4(\sigma+1)}{(\sigma+2)^2}\int_0^\tau \left\|(\theta^{1+\sigma/2})_\tau(s)\right\|^2_{L^2(\Omega)} ds + \frac{1}{2(\sigma+1)}\left\|(\theta^{\sigma+1})_x(\tau)\right\|^2_{L^2(\Omega)} +$$

$$+\frac{\sigma+1}{\sigma(\sigma+2)}\|\theta(\tau)\|^{\sigma+2}_{L^{\sigma+2}(\Omega)} = \frac{1}{2}\|\theta(\tau)\|^{2(\sigma+1)}_{L^{2(\sigma+1)}(\Omega)} + \frac{1}{2(\sigma+1)}\|(\theta_0^{\sigma+1})_x\|^2_{L^2(\Omega)} -$$

$$-\frac{1}{2}\|\theta_0\|^{2(\sigma+1)}_{L^{2(\sigma+1)}(\Omega)} + \frac{\sigma+1}{\sigma(\sigma+2)}\|\theta_0\|^{\sigma+2}_{L^{\sigma+2}(\Omega)},\ \tau > 0. \tag{30}$$

By (26) the right-hand side of (30) is bounded from above, from which we obtain the estimates (27)–(29) (for details see § 2 in Ch. VII). □

*c) Passage to the limit* $\tau \to \infty$. Thus, the Cauchy problem (23), (22) is equivalent to the boundary value problem with the condition

$$\theta(\tau, x) = 0, \ \tau \geq 0, \ x \in \partial\Omega = \overline{\Omega}\setminus\Omega, \tag{22'}$$

and the estimate (26) ensures its global solvability.

Stabilization of $\theta(\tau, x)$ for $\tau = \tau_i \to \infty$ to a stationary solution in the weak sense[8] follows from the estimates (27)–(29), which ensure boundedness of the sequence $\theta_i^{\sigma+1}(\tau, x) = \theta^{\sigma+1}(\tau + \tau_i, x)$, $n = 1, 2, \ldots$ in $H^1((0, 1) \times \Omega)$ (see § 2, Ch. VII). By compactness of the embedding $H^1 \subset L^2$ and the estimate (28), this allows us to choose from any sequence $\tau_i \to \infty$ a subsequence (which we also denote by $\tau_i$), such that $\theta_i^{\sigma+1}(\tau, x) \to \overline{\theta}^{\sigma+1}(x)$ as $\tau_i \to \infty$ in $L^2((0, 1) \times \Omega)$. See subsection 6, § 3.

Passage to the limit in equation (23) is also effected by using the estimate (29), as well as the fact that (22), (22′) admits the Liapunov function

$$V(\theta)(\tau) = \int_\Omega \left\{ \frac{1}{2(\sigma+1)} \left(\theta^{\sigma+1}\right)_x^2 + \frac{\sigma+1}{\sigma(\sigma+2)}\theta^{\sigma+2} - \frac{1}{2}\theta^{2\sigma+2} \right\} dx,$$

which is non-increasing in $\tau$ on any solution of the problem. By formal computations, we have

$$\frac{d}{d\tau}V(\theta)(\tau) = -\frac{4(\sigma+1)}{(\sigma+2)^2}\int_\Omega \left(\theta^{(\sigma+2)/2}\right)_\tau^2 dx \leq 0.$$

Stabilization in $C(\Omega)$ follows from stronger estimates; using the method of Bernstein, it is not hard to show that $|(\theta^{\sigma+1}(\tau, x))_x| \leq \text{const} < \infty$ everywhere in $\mathbf{R}_+ \times \mathbf{R}$. By (26) this means that the trajectory $\{\theta^{\sigma+1}(\tau, x) \mid \tau > 0\}$ is compact in $C(\Omega)$.

Thus, $\theta(\tau, x) \to \overline{\theta}(x)$ for $\tau = \tau_i \to \infty$, where $\overline{\theta}$ is some stationary solution of equation (23), and $\overline{\theta} \in C_0(\overline{\Omega})$. Then, first of all, (25) means that $\overline{\theta} \not\equiv 0$, and, secondly, by (9), (10) $\overline{\theta}(x)$ is an even function, which is non-increasing in $x > 0$. Now, since $\overline{\theta}$ is a function with compact support (by (24) $\operatorname{supp}\overline{\theta} \subset [-l_0 - L_S, l_0 + L_S]$), from the uniqueness of the stationary solution $\theta_S \not\equiv 0$ (see subsection 2, § 1), we have $\overline{\theta}(x) = \theta_S(x)$. Stabilization of $\theta(\tau, x)$ to $\theta_S(x)$ as $\tau \to \infty$ (that is, on any sequence $\tau_i \to \infty$) also follows from uniqueness of the stationary solution $\overline{\theta} = \theta_S(x)$ with the required properties. This concludes the proof. □

Let us present a corollary of (17).

[8] See examples of analysis of degenerate equations in [20, 308, 359].

**Corollary.** *In the conditions of Theorem 2 $u(t,x) \to \infty$ as $t \to T_0^-$ at any point of the domain $x \in (-L_S/2, L_S/2)$.*

Thus, inside the "fundamental" localization domain $\{|x| < L_S/2\}$, a non-trivial solution of the problem (1), (2), where $\beta = \sigma + 1$ and $u_0$ satisfies (9), (10), grows without bound as $t \to T_0^-$. Here condition (17) does not preclude unbounded growth of the solution outside the localization domain; this growth has to be at a rate $o((T_0 - t)^{-1/\sigma})$, i.e at a slower than the self-similar rate.

In numerical computations a more striking phenomenon was observed: for practically all non-monotone initial perturbations $u_0(x)$ in the process of evolution, a thermal structure was formed in a neighbourhood of an extremum point of $u_0(x)$, which developed as $t \to T_0^-(u_0) < \infty$ as the self-similar solution (13), (14); furthermore, outside the localization domain the solution was bounded from above uniformly in $t \in (0, T_0)$ (see, for example, Figure 37). It is of interest that an optimal result of this sort can be obtained by combining Theorem 2 and Theorem 3 of § 4.

**Theorem 3.** *Let $\beta = \sigma + 1$,* meas supp $u_0 < L_S$, *and assume that conditions* (9), (10) *and the condition of Theorem* 3 *in* § 4 *hold. Then $u(t,x) \to \infty$ as $t \to T_0^-(u_0) < \infty$ at all points of the localization domain $\omega(T_0^-) = \{|x| < L_S/2\}$ and $u(t,x) \equiv 0$ everywhere in $[0, T_0) \times \{|x| \geq L_S/2\}$. At all points $x \in \mathbf{R}$ the solution approaches the self-similar one:*

$$(T_0 - t)^{1/\sigma} u(t,x) \to \theta_S(x),\ t \to T_0^-. \tag{31}$$

*Proof.* From Theorem 3 in § 4 it follows that $u \equiv 0$ in $[0, T_0) \times \{|x| \geq L_S/2\}$, while from Theorem 2 (see (17)) follows (31), and therefore the fact that $u(t,x) \to \infty$, $t \to T_0^-$ in $\{|x| < L_S/2\}$. □

## 4 On asymptotic stability of self-similar solutions of the HS-regime, $1 < \beta < \sigma + 1$

We shall consider the Cauchy problem (1), (2) for $1 < \beta < \sigma + 1$ with initial function satisfying conditions (9), (10). Asymptotic stability of the self-similar solution means that the similarity representation (11) satisfies (12), where $\theta_S(\xi) \not\equiv 0$ is the unique non-trivial compactly supported solution of the ordinary differential equation (8). Existence of the compactly supported function $\theta_S$ has been established in Theorem 2 in subsection 3 of § 1; uniqueness will be proved below. We start with some auxiliary estimates.

**Lemma 3.** *Under the above assumptions*

$$\operatorname{supp} u(t,x) \subset [-l_0 - \xi_0 T_0^m - \xi_0(T_0 - t)^m,\ l_0 + \xi_0 T_0^m + \xi_0(T_0 - t)^m], \tag{32}$$

*where* $\xi_0 = \text{meas}\{\xi > 0 \mid \theta_S(\xi) > 0\} < \infty$, $m = |\beta - (\sigma+1)|/|2(\beta-1)| < 0$;

$$\sup_{x\in\mathbf{R}} u(t,x) > (\beta-1)^{-1/(\beta-1)}(T_0-t)^{-1/(\beta-1)}; \tag{33}$$

*there exists a constant* $\theta_* > \theta_H$, *such that*

$$\sup_{x\in\mathbf{R}} u(t,x) < \theta_*(T_0-t)^{-1/(\beta-1)}. \tag{34}$$

*Proof.* Estimate (33) was obtained in Theorem 1, inequality (34) is derived by the method used in Lemma 1 to analyze the S-regime. Existence of the unbounded subsolution $v(t,x)$ appropriate in this case was established in § 1. The magnitude of $\theta_*$ here can always be chosen such that $u_0(x)$ and $v(0,x)$ do not intersect. The estimate (32) of the length of the support follows from Propositions 1 and 2 of § 4 (using an argument as in the proof of Theorem 1′ in § 4). □

**Remark.** In the course of proof of Lemma 3 we established a stronger result: *for* $1 < \beta < \sigma+1$ *and any initial function* $u_0(x)$ *of compact support we have the estimates*

$$\text{meas supp}_{\pm}\, u(t,x) = \xi_0(T_0-t)^m + O(1),\ t \to T_0, \tag{32′}$$

*where* $\text{supp}_+ u(t,x) = \{x > 0 \mid u(t,x) > 0\}$, $\text{supp}_- u(t,x) = \{x < 0 \mid u(t,x) > 0\}$.

Let us show that (32′) implies the following important claim (which could not be proved in § 1 by analyzing an ordinary differential equation):

**Corollary.** *Let* $1 < \beta < \sigma+1$. *Then the compactly supported solution of equation* (8) *is even and unique.*

*Proof.* If $\theta_S \not\equiv 0$ is some compactly supported solution of equation (8), then the corresponding self-similar solution $u_S$ (see (7)) satisfies the condition

$$\text{meas supp}_+ u_S(t,x) \equiv (T_0-t)^m\, \text{meas supp}_+ \theta_S(\xi);$$

so that by (32′) $\text{supp}_+\theta_S = \text{supp}_-\theta_S = \xi_0$. Equation (8) is invariant with respect to the transformation $\xi \to -\xi$, and, as can be easily verified by a local analysis, admits a unique nontrivial extension from the point $\xi = \xi_0$ into the domain $\{\xi < \xi_0\}$ with $\theta_S^{\sigma+1}(\xi_0) = \left(\theta_S^{\sigma+1}\right)'(\xi_0) = 0$. Therefore $\theta_S(\xi)$ is an even solution. Now, if there exist two different solutions with compact support $\theta_S^1$ and $\theta_S^2$, then (32′) ensures that their supports are the same and therefore $\theta_S^1 \equiv \theta_S^2$, which completes the proof. □

From Lemma 3 we deduce the following estimates of the similarity representation (11):

$$\operatorname{supp}\theta(\tau,\xi) \subset [-\xi_0 - (l_0 + \xi_0 T_0^m)T_0^{-m}\exp\{m\tau\},\ \xi_0 + (l_0 + \xi_0 T_0^m)T_0^{-m}\exp\{m\tau\}], \tag{35}$$

$$\sup_{\xi\in\mathbf{R}} \theta(\tau,\xi) > \theta_H = (\beta - 1)^{-1/(\beta-1)}, \tag{36}$$

$$\sup_{\xi\in\mathbf{R}} \theta(\tau,\xi) < \theta_*;\ \tau = -\ln(1 - t/T_0) \in [0,\infty). \tag{37}$$

Let us note that from (35) and from the estimates of Theorem 6 in § 4, it follows that

$$\operatorname{meas}|\operatorname{supp}\theta(\tau,\xi)\backslash(-\xi_0,\xi_0)| \to 0,\ \tau\to\infty. \tag{35'}$$

Let us consider now the equivalent boundary value problem:

$$\theta_\tau = (\theta^\sigma\theta_\xi)_\xi - m\theta_\xi\xi - [\theta/(\beta-1)] + \theta^\beta,\ \tau > 0,\ \xi\in\Omega. \tag{38}$$

$$\theta(0,\xi) = \theta_0(\xi) \equiv T_0^{1/(\beta-1)}u_0(\xi T_0^m),\ \xi\in\Omega, \tag{39}$$

$$\theta(\tau,\xi) = 0,\ \tau > 0,\ \xi\in\partial\Omega. \tag{40}$$

Here $\Omega$ is a bounded domain in $\mathbf{R}$, such that $\overline{\operatorname{supp}}\,\theta(\tau,\xi)\subset\Omega$ for any $\tau\geq 0$ (such $\Omega$ exists in view of (35)). The estimate (37) ensures global solvability of the problem, while (36), by force of an easily derived uniform in $[1,\infty)\times\mathbf{R}$ bound for $(\theta^{\sigma+1})'_\xi$, precludes stabilization as $\tau\to\infty$ to the trivial stationary solution $\theta\equiv 0$.

Therefore, since we have shown that the admissible non-trivial compactly supported solution of equation (38) is unique, uniform in $\mathbf{R}$ stabilization to it as $\tau\to\infty$ would follow from existence of a Liapunov function with appropriate properties for the problem (38)–(40). Such a function can be formally constructed using the general approach of [42, 383]. However, it cannot be written down explicitly and admits a representation in terms of a two-parameter family of solutions of the ordinary differential equation (8). This makes verification of the necessary properties of such a Liapunov function difficult, and therefore we do not consider this problem here; see Remarks.

**Remark.** Conditions (9), (10) were not used in the derivation of (32), (33). Let us show that (34) (or, which amounts to the same, (37)) for $1<\beta<\sigma+1$ also holds without these restrictions. Indeed, let us consider a solution $\theta(\xi;\mu)$ of the stationary equation (38) for $\xi>0$, satisfying the conditions $\theta(0;\mu)=\mu>0$, $\theta'_\xi(0;\mu)=0$. From the analysis contained in the proof of Theorem 2 in subsection 3 of § 1 it follows that there are sufficiently large $\mu>0$, such that $\theta(\xi;\mu)$ vanishes at a point $\xi=\xi_\mu$. We also have that $\xi_\mu\to\infty$ as $\mu\to\infty$. Let us choose $\mu>0$

so large that $\Omega \subset (-\xi_\mu, \xi_\mu)$ and $\theta_0(\xi) \le \theta(|\xi|;\mu)$ in $\Omega$. Then by the Maximum Principle $\theta(\tau,\xi) \le \theta(|\xi|;\mu)$ in $\mathbf{R}_+ \times \Omega$, i.e., the problem (38)–(40) is globally solvable and estimate (37) holds. Thus we have proved the following general statement:

**Proposition 1.** *Let* $1 < \beta < \sigma + 1$, *let* $u_0(x)$ *be an arbitrary compactly supported function and* $T_0 = T_0(u_0) < \infty$ *be the time of existence of the solution of the Cauchy problem* (1), (2). *Then for all* $t \in [0, T_0)$ *we have the estimates*

$$\sup_{x \in \mathbf{R}} u(t,x) > (\beta - 1)^{-1/(\beta-1)}(T_0 - t)^{-1/(\beta-1)};$$

*there exists* $\theta_* > 0$, *such that* $\sup_{x\in\mathbf{R}} u(t,x) < \theta_*(T_0 - t)^{-1/(\beta-1)}$;

$$\operatorname{meas}\operatorname{supp} u(t,x) = 2\xi_0(T_0 - t)^{[\beta-(\sigma+1)]/[2(\beta-1)]} + O(1),\ t \to T_0^-.$$

## 5 On stability of the self-similar LS-regime, $\beta > \sigma + 1$

For $\beta > \sigma + 1$, the self-similar functions $\theta_S(\xi)$ (let us note, that in general, there is more than one) are strictly positive in $\mathbf{R}$ (see § 1). The similarity representation (11) has compact support in $\xi$, but as $t \to T_0^-$ the size of the support goes to infinity:

$$\operatorname{meas}\operatorname{supp}\theta(t,\xi) \sim (T_0 - t)^{-[\beta-(\sigma+1)]/[2(\beta-1)]} \to \infty,\ t \to T_0.$$

It is not particularly hard to show that under the assumptions (9), (10), $\theta(t,\xi)$ is uniformly bounded; for $\beta > \sigma + 1$ we have the estimate (34) and, therefore (37). This is done as in the case $\beta \le \sigma + 1$, using the results of § 1 (see Lemma 4). By Theorem 1 we also have the lower bound (36), so that if $\theta(\tau,\xi) \to \bar\theta(\xi)$ as $\tau \to \infty$, then $\bar\theta \not\equiv 0$.

However, the following difficulty arises in the analysis of the behaviour of $\theta(\tau,\xi)$ as $\tau = -\ln(1 - t/T_0) \to \infty$. Unlike the cases of HS- and S-regimes nothing so far stops $\theta(\tau,\xi)$ from stabilizing to the spatially homogeneous solution[9] of equation (38), $\bar\theta \equiv (\beta - 1)^{-1/(\beta-1)}$. That would mean that the asymptotic behaviour of the blow-up process does not follow a self-similar pattern. Sufficient conditions of non-triviality of the limiting function $\bar\theta$ ($\bar\theta \not\equiv (\beta-1)^{-1/(\beta-1)}$) are given by Theorem 4, where we have denoted by $\theta_S(\xi)$ the elementary solution of equation (8) constructed in Theorem 3 of § 1.

**Theorem 4.** *Let* $\beta > \sigma + 1$, *conditions* (9), (10) *are satisfied, and* $T_0 = T_0(u_0) < \infty$ *is the blow-up time of an unbounded solution of the problem* (1), (2). *Furthermore, let the initial function* $u_0(x)$ *be such that* $T_0^{1/(\beta-1)}u_0(\xi T_0^m)$ *intersects the*

[9]This occurs, for example, for $\sigma = 0$ (see Remarks).

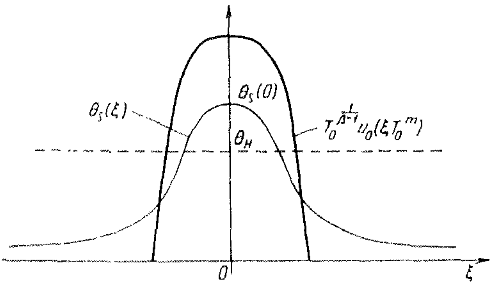


**Fig. 63.**

*function* $\theta_S(\xi)$ *exactly at two points and* $T_0^{1/(\beta-1)}u_0(0) > \theta_S(0)$. *Then we have the estimate*

$$u(t, 0) > \theta_S(0)(T_0 - t)^{-1/(\beta-1)}, \ t \in [0, T_0). \tag{41}$$

*Proof.* In the situation of Figure 63, (41) follows from Proposition 2 in § 4. Indeed, considering the unbounded solutions $u(t, x)$ and $v(t, x) \equiv u_S(t, x)$ we have that $N(0) = 2$. Since $N(t) > 0$ always, and the functions $u$, $v$ are even in $x$, then $N(t) = 2$ for all $t \in [0, T_0)$. This means that

$$\sup_x u \equiv u(t, 0) > \sup_x v \equiv u_S(t, 0),$$

from which (41) follows. □

In the conditions of Theorem 4 $\theta(\tau, 0) > \theta_S(0) > (\beta - 1)^{-1/(\beta-1)}$ for all $\tau \geq 0$, and therefore $\overline{\theta}(\xi) \not\equiv \theta_H$. In § 6 we shall obtain a pointwise estimate, which precludes stabilization of $\theta(\tau, \xi)$ to a spatially homogeneous solution.

Therefore one of the main difficulties which arise in the proof of stabilization of $\theta(\tau, \cdot)$ to $\theta_S(\cdot)$ as $\tau \to \infty$, has to do with lack of a uniqueness theorem for a non-trivial self-similar function $\theta_S$ of the simplest form. Another difficulty, mentioned in subsection 4, is of constructing a good enough Lyapunov function. See Comments.

## § 6 Asymptotics of unbounded solutions of LS-regime in a neighbourhood of the singular point

This whole section is devoted to the proof of effective localization of unbounded self-similar solutions of the Cauchy problem (0.1), (0.2) for $\beta > \sigma + 1$. Below we shall show that under certain restrictions on $u_0(x)$ and $\beta$, the combustion process in the LS blow-up regime leads to unbounded growth of the temperature as $t \to T_0^-$ at one singular point only, that is

$$\operatorname{meas} \omega_L \equiv \operatorname{meas} \{x \in \mathbf{R}^N \mid u(T_0^-, x) = \infty\} = 0$$

(see Figures 43, 44). Let us recall that this is a property of unbounded self-similar solutions of the LS-regime, existence of which was established in § 1 for all $\sigma + 1 < \beta < (\sigma + 1)(N + 2)/(N - 2)_+$. Here we shall consider non-self-similar solutions.

Let us introduce a class of functions $u_0(|x|)$ with compact support, for which we shall show that $\operatorname{meas} \omega_L = 0$. As $u_0$ we shall take functions $U(|x|; U_0)$, which satisfy the stationary equation

$$\frac{1}{r^{N-1}} \left( r^{N-1} U^\sigma U' \right)' + U^\beta = 0, \; r = |x| > 0, \qquad U'_r(0; U_0) = 0, \; U(0; U_0) = U_0, \; U_0 = \text{const} > 0. \tag{1}$$

It was shown in subsection 4.1, § 3, that for $\beta < (\sigma + 1)(N + 2)/(N - 2)_+$, $U(|x|; U_0)$ vanishes at some point $r = r_0(U_0) > 0$. Let us set $U(|x|; U_0) \equiv 0$ for $|x| \geq r_0(U_0)$.

Let $\sigma + 1 < \beta < (\sigma + 1)(N + 2)/(N - 2)_+$. Let us consider the Cauchy problem

$$u_t = \nabla \cdot (u^\sigma \nabla u) + u^\beta, \; t > 0, \; x \in \mathbf{R}^N, \tag{2}$$

$$u(0, x) = U(|x|; U_0), \; x \in \mathbf{R}^N; \; U_0 > 0. \tag{3}$$

Then $u = u(t, |x|)$ is monotone decreasing in $|x|$ and is unbounded; $u(t, 0) \to \infty$ as $t \to T_0^-$ (see Theorem 3 in § 6, Ch. V, and the Remark following it). Moreover, the solution, which has compact support in $x$, is critical:

$$u_t(t, x) \geq 0, \; (t, x) \in (0, T_0) \times \{x \in \mathbf{R}^N \mid u(t, x) > 0\}; \tag{4}$$

see § 2, Ch. V. This means, in particular, that for each $x \in \mathbf{R}^N$ there exists a (finite or infinite) limit

$$u(T_0^-, x) = \lim_{t \to T_0^-} u(t, x).$$

Our goal is to prove that $u(T_0^-, x) < \infty$ in $\mathbf{R}^N \setminus \{0\}$. A lower bound for $u(T_0^-, x)$ is proved relatively easily using the method of stationary states. The following assertion will be proved in § 1, Ch. VII for quite general $u_0 = u_0(|x|)$.

**Theorem 1.** *Let* $\sigma \geq 0$, $\sigma + 1 < \beta < (\sigma + 1)(N + 2)/(N - 2)_+$. *Then*

$$u(T_0^-, x) > C_* |x|^{-2/[\beta - (\sigma+1)]},$$

$$C_* = \left\{ \frac{2N}{\beta - (\sigma + 1)} \left[ \frac{\beta - (\sigma + 1)}{\beta} \right]^{\beta/(\sigma+1)} \right\}^{1/[\beta - (\sigma+1)]}. \tag{5}$$

*for all sufficiently small* $|x| > 0$.

Derivation of an upper bound for $u(T_0^-, x)$, which proves validity of the equality $\operatorname{meas} \omega_L = 0$ and the fact of effective localization itself, is accomplished by comparing $u(t, x)$ with the self-similar solution

$$u_S(t, x) = (T_0 - t)^{-1/(\beta-1)} \theta_S(\xi), \quad \xi = |x|/(T_0 - t)^m, \tag{6}$$

where $m = [\beta - (\sigma + 1)]/[2(\beta - 1)]$ (existence of the function $\theta_S(\xi) \geq 0$ has been established in Theorem 4 of § 1). Solution (6) is effectively localized; in particular, for $\sigma + 1 < \beta \leq (\sigma + 1)N/(N - 2)_+$ (Theorem 5, § 1)

$$\frac{\partial u_S(t, x)}{\partial t} > 0, \; t \in (-\infty, T_0) \times \mathbf{R}^N; \tag{7}$$

$$u_S(t, x) < u_S(T_0, x) \equiv C_S |x|^{-2/[\beta - (\sigma+1)]}, \quad x \in \mathbf{R}^N \setminus \{0\}. \tag{8}$$

Let us state the main result.

**Theorem 2.** *Let* $\sigma \geq 0$, $\sigma + 1 < \beta \leq (\sigma + 1)N/(N - 2)_+$. *Then the solution of the problem* (2), (3) *satisfies the estimate*

$$\begin{gathered} u(t, x) \leq u(T_0^-, x) < C_S |x|^{-2/[\beta-(\sigma+1)]}, \\ t \in (0, T_0), \quad x \in \mathbf{R}^N \setminus \{0\}. \end{gathered} \tag{9}$$

**Remark.** From (5), (9) we immediately obtain the estimate $C_S > C_*$, where $C_S$ is the constant in the asymptotic expansion of the similarity function $\theta_S(\xi) \sim C_S \xi^{-2/[\beta-(\sigma+1)]}$, $\xi \to \infty$ (see subsection 4, § 1).

Let us prove first some auxiliary claims.

**Lemma 1.** *Let* $u = u(t, |x|)$ *and* $u_S = u_S(t, |x|)$ *have the same blow-up time* $t = T_0 < \infty$. *Then the functions* $u_0 \equiv U(r; U_0)$ *and* $u_S(0, r)$ *intersect (in* $r = |x|$*) exactly at one point.*

*Proof.* The functions $u_0(r)$ and $u_S(0, r)$ have to intersect, since the corresponding unbounded solutions have the same blow-up times, and $u_0$ is a function with compact support (see Proposition 2 in § 4).

Let us prove now that

$$u_S(0, r) > U(r; \lambda), \quad r > 0; \; 0 < \lambda \leq u_S(0, 0). \tag{10}$$

It is clear that the condition $U(0; U_0) > u_S(0, 0)$ will follow from that, since in the opposite case $u(t, r)$ and $u_S(t, r)$ will have different blow-up times. More general inequalities of the form of (10) are derived in § 1, Ch. VII. Below we shall briefly discuss the main idea behind the proof.

Let us fix $\lambda \in (0, u_S(0,0)]$. The self-similar solution (6) is defined in $\mathbf{R}^N$ for all $t \in (-\infty, T_0)$, such that, moreover, $u_S \to 0$, $(u_S)_r \to 0$ as $t \to -\infty$ uniformly on every compact set in $\mathbf{R}^N$. Therefore there exists $t_0 \leq 0$, such that $u_S(t_0, |x|)$ intersects $U(r; \lambda)$ only at one point for $r > 0$. However, $u_S$ and $U(r; \lambda)$ are classical solutions of equation (2) in $(t_0, T_0) \times \{|x| < r_0(\lambda)\}$ and $u_S > U = 0$ for $r = r_0(\lambda)$. Therefore the number of intersections of $u_S$ and $U$ cannot increase in $t$, and thus at time $t = t_* \leq 0$, when $u_S(t_*, 0) = U(0; \lambda)$, we must have the inequality $u_S(t_*, r) > U(r; \lambda)$, $r > 0$. By (7), (10) follows from that.

Thus, $U(0; U_0) > u_S(0, 0)$ and the functions $U(r; U_0)$ and $u_S(0, r)$ intersect. We shall show that there is precisely one intersection point. Assume that this is false, and that there are several inersections. Let us consider the family of stationary solutions $\{U(r; \lambda)\}$. For all $\lambda < u_S(0, 0)$ the functions $U(r; \lambda)$ and $u_S(0, r)$ do not intersect (see (10)). Obviously, for sufficiently large $\lambda > 0$ there is only one intersection (this follows from well-known properties of the functions $U(r; \lambda)$ for $\beta < (\sigma + 1)(N + 2)/(N - 2)_+$; see § 3). Therefore by continuous dependence of $U(r; \lambda)$ on $\lambda$ there exists $\lambda = \lambda_* > 0$, such that the curves $u = U(r; \lambda_*)$ and $u = u_S(0, r)$ in the $(u, r)$ plane are tangent at some point $r = r_* > 0$, and at the tangency point we have $u_S = U$, $u_S' = U_r'$, $u_S'' \leq U_{rr}''$. But then $(\partial/\partial t)u_S(0, r_*) \leq 0$, and that contradicts (7). □

The following lemma is a direct corollary of Lemma 1 and Proposition 2, § 4.

**Lemma 2.** *Under the conditions of Lemma 1, $u(t, r)$ and $u_S(t, r)$ intersect exactly at one point for each $t \in [0, T_0)$ for $r > 0$, and therefore*

$$u(t, 0) > u_S(t, 0), \quad t \in [0, T_0).$$

*Proof of Theorem 2.* Let us assume that at some point $t = t_* < T_0$, $r = r_* > 0$, inequality (9) is violated. Then by (4)

$$u(t, r_*) \geq C_S r_*^{-2/[\beta - (\sigma+1)]}, \quad t_* < t < T_0. \tag{11}$$

Let $u_S(t, x)$ blow up at the same time as $u(t, x)$. Let us compare these functions, considering them as solutions of boundary value problems for (2) in the domain $(t_*, T_0) \times \omega_*$, where $\omega_* = \{|x| < r_*\}$, $\partial\omega_*$ is the boundary of $\omega_*$. From (8), (11) we have

$$u_S(t, x) < C_S |x|^{-2/[\beta - (\sigma+1)]} \leq u(t, x) \text{ in } (t_*, T_0) \times \partial\omega_*. \tag{12}$$

From Lemma 2 it follows immediately that

$$u_S(t_*, x) < u(t_*, x), \quad x \in \omega_*. \tag{13}$$

But then $u$, $u_S$ have different blow-up times (see Proposition 3 in § 4). Indeed, from (12), (13) it follows that there exists $\tau \in (0, T_0 - t_*)$, such that $u_S(t_* + \tau, x) \le u(t_*, x)$ in $\overline{\omega}_*$. By (12) and the Maximum Principle it means that $u_S(t + \tau, x) \le u(t, x)$ in $(t_*, T_0 - \tau) \times \omega_*$. Passing in this inequality to the limit as $t \to (T_0 - \tau)^-$, we obtain the inequality $u_S(T_0^-, x) \le u(T_0 - \tau, x)$ in $\omega_*$, which is impossible, since $u_S(T_0^-, 0) = \infty$, $u(T_0 - \tau, 0) < \infty$. □

From Theorems 1, 2 we immediately have

**Theorem 3.** 1. *Let*

$$\sigma + 1 + 2/N \le \beta < (\sigma + 1)(N + 2)/(N - 2)_+ .$$

*Then for any fixed* $p \ge [\beta - (\sigma + 1)]N/2 \ge 1$, $\epsilon > 0$, *solution of the problem* (2), (3) *satisfies the condition*

$$\|u(t, \cdot)\|_{L^p(\{|x| < \epsilon\})} \to \infty, \quad t \to T_0^- . \tag{14}$$

2. *Let* $\sigma + 1 + 2/N < \beta \le (\sigma + 1)N/(N - 2)_+$. *Then for any* $1 \le p < [\beta - (\sigma + 1)]N/2$, $\epsilon > 0$, *and all* $t \in (0, T_0)$ *we have the estimate*

$$\begin{aligned} \|u(t, \cdot)\|_{L^p(\{|x| < \epsilon\})} < \left[2\frac{\pi^{N/2}}{\Gamma(N/2)}\right]^{1/p} C_S \times \\ \times \left(N - \frac{2p}{\beta - (\sigma + 1)}\right)^{-1/p} \epsilon^{N/p - 2/[\beta - (\sigma + 1)]} < \infty. \end{aligned} \tag{15}$$

Let us note the two main requirements on $u_0 = u_0(|x|)$, for which estimate (9) holds. First of all $u_0$ is a critical function, that is, $u_t \ge 0$ almost everywhere in $(0, T_0) \times \mathbf{R}^N$, and, secondly, $u_0(|x|)$ intersects $u_S(0, |x|)$ ($u_S(t, |x|)$ has the same blow-up time $t = T_0 < \infty$) only at a single point $r = |x| > 0$. As far as the first requirement is concerned, no special problems arise here. The family of critical $u_0(|x|)$ includes, in addition to functions $U(|x|, U_0)$ with compact support, for example, smooth functions of the form

$$u_0(|x|) = A(a^2 + |x|^2)^{-1/[\beta - (\sigma + 1)]}, \quad x \in \mathbf{R}^N; \ A > 0, a^2 > 0.$$

It is easily verified that $\nabla \cdot (u_0^\sigma \nabla u_0) + u_0^\beta \ge 0$ in $\mathbf{R}^N$ if $A^{\beta - (\sigma + 1)} \ge 2N/[\beta - (\sigma + 1)]$ (this is sufficient for criticality of the classical solution; see § 1, Ch. V). The functions $u_0(x) = A\exp\{-\alpha|x|^2\}$ are also critical, if

$$A^{\beta - (\sigma + 1)} \ge 2\alpha N \exp\left\{[\beta - (\sigma + 1)]\frac{N}{2(\sigma + 1)}\right\}, \quad \alpha > 0.$$

However, for critical initial functions $u_0(x)$ different from $U(|x|, U_0)$, the question concerning the number of intersections in $|x|$ of the function $u(t, |x|)$ and the self-similar solution $u_S(t, |x|)$ with the same blow-up time is a more difficult one.

Here we would like to stress again that all the assertions of the intersection comparison theorems of § 4 are applicable to comparison of radially symmetric solutions of equation (2) with the same interval of existence. As an example of a fairly general application we shall now derive an exact "self-similar" upper bound.

We shall consider the Cauchy problem for equation (2) with a radially symmetric initial function

$$u(0,x) = u_0(|x|) \geq 0, \quad x \in \mathbf{R}^N, \tag{16}$$

where the compactly supported function $u_0(r)$ is non-increasing in $r = |x| \geq 0$, $\sup|(u_0^\sigma)'| < \infty$. Let $T_0$ be the finite blow-up time of the solution. From the Maximum Principle, which can be applied to the parabolic equation for the derivative $u_r(t,r)$, we conclude that $u(t,r)$ is non-increasing in $r$. First of all, let us note that the elementary intersection comparison with the spatially homogeneous solution $v(t) = \theta_H(T_0 - t)^{-1/(\beta-1)}$ leads to the self-similar lower bound:

$$\sup_{r>0} u(t,r) \equiv u(t,0) > \theta_H(T_0 - t)^{-1/(\beta-1)}, \quad t \in [0,T_0). \tag{17}$$

Indeed, as in the one-dimensional case, solutions $u(t,r)$ and $v(t)$ must intersect for each $t \in [0,T_0)$, otherwise by Proposition 2, § 4 (its proof obviously holds for radially symmetric solutions of the multi-dimensional equation), they will have different blow-up times. The upper bound is proved by comparison with less trivial self-similar solutions. We shall state the most general result, which holds not only for the LS-, but also for the HS- and S-regimes of evolution of unbounded solutions.

**Theorem 4.** *For* $1 < \beta < (\sigma+1)(N+2)/(N-2)_+$ *there exists a constant* $\theta_* > \theta_H$, *so that*

$$u(t,0) < \theta_*(T_0 - t)^{-1/(\beta-1)}, \quad t \in [0,T_0). \tag{18}$$

*Proof.* Proofs of all the three cases, $\beta < \sigma+1$, $\beta = \sigma+1$, and $\beta > \sigma+1$ are similar; see proof of Lemmas 1 and 3 in § 5 for $\beta \leq \sigma+1$. Here we shall consider the case $\beta > \sigma+1$. As solution $v(t,r)$, having the same blow-up time $T_0$ as $u(t,r)$, let us take

$$v(t,r) = (T_0 - t)^{-1/(\beta-1)}\theta(\xi;\mu), \quad \xi = r/(T_0 - t)^m, \tag{19}$$

where the function $\theta(\xi;\mu)$ (solution of problem (4), (17) in § 1) vanishes for all sufficiently large $\mu > \theta_H$ at some point $\xi = \xi_\mu > 0$ (see subsection 4.3 in § 1). Therefore (19) is an unbounded solution of equation (2) in the domain $(0,T_0) \times \{r \leq \xi_\mu(T_0 - t)^m\}$. As shown in subsection 4.3, § 1, for all $\beta < (\sigma+1)(N+2)/(N-2)_+$, $\xi_\mu \to 0$ and $(\theta^\sigma)'_\xi(\xi_\mu;\mu) \to -\infty$ as $\mu \to \infty$. Let

$N_\mu(t)$ be the number of intersections of the solutions $u(t,r)$ and $v(t,r)$ in the domain $\{r \le \xi_\mu(T_0-t)^m\}$. Then under the above restrictions on $u_0$, we conclude that $N_\mu(0)=1$ for any fixed sufficiently large $\mu$. At the same time the following condition clearly holds for the support of the solution: $\overline{\operatorname{supp}}\, v(t,r) \subset \operatorname{supp} u(t,r)$ for $t \in [0,T_0)$. Therefore by Proposition 3, § 4 (it is easy to check that it is valid in this context), it immediately follows that $N_\mu(t) \equiv 1$ (if $N_\mu(t')=0$ at some time $t=t'$, then solutions $u(t,r)$ and $v(t,r)$ would have different blow-up times), from which we obtain (18) with $\theta_* = \mu$. □

Therefore the estimates (17) and (18) show that in the subcritical case $1<\beta<(\sigma+1)(N+2)/(N-2)_+$ the spatial amplitude of radially symmetric solutions grows according to a self-similar law.

**Remark.** From the method of the proof it is easy to see that Theorem 4 is valid not only for the Cauchy problem, but also for the boundary value problem in $(0,T_0)\times B_R$, $B_R=\{|x|<R\}$ ($R=\text{const}>0$) with the boundary condition $u(t,R)=0$ for $t>0$; the initial function satisfies the same assumptions.

## § 7 Blow-up regimes, effective localization for semilinear equations with a source

In this section we study unbounded, as well as some classes of global, solutions of the Cauchy problem for semilinear parabolic equations of the form

$$u_t = \Delta u + Q(u),\ t>0,\ x\in \mathbf{R}^N, \tag{1}$$

$$u(0,x)=u_0(x)\ge 0,\ x\in\mathbf{R}^N;\ u_0\in C(\mathbf{R}^N),\ \sup u_0<\infty, \tag{2}$$

which describe combustion processes in a medium with a constant heat conductivity coefficient $k(u)\equiv 1$. It is assumed that $Q(u)>0$ for $u>0$ and that for all $s>0$

$$F(s)=\int_s^\infty \frac{d\eta}{Q(\eta)}<\infty, \tag{3}$$

and that, furthermore, $F(0)=\infty$ (this property is necessary for uniqueness of solutions of the Cauchy problem; see § 2, Ch. I).

Equation (1) with a source term describes processes with an infinite speed of propagation of perturbations, and if $u_0\not\equiv 0$, then $u(t,x)>0$ wherever the solution is defined. Therefore heat localization in strict sense is impossible here, unlike the case of §§ 1, 4, and we have to use the concept of *effective localization*. There will be two directions of inquiry: first of all, we shall clarify the conditions for occurrence of unbounded solutions and, secondly, we shall establish conditions for their localization (or lack thereof).

The main results will be obtained by applying methods, many of which are fitted to the analysis of semilinear equations of the form (1). This has to do with being able to invert the operator $(\partial/\partial t - \Delta)$, as a result of which the problem (1), (2) is reduced to an integral equation of a sufficiently simple form.

## 1 A general result of non-existence of global solutions

We shall start the study by deriving conditions for unboundedness of solutions of the problem (1), (2) with a general source term $Q(u)$. A great advantage of the semilinear equation (1) in comparison with quasilinear ones is, in particular, the fact that the solution of the corresponding equation without a source term,

$$v_t = \Delta v,\ t > 0,\ x \in \mathbf{R}^N;\ v(0,x) = u_0(x),\ x \in \mathbf{R}^N, \tag{4}$$

can be written down in terms of a heat potential

$$v(t,x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbf{R}^N} \exp\left\{-\frac{|y|^2}{4t}\right\} u_0(x+y)\,dy. \tag{5}$$

It turns out that one can effectively compare the solution of the problem (4) with the solution of the original problem (1), (2).

Let $E(p,\tau)$ be a sufficiently smooth function, which is monotone increasing in $p \ge 0$, $E(p,\tau) \ge 0$ for all admissible $p \ge 0$, $\tau \ge 0$; $E(0,\tau) = 0$ and $E(p,0) \equiv p$. The function $E$ has been introduced for an operator (functional) comparison of solutions of equations (1) and (4). By the change of variables

$$u(t,x) = E(U(t,x),t), \tag{6}$$

in terms of the new function $U$, equation (1) takes the form

$$U_t = \Delta U + \frac{E''_{UU}}{E'_U}|\nabla U|^2 + \frac{Q(E) - E'_t}{E'_U}, \tag{7}$$

and $U(0,x) = u_0$ in $\mathbf{R}^N$ by the identity $E(p,0) \equiv p$. Then, comparing (7) with the linear equation (4) (see § 1, Ch. I), by the Maximum Principle we have that in order to be able to compare their solutions, that is, in order that we have the inequality

$$U(t,x) \ge v(t,x),\ t > 0,\ x \in \mathbf{R}^N,$$

it is sufficient for the function $E(p,\tau)$ to satisfy the conditions

$$E''_{pp}(p,\tau) \ge 0,\ \ Q(E(p,\tau)) - E'_\tau(p,\tau) \ge 0. \tag{8}$$

**Lemma 1.** *Let $Q(u)$ be a convex function in $\mathbf{R}_+$, that is,*

$$Q''(u) \geq 0, u > 0. \tag{9}$$

*Then*

$$E(p,\tau) = F^{-1}(F(p) - \tau), \tag{10}$$

*where $F^{-1}$ is the function inverse to (3), is a solution of the system of inequalities (8).*

*Proof.* The function (10) transforms the second inequality in (8) into an identity. Let us write the first one in an equivalent form:

$$F''(p) + Q'(F^{-1}(F(p) - \tau))F'^2(p) \geq 0.$$

It is satisfied for $\tau = 0$. Therefore it holds for all $\tau > 0$ ($\tau < F(p)$) by convexity of $Q$ and monotonicity of $F$. □

Operator (10) is the identity for $\tau = 0$.

**Lemma 2.** *Let $Q(u)$ be a convex function. Then for the problem (1), (2) we have the lower bound*

$$u(t,x) \geq F^{-1}[F(v(t,x)) - t], \; t > 0, \;\; x \in \mathbf{R}^N, \tag{11}$$

*where $v(t,x)$ is determined from (4).*

Since $F$ is a decreasing function, inequality (11) is equivalent to the following one:

$$F(v(t,x)) - F(u(t,x)) \geq t, \;\; t > 0, x \in \mathbf{R}^N. \tag{12}$$

In the case of a source term of power type, $Q(u) = u^\beta$, $\beta > 1$, we have $F(s) = s^{1-\beta}/(\beta - 1)$ and (12) assumes the form

$$v^{1-\beta}(t,x) - u^{1-\beta}(t,x) \geq (\beta - 1)t. \tag{13}$$

These inequalities come in handy for determining conditions of global insolvability of the Cauchy problem (1), (2).

**Theorem 1.** *Let $Q''(u) \geq 0$ for $u > 0$, and assume that the limit*

$$\lim_{s \to 0^+} \frac{s^{1+2/N}}{Q(s)} = \nu < \infty \tag{14}$$

*exists. Then for any initial functions, such that*

$$\|u_0\|_{L^1(\mathbf{R}^N)} > [2\pi N\nu]^{N/2}, \tag{15}$$

*the Cauchy problem (1), (2) has no global solutions.*

*Proof.* First of all let us note that $F^{-1}(0) = \infty$. Therefore it follows immediately from (12) that $u(t, x)$ is unbounded, if we can find $t_* > 0$ and $x_* \in \mathbf{R}^N$, such that $F(v(t_*, x_*)) - t_* \le 0$, or, equivalently,

$$F(v(t_*, x_*))/t_* \le 1. \tag{16}$$

Let us set $x_* = 0$. From (5) and the assumption $u_0 \in L^1(\mathbf{R}^N)$ we obtain

$$v(t, 0) \simeq (4\pi t)^{-N/2} \|u_0\|_{L^1(\mathbf{R}^N)}, \quad t \to \infty,$$

and therefore, resolving the indeterminacy in the expression $F(v(t, 0))/t$ as $t \to \infty$, we have

$$\lim_{t\to\infty} \frac{F(v(t, 0))}{t} = 2\pi N \|u_0\|^{2/N}_{L^1(\mathbf{R}^N)} \lim_{s\to 0^+} \frac{s^{1+2/N}}{Q(s)} = 2\pi N \|u_0\|^{2/N}_{L^1(\mathbf{R}^N)} \nu.$$

Therefore by (15) for this class of initial functions (16) holds for $x_* = 0$ and some sufficiently large $t_*$, which entails unboundedness of solutions of the Cauchy problem (1), (2). □

**Corollary.** *Let $\nu = 0$ in (14). Then for any $u_0 \not\equiv 0$, solution of the problem (1), (2) is unbounded.*

In the case $\nu > 0$ Theorem 1 defines a certain minimal initial energy needed for the occurrence of finite time blow-up: $E_{min} = (2\pi N\nu)^{N/2}$. In fact, for $\nu \in \mathbf{R}_+$ in many cases all non-trivial solutions of the problem are unbounded. In the sequel this will be demonstrated for the example of a power type source term, $Q(u) = u^\beta$. Let us note that using the inequality (16), we could establish conditions of global insolvability also for $\nu = \infty$ (in which case conclusions of the theorem to some extent indicate the possibility of existence of a class of global solutions).

## 2 Equation with a power type nonlinearity $u_t = \Delta u + u^\beta$

In this subsection we present a detailed analysis of unbounded and global solutions of the Cauchy problem for an equation with a power type source term:

$$\begin{gathered} \mathbf{A}(u) \equiv u_t - \Delta u - u^\beta = 0, \quad t > 0, \; x \in \mathbf{R}^N, \\ u(0, x) = u_0(x) \ge 0, \quad x \in \mathbf{R}^N; \; \beta > 1. \end{gathered} \tag{17}$$

Some of the results are the analogues of those obtained in § 3 for quasilinear equations; therefore they are stated without proofs. Observe that from Theorem 1 we immediately have that all solutions $u(t, x) \not\equiv 0$ are unbounded for $\beta \in (1, 1 + 2/N)$. Therefore globally existing solutions are possible only for $\beta \ge 1 + 2/N$ (in fact they do not exist for $\beta = 1 + 2/N$ either).

We shall start the study of the problem (17) by constructing unbounded sub- and global supersolutions. These provide explicit conditions of local or global solvability.

*1 Conditions of global insolvability of the problem*

Construction of unbounded subsolutions of problem (17) allows us to derive a sharp upper bound on the time of existence of a solution, which due to the technique of the proof was not obtained in Theorem 1.

Let us consider in $(0, T) \times \mathbf{R}^N$ the function

$$u(t, x) = (T - t)^{-1/(\beta-1)}\theta(\xi), \quad \xi = |x|/(T - t)^{1/2}. \tag{18}$$

Let $\theta \in C^2([0, \infty))$, $\theta'(0) = 0$. For the function (18) to be a subsolution of equation (17), it is enough to satisfy the inequality $\mathbf{A}(u) \leq 0$ in $(0, T) \times \mathbf{R}^N$. Substitution of (18) into (17) gives us the following condition:

$$\frac{1}{\xi^{N-1}}\left(\xi^{N-1}\theta'\right)' - \frac{1}{2}\theta'\xi - \frac{1}{\beta-1}\theta + \theta^\beta \geq 0, \; \xi > 0. \tag{19}$$

We shall seek the function $\theta$ in the form $\theta(\xi) = A\exp\{-\alpha\xi^2\}$, where $A > 0$, $\alpha > 0$ are constants.

Then from (19) we obtain the inequality

$$\alpha(4\alpha + 1)\xi^2 + A^{\beta-1}\exp\{\alpha(1 - \beta)\xi^2\} \geq 2\alpha N + 1/(\beta - 1). \tag{20}$$

It is easy to see that it is satisfied for any

$$\xi \geq \xi_* = \left[\frac{2\alpha N + 1/(\beta - 1)}{\alpha(4\alpha + 1)}\right]^{1/2}.$$

and in order that (20) holds for the remaining $\xi \in [0, \xi_*]$, it is sufficient that the inequality

$$A^{\beta-1}\exp\{\alpha(1 - \beta)\xi_*^2\} \geq 2\alpha N + 1/(\beta - 1) \tag{21}$$

be satisfied.

**Theorem 2.** *Let the initial function $u_0$ in* (17) *be such that*

$$u_0(x) \geq T^{-1/(\beta-1)}A\exp\{-\alpha|x|^2T^{-1}\}, \; x \in \mathbf{R}^N,$$

*where $T, \alpha, A$ are positive constants, and $\alpha, A$ satisfy inequality* (21). *Then the solution of problem* (17) *exists for time not exceeding $T$.*

For convenience, we state an immediate corollary of Theorem 1.

**Theorem 3.** *Let* $1 < \beta < 1 + 2/N$, $u_0 \not\equiv 0$. *Then the solution of problem* (17) *is unbounded.*

It is not hard to obtain this result by reasoning as in § 3, subsection 2, that is, by comparing the family of subsolutions (18) with an arbitrarily small fundamental solution of the heat equation. In the case $\beta < 1 + 2/N$ this procedure is comparatively simple. In the critical case $\beta = 1 + 2/N$ it is much harder to do, and therefore it is more convenient to construct an unbounded subsolution by an iterative method, as is done in the proof of the following assertion.

**Theorem 4.** *Let* $\beta = 1 + 2/N$, $u_0 \not\equiv 0$. *Then the problem* (17) *does not have a global solution.*

*Proof.* We carry out the proof for $N = 2$, that is, $\beta = 1 + 2/N = 2$. With slight modifications the same argument works for any $N$.

First of all let us note that for any initial function $u_0 \not\equiv 0$ we can always find constants $t_0$, $A_0$, $\alpha_0$, such that $u(t_0, x) \geq A_0 \exp\{-\alpha_0|x|^2\}$ in $\mathbf{R}^2$. Therefore by the comparison theorem, it is sufficient to prove the claim for functions of the form

$$u_0(x) = A_0 \exp\{-\alpha_0|x|^2\}, \quad x \in \mathbf{R}^2. \tag{22}$$

The Cauchy problem (17), (22) for $N = 2$, $\beta = 2$, is equivalent to the following integral equation:

$$u(t, x) = \mathbf{P}(u) \equiv (4\pi t)^{-1} \int_{\mathbf{R}^2} \exp\left\{-\frac{|x - y|^2}{4t}\right\} u_0(y)\, dy +$$
$$+ \int_0^t [4\pi(t - \tau)]^{-1} \int_{\mathbf{R}^2} \exp\left\{-\frac{|x - y|^2}{4(t - \tau)}\right\} (u(\tau, y))^2\, dy\, d\tau. \tag{23}$$

Let us form the recurrent sequence of functions

$$U_1(t, x) = \mathbf{P}(0); \quad U_{n+1}(t, x) = \mathbf{P}(U_n(t, x)), \quad n = 1, 2, \ldots .$$

From (23) it follows immediately that for any $n$

$$u(t, x) \geq U_n(t, x), \quad t > 0, x \in \mathbf{R}^2.$$

Therefore if $\{U_n\}$ diverges at least at one point, the original problem has no global solution. Let us show that this is indeed the case.

First of all we have

$$U_1(t, x) = \mathbf{P}(0) = \nu_1(t)E(t, x),$$

where $\nu_1(t) = A_0/(1 + 4\alpha_0 t)$, $E(t, x) = \exp\{-\alpha_0|x|^2/(1 + 4\alpha_0 t)\}$. Let us now estimate other terms in the sequence.

Let us prove by induction that

$$U_n(t, x) \geq \sum_{k=1}^{n} \nu_k(t) E^k(t, x), \quad n = 2, 3, \dots, \tag{24}$$

where functions $\nu_k \geqslant 0$ will be defined below ($\nu_1$ we already know). From (23) we easily obtain

$$U_{n+1}(t, x) \geq \nu_1 E +$$

$$+ \int_0^t d\tau \int_{\mathbf{R}^2} [4\pi(t-\tau)]^{-1} \exp\left\{-\frac{|x-y|^2}{4(t-\tau)}\right\} \left(\sum_{k=1}^{n} \nu_k(\tau) E^k(\tau, y)\right)^2 dy \geq$$

$$\geq \nu_1 E + \sum_{k=1}^{n} \int_0^t \sum_{l=1}^{k} \nu_l(\tau) \nu_{k+1-l}(\tau)\, d\tau \times$$

$$\times \int_{\mathbf{R}^2} [4\pi(t-\tau)]^{-1} \exp\left\{-\frac{|x-y|^2}{4(t-\tau)}\right\} E^{k+1}(\tau, y)\, dy. \tag{25}$$

The inner integrals in spatial variables are easily calculated for each $k = 1, 2, \dots, n$ and are equal to

$$\frac{1 + 4\alpha_0 \tau}{1 + 4\alpha_0 t - 4k\alpha_0 \tau} \exp\left\{-\frac{(k+1)\alpha_0 |x|^2}{1 + 4(k+1)\alpha_0 t - 4k\alpha_0 \tau}\right\} \geq$$

$$\geq \frac{1 + 4\alpha_0 \tau}{(k+1)(1 + 4\alpha_0 t)} E^{k+1}(t, x).$$

Therefore from (25) we obtain the estimate

$$U_{n+1}(t, x) \geq$$

$$\geq \nu_1 E + \sum_{k=1}^{n} \frac{E^{k+1}(t, x)}{(k+1)(1 + 4\alpha_0 t)} \int_0^t \sum_{l=1}^{k} \nu_l(\tau) \nu_{k+1-l}(\tau)(1 + 4\alpha_0 \tau)\, d\tau,$$

that is, in (24) we can set

$$\nu_{k+1}(t) = \frac{1}{(k+1)(1 + 4\alpha_0 t)} \int_0^t \sum_{l=1}^{k} \nu_l(\tau) \nu_{k+1-l}(\tau)(1 + 4\alpha_0 \tau)\, d\tau. \tag{26}$$

Let us show that hence we can obtain the following inequalities:

$$\nu_k(t) \geq \frac{k}{6^{k-1}(1 + 4\alpha_0 t)} \left(\frac{A_0}{4\alpha_0}\right)^k 4\alpha_0 \ln^{k-1}(1 + 4\alpha_0 t). \tag{27}$$

This estimate is valid for $k = 1$. Let it hold for all $1 \leq k \leq M$, and let us show that (27) holds also for $k = M + 1$. From (26) we have that

$$v_{M+1}(t) \geq \frac{1}{(M+1)(1+4\alpha_0 t)} \left(\frac{A_0}{4\alpha_0}\right)^{M+1} (4\alpha_0)^2 \times$$
$$\times \int_0^t \frac{\ln^{M-1}(1+4\alpha_0\tau)}{1+4\alpha_0\tau} \cdot 6^{-M+1}\, d\tau \sum_{l=1}^{M} l(M+1-l). \tag{28}$$

It is not hard to verify that $\sum_{l=1}^{M} l(M+1-l) = M(M+1)(M+2)/6 > M(M+1)^2/6$, so that (27) follows immediately from the inequality (28) for $k = M + 1$.

Thus, as $n \to \infty$ we obtain the inequality

$$u(t, x) \geq \frac{A_0 E(t, x)}{1 + 4\alpha_0 t} \sum_{k=1}^{\infty} k z^{k-1}, \tag{29}$$

where

$$z = z(t, x) = \frac{A_0 E(t, x)}{24\alpha_0} \ln(1 + 4\alpha_0 t) > 0.$$

However, the series in (29) diverges for $z = 1$, for example, for $x = 0$ ($E(t, 0) = 1$), $t = t_*$, where

$$\frac{A_0}{24\alpha_0} \ln(1 + 4\alpha_0 t_*) = 1,$$

that is,

$$t_* = \frac{1}{4\alpha_0}\left(\exp\left\{\frac{24\alpha_0}{A_0}\right\} - 1\right).$$

Therefore solution of the problem with an initial function of the form (22) exists for time not exceeding $t_*$. □

Summing the series in (29), we can obtain an explicit form of the presumed "subsolution" $u_-(t, x)$, which has been constructed by the iterative procedure for the critical case $\beta = 1 + 2/N$, $N = 2$. It has quite an unusual spatio-temporal structure:

$$u(t, x) \geq u_-(t, x) =$$

$$= \frac{A_0}{1 + 4\alpha_0 t} \exp\left\{-\frac{\alpha_0 |x|^2}{1 + 4\alpha_0 t}\right\} \left\{1 - \frac{A_0}{24\alpha_0} \exp\left\{-\frac{\alpha_0 |x|^2}{1 + 4\alpha_0 t}\right\} \ln(1 + 4\alpha_0 t)\right\}^{-2},$$

$$0 < t < t_*, \quad x \in \mathbf{R}^2.$$

Thus, if $1 < \beta \leq 1 + 2/N$ all non-trivial solutions of the problem (17) are unbounded.

*2 Existence time of elementary perturbations*

Using inequality (13), derived in subsection 1, it is possible to obtain explicit upper bounds on the time of existence of unbounded solutions. In the case of power type nonlinearity, this inequality has the form

$$\int_{\mathbf{R}^N} \exp\left\{-\frac{|y|^2}{4t}\right\} u_0(x+y)\,dy \geq M t^{N/2-1/(\beta-1)}, \tag{30}$$

where $M = (4\pi)^{N/2}(\beta-1)^{-1/(\beta-1)}$. If this inequality is satisfied at a point $(t_*, x_*)$, then the unbounded solution exists for time not exceeding $t_*$.

Let $u_0(x)$ be an elementary perturbation: $u_0(x) = \delta > 0$ for all $|x| < a < \infty$, $u_0(x) \equiv 0$ for $|x| \geqslant a$. Using the estimate $\exp\{-|y|^2/(4t)\} \geq (1-|y|^2/(4t))_+$, we obtain a lower bound for the integral in (30):

$$\begin{aligned} I(t,x) &\geq \int_{|y|<l(t)} \left(1-\frac{|y|^2}{4t}\right) u_0(x+y)\,dy, \\ l(t) &= \min\{a, 2t^{1/2}\}. \end{aligned} \tag{31}$$

Clearly, in this case we can set $x_* = 0$. Let us assume initially that $t_* > a^2/4$. Then $l(t_*) = a$, and the upper bound for the time of existence of the solution is determined from the equation

$$\|u_0\|_{L^1(\mathbf{R}^N)} - \frac{M_1}{t} = M t^{N/2-1/(\beta-1)}, \quad M_1 = \frac{a^{N+2}\pi^{N/2}}{2\Gamma(N/2)(N+2)}\delta.$$

For certain $\beta$ this equation can be solved exactly.

For example, for $\beta = 1 + 2/N$

$$t_* = \frac{M_1}{\|u_0\|_{L^1(\mathbf{R}^N)} - M}, \quad \|u_0\|_{L^1(\mathbf{R}^N)} > M.$$

This formula is correct if $t_* > a^2/4$, that is,

$$M < \|u_0\|_{L^1(\mathbf{R}^N)} < 4M_1/a^2 + M.$$

If $\beta = (4+N)/(2+N)$, then

$$t_* = \frac{M+M_1}{\|u_0\|_{L^1(\mathbf{R}^N)}}, \quad \|u_0\|_{L^1(\mathbf{R}^N)} < \frac{4}{a^2}(M+M_1).$$

Let us note that this estimate shows that for $\beta = (4+N)/(2+N) < 1+2/N$ solutions corresponding to elementary initial perturbations with arbitrarily low energy, are unbounded. The same applies to the case $\beta = (3+N)/(1+N)$, when

$$t_* = \left\{ \frac{M}{2\|u_0\|_{L^1(\mathbf{R}^N)}} + \left[ \left( \frac{M}{2\|u_0\|_{L^1(\mathbf{R}^N)}} \right)^2 + M_1 \right]^{1/2} \right\}^2$$

(here we must have $t_* > a^2/4$, that is, the energy $\|u_0\|_{L^1(\mathbf{R}^N)}$ must not be too high). This estimate is valid, for example, if $\|u_0\|_{L^1(\mathbf{R}^N)} \le 2M/a$.

Let us now consider the case $t^* \le a^2/4$. Then $l(t) = 2t^{1/2}$ in (31), and the solution of the inequality (30) has the form

$$t^* = \left[ \frac{M}{M_2\|u_0\|_{L^1(\mathbf{R}^N)}} \right]^{\beta-1}, M_2 = \frac{2^{N+1}}{(N+2)a^N}.$$

The estimate $T_0 \le t_*$ holds if $t^* \le a^2/4$, that is, if

$$\|u_0\|_{L^1(\mathbf{R}^N)} \ge \frac{M}{M_2} \left( \frac{a^2}{4} \right)^{-1/(\beta-1)}.$$

*3 Global solutions for $\beta > 1 + 2/N$*

We shall seek a bounded supersolution of equation (17) in the form

$$u_+(t,x) = (T+t)^{-1/(\beta-1)}\theta_+(\xi),\ \xi = |x|/(T+t)^{1/2}, \tag{32}$$

where $\theta_+(\xi) = A\exp\{-\alpha\xi^2\}$ and $T, A, \alpha$ are positive constants. Substitution of (32) in the condition $\mathbf{A}(u_+) \ge 0$ results in an inequality, which can be brought to the form

$$\alpha(4\alpha-1)\xi^2 + A^{\beta-1}\exp\{\alpha(1-\beta)\xi^2\} \le 2\alpha N - 1/(\beta-1),\ \xi \in \mathbf{R}_+. \tag{33}$$

From this we obtain the restrictions on the parameters $A$ and $\alpha$. First of all, the right-hand side of (33) must be positive, that is

$$\alpha > \frac{1}{2N(\beta-1)}. \tag{34}$$

Secondly,

$$A \le \left( 2\alpha N - \frac{1}{\beta-1} \right)^{1/(\beta-1)},\ \alpha \le \frac{1}{4}, \tag{35}$$

and from these inequalities we have the restriction $\beta > 1 + 2/N$. Thus we have proved

**Theorem 5.** *Let $\beta > 1 + 2/N$ and let the initial function $u_0$ be such that*

$$u_0(x) \leq T^{-1/(\beta-1)} A \exp\{-\alpha|x|^2 T^{-1}\}, \ x \in \mathbf{R}^N, \tag{36}$$

*where $T$, $A$, $\alpha$ are constants, the two last ones satisfying inequalities* (34), (35). *Then problem* (17) *has a global solution, and furthermore*

$$u(t, x) \leq (T + t)^{-1/(\beta-1)} A \exp\left\{-\frac{\alpha|x|^2}{T+t}\right\} \ \ in \ \mathbf{R}_+ \times \mathbf{R}^N. \tag{37}$$

In conclusion, let us observe that the stable set $\mathcal{W}$ constructed here consists of functions $u_0$ which decay exponentially as $|x| \to \infty$. As in subsections 4, 5 of § 3, for $\beta > 1 + 2/N$ we could construct a different set $\mathcal{W}$ with a weaker (power) decay rate of $u_0(x)$ at infinity. In this case the boundary of $\mathcal{W}$ consists of global self-similar solutions of equation (17), which will be considered in subsection 2.6.

### *4 Effective localization of unbounded solutions. LS-regime of combustion*

In this subsection we move on to a description of particular properties of unbounded solutions of problem (17): their spatio-temporal structure for times close to the blow-up time. A fundamental property of blow-up regimes, which does not depend on precise initial functions, is the property of localization. Equation (17) describes processes with infinite speed of propagation of perturbations, therefore, as in the case of the boundary value problem for the heat equation without a source term (see § 4 of Ch. III), we shall introduce the concept of effective localization of combustion.

**Definition.** An unbounded solution of the Cauchy problem (17) is called *effectively localized* if it goes to infinity as $t \to T_0^-$ ($T_0 < \infty$ is the time of existence of the solution) on a bounded set

$$\omega_L = \{x \in \mathbf{R}^N \mid u(T_0, x) \equiv \overline{\lim_{t \to T_0}} u(t, x) = \infty\},$$

which we shall call the *localization domain*.

If, on the other hand, $\omega_L$ is an unbounded domain (for example, $\omega_L = \mathbf{R}^N$) then we say that there is no effective localization.

For our purposes the above definition is sufficient. In the general case the following blow-up set should be considered: $B_L = \{x \in \mathbf{R}^N \mid \exists t_n \to T_0 \text{ and } x_n \to x, \text{ such that } u(t_n, x_n) \to \infty \text{ as } n \to \infty\}$, which by definition of an unbounded solution is non-empty for bell-shaped data.

In the following an effectively localized combustion process will be called simply localized. In the one-dimensional case it is convenient to introduce *localization depth*

$$L_T = \operatorname{meas}\{x \in \mathbf{R} \mid u(T_0^-, x) = \infty\}$$

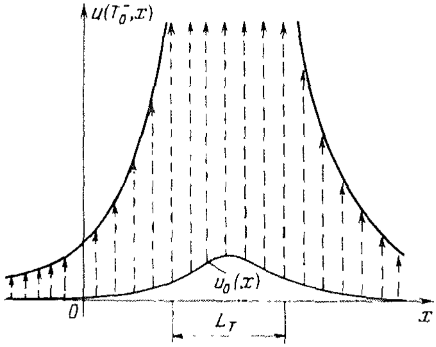


**Fig. 64.** Effective localization ($L_T$ is the localization depth)

(that is, $L_T$ is the extent of the domain in which the solution grows without bound as $t \to T_0$; see Figure 64). If $u(t, x)$ becomes infinite at one point, then $L_T = 0$, which corresponds to the LS blow-up regime of combustion.

Evolution of unbounded solutions of the problem (17) proceeds for $\beta > 1$, as a rule, precisely in the LS-regime and $u(t, x) \to \infty$ on a set $\omega_L$ of measure zero. This is indicated, for example, by the estimates obtained in Theorem 2, in which we derived unbounded subsolutions that do evolve in the LS-regime. And, of course, this conclusion is corroborated by numerical computations. In Figure 65 we present results of one such computation. It is clearly seen that in the blow-up process there arises a spatio-temporal structure with ever decreasing half-width and a conspicuous unique maximum in $x$ of the spatial profile.

Let us consider the spatio-temporal structure of unbounded solutions for times close to blow-up time. For that, by analogy with the quasilinear case (§ 1), we can consider unbounded self-similar solutions

$$u_S(t, x) = (T_0 - t)^{-1/(\beta-1)}\theta_S(\xi), \ \xi = |x|/(T_0 - t)^{1/2}, \tag{38}$$

where the function $\theta_S(\xi) > 0$ satisfies the ordinary differential equation

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}\theta_S')' - \frac{1}{2}\theta_S'\xi - \frac{1}{\beta-1}\theta_S + \theta_S^\beta = 0, \ \xi > 0; \tag{39}$$

$$\theta_S'(0) = 0, \ \theta_S(\infty) = 0. \tag{40}$$

If we assume that the self-similar solution $u_S$ describes characteristic properties of LS blow-up regimes, then the amplitude of the solution $u_m(t)$ and the half-width of a symmetric domain of intensive combustion $l_{ef}(t)$ can be estimated as $t \to T_0$ according to

$$u_m(t) \equiv \sup_x u(t, x) \sim (T_0 - t)^{-1/(\beta-1)}, \ l_{ef}(t) \sim (T_0 - t)^{1/2}. \tag{41}$$

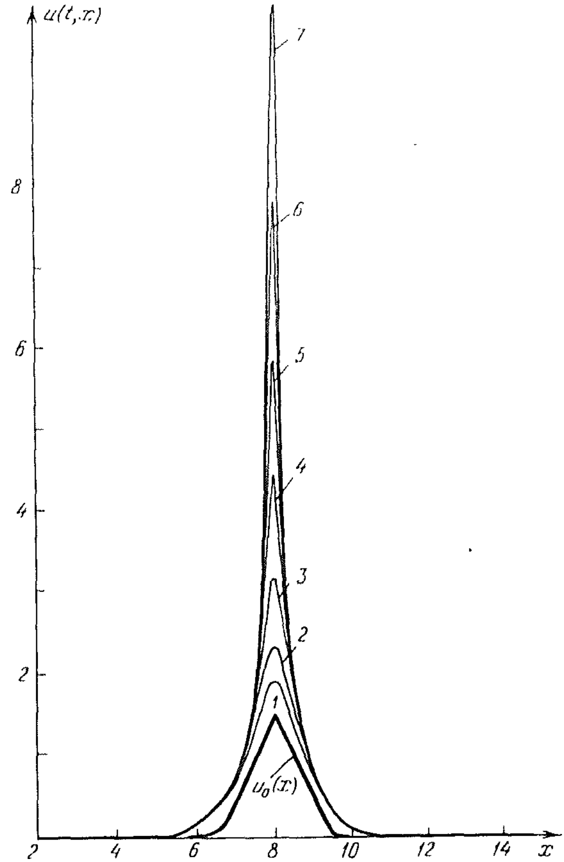


**Fig. 65.** Numerical solution of the problem (17) for $\beta = 4$, $N = 1$; 1: $t_1 = 0.236$, 2: $t_2 = 0.269$, 3: $t_3 = 0.291$, 4: $t_4 = 0.299$, 5: $t_5 = 0.3006$, 6: $t_6 = 0.3018$, 7: $t_7 = 0.3022$

There is, however, one significant difference between this and the quasilinear case.

**Proposition 1.** *Let* $N = 1$. *Then for any* $\beta > 1$ *the problem* (39), (40) *has no solutions* $\theta_S(\xi) > 0$.

For the case $\beta = 3$ it has been proved in [219]; for arbitrary $\beta > 1$ it has been established in [3] (see also [1, 2], where $1 < \beta < 3$). In the course of the proof (see [3]) it is shown that every solution of equation (39) in the domain

$\{\xi > \xi_0 \geq 0\}$, satisfying conditions

$$\theta(\xi_0) = \mu > \theta_H = (\beta - 1)^{-1/(\beta-1)}, \ \theta'(\xi_0) = 0 \tag{42}$$

vanishes at some point $\xi = \xi_1 > \xi_0$, and is monotone on $(\xi_0, \xi_1)$. Therefore (39), (42) has no non-monotone positive solutions having a point of minimum. We remind the reader that in the proof of Theorem 3, subsection 4, § 1, we made essential use of non-monotonicity of solutions.

It is appropriate to recall that the analysis of that subsection concerning the self-similar equation (39), linearized around the homothermic solution $\theta \equiv (\beta - 1)^{-1/(\beta-1)}$, shows that it has no non-monotone solutions.

Let us observe that this argument correctly indicates non-existence of non-trivial solutions of the problem (39), (40) for any $1 < \beta \leq (N+2)/(N-2)_+$ [197]. Therefore for those values of $\beta$ asymptotic evolution of unbounded solutions follows non-self-similar patterns. In distinction to (41), the half-width of the combustion domain changes as $t \to T_0^-$ according to $l_{ef}(t) \sim (T_0 - t)^{1/2}|\ln(T_0 - t)|^{1/2}$ (see Remarks).

In the case $\beta > (N+2)/(N-2)_+$ the problem (39), (40) may have non-trivial solutions. Then as $t \to T_0^-$, the solution evolves according to the self-similar laws of (41).

**Example.** Let $\beta = 2$. Then for all $N \in (6, 16)$ (note that here $\beta > 1 + 2/N$), there exists a solution of the problem (39), (40)

$$\theta_S(\xi) = A_N/(a_N + \xi^2)^2 + B_N/(a_N + \xi^2), \tag{43}$$

where $a_N, A_N, B_N$ are positive constants:

$$\begin{aligned} a_N &= 2[10(1 + N/2)^{1/2} - (N + 14)], \\ A_N &= 48[10(1 + N/2)^{1/2} - (N + 14)], \\ B_N &= 24[(1 + N/2)^{1/2} - 2]. \end{aligned}$$

Non-existence of self-similar solutions in the one-dimensional case requires a substantial modification of the method of proof of effective localization for unbounded solutions.

### *5 Proof of effective localization in the one-dimensional case*

Thus, for $N = 1$ the problem (39), (40) has no solutions. However, (39) admits solutions $\theta_S(\xi)$ with the asymptotics

$$\theta_S = \theta_*(\xi) = C\xi^{-2/(\beta-1)}(1 + \omega(\xi)), \ \xi \to \infty, \tag{44}$$

where $\omega(\xi) \to 0$ as $\xi \to \infty$ and $C > 0$ is a constant. Let us fix a $C = C_*$ in (44) and extend the solution into the domain of sufficiently small $\xi$. Then, obviously,

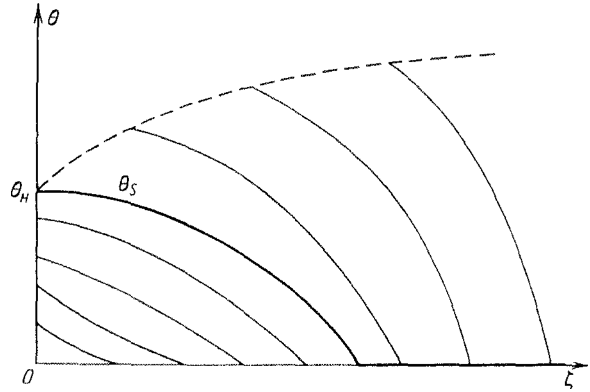


**Fig. 68.** Solutions of equation (76) for $1 < \beta < 2$

Therefore for $\beta > 2$ the solution is strictly positive in $\mathbf{R}_+$. □

Let us also observe that for all $\beta > 1$, the required solution $\theta_S$ is not smaller than the decreasing solution of the equation without a source,

$$f'^2 - \frac{\beta - 2}{2(\beta - 1)} f'\zeta - \frac{1}{\beta - 1} f = 0, \ \zeta > 0, \ f(0) = \theta_H,$$

which was considered in detail in § 4, Ch. III. There it was shown that the function $f(\zeta) \geqslant 0$ is determined from a certain algebraic equality, which allows us to derive lower bounds for $\theta_S(\zeta)$. Hence, for example in the case $1 < \beta < 2$, we have

$$\text{meas supp}\, \theta_S > \text{meas supp}\, f = 2\left(\frac{2 - \beta}{\beta - 1}\right)^{(2-\beta)/[2(\beta-1)]}.$$

Quite a good understanding of qualitative properties of the solution $\theta_S$ can be gleaned from considering Figures 68, 69, which show the field of integral curves of the equation

$$\theta' = \frac{\beta - 2}{4(\beta - 1)}\zeta - \left\{\left[\frac{\beta - 2}{4(\beta - 1)}\right]^2 \zeta^2 + \frac{1}{\beta - 1}\theta - \theta^\beta\right\}^{1/2}, \tag{76}$$

which is equivalent to (70). The thick line denotes the solution $\theta_S \geq 0$ with the asymptotics (72), which satisfies conditions (71).

Here it has to be said that for $\beta > 2$, apart from a solution of the form (72), there exists another family of admissible positive solutions with different asymptotics:

$$\theta_S(\zeta) \simeq (\beta - 1)^{-1/(\beta-1)} - C\zeta^{[2(\beta-1)]/(\beta-2)}, \ \zeta \to 0; \ C = \text{const} > 0. \tag{72'}$$

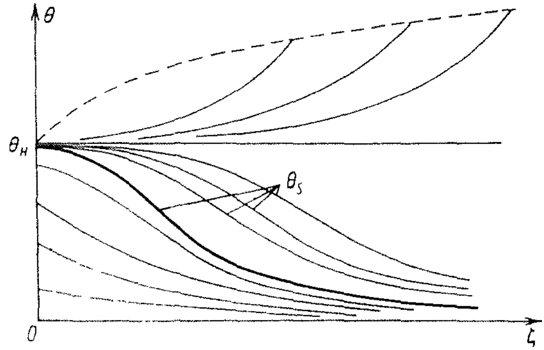


**Fig. 69.** Solutions of equation (76) for $\beta > 2$. Solutions of the problem (70), (71) are not unique

which is not analytic at $\zeta = 0$ and is shown in Figure 69. Numerical simulations of the problem (56), (57) indicate that the spatio-temporal structure of a.s.s. (68) does correctly describe the asymptotic properties of unbounded solutions and that the function (72) is realized in the LS-regime as $t \to T_0^-$.

Validity of this "slicing" of equation (56) and passage to a first order equation has been checked by numerous numerical experiments, which demonstrated asymptotic convergence of the unbounded solution of problem (56), (57) to a.s.s. (68). In numerical computations, similarity representation of the solution $U(t, x)$ was determined from the formula

$$\theta(t, \zeta) = \theta_s(0) \frac{U\left(t, \zeta \|U\|_{C_x}^{(2-\beta)/2} \cdot \theta_s^{(\beta-2)/2}(0)\right)}{\|U\|_{C_x}}, \tag{77}$$

where $\|U\|_{C_x} = \sup_x U(t, x)$, $\theta_s(0) = (\beta - 1)^{-1/(\beta-1)}$. As always, the spatio-temporal structure of a.s.s. (68) is implicit in the representation (77), and if $u(t, x)$ evolves according to the rules of (68), we must have the limiting equality

$$\lim_{\|U\|_{C_x} \to \infty} \theta(t, \zeta) = \theta_s(\zeta). \tag{78}$$

We emphasize that in this case it is hard to accomplish numerically the usual similarity normalization as, for example, in § 4, Ch. III, since here the blow-up time is not known *a priori*.

Convergence in the sense of (78) to a.s.s. (68) for sufficiently general initial functions has been established numerically for different $\beta \in (1, 2]$. For $\beta > 2$ (LS-regime), as we know, the similarity function $\theta_s = \theta_s(|\xi|)$ in (68) is not

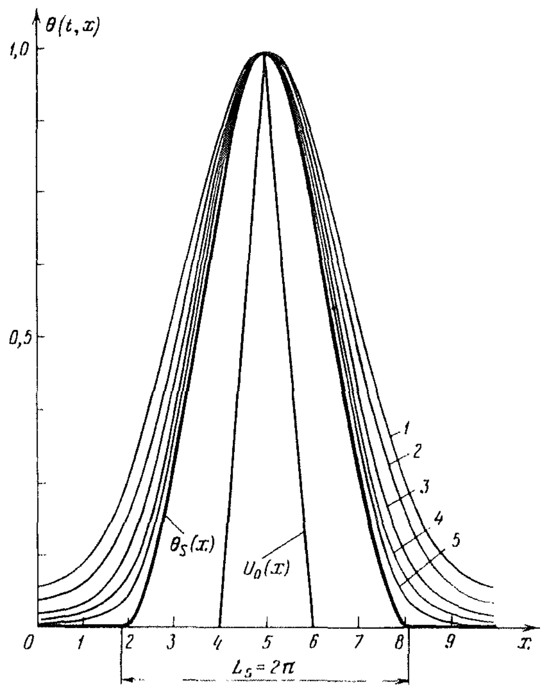


**Fig. 70.** Numerical verification of asymptotic stability in the sense of (78) of a.s.s. (68) for $\beta = 2$ ($\zeta \equiv x$), $N = 1$; 1: $t_1 = 2.82$, 2: $t_2 = 3.03$, 3: $t_3 = 3.12$, 4: $t_4 = 3.16$, 5: $t_5 = 3.18$

uniquely defined. As an example (Figure 70) we show the results of similarity transformation (77) of the solution of problem (56), (57) for $\beta = 2$, when $\theta_s$ has a very simple form; see (74). Convergence (78) is clearly seen already when the solution grows by a factor of 10-20.

## 4 Three types of unbounded solutions

From (68) it follows that the spatio-temporal structure of a.s.s. depends on the sign of the difference $\beta - 2$. Thus, if $\beta < 2$, then $U_s(t, x) \to \infty$ as $t \to T_0^-$ simultaneously in the whole space; this is the HS-regime, and there is no localization. Thus we have found a semilinear equation with unbounded HS-solutions (recall that the equation $u_t = \Delta u + u^\beta$ does not admit such solutions). Figure 71 shows the results of numerical computation of the one-dimensional problem (56),

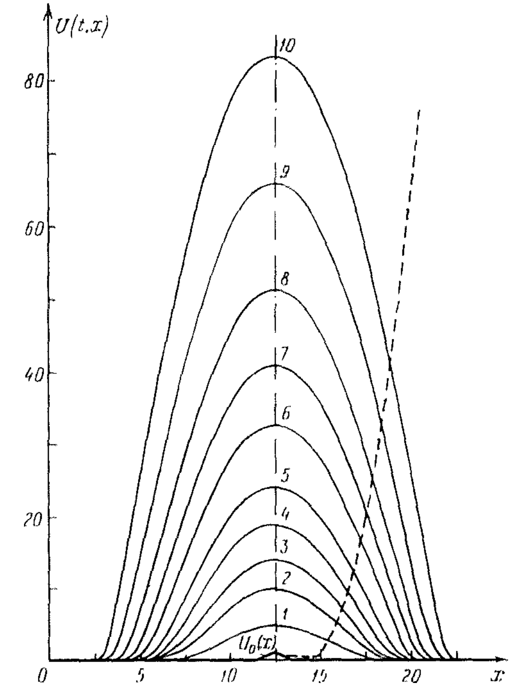


**Fig. 71.** Numerical solution of the problem (56), (57) for $\beta = 1.35$ (HS-regime), $N = 1$; 1: $t_1 = 2.10$, 2: $t_2 = 2.87$, 3: $t_3 = 3.00$, 4: $t_4 = 3.11$, 5: $t_5 = 3.19$, 6: $t_6 = 3.28$, 7: $t_7 = 3.34$, 8: $t_8 = 3.40$, 9: $t_9 = 3.45$, 10: $t_{10} = 3.51$

(57) for $\beta < 2$, which are in good agreement with (68), in particular, as regards the change in the amplitude and half-width of the solution.

For $\beta = 2$, as follows from (68) and numerical results, the combustion process evolves in the S-regime: $u(t, x)$ becomes infinite on a bounded set. In the case of a symmetric elementary initial perturbation, the localization domain of the S-regime is the support of the function (74): $\omega_L = \{|x| < \pi\}$ (Figure 72). Similarity transformation of this computation is shown in Figure 70. If localization domains corresponding to different perturbations are disjoint, combustion inside each one of them proceeds almost independently as $t \to T_0^-$ (Figure 73).

**Remark.** Proving localization of sufficiently arbitrary unbounded solutions of equation (56) for $\beta \geqslant 2$ is an interesting mathematical problem. In particular, in the one-dimensional case for $\beta = 2$, one could use to that end the method of intersection comparison of the solution under consideration and an interesting exact non-invariant solution (it is not invariant with respect to Lie-Bäcklund transforma-

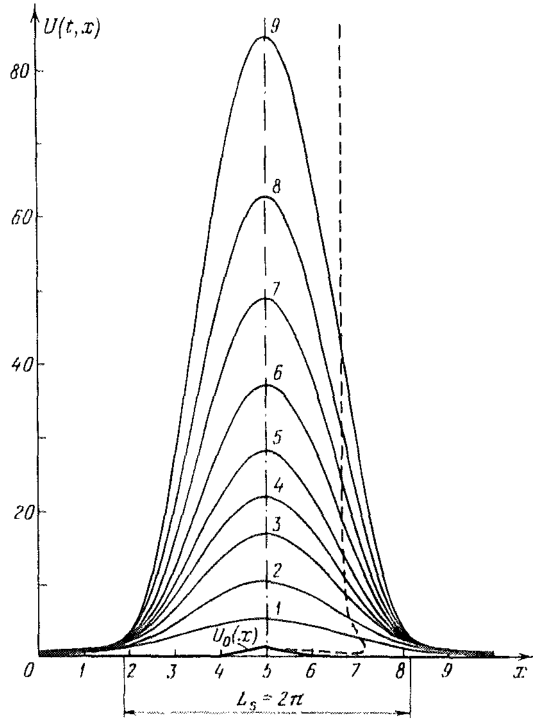


**Fig. 72.** Numerical solution of the problem (56), (57) for $\beta = 2$ (S-regime), $N = 1$; 1: $t_1 = 3.007$, 2: $t_2 = 3.107$, 3: $t_3 = 3.142$, 4: $t_4 = 3.156$, 5: $t_5 = 3.165$, 6: $t_6 = 3.173$, 7: $t_7 = 3.179$, 8: $t_8 = 3.183$, 9: $t_9 = 3.187$

tion groups). This exact solution, which is $2\pi$-periodic in space, has the following form:

$$U_*(t, x) = \phi(t)|\psi(t) + \cos x|,$$

where the functions $\phi(t)$, $\psi(t)$ satisfy the system of nonlinear ordinary differential equations

$$\phi' = -\phi + 2\phi^2\psi, \psi' = \psi + \phi - \phi\psi^2, \ t > 0,$$

as can be easily checked. In fact, this dynamical system is precisely the semilinear parabolic equation on the linear subspace $\mathcal{L}\{1, \cos x\}$, which is invariant under the nonlinear operator $U_{xx} + (U_x)^2 + U^2$. This system is equivalent to the first order equation

$$d\psi/d\phi = (\psi + \phi - \phi\psi^2)/(2\phi^2\psi - \phi).$$

It is easy to show that this equation has a family of trajectories corresponding to various unbounded solutions of equation (56) for $N = 1$, $\beta = 2$, with the following asymptotic behaviour close to the blow-up time $t = T_0$:

$$\phi(t) = (1/2)(T_0 - t)^{-1}[1 + O((T_0 - t)|\ln(T_0 - t)|)],$$

$$\psi(t) = 1 + O((T_0 - t)|\ln(T_0 - t)|).$$

Therefore, as $t \to T_0$

$$U_*(t, x) = (T_0 - t)^{-1}\cos^2(x/2) + O(|\ln(T_0 - t)|),$$

that is, as the blow-up time is approached, the exact solution $U_*(t, x)$ converges uniformly (after the corresponding similarity transformation) to the approximate self-similar solution (68), (74). However, in the general case the problems of localization for $\beta = 2$, $N > 1$, remain for the most part open.

If $\beta > 2$, then (68) ensures unbounded growth of the solution at one point only. This is the localized LS-regime; a specific example is shown in Figure 74. It is clearly seen that everywhere, apart from one singular point, the solution $U(t, x)$ of (56), (57), is bounded from above uniformly in $t$ by some limiting profile $U(T_0^-, x) < \infty$, $x \neq 3$.

In those figures, dashed lines indicate the motion of the half-width $x_{ef}(t)$ of the thermal structure, combustion of which is initiated by the same perturbation. First the amplitude of the solution $U(t, x)$ becomes smaller and the half-width increases, which corresponds to the process of spread of the non-resonance perturbation. Then, approaching finite time blow-up, $x_{ef}(t)$ starts to change in accordance with a.s.s. (68): $x_{ef}(t) \sim (T_0 - t)^{(\beta-1)/[2(\beta-1)]}$, $t \to T_0$. In particular, for $\beta = 2$ (S-regime) the half-width stabilizes, which can be clearly seen in Figure 72.

Conclusions concerning stability of a.s.s., which we presented above, are in good agreement with the qualitative non-stationary averaging theory. As in § 2, we shall seek an approximate solution $U_*(t, x)$ in the form $U_*(t, x) = \psi(t)\mu(\xi)$, $\xi = x/\phi(t)$; $t > 0$, $\xi \in \mathbf{R}^N$, and let us demand that $U_*$ satisfy the conservation laws

$$\int_{\mathbf{R}^N} \mathbf{D}(U_*)\,dx = 0, \quad \int_{\mathbf{R}^N} U_*\mathbf{D}(U_*)\,dx = 0.$$

Then we arrive at a system of ordinary differential equations for the amplitude $\psi(t)$ and the half-width $\phi(t)$:

$$\begin{aligned} (\psi\phi^N)' &= \nu_1\psi^2\phi^{N-2} + \nu_2\psi^\beta\phi^N, \\ (\psi^2\phi^N)' &= -\nu_3\psi^2\phi^{N-2} + \nu_4\psi^3\phi^{N-2} + \nu_5\psi^{\beta+1}\phi^N, \end{aligned} \tag{79}$$

where $\nu_i$ ($i = 1, 2, \ldots, 5$) are some functionals of $\mu$ (their exact form can be easily written down). From the system we pass to the single equation

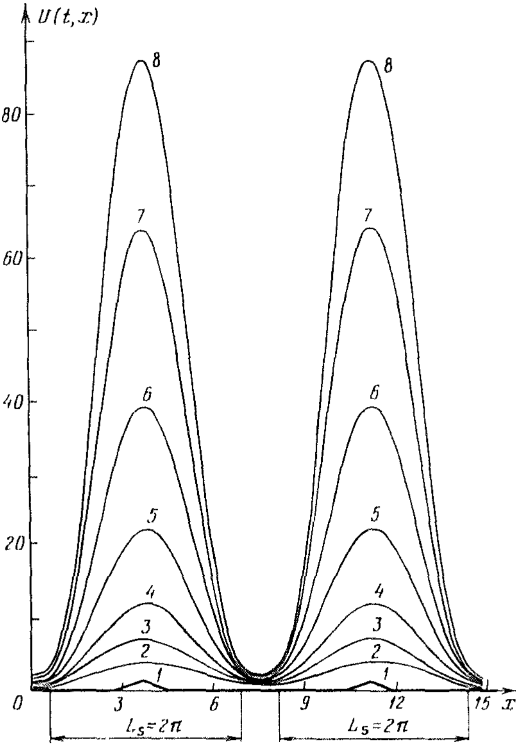


**Fig. 73.** Independent combustion of thermal structures for $\beta = 2$ (S-regime), $N = 1$; 1: $t_1 = 0$, 2: $t_2 = 2.915$, 3: $t_3 = 3.026$, 4: $t_4 = 3.079$, 5: $t_5 = 3.115$, 6: $t_6 = 3.133$, 7: $t_7 = 3.1421$, 8: $t_8 = 3.1459$

$$\frac{d\psi}{d\phi} = -N\frac{\psi}{\phi}\left\{\frac{a_1\psi^{\beta-1}\phi^2 - b_1\psi - 1}{a_2\psi^{\beta-1}\phi^2 - b_2\psi - 1}\right\},\quad \psi > 0,\ \phi > 0, \tag{80}$$

where $a_1, a_2, b_1, b_2$ are some constants, which we take to be positive based on natural requirements on the behaviour of the trajectories.

Equation (80) is more complicated than the one considered in § 2. The main difference is essentially the following: (80) contains three independent critical values of the parameter $\beta$, which "control" the general structure of the phase plane. First is $\beta = 2$; the criterion $\beta \gtrless 2$ determines the presence, or lack, of localization of unbounded solutions (in the cases $\beta \geq 2$ and $\beta < 2$ the behaviour of the integral curves is completely different). Secondly, $\beta = 1 + 2/N$; for $\beta < 1 + 2/N$ equation (80) has no globally defined trajectories, while on the other hand, for $\beta > 1 + 2/N$ there are trajectories to which there correspond global solutions of the original

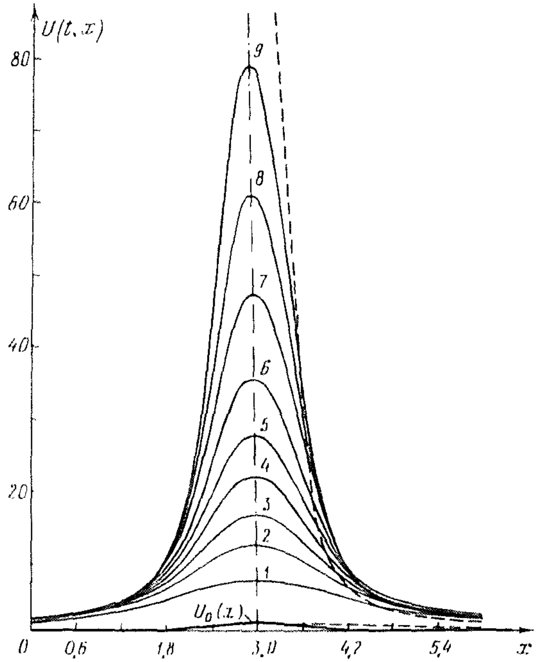


**Fig. 74.** Numerical solution of the problem (56), (57) for $\beta = 2.50$ (LS-regime). $N = 1$; 1: $t_1 = 4.257$, 2: $t_2 = 4.275$, 3: $t_3 = 4.280$, 4: $t_4 = 4.283$, 5: $t_5 = 4.285$, 6: $t_6 = 4.286$, 7: $t_7 = 4.2873$, 8: $t_8 = 4.2879$, 9: $t_9 = 4.2883$

problem. Finally, the third critical value is $\beta = 2 + 2/N$. For large amplitudes $\psi$ we can neglect constant terms in the numerator and the denominator of (80). As a result we have the approximate equation

$$\frac{d\psi}{d\phi} \simeq -N\frac{\psi}{\phi}\left\{\frac{a_1\psi^{\beta-2}\phi^2 - b_1}{a_2\psi^{\beta-2}\phi^2 - b_2}\right\},$$

which is the same as the one considered in § 2 for $\sigma = 1$. Therefore the phase plane behaviour depends on the condition $\beta \gtrless \sigma + 1 + 2/N = 2 + 2/N$. For $\beta < 2 + 2/N$ all unbounded trajectories converge to the "separatrix" generated by a.s.s. (68):

$$\psi \simeq \phi^{-2/(\beta-2)}, \quad \psi \to \infty;$$

$$\psi(t) \sim (T_0 - t)^{-1/(\beta-1)}, \quad \phi(t) \sim (T_0 - t)^{(\beta-2)/[2(\beta-1)]} \tag{81}$$

as $t \to T_0^-$. On the other hand, if $\beta \geq 2 + 2/N$, then the asymptotics of unbounded solutions as $t \to T_0^-$ is a non-self-similar one (see § 2).

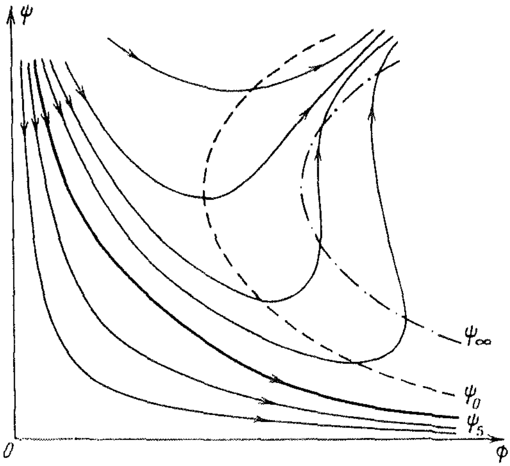


**Fig. 75.** Integral curves of equation (80) for $\beta \in (1, 2)$, $\beta > 1 + 2/N$

Most of these results are in good qualitative agreement with the conclusions we arrived at earlier. The estimates (81) give us, in addition, quantitative agreement for $1 < \beta < 2 + 2/N$, which supplements the evidence that the construction of unbounded a.s.s. (68) is valid.

Figures 75-77 show schematically the fields of integral curves of equation (80) in, respectively, the cases $1 < \beta < 2$ (HS blow-up regime), $\beta = 2$ (S-regime), $\beta > 2$ (LS-regime). In all three figures the parameters $\beta$, $N$ have been so chosen, that, first, $\beta > 1 + 2/N$, so that there is a class of global trajectories, and, second, $\beta < 2 + 2/N$, that is, unbounded trajectories behave according to (81). We denote by $\psi_S$ the separatrix, which separates families of global and unbounded trajectories, and by $\psi_0$ and $\psi_\infty$, the isoclines of zero and infinity, respectively.

Thus, localization of unbounded solutions of the problem (56), (57) occurs for $\beta = 2$ (S-regime) and $\beta > 2$ (LS-regime), while for $\beta < 2$ there is no localization. Clearly, this classification remains the same if we go back to the original problem (54), (55). Setting $u_s = \exp\{U_s\} - 1$, we obtain the following expression for a.s.s. of equation (54):

$$u_s(t, x) = \exp\{(T_0 - t)^{-1/(\beta-1)}\theta_s(\xi)\} - 1,$$
$$\xi = |x|/(T_0 - t)^{(\beta-2)/[2(\beta-1)]}. \tag{82}$$

It is not hard to see that it satisfies the following nonlinear first order equation of Hamilton-Jacobi type:

$$\partial u_s/\partial t = |\nabla u_s|^2/(1 + u_s) + (1 + u_s)\ln^\beta(1 + u_s). \tag{83}$$

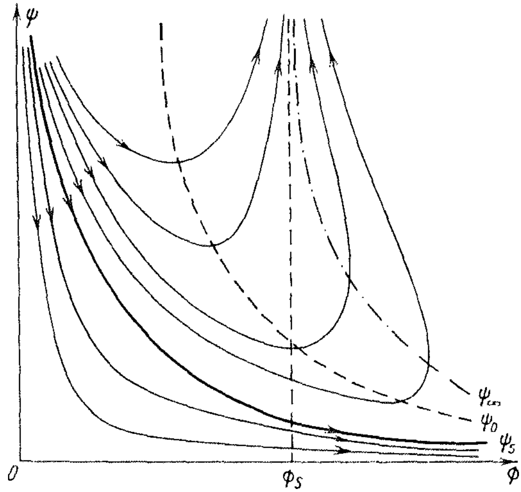


**Fig. 76.** Integral curves of equation (80) for $\beta = 2$ (the case $\beta > 1 + 2/N$ for $N > 2$), $\phi_S = (b_2/a_2)^{1/2}$

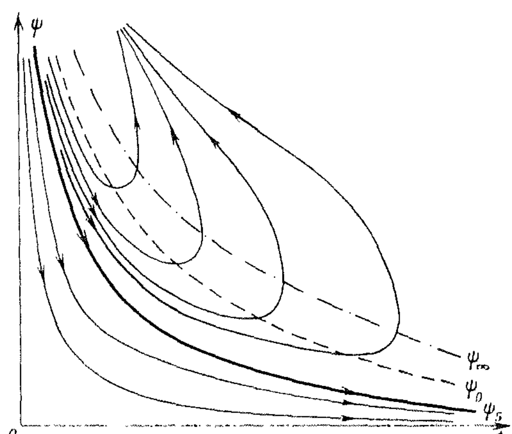


**Fig. 77.** Integral curves of equation (80) for $\beta > 2$ (the case $1 + 2/N < \beta < 2 + 2/N$)

Let us note that in the original variables $t, x, u$ the evolution of the blow-up process looks different. In particular, from (82) it is not hard to obtain an expression for the dependence on time of the half-width of the structure, $l_{ef}(t)$.

for all $\beta > 1$.

$$l_{ef}(t) \sim (T_0 - t)^{1/2}, t \to T_0^- . \tag{84}$$

In this sense there is no difference between the three finite time blow-up regimes. At the same time, for $\beta < 2$ the solutions are not localized and $u_s(t, x) \to \infty$ in $\mathbf{R}^N$, $t \to T_0^-$. On the other hand, for $\beta \geq 2$ solutions are localized. The estimate (84) holds in all the cases. Therefore in numerical simulations of the original problem (54), (55) the difference between the HS- and S- . LS-regimes becomes apparent only once the amplitude of the unbounded solution has grown significantly (by order of tens of hundreds). The logarithmic change of variable $U = \ln(1 + u)$ removes this inconvenience (see Figures 71, 72, where in order to identify the blow-up regime it is sufficient for the amplitude to grow by a factor of 5-10).

### 5 *Global a.s.s.*

For $\beta > 1 + 2/N$ there exist global a.s.s. of the problem (54), (55). From the method of construction of bounded supersolutions in Theorem 9 it follows that at the asymptotic stage we have to neglect the term $|\nabla U|^2$ in equation (56). Therefore the global a.s.s. $U_s$ satisfy the parabolic equation

$$\partial U_s/\partial t = \Delta U_s + U_s^\beta, \ t > 0, \ x \in \mathbf{R}^N,$$

asymptotic properties of the solutions of which are well known: see subsection 2.6.

Going back to the original notation, we see that the global a.s.s. $u_s = \exp\{U_s\} - 1$ satisfies the following parabolic equation:

$$\partial u_s/\partial t = \Delta u_s - |\nabla u_s|^2/(1 + u_s) + (1 + u_s)\ln^\beta(1 + u_s). \tag{85}$$

Therefore equation (54) has the following interesting property: asymptotic behaviour of its unbounded and global solutions is described by vastly dissimilar nonlinear equations of different orders, (83) and (85), respectively.

## Remarks and comments on the literature

The first qualitative and numerical results for unbounded solutions of the problem (0.1), (0.2) were obtained in [349, 353, 391, 89, 92, 268, 276]. These papers also contain a preliminary analysis of unbounded self-similar solutions and first formulate the concepts of localization in finite time blow-up regimes in heat conducting media with volumetric energy production.

**§ 1.** Numerical examples of evolution of S-, HS-, and LS-regimes are taken from [391, 92, 353]. The idea of describing non-monotonicity of the functions $\theta_S(\xi)$ for $\beta > \sigma + 1$ close to the homothermic solution by linearizing the equation, appears in [349, 89] (see also [90, 267, 268], which present numerous examples of numerical construction of the families of solutions $\{\theta_S\}$ for various $\sigma > 0$ and $\beta > \sigma + 1$, $N = 1$). This idea was then exploited in [1, 2], where existence of a finite set of self-similar functions $\theta_S(\xi)$ of the LS-regime is proved for $N = 1$ (the methods of [1, 2] are different from those of § 1). The asymptotic expansion (33) can be proved by the methods of [210] and [39, 40, 41, 370] (for $N = 1$ it was done in [1, 2]).

**§ 2.** The idea of methods of non-stationary averaging is due to the authors of [91]; for more on this method see also [89, 90, 268]. Let us note that simplicity, constructiveness and sufficient trustworthiness of the method make it applicable in a number of other problems, for example, in the study of nonlinear problems of thermochemistry [56]. There are reasons to consider it as a version of the method of radially spherical decomposition of the function space (as opposed to the method of spherical decomposition [335, 336]).

**§ 3.** In the presentation of subsections 1-3 we mainly follow [152]. We note that for $\sigma = 0$ Theorem 2 gives the familiar result [296, 112] (the same is true for Theorem 3), though, of course, the proof in the quasilinear case is substantially different from the semilinear case. Theorem 2 is also true for the critical exponent $\beta = \sigma + 1 + 2/N$ [138]. Let us briefly mention a modified simpler argument based on inequality (19) (see a slightly different approach in [244]). As in the proof of Theorem 2, there exists a solution $v_S$ of (10) such that $u \geqslant v_S$ everywhere. Integrating equation (1) with $\beta = \sigma + 1 + 2/N$ over $\mathbf{R}^N$ we conclude that

$$\frac{d}{dt}\int u(t,x)\,dx = \int u^{\beta}(t,x)\,dx \geqslant \int v_S^{\beta}(t,x)\,dx = \frac{c}{(T_1+t)}.$$

Hence, $\int u(t,x)\,dx \geqslant c\ln(T_1+t) \to \infty$ as $t \to \infty$. Therefore, there exists $t \gg 1$ such that $M_1 = \int u(t,x)\,dx \gg 1$. By comparison we have $u(t,x) \geqslant v(t,x)$ for $t > t_1$, where $v(t,x)$ solves (10) with initial data $u(t_1,x)$. It follows from asymptotic stability of the self-similar solution (11) ([107]) that for $t_2 \gg t_1$ there holds $v(t_2,x) \sim v_s(t_2,x)$ where $v_s$ has the same mass $M_1 \gg 1$. Therefore, the constant $\eta_0$ in (12) satisfies $\eta_0 = \eta_0(M_1) \gg 1$. Finally, the profile $v_s(t_2,x)$ satisfies the property (19), and hence $u(t,x)$ blows up in a finite time.

The survey [290] (see also [292]) contains a large number of nonlinear equations and systems thereof, for which there exists a critical value of the parameter of the source term (in the sense of Fujita [112]).

The assertions of subsection 4 were obtained for $\sigma = 0$ in [210]. The basis of the proof of Lemma 1 is the well known Pohozaev inequality [332, 333]; the scheme of the proof is taken from [210]. The family of solutions (50) was discovered in [298], where the equation $\Delta u - u^{(N+2)/(N-2)} = 0$ was considered.

Existence of such a family has to do with invariance of the elliptic equation with respect to a conformal transformation; see the bibliography in [221], as well as [220]. Results of subsections 5-6 are obtained using the methods of [125, 127, 162]. In the passage to the limit $\tau \to \infty$ in (75) we follow an idea from [178, 179] where a more detailed analysis is given.

**§ 4.** Intersection comparison theorems of subsection 1 were first applied to study unbounded solutions of equation (1) in [129] (see also [130], where Theorem 1′ (subsection 2) is proved). The subtler Theorem 1 is proved in [132, 139]. Let us note that intersection comparison with the explicit solution $u_*(t, x)$ for $\beta = \sigma + 1$ establishes the following upper bound for an arbitrary unbounded solution: $u(t, x) \leq \sup_x u_*(t, x) \equiv |\phi(t)(\psi(t) + 1)|^{1/\sigma}$ in $(0, T_0)$ (see [135]), which is optimal, since it is attained on the solution $u_*(t, x)$. Furthermore, the same method is used to prove results concerning the structure of the blow-up set $B_L$ (see [140]): if $x_1, x_2 \in V_L \equiv \mathbf{R}\backslash B_L$ and $|x_1 - x_2| < L_S$, then $[x_1, x_2] \subset V_L$. It is found that meas $B_L \geq L_S$ for any initial function with compact support.

An essentially different example of combustion in the S-regime is presented in [109] (see also [14]), where the boundary problem with zero Dirichlet conditions on the boundary is considered for the equation $u_t = u^2(u_{xx} + u)$. The authors prove localization of the unbounded solution in the localization domain $B_l = \{|x| \leq \pi/2\}$. Unlike the example considered above, the problem of asymptotic behaviour of the solution in $B_L$ close to the blow-up time, remains largely open. It is proved that $u(t, x) \approx A(t)\cos x$ in $B_L$, but the behaviour of the amplitude $A(t)$ as $t \to T_0$ is at this stage unknown. Note that the conjecture [109] $A(t) \sim (T-t)^{-1/2}g(t)$ with $g(t) = |\ln|\ln(T-t)||^{1/2}$, $t \to T$, seems to be true, since the factor $g(t)$ is a natural one for the equation without a source term, $v_t = v^2 v_{xx}$, which governs the process in a small neighbourhood of the end point of $B_L$, $x = \pi/2$, where $|u^2 u_{xx}| \gg u^3$. We invoke now a formal matching argument. There exists an explicit solution of the log-travelling wave type, $v_*(t, x) = (T-t)^{-1/2} f(\eta)$, $\eta = x - \pi/2 + \lambda \ln(T-t)$ (cf. Ch. II, § 12), where $f$ solves the ODE $f^2 f'' + \lambda f' - f/2 = 0$. Since $f(\eta) \sim (-\eta)|\ln(-\eta)|^{1/2}$ as $\eta \to -\infty$, by slightly perturbing $v_*$ (for instance, by assuming that $\lambda = \lambda(t)$ is a slowly decaying function) we have the following *outer* expansion for $x \approx \pi/2$:

$$v(t, x) \approx \lambda(t)(T-t)^{-1/2}|\ln(T-t)|g(t)|1 - (x - \pi/2)/\lambda(t)|\ln(T-t)||.$$

Then a matching procedure with the *inner* expansion $u(t, x) \approx A(t)\cos x$ for $x \approx \pi/2$ (for instance, by taking $u_x \approx v_x$ at $x = \pi/2$) yields $A(t) = (T-t)^{-1/2}g(t)$.

Proof of localization of unbounded solutions for $\beta > \sigma + 1$ (subsection 3) is given in [129]. The presentation of subsection 4 uses results of [130].

As we mentioned earlier, Proposition 1, on the number of spatial intersections (or number of changes of sign of the difference of two solutions) is a natural consequence of the Strong Maximum Principle for linear parabolic equations and has been known for a long time; see the first quite general results by [368] and [316, 355], as well as various examples of analysis of the zero set of solutions of parabolic equations in [303, 315, 13, 171, 175, 180, 263] and others. Among the general results contained in these papers, let us note those [13] and [263], where the zero set of solutions of linear parabolic equations is studied under quite weak restrictions on coefficients.

The idea of intersection comparison turned out to be very fruitful in the analysis of unbounded solutions of a wide class of equations. The above results were obtained by studying the variation with time of the number of intersections. More subtle results, obtained for different types of equations with a source in [170, 171, 175, 180], deal with analysis not only of the number, but also of the character of points of intersection. One of the main results of these papers is the following conclusion, which has a simple geometric interpretation: in certain conditions a point of "inflection" with a stationary solution can arise, as the solution evolves, from at least three intersection points. Such a comparison with a family of stationary solutions allows us, for example, to show that for an equation of the general form $u_t = (\phi(u))_{xx} + Q(u)$, a solution that has become sufficiently large at a point $x = x_0$ at time $t = t_0$, can only increase in time : $u_t(t, x_0) \geq 0$ for $t > t_0$; see [175, 180]. Other applications of intersection comparison with radial stationary solutions in the multi-dimensional case can be found in [137]. [164] contains a general description of applications of the method of stationary states (m.s.s.) in the study of unbounded solutions of nonlinear parabolic equations and systems; see also other applications in [172, 174] (for other details see § 1, Ch. VII). M.s.s. provides sufficient conditions for absence of localization of unbounded solutions with arbitrary coefficients.

§ 5. The main results were obtained in [130]; see also [129]. Everywhere in § 5 we are concerned with the determination of an attracting set $\mathcal{W}$ of an unstable stationary solution. In the case of equation (23) it has the form $\mathcal{W} = \{\theta_0 = T_0^{1/\sigma} u_0(x)$, where $u_0 \geq 0$ satisfies (9), (10) and $0 < T_0 < \infty$ is the blow-up time of the solution of the problem (1), (2) with the given initial function $u_0(x)\}$. It is of interest that $\mathcal{W}$ is unbounded and contains functions $\theta_0$, the difference of which with $\theta_S$ in $\mathbf{R}$ can be arbitrarily large. It is important to note that $\mathcal{W}$ is an infinite-dimensional set, which, of course, is not dense in $L^2$.

As of now there are relatively few complete results concerning the structure of attracting sets of unstable stationary solutions of nonlinear parabolic equations. In this direction, let us mention [226, 158, 169, 170, 197, 314]; results of [47] (fast diffusion equation in a bounded domain) are discussed in comments to Ch. II. A large number of papers deals with asymptotic stability of regular (without points of "singularity" in time) solutions of parabolic equations; see, for example [42, 378, 158, 162, 184, 383, 241, 242, 10, 11, 20, 21, 22, 50, 107, 208, 213, 234, 235, 308, 344, 356, 359] and references contained therein. We should also like to mention the interesting ideas of [8] and [284] concerning stabilization without constructing Liapunov functions. The problem that arises most frequently in this context is that of finding an attracting set that contains a neighbourhood of the stationary solution (that is, is dense). In [158] we obtain conditions for asymptotic stability of unbounded self-similar solutions of quasilinear equations with a source, for which boundary value problems in bounded domains with moving boundaries were formulated. This ensured asymptotic stability of the corresponding stationary solutions. Let us note that the estimate (20) (or (26)), which plays an important part in the proof of Theorem 2 dealing with stabilization, holds also without the restrictions of the form (9), (10) on the compactly supported function $u_0(x)$ (see [130, 171]). Asymptotic stability of self-similar profiles for $\beta > \sigma + 1$ is proved in [141].

**§ 6.** Results presented here are contained in [131, 123]. Statement 1 of Theorem 3, dealing with the semilinear equation for $\sigma = 0$, $p > (\beta - 1)N/2$, has been proved in [379] (see also [26, 213] and comments on Ch. VII).

Of utmost importance in demonstrating localization in blow-up regimes is the derivation of upper bounds for the solution $u(t, x)$. In § 6 we present an approach based on intersection comparison with a localized self-similar solution with the same blow-up time. It is, however, not without disadvantages. In particular, its use throws up a restriction on the maximal value of the source parameter $\beta$. In this context, very effective is an idea that first appears in [108], where it is used to prove blow-up at a single point for semilinear equations $u_t = \Delta u + Q(u)$. A certain modification of this approach (see [172, 173]), applied to radial solutions of quasilinear equations of general form, $u_t = \Delta\phi(u) + Q(u)$, which consists of deriving conditions under which $w(t, x) \equiv r^{N-1}\phi'(u)u_r + r^N F(u) \leq 0$ in $(0, T_0) \times \mathbf{R}_+$ for a special "optimal" choice of the non-negative function $F(u)$ (it satisfies a certain ordinary differential equation), allows [173] to prove an estimate of the form (9) for a special class of $u_0$ for arbitrary $\beta > \sigma + 1$. The same approach to equations of general form [172, 177] provides conditions of localization of unbounded solutions in terms of the coefficients $\phi(u)$, $Q(u)$. It is interesting that practically in all cases this approach gives upper bounds that coincide precisely with the real asymptotic behaviour of a wide class of unbounded solutions.

Questions of *complete* blow-up, i.e., of possible extension of a blow-up solution for $t > T$ via construction of a certain minimal solution (as the limit of solutions to truncated equations) have been considered in [27] for semilinear equations $u_t = \Delta u + Q(u)$. A criterion of *incomplete* blow-up for a general quasilinear equation $u(t) = (\phi(u))_{xx} + Q(u)$ has been derived in [193] via intersection comparison with the set of travelling wave solutions. For equation (2), $N = 1$, it is $p + \sigma \leqslant 1$, $\sigma \in (-1, 0)$.

**§ 7.** In the presentation of most of the results of subsections 1 and 2.1–2.3 we follow [150]. Inequality (12) was obtained earlier by a different method in

[112, 114], where Theorem 1 is proved. Theorem 3 is proved in [112] (on this see also [296, 254]). For the cases $N = 1$ and $N = 2$ Theorem 4 is proved in [212]; a generalization of the method of [212], which is used in subsection 2.1, to cover the case of arbitrary $N \geq 1$, is contained in [22, 254]. For $\beta = 3$ Proposition 1 was proved in [219]; the case of arbitrary $\beta \in (1, \infty)$ was considered in [3, 2]. Solution (43) was constructed in [158], using the ideas of [34], where a solution of a similar structure was found for an equation (39) with a sink $-\theta^2$ ($\beta = 2$), instead of a source.

In the conditions of Proposition 1 the similarity representation $\theta(t, \xi) \equiv (T_0 - t)^{1/(\beta-1)} u(t, \xi(T_0 - t)^{1/2})$ stabilizes, for a large class of $u_0(x)$, as $t \to T_0^-$ on any set $\{|x| < c(T_0 - t)^{1/2}\}$ to the unique non-trivial solution $\theta_H = (\beta - 1)^{-1/(\beta-1)}$ of equation (39) in $\mathbf{R}$ [169, 170, 171]. In [170, 171] the estimate $u(t, x) \leq \mu_*(T_0 - t)^{-1/(\beta-1)}$ in $(0, T_0) \times \mathbf{R}$ was obtained under the following restrictions on $u_0$: $\sup u_0 < \infty$, $u_0$ is uniformly Lipschitz continuous in $\mathbf{R}$; these conditions are weaker than the ones in [380, 381]. The results of [170], as well as of [169], were obtained by applying comparison theorems for different solutions $u$ and $v$, based not only on the time dependence of the number of their spatial intersections, but also on the nature of those intersections (for example, in [170] under certain restrictions, a theorem of the following form is proved; if $w(t_0, x_0) \equiv u(t_0, x_0) - v(t_0, x_0) = 0$, then $w_x(t_0, x_0) \neq 0$). These theorems are fairly general; they hold true for a large class of quasilinear (degenerate) parabolic equations, including the multi-dimensional case, $u = u(t, |x|)$. See also general results for linear parabolic equations in [13] and [263].

Let us note that if there are lower and upper bounds for $u(t, x)$, the stabilization $\theta(t, \xi) \to \theta_H$, which is uniform on all compact sets in $\mathbf{R}$, follows from the results of [197]; they also consider the multi-dimensional case. In [197] it is shown that under these conditions, stabilization occurs for any $1 < \beta \leq \beta_* = (N + 2)/(N - 2)_+$, which has to do with non-existence of non-trivial solutions $\theta(\xi) \not\equiv \theta_H$ of the elliptic equation

$$\Delta_\xi \theta - \frac{1}{2}\nabla_\xi \theta \cdot \xi - \theta/(\beta - 1) + \theta^\beta = 0, \quad \xi \in \mathbf{R}^N$$

(for $\theta = \theta(|\xi|)$ it becomes (39)), if $1 < \beta \leq \beta_*$. This is proved in [197] by deriving Pohozaev type inequalities [332, 333]. Estimates of $\sup_x u(t, x)$, as well as the structure of the blow-up set, were later studied in [198, 199]. Results concerning existence of non-trivial self-similar functions $\theta_S \not\equiv \text{const}$ (see (39)) in the supercritical case $\beta > \beta_*$ were obtained in [286, 287], where it is shown that for

$$\beta_* < \beta < \beta^* \equiv \frac{N - 2(N-1)^{1/2}}{[N - 4 - 2(N-1)^{1/2}]_+}, \quad N \geq 3,$$

there exists an infinite number of solutions [286] (see also the exact solutions [158] for $6 < N < 16$, $\beta = 2$ and the existence theorem of [370] for $N = 3$,

$6 \leq \beta \leq 12$), while for $\beta^* \leq \beta < 1 + 6/(N-10)_+$ [287] ranges of $\beta$ with any finite number of solutions are determined. In this connection, let us mention the result of [39, 40], who show that for $\beta > \beta^*$ the asymptotic behaviour of a solution as $t \to T_0^-$ is not self-similar if it satisfies everywhere the condition $u_t \geq 0$, that is, in this case we can say that the non-trivial self-similar solution in unstable in this class. The proof uses results of intersection comparison with a singular stationary solution. The resulting non-self-similar asymptotics will be presented below.

A more accurate qualitative analysis shows that the spatio-temporal structure of $u(t,x)$ as $t \to T_0^-$ is described by a.s.s. of the form $f_A(t,\eta) = (T_0 - t)^{-1/(\beta-1)} f_*(\eta)$, $\eta = x/(T_0 - t)^{1/2} |\ln(T_0 - t)|^{1/2}$, where the function $f_*(\eta) > 0$ satisfies in $\mathbf{R}$ the first order equation $-f'\eta/2 - f/(\beta-1) + f^\beta = 0$, $f(\pm\infty) = 0$. It has a whole family of non-trivial solutions $f(\eta) = (\beta - 1 + C\eta^2)^{-1/(\beta-1)}$, $C = \text{const} > 0$. Such an a.s.s. was first introduced in [219]. A similar phenomenon of convergence to a self-similar solution of a first order Hamilton-Jacobi equation occurs for solutions of the porous medium equation with strong absorption [188].

From the requirement of analyticity of the corresponding similarity representation

$$f(t,\eta) = (T_0 - t)^{1/(\beta-1)} u(t, \eta(T_0 - t)^{1/2} |\ln(T_0 - t)|^{1/2})$$

at the point $t = T_0$, $\eta = 0$, it follows that as $t \to T_0^-$ only one solution in the family $\{f\}$ is realized: to it corresponds $C = C_* = (\beta - 1)^2/(4\beta)$, and we have the stabilization $f(t,\eta) \to f_*(\eta)$ as $t \to T_0^-$, which was verified numerically. These conclusions for $\beta = 3$ were derived in [219] (which contains some results for $N = 2$); the analysis of arbitrary $\beta > 1$ was performed in [169, 170].

A rigorous justification of the non-self-similar asymptotics mentioned above has been carried out by different methods in [36, 215, 216] (see also the results of [97, 143]). Upper bounds, which are exactly the same as this asymptotic behaviour, have been derived earlier by [172, 177]. A similar situation occurs in the semilinear equation with an exponential source, $u_t = \Delta u + e^u$. The first qualitative result concerning non-self-similar asymptotics was obtained in [78]; see also [79]. Non-existence of non-trivial self-similar solutions of the form $u_S(t,x) = -\ln(T_0 - t) + \theta_S(x/(T_0 - t)^{1/2})$ was established in [37, 87] for $N = 1, 2$. Such solutions can exist for $N \geqslant 3$ [88]. The whole spectrum of results obtained here is presented in [40]. Justification of the non-self-similar asymptotics as $t \to T_0^-$ is carried out in [52, 36, 215, 216, 217]; see also [143] (whose approach is also used in the problem for a semilinear equation with strong absorption in the study of the total extinction phenomenon in finite time [218, 144]).

Theorem 6 is proved in [169]. In [170] it is shown that the criticality of the solution condition ($u_t \geq 0$ in $(0, T_0) \times \mathbf{R}$) can be dispensed with. In [170, 171] it is removed by the following quite general result: there exists a constant $M_k > 0$, such that if $u(t_0, x_0) > M_k$, then $u_t(t, x_0) \geq 0$ for all $t \in [t_0, T_0)$. In [380], under fairly severe restrictions on $u_0(x)$ and $\beta > 2$, it is shown that the unbounded solution $u(t, |x|)$ of equation (17) satisfies the condition meas $\omega_L = 0$, so that $u(T_0^-, x) = \infty$

only at one point. As we already mentioned, a very effective approach to proofs of localization is that of [108], which was used in a wide variety of equations; see the references in [40, 104, 133, 164, 172, 173, 177]; for its applications in a problem of total extinction see [106, 144]. A special role in the study of semilinear heat equations with a source is played by methods of analysis of the set of zeros in spaces, or, which is the same, by intersection comparison methods. This approach allows us to obtain reasonably complete results concerning the structure of the blow-up set for semilinear equations with a source (see [62, 104, 115, 65]) and of the total extinction set [66].

Theorem 7 is taken from [210]. The main results of subsection 3 were obtained in [150, 127, 347]. The exact solution presented here for $\beta = 2, N = 1$ was constructed in [134, 176], convergence as $t \to T_0$ to the self-similar solution of a Hamilton-Jacobi equation and localization for a large class of equations were established in [189] (a similar asymptotic technique based on a general stability theorem for perturbed dynamical systems [190] can be used for $\beta < 2$ and $\beta > 2$). There it is shown, for example, that meas $B_l = 2\pi$. Absence of localization in the boundary value problem in a bounded domain for $1 < \beta < 2$ was first demonstrated by [281]; localization at one point for $\beta > 2$ and upper bounds corresponding to spatio-temporal structure of a.s.s. were established in [177]; see also the references there. The asymptotic behaviour of blow-up in the three parameter ranges, $\beta < 2$, $\beta = 2$ and $\beta > 2$ for a more general quasilinear heat equation has been established in [192]. Proof of convergence of some classes of global solutions of heat equations with a source to a.s.s. which satisfy nonlinear Hamilton-Jacobi type equations can be found in [160].

Let us note that for equations of other types the problem of determining the blow-up set is formulated in a different way. The structure of the so-called degeneracy surface in $(t, x)$ space was studied in [60, 61, 110] for hyperbolic equations $u_{tt} = \Delta u + F(u)$ and in [111] for Hamilton-Jacobi equations $u_t + H(D_x u) = F(u)$.

We do not consider here in detail questions of fine structure of quasilinear parabolic equations. In this context, we mention [89, 90, 267, 268, 270, 271, 272, 274, 276, 349] (see also §§ 3, 4, Ch. VII). Multi-dimensional non-symmetric eigenfunctions of nonlinear elliptic problems, which arise in the construction of unbounded self-similar solutions, have up till now been studied only numerically [274, 276]. They can have a varied spatial structure; for example, a "star-shaped" localization domain [274]. Group-theoretic analysis of multi-dimensional nonlinear heat equations with a source was carried out in [83, 84, 85]. General ideas concerning the role of eigenfunctions of nonlinear continua in mathematical physics are developed in [267, 268, 275]. For applications, consult [267, 392, 350], as well as the survey of [269], which contains, in particular, a bibliography of applications of blow-up processes in the theory of self-organization of nonlinear systems.

Interesting properties are also exhibited by unbounded solutions of a different parabolic equation with power form nonlinearities:

$$u_t = \nabla \cdot (|\nabla u|^{\sigma} \nabla u) + u^{\beta};\ \sigma > 0, \beta > 1;\ u = u(t, x) \geq 0.$$

Conditions for unboundedness of solutions of the boundary value problem were obtained, for example, in [371, 293] (see the survey of [157]). The Cauchy problem was considered in [128], where it is shown that for $\beta \in (1, \sigma+1+(\sigma+2)/N)$ all the non-trivial solutions $u \not\equiv 0$ are unbounded (it is also true for $\beta = \sigma+1+(\sigma+2)/N$ [138]), while for $\beta > \sigma + 1 + (\sigma + 2)/N$ there is a class of small global solutions. There it is also shown that for $\beta \geq \sigma + 1$ unbounded solutions are localized, while for $1 < \beta < \sigma + 1$ there is no localization. The localization property for $\beta = \sigma + 1$ (S-regime) is illustrated by the separable self-similar solution constructed in [128]: $u_S = (T_0 - t)^{-1/\sigma}\theta(x) > 0$ for $|x| < L_S/2$, $\theta = 0$ for $|x| \geq L_S/2$, where $L_S$ is the fundamental length of the S-regime: $L_S = \pi(\sigma + 1)^{1/(\sigma+2)}[\sigma \sin(\pi/(\sigma + 2))]^{-1}$. Fine structure of localized self-similar solutions for $\beta > \sigma + 1$ (LS-regime), $u_S = (T_0 - t)^{-1/(\beta-1)}\theta(\xi)$, $\xi = x/(T_0 - t)^m$, $m = [\beta - (\sigma + 1)]/[(\sigma + 2)(\beta - 1)]$, was studied in [155], where the elliptic problem for the function $\theta(\xi) \geq 0$ is considered:

$$\nabla \cdot (|\nabla \theta|^{\sigma} \nabla \theta) - m \nabla \theta \cdot \xi - \frac{1}{\beta - 1}\theta + \theta^{\beta} = 0,\ \xi \in \mathbf{R}^N.$$

It is shown that even in the symmetric case, $\theta = \theta(|\xi|)$, it has quite a complicated spectrum of solutions, which consists, roughly speaking, of four families of solutions: three discrete (two countable) families and one discrete continuum of solutions. Existence theorems for self-similar solutions are proved in [156]. Localization of unbounded solutions for $\beta = \sigma + 1$ and an estimate for the thermal front of the compactly supported solution, $h_+(t) \leq h_+(0) + L_S$, are proved in [134] by intersection comparison with the above exact self-similar solution. Asymptotic stability of the self-similar solution is proved by the methods of § 5. Blow-up at a single point for $\beta > \sigma + 1$ has been established in [133] (for $N = 1$) and in [181] (arbitrary $N \geq 1$).

## Open problems

1. (§ 1) Show that the number of positive solutions of the problem (5)–(7) is finite for $1 < \beta < (\sigma + 1)N/(N - 2)_+$. Are the predictions of the linearization procedure in subsection 4.1 correct?
2. (§ 1) Prove existence of solutions for the self-similar problem (5)–(7) for $\beta \geq (\sigma + 1)(N + 2)/(N - 2)_+$.

3. (§ 1) Prove existence of radially non-symmetric solutions of the elliptic equation (2) in $\mathbf{R}^N$ for $\beta > \sigma+1$, $\sigma > 0$, $N > 1$ (for $\sigma = 0$, $1 < \beta \leq (N+2)/(N-2)_+$ they do not exist [197]). Determine the number and spatial structure of such solutions (qualitative and numerical analyses are carried out in [274]).
4. (§ 1) Prove uniqueness of the strictly monotone solution constructed in Theorem 4 (it is important for the stability results discussed in § 5) and of any symmetric solution having a given number of maxima.
5. (§ 4) Demonstrate localization of unbounded solutions of the Cauchy problem for $u_t = \nabla \cdot (u^\sigma \nabla u) + u^\beta$, $t > 0$, $x \in \mathbf{R}^N$ for $\beta \geq \sigma + 1$, $\sigma > 0$ in the case of arbitrary initial functions $u_0$. Is it possible to derive, as in the one-dimensional case (§ 4), an estimate of $\operatorname{supp} u(T_0^-, x)$ in terms of $\operatorname{supp} u_0$ and the time of existence of the solution?
6. (§ 4) Prove effective localization in the case $\beta \geq \sigma + 1$ for arbitrary (non-compactly supported) $u_0(x) \to 0$, $|x| \to \infty$.
7. (§ 5) Prove asymptotic stability of unbounded self-similar solutions of the LS-regime, $\beta > \sigma + 1$ for $N > 1$ (for $N = 1$ see [141]).
8. (§ 7) Prove that the asymptotic behaviour of blow-up solutions of equation (17) is stable with respect to "small" nonlinear perturbations of the equation, when it becomes $u_t = \nabla \cdot (k(u)\nabla u) + Q(u)$. For which $k(u)$ is there nonsymmetric single point blow-up (see [167] for such examples)?
9. (§ 7) Justify in the general case the asymptotics of unbounded solutions of the problem (17), $u(t, x) \simeq (T_0 - t)^{-1/(\beta-1)} f_*(x/(T_0 - t)^{1/2}|\ln(T_0 - t)|^{1/2})$ as $t \to T_0^-$, which was suggested, at a qualitative level, in [219, 169] (this problem is partially solved in [36, 97] and [215, 216]). What is the structure of the attracting set of non-trivial self-similar solutions, which exist in the supercritical case $\beta > (N + 2)/(N - 2)_+$?
10. (§ 7) Prove effective localization of arbitrary unbounded solutions of the Cauchy problem for the equation $u_t = \Delta u + (1 + u) \ln^\beta(1 + u)$, $t > 0$, $x \in \mathbf{R}^N$, for $\beta \geq 2$ (see the partial results of [189] for $\beta = 2$ and [177] for $\beta > 2$ and also [192]).
11. (§ 7) Determine conditions, under which asymptotic behaviour of nonsymmetric unbounded solutions of the problem (54), (55) is described by the degenerate a.s.s. (68) (see the result of [189] for the case $\beta = 2$ and general analysis of symmetric solutions in [192]).

# Chapter V

# Methods of generalized comparison of solutions of different nonlinear parabolic equations and their applications

In this chapter we prove comparison theorems for solutions of different parabolic equations, based on special poinwtise estimates of the highest order spatial derivative of the majorizing solution in terms of the lower order derivatives. Derivation of such estimates is done under conditions of criticality of the problem (§ 1, 2). In § 3 we consider the more general $\psi$-criticality conditions. In § 4, 5, using an operator version of the comparison theorem, we study the heat localization phenomenon in media with an arbitrary thermal conductivity. In § 6 the results of § 1-3 are used to study unbounded solutions of quasilinear parabolic equations. In § 7 we obtain conditions for criticality of finite difference solutions. Using these conditions, we prove a direct comparison theorem for implicit finite difference methods for different nonlinear heat equations.

## § 1 Criticality conditions and a direct solutions comparison theorem

### 1 Formulation of the boundary value problem and the Cauchy problem

Let $\Omega$ be an arbitrary domain (not necessarily bounded) in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$. For a nonlinear parabolic equation

$$u_t = \mathbf{A}(u) = L(u, |\nabla u|, \Delta u) \tag{1}$$

(here $|\nabla u| = |\text{grad}_x u|$), let us consider the boundary value problem in $\omega_T = (0, T) \times \Omega$ with the conditions

$$u(0, x) = u_0(x) \geq 0, \quad x \in \Omega; \tag{2}$$

$$u(t,x) = u_1(t,x) \geq 0,\ 0 < t < T, x \in \partial\Omega. \tag{3}$$

Let us assume that $u_0(x) \to 0$, $u_1(t,x) \to 0$ as $|x| \to \infty$ and $u(t,x) \to 0$ as $|x| \to \infty$ for any $0 < t < T$. The problem (1)–(3) includes the Cauchy problem as a particular case (then simply $\Omega = \mathbf{R}^N$ and (3) should be omitted).

The function $L(p,q,r)$ in (1) is defined and once differentiable in all its arguments; furthermore $\partial L/\partial r > 0$ in $\mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}$, which means that the equation is parabolic.

We shall also assume that there exists a real valued function $r = l(p,q,Y)$ which satisfies the identity

$$L(p,q,l(p,q,Y)) \equiv Y,\ (p,q,Y) \in \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}. \tag{4}$$

The function $l$ is differentiable in all its arguments in view of the smoothness of $L$ and the parabolicity condition. From (4) we obtain the following identities:

$$\begin{gathered} L_1(p,q,l(p,q,l(p,q,Y)) + L_3(p,q,l(p,q,Y))l_1(p,q,Y) \equiv 0, \\ L_3(p,q,l(\cdot))l_3(\cdot) \equiv 1, \\ L_2(p,q,l(\cdot)) + L_3(p,q,l(\cdot))l_2(\cdot) \equiv 0 \end{gathered} \tag{5}$$

(here and below we are using the notation $L_1 = \partial L/\partial p$, $l_1 = \partial l/\partial p$, $L_2 = \partial L/\partial q$, $L_3 = \partial L/\partial r$ and so forth).

Let us set

$$l_0(p,q) = l(p,q,0). \tag{6}$$

By (4) the function $l_0$ satisfies the identity

$$L(p,q,l_0(p,q)) \equiv 0,\ (p,q) \in \mathbf{R}_+ \times \mathbf{R}_+. \tag{4'}$$

The above requirements are satisfied, for example, by the quasilinear operator

$$\mathbf{A}(u) = K(u,|\nabla u|)\Delta u + N(u,|\nabla u|), \tag{7}$$

where $K(p,q) > 0$, $N(p,q)$ are given sufficiently smooth functions. In this case

$$l(p,q,Y) = [Y - N(p,q)]\frac{1}{K(p,q)},\quad l_0(p,q) = -\frac{N(p,q)}{K(p,q)}. \tag{8}$$

For the operator of nonlinear heat conduction with a source,

$$\mathbf{A}(u) = \nabla \cdot (k(u)\nabla u) + Q(u) \equiv k(u)\Delta u + k'(u)|\nabla u|^2 + Q(u),\ k > 0, \tag{9}$$

the analogous expressions have the form

$$\begin{gathered} l(p,q,Y) = [Y - k'(p)q^2 - Q(p)]\frac{1}{k(p)}, \\ l_0(p,q) = -\left[\frac{k'(p)}{k(p)}\right]q^2 - \frac{Q(p)}{k(p)}. \end{gathered} \tag{10}$$

We shall assume that in $\omega_T$ there exists a positive classical solution of problem (1)–(3), and that it is unique.

Everywhere below we shall take the following restrictions on the function $L$ to hold:

a) the operator $\mathbf{A}$ is parabolic: $\partial L/\partial r > 0$;

b) there exist functions $l$, $l_0$ satisfying, respectively, the identities (4), (4′).

## 2 Conditions for criticality of the problem

**Definition.** We shall call a problem (1)–(3) and its solution $u(t,x)$ *critical* if everywhere in $\omega_T$ it satisfies the condition

$$u_t(t,x) \geq 0. \tag{11}$$

Condition (11) will be used to derive a pointwise estimate of the highest order spatial derivative (Laplacian) of the solution, $\Delta u(t,x)$, in terms of the lower order derivatives, $|\nabla u(t,x)|$ and $u(t,x)$. Indeed, by (1), condition (11) is equivalent to the inequality

$$L(u(t,x), |\nabla u(t,x)|, \Delta u(t,x)) \geq 0 \text{ in } \omega_T. \tag{11'}$$

However, $L_3(p,q,r) > 0$, and therefore the inequality (11′) can be solved for $\Delta u$.

This leads to the estimate

$$\Delta u(t,x) \geq l_0(u(t,x), |\nabla u(t,x)|) \text{ in } \omega_T. \tag{12}$$

In particular, for the operator (7) we obtain

$$\Delta u \geq -\left\{\frac{N(u,|\nabla u|)}{K(u,|\nabla u|)}\right\} \text{ in } \omega_T, \tag{13}$$

while for the operator (9) we have

$$\Delta u \geq -\left\{\frac{k'(u)}{k(u)}|\nabla u|^2 + \frac{Q(u)}{k(u)}\right\} \text{ in } \omega_T. \tag{14}$$

Pointwise estimates of the form (12)–(14) are the basis of the approach to comparison of solutions of different parabolic equations we propose below.

For simplicity, let us assume initially that $u_0 \in C^2(\Omega)\cap C(\overline{\Omega})$, $u_1 \in C^{1,0}_{tx}([0,T)\times \partial\Omega)$, $u \in C^{2,4}_{tx}(\omega_T)\cap C^{1,2}_{tx}(\overline{\omega_T})$. This allows us to differentiate equation (1) once in $t$ in $\omega_T$.

**Theorem 1.** *For criticality of the problem* (1)–(3) *it is necessary and sufficient that*

$$\mathbf{A}(u_0) \equiv L(u_0, |\nabla u_0|, \Delta u_0) \geq 0, \quad x \in \Omega, \tag{15}$$

$$\partial u_1(t, x)/\partial t \geq 0, \quad (t, x) \in (0, T) \times \partial\Omega. \tag{16}$$

*Proof.* Necessity of the conditions of the theorem is obvious. Let us prove sufficiency. Let us set $u_t(t, x) = z$ in $\omega_T$. Then $L(u, |\nabla u|, \Delta u) = z$ and therefore $\Delta u = l(u, |\nabla u|, z)$. From (1) it follows that the function $z$ satisfies in $\omega_T$ the equation

$$z_t = L_1(u, |\nabla u|, l(u, |\nabla u|) z))z + L_2(u, |\nabla u|, l(u, |\nabla u|, z))(\nabla u \cdot \nabla z/|\nabla u|) + \\ + L_3(u, |\nabla u|, l(u, |\nabla u|, z))\Delta z. \tag{17}$$

Here $\nabla u \cdot \nabla z$ is the scalar product of two vectors obtained as a result of differentiation: $|\nabla u|_t = (\nabla u \cdot \nabla z)/|\nabla u|$. Formally this equation is a linear homogeneous parabolic equation with bounded coefficients, which is ensured by sufficient smoothness of the solution $u(t, x)$ and of the function $L$. Therefore by the Maximum Principle (see § 1, Ch. I) $z \geq 0$ everywhere in the domain $\omega_T$ as soon as $z \geq 0$ on its parabolic boundary $\gamma_T = \{t \in (0, T), x \in \partial\Omega\} \cup \{t = 0, x \in \Omega\}$. This completes the proof. □

In the following, inequalities (15), (16) will be called the *criticality conditions* for the boundary data of the problem (1)–(3).

**Remark 1.** It follows from the theorem that criticality of the problem does not impose any restrictions on the elliptic operator of equation (1) (if it does not depend explicitly on the variable $t$): it is fully defined by properties of the boundary data. For an operator of a more general form,

$$\mathbf{A}(u) = L(u, |\nabla u|, \Delta u;\ t, x)$$

the same statement is no longer true.

Following the proof of Theorem 1, it is not hard to see that in this case for criticality of the problem we need in addition the following inequality:

$$\frac{\partial}{\partial t} L(p, q, r;\ t, x)|_{r=l_0(p,q;\ t,x)} \geq 0, \quad (p, q) \in \mathbf{R}_+ \times \mathbf{R}_+, \quad (t, x) \in \omega_T. \tag{18}$$

For example, if

$$\mathbf{A}(u) = \nabla \cdot [k(u;\ t)\nabla u] + Q(u;\ t),$$

then condition (18) can be written in the following form:

$$\left(\frac{\partial^2 k}{\partial t \partial p} - \frac{\partial k}{\partial p}\frac{\partial k}{\partial t}\frac{1}{k}\right) q^2 + \frac{\partial Q}{\partial t} - \frac{\partial k}{\partial t}\frac{Q}{k} \geq 0.$$

In view of the fact that the quantities $p$ and $q$ vary independently, this relation decomposes into the two inequalities

$$\frac{\partial}{\partial t}\left[\frac{k'_p(p;\ t)}{k(p;\ t)}\right] \geq 0, \quad \frac{\partial}{\partial t}\left[\frac{Q(p;\ t)}{k(p;\ t)}\right] \geq 0.$$

**Remark 2.** Let us show that the smoothness restrictions on the solution $u$ used in the proof, may be weakened substantially. Under the natural assumption that $u \in C^{1,2}_{t,x}(\omega_T) \cap C(\overline{\omega}_T)$, the proof follows exactly the same lines, with the difference that instead of the function $z = u_t(t, x)$ we consider the finite difference

$$z(t, x) = \frac{1}{\tau}[u(t+\tau, x) - u(t, x)], \quad (t, x) \in \omega_{T-\tau},$$

where $\tau \in (0, T)$ is a fixed constant.

Then a parabolic equation of the form (17) satisfied by $z$ can be derived in a similar manner. As far as boundary data are concerned, in this case under the same conditions (15), (16), for $(t, x) \in (0, T-\tau) \times \partial\Omega$ we have

$$z(t, x) = \frac{1}{\tau}[u_1(t+\tau, x) - u_1(t, x)] \geq 0.$$

Furthermore, for $t = 0$

$$z(0, x) = [u(\tau, x) - u_0(x)]/\tau, \ x \in \Omega.$$

However the function $v(t, x) \equiv u_0(x)$ is, in view of (15), a subsolution of the equation (1); in addition, $v = u_1(0, x) \leq u_1(t, x)$ on $\partial\Omega$. Therefore $u \geq v$ on $\gamma_T$, and thus $u(t, x) \geq v(t, x) \equiv u_0(x)$ in $\omega_T$. The last condition is equivalent to the condition $z(0, x) \geq 0$ in $\Omega$ for any $\tau \in (0, T)$.

Thus the function $z$ satisfies a parabolic equation in $\omega_{T-\tau}$ and $z \geq 0$ on the parabolic boundary $\gamma_{T-\tau}$. Then $z \geq 0$ in $\omega_{T-\tau}$, from which it follows that for all $(t, x) \in \omega_T$

$$u_t(t, x) = \lim_{\tau \to 0^+} \frac{u(t+\tau, x) - u(t, x)}{\tau} \geq 0.$$

Obviously, using this method of proof, we can replace condition (16) by a weaker one: $u_1(t, x)$ does not decrease in $t$ on $\partial\Omega$. We can also weaken the smoothness requirements on the initial function $u_0(x)$.

## 3 A theorem on direct comparison of solutions

Let us consider in $\omega_T$ boundary value problems for two different parabolic equations ($\nu = 1, 2$):

$$u_t^{(\nu)} = \mathbf{A}^{(\nu)}(u^{(\nu)}) = L^{(\nu)}(u^{(\nu)}, |\nabla u^{(\nu)}|, \Delta u^{(\nu)}); \tag{19}$$

$$u^{(\nu)}(0,x) = u_0^{(\nu)}(x) \geq 0,\ \ x \in \Omega;\ u_0^{(\nu)} \in C(\Omega); \tag{20}$$

$$u^{(\nu)}(t,x) = u_1^{(\nu)}(t,x),\ \ 0 < t < T,\ x \in \partial\Omega; \tag{21}$$

$$u_1^{(\nu)} \in C([0,T) \times \partial\Omega).$$

The functions $L^{(\nu)}(p,q,r)$ are assumed to be sufficiently smooth. As in subsection 1, we denote by $l^{(2)}(p,q,Y)$ ($l_0^{(2)} = l^{(2)}(p,q,0)$) the solution of the equation

$$L^{(2)}(p,q,l^{(2)}) = Y,\ (p,q,Y) \in \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}.$$

Let there be positive classical solutions of these problems in $\omega_T$, and assume that $u^{(2)} \geq u^{(1)}$ on $\gamma_T$. In the following assertion we state two sufficient conditions for a direct comparison of the solutions of the problems (19)–(21), under which $u^{(2)} \geq u^{(1)}$ everywhere in $\omega_T$. Let us emphasize that we are talking about comparing solutions of two substantially different equations.

**Theorem 2.** *Let $u^{(2)} \geq u^{(1)}$ on $\gamma_T$, that is,*

$$\begin{aligned} u_0^{(2)}(x) &\geq u_0^{(1)}(x),\ \ x \in \Omega, \\ u_1^{(2)}(t,x) &\geq u_1^{(1)}(t,x),\ \ 0 < t < T,\ x \in \partial\Omega. \end{aligned} \tag{22}$$

*In addition, let the solution of the problem (19)–(21) for $v = 2$ be critical (this means that $u_t^{(2)} \geq 0$ in $\omega_T$), and that for all $(p,q,r) \in \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}$ we have the inequalities*

$$\frac{\partial}{\partial r}[L^{(2)}(p,q,r) - L^{(1)}(p,q,r)] \geq 0, \tag{23}$$

$$L^{(1)}(p,q,l_0^{(2)}(p,q)) \leq 0. \tag{24}$$

*Then $u^{(2)} \geq u^{(1)}$ everywhere in $\omega_T$.*

*Proof.* Let us set $u^{(2)} - u^{(1)} = z$. Then the function $z$ satisfies in $\omega_T$ the equation

$$z_t = L^{(2)}(u^{(2)}, |\nabla u^{(2)}|, \Delta u^{(2)}) - L^{(1)}(u^{(2)} - z, |\nabla u^{(2)} - \nabla z|, \Delta u^{(2)} - \Delta z), \tag{25}$$

and, by (22), $z \geq 0$ on $\gamma_T$. Linearizing the right-hand side of (25) with respect to the function $z$ and its derivatives, we obtain for $z$ a linear parabolic equation with bounded coefficients:

$$\begin{aligned} z_t - L_3^{(1)}(u^{(2)}, |\nabla u^{(2)}|, v_3)\Delta z - L_2^{(1)}(u^{(2)}, |\nabla v_2|, \Delta u^{(2)})(\nabla v_2 \cdot \nabla z)/|\nabla v_2| - \\ - L_1^{(1)}(v_1, |\nabla u^{(2)}|, \Delta u^{(2)})z = \\ = L^{(2)}(u^{(2)}, |\nabla u^{(2)}|, \Delta u^{(2)}) - L^{(1)}(u^{(2)}, |\nabla u^{(2)}|, \Delta u^{(2)}). \end{aligned} \tag{26}$$

where the bounded smooth functions $v_j$, $j = 1, 2, 3$ (some average values), depend on the solutions $u^{(1)}$, $u^{(2)}$.

Let us consider the right-hand side of (26). By criticality of the solution $u^{(2)}$, we have the pointwise estimates (see subsection 2)

$$\Delta u^{(2)} \geq l_0^{(2)}(u^{(2)}, |\nabla u^{(2)}|) \text{ in } \omega_T. \tag{27}$$

Condition (23) means that the function $L^{(2)}(p, q, r) - L^{(1)}(p, q, r)$ is non-decreasing in its third argument. Therefore, using (27), we obtain

$$\mathbf{A}^{(2)}(u^{(2)}) - \mathbf{A}^{(1)}(u^{(2)}) \geq L^{(2)}(u^{(2)}, |\nabla u^{(2)}|, l_0^{(2)}(u^{(2)}, |\nabla u^{(2)}|)) -$$
$$- L^{(1)}(u^{(2)}, |\nabla u^{(2)}|, l_0^{(2)}(u^{(2)}, |\nabla u^{(2)}|)).$$

However, by definition $L^{(2)}(p, q, l_0^{(2)}(p, q)) \equiv 0$. Hence by (24)

$$\mathbf{A}^{(2)}(u^{(2)}) - \mathbf{A}^{(1)}(u^{(2)}) \geq -L^{(1)}(u^{(2)}, |\nabla u^{(2)}|, l_0^{(2)}(u^{(2)}, |\nabla u^{(2)}|)) \geq 0.$$

Therefore from (26) we have that

$$z_t - L_3^{(1)}\Delta z - L_2^{(1)}\frac{(\nabla v_2 \cdot \nabla z)}{|\nabla v_2|} - L_1^{(1)} z \geq 0$$

everywhere in $\omega_T$. Since $z \geq 0$ on $\gamma_T$, invoking the Maximum Principle, we conclude that $z \geq 0$ in $\omega_T$, that is $u^{(2)} \geq u^{(1)}$ everywhere in that domain. □

Let us see what form the comparison conditions (23), (24) take in the case of particular parabolic operators.

**Example 1.** The inequality (23) depends, in general, on the three variables $p, q, r$. However, for quasilinear operators of the form

$$\mathbf{A}^{(\nu)}(u^{(\nu)}) = K^{(\nu)}(u^{(\nu)}, |\nabla u^{(\nu)}|)\Delta u^{(\nu)} + N^{(\nu)}(u^{(\nu)}, |\nabla u^{(\nu)}|), \quad \nu = 1, 2,$$

it depends, as does (24), only on $p, q$. The inequalities (23), (24) in this case have the form

$$\begin{gathered} K^{(2)}(p, q) - K^{(1)}(p, q) \geq 0, \\ K^{(2)}(p, q)N^{(1)}(p, q) - K^{(1)}(p, q)N^{(2)}(p, q) \leq 0. \end{gathered} \tag{28}$$

**Example 2.** For the nonlinear heat equation with a source,

$$u_t^{(\nu)} = \nabla \cdot (k^{(\nu)}(u^{(\nu)})\nabla u^{(\nu)}) + Q^{(\nu)}(u^{(\nu)}), \tag{29}$$

due to independence of $p$ and $q$ in the second of the inequalities (28), these inequalities can be written as

$$k^{(2)}(p) - k^{(1)}(p) \geq 0, \tag{30}$$

$$k^{(2)'}(p)k^{(1)}(p) - k^{(1)'}(p)k^{(2)}(p) \geq 0, \tag{31}$$

$$Q^{(2)}(p)k^{(1)}(p) - Q^{(1)}(p)k^{(2)}(p) \geq 0. \tag{32}$$

The inequality (31) can be put into a more compact form:

$$[k^{(2)}(p)/k^{(1)}(p)]' \geq 0. \tag{31'}$$

**Example 3.** Let the functions $v^{(\nu)}$ satisfy the equations

$$v_t^{(\nu)} = \mathbf{B}^{(\nu)}(v^{(\nu)}) \equiv a^{(\nu)}(v^{(\nu)})\Delta v^{(\nu)} + b^{(\nu)}(v^{(\nu)}), \quad \nu = 1, 2. \tag{33}$$

The comparison conditions (23), (24) (or (28)) of solutions of equations (33) have the form

$$\begin{aligned} a^{(2)}(p) - a^{(1)}(p) &\geq 0, \\ b^{(2)}(p)a^{(1)}(p) - b^{(1)}(p)a^{(2)}(p) &\geq 0, \end{aligned} \tag{34}$$

and look much simpler than (30)–(32) (at least they do not contain derivatives of the functions entering them).

At the same time equations (29) can be reduced to the form (33) by simple transformations. Indeed, let us set

$$H^{(\nu)}(p) = \int_0^p k^{(\nu)}(\eta)\, d\eta, \; p > 0, \;\; \nu = 1, 2,$$

and denote by $h^{(\nu)}$ the functions inverse to $H^{(\nu)}$, so that $H^{(\nu)}(h^{(\nu)}(p)) \equiv p$ ($h^{(\nu)}$ exist at least for all small enough $p$ by monotonicity of $H^{(\nu)}$; the latter is ensured by the conditions $k^{(\nu)} > 0$).

Let us make a change of variable in the equations (29),

$$u^{(\nu)} = h^{(\nu)}(v^{(\nu)}), \quad \nu = 1, 2. \tag{35}$$

Then, taking into account the fact that $k^{(\nu)}(h^{(\nu)}(p))h^{(\nu)'}(p) \equiv 1$, we obtain for the functions $v^{(\nu)}$ the equations

$$v_t^{(\nu)} = k^{(\nu)}(h^{(\nu)}(v^{(\nu)}))\Delta v^{(\nu)} + Q^{(\nu)}(h^{(\nu)}(v^{(\nu)}))k^{(\nu)}(h^{(\nu)}(v^{(\nu)})), \quad \nu = 1, 2,$$

which are the same as (33), if we set in those equations

$$\begin{aligned} a^{(\nu)}(p) &= k^{(\nu)}(h^{(\nu)}(p)), \\ b^{(\nu)}(p) &= Q^{(\nu)}(h^{(\nu)}(p))k^{(\nu)}(h^{(\nu)}(p)). \end{aligned} \tag{36}$$

From (35) it follows that the inequality $v^{(2)} \geq v^{(1)}$ is equivalent to the inequality

$$H^{(2)}(u^{(2)}) \geq H^{(1)}(u^{(1)}) \text{ in } \omega_T. \tag{37}$$

As a result we obtain that under the conditions

$$k^{(2)}(h^{(2)}(p)) - k^{(1)}(h^{(1)}(p)) \geq 0, \tag{38}$$

$$Q^{(2)}(h^{(2)}(p)) - Q^{(1)}(h^{(1)}(p)) \geq 0, \quad p > 0, \tag{39}$$

and also under the other conditions of Theorem 2, in particular the inequality

$$H^{(2)}(u^{(2)}) \geq H^{(1)}(u^{(1)}) \text{ on } \gamma_T, \tag{40}$$

(37) holds.

The comparison conditions (38), (39) can be written as:

$$k^{(2)}(p) - k^{(1)}[h^{(1)}(H^{(2)}(p))] \geq 0, \tag{38'}$$

$$Q^{(2)}(p) - Q^{(1)}[h^{(1)}(H^{(2)}(p))] \geq 0, \quad p > 0. \tag{39'}$$

Then the inequality (37) can be written in the following form:

$$u^{(2)} \geq h^{(2)}[H^{(1)}(u^{(1)})] \text{ in } \omega_T. \tag{37'}$$

Therefore, by comparing not the solutions $u^{(i)}$ themselves, but rather some nonlinear functions of these solutions (see (37) or (37′)), we managed to simplify the comparison conditions considerably: instead of the three inequalities (30)–(32), only two remain: either (38), (39), or (38′), (39′). We shall call this generalized comparison method the *operator* or the *functional* comparison method. It will be considered in more detail in the following section.

**Remark.** It is not hard to see that the comparison conditions (30)–(32) will be satisfied if and only if the functions $k^{(1)}$ and $Q^{(1)}$ can be represented as

$$k^{(1)}(p) = k^{(2)}(p)[1 + \mu(p)]^{-1},$$
$$Q^{(1)}(p) = Q^{(2)}(p)[1 + \mu(p)]^{-1}[1 + \nu(p)]^{-1},$$

where $\mu, \nu$ are arbitrary smooth non-negative functions; in addition $\mu$ is a non-decreasing function.

## § 2 The operator (functional) comparison method for solutions of parabolic equations

In this section we prove a more general comparison theorem for solutions of two different nonlinear parabolic equations than that of in § 1. We present the material using degenerate equations, which, as is well known (see § 3, Ch. 1), do not necessarily have a classical solution. We shall consider in most detail the one-dimensional case, though all the results hold for equations in many independent variables (corresponding examples are given below).

## 1 Criticality conditions for solutions of degenerate parabolic equations

Let us consider in $\omega_T = (0, T) \times \mathbf{R}_+$ the boundary value problem

$$u_t = (k(u)u_x)_x \equiv (\phi(u))_{xx}; \tag{1}$$

$$u(0, x) = u_0(x) \geq 0, x > 0; \ \sup u_0 < \infty, \sup |[\phi(u_0)]_x| < \infty; \tag{2}$$

$$u(t, 0) = u_1(t) > 0, 0 < t < T. \tag{3}$$

Let the equation (1) be degenerate for $u = 0$, that is $k(0) = 0$ and $k \in C^2(\mathbf{R}_+) \cap C([0, \infty))$. The solution of the problem is classical in

$$P_T[u] = \{(t, x) \in \omega_T \mid u(t, x) > 0\}$$

and it can happen that at points of

$$S_T[u] = \overline{P}_T[u] \backslash P_T[u] \backslash \partial\omega_T$$

(degeneracy points) not all the derivatives in (1) are defined. There the function $k(u(t, x))u_x(t, x)$ is continuous in $x$ in $\mathbf{R}_+$ for all fixed $t \in (0, T)$.

As in § 1, we shall call the problem (1)–(3) and its solution *critical* if $u_t(t, x) \geq 0$ everywhere in $P_T[u]$.

1.1. For convenience, we shall assume below that $u_0 \in C^2$ everywhere where $u_0 > 0$, and that $u_1 \in C^1([0, T))$ (these restrictions can be weakened substantially). Under the assumptions made, we have the following theorem.

**Theorem 1.** *For criticality of the problem* (1)–(3) *it is necessary and sufficient that*

$$(k(u_0)u_0')' \geq 0, x \in \{x > 0 \mid u_0(x) > 0\}, \tag{4}$$

$$u_1'(t) \geq 0, 0 < t < T. \tag{5}$$

*Proof.* Let us prove sufficiency of conditions (4), (5). Let us make the preliminary observation that a critical initial function $u_0(x)$, bounded in $\overline{\mathbf{R}}_+$, is non-increasing. Indeed, by (4), it cannot reach a positive maximum in $\mathbf{R}_+$. Let us assume that at a point $x^* \in \mathbf{R}_+$, where $u_0(x^*) > 0$, it is increasing, that is $u_0'(x^*) > 0$. Then $u_0(x) > 0$, $u_0'(x) > 0$ for all $x > x^*$, and we have from (4) that

$$k(u_0(x))u_0'(x) \geq k(u_0)u_0'|_{x=x^*} > 0, \ x > x^*,$$

Therefore for all $x > x^*$

$$\int_{u_0(x^*)}^{u_0(x)} k(\eta)\,d\eta \geq k(u_0)u_0'|_{x=x^*}(x - x^*),$$

that is, $u_0(x)$ grows without bound as $x \to \infty$, which contradicts the assumption that $u_0 \in C(\overline{\mathbf{R}}_+)$ $(\sup u_0 < \infty)$.

Thus the initial function $u_0$ is non-increasing, and we are entitled to conclude that the set of degeneracy points $S_t[u]$ lies on only one curve $(0, T) \times \{x = \zeta(t)\}$, such that furthermore

$$k(u(t, \zeta(t)))u_x(t, \zeta(t)) = 0, 0 < t < T. \tag{6}$$

The function $x = \zeta(t)$ is nondecreasing and continuous in $[0, T)$ (see, for example, [18, 252, 328]).

Theorem 1 can be established in a number of ways. Below we briefly present one of the proofs, which makes substantial use of the property (6) and the assumption that $u \in C^{2,4}(P_T[u])$.

Let us set $z = u_t = (k(u)u_x)_x$. The function $z$ satisfies in $P_T[u]$ the formally linear parabolic equation

$$z_t = [k(u)z]_{xx}. \tag{7}$$

Then $z(0, x) \geq 0$ in $\{0 < x < \zeta(0)\}$ by (4) and, as follows from (5), we can assume that $z(t, 0) \geq 0$ for $t \in (0, T)$. It remains to verify that, roughly speaking, $z \geq 0$ near the curve $(0, T) \times \{x = \zeta(t)\}$, the right lateral boundary of $P_T[u]$. This follows directly from the equality $z = (k(u)u_x)_x$. Integrating this equality over a small interval $(\zeta(t) - \epsilon, \zeta(t))$, $\epsilon > 0$ (it is not hard to check that this makes sense), by (6) we have

$$\int_{\zeta(t)-\epsilon}^{\zeta(t)} z(t, x)\,dx = -(\phi(u))_x|_{(t,\zeta(t)-\epsilon)}, \phi(u) \equiv \int_0^u k(\eta)\,d\eta. \tag{6'}$$

Since $\phi(u(t, x)) > 0$ on $(\zeta(t) - \epsilon, \zeta(t))$ and $(\phi(u(t, x)))_x \to 0$ as $x \to \zeta^-(t)$, we can always find an arbitrarily small $\epsilon > 0$, such that $(\phi(u))_x < 0$ at the point $(t, \zeta(t) - \epsilon)$ and therefore

$$\int_{\zeta(t)-\epsilon}^{\zeta(t)} z(t, x)\,dx > 0.$$

Therefore at any arbitrarily small left half-neighbourhood of the point $x = \zeta(t)$ we can find $x_*(t) \in (\zeta(t) - \epsilon, \zeta(t))$, such that $z(t, x_*(t)) > 0$.

The function $z$ is a classical solution of equation (7) in the domain $(0, T) \times \{0 < x < x_*(t)\}$, such that, furthermore, $z \geq 0$ on its parabolic boundary. Then, by the Strong Maximum Principle, $z \geq 0$ at all interior points of the domain

$(0, T) \times \{0 < x < x_*(t)\}$. Since $\epsilon > 0$ can be arbitrarily small, we obtain that $z \equiv u_t \geq 0$ in $P_T[u]$. □

**Remark 1.** It is not hard to show that we can take the set $(0, T) \times \{x = x_*(t)\}$ to be a continuous curve. Moreover, under the conditions of the theorem $u_t > 0$ in a neighbourhood $(0, T) \times \{\zeta(t) - \delta < x < \zeta(t)\}$ (see [252]).

**Remark 2.** Using the same method, it possible to prove that under the conditions of Theorem 1

$$u_x(t, x) \leq 0 \text{ in } P_T[u]. \tag{8}$$

We note that the functions $z = u_t$ and $z = u_x$ satisfy in $P_T[u]$ the same linear parabolic equation (7).

1.2. An assertion, similar to Theorem 1, is true for a degenerate parabolic equation more general than (1),

$$u_t = \nabla \cdot (k(u)\nabla u) + Q(u) \equiv \Delta\phi(u) + Q(u), \tag{9}$$

where $Q \in C^1([0, \infty))$, $Q(0) = 0$, is a given function. Let $\Omega$ be an arbitrary domain in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$. For the equation (9) let us consider, for example, the boundary value problem (or the Cauchy problem if $\Omega = \mathbf{R}^N$) with the conditions

$$u(0, x) = u_0(x) \geq 0, \ x \in \Omega; \tag{10}$$

$$u(t, x) = 0, \ 0 < t < T, x \in \partial\Omega. \tag{11}$$

For our aims, the following assertion concerning the criticality of solutions of the problem (9)–(11), which is far from being optimal in terms of requirements on $u_0(x)$, will be sufficient.

**Theorem 2.** *Let $Q \in C^1([0, \infty))$, $Q(0) = 0$ and $Q'(u) \geq 0$ for $u > 0$. Let the domain $\Omega_0 = \operatorname{supp} u_0 \subset \Omega$ have a smooth boundary $\partial\Omega_0$ and $u_0 \in C^2(\Omega_0) \cap C(\overline{\Omega})$. Then for criticality of $u(t, x)$ it is both necessary and sufficient that*

$$\Delta\phi(u_0) + Q(u_0) \geq 0, \ x \in \Omega_0. \tag{12}$$

*Proof.* As our point of departure we take the fact that the generalized solution $u(t, x)$ can be obtained as the limit as $\epsilon \to 0^+$ of a sequence of strictly positive classical solutions $u_\epsilon$ of equation (9) in $\omega_T = (0, T) \times \Omega$ with the conditions

$$u_\epsilon(0, x) = u_{\epsilon 0}(x) \equiv \phi^{-1}(\phi(u_0) + \epsilon) \to u_0$$

uniformly in $\Omega$ as $\epsilon \to 0^+$, $u_\epsilon = \phi^{-1}(\epsilon)$ on $(0, T) \times \partial\Omega$. Let us fix a small enough $\tau > 0$ and let us consider the function $z_\epsilon(t, x) = [u_\epsilon(t + \tau, x) - u_\epsilon(t, x)]/\tau$, which satisfies in $\omega_{T-\tau}$ the linear parabolic equation

$$(z_\epsilon)_t = \Delta(az_\epsilon) + bz_\epsilon,$$

where we have denoted by $a, b$ the smooth functions

$$a = \int_0^1 \phi'(\eta u_\epsilon(t+\tau, x) + (1-\eta)u_\epsilon(t, x))\, d\eta,$$

$$b = \int_0^1 Q'(\eta u_\epsilon(t+\tau, x) + (1-\eta)u_\epsilon(t, x))\, d\eta.$$

Furthermore, $z_\epsilon = 0$ on $[0, T-\tau] \times \partial\Omega$.

Let us consider the function $z_\epsilon(0, x) \equiv [u_\epsilon(\tau, x) - u_{\epsilon 0}(x)]/\tau$ in $\Omega$. Since by the Maximum Principle $u_\epsilon \geq \phi^{-1}(\epsilon)$ in $\omega_T$ (recall that $Q(u) \geq 0$ for all $u \geq 0$), then for all $x \in \Omega\backslash\Omega_0$ we have $z_\epsilon(0, x) \equiv [u_\epsilon(\tau, x) - \phi^{-1}(\epsilon)]/\tau \geq 0$. Furthermore, let us consider the function $v(t, x)$, a classical solution of equation (9) in $(0, T) \times \Omega_0$ with the conditions $v(0, x) = u_{\epsilon 0}(x)$ in $\Omega_0$, $v = \phi^{-1}(\epsilon)$ on $(0, T) \times \partial\Omega_0$. It is critical, since by (12)

$$\Delta\phi(u_{\epsilon 0}) + Q(u_{\epsilon 0}) \equiv \Delta\phi(u_0) + Q(\phi^{-1}(\phi(u_0) + \epsilon)) \geq \Delta\phi(u_0) + Q(u_0) \geq 0 \text{ in } \Omega_0,$$

and therefore $v(t, x) \geq v(0, x)$ in $(0, T) \times \Omega_0$. From the comparison theorem we then obtain $u_\epsilon \geq v \geq u_{\epsilon 0}$ in $(0, T) \times \Omega_0$. Therefore $z_\epsilon(0, x) \geq 0$ in $\Omega_0$ for any $\tau \in (0, T)$.

Thus, $z_\epsilon \geq 0$ on the parabolic boundary of the domain $\omega_{T-\tau}$, and by the Maximum Principle $z_\epsilon \geq 0$ in $\omega_{T-\tau}$. Hence, by passing to the limit $\epsilon \to 0^+$ and $\tau \to 0^+$, we obtain that $u(t, x)$ does not decrease in $t$ in $\omega_T$ and therefore $u_t \geq 0$ in $P_T[u]$. Necessity of condition (12) is obvious. □

## 2 An operator comparison of solutions theorem

We shall first demonstrate the possibilities of the operator method of comparison using relatively simple equations. The comparison theorem we obtain in the process will be used in § 4 in the study of the localization in boundary blow-up regimes.

Let us consider in $\omega_T$ boundary value problems for two different (degenerate) parabolic equations ($\nu = 1, 2$):

$$u_t^{(\nu)} = [k^{(\nu)}(u^{(\nu)})u_x^{(\nu)}]_x \equiv (\phi^{(\nu)}(u^{(\nu)}))_{xx}; \tag{13}$$

$$u^{(\nu)}(0, x) = u_0^{(\nu)}(x) \geq 0, \; x > 0; \tag{14}$$

$$u^{(\nu)}(t, 0) = u_1^{(\nu)}(t) > 0, \; 0 < t < T. \tag{15}$$

Let the functions in the statement of the problems (13)–(15) satisfy all the requirements of subsection 1 concerning the functions $k$, $u_0$, $u_1$ in the statement of problem (1)–(3), and assume that there are in $\omega_T$ non-negative generalized solutions of the problem we are considering.

Let us introduce a function $E(p)$, which is twice continuously differentiable for all $p > 0$, such that, moreover, $E(0) = 0$, $E(\infty) = \infty$ and $E'(p) > 0$ for all $p > 0$. The last condition means that $E$ is a bijection $\overline{\mathbf{R}}_+ \mapsto \overline{\mathbf{R}}_+$. Therefore the inverse mapping $E^{-1}(p)$ is defined on $\mathbf{R}_+$; it satisfies all the requirements made on the function $E(p)$.

Let $u^{(2)} \geq E^{-1}(u^{(1)})$ on $\gamma_T$. The problem of the *operator* (functional) comparison of solutions $u^{(2)}$ and $u^{(1)}$ is to determine conditions under which $u^{(2)} \geq E^{-1}(u^{(1)})$ everywhere in $\omega_T$. In the following theorem we use to that end pointwise estimates of the highest order derivative of the majorizing solution, which follow from its criticality.

**Theorem 3.** *Let $u^{(2)} \geq E^{-1}(u^{(1)})$ on $\gamma_T$, that is,*

$$
\begin{aligned}
u_0^{(2)}(x) &\geq E^{-1}(u_0^{(1)}(x)), \quad x \in \mathbf{R}_+, \\
u_1^{(2)}(t) &\geq E^{-1}(u_1^{(1)}(t)), \quad 0 < t < T.
\end{aligned} \tag{16}
$$

*Moreover, let the solution of problem* (13)–(15) *be critical for* $\nu = 2$ *and assume that for all* $p > 0$ *the conditions*

$$k^{(2)}(p) - k^{(1)}(E(p)) \geq 0, \tag{17}$$

$$[k^{(2)}(p)/k^{(1)}(E(p))E'(p)]' \geq 0, \tag{18}$$

*hold. Then $u^{(2)} \geq E^{-1}(u^{(1)})$ everywhere in $\omega_T$.*

*Proof.* Let us set $E^{-1}(u^{(1)}) = V^{(1)}$. The function $V^{(1)}$ satisfies in $\omega_T$ the equation

$$
\begin{aligned}
V_t^{(1)} &= L^{(1)}(V^{(1)}, |V_x^{(1)}|, V_{xx}^{(1)}) \equiv \\
&\equiv k^{(1)}(E(V^{(1)}))V_{xx}^{(1)} + \frac{[k^{(1)}(E(V^{(1)}))E'(V^{(1)})]'}{E'(V^{(1)})}(V_x^{(1)})^2,
\end{aligned} \tag{19}
$$

and, by (16), $u^{(2)} \geq V^{(1)}$ on $\gamma_T$. The solution $u^{(2)}(t, x)$ of the equation

$$u_t^{(2)} = L^{(2)}(u^{(2)}, |u_x^{(2)}|, u_{xx}^{(2)}) \equiv k^{(2)}(u^{(2)})u_{xx}^{(2)} + k^{(2)\prime}(u^{(2)})(u_x^{(2)})^2 \tag{20}$$

is critical, that is, $u_t^{(2)} \geq 0$ in $P_T[u^{(2)}]$, which ensures that in $P_T[u^{(2)}]$ we have the pointwise bound

$$u_{xx}^{(2)} \geq -\frac{k^{(2)\prime}(u^{(2)})}{k^{(2)}(u^{(2)})}(u_x^{(2)})^2. \tag{21}$$

It is not hard to see that the inequalities (17), (18) are the conditions for direct comparison of solutions of parabolic equations (19), (20), if we have the estimate (21) (see Theorem 2 of § 1).

Let us use the fact that generalized solutions of the problems (13)–(15) can be obtained as limits as $k \to \infty$ of sequences of classical strictly positive and bounded solutions $\{u_k^{(\nu)}\}$ of the corresponding equations. It is not hard to see that monotone decreasing with $k$ sequences of infinitely differentiable functions $\{u_k^{(2)}(0, x)\}$ and $\{u_k^{(2)}(t, 0)\}$, converging to the functions $u_0^{(2)}(x)$ and $u_1^{(2)}(t)$, can be chosen to be critical.

Existence of the sequence $\{u_k^{(2)}(t, 0)\}$ with the required properties is obvious. As far as the initial function is concerned, this problem reduces to approximation of a piecewise smooth convex function $U_0^{(2)}(x) = \phi^{(2)}(u_0^{(2)}(x))$ $(U_0^{(2)''} \geq 0$ at all points where $U_0^{(2)} > 0)$ by a sequence of smooth convex positive functions $\{\phi^{(2)}(u_k^{(2)}(0, x))\}$ uniformly bounded away from zero. Clearly, this can always be done.

In the construction of the approximating smooth solutions $\{u_k^{(\nu)}\}$ it is not hard to have $u_k^{(2)} \geq V_k^{(1)} = E^{-1}(u_k^{(1)})$ on $\gamma_T$ for any $k = 1, 2, \ldots$. The functions $u_k^{(2)}$ are critical and satisfy the inequality (21). Therefore by Theorem 2 $u_k^{(2)} \geq V_k^{(1)}$ in $\omega_T$ for each $k = 1, 2, \ldots$. Hence by passing to the limit $k \to \infty$ we have that $u^{(2)} \geq V^{(1)}$ in $\omega_T$. □

**Corollary.** *Let the function $E$ be such that*

$$k^{(1)}(E(p)) = k^{(2)}(p), \quad p > 0, \tag{22}$$

*and $u^{(2)} \geq E^{-1}(u^{(1)})$ on $\gamma_T$. Let us have, furthermore, that*

$$E''(p) \leq 0, \quad p > 0. \tag{23}$$

*Then $u^{(2)} \geq E^{-1}(u^{(1)})$ everywhere in $\omega_T$.*

The inequality (23) is equivalent to the following one:

$$\left[\frac{k^{(2)'}(k^{(2)-1}(p))}{k^{(1)'}(k^{(1)-1}(p))}\right]' \geq 0 \quad (k^{(\nu)'}(p) \neq 0, \, p > 0). \tag{23'}$$

Let us note that in the conditions of the corollary there is no assumption of criticality of $u^{(2)}$. Validity of the inequality $u^{(2)} \geq E^{-1}(u^{(1)}) \equiv V^{(1)}$ in $\omega_T$ follows from a direct comparison of equations (19) and (20). The first of these can be written in the form

$$V_t^{(1)} = (k^{(2)}(V^{(1)})V_x^{(1)})_x + \frac{k^{(2)}(V^{(1)})E''(V^{(1)})}{E'(V^{(1)})}(V_x^{(1)})^2.$$

from which, if (23) holds, it follows that $V^{(1)}$ is a subsolution of the problem (13)–(15) for $\nu = 2$, and since $u^{(2)} \geq V^{(1)}$ on $\gamma_T$, $u^{(2)} \geq V^{(1)}$ everywhere in $\omega_T$.

In the proof of Theorem 3 we used our ability to approximate a critical initial function $u_0^{(2)}(x)$ by a sequence of positive smooth critical functions. Under the assumption that this can be done (see Theorem 2), the comparison theorem is valid in the case of boundary value problems for parabolic equations with a source:

$$u_t^{(\nu)} = \nabla \cdot (k^{(\nu)}(u^{(\nu)})\nabla u^{(\nu)}) + Q^{(\nu)}(u^{(\nu)}), \quad \nu = 1, 2, \tag{24}$$

where $Q^{(\nu)} \in C^1([0, \infty))$ are given functions.

**Theorem 4.** *Let $u^{(2)} \geq E^{-1}(u^{(1)})$ on $\gamma_T$, and let the solution $u^{(2)}$ be critical, that is, $u_t^{(2)} \geq 0$ in $P_T[u^{(2)}]$. Assume, moreover, that we have the inequalities*

$$k^{(2)}(p) - k^{(1)}(E(p)) \geq 0, \tag{25}$$

$$\{k^{(2)}(p)/[k^{(1)}(E(p))E'(p)]\}' \geq 0, \tag{26}$$

$$Q^{(2)}(p)k^{(1)}(E(p)) - Q^{(1)}(E(p))k^{(2)}(p)/E'(p) \geq 0. \tag{27}$$

*Then $u^{(2)} \geq E^{-1}(u^{(1)})$ everywhere in $\omega_T$.*

*Proof.* To prove the theorem, it is convenient to write the equation satisfied by the function $V^{(1)} = E^{-1}(u^{(1)})$ in the form

$$V_t^{(1)} = \nabla \cdot [k^{(1)}(E)\nabla V^{(1)}] + \frac{k^{(1)}(E)E''}{E'}|\nabla V^{(1)}|^2 + \frac{Q^{(1)}(E)}{E'}. \tag{28}$$

Then the inequalities (25)–(27) are the conditions for a direct comparison of solutions of equations (24) for $\nu = 2$ and (28) (see Theorem 2 in § 1). □

Using the Maximum Principle it is not hard to check the validity of the following simplest possible version of the operator comparison theorem:

**Corollary.** *Let the function $E$ be such that the inequality (22) holds, and let $u^{(2)} \geq E^{-1}(u^{(1)})$ on $\gamma_T$. Assume, moreover, that for all $p > 0$*

$$E''(p) \leq 0, Q^{(2)}(p) \geq Q^{(1)}(E(p))/E'(p).$$

*Then $u^{(2)} \geq E^{-1}(u^{(1)})$ in $\omega_T$.*

## § 3 $\psi$-criticality conditions

In this section we present one possible generalization of the concept of criticality of a solution, and derive a new class of pointwise estimates of the highest order derivative of a solution of a quasilinear parabolic equation in terms of lower order ones. These results can be used to prove more general comparison theorems for solutions of different equations; they can also be applied in other areas.

## 1 Definition of a $\psi$-critical problem

Let $\Omega$ be a bounded domain in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$. In this section we consider a boundary value problem for a quasilinear parabolic equation:

$$u_t = \mathbf{A}(u) = \nabla \cdot (k(u)\nabla u) + Q(u),\ t > 0,\ x \in \Omega; \tag{1}$$

$$u(0, x) = u_0(x) \geq 0,\ x \in \Omega; \tag{2}$$

$$u(t, x) = 0,\ 0 < t < T,\ x \in \partial\Omega. \tag{3}$$

For simplicity we shall assume that the positive in $\omega_T$ solution of the problem (1)–(3) is classical.

Suppose we are given a function $\psi(p)$, which is twice continuously differentiable for $p > 0$, $\psi(0) = 0$, $\psi \in C([0, \infty))$.

**Definition.** The problem (1)–(3) and its solution will be called *$\psi$-critical (with respect to a given function $\psi$)*, if everywhere in $\omega_T$ we have the inequality

$$u_t(t, x) - \psi(u(t, x)) \geq 0. \tag{4}$$

In accordance with the definition of § 1, we shall call a zero-critical problem (that is, $\psi$-critical with respect to $\psi \equiv 0$) simply critical. From inequality (4) follow more general estimates of the highest order spatial derivative of the solution in terms of the lower order ones, than those obtained in § 1:

$$\Delta u \geq -\frac{k'(u)}{k(u)}|\nabla u|^2 + \frac{\psi(u) - Q(u)}{k(u)} \text{ in } \omega_T. \tag{5}$$

Using estimates of the form (5) with a sufficiently general function $\psi$ widens the scope of the direct and operator methods of solution comparison.

## 2 Sufficient conditions for $\psi$-criticality of a problem

Let $u_0 \in C^2(\Omega) \cap C(\bar{\Omega})$, $u \in C^{2,4}_{t,x}(\omega_T) \cap C^{1,2}_{t,x}(\bar{\omega}_T)$. Let us note that the smoothness requirements on the solution in $\omega_T$ (and, in part, on the initial function $u_0$) can in principle be weakened. Under these assumptions we have

**Theorem 1.** *For $\psi$-criticality of the problem* (1)–(3) *it is sufficient that the function $u_0$ satisfies the condition*

$$\mathbf{A}(u_0(x)) - \psi(u_0(x)) \geq 0,\ x \in \Omega, \tag{6}$$

*and that for all $p > 0$ we have the inequalities*

$$[(k\psi)'/k]'(p) \geq 0, \tag{7}$$

$$[k'\psi^2 - Q^2(k\psi/Q)'](p) \geq 0. \tag{8}$$

The inequality (8) can be written in the more compact form

$$\psi^2(p)\left[\frac{1}{k}\left(\frac{Q}{\psi}-1\right)\right]'(p)\geq 0,\ p>0. \tag{8'}$$

Let us note that (8′) makes sense also for values $p > 0$ where $\psi(p) = 0$ (similarly, (8) is defined at points where $Q = 0$).

*Proof.* Let us set $z = u_t - \psi(u)$. From (6) and the condition $\psi(0) = 0$, it follows that $z \geq 0$ on $\gamma_T$. Using the equalities

$$u_t = z + \psi(u),\ \Delta u = \frac{1}{k(u)}\{z + \psi(u) - Q(u) - k'(u)|\nabla u|^2\}, \tag{9}$$

it is not hard to obtain the parabolic equation satisfied by $z$:

$$\begin{aligned} &z_t - k(u)\Delta z - 2k'(u)\nabla u\cdot\nabla z - \\ &-z\left\{\frac{k'(u)}{k(u)}z + \frac{k'(u)}{k(u)}(2\psi(u) - Q(u)) + Q'(u) + k(u)\left[\frac{k'(u)}{k(u)}\right]'|\nabla u|^2\right\} = \\ &= \left\{k'(u)\psi^2(u) - Q^2(u)\left[\frac{k(u)\psi(u)}{Q(u)}\right]'\right\}\frac{1}{k(u)} + k(u)\left[\frac{(k(u)\psi(u))'}{k(u)}\right]'|\nabla u|^2. \end{aligned} \tag{10}$$

To derive this equation, it suffices to notice that $z_t = u_{tt} - \psi'(u)u_t$ and then to determine from equation (1) the derivative $u_{tt}$, simplifying by using the equalities (9).

Conditions (7), (8) guarantee that the right-hand side of equation (10) is non-negative, which by the Maximum Principle ensures non-negativity of the function $z = u_t - \psi(u)$ in $\omega_T$ if $z \geq 0$ on $\gamma_T$. □

**Remark 1.** In the case $\psi \equiv M = \text{const} < 0$, criticality conditions for an operator **A** have the following form: $(k'/k)' \leq 0$, $|(Q/M - 1)/k|' \geq 0$. They hold, for example, for $k(u) = (1 + u)^\sigma$, $\sigma > 0$, and $Q \equiv 0$. Therefore the solution of the equation $u_t = \nabla\cdot((1 + u)^\sigma\nabla u)$, which satisfies conditions (2), (3), has the following interesting property

$$u_t(t,x) \geq \inf_{x\in\Omega} u_t(0,x) \equiv \inf_{x\in\Omega}\{\nabla\cdot((1 + u_0)^\sigma\nabla u_0)\}(x),$$

that is, for $t > 0$ the solution cannot decrease in $t$ faster than it did at the initial time.

**Remark 2.** Let $\psi(p) > 0$ for $p > 0$. Then from (4) it follows that everywhere in $\omega_T$ the solution of a $\psi$-critical problem is related to the initial function in the following way:

$$\int_{u_0(x)}^{u(t,x)} \frac{d\eta}{\psi(\eta)} \geq t, \ 0 < t < T. \tag{11}$$

This inequality provides us with a pointwise lower bound from for the solution (for an application of this kind of bound, see § 6).

**Remark 3.** As we already mentioned above, the smoothness requirement $u \in C_{t,x}^{2,4}(\omega_T) \cap C_{t,x}^{1,2}(\overline{\omega}_T)$ can be weakened, if we prove the claim first for the function

$$z(t,x) = \frac{1}{\tau}[u(t+\tau,x) - u(t,x)] - \psi(u(t,x)), \tau \in (0,T),$$

and then pass in the inequality $z \geq 0$ in $\omega_{T-\tau}$ to the limit as $\tau \to 0^+$.

**Remark 4.** Theorem 1 holds also in the case of generalized solutions of the problem (1)–(3). If, for example, we use the regularization of the proof of Theorem 2, § 2, then under the assumption $\phi(u_0) \in C^2(\overline{\Omega})$, $\Delta\phi(u_0) + Q(u_0) \geq \psi(u_0)$ in $\Omega$, for initial functions $u_{\epsilon 0} = \phi^{-1}(\phi(u_0) + \epsilon) \geq \phi^{-1}(\epsilon)$ in $\Omega$ we obtain

$$\Delta\phi(u_{\epsilon 0}) + Q(u_{\epsilon 0}) - \psi(u_{\epsilon 0}) \equiv \Delta\phi(u_0) + Q(u_{\epsilon 0}) - \psi(u_{\epsilon 0}) \geq$$

$$\geq Q(u_{\epsilon 0}) - Q(u_0) - [\psi(u_{\epsilon 0}) - \psi(u_0)] = o(1)$$

as $\epsilon \to 0^+$ in $\Omega$. Here $(u_\epsilon)_t - \psi(u_\epsilon) \equiv -\psi(\phi^{-1}(\epsilon)) = o(1)$ as $\epsilon \to 0^+$ on $(0,T) \times \partial\Omega$. Therefore if inequalities (7), (8) hold, by the Maximum Principle we have that

$$(u_\epsilon)_t - \psi(u_\epsilon) \geq \inf_{\gamma_t}\{(u_\epsilon)_t - \psi(u_\epsilon)\} = o(1)$$

as $\epsilon \to 0^+$ in $\omega_T$. Now, passing to the limit $\epsilon \to 0^+$ we derive the inequality $u_t - \psi(u) \geq 0$ in $P_T[u]$.

It is possible to give a different proof. Let us consider, for example, the case of radially symmetric solutions, $u = u(t,r)$, $r = |x|$. Let $P_T[u] = (0,T) \times \{0 \leq |x| < \zeta(t)\}$. As in the proof of Theorem 1, § 2, we use continuity of the derivative: $(\phi(u))_r' \to 0$ as $r \to \zeta^-(t)$. Then we can find in $P_T[u]$ a subset $(0,T) \times \{0 \leq r < r_*(t)\}$, such that, first, $r_*(t) \in (\zeta(t) - \epsilon, \zeta(t))$ for all $t \in (0,T)$, and, second, $u_t - \psi(u) \geq -\delta$ at the point $(t, r_*(t))$, where $\epsilon > 0$, $\delta > 0$ can be arbitrarily small. In the derivation of the last inequality, the value $r = r_*(t)$ is chosen from the condition $u_t(t, r_*(t)) > 0$, while the estimate $u_t - \psi(u) \geq -\delta$ for $r = r_*(t)$ follows from the conditions $r_*(t) \in (\zeta(t) - \epsilon, \zeta(t))$, $\psi(0) = 0$. Therefore the argument of the proof of Theorem 1 shows that by the Maximum Principle $z \equiv u_t - \psi(u) \geq -\delta$ in all interior points of the set $(0,T) \times \{0 \leq r < r_*(t)\}$. Hence, letting $\epsilon$ and $\delta$ go to zero, we obtain that $z \geq 0$ in $P_T[u]$.

**Remark 5.** It is not hard to extend the proof of Theorem 1 to the Cauchy problem. In this case $u_0(x)$ must be such that $z = u_t - \psi(u) \to 0$ as $|x| \to \infty$ for all

$t \in (0, T)$. Under the restrictions of Remark 4, the same statement also holds if $u$ is a generalized solution of the Cauchy problem with compact support.

## § 4 Heat localization in problems for arbitrary parabolic nonlinear heat equations

The main result of this section is the proof of existence of the heat localization phenomenon in media with arbitrary dependence of the thermal conductivity coefficient on temperature. In § 5 we obtain a slightly less general result on non-existence of localization.

Proofs of these assertions are based on the operator comparison theorem formulated in § 2. A different approach to the study of the heat localization phenomenon in general media is suggested in Ch. VI.

### 1 Formulation of the problem

We are going to consider in $\omega_T = (0, T) \times \mathbf{R}_+$, $T < \infty$, the first boundary value problem for a degenerate parabolic equation:

$$u_t = (k(u)u_x)_x \equiv (\phi(u))_{xx}; \tag{1}$$

$$u(0, x) = u_0(x) \geq 0, \ x \in \mathbf{R}_+; \ u_0 \in C(\mathbf{R}_+), \ \sup u_0 < \infty; \tag{2}$$

$$u(t, 0) = u_1(t) > 0, \ 0 < t < T; \ u_1 \in C([0, T)), \tag{3}$$

where the boundary function $u_1(t)$ blows up in finite time:

$$u_1(t) \to \infty, t \to T^- . \tag{4}$$

The function $k(u)$ (thermal conductivity coefficient) is sufficiently smooth: $k \in C^2(\mathbf{R}_+) \cap C([0, \infty))$ and is positive for $u > 0$, $k(0) = 0$. Moreover, we shall assume that the inequality

$$\int_0^1 \frac{k(u)}{u}\, du < \infty \tag{5}$$

holds; this is a necessary and sufficient condition for finite speed of propagation of disturbances in processes described by equation (1) (see § 3, Ch. I). We shall also assume that $k'(u) > 0$ for all sufficiently small $u > 0$. The initial function $u_0(x)$ will be taken to have compact support, which by (5) ensures that the solution of the problem (1)–(3) has compact support in $x$ for each $0 < t < T$. Furthermore, let $\sup |[\phi(u_0)]_x| < \infty$.

Under these assumptions there exists in $\omega_\tau = (0, \tau) \times \mathbf{R}_+$, $\tau < T$, a unique nonnegative generalized solution of the problem (1)–(3). We remind the reader (see § 1, Ch. III) that by definition problem (1)–(3) with the given boundary regime with blow-up will exhibit *heat localization* if there exists a constant $l^* < \infty$, such that

$$\text{meas supp } u(t, x) \leq l^*, 0 \leq t < T. \tag{6}$$

The smallest possible value of $l^*$ in (6) is called the *localization depth.*

If on the other hand meas supp $u(t, x) \to \infty$ as $t \to T^-$, then *there is no heat localization in the problem* (in this case the thermal wave heats to infinite temperature the whole half-space $x > 0$).

In this subsection we solve the following problem: *given a (sufficiently arbitrary) thermal conductivity coefficient $k(u)$ in equation* (1) *find the classes of boundary regimes with blow-up $\{u_1(t)\}$ which lead to heat localization.*

To that end we use the method of generalized comparison of solutions of two different parabolic equations (see § 2).

## 2 Sufficient conditions for heat localization

The main result of this subsection is

**Theorem 1.** *Let the thermal conductivity coefficient $k(u)$ satisfy for some $\alpha =$* const $> 0$ *the condition*

$$[k^\alpha]'(0) < \infty. \tag{7}$$

*Then there exist boundary blow-up regimes, which lead to heat localization in the problem* (1)–(3).

Therefore the existence of the heat localization effect is independent of the behaviour of $k(u)$ as $u \to \infty$. Naturally, the form of the boundary regime with blow-up, which leads to localization is primarily determined by the behaviour of the thermal conductivity at high temperatures. Sharp estimates for classes of localized boundary regimes will be obtained below.

2.1. Let us consider first the *case of unbounded coefficients* $k$, when

$$k(u) \to \infty, u \to \infty. \tag{8}$$

In this case the localization effect will be analyzed using the operator comparison method for solutions of equation (1) and an equation with power nonlinearity,

$$u_t = (u^\sigma u_x)_x, \tag{9}$$

where $\sigma > 0$ is a constant.

The operator method will be used to compare the solution of the problem (1)–(3) and the separable solution of equation (9) (S-regime):

$$u_{(\sigma)}(t,x) = (T-t)^{-1/\sigma}(1 - x/x_0)_+^{2/\sigma},\ x_0 = [2(\sigma+2)/\sigma]^{1/2}. \tag{10}$$

This solution was studied in detail in Ch. III, §§ 2, 3. It graphically illustrates the heat localization property. Here $l^* = x_0$. Note that the function $u_{(\sigma)}$ is critical, since $(\partial/\partial t)u_{(\sigma)}(t,x) \geq 0$ almost everywhere in $\omega_T$.

Given a thermal conductivity coefficient $k$ in (1), let us find out which functions (operators) $E$ ensure that if the inequality $u_{(\sigma)}(t,x) \geq E^{-1}(u(t,x))$ holds on $\gamma_T$, then it holds in $\omega_T$. It follows from Theorem 3, § 2, that for that it suffices to find at least for one $\sigma > 0$ a solution $E(p)$ of the system of ordinary differential inequalities

$$p^{\sigma} - k(E(p)) \geq 0,\ p > 0, \tag{11}$$

$$\left[\frac{p^{\sigma}}{k(E(p))E'(p)}\right]' \geq 0,\ p > 0. \tag{12}$$

We remind the reader that the mapping $E : \mathbf{R}_+ \mapsto \mathbf{R}_+$ must be bijective and monotone, that is $E'(p) > 0$ for $p > 0$, $E(0) = 0$, $E(\infty) = \infty$.

Inequalities (11), (12) follow directly from the comparison conditions (17), (18) of Theorem 3, § 2, if we set there $k^{(1)}(u) = k(u)$, $k^{(2)}(u) = u^{\sigma}$.

The following assertion gives necessary conditions for solvability of the system of inequalities (11), (12).

**Lemma 1.** *Let the thermal conductivity coefficient satisfy* (7) *for some* $\alpha > 0$. *Then for any* $0 < \sigma \leq \sigma_0 = 1/\alpha$ *there exists a solution* $E$ *of the system of inequalities* (11), (12).

*Proof.* For convenience let us set $E^{-1} = H$. Then the inequalities (11), (12) take the form

$$k(p) \leq H^{\sigma},\ p > 0, \tag{11'}$$

$$\left[\frac{k(p)}{H^{\sigma}(p)H'(p)}\right]' \leq 0,\ p > 0. \tag{12'}$$

Let us set

$$\frac{k(p)}{H^{\sigma}H'(p)} \equiv \frac{1}{\omega(p)},\ p > 0.$$

Then, clearly, inequality (12) will be satisfied if

$$\omega(p) > 0,\ \omega'(p) \geq 0,\ p > 0. \tag{13}$$

The function $H(p)$ has the following form:

$$H(p) = \left[(1+\sigma)\int_0^p k(\eta)\omega(\eta)\,d\eta\right]^{1/(\sigma+1)},\ p > 0. \tag{14}$$

By condition (7) and the assumption $\sigma \le \sigma_0$, we have

$$\left[k^{1/\sigma}\right]'(0) = \frac{1}{\sigma\alpha}\left\{k^{1/\sigma - \alpha}\left[k^\alpha\right]'\right\}(0) < \infty.$$

Therefore we can always find a smooth function $\omega(p)$, satisfying the inequalities (13), such that

$$\omega(p) \ge \left[k^{1/\sigma}(p)\right]', \quad p > 0, \tag{15}$$

since the expression in the right-hand side is bounded for all $p \ge 0$.

Let us substitute into (14) an arbitrary function $\omega$, which satisfies conditions (13), (15), and let us show that the operator (14) constructed in this way is a solution of the system of inequalities (11′), (12′). Indeed, in the new notation (11′) takes the form

$$\int_0^p k(\eta)\left\{\left[k^{1/\sigma}(\eta)\right]' - \omega(\eta)\right\} d\eta \le 0$$

and by (15) is satisfied for all $p > 0$.

Next, from (14) it immediately follows that $H(0) = 0$, $H'(p) > 0$ for $p > 0$ (the latter inequality is ensured by the first condition in (13)). Moreover, using (15), we have from (14) that

$$H(p) \ge \left[(1+\sigma)\int_0^p k(\eta)\left[k^{1/\sigma}(\eta)\right]' d\eta\right]^{1/(1+\sigma)} \equiv k^{1/\sigma}(p).$$

Hence by (8) $H(\infty) = \infty$.

Thus the function $H : \mathbf{R}_+ \to \mathbf{R}_+$, defined by (14) satisfies the system (11′), (12′). Therefore $E = H^{-1} : \mathbf{R}_+ \to \mathbf{R}_+$ is a solution of the original system (11), (12). □

**Remark 1.** It is not hard to see that the claim of the lemma is still valid without the restriction (8) on the coefficient $k$, since no conditions on the rate of increase of the function $\omega(p)$ (apart from (13), (15)) arise in the course of the proof. Therefore we can always choose the function $\omega$ so that the integral in the right-hand side of (14) grows without bound as $p \to \infty$, which ensures that $H(\infty) = \infty$, or, which is the same, that $E(\infty) = \infty$.

**Remark 2.** Let us show that from (7) it follows that condition (5) holds. Indeed, by (7) there exists a constant $M > 0$, such that $k(u) \le Mu^{1/\alpha}$ for all $0 \le u \le 1$, and therefore

$$\int_0^1 \frac{k(u)}{u}\, du \le M\alpha < \infty.$$

Let us note that there exist coefficients $k$ for which condition (7) does not hold for any $\alpha > 0$ and system (11), (12) has no solutions. This is true, for example, for a coefficient $k$ which has for $u \in (0, \epsilon]$, $\epsilon < 1$, the form $k(u) = (-\ln u)^{\mu}$, where $\mu < -1$ is a constant. At the same time condition (5) for finite speed of propagation of perturbations is satisfied.

Let us observe also that solvability of the system of inequalities (11), (12), which defines conditions for heat localization in this problem, depends only on the behaviour of the thermal conductivity coefficient $k(u)$ for small $u \to 0^+$.

From the method of proof of Lemma 1 we immediately have

**Corollary.** *Let condition* (8) *hold, and assume that there exists a constant* $\alpha > 0$, *such that*

$$[k^{\alpha}(p)]'' \geq 0, \quad p > 0. \tag{16}$$

*Then* $E = k^{-1}(p^{1/\alpha})$, *where* $k^{-1}$ *is the function inverse to* $k$ *(*$k^{-1}$ *exists due to monotonicity of* $k$*; this follows from (16)), is a solution of the system of inequalities* (11), (12) *for* $\sigma = 1/\alpha$.

Now using Lemma 1 and the operator comparison theorem from § 2, we can formulate sufficient conditions for heat localization in the problem (1)–(3).

**Theorem 2.** *Let the thermal conductivity coefficient* $k(u)$ *satisfy* (7) *for some* $\alpha > 0$*; let* $E$ *be some solution of the system of inequalities* (11), (12), *corresponding to a fixed* $\sigma \in (0, 1/\alpha]$. *Moreover, let the boundary conditions of problem* (1)–(3) *satisfy the inequalities*

$$u_0(x) \leq E\left[T^{-1/\sigma}\left(1 - \frac{x}{x_0}\right)_+^{2/\sigma}\right], \quad x > 0; \tag{17}$$

$$u_1(t) \leq E\left[(T-t)^{-1/\sigma}\right], \quad 0 < t < T. \tag{18}$$

*Then everywhere in* $\omega_T$ $u(t,x) \leq E[u_{(\sigma)}(t,x)]$, *where* $u_{(\sigma)}$ *is defined in* (10). *Therefore there is heat localization with depth* $l^* \leq x_0$ *in problem* (1)–(3).

**Remark.** In § 4, Ch. III it was shown that solution of the problem for equation (9) with the boundary regime $u(t,0) = (T-t)^{-1/\sigma}$, $t \in (0,T)$ and an initial function $u(0,x) \in C(\overline{\mathbf{R}}_+)$ is bounded uniformly in $t \in (0,T)$ for all $x > x_0 = [2(\sigma+2)/\sigma]^{1/2}$. Using this result, it is not hard to show that the restriction (17) on $u_0(x)$ in the theorem is not essential: if all the other conditions hold, in order to have localization, it is sufficient for $u_0$ to be a function with compact support. A method to prove this type of assertion will be presented below.

The result of Theorem 2 proves Theorem 1 in the case of unbounded thermal conductivity coefficients $k$, which satisfy (8).

Let us consider now some specific examples of the use of Theorem 2 (all the coefficients of the examples below satisfy the condition of finite speed of propagation of perturbations).

**Example 1.** Let $k(u) = u^\sigma/[1+\mu(u)]$, $\sigma > 0$, where $\mu(u)$ is an arbitrary smooth function, satisfying $\mu(u) \geq 0$, $\mu'(u) \geq 0$, $u > 0$.

In this case a solution of the system of inequalities (11), (12) is the identity transformation $E(p) \equiv p$. This is equivalent to an application of the direct comparison theorem to solutions of equations (1), (9) (see Theorem 2 in § 1 and Example 1 considered there). Therefore by Theorem 2, there is heat localization with depth $l^* \leq x_0 = [2(\sigma+2)/\sigma]^{1/2}$ in problem (1)–(3) with boundary conditions that satisfy the inequalities

$$u_0(x) \leq u_{(\sigma)}(0, x), \ x \in \mathbf{R}_+; \ u_1(t) \leq (T-t)^{-1/\sigma}, \ 0 < t < T.$$

**Example 2.** Let $k(u) = [e^u - 1]^\lambda$, where $\lambda > 0$ is a fixed constant. In this case inequality (16) holds for $\alpha = 1/\lambda$, and therefore a solution of the system of inequalities (11), (12) with $\sigma = \lambda$ is the transformation

$$E(p) = k^{-1}(p^\lambda) \equiv \ln(1+p), \ p > 0.$$

Then from Theorem 2 we conclude that boundary conditions that satisfy the inequalities

$$\begin{aligned} u_0(x) &\leq \ln[1 + u_{(\lambda)}(0, x)], \ x \in \mathbf{R}_+; \\ u_1(t) &\leq \ln[1 + (T-t)^{-1/\lambda}], \ 0 < t < T, \end{aligned}$$

ensure occurrence of heat localization in the problem (1)–(3) with the depth $l^* \leq [2(\lambda+2)/\lambda]^{1/2}$ (let us observe that here $u(t,x) \leq \ln[1 + u_{(\lambda)}(t,x)]$ everywhere in $\omega_T$).

**Example 3.** Let us consider the coefficient $k(u) = u\exp\{u^2\}$, which satisfies condition (16) for $\alpha = 1$. Therefore the transformation $E(p) = k^{-1}(p)$ is the required one, and if inequalities (17), (18) hold, there is heat localization in the problem (1)–(3) with the depth $l^* \leq \sqrt{6}$. Let us estimate the asymptotic behaviour as $t \to T^-$ of the boundary regime leading to localization. Since

$$k^{-1}(u) \simeq \ln^{1/2} u - \frac{1}{4}\frac{\ln \ln u}{\ln^{1/2} u}, \ u \to \infty,$$

for localization it is enough that

$$u_1(t) \lesssim |\ln(T-t)|^{1/2} - \frac{1}{4}\frac{\ln|\ln(T-t)|}{|\ln(T-t)|^{1/2}}, \ t \to T^-.$$

**Example 4.** Let $k(u) = \exp\{e^u - 1\} - 1$. Here the required operator $E$ corresponding to $\sigma = 1$ is

$$E(p) = k^{-1}(p) = \ln|1 + \ln(1 + p)|,$$

and therefore if $u_0(x) \le E[u_{(1)}(0, x)]$ in $\mathbf{R}_+$ and

$$u_1(t) \le \ln\left\{1 + \ln|1 + (T - t)^{-1}|\right\}, \quad 0 < t < T,$$

then heat localization with the depth $l^* \le \sqrt{6}$ occurs in the problem (1)–(3).

Using the methods of the theory of a.s.s. (see Ch. VI), it can be shown that the estimates of localized blow-up regimes obtained in Examples 2-4, are optimal and cannot be improved.

Let us consider now an example for which this is not the case.

**Example 5.** Let $k(u) = \ln^\lambda(1+u)$, $\lambda > 0$ is a fixed constant. In this case condition (7) is satisfied, for example, for $\alpha = 1/\lambda$, and therefore we conclude from Lemma 1 that for any $\sigma \in (0, \lambda]$ there exists a solution of the system of inequalities (11), (12). Following the proof of Lemma 1, let us construct the desired $E$.

Let us fix arbitrary $\sigma \in (0, \lambda]$. Inequality (15) is equivalent to the inequality

$$\omega(p) \ge \frac{\lambda}{\sigma} \frac{\ln^{\lambda/\sigma - 1}(1 + p)}{1 + p}, \quad p > 0. \tag{19}$$

The function in the right-hand side is bounded from above by the quantity

$$C_{\lambda\sigma} = \frac{\lambda}{\sigma}\left(\frac{\lambda}{\sigma} - 1\right)^{\lambda/\sigma - 1} \exp\left\{1 - \frac{\lambda}{\sigma}\right\}, \quad \sigma < \lambda, C_{\lambda\lambda} = 1.$$

Hence, taking into account conditions (13), we conclude that to achieve the maximal growth rate of $E(p)$ as $p \to \infty$ (and therefore the maximal admissible growth rate of boundary blow-up regimes $u_1(t) \le E|(T - t)^{-1/\sigma}|$ as $t \to T^-$), it is necessary to set $\omega(p) \equiv C_{\lambda\sigma}$ in (14). Recall that $H = E^{-1}$, and therefore the slower $H(p)$ grows as $p \to \infty$, the faster will $E(p)$ grow.

Thus, from (14) for $\sigma \in (0, \lambda]$ we obtain

$$H(p) \equiv E^{-1}(p) = \left[(1 + \sigma)C_{\lambda\sigma} \int_0^p \ln^\lambda(1 + \eta)\, d\eta\right]^{1/(\sigma+1)}.$$

Hence

$$E^{-1}(p) \simeq a_{\lambda\sigma}(p \ln^\lambda p)^{1/(\sigma+1)}, \; a_{\lambda\sigma} = [(1 + \sigma)C_{\lambda\sigma}]^{1/(1+\sigma)},$$

$$E(p) \simeq a_{\lambda\sigma}^{-(1+\sigma)}(1 + \sigma)^{-\lambda} p^{1+\sigma} \ln^{-\lambda} p$$

for sufficiently large $p$.

From Theorem 2 we conclude that in this problem localization is produced by any boundary blow-up regimes that satisfy as $t \to T^-$ the condition

$$u_1(t) \le E[(T-t)^{-1/\sigma}] \simeq a_{\lambda\sigma}^{(1+\sigma)}[\sigma/(\sigma+1)]^{\lambda}(T-t)^{-(1+\sigma)/\sigma}|\ln(T-t)|^{-\lambda}. \quad (20)$$

Recall that here the value of the parameter $\sigma \in (0, \lambda]$ is arbitrary. In particular, decreasing $\sigma$ we have that any power law boundary blow-up regimes

$$u_1(t) = (T-t)^n, \ 0 < t < T; \ n = \text{const} < 0, \quad (21)$$

will be localized.

However, the right-hand side of (20) does not allow rigorous passage to the limit as $\sigma \to 0^+$ (because, among others, the estimate $l^* \le x_0 = [2(\sigma+2)/\sigma]^{1/2}$ does not make any sense then) and therefore we cannot obtain in this way the precise boundary of localized regimes.

Such a boundary will be determined in § 2, Ch. VI by constructing approximate self-similar solutions of this equation. To this boundary there corresponds a function of exponential form,

$$u_1(t) = \exp\{(T-t)^{-1/(1+\lambda)}\}, \ 0 < t < T,$$

which agrees on the whole with the fact that the exponent $n < 0$ in the family (21) of localized regimes is arbitrary.

2.2. Let us now consider the *case of bounded coefficients* $k$. Without loss of generality we shall assume that

$$k(p) \le 1, \ p > 0. \quad (22)$$

In § 4, Ch. III we studied the action of boundary blow-up regimes on a medium with constant thermo-physical properties, diffusion of heat in which is described by the linear equation

$$v_t = v_{xx}, \ 0 < t < T, \ x \in \mathbf{R}_+; \quad (23)$$

$v(0,x) \equiv 0$ (which is not essential by the superposition principle).

It was shown, in particular, that the boundary blow-up regime

$$v(t,0) = \exp\{(T-t)^{-1}\}, \ 0 < t < T, \quad (24)$$

leads to effective heat localization with depth $L^* = 2$. This means that $v(t,x) \to \infty$ as $t \to T^-$ for all $0 \le x \le 2$, while for $x > 2$ the solution is bounded from above uniformly in $t \in (0, T)$:

$$v(t,x) < v(T^-,x) = \\ = \pi^{-1/2}\left[1-\left(\frac{2}{x}\right)^2\right]^{-1/2}\int_{(x^2-4)/(4T)}^{\infty} e^{-\eta}\eta^{-1/2}\,d\eta < \infty. \quad (25)$$

This result will be used in the operator comparison of solutions of equations (1) and (23); comparison conditions are inequality (22) and the inequality

$$[k(E(p))E'(p)]' \leq 0, \quad p > 0 \tag{26}$$

(this is equivalent to (12) if $\sigma = 0$).

Setting $H = E^{-1}$, we rewrite (26) as

$$[H'(p)/k(p)]' \geq 0, \quad p > 0.$$

Hence

$$E^{-1}(p) \equiv H(p) = \int_0^p k(\eta)\omega(\eta)\,d\eta, \quad p > 0, \tag{27}$$

where $\omega(p)$ is an arbitrary bounded function, which satisfies (13) and the condition

$$\int_0^\infty k(\eta)\omega(\eta)\,d\eta = \infty. \tag{28}$$

The restriction (28) ensures that $E(\infty) = \infty$.

From (27) it follows that in the case

$$\int_0^\infty k(\eta)\,d\eta = \infty, \tag{28'}$$

in order to have the maximal growth rate of $E(p)$ (or minimal for $E^{-1}(p)$) as $p \to \infty$, we have to require that the non-decreasing function $\omega$ be bounded in $\mathbf{R}_+$, for example, by setting $\omega \equiv 1$.

Thus, if conditions (13), (28) hold, operator $E$ in (27) guarantees comparison of solutions of equations (1) and (23). Without loss of generality we can consider only the case $u_0 \equiv 0$ in $\mathbf{R}_+$. Then, since the boundary condition (24) is critical, we conclude from the operator comparison theorem (see § 2) that the boundary blow-up regime

$$u_1(t) \leq E\{\exp\{(T-t)^{-1}\}\}, \quad 0 < t < T, \tag{29}$$

leads to effective heat localization in the problem (1)–(3) with depth $L^* \leq 2$, such that, furthermore

$$u(t, x) < E[v(T^-, x)], \quad 0 < t < T, \quad x > 2. \tag{30}$$

In the next theorem we "pass" from effective heat localization to localization in the strict sense.

**Theorem 3.** *Assume that in the problem (1)–(3) $u_0(x) \equiv 0$, and that the boundary blow-up regime satisfies condition (29), where $E: \mathbf{R}_+ \to \mathbf{R}_+$ is a solution of the system of inequalities (22), (26). Then there is heat localization in the problem, and there exists a constant $l^* > 0$, such that*

$$u(t, x) = 0 \text{ in } (0, T) \times \{x > l^*\}. \tag{31}$$

*Proof.* Let us fix arbitrary $x^* > 2$. Then it follows from (30) that $u(t, x^*) < E|v(T^-, x^*)| < \infty$ for any $0 < t < T$, and therefore by the comparison theorem for solutions of parabolic equations (see § 1, 3, Ch. I) in $(0, T) \times \{x > x^*\}$ the function $u(t, x)$ does not exceed the solution of the problem

$$
\begin{gathered}
U_t = (k(U)U_x)_x, \quad 0 < t < T, \quad x > x^*, \\
U(0, x) = 0, \quad x > x^*; \quad U(t, x^*) = E|v(T^-, x^*)|, \quad 0 < t < T.
\end{gathered} \tag{32}
$$

The solution of this problem is a self-similar one (see § 3, Ch. II) and has the form $U(t, x) = f(\zeta)$, where $\zeta = (x - x^*)/t^{1/2}$. The function $f$ is determined from the following boundary value problem for an ordinary differential equation:

$$
\begin{gathered}
(k(f)f')' + \frac{1}{2} f'\zeta = 0, \quad \zeta > 0; \\
f(0) = E|v(T^-, x^*)|, \quad f(\infty) = 0.
\end{gathered} \tag{33}
$$

Existence of a solution of the problem (33) for any finite $f(0) > 0$ has been established in [23, 68] (see § 1, Ch. I). There it is also shown that under condition (5), the function $f(\zeta)$ has compact support, that is, $f(\zeta) = 0$ for all $\zeta \geq \zeta_0$ ($\zeta_0 = \zeta_0(x^*) < \infty$ depends on the choice of $x^* > 2$).

Thus, everywhere in $(0, T) \times \{x > x^*\}$ we have the inequality

$$
u(t, x) \leq U(t, x) = f\left(\frac{x - x^*}{t^{1/2}}\right).
$$

Hence we immediately infer that

$$
\begin{aligned}
\operatorname{meas} \operatorname{supp} u(t, x) &\leq x^* + \operatorname{meas} \operatorname{supp} U(t, x) = \\
&= x^* + \zeta_0(x^*)t^{1/2} < x^* + \zeta_0(x^*)T^{1/2} < \infty.
\end{aligned}
$$

Therefore the problem (1)–(3) exhibits heat localization in the sense of (6), while for localization depth we have the estimate

$$
l^* \leq \inf_{x^* > 2} \left\{ x^* + \zeta_0(x^*)T^{1/2} \right\} < \infty, \tag{34}
$$

which completes the proof. □

**Remark.** In § 4, Ch. III it was shown that the opposite "passage," from strict (for a function $u_0(x)$ of compact support) to effective localization (when $u_0(x) \in C(\mathbf{R}_+)$ is an arbitrary function) is also possible. It is made by deriving a special energy estimate for the difference of two solutions of equation (1) which correspond to the same boundary condition.

Let us note that there is no restriction (7) on the thermal conductivity coefficient in Theorem 3: it is sufficient that the condition for finite speed propagation of perturbations holds. Let us consider some examples.

**Example 6.** Let $k(u) = 2\pi^{-1} \arctan u$. Then condition (22) is satisfied (inequality (5) also holds). Setting $\omega = 1$ in (27), we obtain

$$E^{-1}(p) = \int_0^p k(\eta)d\eta = p + \frac{2}{\pi} p \left[\arctan p - \frac{\pi}{2}\right] - \frac{1}{\pi} \ln(1 + p^2) \simeq$$
$$\simeq p - \tfrac{2}{\pi} \ln p$$

for sufficiently large $p$. Hence $E(p) \simeq p + 2/\pi \ln p$, $p \to \infty$, and by Theorem 3 we conclude that localization with depth (34) is produced by boundary regimes, which satisfy the estimate

$$u_1(t) \leq E\{\exp\{(T-t)^{-1}\}\} \simeq \exp\{(T-t)^{-1}\} + \frac{2}{\pi}(T-t)^{-1}$$

as $t \to T^-$.

**Example 7.** Let us consider the coefficient $k(u) = u[1 + 2u\ln(1 + u)]^{-1}$. In this case condition (22) holds, since $2u\ln(1 + u) \geq 2u^2/(1 + u)$ for all $u > 0$ and therefore

$$k(u) \leq u[1 + 2u^2/(1 + u)]^{-1} \equiv u(1 + u)[1 + u(1 + 2u)]^{-1} < 1$$

for any $u > 0$. Equality (28′) also holds. Then from (27) for $\omega \equiv 1$ we have

$$E^{-1}(p) = \int_0^p \frac{\eta d\eta}{1 + 2\eta\ln(1 + \eta)} \simeq \frac{1}{2}\frac{p}{\ln p}, \quad p \to \infty.$$

Hence $E(p) \simeq 2p\ln p$, and therefore the localized regimes satisfy

$$u_1(t) \leq E\{\exp\{(T-t)^{-1}\}\} \simeq 2(T-t)^{-1}\exp\{(T-t)^{-1}\}, \quad t \to T^-.$$

**Example 8.** Let $k(u) = 2u/(1 + u^2)$. In this case conditions (22), (28′) hold. Setting in (27) $\omega \equiv 1$, we have $E^{-1}(p) = \ln(1 + p^2)$, $E(p) = (e^p - 1)^{1/2}$. Therefore any boundary regime of the form

$$u_1(t) \leq \{\exp\{\exp\{(T-t)^{-1}\}\} - 1\}^{1/2} \simeq \exp\left\{\frac{1}{2}\exp\{(T-t)^{-1}\}\right\}, \quad t \to T^-,$$

leads to strict localization with depth (34) (and to effective localization with $L^* \leqslant 2$).

Estimates of localized boundary blow-up regimes obtained in Examples 7, 8 are not optimal. Sharp estimates for these cases will be derived in Ch. VI.

**Example 9.** Let $k(u) = u/(1+u^3)$. Then condition (28′) is not satisfied. Choosing $\omega(p) = p$ (conditions (13) are satisfied), we obtain from (27)

$$E^{-1}(p) = \frac{1}{3}\ln(1+p^3),\, E(p) = \left(e^{3p}-1\right)^{1/3} \simeq e^p,\ p \to \infty,$$

and therefore the localized regimes are

$$u_1(t) \le \exp\{\exp\{(T-t)^{-1}\}\},\ t \to T^-.$$

Setting now $\omega(p) \simeq p/\ln p$ as $p \to \infty$ (conditions (13) are still satisfied), we obtain from (27)

$$E^{-1}(p) \simeq \ln\ln p,\, E(p) \simeq \exp\{e^p\},\ \ p \to \infty,$$

and localization is produced by blow-up regimes

$$u_1(t) \le \exp\{\exp\{\exp\{(T-t)^{-1}\}\}\},\ t \to T^-.$$

As $\omega$ we could also take the function $\omega(p) \simeq p/|\ln p(\ln\ln p)|$ as $p \to \infty$ and so on.

Proceeding in this fashion, we conclude that in this case all the blow-up regimes of the form

$$u_1(t) < \exp\{\exp \ldots \{\exp\{(T-t)^{-1}\}\}\ldots\},\ t \to T^-,$$

with any finite number of exponents in the right-hand side, will be localized.

In § 2, Ch. VI we shall obtain results showing that under the condition

$$\int_1^\infty \frac{k(\eta)}{\eta}\, d\eta < \infty \tag{35}$$

all boundary blow-up regimes are localized. (Actually, this can be proved by comparison with a solution of travelling wave type, which, if (35) holds, blow up in finite time; see § 3, Ch. I). It is not hard to see that the coefficient $k$ of Example 9 satisfies this condition.

## 3 Effective heat localization

All the results of the previous subsection can be used to analyse effective localization.

A solution of problem (1)–(3), which blows up in finite time, is called *effectively localized* if it becomes infinite as $t \to T^-$ on a set of finite measure:

$$L^* = \operatorname{meas}\{x \in \mathbf{R}_+ \mid \overline{\lim_{t \to T}}\, u(t,x) = \infty\} < \infty \tag{36}$$

($L^*$ is the localization depth).

In this definition there is no requirement on the initial function $u_0(x)$ to have compact support; it is only necessary for it to be bounded in $\mathbf{R}_+$. Moreover, condition (5), of finite speed of propagation of perturbations, is not necessary.

To study effective localization, it is not hard to modify Theorems 2, 3, as well as the results obtained in Examples 1-9. Then analysis of unbounded coefficients $k(u)$ uses the operator comparison methods and the derivations of § 4, Ch. III. As a result, for the localization depth we obtain the estimate $L^* \le [2(\sigma+2)/\sigma]^{1/2}$, where $\sigma \in (0, 1/\alpha]$ is the parameter in the system of inequalities (11), (12).

The case of bounded coefficients is analyzed as in subsection 2. Note that the boundary blow-up regimes mentioned in Examples 6-9 lead to effective localization with depth $L^* \le 2$.

**Example 10.** Let $k(u) = 1/(1+u)$. Setting $\omega \equiv 1$ in (27), we obtain $E(p) = e^p - 1$. Hence the boundary regimes

$$u_1(t) \lesssim \exp\{\exp\{(T-t)^{-1}\}\},\ t \to T^-,$$

lead to effective localization with depth $L^* \le 2$ (this upper bound for localized boundary regimes is not optimal; see § 2, Ch. VI).

## 4 Heat localization in the Cauchy problem

The solution of the Cauchy problem for equation (1) with an initial function of compact support

$$u(0,x) = u_0(x) \ge 0,\ \ x \in \mathbf{R};\ u \in C(\mathbf{R}),\ u_0 \not\equiv 0, \tag{37}$$

is called *localized* if its support is stationary for some finite time, that is, there exists $T^* \in (0, \infty)$, such that

$$\operatorname{supp} u(t,x) = \operatorname{supp} u_0(x),\ \ 0 < t < T^* \tag{38}$$

(here, naturally, we assume that condition (5) is satisfied).

It is well known (see Remarks) that stationarity of a front point of a generalized solution of equation (1) is determined by the asymptotics of the initial function in a neighbourhood of that point. However, the magnitude of the localization time $T^*$, which is of physical importance, depends on the "global" spatial structure of the function $u_0(x)$. This dependence is reflected in the theorem stated below.

**Theorem 4.** *Let the coefficient $k$ satisfy condition* (7) *for some $\alpha > 0$, and let $E$ be a solution of the system of inequalities* (11), (12), *which corresponds to a fixed $\sigma \in (0, 1/\alpha]$. Let $u_0(x)$ satisfy the inequalities*

$$0 < u_0(x) \leq E[u_m(1 - |x|/x_m)_+^{2/\sigma}], \quad x \in \mathbf{R},$$

*where $u_m$, $x_m$ are fixed positive constants. Then the solution of the Cauchy problem* (1), (37) *is localized in the domain $\{|x| < x_m\}$ and for localization time we have the estimate $T^* \geq \sigma x_m^2/[2u_m^\sigma(\sigma + 2)]$.*

The validity of this statement is deduced from Theorem 2 using a technique applied in § 3, Ch. III to a similar analysis of equation (9). To illustrate Theorem 4, let us consider

**Example 11.** Let $k(u) = e^u - 1$. In this case a solution of the system of inequalities (11), (12) for $\sigma = 1$ is the function $E(p) = \ln(1 + p)$. Then from Theorem 4 we have that the solution generated by the initial function

$$u_0(x) = \ln\{1 + u_m(1 - |x|/x_m)_+^2\}, \quad x \in \mathbf{R},$$

is localized in the domain $\{|x| < x_m\}$ for time not less than $x_m^2/(6u_m)$.

## § 5 Conditions for absence of heat localization

### 1 Formulation of the problem

As in § 4, we shall consider in $\omega_T$ the first boundary value problem for a degenerate parabolic equation:

$$u_t = (k(u)u_x)_x; \tag{1}$$

$$u(0, x) \equiv 0, \quad x \in \mathbf{R}_+; \quad u(t, 0) = u_1(t) > 0, \quad t \in (0, T), \tag{2}$$

where the boundary function $u_1 \in C^1([0, T))$, $u_1' \geq 0$, blows up in finite time.

Let all the restrictions imposed on the function $k(u)$ in subsection 1 of § 4 hold. In particular, we assume that the condition for finite speed of propagation of perturbations is satisfied:

$$\int_0^1 \frac{k(\eta)}{\eta}\, d(\eta) < \infty. \tag{3}$$

There will be no localization in the problem (1), (2) if

$$\text{meas supp}\, u(t, x) \to \infty, \quad t \to T^- . \tag{4}$$

that is, as $t \to T^-$ heat penetrates arbitrarily far from the boundary $x = 0$.

Let us remark that (4) is equivalent to the condition

$$u(t, x) \to \infty \text{ in } \mathbf{R}_+, \ t \to T \tag{5}$$

(the truth of this statement is proved by the method used in the proof of Theorem 3 of § 4).

## 2 Sufficient conditions for the absence of heat localization

Let us denote by $u_{(\sigma)}(t, x)$ the solution of the equation with a power law nonlinearity

$$u_t = (u^\sigma u_x)_x, \ \sigma = \text{const} > 0, \tag{6}$$

which satisfies the boundary condition

$$u_{(\sigma)}(t, 0) = (T - t)^n, \ 0 < t < T; \ n = \text{const} < -1/\sigma, \tag{7}$$

$$u_{(\sigma)}(0, x) \in C(\mathbf{R}_+).$$

In § 2, 3 of Ch. III it is shown that the function $u_{(\sigma)}$ is not localized, and that there exists a constant $a_0 > 0$, such that

$$\text{meas supp } u_{(\sigma)}(t, x) \geq a_0(T - t)^{(1+n\sigma)/2} \to \infty, \ t \to T \ . \tag{8}$$

This result will be used in the comparison of solutions of equations (6) and (1). Below we shall assume that the conditions

$$k'(u) > 0, u > 0; \ k(\infty) = \infty, \tag{9}$$

are satisfied. Conditions for the absence of localization in the case of bounded coefficients $k(u)$ will be obtained by a different method in Ch. VI.

Suppose we are given an arbitrary coefficient $k \in C^2(\mathbf{R}_+) \cap C([0, \infty))$, which satisfies conditions (3), (9). Let us find what functions $E$ enable us to apply operator comparison methods to the solution $u_{(\sigma)}$ of equation (6) and the solution of the original problem (1), (2), that is, when does the inequality $u(t, x) \geq E^{-1}[u_{(\sigma)}(t, x)]$ hold everywhere in $\omega_T$.

Since the solution $u(t, x)$ is critical, from Theorem 3, § 2 we have that to that end we must find the solution $E(p) : \ \mathbf{R}_+ \mapsto \mathbf{R}_+$ of the system of ordinary differential inequalities

$$k(p) - E^\sigma(p) \geq 0, \ p > 0, \tag{10}$$

$$\left[\frac{k(p)}{E^\sigma(p)E'(p)}\right]' \geq 0, \ p > 0. \tag{11}$$

These inequalities coincide with the comparison conditions (17), (18) of Theorem 3 of § 2, if we set there $k^{(1)}(u) = u^\sigma$, $k^{(2)}(u) = k(u)$. Sufficient conditions for solvability of the system (10), (11) are given by the following

**Lemma 1.** *Assume that conditions* (9) *are satisfied and that there exists a constant* $\alpha > 0$, *such that*

$$[k^\alpha]'(0) > 0. \tag{12}$$

*In addition, let the function* $[k^\alpha]''(u)$ *have for* $u > 0$ *a finite number of zeros. Then for* $\sigma = 1/\alpha$ *there exists a solution* $E$ *of the system of inequalities* (10), (11).

*Proof.* Let us set $k(p)/[E^\sigma(p)E'(p)] \equiv 1/\omega(p)$. The inequality (11) will be satisfied if

$$\omega(p) > 0, \ \omega'(p) \leq 0, \ p > 0. \tag{13}$$

Then

$$E(p) = \left[(1+\sigma) \int_0^p k(\eta)\omega(\eta)\, d\eta\right]^{1/(1+\sigma)}.$$

Inequality (10) then takes the form

$$\int_0^p k(\eta) \left\{ [k^{1/\sigma}(\eta)]' - \omega(\eta) \right\} d\eta \geq 0, \ p > 0. \tag{14}$$

Assumption (12) for $\alpha = 1/\sigma$ enables us to construct the function $\omega(p)$ satisfying conditions (13) and the inequality $\omega(p) \leq [k^{1/\sigma}]'(p)$ for all sufficiently small $p > 0$. The second condition of the lemma means that the function $[k^{1/\sigma}]'(p)$ is monotone for all sufficiently large $p > p_* > 0$ (this condition, obviously, is not optimal). Therefore there exists $\lim_{p \to \infty} [k^{1/\sigma}]'(p) = \kappa$. If $\kappa > 0$, we can set for $p > p_*$ $\omega(p) = \inf_{p \in (0, p_*)}\{[k^{1/\sigma}]'(p)\}$. If on the other hand $\kappa = 0$, we set $\omega(p) = [k^{1/\sigma}]'(p)$ for $p > p_*$. In both cases such an extension of the function $\omega(p)$ for large $p > 0$, while preserving conditions (13), (14), allows us to achieve the equality $E(\infty) = \infty$. □

**Remark.** There exist coefficients $k$, for which condition (12) is not satisfied for any $\alpha > 0$. This is true, for example, for the function $k(u) = \exp\{-u^\nu\}$, $\nu = \text{const} < 0$.

From the method of proof of the lemma we deduce

**Corollary.** *Let conditions* (9) *hold, and let there exist a constant* $\alpha > 0$, *such that*

$$[k^\alpha(p)]'' \leq 0, \ p > 0. \tag{15}$$

*Then the function* $E = k^\alpha(p)$ *is a solution of the system of inequalities* (10), (11) *for* $\sigma = 1/\alpha$.

It is not hard to see that if (9), (15) hold, then the function $k$ defines a bijective mapping $[0,\infty) \to [0,\infty)$. Therefore the mapping $E = k^\alpha$ has the same properties.

Using Lemma 1 and the operator comparison theorem, we state

**Theorem 1.** *Let $k(\infty) = \infty$, and suppose, furthermore, that condition* (12) *is satisfied for some $\alpha > 0$. Let $E$ be a solution of the system of inequalities* (10), (11) *for $\sigma = 1/\alpha$. If for sufficiently large $t < T$ we have*

$$u_1(t) \geq E^{-1}[(T-t)^n], \quad n = \text{const} < -1/\sigma, \tag{16}$$

*then there is no heat localization in the problem* (1), (2): *$u(t,x) \to \infty$ everywhere in $\mathbf{R}_+$ as $t \to T^-$, and there exists a constant $\overline{a}_0 > 0$, such that*

$$\text{meas supp}\, u(t,x) \geq \overline{a}_0(T-t)^{(1+n\sigma)/2} \to \infty, \quad t \to T^-. \tag{17}$$

*Proof.* The proof is based on comparing in $(\tau, T) \times \mathbf{R}_+$ the solution $u(t,x)$ of the problem (1), (2) with $v_\nu \equiv \nu u_{(\sigma)}(t, x\nu^{-\sigma/2})$, a self-similar solution of equation (6). The constants $\tau \in (0,T)$, $\nu > 0$, are chosen from the condition $u(\tau, x) \geq v_\nu(\tau, x)$ in $\mathbf{R}_+$. Since $v_\nu(\tau, x) \to 0$ and $\text{supp}\, v_\nu(\tau, x) \to \{0\}$ as $\nu \to 0^+$, this can always be achieved. Then the claim of the theorem follows from the inequality $u \geq E^{-1}(v_\nu)$ in $(\tau, T) \times \mathbf{R}_+$. Furthermore, in (17) $\overline{a}_0 = a_0\nu^{\sigma/2}$, where $a_0 = a_0(n,\sigma) > 0$ is the constant in (8). □

**Example 1.** Let $k(u) = u^\sigma[1 + \mu(u)]$, $\sigma > 0$, where $\mu \in C^2(\mathbf{R}_+)$ satisfies the conditions $\mu \geq 0$, $\mu' \geq 0$. In this case a solution of the system (10), (11) is $E(p) \equiv p$, which is equivalent to applying the direct solution comparison theorem to equations (6), (1) (see Theorem 2, § 1). From Theorem 1 we then obtain that boundary regimes $u_1(t) \geq (T-t)^n$, $t \to T^-$, where $n < -1/\sigma$, do not lead to heat localization.

**Example 2.** Let us consider the coefficient $k(u) = \ln^\lambda(1+u)$, where $\lambda > 0$ is a fixed constant. Then for $\alpha = 1/\lambda$, condition (15) holds, and therefore the function

$$E(p) = k^{1/\lambda}(p) \equiv \ln(1+p), \quad E^{-1}(p) = e^p - 1,$$

is a solution of the system of inequalities (10), (11), which corresponds to $\sigma = 1/\alpha = \lambda$. Thus there is no heat localization in the problem (1), (2) if

$$u_1(t) \geq \exp\{(T-t)^n\} - 1, \quad t \to T^-, \tag{18}$$

where $n < -1/\lambda$. There exists a constant $\overline{a}_0 > 0$, such that

$$\text{meas supp}\, u(t,x) \geq \overline{a}_0(T-t)^{(1+n\lambda)/2}, \quad t \to T^-. \tag{19}$$

**Example 3.** Let $k(u) = \ln[1 + \ln(1 + u)]$. Since $k''(u) \le 0$ in $\mathbf{R}_+$, the required operator is the function

$$E(p) = k(p),\ E^{-1}(p) = \exp\{e^p - 1\} - 1,$$

which satisfies (10), (11) for $\sigma = 1$. Therefore boundary regimes

$$u_1(t) \ge \exp\{\exp\{(T - t)^n\} - 1\} - 1,\ t \to T^-, \tag{20}$$

for $n < -1$ lead to absence of heat localization:

$$\text{meas supp}\, u(t, x) \ge \bar{a}_0(T - t)^{(1+n)/2} \to \infty,\ t \to T^-. \tag{21}$$

The lower bounds of non-localized boundary blow-up regimes we have computed in Examples 2, 3 are not optimal. In § 2, Ch. VI, by constructing a.s.s. for equations under consideration, we shall establish sharp bounds for such regimes. In particular, it will be shown that in Example 2 there is no localization for any $n < -1/(1 + \lambda)$ in (18), and unlike (19)

$$\text{meas supp}\, u(t, x) \ge b_0(T - t)^{[1+n(1+\lambda)]/2} \to \infty,\ t \to T^-,$$

where $b_0 > 0$ is a constant, which depends only on $n$, $\lambda$.

In Example 3 absence of localization is caused by boundary blow-up regimes that are weaker than (20):

$$u_1(t) \ge \exp\left\{(T - t)^n |n \ln(T - t)|^{-1}\right\},\ t \to T^-, \tag{22}$$

where $n < -1$. Furthermore, estimate (21) holds for some $\bar{a}_0 = \bar{a}_0(n) > 0$. Let us stress that the limiting exponents in these non-localized boundary regimes, $n = -1/(1 + \lambda)$ in (18) and $n = -1$ in (22), are sharp and cannot be replaced by larger ones.

**Example 4.** Let $k(u) = ue^u$. In this case condition (12) holds for every $\alpha \in (0, 1]$ (note that (15) does not hold for any $\alpha > 0$). Therefore it follows from Lemma 1 that for any $\sigma = 1/\alpha \ge 1$ there exists a suitable solution of the system of inequalities (10), (11). For example, let us set $\sigma = 1$. Then conditions (13), (14) are satisfied for $\omega \equiv 1$ and the transformation $E$ has the form

$$E(p) = \left[2\int_0^p \eta e^\eta d\eta\right]^{1/2} \simeq [2pe^p]^{1/2},\ p \to \infty.$$

Hence $E^{-1}(p) \simeq 2 \ln p - \ln(4 \ln p)$ for large $p$, and therefore under the influence of boundary regimes of the form

$$u_1(t) \ge 2 \ln|(T - t)^{-1}| - \ln\left\{4 \ln|(T - t)^{-1}|\right\},\ t \to T^-,$$

there will be no heat localization.

## § 6 Some approaches to the determination of conditions for unboundedness of solutions of quasilinear parabolic equations

In this section the methods of §§ 1-3 are used to derive conditions of global insolvability of quasilinear parabolic equations of the form

$$u_t = \nabla \cdot (k(u)\nabla u) + Q(u) \equiv \Delta\phi(u) + Q(u), \tag{1}$$

where $k$, $Q$ are sufficiently smooth non-negative functions, such that $Q(u) > 0$ for $u > 0$ and $Q(u) \geq 0$.

We consider two problems for equation (1): a boundary value problem for $t > 0$, $x \in \Omega$ ($\Omega$ is a bounded domain in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$) with the conditions

$$u(0, x) = u_0(x) \geq 0, \ x \in \Omega; \ u_0 \in C(\overline{\Omega}), \tag{2}$$

$$u(t, x) = 0, \ t > 0, \ x \in \partial\Omega, \tag{3}$$

and the Cauchy problem with the initial condition

$$u(0, x) = u_0(x) \geq 0, x \in \mathbf{R}^N; \ u_0 \in C(\mathbf{R}^N). \tag{2'}$$

It is assumed that the function $Q$ satisfies the inequality

$$\int_1^\infty \frac{d\eta}{Q(\eta)} < \infty, \tag{4}$$

which, as we know (see § 2, Ch. I), is a necessary condition for existence of unbounded solutions of the problems we are considering. In the following we shall use extensively results obtained for the equation with power type nonlinearities,

$$u_t = \nabla \cdot (u^\sigma \nabla u) + u^\beta, \ \sigma > 0, \ \beta > 1, \tag{5}$$

which appear in § 3, Ch. IV.

### 1 A method based on $\psi$-criticality of the problem

Let us consider the boundary value problem (1)–(3). A solution of the problem is called *$\psi$-critical*, if everywhere in the domain we have

$$u_t(t, x) - \psi(u(t, x)) \geq 0. \tag{6}$$

Sufficient conditions of $\psi$-criticality of the problem were established in Theorem 1, § 3. Assuming that the solution is sufficiently regular (it is also assumed that

$\psi \in C^2((0, \infty)) \cap C([0, \infty))$, $\psi(0) = 0$, $u_0 \in C^2(\Omega)$), these conditions have the form

$$\mathbf{A}(u_0) \equiv \nabla \cdot (k(u_0)\nabla u_0) + Q(u_0) \geq \psi(u_0), \quad x \in \Omega; \tag{7}$$

$$[(k\psi)'/k]'(p) \geq 0, \quad p > 0, \tag{8}$$

$$[k'\psi^2 - Q^2(k\psi/Q)'](p) \geq 0, \quad p > 0. \tag{9}$$

We shall use the inequality (6) to determine conditions for unboundedness of solutions to the problem. Let the function $\psi$ be positive in $\mathbf{R}_+$ and

$$\int_1^\infty \frac{d\eta}{\psi(\eta)} < \infty. \tag{10}$$

Then, if the solution of the problem is $\psi$-critical, it follows from (6) that the function $u_m(t) = \max_{x\in\Omega} u(t, x)$ will be for all $t > 0$ not smaller than the solution $Y(t)$ of the following Cauchy problem for the ordinary differential equation,

$$\frac{dY}{dt}(t) - \psi(Y(t)) = 0, \quad t > 0, \tag{11}$$

$$Y(0) = u_m(0) = \max_{x\in\Omega} u_0(x) > 0. \tag{12}$$

By (10) the function $Y(t)$ is defined on the bounded interval $(0, t^*)$, where

$$t^* = \int_{u_m(0)}^\infty \frac{d\eta}{\psi(\eta)} < \infty. \tag{13}$$

Hence it follows that the original problem (1)–(3) has no global solutions and that there exists $T_0 \leq t^*$, such that

$$\overline{\lim}_{t\to T_0} \max_{x\in\Omega} u(t, x) = \infty. \tag{14}$$

As $u_0$ we can take any non-trivial non-negative solution of the boundary value problem for the quasilinear elliptic equation

$$\nabla \cdot (k(u_0)\nabla u_0) + Q(u_0) = \psi(u_0), \quad x \in \Omega; \quad u_0|_{\partial\Omega} = 0. \tag{15}$$

In the one-dimensional case this equation can be integrated in quadratures, which allows us to give a reasonably detailed description of the spatial structure of its solutions (see subsection 2). For $N \geq 2$ the question of solvability of the problem (15) is an interesting and sufficiently complicated problem in its own right (see § 3, Ch. IV).

Let us indicate another application of the inequality (6). If (10) holds then from (6) we can deduce the following upper bound for the unbounded solution (it is derived by integrating (6) over $(t, T_0)$, where $T_0$ is the time of existence of the unbounded solution):

$$\int_{u(t,x)}^{\infty} \frac{d\eta}{\psi(\eta)} \geq T_0 - t,\ t \in (0, T_0),\ \ x \in \Omega. \tag{16}$$

**Example 1.** Let $k(p) \equiv 1$, and let the function $Q$ be convex: $Q''(p) \geq 0$ for all $p > 0$. Then as $\psi$ we can take $\psi(p) = \nu Q(p)$, where $\nu \in (0, 1)$ is a constant. Indeed, inequalities (8), (9) are satisfied, while (10) holds in view of (4). As an initial function satisfying (7), we can take a solution of the boundary value problem

$$\Delta u_0 + (1 - \nu)Q(u_0) = 0,\ \ x \in \Omega;\ u_0|_{\partial\Omega} = 0. \tag{17}$$

This problem does not always have a non-trivial solution. For example, if $\Omega \in \mathbf{R}^N$, $N \geq 3$, is star-shaped with respect to some point (in particular, if it is convex) and $Q(u) = u^\beta$, then for $\beta \geq (N+2)/(N-2)$ there are no solutions (see § 3, Ch. IV). At the same time, in the case of annular domains $\Omega = \{0 < a < |x| < b < \infty\}$, a solution exists for all $\beta > 1$ (see the Comments section).

Thus, let the initial function $u_0$ satisfy (17). Then it follows from (13) that the solution of the problem grows without bound in finite time not longer than

$$t^* = \nu^{-1} \int_{u_m(0)}^{\infty} \frac{d\eta}{Q(\eta)} < \infty;\ u_m(0) = \max u_0.$$

Let us observe that $\psi(p) = \nu Q(p)$ satisfies the inequality (9) also for arbitrary non-increasing coefficients $k$ (when $k'(p) \leq 0$ for $p > 0$).

Let us see now what can be deduced from the estimate (16). Let the solution be $\psi$-critical with respect to the function $\psi(p) = \nu Q(p)$, $\nu \in (0, 1)$. Then

$$\int_{u(t,x)}^{\infty} \frac{d\eta}{Q(\eta)} \geq \nu(T_0 - t) \text{ in } (0, T_0) \times \Omega.$$

In particular,

1) if $Q(u) = u^\beta$, $\beta > 1$, then

$$u(t, x) \leq [\nu(\beta - 1)]^{-1/(\beta-1)}(T_0 - t)^{-1/(\beta-1)};$$

2) if $Q(u) = (1 + u)\ln^\beta(1 + u)$, $\beta > 1$ ($Q'' \geq 0$ in $\mathbf{R}_+$), then

$$u(t, x) \leq \exp\left\{[\nu(\beta - 1)]^{-1/(\beta-1)}(T_0 - t)^{-1/(\beta-1)}\right\} - 1;$$

3) if $Q(u) = e^u$, then

$$u(t, x) \leq \ln|\nu^{-1}(T_0 - t)^{-1}|,\ t \in (0, T_0).$$

All these estimates are sharp in their dependence on $(T_0 - t)$. This is demonstrated by comparing $u(t, x)$ with the spatially homogeneous solution $v$: $v'(t) = Q(v(t))$, $v(T_0^-) = \infty$. By the comparison theorem of § 4, Ch. IV, $u(t, x)$ must intersect $v(t)$ for any $t \in (0, T_0)$, and therefore $\sup_x u(t, x) > v(t)$ in $(0, T_0)$, which produces the following lower bounds for the amplitude:

1) $\sup_x u > (\beta - 1)^{-1/(\beta-1)}(T_0 - t)^{-1/(\beta-1)}$;

2) $\sup_x u > \exp\{(\beta - 1)^{-1/(\beta-1)}(T_0 - t)^{-1/(\beta-1)}\} - 1$;

3) $\sup_x u > \ln[(T_0 - t)^{-1}]$, $t \in (0, T_0)$.

Let us consider two examples of degenerate parabolic equations (the possibilities of derivation of $\psi$-criticality conditions are discussed in § 3).

**Example 2.** Let us consider equation (5) for $\beta > \sigma + 1$. Let us set $\psi(p) = \nu p^\alpha$, where $\nu > 0$, $\alpha \in (1, \beta - \sigma]$ are constants. Inequalities (8), (9), which reduce, respectively, to $(\alpha - 1)(\alpha + \sigma) \geq 0$ and $\sigma\nu - (\alpha + \sigma - \beta)p^{\beta-\alpha} \geq 0$, are satisfied. Since $\alpha > 1$, condition (10) holds as well. Choosing the initial function in the form $u_0 = [(\sigma + 1)v_0]^{1/(\sigma+1)}$, where $v_0 \not\equiv 0$ is a solution of the problem

$$\Delta v_0 + [(\sigma + 1)v_0]^{\beta/(\sigma+1)} - \nu[(\sigma + 1)v_0]^{\alpha/(\sigma+1)} = 0 \text{ in } \Omega,\ v_0 = 0 \text{ on } \partial\Omega,$$

we have that for some $T_0 \leq t^* = [\max u_0]^{1-\alpha}/[\nu(\alpha - 1)]$ (14) is satisfied.

**Example 3.** Let equation (1) have the form

$$u_t = \nabla \cdot (\ln(1 + u)\nabla u) + (1 + u)\ln^\beta(1 + u), \tag{18}$$

where $\beta > 2$ is a constant. Let us take $\psi(p) = \nu(1 + p)\ln^\alpha(1 + p)$, $\nu > 0$, $\alpha \in (1, \beta - 1)$ (for $\alpha > 1$ the integral in (10) converges). Conditions (8) and (9) assume, respectively, the form $\alpha^2 - 1 \geq 0$ and $\nu - (\alpha + 1 - \beta)\ln^{\beta-\alpha}(1 + p) \geq 0$, and since by assumption $\alpha \in (1, \beta - 1)$, hold for $p > 0$. Therefore if $u_0$ satisfies inequality (7), the solution of the problem becomes unbounded after time $T_0$ which is not larger than

$$t^* = [\ln[1 + \max u_0]]^{1-\alpha}/[\nu(\alpha - 1)] < \infty.$$

There is a close connection between the results of Examples 2, 3. Let us make the change of variables $u = e^U - 1$ in (18). Then the function $U$ satisfies the equation

$$U_t = \nabla \cdot (U\nabla U) + U^\beta + U|\nabla U|^2, \tag{19}$$

which differs from (5) for $\sigma = 1$ by the presence of the additional non-negative term $U|\nabla U|^2$ in the right-hand side. Therefore solutions of equations (19) and (5) for $\sigma = 1$ are related by $U(t, x) \geq u(t, x)$ if it holds for $t = 0$ (here we are in fact using the simplest version of the operator comparison theorem). This argument allows us to derive unboundedness conditions for solutions of one equation from similar conditions for another one (see subsection 3).

## 2 Unbounded solutions of the Cauchy problem with a critical initial function

In this subsection we show that criticality of solutions of a wide range of equations (1) frequently leads to global insolvability of the Cauchy problem.

An initial function and the solution of the problem (1), (2′) are called *critical*, if $u_t \geq 0$ in $P_T[u]$. Thus $u$ does not decrease in $t$ in $(0, T) \times \mathbf{R}^N$. As can be seen from the results of § 2, for criticality of a solution it is, in general, sufficient that the initial function satisfies the inequality

$$\nabla \cdot (k(u_0)\nabla u_0) + Q(u_0) \geq 0, \quad x \in \{x \in \mathbf{R}^N \mid u_0(x) > 0\}. \tag{20}$$

In particular, a critical function is any non-negative solution of the boundary problem

$$\nabla \cdot (k(u_0)\nabla u_0) + Q(u_0) = 0, \; x \in \Omega; \; u_0|_{\partial\Omega} = 0 \tag{21}$$

($\Omega \subset \mathbf{R}^N$ is an arbitrary bounded domain with a smooth boundary $\partial\Omega$) extended by zero into $\mathbf{R}^N \setminus \Omega$.

When $\Omega$ is a ball, all radially symmetric solutions of the problem (21) can be determined from the equation

$$\left(k(u_0)u_0'\right)' + \frac{N-1}{r}k(u_0)u_0' + Q(u_0) = 0, \quad r = |x| > 0, \tag{22}$$

and the boundary conditions

$$u_0(0) = u_m, \; u_0'(0) = 0, \tag{23}$$

where $u_m > 0$ is an arbitrary constant.

In the one-dimensional case equation (22) can be integrated and the solution of the problem (22), (23) has the form

$$u_0(x) = X_{kQ}^{-1}(x), \tag{24}$$

where $X_{kQ}^{-1}$ is the function inverse to

$$X_{kQ}(s) = \int_s^{u_m} \frac{k(\eta)\, d\eta}{\left[2\int_\eta^{u_m} k(\zeta)Q(\zeta)\, d\zeta\right]^{1/2}}, \quad s \in (0, u_m). \tag{25}$$

Hence it follows that the function (24) is defined and strictly positive for all $|x| < r_0(u_m)$, where

$$r_0(u_m) = \frac{1}{2}\text{meas supp}\, u_0 = X_{kQ}(0). \tag{26}$$

Equalities (24)–(26) give us an idea about the nature of the dependence of the spatial structure of a critical initial function $u_0$ on the magnitude of its maximum $u_m$ and the coefficients $k$, $Q$.

For arbitrary $N \geq 1$, equation (22) can be integrated, for example, if $Q(u) = \nu\phi(u)$, $\nu = \text{const} > 0$, where

$$\phi(u) = \int_0^u k(\eta)\,d\eta, \quad u > 0. \tag{27}$$

Then by a change of variable $u_0 \to \phi(u_0)$ it reduces to a linear equation, the solution of which is the function $r^{(2-N)/2}J_{(N-2)/2}(\nu^{1/2}r)$, where $J_{(N-2)/2}$ is the Bessel function.

Let us now state the main results. Let us fix an arbitrary $R > 0$. We denote by $\Omega_R$ the ball $\{|x| < R\}$ and introduce the function

$$\psi_R(x) = C_0 r^{(2-N)/2}J_{(N-2)/2}(\lambda^{1/2}r), \tag{28}$$

which is positive in $\Omega_R$; here

$$\lambda = |z_N^{(1)}/R|^2, \tag{29}$$

and $z_N^{(1)}$ is the first (smallest) positive root of the Bessel function $J_{(N-2)/2}$. The constant $C_0$ is determined by the condition $\|\psi_R\|_{L^1(\Omega_R)} = 1$. It is not hard to verify that $\psi_R(x)$ solves the problem

$$\Delta\psi_R + \lambda\psi_R = 0, \; \psi_R = 0 \text{ on } \partial\Omega_R. \tag{30}$$

We set $\psi_R = 0$ in $\mathbf{R}^N \setminus \Omega_R$.

Let us consider the function

$$H_R(t) = (u(t,x), \psi_R) \equiv \int_{\Omega_R} u(t,x)\psi_R(x)\,dx. \tag{31}$$

Taking scalar products in $L^2(\Omega_R)$ of both sides of equation (1) with $\psi_R$ and assuming for simplicity that the function $H_R(t)$ is differentiable in $t$, we obtain the equality

$$dH_R/dt = (\Delta\phi(u), \psi_R) + (Q(u), \psi_R), \quad t > 0. \tag{32}$$

**Lemma 1.** *For all admissible $t > 0$*

$$(\Delta\phi(u), \psi_R) \geq -\lambda(\phi(u), \psi_R). \tag{33}$$

*Proof.* If the solution $u > 0$ is a classical one, (33) is immediately verified. Indeed, using Green's formula, we have

$$(\Delta\phi(u), \psi_R) = (\phi(u), \Delta\psi_R) + \int_{\partial\Omega_R} \left(\psi_R \frac{\partial\phi(u)}{\partial n} - \phi(u)\frac{\partial\psi_R}{\partial n}\right) ds, \tag{34}$$

where $\partial/\partial n$ is the derivative in the outer normal direction to $\partial\Omega_R$. However, $\psi_R = 0$, $\partial\psi_R/\partial n \le 0$ on $\partial\Omega_R$ (the last statement follows from positivity of $\psi_R$ in $\Omega_R$). Therefore, taking into account (30), we obtain from (34)

$$(\Delta\phi(u), \psi_R) \ge (\phi(u),\ \Delta\psi_R) = -\lambda(\phi(u), \psi_R).$$

If, on the other hand, $u$ is a generalized solution, then the estimate (33) is proved by a standard regularization argument. □

Using (33) to estimate the right-hand side of (32), we deduce the inequality

$$dH_R/dt \ge -\lambda(\phi(u), \psi_R) + (Q(u), \psi_R),\ t > 0, \tag{35}$$

which forms the basis of the following analysis.

**Lemma 2.** *Assume that the function $Q$ satisfies condition* (4), *as well as the inequality*

$$Q''(u) \ge 0,\ \ u > 0, \tag{36}$$

*and that there exists a constant $\mu > 0$, such that*

$$\phi(u) \le \mu Q(u) \text{ in } \mathbf{R}_+. \tag{37}$$

*Then the solution of the Cauchy problem* (1), (2′) *is unbounded and exists at most for time*

$$t^* = \min_{R > z_N^{(1)}\mu^{1/2}} \left\{ \frac{R^2}{R^2 - (z_N^{(1)})^2\mu} \int_{H_R(0)}^{\infty} \frac{d\eta}{Q(\eta)} \right\} < \infty. \tag{38}$$

*Proof.* Let us choose the constant $R > z_N^{(1)}\mu^{1/2}$ such that $H_R(0) > 0$ (this can always be done if $u_0 \not\equiv 0$). By (37), from inequality (35) we have

$$\frac{dH_R}{dt} \ge \frac{R^2 - (z_N^{(1)})^2\mu}{R^2}(Q(u), \psi_R),\ t > 0. \tag{39}$$

Hence, using Jensen's inequality for convex functions [211], $(Q(u), \psi_R) \ge Q[(u, \psi_R)] = Q(H_R)$ (recall that $\|\psi_R\|_{L^1(\Omega_R)} = 1$), we derive the estimate

$$\frac{dH_R}{dt} \ge \frac{R^2 - (z_N^{(1)})^2\mu}{R^2} Q(H_R),\ t > 0.$$

From this estimate, in its turn, we obtain

$$\int_{H_R(0)}^{H_R(t)} \frac{d\eta}{Q(\eta)} \ge \frac{R^2 - (z_N^{(1)})^2\mu}{R^2} t,\ t > 0. \tag{40}$$

Hence $H_R(t)$ is unbounded on $(0, t^*]$, which, in view of $H_R(t) \le \max_{x\in\Omega} u(t, x)$, ensures that (14) holds. □

**Theorem 1.** *Let conditions* (4), (36) *hold, and assume that there exist constants* $\mu > 0$, $h > 0$, *such that for all* $u > h$ *inequality* (37) *holds. Let the initial function* $u_0$ *be critical, and that, moreover,* $u_0(x) \geqslant h$ *for all* $|x| \leqslant a$, *where the constant* $a > 0$ *is such that*

$$1 - \mu(z_N^{(1)}/a)^2 > 0. \tag{41}$$

*Then the solution of the Cauchy problem* (1), (2′) *exists at most for time*

$$t^* = \frac{a^2}{a^2 - (z_N^{(1)})^2\mu} \int_{u_0(0)}^{\infty} \frac{d\eta}{Q(\eta)} < \infty. \tag{42}$$

*Proof.* By criticality of the initial function, we have the inequality $u(t, x) \geq h$ for all $|x| \leq a$ and admissible $t > 0$. Let us take $R = a$. Then $\phi(u) \leq \mu Q(u)$ everywhere in $\Omega_a$. Therefore from (39) follows validity of the inequality

$$\frac{dH_a}{dt} \geq \frac{a^2 - (z_N^{(1)})^2 \mu}{a^2} (Q(u), \psi_a), \quad t > 0,$$

which by (41) ensures global insolvability of the problem (see the proof of Lemma 2). □

A stronger result than that of Theorem 1 will be obtained for $N = 1$. Here we shall assume that the condition

$$\int_0^1 \frac{k(\eta)}{\eta}\, d\eta < \infty \tag{43}$$

holds, which ensures that the solution of the problem has compact support in $x$ if meas supp $u_0(x) < \infty$ (see § 3, Ch. I).

Let us denote by

$$v_S(t, x) = f(\zeta), \quad \zeta = x/t^{1/2}, \tag{44}$$

the self-similar solution of the equation

$$v_t = (k(v)v_x)_x, \; t > 0, \quad x \in \mathbf{R}_+, \tag{45}$$

which satisfies

$$v_S(t, 0) = u_m, \; v_S(t, \infty) = 0, \quad t > 0 \tag{46}$$

(here $u_m > 0$ is a fixed constant).

The function $f(\zeta)$ is determined from the following boundary value problem for an ordinary differential equation:

$$(k(f)f')' + \frac{1}{2} f'\zeta = 0, \quad \zeta > 0; \quad f(0) = u_m, \; f(\infty) = 0, \tag{47}$$

where $f' = df/d\zeta$. Existence and uniqueness of the solution of problem (47) for sufficiently arbitrary coefficients $k$ have been established in [23, 24, 68]. There it is also shown that if (43) holds, the function $f(\zeta)$ has compact support: $\zeta_0 = \text{meas supp} f < \infty$ and wherever it is positive, it is strictly decreasing. This guarantees the existence of a point $\zeta = \zeta_m \in \mathbf{R}_+$, such that $f(\zeta_m) = u_m/2$.

**Theorem 2.** *Assume that the conditions* (4), (36), (43) *are satisfied and that the function $Q/\phi$ is non-decreasing, that is,*

$$Q'(u)\phi(u) - Q(u)k(u) \geq 0, \quad u > 0. \tag{48}$$

*Let the initial function $u_0$ be critical*[1], *and* $\max u_0(x) = u_m > 0$. *Then for $N = 1$ the solution of the Cauchy problem* (1), (2′) *exists at most for time*

$$t^* = 2\frac{\phi(u_m/2)}{Q(u_m/2)}\left(\frac{z_N^{(1)}}{\zeta_m}\right)^2 + 2\int_{u_m/2}^{\infty} \frac{d\eta}{Q(\eta)} < \infty. \tag{49}$$

*Proof.* For definiteness, let $\sup u_0$ be attained at $x = 0$, that is, $u_0(0) = u_m$. Then by criticality of the initial function we have the inequality $u(t, 0) \geq u_m$ for all admissible $t > 0$. Comparing equation (1) (for $N = 1$) and (45), we see that

$$u(t, x) \geq f(|x|/t^{1/2}) \tag{50}$$

everywhere in the domain of definition of the solution of the Cauchy problem we are considering. The validity of this conclusion follows from the Maximum Principle; also taken into account are condition (46) and the assumption (43); in addition, we make use of the fact that $v_S(t, |x|) \to 0$ as $t \to 0^+$ everywhere in $\mathbf{R}\setminus\{0\}$.

Let us set $h_m = u_m/2$, $\mu_m = \phi(h_m)/Q(h_m)$. Then by (48) for all $u \geq h_m$ inequality (37) holds for $\mu = \mu_m$. From (50) we have that for all $t \geq t_1^*$, where

$$t_1^* = 2\frac{\mu_m}{\zeta_m^2}(z_N^{(1)})^2 = 2\frac{\phi(u_m/2)}{Q(u_m/2)\zeta_m^2}(z_N^{(1)})^2, \tag{51}$$

on the interval $|x| \leq a_m = \zeta_m(t_1^*)^{1/2}$ we have the inequality $u(t, x) \geq h_m = u_m/2$. This choice of the constants $h_m$, $\mu_m$, $a_m$ ensures that (41) holds (here $\mu_m(z_N^{(1)}/a_m)^2 = 1/2 < 1$). By Theorem 1 this guarantees unbounded growth of the function $H_{a_m}(t_1^* + t)$ in time which does not exceed

$$t_2^* = \frac{a_m^2}{a_m^2 - \mu_m(z_N^{(1)})^2}\int_{H_{a_m}(t_1^*)}^{\infty} \frac{d\eta}{Q(\eta)} \leq 2\int_{u_m/2}^{\infty} \frac{d\eta}{Q(\eta)} < \infty \tag{52}$$

[1] Without loss of generality we can take $u_0$ to be defined by (24). It is not hard to show that (24) is the minimal critical initial function having a maximum $u_m > 0$.

(in the derivation of the last inequality we used the obvious estimate $H_{u_m}(t_1^*) > u_m/2$). Adding (51) and (52) together, we arrive at the required result. □

The conditions of Theorem 2 are satisfied, for example, by coefficients of equation (5) for $\beta \geq \sigma + 1$ (this restriction is connected with (48)). It is known, however (see § 3, Ch. IV), that for $1 < \beta < \sigma + 3$ all non-negative non-trivial solutions of the Cauchy problem (5), (2′) are unbounded in the case $N = 1$. Hence we have

**Theorem 3.** *Let the initial function $u_0 \not\equiv 0$ of the problem* (5), (2′) *for $\beta > 1$, $N = 1$, be critical. Then the problem has no global solutions.*

This theorem can be extended to the multi-dimensional problem (5), (2′) under the condition $1 < \beta < (\sigma+1)(N+2)/(N-2)_+$. Let us briefly explain the method of proof.

If $u_0(x)$ is a critical initial function with compact support, then assuming that $u(t,x)$ is a global solution leads to the following conclusion: $u(t,x) \to \infty$ in $\mathbf{R}^N$ as $t \to \infty$. If that is not the case, then two possibilities arise: either $u(t,x)$ is bounded uniformly in $t$ in $\mathbf{R}^N$, or $u(t,x)$ stabilizes from below as $t \to \infty$ to a singular stationary solution $\bar{u}_*(x)$, defined, for example, in $\mathbf{R}^N \setminus \{0\}$, and $\bar{u}_*(0) = \infty$ (such solutions exist for $\beta > (\sigma+1)N/(N-2)_+$; see [201, 227]). By monotonicity of $u(t,x)$ in $t$ (which implies existence of a Liapunov function $-\int_K u(t,x)\,dx$ with an arbitrary compact subset $K \subset \mathbf{R}^N$, and hence the estimate $\int_K |u_t(t)| \in L^1((1,\infty))$, which would be sufficient to pass to the limit as $t = t_k \to \infty$), the first assumption leads by standard arguments to the conclusion that $u(t,x)$ stabilizes from below to a stationary solution $u_*(x) > 0$ in $\mathbf{R}^N$, which does not exit for $\beta < (\sigma+1)(N+2)/(N-2)_+$. (For the case $u_* = u_*(|x|)$ this has been established in § 3, Ch. IV; non-existence of asymmetric in $|x|$ solutions of the equation $\Delta u_*^{\sigma+1} + u_*^{\beta} = 0$ in $\mathbf{R}^N$ has been proved in [201].)

The second assumption reduces to the first one. It means that $u \to \infty$ as $t \to \infty$ only at the point $x = 0$. However, $u(t,x) < \bar{u}_*(x)$ in $\mathbf{R}_+ \times \mathbf{R}^N$, since $u_t > 0$ in $P_\infty|u|$ by the Maximum Principle [101] ($z = u_t > 0$ in a neighbourhood of any point where the equation for $z$ is uniformly parabolic). Therefore there exists $x_0 \in \mathbf{R}^N$, such that $u(t,x) \leq \bar{u}_*(x+x_0)$, and therefore $u(t,0) \leq \bar{u}_*(x_0) < \infty$ for all $t > 0$, which means that the solution $u$ is uniformly bounded in $\mathbf{R}_+ \times \mathbf{R}^N$.

Thus, if $u$ is a global critical solution with compact support, then $u \to \infty$ in $\mathbf{R}^N$ as $t \to \infty$. Therefore sooner or later $u(t,x)$ will satisfy the conditions of Theorem 1 (the case $\beta \geqslant \sigma + 1$) and eventually will become unbounded. For $1 < \beta < \sigma + 1$ every solution $u \not\equiv 0$ of the Cauchy problem for (5) is unbounded (see § 3, Ch. IV).

## 3 Application of generalized comparison theorems

In this subsection unbounded solutions of equation (1) are studied by comparing them with various solutions of equation (5), properties of which were studied in detail in § 3, Ch. IV.

For convenience, let us introduce the function

$$U_{\sigma\beta}(t, x; T) = (T-t)^{-1/(\beta-1)} A[1 - |x|^2 (T-t)^{-[\beta-(\sigma+1)]/(\beta-1)}/a^2]_+^{1/\sigma},$$

$$0 \le t < T, \ x \in \mathbf{R}^N; \ \sigma > 0, \ \beta > 1, \tag{53}$$

where $T, a, A$ are some positive constants, the latter two of which satisfy the relations

$$\frac{4}{\sigma} \frac{A^\sigma}{a^2} > \frac{\sigma + 1 - \beta}{\beta - 1},$$

$$A^{\beta-1} \ge \left( \frac{1}{\beta - 1} + \frac{2N}{\sigma} \frac{A^\sigma}{a^2} \right) \left\{ \frac{1 + 2(N + 2/\sigma) A^\sigma / a^2}{[\beta - (\sigma + 1)]/(\beta - 1) + 4A^\sigma/(\sigma a^2)} \right\}^{(\beta+\sigma-1)/\sigma}. \tag{54}$$

In § 3, Ch. IV it is shown (Theorem 1) that if the inequalities (54) hold, the function (53) is an unbounded subsolution of equation (5). We shall use the fact that for $1 < \beta < \sigma + 1 + 2/N$ all the non-trivial solutions of the Cauchy problem (5), (2′) are unbounded (Theorem 2, § 3, Ch. IV).

3.1. First we use the direct solutions comparison theorem (Theorem 2, § 1).

**Theorem 3.** *Let the coefficients $k, Q$ in (1) satisfy for all $u > 0$ the inequalities*

$$k(u) \ge u^\sigma, \ (k(u)/u^\sigma)' \ge 0, \ Q(u) \ge u^{\beta-\sigma} k(u), \tag{55}$$

*where $\sigma > 0$, $\beta \in (1, \sigma + 1 + 2/N)$ are fixed constants. Assume that the initial function $u_0(x) \not\equiv 0$ in (2′) is critical. Then the Cauchy problem (1), (2′) has no global solutions.*

**Example 4.** Let us consider the equation

$$u_t = \nabla \cdot (u e^{u^2} \nabla u) + u^\beta e^{u^2}, \ \beta > 1. \tag{56}$$

It is not hard to see that conditions (55) are satisfied with $\sigma = 1$. Then it follows from Theorem 4 that in the case of $1 < \beta < 2(N+1)/N$ any critical initial function $u_0 \not\equiv 0$ generates an unbounded solution of the problem.

Let us consider in more detail the problem (56), (2′) for $N = 1$. Here

$$\phi(u) \equiv \int_0^u k(\eta)\,d\eta = \frac{1}{2}(e^{u^2} - 1),\ u > 0.$$

It is easy to check that

$$\operatorname{sgn}\{[Q/\phi]'(u)\} = \operatorname{sgn}\left[\beta u^{\beta-1}(e^{u^2} - 1) - 2u^{\beta+1}\right] \geq \operatorname{sgn}[(\beta-2)u^{\beta+1}] = \operatorname{sgn}(\beta-2)$$

Hence it follows that for $\beta \geq 2$ inequality (48) holds, and then, using Theorem 2, we conclude that for $N = 1$ any solution of the problem (56), (2′) corresponding to a critical initial function (defined, for example, by (24)), is unbounded.

3.2. Below we apply to the Cauchy problem (1), (2′) the operator comparison method in the form presented in the corollary to Theorem 4, § 2. The following claim is proved using the results of § 3, Ch. IV.

**Theorem 4.** *Let there exist a monotone increasing function* $E : \bar{\mathbf{R}}_+ \to \bar{\mathbf{R}}_+$, *such that*

$$E''(u) \leq 0,\ u > 0; \tag{57}$$

$$k(u) = [E(u)]^\sigma,\ \ Q(u) \geq [E(u)]^\beta / E'(u),\ \ u > 0;\ \sigma > 0,\ \beta > 1. \tag{58}$$

*Then:* 1) *if* $1 < \beta < \sigma + 1 + 2/N$ *and* $u_0 \not\equiv 0$, *the Cauchy problem* (1), (2′) *has no global solutions*[2];

2) *if*

$$u_0(x) \geq E^{-1}\{U_{\sigma\beta}(0, x; T)\},\ \ x \in \mathbf{R}^N, \tag{59}$$

*for some* $T < \infty$ ($E^{-1}$ *is the function inverse to* $E$), *then the solution of the problem exists for time* $T_0 \leq T$, *and for all* $0 < t < T_0$ *we have the estimate*

$$u(t, x) \geq E^{-1}\{U_{\sigma\beta}(t, x; T)\},\ \ x \in \mathbf{R}^N. \tag{60}$$

Let us consider some examples.

**Example 5.** Let $E(u) = \ln(1 + u)$. Condition (57) is satisfied, and examples of functions $k, Q$ which fulfill the requirements (58) are the coefficients of the equation

$$u_t = \nabla \cdot (\ln^\sigma(1 + u)\nabla u) + (1 + u)\ln^\beta(1 + u). \tag{61}$$

Therefore for $1 < \beta < \sigma + 1 + 2/N$ all non-trivial solutions of this equation are unbounded. In the case $\beta \geq \sigma + 1 + 2/N$ a solution will be unbounded if the inequality (59) holds; here it has the form

$$u_0(x) \geq \exp[U_{\sigma\beta}(0, x; T)] - 1,\ \ x \in \mathbf{R}^N.$$

[2] In this case the interval of time of existence of the problem satisfies the upper bound obtained in Theorem 2 of § 3, Ch. IV.

In § 7, Ch. IV it is shown that for $\sigma = 0$, $\beta > 1 + 2/N$ for sufficiently small initial functions $u_0$ equation (61) has global solutions.

More detailed information concerning unbounded solutions of equations of the type of (61) is obtained in § 7, Ch. IV by a different method.

**Example 6.** Now let $E(u) = u/\ln(e^2+u)$. It is not hard to check that the function $E$ is monotone increasing and concave in $\mathbf{R}_+$. Then conditions (58) are satisfied by the functions

$$k(u) = u^\sigma/\ln^\sigma(e^2+u),\ Q(u) \geq 2u^\beta \ln^{1-\beta}(e^2+u).$$

Therefore the Cauchy problem for equation (1) with such coefficients does not have global solutions in the case $1 < \beta < \sigma + 1 + 2/N$, $u_0 \not\equiv 0$.

As other examples of functions $E : \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$, which satisfy (57), we could take, say, $E(u) = u^{1/2}\ln(e^4+u)$, $E(u) = \exp\{\ln^{1/2}(1+u)\} - 1$ and so on.

## § 7 Criticality conditions and a comparison theorem for finite difference solutions of nonlinear heat equations

In this section we show that the assertions concerning criticality and the comparison theorem can be extended without requiring major modifications to cover the case of finite difference solutions of the same parabolic equations, that is, the solutions of implicit difference schemes constructed using an approximation of the parabolic operator in divergence (conservative) form. This indicates that quite subtle properties of the heat transfer process are shared by its correctly constructed finite difference approximation. This fact is of supreme importance for us, since at all stages of the investigation of nonlinear processes of heat conduction and combustion we use numerical methods extensively. The theory of comparison of solutions of different parabolic equations is one of the main tools of our study. Therefore it is important that this theory can be used almost as freely at the level of finite difference solutions.

Below we present our analysis using as an example the nonlinear heat equation

$$u_t = (\phi(u))_{xx}, \tag{1}$$

for which we shall consider a boundary value problem in $(0, T) \times (0, l)$, where $T$, $l$ are fixed positive constants, with the conditions

$$u(0, x) = u_0(x) \geq 0,\ \ x \in [0, l]; \tag{2}$$

$$u(t, 0) = u_1(t) \geq 0,\ u(t, l) = u_2(t) \geq 0,\ \ 0 \leq t \leq T. \tag{3}$$

The non-negative function $\phi \in C^2(\mathbf{R}_+) \cap C([0, \infty))$ is strictly increasing: $\phi'(u) > 0$ for $u > 0$, $\phi(0) = 0$. The functions $u_0$, $u_1$, $u_2$ are taken to be sufficiently smooth, $u_0(0) = u_1(0)$, $u_0(l) = u_2(0)$.

Let us introduce a spatial grid

$$\omega_h = \{x_k = kh, h > 0;\ k = 1, 2, \ldots, M-1;\ hM = l\}$$

and a (non-uniform) grid in time $\omega_\tau$, generated by a system of time intervals

$$\{\tau_j, j = 0, 1, \ldots, N;\ \sum \tau_j = T\}.$$

Let us denote by

$$u_{0h} = u_0(kh),\ 0 < k < M,$$

$$u_{i\tau} = u_i\left(\sum_{j=0}^{n} \tau_j\right),\ 0 \le n \le N,\ i = 1, 2,$$

the corresponding grid functions, which coincide with $u_0(x)$ and $u_i(t)$ at the nodes of the grids $\omega_h$ and $\omega_\tau$, respectively.

Corresponding to the problem (1)–(3), let us set up the implicit (nonlinear) difference scheme:

$$(v_k)_t \equiv \frac{\hat{v}_k - v_k}{\tau_j} = (\phi(\hat{v}_k))_{\bar{x}x},\ (t, x) \in \omega_{\tau h} = \omega_\tau \times \omega_h; \tag{4}$$

$$v_k^0 = u_{0h} \ge 0,\ x \in \omega_h;\ v_0^0 = u_0(0),\ v_M^0 = u_0(l); \tag{5}$$

$$\hat{v}_0 = u_{1\tau} \ge 0,\ \hat{v}_M = u_{2\tau} \ge 0,\ t \in \omega_\tau. \tag{6}$$

Here we have introduced notation which is standard in the theory of difference schemes [346]: $\hat{v} = v_k^{j+1}$, $v_k = v_k^j$ is the unknown function, $(v_k)_{\bar{x}} = (v_k - v_{k-1})/h$, $(v_k)_x = (v_{k+1} - v_k)/h$ are first order difference operators, so that

$$\rho_h(\hat{v}_{k-1}, \hat{v}_k, \hat{v}_{k+1}) \equiv (\phi(\hat{v}_k))_{\bar{x}x} = \frac{1}{h^2}[\phi(\hat{v}_{k-1}) - 2\phi(\hat{v}_k) + \phi(\hat{v}_{k+1})]. \tag{7}$$

The problem (4)–(6) is a system of nonlinear algebraic equations. Questions of existence and uniqueness of the solution of the resulting finite difference problem are considered in § 5, Ch. VII. Iterative methods of solution of similar nonlinear finite difference problems are considered in detail in [346].

We shall assume that the solution of the difference problem (4)–(6) is defined everywhere in $\omega_{\tau h}$, and that at each $j$-th step in time the system of equations is solvable for all $0 < \tau \le \tau_j$. We shall also assume that in some class of grid functions $v_k^{j+1}$ close to $v_k^j$ the mapping $\tau \to v_k^{j+1}$ is a bijection and continuous in $C(\omega_h)$ and that $v_k^{j+1} \to v_k^j$ as $\tau \to 0^+$ in $\omega_h$. The last requirement is natural; the two preceding conditions are not restrictive and are satisfied in a more general setting (see § 5, Ch. VII).

## 1 A Maximum Principle

Let us denote by $\partial\omega_{\tau h}$ the parabolic boundary of $\omega_{\tau h}$, that is, $\partial\omega_{\tau h} = \{t = 0, x \in \bar{\omega}_h\} \cup \{t \in \omega_\tau, x = 0\} \cup \{t \in \omega_\tau, x = l\}$, $\overline{\omega}_h = \{x = kh, k = 0, \ldots, M\}$. In the following lemma (it is repeatedly used below) we obtain restrictions on the finite difference approximation of the parabolic operator, under which the solution of the problem cannot take negative values, that is, it satisfies a weak Maximum Principle.

**Lemma 1.** *Let the grid function $z_k^j$ be the solution of the problem*

$$(z_k)_t \equiv \frac{\hat{z}_k - z_k}{\tau_j} = \theta_h(\hat{z}_{k-1}, \hat{z}_k, \hat{z}_{k+1}) \text{ in } \omega_{\tau h}, \tag{8}$$

*where the function $\theta_h(a, b, c)$, which is continuous on* $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$, *is such that*

$$\theta_h(a, b, c) \geq \theta_h(b, b, b) \geq 0, \; a \geq b, \; c \geq b. \tag{9}$$

*Let $z_k^j \geq 0$ on $\partial\omega_{\tau h}$. Then $z_k^j \geq 0$ in $\omega_{\tau h}$.*

*Proof.* Let $J = \max\{j \mid z_k^i \geq 0, x \in \omega_h;\ 0 \leq i \leq j\} < N$. Then there exists $x \in \omega_h$ (a point of negative minimum in $x$ in $\omega_h$), such that $z_k^{J+1} < 0$, $z_{k-1}^{J+1} \geq z_k^{J+1}$, $z_{k+1}^{J+1} \geq z_k^{J+1}$. Therefore, using (8), we obtain from (9)

$$\frac{z_k^{J+1} - z_k^J}{\tau_J} = \theta_h\left(z_{k-1}^{J+1}, z_k^{J+1}, z_{k+1}^{J+1}\right) \geq \theta_h\left(z_k^{J+1}, z_k^{J+1}, z_k^{J+1}\right) \geq 0. \tag{10}$$

Therefore $z_k^{J+1} \geq z_k^J$, which contradicts the choice of the number $J$. □

**Corollary.** *The solution of the problem* (4)–(6) *satisfies the weak Maximum Principle: $v_k^j \geq 0$ in $\omega_{\tau h}$. If $v_k^0 > 0$ in $\omega_h$, then $v_k^j > 0$ in $\omega_{\tau h}$.*

*Proof.* Let us consider the system of equations

$$(z_k)_t \equiv \frac{\hat{z}_k - z_k}{\tau_j} = \theta_h(\hat{z}_{k-1}, \hat{z}_k, \hat{z}_{k+1}) \equiv (\psi(\hat{z}_k))_{\bar{x}x}, (t, x) \in \omega_{\tau h}. \tag{11}$$

with the boundary conditions (5), (6), where $\psi(v) = \phi(|v|)\operatorname{sgn} v$. It is not hard to check that for this problem all the conditions of Lemma 1 are satisfied, so that $z_k^j \geq 0$ in $\omega_{\tau h}$. However, for $\hat{z}_k \geq 0$ equation (11) is the same as (4), so that by uniqueness of solutions (see § 5, Ch. VII) $v_k^j \equiv z_k^j \geq 0$ in $\omega_{\tau h}$. The second assertion follows immediately from the analysis of equation (10) with right-hand side (11). □

## 2 Sufficient conditions for criticality of the finite difference solution

**Definition.** A solution of the problem (4)–(6) will be called *critical* if for all $t \in \omega_\tau$

$$\hat{v}_k - v_k \geq 0, \quad x \in \omega_h. \tag{12}$$

In the case of the problem (4)–(6), the theorem concerning criticality of the solution is to all intents and purposes the same as its differential analogue of § 1.

**Theorem 1.** *Criticality of solution of the problem* (4)–(6) *is ensured by the inequalities*

$$(\phi(u_{0h}))_{\bar{x}x} > 0, \quad x \in \omega_h; \tag{13}$$

$$u_1'(t) \geq 0, \; u_2'(t) \geq 0, \quad t \in [0, T]. \tag{14}$$

Of course, (14) can be replaced by a condition of non-decreasing of the functions $u_i(t)$ $(i = 1, 2)$.

*Proof.* Let us set $\hat{z}_k = (\hat{v}_k - v_k)/\tau_j$. Then $\hat{z}_k$ is a solution of the problem

$$\frac{\hat{z}_k - z_k}{\tau_j} = \theta_h(\hat{z}_{k-1}, \hat{z}_k, \hat{z}_{k+1}) \equiv$$

$$\equiv \frac{1}{\tau_j}[\rho_h(v_{k-1} + \tau_j \hat{z}_{k-1}, v_k + \tau_j \hat{z}_k, v_{k+1} + \tau_j \hat{z}_{k+1}) - \tag{15}$$

$$- \rho_h(v_{k-1}, v_k, v_{k+1})], \quad (t, x) \in \omega_{\tau h};$$

$$z_k^0 = \rho_h(v_{k-1}^0, v_k^0, v_{k+1}^0), \quad x \in \omega_h; \quad z_0^0 = z_M^0 = 0; \tag{16}$$

$$\hat{z}_0 = \frac{\hat{u}_{1\tau} - u_{1\tau}}{\tau_j}, \; \hat{z}_M = \frac{\hat{u}_{2\tau} - u_{2\tau}}{\tau_j}, \quad t \in \omega_\tau. \tag{17}$$

To define the initial function in (16), we have introduced an additional fictitious time level with index $j = -1$; $\tau_{-1} > 0$,

$$v_k^{-1} = v_k^0 - \tau_{-1}\rho_h(v_{k-1}^0, v_k^0, v_{k+1}^0)$$

in $\omega_h$.

Let $J = \max\{j \mid z_k^i > 0, x \in \omega_h; \; 0 \leq i \leq j\} < N$. Since $z_k^J > 0$ in $\omega_h$, we deduce from (4) that $(\phi(v_k^J))_{\bar{x}x} \equiv z_k^J > 0$ in $\omega_h$. Let us consider (15) for $j = J$. By assumption, $\hat{z}_k \tau_j \equiv \hat{v}_k - v_k \to 0$ as $\tau_j \to 0^+$ in $C(\omega_h)$. Therefore there exists $\tau_j > 0$, such that $\tau_j \hat{z}_k = 0$ for some $x \in \omega_h$ and $\tau_j \hat{z}_{k\pm1} \geq 0$ ($\tau_j > 0$ by the condition $(\phi(v_k))_{\bar{x}x} > 0$ in $\omega_h$). Then from equations (15), (7) we arrive at a contradiction, since its right-hand side $\theta_h(\hat{z}_{k-1}, 0, \hat{z}_{k+1})$ is non-negative by the conditions $\phi'(u) > 0$ for $u > 0$, $\hat{z}_{k\pm1} \geq 0$. □

## 3 A comparison theorem

Let us consider in $\omega_{\tau h}$ the two finite difference problems corresponding to boundary value problems for two different ($\nu = 1, 2$) nonlinear parabolic equations with operators of the form (1):

$$(v_k^{\nu})_t = \mathcal{L}_h^{(\nu)} \hat{v}_k^{(\nu)} \equiv (\phi^{(\nu)}(\hat{v}_k^{(\nu)}))_{\bar{x}x}; \tag{18}$$

$$v_k^{(\nu)0} = u_{0h}^{(\nu)} > 0, \ x \in \omega_h; \tag{19}$$

$$\hat{v}_0^{(\nu)} = \hat{u}_{1\tau}^{(\nu)} \geq 0, \ \hat{v}_M^{(\nu)} = \hat{u}_{2\tau}^{(\nu)} \geq 0, \ t \in \omega_\tau. \tag{20}$$

The following assertion is quite similar to the comparison theorem of § 1. The methods of proof are, however, significantly different.

**Theorem 2.** *Let*

$$v_k^{(2)0} > v_k^{(1)0}, \ x \in \omega_h; \ u_i^{(2)}(t) \geq u_i^{(1)}(t), \ 0 \leq t \leq T, \ i = 1, 2, \tag{21}$$

*and let the boundary conditions of the problem for $\nu = 2$ be critical:*

$$\mathcal{L}_h^{(2)} v_k^{(2)0} \equiv (\phi^{(2)}(v_k^{(2)0}))_{\bar{x}x} > 0, \ x \in \omega_h;$$

$$u_i^{(2)}(t)' \geq 0, \ 0 \leq t \leq T, \ i = 1, 2.$$

*Assume that for all $p > 0$*

$$\phi^{(2)\prime}(p) - \phi^{(1)\prime}(p) \geq 0, \tag{22}$$

$$\left[\phi^{(2)\prime}(p)/\phi^{(1)\prime}(p)\right]' \geq 0. \tag{23}$$

*Then $\hat{v}_k^{(2)} > \hat{v}_k^{(1)}$ in $\omega_{\tau h}$.*

*Proof.* Setting $\hat{z} = \hat{v}_k^{(2)} - \hat{v}_k^{(1)}$, we obtain for this grid function the problem

$$\frac{\hat{z}_k - z_k}{\tau_j} = \theta_h(\hat{z}_{k-1}, \hat{z}_k, \hat{z}_{k+1}) \equiv \mathcal{L}_h^{(2)} \hat{v}_k^{(2)} - \mathcal{L}_h^{(1)} \hat{v}_k^{(1)},$$

and by (21) $\hat{z}_k \geq 0$ on $\partial\omega_{\tau h}$; $z_h^0 > 0$ in $\omega_h$. Using the method of proof of Theorem 1, we have that $\hat{z}_k > 0$ in $\omega_h$ if $\theta_h(\hat{z}_{k-1}, 0, \hat{z}_{k+1}) \geq 0$ where $\hat{z}_{k+1} \geq 0$ (here we have introduced the notation $\hat{z}_k = z_k^{J+1}$, $J = \max\{j \mid z_k^i > 0 \text{ in } \omega_h;\ 0 \leq i \leq j\}$.

Since

$$\theta_h(\hat{z}_{k-1}, 0, \hat{z}_{k+1}) \geq \theta_h(0, 0, 0) \equiv \mathcal{L}_h^{(2)} \hat{v}_k^{(2)} - \mathcal{L}_h^{(1)} \hat{v}_k^{(2)},$$

the theorem will be proved if

$$\begin{aligned} \mathcal{L}_h^{(2)} \hat{v}_k^{(2)} - \mathcal{L}_h^{(1)} \hat{v}_k^{(2)} \equiv \frac{1}{h^2}\{&\phi^{(2)}(\hat{v}_{k-1}^{(2)}) - \phi^{(1)}(\hat{v}_{k-1}^{(2)}) + \\ &+ \phi^{(2)}(\hat{v}_{k+1}^{(2)}) - \phi^{(1)}(\hat{v}_{k+1}^{(2)}) - 2[\phi^{(2)}(\hat{v}_k^{(2)}) - \phi^{(1)}(\hat{v}_k^{(2)})]\} \geq 0. \end{aligned} \tag{24}$$

From the criticality of the solution $\hat{v}_k^{(2)}$ it follows that in $\omega_{\tau h}$ we have the inequality

$$\mathcal{L}_h^{(2)}\hat{v}_k^{(2)} = \frac{1}{h^2}[\phi^{(2)}(\hat{v}_{k-1}^{(2)}) + \phi^{(2)}(\hat{v}_{k+1}^{(2)}) - 2\phi^{(2)}(\hat{v}_k^{(2)})] \geq 0.$$

Therefore

$$\hat{v}_k^{(2)} \leq (\phi^{(2)})^{-1}\left(\frac{\phi^{(2)}(\hat{v}_{k-1}^{(2)}) + \phi^{(2)}(\hat{v}_{k+1}^{(2)})}{2}\right), \tag{25}$$

where $(\phi^{(2)})^{-1}$ is the function inverse to $\phi^{(2)}$.

Since $\phi^{(2)} - \phi^{(1)}$ does not decrease in $\mathbf{R}_+$ (see (22)), using (25) we obtain

$$\begin{aligned}\phi^{(2)}(\hat{v}_k^{(2)}) - \phi^{(1)}(\hat{v}_k^{(2)}) \leq \frac{1}{2}\left[\phi^{(2)}(\hat{v}_{k-1}^{(2)}) + \phi^{(2)}(\hat{v}_{k+1}^{(2)})\right] - \\ - \phi^{(1)}\left((\phi^{(2)})^{-1}\left(\frac{1}{2}\left(\phi^{(2)}(v_{k-1}^{(2)}) + \phi^{(2)}(\hat{v}_{k+1}^{(2)})\right)\right)\right).\end{aligned}$$

Substituting this estimate into (24), we obtain

$$\begin{aligned}\mathcal{L}_h^{(2)}\hat{v}_k^{(2)} - \mathcal{L}_h^{(1)}\hat{v}_k^{(2)} \geq \frac{2}{h^2}\left\{\phi^{(1)}\left((\phi^{(2)})^{-1}\left(\frac{\phi^{(2)}(\hat{v}_{k-1}^{(2)}) + \phi^{(2)}(\hat{v}_{k+1}^{(2)})}{2}\right)\right) - \right. \\ \left. - \frac{1}{2}[\phi^{(1)}(\hat{v}_{k+1}^{(2)}) + \phi^{(1)}(\hat{v}_{k-1}^{(2)})]\right\}.\end{aligned} \tag{26}$$

Let us introduce the notation $\phi^{(2)}(\hat{v}_{k+1}^{(2)}) = w_{k+1}$. Then the last inequality can be written in the form

$$\begin{aligned}\mathcal{L}_h^{(2)}\hat{v}_k^{(2)} - \mathcal{L}_h^{(1)}\hat{v}_k^{(2)} \geq \frac{2}{h^2}\left\{\phi^{(1)}\left((\phi^{(2)})^{-1}\left(\frac{w_{k-1} + w_{k+1}}{2}\right)\right) - \right. \\ \left. - \frac{1}{2}\left[\phi^{(1)}((\phi^{(2)})^{-1}(w_{k-1})) + \phi^{(1)}((\phi^{(2)})^{-1}(w_{k+1}))\right]\right\}.\end{aligned} \tag{27}$$

However, inequality (23) ensures that the function $\phi^{(1)}((\phi^{(2)})^{-1}(p))$ is concave, since

$$\left[\phi^{(1)}((\phi^{(2)})^{-1}(p))\right]'' = \left[\phi^{(2)'}(\eta)\right]^{-1}\left[\phi^{(1)'}(\eta)/\phi^{(2)'}(\eta)\right]' \leq 0,$$

where $\eta = (\phi^{(2)})^{-1}(p)$. Since $f((\eta_1 + \eta_2)/2) \geq [f(\eta_1) + f(\eta_2)]/2$ for any concave function $f(\eta)$ and any $\eta_1, \eta_2 > 0$, we have that the right-hand side of (27) is non-negative, which completes the proof. □

## Remarks and comments on the literature

Results of § 1, as well as of subsection 1 of § 2, are presented in detail in [147, 148, 151]. Comparison theorems proved here, which are based on special pointwise estimates of the highest order spatial derivative of the majorizing solution, can be considered as generalizations of well-known assertions concerning the relations among subsolutions and supersolutions of equations or systems of equations of parabolic type (see, for example, [101, 338, 361, 378]). Earlier criticality theorems for solutions of semilinear parabolic equations were used in [356, 357, 378]. Criticality conditions for a generalized solution of a scalar quasilinear heat equation were obtained in [295]. Particular comparison theorems for solutions of specific quasilinear parabolic equations, proof of which uses the sign of the second spatial derivative of one of the solutions, can be found in [252]. Slightly after [147, 148, 151], the same method was used in [45] to establish the comparison theorem for operators (33) with $b^{(i)} \equiv 0$ (see § 1); the comparison condition then has the form of the first inequality (34).

The operator comparison theorem, particular cases of which are Theorems 3, 4 of § 2, stated for sufficiently arbitrary nonlinear parabolic equations, which uses estimates following from $\psi$-criticality of the majorizing solution, was proved in [117, 118] (in [118] the results are presented using an example of quasilinear equations with a source).

Sufficient conditions of $\psi$-criticality of solutions (Theorem 1, § 3) of problems in one space variable were obtained in [117, 118]. In [117] in the case of the Cauchy problem the dependence $\psi = \psi(u, u_x)$ was analyzed (the setting of the Cauchy problem allows us to determine the sign of the function $z(0, x) = u_t(0, x) - \psi(u_0(x), u_0'(x))$). Later concepts equivalent to $\psi$-criticality were introduced in [364, 365]; these papers contain applications to unbounded solutions See other applications to explicit solutions in [139].

Most of the results of § 4, 5 are contained in [146, 154]. A different approach to the analysis of the localization phenomenon is used in Ch. VI. Theorem 4 of § 4, concerning heat localization in the Cauchy problem, was proved in [146]. Conditions for immobility for a finite length of time of the front point of generalized solutions of equations with $k(u)$ not of power type, were studied in [252]. For $k(u) = u^\sigma$, $\sigma > 0$ such studies are to be found in [58, 232, 248]. For the multidimensional equation some results in the same direction are contained in [59]. There (see also [328]) the authors also study the properties of the degeneracy surface, which corresponds to an arbitrary generalized solution of the Cauchy problem for the equation $u_t = \Delta u^{\sigma+1}$. Most of these results are analyzed from a general standpoint in the monograph [103].

The method of analyzing unbounded solutions of parabolic equations with a source (subsection 1, § 6), based on $\psi$-criticality conditions of the problem, was presented in [120, 125]. To establish upper bounds a similar approach was used independently by [108] and in a large number of consequent papers. In proving assertions of § 2, we use a method (called in the terminology of [289] the method of eigenfunctions), which was also applied for a similar analysis of boundary value problems in bounded domains for semilinear ($k(u) \equiv 1$) [243, 289], quasilinear [120, 121, 125, 225] parabolic equations and systems thereof [161].

Proof of Theorems 1–3 in § 6, which uses a new idea, viz., criticality of the initial function, is given in [120, 127]. Part of the results of subsection 3 of § 6 is to be found [150]. For other methods applicable to the study of unbounded solutions see Ch. IV, VII. For short surveys concerning unbounded solutions of evolution problems see [157, 289, 334].

The results of § 7 are a particular case of the statements proved in [126], where criticality of finite difference solutions has been established for parabolic equations of general form $u_t = L(u, u_x, u_{xx})$. [126] also contains comparison theorems for solutions of implicit difference schemes for two different equations of the form $u_t^{(\nu)} = L(u^{(\nu)}, u_{xx}^{(\nu)})$.

# *Chapter VI*

# Approximate self-similar solutions of nonlinear heat equations and their applications in the study of the localization effect

## § 1 Introduction. Main directions of inquiry

In this chapter we propose a general approach to the study of asymptotic behaviour of solutions of quasilinear parabolic equations

$$u_t = (k(u)u_x)_x;\ k(u) > 0,\ u > 0, \tag{1}$$

where $k \in C^2((0,\infty)) \cap C([0,\infty))$. For this equation we shall consider a boundary value problem in $\omega_T = (0,T) \times \mathbf{R}_+$, $T < \infty$, with the initial condition $u(0,x) = u_0(x) \geq 0$ in $\mathbf{R}_+$, and with the fixed boundary behaviour $u(t,0) = u_1(t) > 0$, $t \in (0,T)$, which shows blow-up behaviour:

$$u_1(t) \to \infty,\ t \to T^- . \tag{2}$$

We shall be interested in the asymptotic properties of solutions of the problem under consideration, which are expressed at times sufficiently close to the blow-up time $t = T^-$, and in particular the restrictions on $u(t,0) = u_1(t)$ under which the problem admits or does not admit localization of heat (understood in either the strict or the effective sense; see § 1, Ch. III).

We want to undertake such a study for sufficiently arbitrary boundary regimes $u_1(t)$, and for a wide class of coefficients $k(u)$ as well.

As we have mentioned already (see Ch. I–III), an efficient method of studying such problems consists of constructing and analyzing self-similar or some other invariant solutions of equation (1), which satisfy some ordinary differential equations. These particular solutions have a simple spatio-temporal structure, which defines the form of the boundary condition $u(t,0) = u_1(t)$ and supply us with the necessary information concerning the asymptotic behaviour of the process. Reasonably detailed information concerning invariant solutions of equation (1) is contained in Ch. I–III.

However, the group classification of equation (1) performed in [321, 322] shows that the number of solutions of this equation invariant with respect to a Lie group of point transformations is not large. The more general Lie-Bäcklund transformations do not significantly enlarge the class of invariant solutions; new possibilities here arise only for $k(u) = (1+u)^{-2}$ (see [221, 51, 262]). For $k(u)$ not of power ($k(u) \neq u^{\sigma}$, $\sigma = \text{const}$) or not of exponential ($k(u) \neq e^{u}$) type, equation (1) admits only two types of nontrivial invariant solutions: $u_S(t,x) = f_S(x/(1+t)^{1/2})$ and $u_S(t,x) = f_S(x-t)$. Of these only the second one, in the case when the integral $\int_1^{\infty} (k(\eta)/\eta)\, d\eta$ converges, is generated by the boundary blow-up regime. Equations (1) with a power law or exponential nonlinearity admit other invariant solutions of interest for us (see § 3, Ch. II).

The fact that certain parabolic equations of the form (1) admit a wider class of invariant solutions provided one of the main stimuli to the development of the special comparison theory for solutions of various nonlinear parabolic equations (see Ch. V). However, the upper or lower bounds of solutions of equation (1) of general type, obtained in the framework of comparison are frequently not sharp enough, and thus do not allow us to describe the asymptotic stage of evolution of the process. Using the theorems proved in Ch. V, it is not always possible to single out, using upper and lower bounds, a sufficiently narrow "corridor" of the solution's evolution in time (narrow enough to enable us to speak about a correctly determined asymptotic behaviour of the solution). This is mainly connected to the paucity of invariant solutions of equation (1).

To determine asymptotic behaviour of solutions of equation (1), we use in the present chapter approximate self-similar solutions (a.s.s.), which, though they do not satisfy equation (1), do describe the asymptotic properties of its solutions correctly. In each of the following sections we shall describe a different method of construction of a.s.s. The main problem is that of determining the principal (or we can say, defining) operator in the right-hand side of the equation, which dominates the fully developed stage of evolution of a boundary regime with blow-up. Of particular interest are the results of § 2, where we determine a class of coefficients $\{k(u)\}$, for which the defining operator is a first order operator, and finally the asymptotics of a heat transfer process is described by invariant solutions of an equation of Hamilton-Jacobi type.

It has to be noted in particular that every non-trivial self-similar solution of the nonlinear heat equation (1) is, as a rule, asymptotically stable with respect to small perturbations not only of boundary conditions, which is quite natural, but also of the equation itself (that is, to perturbations in the coefficient $k(u)$ with respect to the corresponding invariant dependencies). It has to be said that in the latter case the term "small" does not have to be taken literally, since frequently an a.s.s. obtained from an invariant solution as a result of a small perturbation of $k(u)$, does not look anything like it.

As discovered in [184, 185, 186, 187], the set of sufficiently "regular" asymptotic behaviours of solutions of equation (1), which grow unboundedly, can be subdivided into three classes, depending on the character of growth of $k(u)$ for large $u$; each of these classes consists of three subclasses, ordered by the form of the boundary functions $\{u_1(t)\}$. The first class, $k(u)$ of "weakly linear" form, is considered in § 2; the second class, of $k(u)$ "close" to power law dependence, is studied in § 3. In § 4 we propose another method of constructing a.s.s., applicability of which is perhaps more restricted; this study leads, nonetheless, to intriguing general conclusions.

In this chapter we do not consider the third class, of nonlinearities $k(u)$ close to exponential, since in the analysis of asymptotic stability of the corresponding a.s.s., a boundary value problem in a bounded domain with moving boundaries has to be considered [186], so that this case cannot be applied in the study of heat localization in half-space.

Questions related to construction of a.s.s. connected with usual boundary regimes without blow-up: $u_1(t) \to \infty$ as $t \to \infty$, are not considered here. In this regard, see the papers [184, 185, 186, 187]; in the last of these a classification of such a.s.s. in the "plane of boundary value problems" is made.

## § 2 Approximate self-similar solutions in the degenerate case

### 1 Statement of the problem

Let us consider the first boundary value problem:

$$u_t = (k(u)u_x)_x, \quad (t, x) \in \omega_T = (0, T) \times \mathbf{R}_+; \tag{1}$$

$$u(0, x) = u_0(x) \geq 0, \quad x \in \mathbf{R}_+, \quad u_0 \in C(\mathbf{R}_+), \quad \sup u_0 < \infty; \tag{2}$$

$$u(t, 0) = u_1(t) > 0, \quad 0 \leq t < T;$$

$$u_1(t) \to \infty, \quad t \to T^- \ ; \quad u_1 \in C^1([0, T)). \tag{3}$$

We shall take the function $u_1$ in (3) to be monotone increasing.

In this section we construct a.s.s. for a large class of equations (1) with non-power law nonlinearities. These a.s.s. are shown to satisfy certain first order quasilinear equations.

The construction of a.s.s. is done under the following restrictions on the coefficient $k$: $k'(u) \neq 0$ for $u > 0$,

$$[k(s)/k'(s)]' \to \infty, \quad s \to \infty, \tag{4}$$

$$\int_0^\infty \frac{k(\eta)}{\eta+1}\, d\eta = \infty. \tag{5}$$

Condition (5) places a restriction from below on the behaviour of $k(u)$ for large $u$, while condition (4) restricts its behaviour from both above and below. In particular, it follows from (4) that for any $\alpha > 0$ and all sufficiently large $s > 0$, we have the estimates

$$s^{-\alpha} < k(s) < s^{\alpha}. \tag{4'}$$

In the following we shall need the function $E$ defined by the equality

$$\int_0^{E(s)} \frac{k(\eta)}{\eta+1}\, d\eta = s, \;\; s \geq 0. \tag{6}$$

The function $E$ is positive and strictly increasing in $\mathbf{R}_+$, $E \in C^3((0,\infty)) \cap C([0,\infty))$, $E(0) = 0$, and $E(\infty) = \infty$ (the latter is assured by condition (5)). Therefore $E$ is a one-to-one mapping $\overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ and there exists $E^{-1} : \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$, a monotone function inverse to $E$.

For all $u > 0$ let us define the function

$$p_k(u) = \max_{\eta \in [0,u]} k(E(\eta)); \tag{7}$$

from (4) it follows that

$$p_k(u)/u \to 0, \;\; u \to \infty \tag{8}$$

(we shall need this result below).

Some typical coefficients $k(u)$, which satisfy conditions (4), (5), are shown in Table 1. There we also give the leading terms of the asymptotic expansions of the functions $E(u)$ of (6) for large $u$. These are needed in the determination of the form of a.s.s.

**Table 1**

| $k(u) =$ | $E(u) \approx$ |
|---|---|
| $\exp\{\ln^\alpha(1+u)\},\ 0 < \alpha < 1$ | $\exp\left\{\ln^{1/\alpha} u \left(1 + \frac{\alpha-1}{\alpha^2}\frac{\ln\ln u}{\ln u}\right)\right\}$ |
| $\ln^\alpha(1+u),\ \alpha > 0$ | $\exp\{[(1+\alpha)u]^{1/(1+\alpha)}\}$ |
| $\ln[1+\ln(1+u)]$ | $\exp\{u/\ln u\}$ |
| $1$ | $e^u$ |
| $\{1+\ln[1+\ln(1+u)]\}^{-1}$ | $\exp\{u \ln u\}$ |
| $\{1+\ln^\alpha(1+u)\}^{-1},\ 0 < \alpha < 1$ | $\exp\{[(1-\alpha)u]^{1/(1-\alpha)}\}$ |
| $\{1+\ln(1+u)\}^{-1}$ | $\exp\{e^u\}$ |
| $[1+\ln(1+u)]^{-1}\{1+\ln[1+\ln(1+u)]\}^{-1}$ | $\exp\{\exp\{e^u\}\}$ |

We shall show that from the point of view of study of the localization phenomenon, the most interesting boundary regimes with blow-up are of the form

$$u(t,0) = u_1(t) = E[(T-t)^n], \;\; 0 < t < T. \tag{3'}$$

where $n < 0$ is a fixed constant. As $E(\infty) = \infty$, the function (3′) describes a regime with blow-up.

## 2 Formal determination of a.s.s.

Below we shall demonstrate that under the assumptions made above the solution of the problem (1)–(3′) converges asymptotically in a special norm to the exact invariant solution $u_s$ of the following first order equation (a Hamilton-Jacobi type equation):

$$u_t = \frac{k(u)}{u+1}(u_x)^2, \quad (t, x) \in \omega_T. \tag{9}$$

The function $u_s$ is an a.s.s. of the original equation (1) and has the form

$$u_s(t, x) = E[(T-t)^n \theta_s(\xi)], \quad \xi = x/(T-t)^{(1+n)/2}. \tag{10}$$

The non-negative function $\theta_s$ is the solution of the boundary value problem for the ordinary differential equation obtained by substituting the expression (10) into (9):

$$\begin{gathered}(\theta_s')^2 - \frac{1+n}{2}\xi\theta_s' + n\theta_s = 0, \quad \xi > 0, \\ \theta_s(0) = 1, \quad \theta_s(\infty) = 0.\end{gathered} \tag{11}$$

Existence and uniqueness of solutions of the problem (11) were established in § 4, Ch. III. There we also studied its properties, and, in particular, obtained the following estimates:

$$\begin{gathered}\theta_s''(\xi) \geq 0, \quad \xi \in [0, \xi_0); \\ q_n = \max_{\xi \in [0, \xi_0)} \theta_s''(\xi) = (1-n)/4 < \infty,\end{gathered} \tag{12}$$

where $\xi_0 = \text{meas}\,\text{supp}\,\theta_s$; moreover, $\xi_0 = \infty$ for $n \in (-1, 0)$ (that is, $\theta_s > 0$ in $\mathbf{R}_+$), $\xi_0 = 2$ for $n = -1$ and

$$\xi_0 = 2(-n)^{n/2}(-1-n)^{-(1+n)/2}$$

for $n < -1$. For $n = -1$ the solution of the problem (11) is the function

$$\theta_s(x) = (1 - x/2)_+^2, \quad x \geq 0 \tag{13}$$

(in this case $\xi \equiv x$). By the condition $\theta_s(0) = 1$, the function (10) satisfies (3′).

Let us denote by $\theta(t, \xi)$ the similarity representation of the solution of the problem (1)–(3′), defined by the spatio-temporal structure of the a.s.s. (10):

$$\theta(t, \xi) = (T-t)^{-n} E^{-1}[u(t, \xi(T-t)^{(1+n)/2})], \quad (t, x) \in \omega_T. \tag{14}$$

### 3 The convergence to a.s.s. theorem

Let us show that the similarity representation of the solution of the problem under consideration converges as $t \to T^-$ to $\theta_s$, which ensures that the solution $u(t,x)$ is close to the a.s.s. (10). Thus we establish asymptotic convergence of non-stationary solutions of equations of different orders: a parabolic one (1), and one of the Hamilton-Jacobi type (9). The physical reason for this sort of degeneration of the original equation in the case of $k(u) \equiv 1$ was discussed in § 4, Ch. III. It is not hard to present the same kind of analysis for $k(u)$ of general form.

**Theorem 1.** *Let conditions* (4), (5) *be satisfied. Then the similarity representation* (14) *of the solution of the problem* (1)–(3′) *for* $n \in ]-1,0)$ *converges as* $t \to T^-$ *to the function* $\theta_s(\xi)$, *the solution of the problem* (11). *Moreover, we have the estimate*

$$\|\theta(t,\cdot) - \theta_s(\cdot)\|_{C(\mathbf{R}_+)} \equiv \sup_{\xi \in \mathbf{R}_+} |\theta(t,\xi) - \theta_s(\xi)| =$$

$$= O\left((T-t)^{-n} \int_0^t \frac{p_k[(T-\tau)^n]}{T-\tau}\, d\tau\right) \to 0,\ t \to T^- . \tag{15}$$

*Proof.* Let us introduce in $\omega_T$ new functions $U$, $U_s$ defined by $U = E^{-1}(u)$, $U_s = E^{-1}(u_s)$. Substituting $u = E(U)$ and $u_s = E(U_s)$, into, respectively, (1) and (9), we obtain the equations

$$U_t = k[E(U)][U_{xx} + U_x^2, \tag{16}$$

$$(U_s)_t = (U_s)_x^2. \tag{17}$$

Let us set $U(t,x) - U_s(t,x) = z(t,x)$. As follows from (16), (17), the function $z$ satisfies the parabolic equation

$$z_t = k[E(U)]z_{xx} + k[E(U)](U_s)_{xx} + z_x(U_x + (U_s)_x) \tag{18}$$

and the conditions

$$z(0,x) = E^{-1}[u_0(x)] - T^n\theta_s(x/T^{(1+n)/2}),\ x \in \mathbf{R}_+;\ z(t,0) = 0,\ t \in (0,T). \tag{19}$$

Below we shall analyse the solution of equation (18) with the help of the Maximum Principle. To justify its use, we make the following remark.

The generalized solution $u(t,x)$ (and therefore $U(t,x)$) of the degenerate equation (1) does not necessarily have the smoothness required for the formal application of the Maximum Principle (see § 1, Ch. I). However the function $u(t,x)$ ($U(t,x)$) can be represented as the limit as $k \to \infty$ of a sequence of smooth, positive in $\omega_T$ solutions $u_k \in C_{tx}^{1,2}(\omega_T)$ ($U_k \in C_{tx}^{1,2}(\omega_T)$).

For $n = -1$ the function $U_s \equiv (T-t)^{-1}(1-x/2)_+^2$ also does not have the requisite smoothness: $U_s \notin C_{tx}^{1,2}(\omega_T)$ (but, which is very important, $U_s \in C^1(\omega_T)$). Therefore we shall be using the fact that the non-negative generalized solution of the first order equation (17) can be obtained as the limit as $\epsilon \to 0^+$ of a sequence of classical positive solutions $U_s^\epsilon$ of parabolic equations

$$(U_s^\epsilon)_t = (U_s^\epsilon)_x^2 + \epsilon (U_s^\epsilon)_{xx}, \ \epsilon > 0, \tag{20}$$

satisfying the same boundary conditions as $U_s$ [257, 260]. Here, since we have that $U_s \in C^1(\omega_T)$, we have the uniform in $\epsilon \in (0,1)$ estimate $|(U_s^\epsilon)_{xx}| \le \text{const}$ in $(\delta, \tau) \times \mathbf{R}_+$; $0 < \delta < \tau < T$ are constants (this is important in the proof of convergence to a.s.s.).

The sequence of functions $z_k^\epsilon = U_k - U_s^\epsilon \in C_{tx}^{1,2}(\omega_T)$ converges uniformly as $k \to \infty$, $\epsilon \to 0^+$, to a function $z$ on each bounded set $\omega_\tau' \subset \omega_\tau = (0, \tau) \times \mathbf{R}_+$, $\tau \in (0, T)$. From the argument above, we shall formally assume that the function $z(t,x)$ is sufficiently smooth. Here we are implicitly assuming that the necessary estimates are first derived for the smooth functions $z_k^\epsilon(t,x)$, and the final result is obtained by passing to the limit as $k \to \infty$, $\epsilon \to 0^+$. Let us note that if equation (20) is used instead of (17), the equation for $z_k^\epsilon$ includes an additional term, which is not essential for the final estimate (15) as $\epsilon \to 0^+$.

Thus, let $z \in C_{tx}^{1,2}(\omega_T) \cap C(\overline{\omega}_T)$. Then from equation (18) by the comparison theorem we conclude that

$$\max_x z(t,x) \le z^+(t), \ \min_x z(t,x) \ge z^-(t),$$

where the smooth functions $z^\pm(t)$ satisfy the inequalities

$$dz^+/dt \le \sup_x k[E(U(t,x))] \sup_x (U_s)_{xx}(t,x), \tag{21}$$

$$dz^-/dt \ge 0, \ 0 < t < T, \tag{22}$$

and moreover $z^+(0) = \max z(0,x) < \infty$, $z^-(0) = \min z(0,x) > -\infty$.

In the derivation of (22) we take into account the first of the inequalities (12). Using the notation (7) and the explicit form of the function $U_s \equiv (T-t)^{-n}\theta_s(\xi)$, we obtain

$$\sup_x k[E(U)] = \sup_{s \in ]0,(T-t)^n]} k[E(s)] = p_k[(T-t)^n],$$

$$\sup_x (U_s)_{xx} = (T-t)^{-1} \max_{\xi \ge 0} \theta_s''(\xi) = (T-t)^{-1} q_n, \ t \to T^-.$$

Substituting these equalities into (21), (22), we derive as $t \to T^-$ the estimates

$$\frac{dz^+}{dt} \le q_n \frac{p_k[(T-t)^n]}{T-t}, \ \frac{dz^-}{dt} \ge 0.$$

Hence by the inequality

$$\|\theta(t,\cdot)-\theta_s(\cdot)\|_C \le (T-t)^{-n}\max\{z^+(t),|z^-(t)|\}, 0<t<T,$$

we deduce the validity of the estimate (15).

We need only to demonstrate convergence of $\theta$ to $\theta_s$ as $t \to T^-$. Resolving the indeterminate in the right-hand side of (15), and taking (8) into account, we obtain

$$\lim_{t\to T^-}\|\theta(t,\cdot)-\theta_s(\cdot)\|_C \le \lim_{t\to T}\frac{q_n}{-n}\frac{p_k[(T-t)^n]}{(T-t)^n}=0.$$

□

Theorem 1 allows us to determine asymptotically exactly the dependence of the depth of penetration of the thermal wave on time. From the convergence estimate (15) it follows that $x_{ef}(t)$ satisfies as $t \to T^-$ the equality $u_s(t, x_{ef}(t)) \simeq 1/(2E[(T-t)^n])$.

Hence, using the specific form of a.s.s. $u_s(t,x)$ (see (10)), we obtain

$$x_{ef}(t) \simeq (T-t)^{(1+n)/2}\theta_s^{-1}\left[\frac{E^{-1}[E[(T-t)^n]/2]}{(T-t)^n}\right], \quad t\to T^-. \tag{23}$$

Here $\theta_s^{-1}$ is the function inverse to $\theta_s$ ($\theta_s^{-1}$ exists on the interval $(0,1)$ in view of the monotonicity of $\theta_s$).

Let us demand in addition that

$$\lim_{s\to\infty}\frac{k(s/2)}{k(s)}=1. \tag{24}$$

Then it is easily verified that $E^{-1}(E(s)/2)/s \to 1$, $s\to\infty$. Since $\theta_s^{-1}(\xi) \simeq (1-\xi)/(-n)^{1/2}$ for small $\xi>0$, we derive from (23) the following estimate for the penetration depth of the thermal wave:

$$x_{ef}(t) \simeq \frac{(T-t)^{(1+n)/2}}{(-n)^{1/2}}\left[1-\frac{E^{-1}[E[(T-t)^n]/2]}{(T-t)^n}\right], \tag{25}$$

which holds for all $t$ sufficiently close to $T^-$.

Let us consider separately the case $n<-1$. For $n<-1$ the function $(U_s)_\xi$ has a jump discontinuity of the first kind at the "front" point $x_0(t)=\xi_0(T-t)^{(1+n)/2}$, so that $U_s \in C(\omega_T)$. Therefore additional difficulties arise in the proof of convergence to a.s.s. for $n<-1$. Below we obtain an estimate of the rate of convergence (15) for $n<-1$ for the case $k(u)\equiv 1$, when $E(s)=\exp s - 1$, so that the similarity representation (14) has the form

$$\theta(t,\xi)=(T-t)^{-n}\ln[1+u(t,\xi(T-t)^{(1+n)/2})], \tag{14'}$$

**Theorem 1′.** *Let $k(u) \equiv 1$. Then for any $n < 0$ the following estimate of the rate of convergence to a.s.s. is valid:*

$$\|\theta(t,\cdot) - \theta_s(\cdot)\|_{C(\mathbf{R}_+)} = O[(T-t)^{-n}|\ln(T-t)|], \ t \to T^- . \tag{15′}$$

*Proof.* The case $n \in [-1, 0)$ was considered in Theorem 1. Let us note for $k \equiv 1$ (15) is the same as (15′), since then $\rho_k(u) \equiv 1$ (see (7)).

Thus, let $n < -1$ and without loss of generality $u_0(x) \equiv 0$. Then the solution of the problem (1)–(3′) has the form

$$u(t,x) = \frac{x}{2\pi^{1/2}} \int_0^t \exp\left\{-\frac{x^2}{4(t-\tau)}\right\} [\exp\{(T-\tau)^n\} - 1](t-\tau)^{-3/2}\, d\tau.$$

The main problem is to estimate $u(t,x)$ on the weak discontinuity surface of a.s.s. (10), $x_0(t) = \xi_0(T-t)^{(1+n)/2}$, on which the a.s.s. $u_s \equiv 0$ and does not have the requisite smoothness. Setting in the last equality $x = \xi_0(T-t)^{(1+n)/2}$, and introducing a new variable of integration by $(T-t)(t-\tau)^{-1} = s$, we have the following estimate:

$$u(t, x_0(t)) \le \frac{\xi_0}{2\pi^{1/2}} \lambda^{n/2} \int_0^\infty \exp\{-\lambda^n P_n(s)\} s^{-1/2}\, ds,$$

where we have introduced the notation $\lambda = T - t \to 0^+$, $t \to T^-$, and $P_n(s)$ stands for the function $\xi_0^2 s/4 - (1+s)^n s^{-n}$.

It is easily checked that $P_n(s)$ is non-negative exactly for $\xi_0 = 2(-n)^{n/2}(-1-n)^{-(n+1)/2}$. Therefore the above integral converges and goes to zero as $\lambda \equiv T-t \to 0^+$. Hence $u(t, x_0(t)) = o((T-t)^{n/2})$ and thus $U(t, x_0(t)) = O[|\ln(T-t)|]$ as $t \to T^-$. Since $u_x(t, x) \le 0$ (see § 2, Ch. V), this estimate holds everywhere in $\{x \ge x_0(t)\}$. In the domain $\{0 < x < x_0(t)\}$, where $U_s \in C^\infty$, the method of proof of Theorem 1 can be applied, which gives us as a result the estimate (15′) of the rate of convergence to a.s.s. □

## 4 Sufficient conditions for absence of localization

**Theorem 2.** *Assume that conditions* (4), (5) *hold. Then there is no localization in the problem* (1)–(3′) *for $n < -1$. The solution grows without bound as $t \to T^-$ everywhere in* $\mathbf{R}_+$. *Furthermore,*

$$\varliminf_{t\to T^-} \frac{E^{-1}(u(t,x))}{(T-t)^n} \ge 1, \ x \in \mathbf{R}_+ . \tag{26}$$

*If equation* (1) *admits finite speed of propagation of perturbations and* meas supp $u_0 < \infty$, *then for the size of the support of the solution we have the estimate*

$$x_*(t) \equiv \text{meas supp}\, u(t, x) \geq (T-t)^{(1+n)/2}(\xi_0 - \epsilon(t)),\ 0 < t < T, \tag{27}$$

*where* $\xi_0 = \text{meas supp}\, \theta_s = 2(-n)^{n/2}(-1-n)^{-(1+n)/2}$ *and the non-negative function* $\epsilon(t) \to 0$, $t \to T^-$. *If* $k \equiv 1$, *then by Theorem* 1$'$ *we have equality in* (26).

*Proof.* Let us denote by $\bar{u}(t,x)$ the solution of problem (1)–(3$'$) in $(0,T) \times (0, x_0(t))$, $x_0(t) = \xi_0(T-t)^{(1+n)/2}$, satisfying $\bar{u}(t, x_0(t)) \equiv 0$, $\bar{u}(0,x) \leq u_0(x)$ in $(0, x_0(0))$. Since $u(0, x_0(t)) \geq 0$ in $(0,T)$, by the comparison theorem $u \geq \bar{u}$ in $(0, T) \times (0, x_0(t))$. But $u_s \in C^{1,2}_{t,x}((0, T) \times (0, x_0(t)))$, $u_s(t, x_0(t)) \equiv 0$. Therefore as in the proof of Theorem 1 , we see that $\theta(t,\xi) \to \theta_s(\xi)$ as $t \to T$ in $C((0,\xi_0))$ with the rate of convergence given by (15) (here $\bar{\theta}(t,\xi)$ is the similarity representation (14) of the solution $\bar{u}$). Then the claim of Theorem 2 follows from the inequality $u \geq \bar{u}$ in $(0, T) \times (0, x_0(t))$; in the derivation of (27) we use the expansion

$$\theta_s(\xi) \simeq -\frac{1+n}{2}\xi_0(\xi_0 - \xi),\ \xi \to \xi_0^-. \tag{28}$$

□

Theorem 2 provides sufficient conditions for the absence of localization in the problem (1)–(3$'$). Unfortunately, Theorem 1 cannot be used to establish the parallel result for presence of localization for $n \geq -1$. In the case $n = -1$ we can prove the following assertion:

**Theorem 3.** *Assume that conditions* (4), (5) *hold. Then the solution of the problem* (1)–(3$'$) *for* $n = -1$ *satisfies the relation*

$$\lim_{t \to T} \frac{E^{-1}(u(t,x))}{(T-t)^{-1}} = \left(1 - \frac{x}{2}\right)_+^2,\ x \in \mathbf{R}_+. \tag{29}$$

**Remark.** If $n \in (-1, 0)$ in (3$'$), then $E^{-1}[u(t,x)] = o((T-t)^n)$, $t \to T^-$, in $\mathbf{R}_+$.

Relation (29), which follows from (15) and (13) for $\theta_s(\xi)$ in the case $n = -1$, means that $u(t,x)$ grows without bound for all $x \in (0,2)$ and $u(t,x) \simeq E[(T-t)^{-1}(1-x/2)_+^2]$ as $t \to T^-$. If, on the other hand, $x \geq 2$, then $u(t,x) = o(E[(T-t)^{-1}])$, which, however, does not ensure uniform boundedness of the solution. At the same time it is clear that a. s. s. (10) (which correctly describes the asymptotic behaviour of the solution of the problem) is localized for $n \geq -1$.

In the case $k(u) \equiv 1$, localization of the solution of (1)–(3′) for $n \geq -1$ is proved in § 4, Ch. III, by analyzing the heat potential. All the arguments above, as well as the results of numerical computations, indicate that for $n \geq -1$ the solution of the problem (1)–(3′) is localized.

## 5 Examples

Let us consider other examples.

**Example 1** (compare with Example 2, § 5, Ch. V). Let $k(u) = \ln^\lambda(1+u)$, where $\lambda > 0$ is a constant. Conditions (4), (5) are satisfied. The transformation $E$ in (6) has the form (see Table 1) $E(u) = \exp\{[(1+\lambda)u]^{1/(1+\lambda)}\} - 1$ and therefore, setting $n_\lambda = n/(1+\lambda)$ in (3′) we obtain

$$u_1(t) = \exp\{(1+\lambda)^{1/(1+\lambda)}(T-t)^{n_\lambda}\} - 1,\ 0 < t < T. \tag{30}$$

To this boundary blow-up regime corresponds the a.s.s.

$$u_s(t,x) = \exp\{(1+\lambda)^{1/(1+\lambda)}(T-t)^{n_\lambda}\theta_s^{1/(1+\lambda)}(\xi)\} - 1, \tag{31}$$

where $\xi = x/(T-t)^{[1+(1+\lambda)n_\lambda]/2}$.

From Theorem 1 it follows that for $n \in [-1, 0)$ the similarity representation (14) converges as $t \to T^-$ to the function $\theta_s$, and that we have the estimate

$$\|\theta(t,\cdot) - \theta_s(\cdot)\|_C = O((T-t)^{-n_\lambda}) \to 0,\ t \to T^-.$$

From the structure of the a.s.s. (31) it can be seen that for $n_\lambda < -1/(1+\lambda)$ there is no localization in the problem, and that $u(t,x) \sim \exp\{(T-t)^{n_\lambda}\},\ t \to T^-$, for any $x \in \mathbf{R}_+$. Equation (1) describes processes with finite speed of propagation of perturbations. Therefore for $n_\lambda < -1/(1+\lambda)$ the size of the support of the solution can be estimated using (27).

For $n_\lambda \geq -1/(1+\lambda)$ a.s.s. (31) are localized. In particular, in the case $n_\lambda = -1/(1+\lambda)$ (S-regime) the following equality holds asymptotically as $t \to T^-$ (see Theorem 3):

$$u(t,x) \simeq \exp\{(1+\lambda)^{1/(1+\lambda)}(T-t)^{-1/(1+\lambda)}(1-x/2)_+^{2/(1+\lambda)}\} - 1.$$

Penetration depth of the thermal wave depends on time thus:

$$x_{ef}(t) \simeq \frac{(1+\lambda)^{\lambda/(1+\lambda)}\ln 2}{(-n_\lambda(1+\lambda))^{1/2}}(T-t)^{[1+n_\lambda(\lambda-1)]/2},\ t \to T^-.$$

Hence it follows that in the case $\lambda > 1$ for $n_\lambda \in [-(\lambda-1)^{-1}, 0)$ in (30) penetration depth decreases to zero as $t \to T^-$. This conclusion also holds true for the boundary HS-regime, which heats up to infinite temperature the whole half-space $\mathbf{R}_+$ (see § 4, Ch. III). For $\lambda = 1$ the behaviour of $x_{ef}(t)$ is practically independent of the parameter $n$: $x_{ef}(t) = O((T-t)^{1/2})$, $t \to T^-$.

The above conclusions confirm that, in general, shrinking of the half-width is not sufficient for localization.

**Example 2** (compare with Example 3, § 5, Ch. V). Let us consider equation (1) with coefficient $k(u) = \ln[1 + \ln(1+u)]$. Since here $E(u) \simeq \exp\{u/\ln u\}$ as $u \to \infty$, from Theorem 2 we deduce that the boundary blow-up regime

$$u_1(t) \simeq \exp\{(T-t)^n/|n \ln(T-t)|\}, \quad t \to T^-, \tag{32}$$

leads for $n < -1$ to absence of localization. Here

$$\operatorname{meas} \operatorname{supp} u(t,x) \gtrsim \xi_0 (T-t)^{(1+n)/2} \to \infty, \quad t \to T^-.$$

If on the other hand $n \geq -1$, then the a.s.s. are localized. To the boundary S-regime ($n = -1$) corresponds an a.s.s. of the following form:

$$u(t,x) \simeq u_s(t,x) \simeq \exp\left\{ \frac{(T-t)^{-1}}{|\ln(T-t)|} \frac{[(1-x/2)_+]^2}{\left\{1 + 2\dfrac{\ln[(1-x/2)_+]}{|\ln(T-t)|}\right\}} \right\},$$

$$t \to T^-; \; 0 < x < 2.$$

From the relation (25) it follows that in this problem ($n \geq -1$) the half-width decreases as $t \to T^-$:

$$x_{ef}(t) \simeq (-n)^{-1/2}(T-t)^{(1+n)/2} \ln|\ln(T-t)| \to 0, \quad t \to T^-.$$

**Example 3.** Let

$$k(u) = [1 + \ln(1+u)]^{-1}. \tag{33}$$

In this case $E(u) = \exp\{e^u - 1\} - 1$, therefore it follows from Theorem 2 that the boundary regime

$$u_1(t) = \exp\{\exp\{(T-t)^n\} - 1\} - 1 \tag{34}$$

ensures for $n < -1$ unbounded growth of the solution of problem (1), (2), (34) everywhere in $\mathbf{R}_+$. On the other hand, if $n \geq -1$, then we should expect heat localization in the problem (at least the corresponding a.s.s. have this property).

**Example 4.** Let the thermal conductivity coefficient have the form

$$k(u) = \frac{3\ln^2[1 + \ln(1+u)]}{1 + \ln(1+u)}.$$

Conditions (4), (5) are then satisfied and the function $E$ defined by (6) has the form

$$E(u) = \exp\{\exp\{u^{1/3}\} - 1\} - 1\}.$$

Therefore in this problem the boundary blow-up regime

$$u_1(t) = \exp\{\exp\{(T-t)^n\} - 1\} - 1\}$$

will produce no localization for $n < -1/3$. For $n \geq -1/3$ the a.s.s. are localized.

## 6 Localization conditions for arbitrary boundary blow-up regimes

In the construction of a.s.s. in the degenerate case we made substantial use of condition (5). If the integral in (5) converges:

$$a_k = \int_0^\infty \frac{k(\eta)}{\eta+1}\, d\eta < \infty, \tag{35}$$

then the function $E$ in (6) is defined on the finite interval $(0, a_k)$, and therefore $E^{-1}$ is uniformly bounded in $\mathbf{R}_+$. Thus it makes no sense to consider the family of boundary blow-up regimes (3′), and no a.s.s. of the form (10) exist here.

To clarify the meaning of the restriction (5), let us consider

**Example 5.** Let the coefficient in equation (1) have the form

$$\begin{aligned} k(u) = {} & [1+\ln(1+u)]^{-1}[1+\ln(1+\ln(1+u))]^{-1} \times \ldots \\ & \ldots \times [1+\ln(1+\ldots+\ln(1+u))\ldots)]^{-1}, \quad u > 0. \end{aligned} \tag{36}$$

In each successive bracket the number of logarithms is increased by one. Let the last bracket contain $M$ logarithms, that is, (36) has $M$ factors (for $M = 1$ the coefficient (36) coincides with (33)).

The transformation $E$ is determined from (6):

$$E(u) = \exp\{\exp\{\ldots\{e^u - 1\}\ldots\} - 1\} - 1\},$$

where in the right-hand side $M + 1$ exponents are used. Therefore from Theorem 2 we conclude that the boundary regime

$$\begin{aligned} u_1(t) &= E[(T-t)^n] = \\ &= \exp\{\exp\ldots\{\exp\{(T-t)^n\} - 1\}\ldots - 1\} - 1, \quad 0 < t < T, \end{aligned} \tag{37}$$

is not localized for $n < -1$. Comparison of (36) and (37) shows that increase in the number of logarithms in the last square bracket of (36) leads to the following situation: to bring about the HS-regime without localization, faster and faster (more rapidly increasing as $t \to T^-$) blow-up regimes are required. As $M \to \infty$ the intensity of these regimes does not have an upper bound (in a certain sense). At the same time as $M$ is increased, the "rate of divergence" of the integral (5) becomes lower. Therefore as condition (35) becomes satisfied, the intensity of minimal boundary regimes that lead to absence of localization and unbounded growth of the solution in the whole space, becomes infinite.

If condition (5) holds, all the types of boundary blow-up regimes are possible: HS-, S-, and LS- regimes (under the restriction (4) this was practically established in Theorems 1, 2). On the other hand, if $k(u)$ satisfies (35), then, apparently, there are no non-localized HS-regimes. Let us state again the result which can be proved by comparison with travelling wave solutions (see § 3, Ch. I).

*If the condition*

$$\int_1^\infty \frac{k(\eta)}{\eta}\, d\eta < \infty \tag{38}$$

*holds, any boundary blow-up regime in the problem* (1)–(3′) *leads to localization.*

Condition (38) is satisfied, for example, by the following coefficients: $k(u) = e^{-u}$, $k(u) = (1+u)^{-\sigma}$, $\sigma > 0$, $k(u) = [1 + \ln(1+u)]^{-\lambda}$ and

$$k(u) = [1 + \ln(1+u)]^{-1} \times \{1 + \ln[1 + \ln(1+u)]\}^{-\lambda}, \ \lambda > 1.$$

## § 3 Approximate self-similar solutions in the non-degenerate case. Pointwise estimates of the rate of convergence

In this section we consider non-degenerate a.s.s. of the nonlinear heat equation, which correspond to given blow-up regimes on the boundary $x = 0$. Unlike the degenerate a.s.s. of § 2, they satisfy (under other conditions on the thermal conductivity coefficient) a second order parabolic equation.

*Main assumptions and formal definition of a.s.s.* As before, we shall consider in $\omega_T = (0, T) \times \mathbf{R}_+$ the first boundary value problem

$$u_t = \mathbf{A}(u) = (k(u)u_x)_x; \tag{1}$$

$$u(0, x) = u_0(x) \geq 0,\ x \in \mathbf{R}_+;\ u_0 \in C(\mathbf{R}_+),\ \sup u_0 < \infty; \tag{2}$$

$$u(t, 0) = u_1(t) > 0,\ t \in (0, T),\ u_1 \in C^2([0, T)), \tag{3}$$

where the function $u_1$ of (3), which blows up in finite time, is taken to be monotone increasing.

Let us introduce the necessary restrictions on the coefficient $k(u) \in C^3((0, \infty)) \cap C([0, \infty))$. We shall assume that $k'(u) > 0$ for $u > 0$, $k(0) = 0$.

$k(\infty) = \infty$. The function $k$ defines a bijective mapping $\mathbf{R}_+ \to \mathbf{R}_+$ and therefore we can define $k^{-1} : \mathbf{R}_+ \to \mathbf{R}_+$, the inverse function to $k$.

Let us set

$$\Phi_k(u) = [k(s)/k'(s)]'|_{s=k^{-1}(u)}. \tag{4}$$

The main condition on $k$ is as follows: there exists a constant $\sigma \in \mathbf{R}_+$, such that[1]

$$\Phi_k(u) \to 1/\sigma, \; u \to \infty. \tag{5}$$

In Table 2 we list some coefficients $k$, which satisfy condition (5). In all the cases the constant $\gamma > 0$ is chosen sufficiently large. In the right column of the table we list the principal terms of the expansion of the function $k^{-1}(u)$ as $u \to \infty$.

**Table 2**

| $k(u) =$ | $k^{-1}(u) \approx$ |
|---|---|
| $u^\sigma \ln^\alpha \ln(\gamma + u)$ | $u^{1/\sigma} \ln^{-\alpha/\sigma} \ln u$ |
| $u^\sigma \ln^\alpha(\gamma + u)$ | $\sigma^{\alpha/\sigma} u^{1/\sigma} \ln^{-\alpha/\sigma} u$ |
| $u^\sigma \exp\{\ln^\alpha(\gamma + u)\}$, $0 < \alpha < 1$ | $u^{1/\sigma} \exp\{-\sigma^{-(\alpha+1)} \ln^\alpha u\}$ |
| $u^\sigma \exp\left\{\dfrac{\ln(\gamma + u)}{\ln^\alpha \ln(\gamma + u)}\right\}$, $\alpha > 0$ | $u^{1/\sigma} \exp\left\{-\sigma^{-2} \dfrac{\ln u}{\ln^\alpha \ln u}\right\}$ |

It will be convenient for us to write the boundary condition (3) in the following way:

$$u(t, 0) = u_1(t) = k^{-1}(\psi^\sigma(t)), \; t \in (0, T), \tag{3'}$$

where $\psi(t)$ is a smooth monotone increasing function, $\psi(t) \to \infty$ as $t \to T^-$.

We shall seek a.s.s. of the problem (1)–(3′) in the form

$$u_s(t, x) = k^{-1}[\psi^\sigma(t)\theta_s^\sigma(\xi)], \; \xi = x/\phi(t), \tag{6}$$

where the unknown functions $\theta_s(\xi)$, $\phi(t)$ are determined from conditions of convergence of the solution $u(t, x)$ to the a.s.s. $u_s(t, x)$. It is assumed that $\theta_s(0) = 1$, so that a.s.s. (6) satisfies the boundary condition (3′).

Let us make some necessary preliminary computations. Let us set

$$U_s(t, x) = k(u_s(t, x)) \equiv \psi^\sigma(t)\theta_s^\sigma(\xi). \tag{7}$$

The function $U_s$ satisfies the equation

$$(U_s)_t = \sigma(\psi^{\sigma-1}\psi')(t)\theta_s^\sigma(\xi) - \left(\psi^\sigma \frac{\phi'}{\phi}\right)(t)\xi\frac{d\theta_s^\sigma(\xi)}{d\xi}. \tag{8}$$

---

[1]For $\sigma = 0^+$ relation (5) is equivalent to the equality $\Phi_k(\infty) = \infty$. This case was analysed in § 2.

or, equivalently,

$$(U_s)_t = \sigma\left(\frac{\psi'}{\psi}\right)(t)U_s - \left(\frac{\phi'}{\phi}\right)(t)x\frac{\partial U_s}{\partial x}. \tag{8'}$$

Let us introduce another piece of notation:

$$U(t,x) = k(u(t,x)),\ (t,x)\in\omega_T. \tag{9}$$

For $U$ we obtain the equation

$$U_t = \mathbf{D}_\sigma(U) + [\Phi_k(U) - 1/\sigma]U_x^2, \tag{10}$$

where

$$\mathbf{D}_\sigma(U) = UU_{xx} + \frac{1}{\sigma}U_x^2. \tag{11}$$

Let us observe that $U_s(t,0) \equiv U(t,0) = \psi^\sigma(t)$ for $t\in(0,T)$. Below we shall analyse the equation satisfied by the difference $z = U - U_s$ using the Maximum Principle, and we shall derive conditions such that

$$\|\theta^\sigma(t,\cdot) - \theta_s^\sigma(\cdot)\|_{C(\mathbf{R}_+)} \to 0,\ t\to T^-, \tag{12}$$

where we have denoted by $\theta(t,\xi)$ the similarity representation of the solution of the problem (1)–(3'):

$$\theta(t,\xi) = \frac{1}{\psi(t)}U^{1/\sigma}(t,\xi\phi(t)) \equiv \frac{1}{\psi(t)}k^{1/\sigma}(u(t,\xi\phi(t))),\ t\in(0,T),\ \xi\ge 0. \tag{13}$$

The convergence condition (12) ensures that as $t\to T^-$ the solution of the problem under consideration and a.s.s. (6) have the same properties. Then the function $\phi(t)$ determines, modulo a constant factor, the dependence of the depth of penetration of the thermal wave $x_{ef}(t)$ on time.

Under the above assumptions, the construction of a.s.s. uses exact self-similar solutions of the equation with power type nonlinearity:

$$u_t = (u^\sigma u_x)_x \tag{14}$$

(here the constant $\sigma > 0$ is the same as in condition (5)). We shall need two types of self-similar solutions of equation (14) (see § 3, Ch. II):

$$\text{I.}\qquad u_S(t,x) = (T-t)^n\theta_s(\xi),\ \xi = x/(T-t)^{(1+n\sigma)/2},$$
$$t\in(0,T);\ n=\text{const}<0; \tag{15}$$

$$\text{II.}\qquad u_S(t,x) = e^t\theta_s(\xi),\qquad \xi = x/\exp\{\sigma t/2\},\ t>0. \tag{16}$$

Here the functions $\theta_s(\xi) \geq 0$ satisfy ordinary differential equations obtained by substituting $u_S(t, x)$ into (14):

$$\text{I.} \qquad (\theta_s^\sigma \theta_s')' - \frac{1+n\sigma}{2}\theta_s'\xi + n\theta_s = 0,\ \xi > 0; \tag{17}$$

$$\text{II.} \qquad (\theta_s^\sigma \theta_s')' + \frac{\sigma}{2}\theta_s'\xi - \theta_s = 0, \qquad \xi > 0, \tag{18}$$

as well as the boundary conditions

$$\theta_s(0) = 1,\ \theta_s(\infty) = 0. \tag{19}$$

Under these assumptions on $k(u)$ the method of constructing a.s.s. in each specific case depends on the form of the boundary regime.

## 1 Approximate self-similar solutions of type I

In this subsection we construct a.s.s. of the problem (1)–(3′) with function $\theta_s$ satisfying equation (17). The problem (17), (19) was studied in detail in § 2, Ch. III, where it was shown that for any $n < 0$ its solution exists and is unique. For $n < -1/\sigma$ the function $\theta_s$ has compact support: $\xi_0 = \operatorname{meas}\operatorname{supp}\theta_s < \infty$, while for $n \in (-1/\sigma, 0)$ we have $\theta_s(\xi) > 0$ everywhere in $\mathbf{R}_+$. Moreover,

$$\theta_s(\xi) = C\xi^{2n/(1+n\sigma)} + \dots,\ \xi \to \infty;\ C = C(n,\sigma) = \text{const} > 0. \tag{20}$$

In the case $n = -1/\sigma$ the solution can be written down explicitly:

$$\theta_s(\xi) = \left[\left(1 - \frac{\xi}{\xi_0}\right)_+\right]^{2/\sigma},\ \xi > 0;\ \xi_0 = \left[\frac{2(\sigma+2)}{\sigma}\right]^{1/2}. \tag{21}$$

To prove convergence to a.s.s. we shall need

**Lemma 1.** *Let $\theta_s$ be the solution of the problem* (17), (19). *Then*

$$q_n = \sup_{\xi \in [0,\xi_0)} \frac{d^2\theta_s^\sigma}{d\xi^2}(\xi) < -n\sigma,\ \xi_0 = \operatorname{meas}\operatorname{supp}\theta_s. \tag{22}$$

**Remark.** For $n = -1/\sigma$ we have (see (21)) $q_n = 2/\xi_0^2 = \sigma/(\sigma+2) < 1$.

*Proof.* Let us set $g_s(\xi) = \theta_s^\sigma(\xi)$. The function $g_s$ satisfies the equation

$$g_s g_s'' + \frac{1}{\sigma}(g_s')^2 - \frac{1+n\sigma}{2}g_s'\xi + n\sigma g_s = 0. \tag{23}$$

Let us assume that there exists a point $\xi = \xi^* \in \mathbf{R}_+$, such that $g_s''(\xi^*) = -n\sigma$. Let us show that this leads to a contradiction.

Let us consider first the case $n < -1/\sigma$. Then $\theta_s(\xi) \equiv 0$ for $\xi \geq \xi_0 =$ meas supp $g_s$, and $(g_s^{1/\sigma} g_s')(\xi_0) = 0$. It is not hard to show using the Banach contraction mapping theorem that as $\xi \to \xi_0^-$ we have the expansion

$$g_s(\xi) \equiv \theta_s^\sigma(\xi) = -\frac{1+n\sigma}{2}\sigma\xi_0(\xi_0 - \xi) + \frac{1-n\sigma}{4(\sigma+1)}\sigma(\xi_0 - \xi)^2 + \dots . \tag{24}$$

Hence $g_s''(\xi_0^-) = (1-n\sigma)\sigma/[2(\sigma+1)] < -n\sigma$. Therefore $\xi^* < \xi_0$, i.e. $g_s(\xi^*) > 0$. Then, setting in (23) $\xi = \xi^*$, we obtain

$$g_s'(\xi^*) = \sigma\frac{1+n\sigma}{2}\xi^*. \tag{25}$$

Let us rewrite (23) in an equivalent form:

$$(g_s^{1/\sigma} g_s')' - \frac{1+n\sigma}{2} g_s^{1/\sigma - 1} g_s'\xi + n\sigma g_s^{1/\sigma} = 0, \ \xi > 0. \tag{26}$$

Integrating both sides of the equation over the interval $(\xi^*, \xi_0)$ we obtain

$$\begin{aligned} -g_s^{1/\sigma}(\xi^*)\left[g_s'(\xi^*) - \sigma\frac{1+n\sigma}{2}\xi^*\right] &+ \\ + \sigma\left(\frac{1+n\sigma}{2} + n\right)\int_{\xi^*}^{\xi_0} g_s^{1/\sigma}(\eta)\,d\eta &= 0. \end{aligned} \tag{27}$$

Hence, taking (25) into account, we arrive at a contradiction, since

$$\int_{\xi^*}^{\xi_0} g_s^{1/\sigma}(\eta)\,d\eta > 0.$$

Now let $n > -1/\sigma$. Then $g_s > 0$ in $\mathbf{R}_+$ and $g_s''(\infty) = 0$. We have to consider two cases. The first case is $n \in (-1/\sigma, -1/(\sigma+2))$, when $g_s^{1/\sigma} \in L^1(\mathbf{R}_+)$ (see the asymptotics (20)). A contradiction in this case is obtained by integrating (26) over $(\xi^*, \infty)$. The second case is $n \geq -1/(\sigma+2)$. Then, integrating (26) from $\xi = 0$ to $\xi = \xi^*$, and using (25), we obtain the equality

$$\frac{\sigma(\sigma+2)}{2}\left(n + \frac{1}{\sigma+2}\right)\int_0^{\xi^*} g_s^{1/\sigma}(\eta)\,d\eta = g_s'(0),$$

which cannot possibly hold in view of the conditions $n \geq -1/(\sigma+2)$, $g_s'(0) < 0$. □

**Theorem 1.** *Let condition* (5) *hold, and let the function* $\psi$ *in* (3') *satisfy the condition*

$$(\psi/\psi')'(t) \to \frac{1}{n}, t \to T \ ; \ n = const \in [-1/\sigma, 0). \tag{28}$$

*Then the problem* (1)–(3′) *has a.s.s.* (6), *where*

$$\phi(t) = |-n(\psi^{\sigma+1}/\psi')(t)|^{1/2}, t \in (0, T), \tag{29}$$

*the function* $\theta_s(\xi)$ *is the solution of the problem* (17), (19) *and the limiting equality* (12) *holds.*

Let us make some preliminary computations and see what equation is satisfied by a.s.s. (6) under the conditions of Theorem 1. From (17) it follows that

$$\theta_s^\sigma = -\frac{1}{n\sigma}\theta_s^\sigma \frac{d^2\theta_s^\sigma}{d\xi^2} - \frac{1}{n\sigma^2}\left(\frac{d\theta_s^\sigma}{d\xi}\right)^2 + \frac{1+n\sigma}{2n\sigma}\frac{d\theta_s^\sigma}{d\xi}\xi. \tag{30}$$

Substituting this equality into (8), we derive the following equation for the function $U_s$:

$$(U_s)_t = -\frac{1}{n}(\psi^{\sigma-1}\psi')(t)\theta_s^\sigma \frac{d^2\theta_s^\sigma}{d\xi^2} - \frac{1}{n\sigma}(\psi^{\sigma-1}\psi')(t)\left(\frac{d\theta_s^\sigma}{d\xi}\right)^2 +$$
$$+ \frac{1+n\sigma}{2n}(\psi^{\sigma-1}\psi')(t)\frac{d\theta_s^\sigma}{d\xi}\xi - \left(\psi^\sigma \frac{\phi'}{\phi}\right)(t)\frac{d\theta_s^\sigma}{d\xi}\xi.$$

Setting now $\theta_s^\sigma = U_s/\psi^\sigma$, $\xi = x/\phi$, we have

$$(U_s)_t = -\frac{1}{n}\left(\frac{\psi'\phi^2}{\psi^{\sigma+1}}\right)(t)\,\mathbf{D}_\sigma(U_s) + G(t)(U_s)_x x,$$

where $G(t) = [\ln(\psi^l/\phi)]'(t)$, $l = (1+n\sigma)/(2n)$ and $\mathbf{D}_\sigma$ is the operator (11). However, by the choice of $\phi$ (see (29)) $-\frac{1}{n}\left(\frac{\psi'\phi^2}{\psi^{\sigma+1}}\right)(t) \equiv 1$. Therefore the equation for $U_s$ has the form

$$(U_s)_t = \mathbf{D}_\sigma(U_s) + G(t)(U_s)_x x. \tag{31}$$

Setting in accordance with (7) $U_s = k(u_s)$, we deduce that under the conditions of the theorem a.s.s. (6) satisfies the equation

$$(u_s)_t = \mathbf{A}(u_s) + \left[\frac{1}{\sigma} - \Phi_k(k(u_s))\right]k'(u_s)(u_s)_x^2 + G(t)(u_s)_x x, \tag{32}$$

which differs from the original equation (1) by two additional terms in the right-hand side.

**Remark.** It is interesting to note that a different equation can be derived for an equivalent a.s.s. For example, from (17) we have the equality (here $n \neq -1/\sigma$)

$$\frac{d\theta_s^\sigma}{d\xi}\xi = \frac{2}{1+n\sigma}\left[\theta_s^\sigma \frac{d^2\theta_s^\sigma}{d\xi^2} + \frac{1}{\sigma}\left(\frac{d\theta_s^\sigma}{d\xi}\right)^2\right] + \frac{2n\sigma}{1+n\sigma}\theta_s^\sigma.$$

Substituting it into (8), we derive for $U_s$ the equation

$$(U_s)_t = -\frac{2}{1+n\sigma}\left(\frac{\phi\phi'}{\psi^{\sigma}}\right)(t)\mathbf{D}_{\sigma}(U_s) + \frac{2n\sigma}{1+n\sigma}\left(\ln\frac{\psi^{l}}{\phi}\right)'(t)U_s, \quad l = \frac{1+n\sigma}{2n}. \tag{33}$$

If instead of (29) we define the function $\phi(t)$ so that

$$-\frac{2}{1+n\sigma}\left(\frac{\phi\phi'}{\psi^{\sigma}}\right)(t) \equiv 1,$$

then, setting in (33) $U_s = k(u_s)$ we have for a.s.s. (6) another equation,

$$(u_s)_t = \mathbf{A}(u_s) +$$

$$+\left[\frac{1}{\sigma} - \Phi_k(k(u_s))\right]k'(u_s)(u_s)_x^2 + \frac{k(u_s)}{k'(u_s)}\frac{2n\sigma}{1+n\sigma}\left(\ln\frac{\psi^{l}}{\phi}\right)'(t),$$

which differs from both (32) and the original equation (1). In the sequel we shall only use equation (32).

*Proof of Theorem* 1. Let us set $z = U - U_s$ in $\omega_T$, $z \in C(\omega_T)$. Then it follows from (10), (31) that $z$ satisfies in $\omega_T$ the parabolic equation

$$z_t = \mathscr{L}_1 z + \mathscr{L}_2 z + \mathcal{M}(t,x) + \mathcal{N}(t,x;z). \tag{34}$$

Here $\mathscr{L}_1$ is the linear elliptic operator obtained by transforming the difference $\mathbf{D}_{\sigma}(U) - \mathbf{D}_{\sigma}(U_s)$:

$$\mathscr{L}_1 z = U z_{xx} + \frac{1}{\sigma}(U_x + (U_s)_x)z_x;\quad \mathscr{L}_2 z = (U_s)_{xx} z; \tag{35}$$

$\mathcal{M}$ is a function of the following form:

$$\mathcal{M}(t,x) = -G(t)(U_s(t,x))_x x; \tag{36}$$

$\mathcal{N}$ is the nonlinear operator

$$\mathcal{N}(t,x;z) = [\Phi_k(U_s + z) - 1/\sigma]((U_s)_x + z_x)^2. \tag{37}$$

Here $z(t,0) \equiv 0$, $M_0 = \sup|z(0,x)| < \infty$.

Below we shall derive upper and lower bounds for $z$ in $\omega_T$ by constructing spatially homogeneous sub- and supersolutions for equation (34).

Thus, let condition (5) hold. Then there exists a continuous function $H(u)$:

$$-H(u) \le \Phi_k(u) - 1/\sigma \le H(u), \quad u > 0, \tag{38}$$

such that

$$H(u) > 0 \text{ and } H(u) \text{ is non-increasing for } u > 0, \tag{39}$$

$$H(u) \to 0, \quad u \to \infty. \tag{40}$$

Let us obtain an upper bound for $z(t, \cdot)$. A lower bound will have the same form. It follows from the form of the operator $\mathcal{N}$ that the function $z$ satisfies the inequality

$$z_t \le \mathcal{L}z + \mathcal{M} + \mathcal{N}_0(t, x; z), \quad \mathcal{L} \equiv \mathcal{L}_1 + \mathcal{L}_2,$$

where $\mathcal{N}_0$ is

$$\mathcal{N}_0(t, x; z) = H(U_s + z)((U_s)_z + z_x)^2.$$

Using the fact that by assumptions (39), (40) $H(U_s + z) \le H(z)$ for all $z \ge 0$, it is not hard to show that $z$ is a subsolution of the equation

$$z_t^+ = \mathcal{L}z^+ + \mathcal{M} + H(z^+)((U_s)_x + z_x^+)^2$$

in the domain $\{z^+ \ge 0\}$. Therefore $z \le z^+$ $(z^+ \ge 0)$ in $\omega_T$ if this inequality holds for $t = 0$ and on the boundary $(0, T) \times \{x = 0\}$.

It is obvious that the function $z^+$ is, in its turn, a subsolution of the parabolic equation

$$w_t^+ = \mathcal{L}_1 w^+ + w^+ \sup_x (U_s)_{xx} + \sup_x |\mathcal{M}| + H(w^+)(w_x^+ + 2(U_s)_x)w_x^+ +$$
$$+ H(w^+)\sup_x (U_s)_x^2, \quad w^+ > 0 \text{ in } \omega_T.$$

Appealing to the usual comparison theorem (with respect to boundary data; see § 1, Ch. I), solution of this equation can be estimated in terms of the spatially homogeneous solution $w(t)$.

Summarizing all the above, we have the following estimate:

$$|z(t, x)| \le w(t) \text{ in } \omega_T, \tag{41}$$

where the function $w$ satisfies the following boundary value problem for an ordinary differential equation:

$$\begin{gathered}\frac{dw}{dt} = w \sup_x (U_s)_{xx} + \sup_x |\mathcal{M}| + H(w) \sup_x (U_s)_x^2, \; t \in (0, T); \\ w(0) = M_0 \equiv \sup z(0, x) < \infty.\end{gathered} \tag{42}$$

Taking into account the specific form of $U_s$ (see (7)), as well as (29), it is not hard to see that for $n \in [-1/\sigma, 0)$

$$\sup_x (U_s)_{xx} \equiv \frac{\psi^\sigma(t)}{\phi^2(t)} \sup \left| [\theta_s^\sigma(\xi)]'' \right| = \frac{q_n}{-n} \frac{\psi'(t)}{\psi(t)};$$

$$\sup_x |\mathcal{M}| \equiv |G(t)| \sup_x |(U_s)_x x| = p_n |G(t)| \psi^\sigma(t),$$

$$p_n = \sup |\xi(\theta_s^\sigma)'| < \infty;$$

$$\sup_x (U_s)_x^2 = \frac{r_n}{-n} \frac{\psi^{2\sigma}(t)}{\phi^2(t)} \equiv \frac{r_n}{-n} \psi^{\sigma-1}(t)\psi'(t),$$

$$r_n = \sup \left[(\theta_s^\sigma)'\right]^2 < \infty.$$

Therefore equation (42) has the form

$$\frac{dw}{dt} = \frac{q_n}{-n} \frac{\psi'}{\psi} w + p_n \psi^\sigma |G| + \frac{r_n}{-n} \psi^{\sigma-1} \psi' H(w). \tag{43}$$

Since for all $t \in (0, T)$ we have by (44) that

$$\|\theta^\sigma(t, \cdot) - \theta_s^\sigma(\cdot)\|_C \equiv \frac{1}{\psi^\sigma(t)} \sup_x |z(t, x)| \leq \frac{w(t)}{\psi^\sigma(t)}. \tag{44}$$

to prove the limit equality (12), it suffices to check that $w(t)/\psi^\sigma(t) \to 0$ as $t \to T^-$.

From (43) it is not hard to obtain an estimate of $w(t)$, for example, on the interval $(T/2, T)$. To estimate the last (nonlinear) term in the right-hand side of (43), let us use the inequality

$$\frac{dw}{dt} \geq \frac{q_n}{-n} \frac{\psi'(t)}{\psi(t)} w,$$

from which it follows that for all $t \in (T/2, T)$

$$w(t) \geq M_1 |\psi(t)|^{q_n/(-n)}, \; M_1 = w(T/2)|\psi(T/2)|^{-q_n/(-n)}.$$

Therefore in view of (39), (40) we have the inequality

$$\frac{dw}{dt} \leq \frac{q_n}{-n} \frac{\psi'}{\psi} w + p_n \psi^\sigma |G| + \frac{r_n}{-n} \psi^{\sigma-1} \psi' H(M_1 \psi^{q_n/(-n)}).$$

Consequently, for all $t \in (T/2, T)$

$$w(t) \leq M_1 \psi^{q_n/(-n)}(t) + p_n \psi^{q_n/(-n)}(t) \int_{T/2}^t \psi^{\sigma - q_n/(-n)}(\tau)|G(\tau)|\, d\tau +$$

$$+ \frac{r_n}{-n} \psi^{q_n/(-n)}(t) \int_{T/2}^t \psi^{\sigma-1-q_n/(-n)}(\tau)\psi'(\tau) H(M_1 \psi^{q_n/(-n)}(\tau))\, d\tau \equiv$$

$$\equiv I_1(t) + I_2(t) + I_3(t).$$

Let us consider the relation

$$0 < w(t)/\psi^{\sigma}(t) = I_1(t)\psi^{-\sigma}(t)+I_2(t)\psi^{-\sigma}(t) + I_3(t)\psi^{-\sigma}(t) \equiv$$
$$\equiv J_1(t) + J_2(t) + J_3(t).$$

By Lemma 1 $q_n < -n\sigma$ and therefore

$$J_1(t) \equiv M_1|\psi(t)|^{q_n/(-n) - \sigma} \to 0, t \to T^- .$$

Let us consider the expression for $J_2(t)$. Consecutively resolving all the indeterminacies that arise as $t \to T^-$, and using the exact form of $G(t)$, as well as conditions (28), (29), we obtain (here $C > 0$ are some constants)

$$\begin{aligned}\lim_{t\to T} J_2(t) &= C \lim_{t\to T} \frac{\psi}{\psi'}|G| = \\ &= C \lim_{t\to T} \frac{\psi}{\psi'}\left|l\frac{\psi'}{\psi} - \frac{\phi'}{\phi}\right| = C \lim_{t\to T} \left|l - \frac{\phi'}{\phi}\frac{\psi}{\psi'}\right| = \\ &= C \lim_{t\to T} \left|l - \frac{1}{2}\left\{\sigma + \left[\frac{\psi}{\psi'}\right]'\right\}\right| = C\left|l - \frac{1}{2}\left(\sigma + \frac{1}{n}\right)\right| = 0.\end{aligned}$$

Similarly, by (40)

$$\lim_{t\to T} J_3(t) = C \lim_{t\to T} H(M_1\psi^{q_n/(-n)}(t)) = C \lim_{u\to\infty} H(u) = 0.$$

Thus $w(t)/\psi^{\sigma}(t) \to 0$ as $t \to T^-$, which, as shown by (44), ensures that (12) holds. This concludes the proof. □

**Remark.** For $n < -1/\sigma$, by (24) the function $(\theta_s^{\sigma})_{\xi}$ experiences a jump at the point $\xi = \xi_0$, that is $U_s$ does not have the smoothness required for the proof by the above method. In this case convergence to the a.s.s. is easily proved for $\overline{u}(t, x)$, the solution of the problem (1), (3′) in the domain $(0, T) \times (0, x_0(t))$ (there $U_s \in C_{t,x}^{1,2}$), satisfying $\overline{u}(t, x_0(t)) \equiv 0$ on the moving right boundary $x_0(t) \equiv \xi_0\phi(t)$. Then using the estimate (22), which holds for any $n < 0$, we obtain the same estimate of the range of convergence $\overline{\theta}^{\sigma}(t, \xi) \to \theta_s^{\sigma}(\xi)$ as $t \to T^-$ in $C((0, \xi_0))$. From this estimate we conclude that there is no localization in the original problem (1)–(3′) for $n < -1/\sigma$ and derive, in particular, a lower bound on the size of the support of the non-localized solution (see proof of Theorem 2, § 2).

Table 3 shows functions $\phi(t)$ corresponding to various boundary regimes (3′), in which $\psi$ satisfies condition (28). Properties of the resulting a.s.s. are largely dependent on the relation between the quantities $n$ and $\sigma$. Thus, for $n < -1/\sigma$ the half-width of the a.s.s. grows without bound as $t \to T^-$, that is, the solution of the problem (1)–(3′) is not localized (HS-regime). On the other hand, if $n > -1/\sigma$

a.s.s. becomes infinite at the single point $x = 0$, which indicates localization of the solution (LS-regime). For $n = -1/\sigma$ both localized and non-localized solutions can exist.

Let us consider the case $n = -1/\sigma$ in more detail. It is easy to see here that condition (28) is satisfied by the function

$$\psi(t) = (T-t)^{-1/\sigma}. \tag{45}$$

Then in (29) $\phi(t) \equiv 1$, so that the half-width is constant (S-regime).

**Table 3**

| $\psi(t) =$ | $\phi(t) \approx$ |
|---|---|
| $(T-t)^n \ln^\alpha \ln\vert 2+(T-t)^{-1}\vert$ | $(T-t)^{(1+n\sigma)/2} \ln^{\alpha\sigma/2} \vert\ln(T-t)\vert$ |
| $(T-t)^n \ln^\alpha\vert 2+(T-t)^{-1}\vert$ | $(T-t)^{(1+n\sigma)/2} \vert\ln(T-t)\vert^{\alpha\sigma/2}$ |
| $(T-t)^n \exp\{\ln^\alpha\{(T-t)^{-1}\}\}$, $0<\alpha<1$ | $(T-t)^{(1+n\sigma)/2} \exp\left\{\frac{\sigma}{2}\vert\ln(T-t)\vert^\alpha\right\}$ |
| $(T-t)^n \exp\left\{\dfrac{\ln[(T-t)^{-1}]}{\ln^\alpha \ln\vert 2+(T-t)^{-1}\vert}\right\}$, $\alpha>0$ | $(T-t)^{(1+n\sigma)/2} \exp\left\{\dfrac{\sigma}{2}\dfrac{\vert\ln(T-t)\vert}{\ln^\alpha\vert\ln(T-t)\vert}\right\}$ |

Thus, if condition (5) holds, to the S-regime in the original problem there corresponds the boundary condition

$$u(t,0) = k^{-1}|(T-t)^{-1}|, \quad t \in (0,T). \tag{46}$$

Then a.s.s. can be written down explicitly:

$$u_S(t,x) = k^{-1}|(T-t)^{-1}(1-x/x_0)_+^2|, \quad x_0 = |2(\sigma+2)/\sigma|^{1/2}.$$

**Example 1.** Let equation (1) be

$$u_t = \left[\frac{u^\sigma}{\ln^\delta(\gamma+u)} u_x\right]_x, \tag{47}$$

where $\delta \neq 0$, $\gamma > 1$, and let in problem (1)–(3)

$$u_1(t) = (T-t)^n \ln^\alpha|1+(T-t)^{-1}|, \quad t \in (0,T); \ \alpha = \text{const} \neq 0. \tag{48}$$

Then it follows from (3′) that to this boundary regime there corresponds the function

$$\psi(t) = k^{1/\sigma}(u_1(t)) \simeq (-n)^{-\delta/\sigma}(T-t)^n \ln^{\alpha-\delta/\sigma}|(T-t)^{-1}|, \quad t \to T^-.$$

which satisfies condition (28). Therefore, by Theorem 1, to the regime (48) there corresponds the a.s.s. with half-width satisfying

$$\phi(t) \simeq (-n)^{-\delta/2}(T-t)^{(1+n\sigma)/2}|\ln(T-t)|^{(\alpha\sigma-\delta)/2}, \; t \to T^- .$$

In particular, for $n = -1/\sigma$ this dependence has the form[2]

$$\phi(t) \simeq (\sigma)^{\delta/2}|\ln(T-t)|^{(\alpha\sigma-\delta)/2}, \; t \to T^- .$$

Hence for $\alpha > \delta/\sigma$ we have that $\phi(t) \to \infty$ as $t \to T^-$, which means that the solution grows without bound on the whole space (HS-regime). If $\alpha < \delta/\sigma$, then $\phi(t) \to 0$ as $t \to T^-$, and the a.s.s. is localized (LS-regime); if $\alpha = \delta/\sigma$ then the S-regime obtains: as $t \to T^-$ the solution has a constant (non-zero) half-width. Substituting into (48) $n = -1/\sigma$, $\alpha = \delta/\sigma$, we see that in the case of equation (47) the boundary blow-up S-regime is

$$u_1(t) = (T-t)^{-1/\sigma}\ln^{\delta/\sigma}[1+(T-t)^{-1}], \; t \in (0,T).$$

## 2 Approximate self-similar solutions of type II

Below we construct a.s.s. of problem (1), (3′) considered in a domain $(0,T) \times (0, x_0(t))$, $x_0(t) \equiv \xi_0\phi(t) > 0$, with an additional boundary condition on the moving right boundary $u(t, x_0(t)) = 0$. In this case in the a.s.s. (6) the function $\theta_s$ satisfies the problem (18), (19) and has compact support: $\text{meas}\,\text{supp}\,\theta_s = \xi_0 < \infty$ (see § 3, Ch. II).

**Lemma 2.** *Let $\theta_s$ be the solution of problem* (18), (19). *Then*

$$q_1 = \sup_{\xi\in[0,\xi_0)} \frac{d^2\theta_s^\sigma}{d\xi^2}(\xi) < \sigma.$$

The proof of this lemma, which is similar to the proof of the previous one, uses the asymptotic expansion

$$\theta_s^\sigma = \frac{\sigma^2\xi_0}{2}(\xi_0-\xi) + \frac{\sigma^2}{4(\sigma+1)}(\xi_0-\xi)^2 + \dots, \; \xi \to \xi_0^- .$$

In particular, we obtain $(\theta_s^\sigma)''(\xi_0^-) = \sigma^2/[2(\sigma+1)] < \sigma$.

[2] Let us note that only in the case $\delta = 0$ (when (47) becomes an equation with power law nonlinearity) and $\alpha = 2/\sigma$ can the expression $\phi(t) \simeq |\ln(T-t)|$, $t \to T^-$, be obtained from an analysis of the exact self-similar solution [184, 321].

**Theorem 2.** *Assume that condition* (5) *holds and that the function* $\psi$ *in* (3′) *satisfies the condition*

$$[\psi/\psi']'(t) \to 0,\ t \to T^-. \tag{49}$$

*Then there exists an a.s.s.* (6), *such that*

$$\phi(t) = \{(\psi^{\sigma+1}/\psi')(t)\}^{1/2},\ t \in (0, T),$$

*the function* $\theta_s(\xi)$ *is the solution of the problem* (18), (19) *and* $\theta^\sigma(t, \xi) \to \theta_s^\sigma(\xi)$ *as* $t \to T^-$ *in* $C((0, \xi_0))$.

The proof of this theorem is identical to the proof of the previous one.

It is easily checked that if condition (49) holds, $\phi(t) \to \infty$ as $t \to T^-$. Therefore under the conditions of the theorem, boundary blow-up regimes in (3′) are HS-regimes.

**Example 2.** Assume that in the problem for equation (47)

$$u_1(t) = \exp\{(T-t)^n\},\ t \in (0, T);\ n < 0.$$

Then the function

$$\psi(t) = k^{1/\sigma}[u_1(t)] = (T-t)^{-n\delta/\sigma} \exp\{(T-t)^n\}$$

satisfies condition (49). Therefore we have the estimate

$$\phi(t) \simeq (-n)^{-1/2}(T-t)^{[1-n(\delta+1)]/2} \exp\left\{\frac{\sigma}{2}(T-t)^n\right\},\ t \to T^-,$$

for the half-width of the solution.

## § 4 Approximate self-similar solutions in the non-degenerate case. Integral estimates of the rate of convergence

In this section we present a different method of constructing a.s.s. for the problem

$$u_t = (k(u)u_x)_x,\ (t, x) \in \omega_T; \tag{1}$$

$$u(0, x) = u_0(x) \geq 0,\ x \in \mathbf{R}_+;\ u_0 \in C(\mathbf{R}_+), \sup u_0 < \infty; \tag{2}$$

$$u(t, 0) = u_1(t) > 0,\ u_1'(t) > 0,\ t \in (0, T). \tag{3}$$

In all the cases considered below, the proofs proceed by deriving some integral estimates of the difference of $u(t, x)$ and the corresponding a.s.s. This analysis in fact establishes "transformation rules" for known invariant solutions of (1) for $k = u^\sigma$, $\sigma = \text{const} \geq 0$, under the transformation of the thermal conductivity coefficient $u^\sigma \to k(u)$.

## 1 Approximate self-similar solutions of nearly linear equations

In this subsection we shall construct a.s.s. of equation (1) with a coefficient $k$, which satisfies

$$k \in C^2((0,\infty)) \cap C([0,\infty)); \ k(u) > 0, k'(u) > 0, u > 0; \tag{4}$$

$$(k/k')'(u) \to \infty, \ u \to \infty. \tag{5}$$

Some functions $k$ satisfying (4), (5), are shown in Table 4.

**Table 4**

| $k(u) =$ |
|---|
| $\ln^\alpha \ln(3+u), \alpha > 0$ |
| $\ln^\alpha(3+u), \alpha > 0$ |
| $\ln(3+u)\ln^\alpha \ln(3+u), \alpha > 0$ |
| $\exp\{\ln^\alpha(1+u), 0 < \alpha < 1$ |
| $\exp\left\{\frac{\ln(1+u)}{\ln^\alpha \ln(3+u)}\right\}, \alpha > 0$ |

It will be shown that for a particular choice of $u_1(t)$ a.s.s. of the problem (1)–(3) can be constructed using the self-similar solutions of the linear heat equation

$$u_t = u_{xx}. \tag{6}$$

This approach is different from that of § 2, where, under condition (5), a.s.s. were determined using invariant solutions of the first order equation $u_t = k(u)u_x^2/(u+1)$. Let us observe that in this section we are considering more general coefficients $k(u)$, since there is no need to impose the restriction (2.5), which was essential in § 2.

We shall need two types of self-similar solutions of equation (6):

I. $$u_S(t,x) = (T-t)^n f_1(\zeta), \zeta = x/(T-t)^{1/2}, \ t \in (0,T); \ n < 0; \tag{7}$$

II. $$u_S(t,x) = e^t f_2(x), t > 0. \tag{8}$$

It is assumed that $f_i(0) = 1$, $f_i(\infty) = 0$, $i = 1,2$. Equations for the functions $f_i$ are obtained by substituting (7), (8) into (6). It is easy to verify that

$$f_1(\xi) = \frac{\Gamma(1/2-n)}{\pi^{1/2}\Gamma(-n)} \int_0^\infty \exp\left\{-\frac{\xi^2 s}{4}\right\} s^{-n-1}(1+s)^{n-1/2}\,ds, \ f_2(x) = e^{-x}.$$

In each case an a.s.s. $u_s$ of the problem (1)–(3) is sought in the form

$$u_s(t,x) = u_1(t)\theta_s(\xi), \ \xi = x/\phi(t), \tag{9}$$

where the non-negative and sufficiently smooth functions $\theta_s(\xi)$, $\phi(t)$, have to be determined. Setting $\theta_s(0) = 1$, we have that the a.s.s. (9) satisfies the boundary condition (3).

Let us denote by $\theta(t, \xi)$ the similarity representation of the solution of the problem:

$$\theta(t, \xi) = \frac{1}{u_1(t)} u(t, \xi\phi(t)), \quad \xi \in \mathbf{R}_+. \tag{10}$$

We shall show that for a particular choice of the functions $u_1, \theta_s, \phi$, the function $\theta(t, \xi)$ converges to $\theta_s(\xi)$ as $t \to T^-$, which ensures asymptotic closeness of the solution $u(t, x)$ and the a.s.s. $u_s(t, x)$.

In the proof in the sequel it is assumed that $u(t, \cdot) \in L^2(\mathbf{R}_+)$ for all $t \in [0, T)$ (the norm in $L^2(\mathbf{R}_+)$ is denoted by $\|\cdot\|_2$). Moreover, without loss of generality, we shall take the initial function $u_0(x)$ to be non-increasing in $\mathbf{R}_+$. Then by monotonicity of the boundary regime, this property will also apply to the solution $u(t, x)$ (see § 2, Ch. V).

**Lemma 1.** *Let the coefficient $k$ satisfy conditions* (4), (5). *Then:*

1) *as $u \to \infty$, $k(u)$ grows slower than any power: for any $\alpha > 0$ for all sufficiently large $u > 0$ we have the inequalities*

$$k(u) < u^{\alpha}, \quad k'(u) < u^{\alpha-1}; \tag{11}$$

2) $k''(u) < 0$ *for all sufficiently large* $u > 0$;

3) *for any* $\xi \in (0, 1)$

$$k(\xi u)/k(u) \to 1, \quad u \to \infty. \tag{11'}$$

*Proof.* The claim 1) follows immediately from (5).

Concavity of $k$ for $u \to \infty$ (claim 2)) follows from the relation

$$(k/k')'(u) \equiv 1 - k''(u)k(u)/[k'(u)]^2 \to \infty, \quad u \to \infty.$$

To prove 3), we shall use the finite increment formula: $k(u) = k(\xi u) + k'(\zeta)u(1-\xi)$, where $\zeta \in (\xi u, u)$. Hence

$$k(\xi u)/k(u) = [1 + k'(\zeta)u(1-\xi)/k(\xi u)]^{-1}.$$

However, by 2) $k'(\zeta) < k'(\xi u)$ for large $u > 0$ and therefore

$$\left[1 + \frac{k'(\xi u)\xi u}{k(\xi u)}\frac{1-\xi}{\xi}\right]^{-1} < \frac{k(\xi u)}{k(u)} < 1,$$

which completes the proof of (11′), since

$$\lim_{u\to\infty} \frac{k'(\xi u)}{k(\xi u)}\xi u \equiv \lim_{s\to\infty} \frac{s}{k(s)/k'(s)} = \lim_{s\to\infty}\left[\left(\frac{k(s)}{k'(s)}\right)'\right]^{-1} = 0.$$

□

*1 Approximate self-similar solutions of type I*

**Theorem 1.** *Let conditions* (4), (5) *hold, and assume that for all $t$ sufficiently close to $T^-$,*

$$u_1(t) = (T - \mu^{-1}(t))^n, \tag{12}$$

*where $n < -1/4$ is a constant, and $\mu^{-1} : [0, T) \to [0, T)$ is the inverse of the monotone function*

$$\mu(t) = T - \int_t^{T} \frac{d\tau}{k[(T-\tau)^n]} \to T^-, \quad t \to T^-. \tag{12'}$$

*Then there exists an a.s.s.* (9), *where*

$$\phi(t) = (T - \mu^{-1}(t))^{1/2}, \quad t \to T^-, \tag{13}$$

$$\theta_s(\xi) \equiv f_1(\xi),$$

*and*

$$\lim_{t \to T^-} \|\theta(t, \cdot) - \theta_s(\cdot)\|_2 = 0. \tag{14}$$

The exact form of the rate of convergence of $\theta(t, \xi)$ to $\theta_s(\xi)$ will come out of the proof of the theorem. Unboundedness and monotonicity of the function (12), (12′) as $t \to T^-$ follow from Lemma 1.

In a number of cases we can write down asymptotically exact expressions for the boundary regime (12), (12′). For example, if

$$\lim_{s \to \infty} \frac{k(s)}{k(s/k^n(s))} = 1$$

(this condition is satisfied by coefficients of lines one to three of Table 4, and by the coefficient of line four, if $\alpha \in (0, 1/2)$), then it is not hard to show that

$$u_1(t) \simeq (T-t)^n k^n[(T-t)^n],$$

$$\phi(t) \simeq (T-t)^{1/2} k^{1/2}[(T-t)^n], \quad t \to T^-.$$

The restriction $n < -1/4$ in (12) can be related to the need for the inclusion $\theta_s \equiv f_1 \in L^2(\mathbf{R}_+)$. Since

$$f_1(\xi) \simeq \frac{\Gamma(1/2 - n)}{2^{2n}\pi^{1/2}} \xi^{2n}, \quad \xi \to \infty,$$

it does not hold for $n \in [-1/4, 0)$.

*Proof of Theorem* 1. Let us define a smooth monotone function $\mu : [0, T) \to [0, T)$ so that

$$u_1(\mu(t)) = (T-t)^n, \quad t \to T^-, \tag{15}$$

Then the function $u(\mu(t), x)$ satisfies the equation

$$u_t = \mu'(t)(k(u)u_x)_x \tag{1'}$$

and by (12), the boundary condition

$$u(\mu(t), 0) = (T-t)^n, \quad t \to T^-, \tag{15'}$$

Since, as follows from (12), (13), $\phi(t) = |u_1(t)|^{1/(2n)}$, the equality (15) also means that $\phi(\mu(t)) = (T-t)^{1/2}$, $t \to T^-$, and therefore as $t \to T^-$

$$u_s(\mu(t), x) = (T-t)^n \theta_s(\xi), \quad \xi = x/(T-t)^{1/2},$$

$$\theta(\mu(t), \xi) = (T-t)^{-n} u(\mu(t), \ \xi(T-t)^{1/2}).$$

Observe that $u_s(\mu(t), x)$ is precisely the self-similar solution (7) of equation (6).

Let us set $w(t, x) = u(\mu(t), x) - u_s(\mu(t), x)$. Then under the above assumptions $w(t, 0) = 0$, $w(t, x) \to 0$ as $x \to \infty$ and $w(t, \cdot) \in L^2(\mathbf{R}_+)$ as $t \to T^-$. Taking the scalar product with $w$ of both sides of the equation

$$w_t = [\mu'(t)k(u)u_x - (u_s)_x]_x \tag{16}$$

and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t, \cdot)\|_2^2 = -(\mu'(t)k(u)u_x - (u_s)_x, w_x). \tag{17}$$

Let us denote by $G(s; t)$ the function

$$G(s; t) = \int_0^s \{|\mu'(t)k(\eta)|^{1/2} - 1\}^2 d\eta, \quad s > 0, \ t \in (0, T). \tag{18}$$

Using the identity

$$(\mu'(t)k(u)u_x - (u_s)_x)(u_x - (u_s)_x) \equiv$$

$$\equiv \{|\mu'(t)k(u)|^{1/2}u_x - (u_s)_x\}^2 - (u_s)_x[G(u; t)]_x,$$

we derive from (17) the estimate

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 \le ((u_s)_x, [G(u; t)]_x). \tag{19}$$

Since $(u_s)_x < 0$ and by assumption $u_x \le 0$ in $\omega_T$, the right-hand side does not exceed

$$-\sup_x |(u_s)_x(\mu(t), x)| \int_0^\infty [G(u(\mu(t), x); t)]_x \, dx \equiv$$

$$\equiv q_s(T-t)^{n-1/2} G((T-t)^n; t), \ t \to T^-,$$

where $q_s = \max|\theta_s'(\xi)| < \infty$. Then for all $t \in (t_*, T)$, where $t_* < T$ is sufficiently close to $T$, we obtain

$$\|w(t,\cdot)\|_2^2 \le \|w(t_*,\cdot)\|_2^2 + 2q_s\int_{t_*}^{t}(T-\tau)^{n-1/2}G((T-\tau)^n;\ \tau)\,d\tau,\ t_* < t < T.$$

Hence, using the easily verifiable equality

$$\|w(t,\cdot)\|_2^2 = (T-t)^{2n+1/2}\|\theta(\mu(t),\cdot) - \theta_s(\cdot)\|_2^2,\ t_* < t < T,$$

we obtain

$$\begin{aligned}\|\theta(\mu(t),\cdot) - \theta_s(\cdot)\|_2^2 \le (T-t)^{-2n-1/2}\|w(t_*,\cdot)\|_2^2 + \\ + 2q_s(T-\tau)^{-2n-1/2}\int_{t_*}^{t}(T-\tau)^{n-1/2}G((T-\tau)^n;\ \tau)d\tau.\end{aligned} \tag{20}$$

Let us show that the right-hand side of this inequality goes to zero as $t \to T^-$. The first term goes to zero by the assumption $n < -1/4$. Let us consider the second term. Since $\mu'(t) = 1/|k((T-t)^n)|$ as $t \to T^-$, and resolving indeterminacies in (20), we have

$$\begin{aligned}\lim_{t\to T}\|\theta - \theta_s\|_2^2 \le -\frac{2q_s}{2n+1/2}\times \\ \times\lim_{t\to T}(T-t)^{-n}\int_0^{(T-t)^n}\{[\mu'(t)k(\eta)]^{1/2} - 1\}^2\,d\eta.\end{aligned} \tag{21}$$

By the change of variable $\eta = (T-t)^n\zeta$, the right-hand side of the last inequality takes the form

$$-\frac{2q_s}{2n+1/2}\lim_{t\to T}\int_0^1\left\{\left[\frac{k(\zeta(T-t)^n)}{k((T-t)^n)}\right]^{1/2} - 1\right\}^2 d\zeta. \tag{22}$$

Since $k(u)$ is increasing, the integrand is bounded uniformly in $\zeta \in (0,1)$ as $t \to T^-$, and by (11′) goes to zero as $t \to T^-$ for any $\zeta \in (0,1)$. This proves (14), while (20) provides an estimate of the rate of convergence to the a.s.s. □

Under the conditions of Theorem 1, we have that $\phi(t) \to 0$ as $t \to T^-$. Therefore the structure of the a.s.s. (9) indicates that the solution $u(t,x)$ grows without bound only at the one point $x = 0$. This points to localization in this problem, with (12) an LS blow-up regime.

**Example 1.** Take in (1) $k(u) = \ln^\alpha(2+u)$, $\alpha > 0$ (see Table 4). Then it follows from (12), (13) that the boundary regime

$$u_1(t) \simeq (-n)^{\alpha n}(T-t)^n|\ln(T-t)|^{\alpha n},\ t \to T^-.$$

leads to the appearance of a thermal wave, the half-width of which decreases as $t \to T^-$ according to

$$x_{ef}(t) \simeq (-n)^{\alpha/2}\xi^*(T-t)^{1/2}|\ln(T-t)|^{\alpha/2}.$$

The constant $\xi^* \in \mathbf{R}_+$ is such that $f_1(\xi^*) = 1/2$.

## 2 Approximate self-similar solutions of type II

**Theorem 2.** *Let conditions* (4), (5) *hold, and assume that, moreover,*

$$\int_0^\infty \frac{d\eta}{k(e^\eta)} < \infty. \tag{23}$$

*For all $t$ sufficiently close to $T^-$, let*

$$u_1(t) = \exp\{\mu^{-1}(t)\}, \tag{24}$$

*where $\mu^{-1} : (0, T) \to \mathbf{R}_+$ is the inverse of the monotone increasing function $\mu : \mathbf{R}_+ \to (0, T)$, defined for sufficiently large $\tau$ by*

$$\mu(\tau) = T - \int_\tau^\infty \frac{d\eta}{k(e^\eta)}. \tag{25}$$

*Then $\phi(t) \equiv 1$, $\theta_s(x) \equiv f_2(x) = e^{-x}$ and the equality (14) holds.*

*Proof.* The proof is essentially identical to that of Theorem 1. We establish the following estimate:

$$\begin{aligned} \|\theta(\mu(t), \cdot) - \theta_s(\cdot)\|_2^2 \le{}& \|u(t_*, \cdot) - u_s(t_*, \cdot)\|_2^2 e^{-2t} + \\ &+ 2e^{-2t}\int_{t_*}^t e^\tau \left[\int_0^{e^\tau} \{|\mu'(\tau)k(\eta)|^{1/2} - 1\}^2 d\eta\right] d\tau, \quad t_* < t < T. \end{aligned} \tag{26}$$

Hence, by Lemma 1,

$$\lim_{t\to T} \|\theta(t, \cdot) - \theta_s(\cdot)\|_2^2 \le \lim_{t\to T} \int_0^1 \left\{\left[\frac{k(u_1(t)\zeta)}{k(u_1(t))}\right]^{1/2} - 1\right\}^2 d\zeta = 0. \tag{27}$$

(To obtain (27) from (26), it is sufficient to take (25), and, taking (24) into account, transform back, $\mu(t) \to t$.) □

Under the conditions of Theorem 2, the a.s.s. (9) has the form

$$u_s(t, x) = \exp\{\mu^{-1}(t) - x\}.$$

and $u_x(t, x) \to \infty$ in $\mathbf{R}_+$ as $t \to T^-$. It follows from Theorem 2 that the solution $u(t, x)$ will have the same properties, i.e. in this problem there will be no heat localization and the boundary blow-up regime (24) is an HS-regime.

On the other hand, the boundary condition (24) leads to the appearance of a thermal wave with a constant (as $t \to T^-$) half-width $\phi(t) \equiv 1$. Condition (23) ensures that such boundary regimes belong in the class of blow-up regimes. In [187] it is shown that divergence of the integral

$$\int_0^\infty \frac{d\eta}{k(e^\eta)} = \infty$$

leads to solutions with constant half-width being generated by boundary regimes without blow-up, which are defined for all $t > 0$. In this case a.s.s. are defined by the same formula (9). This will be the case for $u_1(t) = \exp\{\mu^{-1}(t)\}$, where we have denoted by $\mu^{-1}$ the inverse of

$$\mu(t) = \int_0^t \frac{d\eta}{k(e^\eta)}, \quad t > 0.$$

Then, if $\phi \equiv 1$ and $\theta_s \equiv e^{-\xi}$ in (9), we have $\theta(t, \cdot) \to \theta_s(\cdot)$ in $L^2(\mathbf{R}_+)$ as $t \to \infty$, where $\theta(t, \xi)$ is the similarity representation of (10).

Therefore the inequality (23), which is equivalent to

$$\int_1^\infty \frac{d\eta}{\eta k(\eta)} < \infty, \tag{28}$$

is a necessary and sufficient condition for a thermal wave with constant penetration depth to be generated by a boundary blow-up regime. The precise form of this regime is determined by (24), (25).

Condition (23) (or, equivalently, (28)) is satisfied by coefficients $k$ in the fourth and fifth lines in Table 4, and, if $\alpha > 1$, also in the second and third lines.

To illustrate the possibilities of Theorem 2, let us consider

**Example 2.** Let $k(u) = \ln(1 + u)\ln^2\ln(3 + u)$. Then condition (23) is satisfied and it follows from (25) that

$$\mu(\tau) \simeq T - 1/\ln\tau, \quad \tau \to \infty;$$

$$\mu^{-1}(t) \simeq \exp\{(T - t)^{-1}\}, \quad t \to T^-.$$

Therefore the boundary blow-up regime

$$u_1(t) \simeq \exp\{\exp\{(T - t)^{-1}\}\}$$

generates a solution with constant (as $t \to T^-$) and non-zero half-width.

## 2 Approximate self-similar solutions of equations with nearly power law coefficients

In this subsection we consider equations (1) with coefficients $k$, which satisfy the condition

$$(k/k')'(u) \to 1/\sigma, \quad u \to \infty; \quad \sigma = \text{const} > 0 \tag{29}$$

(for $\sigma = 0^+$ (29) coincides with (5)). We shall also assume that the following conditions are satisfied:

$$\frac{k(\xi u)}{\xi^\sigma} \text{ is non-increasing in } \xi \in (0,1) \text{ for any } u > 0, \tag{30}$$

$$\frac{k(\xi u)}{\xi^\sigma k(u)} \to 1 \text{ as } u \to \infty \text{ for any } \xi \in (0,1). \tag{30'}$$

All the above requirements are fulfilled, for example, by the coefficients $k(u) = u^\sigma \ln^\alpha(1+u)$, $\alpha > 0$; $k(u) = u^\sigma \exp\{\ln^\alpha(1+u)\}$, $0 < \alpha < 1$; $k(u) = u^{\sigma + 1/\ln\ln(8+u)}$.

In this case the a.s.s. of the problem (1)–(3) are constructed using two types of invariant solutions of the equation with a power law nonlinearity,

$$u_t = (u^\sigma u_x)_x \tag{31}$$

(the constant $\sigma > 0$ here is the same as in condition (29)), which have the form

$$\text{A.} \quad u_S(t,x) = (T-t)^n g_1(\zeta), \quad \zeta = x/(T-t)^{(1+n\sigma)/2},$$
$$t \in (0,T); \quad n < 0; \tag{32}$$

$$\text{B.} \quad u_S(t,x) = e^t g_2(\zeta), \quad \zeta = x/\exp\{\sigma t/2\}, \quad t > 0. \tag{33}$$

The functions $g_1$, $g_2$, which satisfy the boundary conditions $g_i(0) = 1$, $g_i(\infty) = 0$, $i = 1, 2$, are determined from ordinary differential equations obtained by substituting the expressions for $u_S$ into (31) (see § 3).

In each of the cases under consideration, we shall seek a.s.s. in the form (9), and will denote by $\theta(t,\xi)$ the corresponding similarity representation (10) of the solution $u(t,x)$ of the original problem.

We shall establish convergence of $\theta(t,\cdot)$ to $\theta_s(\cdot)$ in the norm of the space $h^{-1}(\mathbf{R}_+)$. All the functions $u$ in $L^1(\mathbf{R}_+)$, which satisfy the conditions

$$\int_x^\infty u(y)dy \in L^2(\mathbf{R}_+), \quad \left|\int_0^\infty dx \int_x^\infty u(y)dy\right| < \infty,$$

belong in the Hilbert space $h^{-1}(\mathbf{R}_+)$.

The scalar product in $h^{-1}(\mathbf{R}_+)$ has the form

$$(u,v)_{h^{-1}(\mathbf{R}_+)} = \int_0^\infty u(x)\left[\left(-\frac{\partial^2}{\partial x^2}\right)^{-1} v\right](x)dx,$$

where we have denoted by $V = (-\partial^2/\partial x^2)^{-1} v$ the solution of the problem

$$\partial^2 V/\partial x^2 = -v, \ x > 0,$$

which satisfies

$$V(0) = 0, \ |V(\infty)| < \infty$$

(it is easily checked that under the assumptions we have made on functions in $h^{-1}(\mathbf{R}_+)$, a solution of this problem exists). We denote by $\|u\|_{-1}$ the norm in $h^{-1}(\mathbf{R}_+)$:

$$\|u\|_{-1} = (u, u)^{1/2}_{h^{-1}(\mathbf{R}_+)}.$$

It is not hard to see that

$$\|u\|_{-1} = \left\| \left( -\frac{\partial}{\partial x} \right)^{-1} u \right\|_2 \equiv \left\| \int_x^\infty u(y)\,dy \right\|_2 .$$

Below we shall assume that $u_0 \in h^{-1}(\mathbf{R}_+)$, $u(t,\cdot) \in h^{-1}(\mathbf{R}_+)$ for all $t \in (0, T)$. Note that the second condition holds for a generalized solution with compact support $u \in C^{1,2}_{t,x}(P_T|u|)$,

$$P_T|u| = \{t \in (0, T),\ x \in \mathbf{R}_+ \mid u(t, x) > 0\},$$

with a continuous derivative $k(u)u_x$.

The following easily verified assertion will be used in the sequel.

**Lemma 2.** *Let a function $k \in C^2((0, \infty))$ satisfy condition (29), and let $\alpha \in (0, \sigma]$ be an arbitrary constant. Then for all sufficiently large $u > 0$ we have the inequalities*

$$u^{\sigma - \alpha} < k(u) < u^{\sigma + \alpha}, \quad u^{\sigma - 1 - \alpha} < k'(u) < u^{\sigma - 1 + \alpha}. \tag{34}$$

*1 Approximate self-similar solutions of type A*

**Theorem 3.** *Assume that conditions (29), (30), (30′) hold, and that for all $t$ sufficiently close to $T^-$,*

$$u_1(t) = (T - \mu^{-1}(t))^n, \tag{35}$$

*where $n < -3/(3\sigma + 4)$ is a constant and $\mu^{-1} : [0, T) \to [0, T)$ is the function inverse to*

$$\mu(t) = T - \int_t^T \frac{(T - \tau)^{n\sigma}\,d\tau}{k\{(T - \tau)^n\}} \to T^-, \ t \to T^- . \tag{35′}$$

*Then the problem (1)–(3) has the a.s.s. (9), where*

$$\phi(t) = (T - \mu^{-1}(t))^{(1 + n\sigma)/2}, \ t \to T^- , \tag{36}$$

*the function $\theta_s(\xi)$ is the same as $g_1(\xi)$ in (32) and*

$$\lim_{t \to T^-} \|\theta(t, \cdot) - \theta_s(\cdot)\|_{-1} = 0. \tag{37}$$

Monotonicity and unboundedness as $t \to T^-$ of the function $u_1(t)$ in (35) follow from Lemma 2.

The restriction $n < -3/(3\sigma + 4)$ has to do with the condition $\theta_s \equiv g_1 \in h^{-1}(\mathbf{R}_+)$. Since (see § 2, Ch. III)

$$g_1(\xi) \simeq C\xi^{2n/(1+n\sigma)}, \ \xi \to \infty; \ C > 0,$$

this inclusion does not hold for $n \in ]-3/(3\sigma + 4), 0)$.

*Proof of Theorem* 3. Let us change the variable $t$ to $\mu(t)$, where the smooth, monotone function $\mu$ is defined by (15). Then the function $u(\mu(t), x)$ satisfies equation (1′) and the boundary condition (15′). By (15), (35), (36) we have

$$\phi(\mu(t)) = (T - t)^{(1+n\sigma)/2}, \ t \to T^-,$$

and, as is easily seen, for all $t$ sufficiently close to $T^-$

$$u_s(\mu(t), x) = (T - t)^n \theta_s(\xi), \ \xi = x/(T - t)^{(1+n\sigma)/2},$$

$$\theta(\mu(t), \xi) = (T - t)^{-n} u(\mu(t), \ \xi(T - t)^{(1+n\sigma)/2)}).$$

By the equality $\theta_s \equiv g_1$, as $t \to T^-$, the a.s.s. $u_s(\mu(t), x)$ becomes exactly (32), the self-similar solution of equation (31).

The function $w(t, x) = u(\mu(t), x) - u_s(\mu(t), x)$ satisfies the equation

$$w_t = [\mu'(t)k(u)u_x - u_s^\sigma(u_s)_x]_x, \tag{38}$$

and $w(t, 0) = 0$, $w(t, x) \to 0$ as $x \to \infty$, $w(t, \cdot) \in h^{-1}(\mathbf{R}_+)$ as $t \to T^-$. Taking the scalar product of (38) with $(-\partial^2/\partial x^2)^{-1} w$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t, \cdot)\|^2_{-1} = \left(\mu'(t)[F(u)]_{xx} - \frac{1}{\sigma + 1}(u_s^{\sigma+1})_{xx}, \left(-\frac{\partial^2}{\partial x^2}\right)^{-1} w\right), \tag{39}$$

where

$$F(u) = \int_0^u k(\eta)d\eta, \ u \geq 0.$$

We rewrite the right-hand side of (39) in the equivalent form

$$\begin{aligned} &\mu'(t)\left([F(u) - F(u_s)]_{xx}, \left(-\frac{\partial^2}{\partial x^2}\right)^{-1} w\right) + \\ &+ \left(\mu'(t)[F(u_s)]_{xx} - \frac{1}{\sigma + 1}(u_s^{\sigma+1})_{xx}, \left(-\frac{\partial^2}{\partial x^2}\right)^{-1} w\right) = \\ &= -\mu'(t)(F(u) - F(u_s), u - u_s) + \\ &+ \left((u_s)_x[\mu'(t)k(u_s) - u_s^\sigma], \left(\frac{\partial}{\partial x}\right)^{-1} w\right). \end{aligned} \tag{40}$$

Using now the fact that $(F(u)-F(u_s), u-u_s) \geq 0$ and appealing to the Cauchy-Schwarz inequality to estimate the last term in (40), we obtain from (39)

$$\frac{1}{2}\frac{d}{dt}\|w\|_{-1}^2 \leq \|w\|_{-1}\,\big\|(u_s)_x[\mu'(t)k(u_s)-u_s^\sigma]\big\|_2 .$$

Hence

$$\frac{d}{dt}\|w\|_{-1} \leq \|(u_s)_x[\mu'(t)k(u_s)-u_s^\sigma]\|_2. \tag{41}$$

Let us estimate the right-hand side of this inequality:

$$\|(u_s)_x[\mu'(t)k(u_s)-u_s^\sigma]\|_2 =$$

$$= \left\{\int_0^\infty (u_s)_x^2|\mu'(t)k(u_s)-u_s^\sigma|^2dx\right\}^{1/2} = \left\{\int_0^\infty u_s^\sigma (u_s)_x[\Phi(u_s;\,t)]_x dx\right\}^{1/2}.$$

Here we have denoted by $\Phi(s;\,t)$ the function

$$\Phi(s;\,t) = \int_0^s \left[\mu'(t)\frac{k(\eta)}{\eta^\sigma}-1\right]^2 \eta^\sigma\,d\eta,\ s>0,\ t\in(0,T).$$

Since the function $u_s$ is monotone in $x$, we have finally

$$\|(u_s)_x[\mu'(t)k(u_s)-u_s^\sigma]\|_2 \leq$$

$$\leq \left\{-\sup_x |u_s^\sigma(u_s)_x|\int_0^\infty |\Phi(u_s;\,t)|_x\,dx\right\}^{1/2} =$$

$$= q_s^{1/2}(T-t)^{[n(\sigma+2)-1]/4}\Phi^{1/2}[(T-t)^n;\,t],\ t\to T^-,$$

where $q_s = \max|\theta_s^\sigma\,d\theta_s/d\xi| < \infty$. Then we obtain from (41) the following estimate, which is valid for all $t\in(t_*,T)$ which are sufficiently close to $T^-$:

$$\|w(t,\cdot)\|_{-1} \leq \|w(t_*,\cdot)\|_{-1} +$$

$$+ q_s^{1/2}\int_{t_*}^t (T-t)^{[n(\sigma+2)-1]/4}\Phi^{1/2}((T-\tau)^n;\,\tau)\,d\tau.$$

Since

$$\|w(t,\cdot)\|_{-1} = (T-t)^{n+3(1+n\sigma)/4}\|\theta(\mu(t),\cdot)-\theta_s(\cdot)\|_{-1},$$

from that inequality we derive the estimate

$$\|\theta(\mu(t),\cdot)-\theta_s(\cdot)\|_{-1} \leq \|w(t_*,\cdot)\|_{-1}(T-t)^{-\frac{3\sigma+4}{4}\left[n+\frac{3}{3\sigma+4}\right]} +$$

$$+ q_s^{1/2}(T-t)^{-n-\frac{3}{4}(1+n\sigma)}\int_{t_*}^t (T-\tau)^{\frac{n(\sigma+2)-1}{4}}\Phi^{1/2}((T-\tau)^n;\,\tau)d\tau.$$

Hence, since $n < -3/(3\sigma+4)$, resolving the indeterminacy in the right-hand side we obtain

$$\lim_{t\to T^-} \|\theta-\theta_s\|_{-1}^2 \le \frac{q_s}{|n+3(1+n\sigma)/4|^2}\lim_{t\to T^-}\frac{\Phi((T-t)^n;\, t)}{(T-t)^{n(\sigma+1)}}.$$

From (35′) it follows that

$$\mu'(t) = \frac{(T-t)^{n\sigma}}{k|(T-t)^n|},\quad t\to T^- .$$

and therefore

$$\lim_{t\to T^-} \|\theta-\theta_s\|_{-1}^2 \le \frac{q_s}{|n+3(1+n\sigma)/4|^2}\lim_{t\to T^-}\int_0^1 \left\{\frac{k(\zeta(T-t)^n)}{\zeta^\sigma k((T-t)^n)}-1\right\}^2 \zeta^\sigma\, d\zeta.$$

By (30), the integrand is bounded uniformly in $\zeta\in(0,1)$ as $t\to T^-$. Then (37) follows from (30′). □

**Remark.** In § 3 we used a different method to construct a.s.s. of equation (1) for coefficients $k$ and boundary regimes covered by Theorem 3.

It is not hard to derive from (35), (36) sharp estimates for the spatio-temporal structure of a.s.s. Let $k(u) = u^\sigma\kappa(u)$, where the function $\kappa(u) > 0$ grows slower than any power as $u\to\infty$. Then if the condition

$$\lim_{s\to\infty}\frac{\kappa(s)}{\kappa(s/\kappa^n(s))} = 1$$

holds, the functions $u_1(t)$, $\phi(t)$ in (35), (36) admit the asymptotic estimates

$$u_1(t) \simeq \{(T-t)^{(1+n\sigma)}k|(T-t)^n|\}^n,$$

$$\phi(t) \simeq \{(T-t)^{(1+n\sigma)}k|(T-t)^n|\}^{(1+n\sigma)/2},\quad t\to T^- .$$

Let us use the above theorem to study the heat localization phenomenon. The spatio-temporal structure of the a.s.s. (9), defined by (35), (36), indicates that the properties of the solution of the problem depend on the relation between the quantities $\sigma$ and $n$.

If $n < -1/\sigma$ in (35), then $\phi(t)\to\infty$, $t\to T^-$, and therefore the solution grows without bound as $t\to T^-$ everywhere in the half-space $\{x>0\}$ (HS-regime; no localization).

If $n > -1/\sigma$, then $\phi(t)\to 0$ as $t\to T^-$ (LS-regime), which indicates that the process is localized.

The value $n = -1/\sigma$ corresponds to the limiting localized a.s.s. (S-regime); in view of the fact that $\phi\equiv 1$, as $t\to T^-$ it has constant half-width, which is different from zero. Substituting $n = -1/\sigma$, we obtain for the boundary S-regime the expression

$$u_1(t) \simeq \{(T-t)^2 k[(T-t)^{-1/\sigma}]\}^{-1/\sigma}, \quad t \to T^-,$$

or, which is the same,

$$u_1(t) \simeq k^{-1}[(T-t)^{-1}], \quad t \to T^-,$$

where we denote by $k^{-1}$ the function inverse to $k$. Asymptotic equivalence of these two expressions follows from condition (29). In this case the a.s.s. has the relatively simple form

$$u_s(t,x) = u_1(t)\left(1 - \frac{x}{x_0}\right)_+^{2/\sigma}, \quad x_0 = \left[\frac{2(\sigma+2)}{\sigma}\right]^{1/2}.$$

**Example 3.** Let $k(u) = u^\sigma \ln(2+u)$. Then from (35) we obtain that the limiting localized S-regime in this case is

$$u_1(t) \simeq \{\sigma(T-t)^{-1}|\ln(T-t)|^{-1}\}^{1/\sigma}, \quad t \to T^-,$$

which generates a solution with half-width which becomes constant as $t \to T^-$.

*2 Approximate self-similar solutions of type B*

**Theorem 4.** *Assume that conditions* (29), (30), (30′) *hold, and that, in addition,*

$$\int_0^\infty \frac{e^{\sigma\eta}}{k(e^\eta)}\, d\eta < \infty. \tag{42}$$

*Suppose that for all $t$ sufficiently close to $T$*

$$u_1(t) = \exp\{\mu^{-1}(t)\} \tag{43}$$

*where $\mu^{-1} : (0,T) \to \mathbf{R}_+$ is the inverse of the monotone increasing function $\mu(\tau) : \mathbf{R}_+ \to (0,T)$, which for sufficiently large $\tau > 0$ is determined by the formula*

$$\mu(\tau) = T - \int_\tau^\infty \frac{e^{\sigma\eta}}{k(e^\eta)}\, d\eta. \tag{44}$$

*Then*

$$\phi(t) = \exp\left\{\frac{\sigma}{2}\mu^{-1}(t)\right\}, \quad t \to T^-, \tag{45}$$

*$\theta_s(\xi) \equiv g_2(\xi)$ and the equality* (37) *holds.*

In the course of the proof of this theorem, which is similar to the proof of the previous one, we establish the following estimate:

$$\|\theta(\mu(t),\cdot)-\theta_s(\cdot)\|_{-1}\le\|w(t_*,\cdot)\|_{-1}\exp\left\{-t\left(1+\frac{3\sigma}{4}\right)\right\}+$$
$$+q_s^{1/2}\exp\left\{-t\left(1+\frac{3\sigma}{4}\right)\right\}\int_{t_*}^{t}\exp\left\{\frac{\tau}{2}\left(1+\frac{\sigma}{2}\right)\right\}\times$$
$$\times\left\{\int_0^{e^\tau}\left[\mu'(\tau)\frac{k(\eta)}{\eta^\sigma}-1\right]^2\eta^\sigma\,d\eta\right\}^{1/2}d\tau,\ t\to\infty.$$

Hence by the equality

$$\mu'(t)=\frac{e^{\sigma t}}{k(e^t)},\quad t\to\infty,$$

and assumptions (30), (30′) we have the desired result

$$\lim_{t\to T}\|\theta(t,\cdot)-\theta_s(\cdot)\|_{-1}^2\le$$
$$\le\frac{q_s}{(1+3\sigma/4)^2}\lim_{t\to T}\int_0^1\left[\frac{k(u_1(t)\zeta)}{\zeta^\sigma k(u_1(t))}-1\right]^2\zeta^\sigma\,d\zeta=0.$$

**Remark.** If the condition contrary to (42) holds:

$$\int_0^\infty\frac{e^{\sigma\eta}}{k(e^\eta)}\,d\eta=\infty,$$

the solution obtained from a self-similar type B solution by the "transformation" $u^\sigma\to k(u)$, is defined for all $t>0$, that is, it does not blow up in finite time [187].

From Theorem 4 it follows immediately that under the assumptions we have made, the boundary regime (43) leads to absence of localization, i.e., it is an HS-regime.

**Example 4.** Let us consider the coefficient $k(u)=u^\sigma\ln^2(1+u)$. Condition (42) is satisfied, and therefore we deduce from (44) that

$$\mu^{-1}(t)\simeq(T-t)^{-1},\ t\to T^-.$$

Hence the boundary regime

$$u_1(t)\simeq\exp\{(T-t)^{-1}\},\ t\to T^-,$$

generates a thermal wave that moves according to

$$x_{ef}(t)\simeq\xi^*\phi(t)=\xi^*\exp\left\{\frac{\sigma}{2}(T-t)^{-1}\right\},\ t\to T^-;\ g_2(\xi^*)=\frac{1}{2}.$$

## Remarks and comments on the literature

Our exposition of the results of § 2 follows mainly [119, 185]. First studies of the heat equation $u_t = u_{xx}$ using degenerate a.s.s. are reported in [149, 347, 348] (see § 4, Ch. III). Applications of this theory to the study of heat localization (Theorem 2, Section 4) are contained in [154]. The theorems of § 3 are proved in [184]; partial results were obtained earlier in [119]. Results of § 4 are given in [187, 119].

It must be said that at present there are few examples of really non-trivial a.s.s. of nonlinear heat equations. In this regard, let us mention [234], where a.s.s. of the Cauchy problem for $u_t = (k(u)u_x)_x$, $t > 0$, $x \in \mathbf{R}$, with coefficient not of power type, are constructed. A related result has been established in [187] by a different method; an estimate of the rate of convergence to a.s.s. was also derived there (this estimate could not be obtained in the framework of the methods of [234]).

## Open problems

1. (§ 2) Prove that localization of degenerate a.s.s. (10) for $n \geq -1$ implies localization of solutions of the original problem (1)–(3′) (for the case $k \equiv 1$ this is proved in [149, 347, 348]; see § 4, Ch. III).
2. (§ 3) Prove that under the conditions of Theorem 1 for $n \geq -1/\sigma$ solutions of the problem (1)–(3′) are localized (for $n < -1/\sigma$ a.s.s. are not localized, which implies that there is no localization in the original problem).

# Chapter VII

# Some other methods of study of unbounded solutions

In this chapter we conduct a study of unbounded solutions of various nonlinear parabolic problems. In § 1 we study the character of the asymptotics of unbounded solutions of a quasilinear parabolic equation with a source close to the blow-up time. We obtain a nearly optimal condition for the absence of localization.

In § 2 we investigate boundary value problems in bounded domains.

In § 3, 4 we consider parabolic systems of quasilinear equations which admit unbounded solutions.

Most results are obtained using the same approach, *the method of stationary states*. It is based on an analysis of a special family of stationary solutions, which satisfy the equation or system of equations almost everywhere (this is why it is convenient to call them states, thus emphasizing the fact that they are not stationary solutions in the usual sense).

We find that the family of stationary states contains in a certain parametrized form several important properties of the evolution of the non-stationary problem. We stress that the method is applicable to problems with nonlinearities of a sufficiently general form, when the problem admits no appropriate stable similarity or invariant solutions.

In § 5 we study for the most part a nonlinear (implicit) difference scheme for the equation $u_t = (u^{\sigma+1})_{xx} + u^{\beta}$. The most interesting case here is $\beta \geq \sigma + 1$, when the discretized problem admits unbounded solutions.

## § 1 Method of stationary states for quasilinear parabolic equations

This section is entirely devoted to the study of the phenomenon of localization of unbounded solutions of the Cauchy problem for a parabolic equation with a source of general form:

$$u_t = \nabla \cdot (k(u)\nabla u) + Q(u), \ t > 0, x \in \mathbf{R}^N, \tag{1}$$

$$u(0,x) = u_0(x) \geq 0, \ x \in \mathbf{R}^N, \tag{2}$$

where, as usual, $k \geq 0$, $Q \geq 0$ are known sufficiently smooth functions, $\nabla(\cdot) = \operatorname{grad}_x(\cdot)$. The main question we are considering here is to find conditions on the coefficients $k$, $Q$, under which the solutions of the problem are not localized.

An unbounded solution of the problem (1), (2) is said to be *localized*, if it grows to infinity as $t \to T_0^- < \infty$ on a bounded set in $\mathbf{R}^N$, that is the *localization domain*

$$\omega_L = \left\{ x \in \mathbf{R}^N \mid u(T_0^-, x) \equiv \overline{\lim_{t \to T_0}} \ u(t,x) = \infty \right\} \tag{3}$$

is bounded. The function $u(T_0^-, x)$ is called the *limiting distribution* (l.d.) of the solution. If, on the other hand, $\omega_L$ is unbounded (for example, $\omega_L = \mathbf{R}^N$) we say that *there is no localization in the problem*.

Earlier, in Ch. IV, we studied the localization phenomenon in detail using as an example equations with power type nonlinearities,

$$u_t = \nabla \cdot (u^\sigma \nabla u) + u^\beta, \ t > 0, x \in \mathbf{R}^N; \ \sigma > 0, \beta > 1, \tag{4}$$

by analyzing in an appropriate way their self-similar solutions. It was shown that for $\beta < \sigma + 1$ there is no localization, while for $\beta \geq \sigma + 1$ all unbounded solutions are localized. Equation (1) of general form does not admit such self-similar solutions, and therefore the comparison methods developed in Ch. IV are not applicable here.

In this section we propose an approach to determining sufficient conditions for the absence of localization in the case of equation (1) with arbitrary (not power type) coefficients $k$, $Q$. We also study the structure of l.d. of unbounded solutions. This method encompasses also the case $T_0 = \infty$, when $\sup_x u(t,x) \to \infty$ as $t \to \infty$. Therefore we do not pay any special attention to the condition $\int^\infty d\eta/Q(\eta) < \infty$, which, as is well known, is a necessary condition for existence of unbounded solutions (see § 2, Ch. I).

The method we employ is based on the construction of a one-parameter family $\{U\}$ of stationary solutions of equation (1):

$$\nabla \cdot (k(U)\nabla U) + Q(U) = 0. \tag{5}$$

In essence, we shall show that the family $\{U\}$ contains information about several properties of the evolution of solutions of the non-stationary equation (1), which are parametrized in a special form. Actually this assertion is quite natural, since the main part of equation (1) is exactly the stationary operator, which contains all the nonlinear terms responsible for the evolution of a solution.

In § 3 (and partially in § 4) we use this method to analyse a parabolic system of quasilinear equations, and, while studying localization of unbounded solutions of the Cauchy problem, we at the same time determine conditions for global solvability of the problem in a bounded domain. The concepts we present make it possible to give a general formulation of the method of stationary states for the study of nonlinear parabolic problems satisfying the Maximum Principle.

Below we shall take the function $Q(u)$ to be monotone increasing and we shall also assume that $\phi(u) \to \infty$ as $u \to \infty$, where $\phi(u) \equiv \int_0^u k(\eta)\, d\eta$. We shall denote by $\phi^{-1}$ the function inverse to $\phi$. As far as the solution of problem (1), (2) is concerned, we shall assume that it exists, is unique and for all $0 < t < T_0 = T_0(u_0) \le \infty$ belongs to $C_{t,x}^{1,2}$ wherever $u > 0$, and that in addition $k(u)\nabla u$ is continuous in $x$ in $\mathbf{R}^N$ for all $t \in (0, T_0)$.

## 1 Construction of the family of stationary solutions

We shall be interested in the properties of bounded radially symmetric solutions $U$ of equation (5). Each of those, at points of positivity, satisfies the problem

$$\frac{1}{r^{N-1}}\left(r^{N-1}(\phi(U))'\right)' + Q(U) = 0, \quad r = |x| > 0, \tag{6}$$

$$U(0; U_0) = U_0, \quad U_r'(0; U_0) = 0; \tag{7}$$

while at all other points we assume for convenience $U \equiv U(|x|; U_0) = 0$. Here $U_0 > 0$ is an arbitrary constant (the parameter in the family $\{U(|x|; U_0)\}$).

Local solvability of the problem (6), (7) for small $r > 0$ follows from the analysis of the equivalent integral equation

$$\phi(U(r; U_0)) = \phi(U_0) - \int_0^r \xi^{1-N}\, d\xi \int_0^{\xi} \eta^{N-1} Q(U(\eta; U_0))\, d\eta, \quad r > 0. \tag{8}$$

Hence we have that the solution can be extended in $r > 0$ and is strictly monotone in $r$ in the domain $\{r > 0 \mid U(r) > 0\}$.

Let us first derive a lower bound for $U$. Clearly, $(r^{N-1}(\phi(U))')' \ge -Q(U_0) r^{N-1}$ for $r > 0$. Integrating this inequality twice over $(0, r)$ we obtain the following estimate:

$$\begin{gathered} U(r; U_0) \ge U_-(r; U_0) = \phi^{-1}\left[\phi(U_0)\left(1 - \frac{r^2}{r_0^2}\right)_+\right], \\ r_0 = \left[2N\frac{\phi(U_0)}{Q(U_0)}\right]^{1/2}, \quad r \ge 0. \end{gathered} \tag{9}$$

To derive an upper bound for $U$, let us use the relation

$$r^{N-1}(\phi(U))' = -\int_0^r \eta^{N-1} Q(U)\, d\eta \le -\frac{r^N}{N} Q(U), \quad r > 0.$$

Integrating it from 0 to $r$, we obtain the following estimate:

$$U(r; U_0) \le G^{-1}\left(\frac{r^2}{2N} + G(U_0)\right) < G^{-1}\left(\frac{r^2}{2N}\right), \quad r > 0, \tag{10}$$

where $G^{-1}$ is the function inverse to

$$G(u) = \int_u^\infty \frac{k(\eta)\, d\eta}{Q(\eta)}, \quad u > 0.$$

In the particular case $Q(u) = \mu\phi(u)$, $\mu = \text{const} > 0$, the solution $U$ can be written down in explicit form:

$$U(r; U_0) = \phi^{-1}[c_N(U_0) r^{(2-N)/2} J_{(N-2)/2}(\mu^{1/2} r)] \tag{11}$$

for $0 < r < z_N^{(1)} \mu^{-1/2}$, where $z_N^{(1)} > 0$ is the first root of the Bessel function $J_{(N-2)/2}$ and $c_N(U_0) = \phi(U_0)\Gamma(N/2)(2\mu^{-1/2})^{(N-2)/2}$.

The main properties of stationary solutions of equation (4) with power nonlinearities,

$$\frac{1}{r^{N-1}}\left(r^{N-1} U^\sigma U'\right)' + U^\beta = 0, \quad r > 0, \tag{12}$$

were discussed in § 3, Ch. IV. Let us emphasize that they crucially depend on the relation among the parameters $\sigma$, $\beta$, $N$. In particular, for $N \ge 3$, $\beta \ge (\sigma + 1)(N+2)/(N-2)$ the problem (12), (7) has strictly positive solutions in $\mathbf{R}^N$ (in the other cases the solutions have compact support). For $k$, $Q$ not of power type it is also possible to have solutions of the problem (6), (7) that are defined and strictly positive on the whole of $\mathbf{R}^N$. This, for example, occurs if (see [332])

$$\frac{N-2}{2N} u Q(\phi^{-1}(u)) > \int_0^u Q(\phi^{-1}(\eta))\, d\eta, \quad u > 0; \ N > 2 \tag{12'}$$

(the proof of this fact is the same as in the case of power type coefficients; see § 3, Ch. IV).

We shall also need conditions on $k$, $Q$, under which the functions $U(|x|; U_0)$ have compact support in $\mathbf{R}^N$ (that is, they are not stationary solutions in $\mathbf{R}^N$). As in § 3, Ch. IV, it can be shown that for $N = 1$ or $N = 2$ the functions $U(|x|; U_0)$ have compact support, while in the case $N \ge 3$, under the assumption of non-negativity of $U$ we derive the estimate from below

$$U(r; U_0) \ge \phi^{-1}(c/r^{N-2}), \quad r > 1, \ c = \text{const} > 0. \tag{9'}$$

Comparing (9′) with (10) for large $r > 0$ gives us the following sufficient condition for an arbitrary function $U(|x|; U_0)$ to be of compact support:

$$N = 1, 2 \ \text{ or } \lim_{s \to 0} \frac{\phi^{N/(N-2)}(s)}{Q(s)} = 0 \text{ for } N \ge 3. \tag{12''}$$

For equation (12) this criterion imposes the restriction $\beta < (\sigma+1)N/(N-2)_+$ and is practically a condition for non-existence of solutions $U > 0$ in a neighbourhood of $r = \infty$.

In the one-dimensional case equation (6) can be integrated in quadratures, and the solution of the problem can be determined from the relation

$$r = \frac{1}{\sqrt{2}} \int_{U(r,U_0)}^{U_0} \frac{k(\eta)\, d\eta}{\left\{ \int_\eta^{U_0} k(\xi) Q(\xi)\, d\xi \right\}^{1/2}}. \tag{13}$$

It is strictly positive for $0 \le r < x_0(U_0)$, where

$$x_0(U_0) = \frac{1}{\sqrt{2}} \int_0^{U_0} \frac{k(\eta)\, d\eta}{\left\{ \int_\eta^{U_0} k(\xi) Q(\xi)\, d\xi \right\}^{1/2}}.$$

Below we shall assume that the following conditions are imposed on the initial function: $u_0 = u_0(|x|)$ is a function with compact support, $\phi(u_0)$ is uniformly Lipschitz continuous in $\mathbf{R}^N$, $u_0(r_1) \le u_0(r_2)$ for all $0 \le r_2 \le r_1 < \infty$. Then the solution $u(t,x)$ is radially symmetric and by the Maximum Principle does not grow in $|x|$ for all $t \in (0, T_0)$. Therefore $\sup_x u(t,x) \equiv u(t,0)$.

## 2 Sufficient conditions for the absence of localization

**Theorem 1.** *Let the solution of the problem* (1), (2) *be unbounded and*

$$\overline{\lim_{s\to\infty}} |\phi(s)/Q(s)| = \infty. \tag{14}$$

*Then $u(t,x)$ is not localized and $u(t,x) \to \infty$ as $t \to T_0$ everywhere in $\mathbf{R}^N$.*

Thus, if (14) holds, the Cauchy problem exhibits the HS blow-up regime.

**Remark 1.** Condition (14) will necessarily hold if

$$k(s)/Q'(s) \to \infty, \ s \to \infty. \tag{15}$$

**Remark 2.** For the one-dimensional ($N = 1$) case a sufficient condition for the absence of localization can be formulated as follows:

$$\overline{\lim} \int_0^s \frac{k(\eta)\, d\eta}{\left\{ \int_\eta^s k(\xi) Q(\xi)\, d\xi \right\}^{1/2}} = \infty, \ s \to \infty. \tag{16}$$

**Remark 3.** Applied to equation (4), condition (14) (or (16)) takes the form $s^{\sigma+1-\beta} \to \infty$ as $s \to \infty$. Therefore in the case $\beta < \sigma + 1$ unbounded solutions of the Cauchy problem are not localized (for $N = 1$ this result was obtained in § 4, Ch. IV).

To prove the theorem we shall need the following Lemma

**Lemma 1.** *Let there exist $U_0^* > 0$, such that under the conditions of Theorem 1 we have the inequality*

$$u_0(x) \le U(|x|; U_0^*), \quad x \in \mathbf{R}^N. \tag{17}$$

*Then there exists $t_0 = t_0(U_0^*) \in [0, T_0)$, such that for all $t \in (t_0, T_0)$*

$$u(t, x) \ge U(|x|; U_0^*) \text{ in } \mathbf{R}^N. \tag{18}$$

*Proof.* We shall apply the comparison theorem of § 4, Ch. IV (on the "non-increase" of number of spatial intersections of solutions of a parabolic equation). First of all let us note that from (17) it follows, in particular, that $U$ is a function with compact support. In the opposite case, if $U$ is a stationary solution in $\mathbf{R}^N$, by the Maximum Principle $u \le U$ in $\mathbf{R}_+ \times \mathbf{R}^N$, that is, $u(t, x)$ is necessarily bounded.

Thus, $\operatorname{supp} U \subset \mathbf{R}^N$ is a bounded domain. By (17) the number of spatial intersections (in $r$) of $u_0(r)$ and $U(r; U_0^*)$ in $\omega(U_0^*) = \operatorname{supp} U(r; U_0^*)$ is zero. Let $N(t)$ be the number of intersections in $r = |x|$ in $\omega(U_0^*)$ of two different solutions $u(t, r)$ and $U(r; U_0^*)$ of equation (1). By the comparison theorem (see § 4, Ch. IV) $N(t)$ does not exceed the number of changes of sign of the difference $w = u - U$ on the parabolic boundary of the domain $((0, t) \times \omega(U_0^*)$. By assumption, $N(0) = 0$ ($w \le 0$ for $t = 0$ in $\bar{\omega}(U_0^*)$). If $u > 0$ in $(0, T_0) \times \mathbf{R}^N$, then $w > 0$ on $(0, t) \times \partial\omega(U_0^*)$, and therefore $N(t) \le 1$. If, on the other hand, (1) admits finite speed of propagation of perturbations, then by a known property of parabolic equations with a source $\operatorname{supp} u(t_1, r) \subseteq \operatorname{supp} u(t_2, r)$ for $t_1 \le t_2$, that is, $N(t) \le 1$ for all $t \in [0, T_0)$.

Obviously, there exists $t_* \in (0, T_0)$ such that $u(t_*, x) > 0$ on $\partial\omega(U_0^*)$. In the opposite case, if $u \equiv 0$ on $(0, T_0) \times \partial\omega(U_0^*)$, (17) would imply uniform boundedness of $u$ in $(0, T_0) \times \partial\omega(U_0^*)$: $u \le U$.

Then $u(t, x) > 0$ on $\partial\omega(U_0^*)$ for all $t \in (t_*, T_0)$. Let us choose now $t_0 \in [t_*, T_0)$, such that $u(t_0, 0) > U_0^*$ (this is always possible, since by assumption $\overline{\lim}\, u(t, 0) = \infty$, $t \to T_0$ ).

Let us show that $N(t_0) = 0$. Indeed, $u(t_0, r) - U(r; U_0^*) > 0$ for $x = 0$ and $x \in \partial\omega(U_0^*)$. Therefore in the interval $\omega(U_0^*)$ there can only be an even number of intersections of $u(t_0, r)$ and $U(r; U_0^*)$. However, we established earlier that $N(t_0) \le 1$. Therefore $N(t_0) = 0$, so that $u(t_0, r) \ge U(r; U_0^*)$ in $\mathbf{R}^N$. By the Maximum Principle this inequality will hold for all $t \in (t_0, T_0)$. □

*Proof of Theorem 1.* From (9) it follows that if (14) holds, $U(|x|; U_0) \to \infty$ in $\mathbf{R}^N$ for a sequence $U_0^k \to \infty$. Therefore we can make (17) hold for any compactly supported function $u_0(|x|)$. Then, passing in (18) to the limit $U_0^* = U_0^k \to \infty$, we conclude that for any $x \in \mathbf{R}^N$

$$\varliminf_{t \to T_0} u(t, x) \geq \lim_{U_0^k \to \infty} U(|x|; U_0^k) = \infty,$$

that is, the solution grows without bound as $t \to T_0^-$ at the same time everywhere in the whole space. □

From the method of proof of Theorem 1 we immediately derive the following corollaries (they show how the spatial structure of the family of stationary solutions $\{U\}$ describes the features of the evolution of the solutions of the non-stationary problem).

**Corollary 1.** *Let (1) describe processes with a finite speed of propagation of perturbations. Then under the conditions of Theorem 1 the diameter $D(t)$ of the support $\operatorname{supp}_x u(t, x)$ of the solutions satisfies the estimate*

$$D(t) \geq 2\sqrt{2N} \left[\frac{\phi(u(t,0))}{Q(u(t,0))}\right]^{1/2} \to \infty, \ t \to T_0^-.$$

*The half-width of the structure can be bounded from below by*

$$r_{ef}(t) \geq \sqrt{2N} \left\{\frac{\phi(u(t,0))}{Q(u(t,0))} \left[1 - \frac{\phi(u(t,0)/2)}{\phi(u(t,0))}\right]\right\}^{1/2}.$$

In the case of equation (4), $\beta < \sigma + 1$, this estimates have the form

$$D(t) \geq \frac{2\sqrt{2N}}{\sqrt{\sigma + 1}} |u(t,0)|^{(\sigma+1-\beta)/2} \to \infty, \ t \to T_0^-;$$

$$r_{ef}(t) \geq \frac{\sqrt{2N}}{\sqrt{\sigma + 1}} |u(t,0)|^{(\sigma+1-\beta)/2} (1 - 2^{-(\sigma+1)})^{1/2}$$

and agree well with self-similar behaviour (see § 1, Ch. IV).

**Corollary 2.** *Assume that under the conditions of Theorem 1 there exists $U_0^* > 0$, such that $u_0(|x|) \leq U(|x|; U_0)$ in $\mathbf{R}^N$ for all $U_0 \geq U_0^*$. Let $u(t_*, 0) = U_0^*$. Then $u_t(t, 0) \geq 0$ on $(t_*, T_0)$.*

*Proof.* Let us fix an arbitrary $U_0 \geq U_0^*$ and set $t_1 = \inf\{t \in (0, T_0) \mid u(t, x) > 0$ on $\partial\omega(U_0)\}$ and $t_0 = \inf\{t \in (0, T_0) \mid u(t, 0) > U_0\}$. By the Maximum Principle $t_1 \leq t_0$. Let us show that $u(t_0, x) \geq U(|x|; U_0)$ in $\mathbf{R}^N$. Indeed, if that is not so, there exists $t' > t_0$ such that $u(t', 0) > U_0$, $u(t', x) > 0$ on $\partial\omega(U_0)$, and $u(t', x)$ and $U(|x|; U_0)$ have at least two intersections in the interval $\omega(U_0)$. This contradicts the condition $N(t') \leq 1$ (see proof of Lemma 1).

Thus $u(t_0, x) \geq U(|x|; U_0)$ in $\mathbf{R}^N$. But then $u(t, x) \geq U(|x|; U_0)$ in $(t_0, T_0) \times \mathbf{R}^N$ and therefore $u_t(t_0, 0) \geq 0$. Since $U_0 \geq U$ was arbitrary, this proves the claim of the lemma. □

## 3 Some properties of localized solutions

Thus, a necessary condition for localization of unbounded solutions of the problem (1), (2) is the following:

$$\overline{\lim_{s\to\infty}}\ \phi(s)/Q(s) < \infty. \tag{19}$$

In this case the family $\{U\}$ of stationary states allows us to determine certain properties of the limiting temperature distribution $u(T_0^-, x)$.

All the results below rely on the following lemma, which is proved exactly like the previous one.

**Lemma 2.** *Let $u(t, x)$ be an unbounded solution of problem (1), (2), with initial function $u_0$ such that for all sufficiently large $U_0 > U_0^*$ the functions $u_0(|x|)$ and $U(|x|; U_0)$ intersect (in $|x|$) at most at one point. Then for all sufficiently small $r = |x| > 0$ we have the estimate*

$$u(T_0^-, x) \equiv \lim_{t\to T_0^-} u(t, x) \geq \sup_{U_0 > U_0^*} U(|x|; U_0). \tag{20}$$

Inequality (20) allows us to bound from below the size (diameter of $\omega_L$) of the localization domain (3) and to describe detailed behaviour of $u(T_0^-, x)$ in a neighbourhood of the singular point. In particular, from Lemma 2 and (11) we have

**Theorem 2.** *Let $Q(u) = \mu\phi(u)$, where $\mu = const > 0$. Then an unbounded solution of the problem (1), (2) cannot be localized in a ball of diameter less than*

$$D_* = 2z_N^{(1)}\mu^{-1/2}. \tag{21}$$

To prove the theorem, it suffices to check that the functions (11) satisfy conditions of Lemma 2. Therefore for the functions of (11) we obtain from (20)

$$u(T_0^-, x) \geq \sup_{U_0 > U_0^*} U(|x|; U_0) = \infty, \quad 0 \leq |x| < z_N^{(1)}\mu^{-1/2}.$$

As an example we quote a stronger result, which holds in the one-dimensional case.

**Theorem 2′.** *Assume that $N = 1$ and that the folowing conditions hold:*

$$\int_1^{\infty} Q(\eta)k(\eta)\,d\eta = \infty,$$
$$l_* = 2x_0(\infty) \equiv 2 \varlimsup_{s\to\infty} x_0(s) < \infty. \tag{21′}$$

*Then if $u(t,x)$ is unbounded, $\operatorname{meas}\omega_L \equiv \operatorname{meas}\{x \in \mathbf{R} \mid u(T_0^-, x) = \infty\} \geq l_*$.*

*Proof.* For $N = 1$ we obtain from (6), (7) that for any fixed $U_* \in (0, U_0)$ at points $x = x_*$ where $U(|x|; U_0) = U_*$, we have the equality

$$\left[\frac{d}{dx}\phi(U(|x_*|; U_0))\right]^2 = 2\int_{U_*}^{U_0} Q(\eta)k(\eta)\,d\eta \to \infty, \; U_0 \to \infty.$$

Therefore in view of uniform Lipschitz continuity of $\phi(u_0)$, for all sufficiently large $U_0$ the functions $u_0(|x|)$ and $U(|x|; U_0)$ intersect in $r = |x|$ in $\omega(U_0) = \operatorname{supp} U(|x|; U_0)$ at most in one point. Then, as in the proof of Lemma 1, we have that $u(T_0^-, |x|) \geq U(|x|; U_0)$ in $\omega(U_0)$ for all sufficiently large $U_0 > 0$, and, in particular, $\operatorname{meas}\omega_L \geq 2x_0(U_0)$. □

Of course this theorem is also true when $l_* = \infty$ in (21′).

Let us now consider the case

$$\sup_{U_0 > 0} U(|x|; U_0) < \infty, \; |x| > 0. \tag{22}$$

This condition indicates that the localized solution becomes infinite as $t \to T_0^-$ only at one point, that is, it exhibits the LS blow-up regime. Then Lemma 2 allows us to bound from below the asymptotics of behaviour of $u(T_0^-, x)$ as $|x| \to 0$ and to determine the rate of change of the half-width of the localized structure as $t \to T_0^-$.

Let us note that by (9) localization of the solution in LS-regime is possible when

$$\varlimsup_{s\to\infty} \frac{1}{Q(s)} \int_0^s k(\eta)d\eta = 0. \tag{23}$$

In the one-dimensional ($N = 1$) case the necessary condition for the occurrence of LS-regime has the following form:

$$\varlimsup_{s\to\infty} x_0(s) = 0. \tag{23′}$$

If condition (23) (or (23′)) holds, the family of curves $\{U = U(r; U_0)\}$ allows us to construct in the $\{r, U\}$ plane, for all sufficiently large $U_0$, monotone envelopes, tangent to the curves $U = U(r; U_0)$.

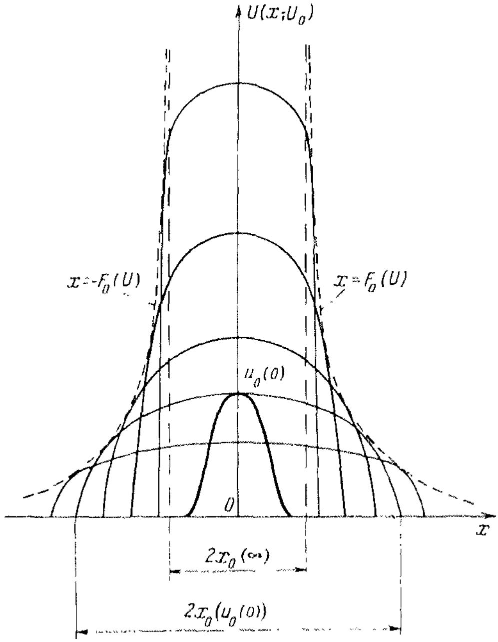


**Fig. 78.** Example of construction of the family of functions $\{U\}$ and envelopes in the S-regime $(0 < x_0(\infty) < \infty)$. Thick line denotes the initial function $u_0(x)$; $N = 1$.

Let us choose a maximal continuous branch $r = F_0(U)$ (see Figure 78). Two cases have to be considered.

1) The envelope $r = F_0(U) \geq 0$ is tangent to the curve $U(r; u_0(0))$ at some point $r^0 > 0$ (see Figure 78). In this case let us set

$$G(x) = \begin{cases} F_0^{-1}(|x|), & |x| < r^0, \\ U(|x|; u_0(0)), & |x| \geq r^0, \end{cases}$$

where $F_0^{-1}$ is the inverse function of $F_0$ (at points where $F_0^{-1}$ is not defined, we set $F_0^{-1} = \infty$).

2) The envelope $r = F_0(U) \geq 0$ has no points in common with the curve $U(r; u_0(0))$, the curve $U = F_0^{-1}(r)$ is defined for all $0 < r < r^*$ and is tangent at the point $r = r^*$ to some curve $U = U(r; U_0^*)$. Then we set

$$G(x) = \begin{cases} F_0^{-1}(|x|), & |x| < r^*, \\ U(|x|; U_0^*), & |x| \geq r^*. \end{cases}$$

From Lemma 2 we immediately obtain

**Theorem 3.** *Assume that condition (23) (or (23′)) is satisfied, and that the solution $u(t,x)$ is unbounded. For all $U_0 \geq U_0^* = u_0(0)$ let the functions $u_0(|x|)$ and $U(|x|; U_0)$ intersect (in $r = |x|$) at most at one point. Then*

$$u(T_0, x) \geq G(x), \quad x \in \mathbf{R}^N. \tag{24}$$

The estimate (24) makes it possible to determine the degree of singularity of the limiting distribution $u(T_0^-, x)$ in a neighbourhood of the point $x = 0$, where the solution $u(t,x)$ grows without bound as $t \to T_0^-$. Here it is convenient to use the estimate (9) for the stationary states $U(r; U_0)$, which we rewrite as

$$\phi(U(r; U_0)) \geq \phi(U_-(r; U_0)) \equiv \phi(U_0)(1 - r^2/r_0^2)_+,$$
$$r \geq 0; \quad r_0 = \left[2N \frac{\phi(U_0)}{Q(U_0)}\right]^{1/2}. \tag{25}$$

First let us present a simpler claim, which follows directly from the estimate (25) and Lemma 2. It has the following form: *under the conditions of Lemma 2 for all $t < T_0$ sufficiently close to the blow-up time $t = T_0$ we have the estimate*

$$u(t,x) \geq U_-(|x|; u(t,0)), \quad |x| \geq 0. \tag{26}$$

Proof of (26) proceeds as in Lemmas 1, 2; let us note that similar statements were frequently used in § 4-6, Ch. IV. From the last inequality we immediately deduce

**Theorem 4.** *Under the conditions of Theorem 3, we have the estimate*

$$\|\phi(u(t,\cdot))\|_{L^1(\mathbf{R}^N)} \geq$$
$$\geq \frac{4(2\pi N)^{N/2}}{N(N+2)\Gamma(N/2)} \frac{|\phi(u(t,0))|^{1+N/2}}{|Q(u(t,0))|^{N/2}}, \quad t \to T_0^-. \tag{27}$$

From this estimate we deduce, for example, the condition for $\|\phi(u(t,\cdot))\|_{L^1(\mathbf{R}^N)}$ to grow to infinity as $t \to T_0^-$:

$$\phi^{1+2/N}(s)/Q(s) \to \infty, \quad s \to \infty.$$

Of course, using (26) we can obtain other integral estimates of unbounded solutions. However, for general $\phi$, $Q$, they look too cumbersome. For some particular cases such estimates will be obtained below.

Let us show how to derive from (26) pointwise estimates of $u(T_0^-, x)$. Set $L(r) = \phi(G(r))$, where $G(r)$ is the envelope of the family of curves $\{U(r; U_0)\}$ (see Theorem 3):

$$G(|x|) = \sup_{U_0 > 0} U(|x|; U_0), \quad |x| > 0.$$

Then if $G_* = G_*(r)$ is the envelope of the family $\{U_-(r; U_0)\}$ of (25), we have clearly that $L_*(r) \equiv \phi(G_*(r)) < L(r)$ for $r > 0$.

The function $L_*(r)$ is determined from the system of equalities

$$L_*(r) = \phi(U_-(r; U_0)), \quad L'_*(r) = [\phi(U_-(r; U_0))]'_r, \tag{28}$$

where $U_0 > 0$ is a parameter. This system is equivalent to the following one:

$$L_*(r) = \phi(U_0) - Q(U_0)\frac{r^2}{2N}, \quad L'_*(r) = -Q(U_0)\frac{r}{N}, \quad r < r_0.$$

Then, by eliminating the parameter $U_0$ we obtain a differential equation for the envelope $L_* = L_*(r)$:

$$L_*(r) = \phi\left[Q^{-1}\left(-\frac{NL'_*(r)}{r}\right)\right] + \frac{r}{2}L'_*(r). \tag{28'}$$

This makes sense for all sufficiently small $r > 0$ (here $Q^{-1}$ is the function inverse to $Q$).

We are interested in monotone decreasing solutions of equation (28′) which satisfy the condition $L_*(0^+) = \infty$. Such a solution is especially easily computed when the coefficients $\phi$, $Q$ in (4) are of power type. Then (28′) takes the form

$$L_*(r) = (\sigma + 1)^{-1}\left(-\frac{NL'_*(r)}{r}\right)^{(\sigma+1)/\beta} + \frac{r}{2}L'_*(r), \quad r > 0. \tag{29}$$

**Theorem 5.** *Let $\sigma + 1 < \beta < (\sigma + 1)(N + 2)/(N - 2)_+$ and let $T_0 < \infty$ be the blow-up time for a solution of the Cauchy problem (4), (2). Then for all sufficiently small $|x| > 0$, we have the estimate*

$$u(T_0^-, x) > G_*(r) \equiv [(\sigma + 1)L_*(r)]^{1/(\sigma+1)} = C_*|x|^{-2/[\beta-(\sigma+1)]}, \tag{29'}$$

*where*

$$C_* = \left\{\frac{2N}{\beta - (\sigma + 1)}\left[\frac{\beta - (\sigma + 1)}{\beta}\right]^{\beta/(\sigma+1)}\right\}^{1/[\beta-(\sigma+1)]}.$$

*If $\beta \geq (\sigma + 1)(N + 2)/(N - 2)_+$, this estimate is valid for all critical functions $u_0(x)$ (that is, such that $u(t, x)$ does not decrease in $t$ in $(0, T_0) \times \mathbf{R}^N$).*

*Proof.* It is easy to check that in the case of equation (4) we have the equality

$$U(|x|; U_0) = U_0 U(U_0^{[\beta-(\sigma+1)]/2}|x|; 1), \quad U_0 > 0.$$

For $\beta < (\sigma + 1)(N + 2)/(N - 2)_+$ the function $U(|x|; 1)$ has compact support, so that $\operatorname{supp} U(|x|; U_0) \to \{0\}$ as $U_0 \to \infty$. Therefore in this case the conditions of Lemma 2 are satisfied. Then, taking into account the fact that $\sup_{U_0 \to U_0^*} U(|x|; U_0) > G_*(|x|)$ in $\mathbf{R}^N \setminus \{0\}$, and computing the precise form of the function $L_*(|x|)$ in (29), we obtain (29′) from (29).

Now let $\beta \geq (\sigma + 1)(N + 2)/(N - 2)_+$. Then $U > 0$ in $\mathbf{R}^N$. Let us fix an arbitrary $U_0 > U_0^* = 2u_0(0)$ and let us consider the ball $B = \{|x| < r_0\}$, where $r_0 > 0$ is such that $U(r_0; U_0) = U_0/2$. Then if $N(t)$ is the number of intersections in $r = |x|$ in the ball $B$ of the functions $u(t, |x|)$ and $U(|x|; U_0)$, then $N(0) = 0$ and by monotonicity of $u(t, r_0)$ in $t$ on $\partial B$, we have that $N(t) \leq 1$ for all $t \in (0, T_0)$. Therefore (20) obtains (see proof of Lemma 1) and, defining $L_*$ by (29), we arrive at (29′). □

**Remark 1.** For $\beta \geq (\sigma + 1)(N + 2)/(N - 2)_+$ the criticality condition on $u_0$ can be replaced by requiring $u_0$ to satisfy the condition of Lemma 2.

**Remark 2.** For equation (4) we can write down the exact expression for the envelope $G = G(|x|)$ of the family of functions $\{U\}$. In its dependence on $|x|$ it is the same as (29′): $G(|x|) = C|x|^{-2/[\beta-(\sigma+1)]}$, where $C = C(\sigma, \beta, N)$ is a constant, and, moreover, $C > C_*$.

Juxtaposition of the nature of the singularity of $u(T_0^-, x)$ with the upper bound derived in § 6, Ch. IV by comparing with the self-similar solution, testifies to the optimality of the estimate (29′).

Inequality (29′) allows us to derive a number of other estimates for unbounded solutions of equation (4).

**Theorem 6.** *Assume that the conditions of Theorem 5 hold. Then for all* $p \geq [\beta - (\sigma + 1)]N/2$

$$\int_{B_\epsilon} u^p(t, x)\, dx \to \infty, \quad t \to T_0^-, \tag{29''}$$

*where* $B_\epsilon = \{|x| < \epsilon\}$ *is a ball of arbitrary radius* $\epsilon > 0$.

It follows from results of § 6, Ch. IV that the restriction $p \geq [\beta - (\sigma + 1)]N/2$ is a necessary and sufficient condition for (29″) to hold for arbitrary $u_0 = u_0(|x|)$ if we demand in addition that $\beta < (\sigma + 1)(N + 2)/(N - 2)_+$.

Let us consider another simple example. Let

$$\phi(u) = e^{\sigma u}, \; Q(u) = e^{\beta u},$$

where $\sigma \geq 0$, $\beta > \sigma$ (for $\beta < \sigma$ the solution cannot be localized, as can be seen from Theorem 1). Here the equation of the envelope $L_*(r)$ has the form

$$L_*(r) = \left(-\frac{N L_*'(r)}{r}\right)^{\sigma/\beta} + \frac{r}{2} L_*'(r), \quad r \to 0^+,$$

Hence we obtain

$$L_*(r) = \left[\frac{\beta - \sigma}{\beta}\left(\frac{2\sigma N}{\beta - \sigma}\right)^{\sigma/\beta}\right]^{\beta/(\beta - \sigma)} r^{-2\sigma/(\beta-\sigma)}, \, r \to 0^+.$$

It is easy to check that in this case $\operatorname{supp} U(|x|; U_0) \to \{0\}$ as $U_0 \to \infty$; therefore by Theorem 3 we have

$$u(T_0, x) > G_*(r) \equiv \frac{1}{\sigma}\ln|L_*(r)| =$$

$$= -\frac{2}{\beta - \sigma}\ln|x| + \frac{\beta}{\sigma(\beta - \sigma)}\ln\left[\frac{\beta - \sigma}{\beta}\left(\frac{2\sigma N}{\beta - \sigma}\right)^{\sigma/\beta}\right]$$

for all sufficiently small $|x| > 0$. The corresponding integral property of unbounded solutions has the form

$$\int_{B_r} \exp\{\gamma u(t, x)\}\, dx \to \infty, \, t \to T_0,$$

for any $\gamma \geq (\beta - \sigma)N/2$.

Using the above approach, which is based on the analysis of singular solutions of the ordinary differential equation (28′), it is not hard to obtain lower bounds for the limiting profile $u(T_0, x)$ in the case of sufficiently general $\phi(u)$, $Q(u)$.

## 4 Necessary and sufficient conditions for localization

In previous subsections we formulated a criterion of localization of unbounded solutions of the Cauchy problem, which determines conditions for the occurrence of HS, S, and LS blow-up regimes in general nonlinear media.

For convenience, let us restate this criterion. Thus, everything depends on the quantity

$$D_* = \sup\left\{r \mid \sup_{U_0 > 0} U(r;\, U_0) = \infty\right\}.$$

If $D_* = \infty$, then there is no localization (Theorem 1, HS-regime). If $D_* \in \mathbf{R}_+$ then only localization in the S-regime (in a domain with a non-zero diameter) is possible. Finally, if $D_* = 0$, then, apparently, the LS-regime is to be expected.

This localization criterion has an especially simple form in the case $N = 1$; $x_0(\infty) = \infty$ leads to the HS-regime, $x_0(\infty) \in \mathbf{R}_+$ to the S-regime, and $x_0(\infty) = 0$ to the LS-regime (see Figure 79); $x_0(\infty)$ is computed from (21′). Here the dependence $x_0(U_0)$ also determines certain properties of evolution of unbounded solutions (see Corollary 1 of Theorem 1).

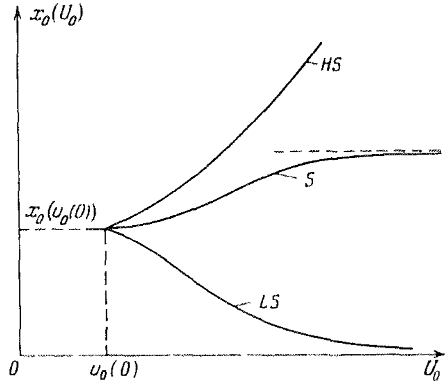


**Fig. 79.** Classification of unbounded solutions according to the criterion $0 \le x_0(\infty) \le \infty$ ($N = 1$)

Numerous estimates, obtained by constructing approximate self-similar solutions, as well as results of numerical computations show that the condition $D_* < \infty$ does indeed entail localization of unbounded solutions for arbitrary coefficients $k$, $Q$, which do not belong to a certain class of weakly nonlinear functions (this class is discussed below).

In the case of equation (4) the criterion leads to a correct result: for $\beta \ge \sigma + 1$ there is localization (if $\beta = \sigma + 1$, $D_* \in \mathbf{R}_+$, and we have the S-regime; for $\beta > \sigma + 1$, the LS-regime), while if $\beta < \sigma + 1$ solutions are not localized (see § 4, Ch. IV). Let us consider more complicated examples. We restrict ourselves to the analysis of the case $N = 1$.

**Example 1.** Let $k(u) = u^\sigma \ln^\mu(2+u)$, $Q(u) = u^{\sigma+1} \ln^\nu(2+u)$, where $\sigma > 0$, $\mu, \nu$ are constants. Then it is not hard to see that for large $U_0$

$$x_0(U_0) \simeq (\ln^{(\mu-\nu)/2} U_0) \frac{\pi}{2\sqrt{\sigma+1}}.$$

Hence we deduce that for $\mu > \nu$ there is no localization (HS-regime), while for $\mu \le \nu$ it occurs, and meas $\omega_L \ge \pi\sqrt{\sigma+1}$ if $\mu = \nu$ (S-regime) and meas $\omega_L = 0$ if $\mu < \nu$ (LS-regime).

**Example 2.** Let $k(u) = (1+u)^\mu e^{\sigma u}$, $Q(u) = (1+u)^\nu e^{\beta u}$, $\sigma \ge 0$, $\beta > 0$, $\mu, \nu$ are constants. In this case as $U_0 \to \infty$

$$x_0(U_0) \simeq U_0^{(\mu-\nu)/2} \exp\left\{\frac{\sigma-\beta}{2} U_0\right\} \left[\frac{\pi}{2(\sigma+\beta)}\right]^{1/2} \frac{\Gamma(\sigma/(\beta+\sigma))}{\Gamma(1/2+\sigma/(\beta+\sigma))}.$$

Hence if $\sigma > \beta$ or if $\sigma = \beta$, $\mu > \nu$, HS-regime obtains, while if $\sigma = \beta$, $\mu = \nu$, we have the S-regime, and if $\sigma < \beta$ or $\sigma = \beta$, $\mu < \nu$, the LS-regime.

Concerning the class of weakly nonlinear functions $k$, $Q$, in which the localization criterion $D_* < \infty$ (or $x_0(\infty) < \infty$ if $N = 1$) is no longer valid, its typical representatives are the family of functions

$$k = k(u), Q = Q_\alpha(u) \equiv \frac{u+1}{k(u)} \left\{ \int_0^u \frac{k(\eta)}{\eta + 1} \, d\eta \right\}^\alpha, \tag{30}$$

where $\alpha = \text{const} > 1$ and the coefficient $k(u)$ satisfies the conditions

$$\int_0^\infty \frac{k(\eta)}{\eta + 1} \, d\eta = \infty, \quad \lim_{u \to \infty} \left[ \frac{k(u)}{k'(u)} \right]' = \infty. \tag{31}$$

A property of unbounded solutions of equation (1) of this class[1] is the fact that their structure is described by a.s.s., which satisfy the nonlinear first order equation [150, 160, 347]

$$v_t = \frac{k(v)}{v+1} |\nabla v|^2 + Q_\alpha(v), \quad t > 0, \; x \in \mathbf{R}^N. \tag{32}$$

These a.s.s. are localized for $\alpha \geq 2$ (for $\alpha = 2$ the S-regime obtains and diam $\omega_L = 2\pi$), while for $\alpha < 2$ there is no localization. At the same time it is not hard to check that the localization criterion $D_* < \infty$ is not applicable here. For example, if $k = 1$, $Q(u) = (1 + u) \ln^\beta (1 + u)$, for $\beta > 1$ we have

$$\lim_{s \to \infty} \frac{1}{Q(s)} \int_0^s k(\eta) \, d\eta = 0,$$

which indicates the presence of the LS-regime (the criterion $0 \leq x_0(\infty) \leq \infty$, $N = 1$, leads to the same conclusion). A correct analysis of such a case is given below.

**Example 3.** Let $k(u) = \ln^\sigma(1 + u)$, $Q(u) = (\sigma + 1)^{-\gamma}(1 + u) \ln^\beta(1 + u)$, $\sigma \geq 0$, $\beta > 1$, $\gamma = (\beta + \sigma)/(\sigma + 1)$. Conditions (30), (31) are satisfied if we set $\alpha = (\beta + \sigma)/(\sigma + 1)$. Hence it follows that there is localization for $\beta \geq \sigma + 2$ (that is, $\alpha \geq 2$), while for $\beta < \sigma + 2$ there is no localization (for $\sigma = 0$ the validity of this result is demonstrated in § 7, Ch. IV).

To conclude, let us observe that there exists a direct connection between the localization property and global solvability of the boundary value problem for equation (1) in a bounded domain.

[1] Let us observe that equation (1) with $k(u) \equiv 1$, $Q(u) = (1 + u) \ln^\beta (1 + u)$ belongs to this class; it was considered in detail in § 7, Ch. IV (that section also contains the method of construction of a.s.s.).

## § 2 Boundary value problems in bounded domains

In this section we consider, in more detail than in Ch. V, blow-up regimes that occur in bounded domains. The need for such a study stems from physical considerations (for example, taking into account heat loss from the boundary of the domain where the process takes place). In the analysis of boundary value problems there arises a whole range of new phenomena not encountered in the Cauchy problem.

Let $\Omega$ be a bounded domain in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$. For the quasilinear parabolic equation

$$u_t = \Delta\phi(u) + Q(u), \quad t > 0, x \in \Omega, \tag{1}$$

we consider the first boundary value problem

$$u(0, x) = u_0(x) \geq 0, \quad x \in \Omega; \; u_0 \in C(\overline{\Omega}), \phi(u_0) \in H_0^1(\Omega), \tag{2}$$

$$u(t, x) = 0, \quad t \geq 0, x \in \partial\Omega. \tag{3}$$

We impose the usual restrictions on the non-negative functions $\phi$, $Q$. In particular, we require the necessary condition for finite time blow-up to hold:

$$\int_1^\infty \frac{d\eta}{Q(\eta)} < \infty. \tag{4}$$

The boundary condition (3) models heat outflow from the domain $\Omega$, in which diffusion and combustion processes take place. Of course, the magnitude of heat loss depends on the intensity of combustion inside $\Omega$.

The following questions arise. Under what conditions on the coefficients $k$, $Q$, initial perturbation $u_0 \not\equiv 0$ and the spatial structure of the domain $\Omega$, will combustion in the problem (1)–(3) lead to finite time blow-up? Conversely, when will the problem have a global solution, defined for all $t > 0$? In other words, we want to find out under what conditions the heat loss at the boundary is able to "extinguish" the vigorous combustion process, and when this will not happen.

For convenience in the exposition below, let us introduce in the space of initial functions $\{u_0 \geq 0 \mid u_0 \in C(\overline{\Omega}), \phi(u_0) \in H_0^1(\Omega)\}$ two sets: the *stable set* $\mathcal{W}$ and the *unstable set* $\mathcal{V}$. These sets are defined as follows: if $u_0 \in \mathcal{W}$, then there exists a global solution with initial data $u_0$; if, on the other hand, $u_0 \in \mathcal{V}$, then the problem (1)–(3) is globally insolvable. Below we present methods of constructive description of the sets $\mathcal{W}$, $\mathcal{V}$. The structure of the unstable set $\mathcal{V}$ was analysed by different means in § 6, Ch. V.

## 1 Equation with power type nonlinearities

In this subsection we consider the problem for equation (1) of the particular form

$$u_t = \Delta u^{\sigma+1} + u^\beta,\ t > 0,\ x \in \Omega;\ \sigma > 0,\ \beta > 1, \tag{5}$$

$$u(0, x) = u_0(x) \geq 0,\ x \in \Omega;\ u(t, x) = 0,\ t \geq 0,\ x \in \partial\Omega. \tag{6}$$

It is assumed that $u_0 \in C(\overline{\Omega})$, $u_0^{\sigma+1} \in H_0^1(\Omega)$. In (5) the heat conductivity coefficient $k(u) = u^\sigma/(\sigma+1)$ reduces to the familiar form $k(u) = u^\sigma$ by the change of spatial coordinate $x \to x/(\sigma+1)^{1/2}$.

It will be shown that the properties of the solution of the problem (5), (6) significantly depend on the relation between the quantities $\sigma$ and $\beta$: as $\beta$ passes through the value $\beta = \sigma + 1$, the structure of the sets $\mathcal{V}$, $\mathcal{W}$ changes drastically. The value $\beta = \sigma + 1$ is critical also for the Cauchy problem for (5) (but for a different reason having to do with the localization phenomenon). The relation between localization in the Cauchy problem and the structure of the sets $\mathcal{V}$, $\mathcal{W}$ for a boundary value problem for the same equation will be discussed in the following.

In the statements of results below, we shall use the fact that the solution of the problem for a parabolic equation which admits negative values of temperature,

$$u_t = \Delta(|u|^\sigma u) + q(u),\ t > 0,\ x \in \Omega, \tag{5'}$$

$$q(u) = \begin{cases} 0, & u < 0, \\ u^\beta, & u \geq 0 \end{cases}$$

((5′) is the same as (5) if $u \geq 0$) satisfies the weak Maximum Principle. Therefore $u(t, x) \geq 0$ almost everywhere (a.e.) in $\Omega$ for all admissible $t > 0$ if $u_0(x) \geq 0$ a.e. in $\Omega$. Analysis of the problem (5), (6) separates naturally into three cases.

*1 Global solvability for $\beta < \sigma + 1$*

**Theorem 1.** *Let $\beta < \sigma + 1$. Then the problem (5), (6) has a global solution and*

$$u^{1+\sigma/2} \in L^\infty(0, T; L^2(\Omega)),\quad \frac{\partial}{\partial t} u^{1+\sigma/2} \in L^2(0, T; L^2(\Omega)), \tag{7}$$

$$u^{1+\sigma} \in L^\infty(0, T; H_0^1(\Omega)). \tag{8}$$

**Remark 1.** From (7) it follows that the mapping $u^{1+\sigma/2} : [0, T] \to L^2(\Omega)$ is continuous (after, possibly, a modification on a set of measure zero), so that the initial condition (6) makes sense.

**Remark 2.** For $\beta < \sigma + 1$ the unstable set $\mathcal{V}$ is empty, i.e., loss of heat at the boundary does not allow finite time blow-up.

The theorem is proved by constructing a global solution by the Galerkin method, using a basis $\{w_j\}$ in $H_0^1(\Omega)$, consisting of eigenfunctions of the problem

$$\Delta w_j + \lambda_j w_j = 0, \quad x \in \Omega; \quad w_j \in H_0^1(\Omega), \; j = 1, 2, \ldots . \tag{9}$$

The eigenvalues $\lambda_j$ can be ordered in an increasing order. Then the eigenvalue $\lambda_1 > 0$ is simple and the corresponding eigenfunction $w_1(x) > 0$ in $\Omega$. The magnitude of $\lambda_1$ can be bounded from below by [282, 283]

$$\lambda_1 \geq \{\kappa(N)\}^{-2}\{\operatorname{meas}\Omega\}^{-2/N}, \tag{10}$$

$$\kappa(1) = \frac{1}{\pi}, \quad \kappa(2) = 2, \quad \kappa(N) = \frac{2(N-1)}{N-2}, \quad N \geq 3. \tag{10$'$}$$

In the particular case when $\Omega$ is a ball, $\Omega = \{|x| < R\}$,

$$\lambda_1 = \left[z_N^{(1)}/R\right]^2, \tag{11}$$

where $z_N^{(1)}$ is the smallest positive root of the Bessel function $J_{(N-2)/2}$.

**Lemma 1** ([282, 283]). *Let the function $v(x)$ be such that $|v|^\sigma v \in H_0^1(\Omega)$. Then we have the estimate*

$$\|v\|_{L^{\beta(\sigma+1)}(\Omega)} \leq C_1 \left\{ \sum_{i=1}^N \left\| \frac{\partial}{\partial x_i}(|v|^\sigma v) \right\|_{L^2(\Omega)}^2 \right\}^{1/[2(\sigma+1)]}, \tag{12}$$

*where $\beta > 1$ is arbitrary for $N = 1, 2$ and $\beta \in (1, (\sigma+1)(N+2)/(N-2))$ for $N \geq 3$. The constant $C_1$ is determined from the formula*

$$C_1 = \left\{ K(N, \sigma, \beta)(\operatorname{meas}\Omega)^{\frac{1}{N} - \frac{\beta-(\sigma+1)}{2(\beta+\sigma+1)}} \right\}^{\frac{1}{\sigma+1}},$$

*where $K(N, \sigma, \beta) = \kappa(N)$ for $N = 1$ and $N \geq 3$, $K(2, \sigma, \beta) = 3/2$ for $\beta \in (1, 2(\sigma+1)]$ and $K(2, \sigma, \beta) = (\beta+\sigma+1)/[2(\sigma+1)]$ for $\beta > 2(\sigma+1)$. If $\beta = \sigma + 1$, then we have the inequality*

$$\|v\|_{L^{2(\sigma+1)}(\Omega)} \leq \lambda_1^{-1/[2(\sigma+1)]} \left\{ \sum_{i=1}^N \left\| \frac{\partial}{\partial x_i}(|v|^\sigma v) \right\|_{L^2(\Omega)}^2 \right\}^{1/[2(\sigma+1)]}. \tag{13}$$

*Proof of Theorem* 1. For every integer $m > 0$ we shall seek an approximate solution of the problem, $u_m$, in the form

$$u_m(t, x) = \left| \sum_{i=1}^m g_{im}(t) w_i \right|^{-\sigma/(\sigma+1)} \sum_{j=1}^m g_{jm}(t) w_j. \tag{14}$$

where the unknown functions $g_{jm} \in C^1([0, T])$ will be determined from the system of equations (here and below, $'$ denotes differentiation in $t$)

$$(u'_m, w_j) + (\nabla(|u_m|^\sigma u_m), \nabla w_j) = (q(u_m), w_j), \quad 1 \le j \le m, \tag{15}$$

and the initial conditions

$$u_m(0) = u_{0m}; \quad |u_{0m}|^\sigma u_{0m} \to u_0^{\sigma+1} \text{ in } H_0^1(\Omega), \quad m \to \infty. \tag{16}$$

Local solvability in the problem (15), (16) follows from the theory of ordinary differential equations (it is not hard to show that the system (15) can be always resolved with respect to the derivatives $g'_{jm}$). Below we shall obtain a priori estimates, which ensure existence of the approximate solution $u_m$ for any $m$ on an interval $[0, T]$ of arbitrary length.

Let us multiply each equation (15) by $g'_{jm}$, sum the resulting equalities in $j$ from 1 to $m$, and then integrate the resulting expression in $t$. We have

$$\begin{aligned}
&\frac{4(\sigma+1)}{(\sigma+2)^2}\int_0^t \|\{|u_m(s)|^{\sigma/2}u_m(s)\}'\|^2_{L^2(\Omega)}\,ds + \\
&+\frac{1}{2}\left\{\sum_{i=1}^N \left\|\frac{\partial}{\partial x_i}(|u_m(t)|^\sigma u_m(t))\right\|^2_{L^2(\Omega)}\right\} - \frac{\sigma+1}{\beta+\sigma+1}\int_\Omega \Phi(u_m(t,x))\,dx = \\
&= \frac{1}{2}\left\{\sum_{i=1}^N \left\|\frac{\partial}{\partial x_i}(|u_{0m}|^\sigma u_{0m})\right\|^2_{L^2(\Omega)}\right\} - \frac{\sigma+1}{\beta+\sigma+1}\int_\Omega \Phi(u_{0m}(x))\,dx,
\end{aligned} \tag{17}$$

where $\Phi(u) = (\max\{0, u\})^{\beta+\sigma+1}$.

Using now the estimate (12) and Young's inequality, we derive the following inequalities (below $c$ stands for various positive constants that do not depend on $m$):

$$\begin{aligned}
&\frac{4(\sigma+1)}{(\sigma+2)^2}\int_0^t \left\|\{|u_m|^{\sigma/2}u_m\}'\right\|^2_{L^2(\Omega)}\,ds + \frac{1}{2}\left\{\sum_{i=1}^N \left\|\frac{\partial}{\partial x_i}(|u_m|^\sigma u_m)\right\|^2_{L^2(\Omega)}\right\} \le \\
&\le c + \frac{\sigma+1}{\beta+\sigma+1}\|u_m\|^{\beta+\sigma+1}_{L^{\beta+\sigma+1}(\Omega)} \le \\
&\le c + C_1^{\beta+\sigma+1}\frac{\sigma+1}{\beta+\sigma+1}\left\{\sum_{i=1}^N \left\|\frac{\partial}{\partial x_i}(|u_m|^\sigma u_m)\right\|^2_{L^2(\Omega)}\right\}^{(\beta+\sigma+1)/[2(\sigma+1)]} \le \\
&\le c + \frac{1}{4}\left\{\sum_{i=1}^N \left\|\frac{\partial}{\partial x_i}(|u_m|^\sigma u_m)\right\|^2_{L^2(\Omega)}\right\}.
\end{aligned} \tag{17'}$$

Hence it follows that the functions $(|u_m|^{\sigma/2}u_m)'$ are bounded in $L^2(0,T;L^2(\Omega))$; $|u_m|^{\sigma/2}u_m$ are bounded in $L^\infty(0,T;L^2(\Omega))$; $|u_m|^\sigma u_m$ in $L^\infty(0,T;H_0^1(\Omega))$; $(|u_m|^\sigma u_m)'$ in $L^2(0,T;L^1(\Omega))$. In particular, we have that $|u_m|^\sigma u_m$ belong to a bounded set, for example, in $W_1^1(0,T;W_1^1(\Omega))$.

In addition to (17) we shall need another identity, to obtain which we multiply each of the equations (15) by $g_{jm}$, sum all the equalities in $j$ from 1 to $m$, and then integrate in $t$. As a result we have

$$\begin{aligned}\frac{1}{\sigma+2}\left\||u_m|^{\sigma/2}u_m\right\|^2_{L^2(\Omega)}-\frac{1}{\sigma+2}\left\||u_{0m}|^{\sigma/2}u_{0m}\right\|^2_{L^2(\Omega)}+\\+\int_0^t\left\{\sum_{i=1}^N\left\|\frac{\partial}{\partial x_i}(|u_m|^\sigma u_m)\right\|^2_{L^2(\Omega)}\right\}ds=\int_0^t\int_\Omega\Phi(u_m(s,x))\,dx\,ds.\end{aligned}\tag{18}$$

From the above estimates it follows that $|u_m|^{\sigma/2}u_m$ are bounded in $H^1(\omega_T)$ ($\omega_T=(0,T)\times\Omega$). However the embedding $H^1(\omega_T)$ into $L^2(\omega_T)$ is compact [296, 362], and therefore from the sequence $u_m$ we can pick a subsequence $u_\mu$, such that $|u_\mu|^{\sigma/2}u_\mu\to|u|^{\sigma/2}u$ in $L^2(\omega_T)$ and a.e. Hence by other estimates we conclude that $q(u_\mu)\to q(u)$ weakly in $L^2(\omega_T)$. Let us set $A(u)=-\Delta(|u|^\sigma u)$. Then it is not hard to see that $A(u_m)$ are bounded in $L^\infty(0,T;H^{-1}(\Omega))$. From that we can conclude that $A(u_\mu)\to\chi$ weak-* in $L^\infty(0,T;H^{-1}(\Omega))$. The proof of the equality $\chi=A(u)$ proceeds as in [296], using (18). Passing now to the limit, and using standard arguments, we have that $u$ is a global generalized solution and satisfies the inclusions (7), (8). To conclude, let us observe that the inclusion (8) also allows us to establish the weak Maximum Principle (for the method of proof consult [125, 371]). □

It is not hard to show that under the conditions of Theorem 1 the global solution $u\not\equiv 0$ stabilizes as $t\to\infty$ to the stationary solution $U>0$ in $\Omega$, $\Delta U^{\sigma+1}+U^\beta=0$ in $\Omega$, $U=0$ on $\partial\Omega$ (existence and uniqueness of the non-trivial function $U$ are established, for example, in [7, 21]). Stabilization of $u^{\sigma+1}(t,\cdot)$ to $U^{\sigma+1}$ in $L^2(\Omega)$ as $t\to\infty$ follows from the existence of the Lyapunov function

$$J(u)(t)=\frac{1}{2}\sum_{i=1}^N\left\|\frac{\partial}{\partial x_i}u^{\sigma+1}(t)\right\|^2_{L^2(\Omega)}-\frac{\sigma+1}{\beta+\sigma+1}\int_\Omega u^{\beta+\sigma+1}(t,x)\,dx,\tag{19}$$

which is non-increasing on trajectories $\{u(t,\cdot),t>0\}$, and, in particular, from the inclusion $(u^{1+\sigma/2})_t\in L^2(\mathbf{R}_+\times\Omega)$ (see the estimate (17′), where $c$ is independent of $T$). (This means that the norm $\|(u^{1+\sigma/2})_t\|^2_{L^2(\Omega)}$ is "small"in a neighbourhood of $t=\infty$, and that is in principle sufficient to prove stabilization to the unique stationary solution $U\not\equiv 0$, $U_t\equiv 0$.) Stabilization to the trivial solution, $u^{\sigma+1}\to 0$ in $L^2(\Omega)$ as $t\to\infty$, is impossible; this is proved using, for example, the arguments used in the proof of Proposition 8, § 4 of Ch. II.

## 2 The case $\beta = \sigma + 1$

**Theorem 2.** *Let $\beta = \sigma + 1$. Then if the domain $\Omega$ is such that $\lambda_1 > 1$, the problem (5), (6) has a global solution, which satisfies (7), (8). Furthermore*

$$\|u(t)\|_{L^{\sigma+2}(\Omega)} = O(t^{-1/\sigma}) \to 0, \quad t \to \infty. \tag{20}$$

*If, on the other hand, $\lambda_1 < 1$, then for any $u_0(x) \not\equiv 0$ the problem (5), (6) has no global solutions*[2] *and there exists $T_0 \in (0, T_*]$, where*

$$T_* = \left\{(1 - \lambda_1)\sigma \|w_1\|^{-\sigma}_{L^1(\Omega)} (u_0, w_1)^{\sigma}\right\}^{-1} < \infty, \tag{21}$$

*such that $(u(t), w_1) \to \infty$ as $t \to T_0$.*

*Proof.* Let us start by establishing the first claim of the theorem. Applying estimate (13) to the equality (17) we derived above, we obtain

$$\begin{aligned} &\frac{4(\sigma+1)}{(\sigma+2)^2} \int_0^t \left\| \{|u_m|^{\sigma/2} u_m\}' \right\|^2_{L^2(\Omega)} ds + \\ &\qquad + \frac{1}{2}(1 - \lambda_1^{-1}) \left\{ \sum_{i=1}^N \left\| \frac{\partial}{\partial x_i} (|u_m|^{\sigma} u_m) \right\|^2_{L^2(\Omega)} \right\} \le c. \end{aligned} \tag{22}$$

Hence by the condition $\lambda_1 > 1$ the functions $(|u_m|^{\sigma/2} u_m)'$ are bounded in $L^2(0, T; L^2(\Omega))$, while $|u_m|^{\sigma} u_m$ are bounded in $L^{\infty}(0, T; H^1_0(\Omega))$.

In a similar manner we have from (18) that

$$\begin{aligned} &\frac{1}{\sigma+2} \frac{d}{dt} \|u_m(t)\|^{\sigma+2}_{L^{\sigma+2}(\Omega)} \le \\ &\qquad \le -(1 - \lambda_1^{-1}) \left\{ \sum_{i=1}^N \left\| \frac{\partial}{\partial x_i} (|u_m(t)|^{\sigma} u_m(t)) \right\|^2_{L^2(\Omega)} \right\}. \end{aligned} \tag{23}$$

From that it follows, in particular, that the functions $u_m$ are bounded in $L^{\infty}(0, T; L^{\sigma+2}(\Omega))$, and that $|u_m|^{\sigma} u_m$ are bounded in $L^2(0, T; H^1_0(\Omega))$. Applying to the right-hand side of the above inequality the estimate (13) and the Hölder inequality, we obtain

$$\frac{1}{\sigma+2} \frac{d}{dt} \|u_m(t)\|^{\sigma+2}_{L^{\sigma+2}(\Omega)} \le -(\lambda_1 - 1)(\text{meas}\,\Omega)^{-\sigma/(\sigma+2)} \|u_m(t)\|^{2(\sigma+1)}_{L^{\sigma+2}(\Omega)},$$

and therefore

$$\|u_m(t)\|_{L^{\sigma+2}(\Omega)} \le \left\{ \|u_{0m}\|^{-\sigma}_{L^{\sigma+2}(\Omega)} + \sigma(\text{meas}\,\Omega)^{-\sigma/(\sigma+2)}(\lambda_1 - 1)t \right\}^{-1/\sigma}.$$

---

[2] Local solvability of the problem in this case will be demonstrated later.

Passing to the limit $m = \mu \to \infty$ is performed as in the proof of Theorem 1. Estimate (20) follows from the last inequality.

Let us move on to prove the second part of the theorem. Integrating (15) for $j = 1$, $\beta = \sigma + 1$, over the interval $(0, t)$, and passing to the limit $m = \mu \to \infty$, we have as a result

$$(u(t), w_1) = (u_0, w_1) + (1 - \lambda_1) \int_0^t (u^{\sigma+1}(s), w_1)\, ds. \tag{23'}$$

Using the Hölder inequality $(u^{\sigma+1}, w_1) \geq \|w_1\|_{L^1(\Omega)}^{-\sigma} (u, w_1)^{\sigma+1}$, we arrive at the following estimate:

$$E(t) \equiv (u(t), w_1) \geq (u_0, w_1) + (1 - \lambda_1)\|w_1\|_{L^1(\Omega)}^{-\sigma} \int_0^t E^{\sigma+1}(s)\, ds.$$

Hence it follows that $E(t) \geq F(t)$ a.e., where $F(t)$ is the solution of the problem

$$F'(t) = (1 - \lambda_1)\|w_1\|_{L^1(\Omega)}^{-\sigma} F^{\sigma+1}(t), \quad t > 0; \quad F(0) = (u_0, w_1),$$

and therefore

$$E(t) \geq F(t) \equiv \left\{\sigma(1 - \lambda_1)\|w_1\|_{L^1(\Omega)}^{-\sigma}(T_* - t)\right\}^{-1/\sigma} \to \infty$$

as $t \to T_*$. □

**Remark.** It follows from (10) that the global solvability condition $\lambda_1 > 1$ will be necessarily satisfied if $\operatorname{meas}\Omega < |\kappa(N)|^{-N}$, i.e. for sufficiently "small" domains $\Omega$.

Let us consider separately the case $\lambda_1 = 1$, $\beta = \sigma + 1$. Then from (22), (23) it follows that the functions $(|u_m|^{\sigma/2}u_m)'$ and $|u_m|^{\sigma/2}u_m$ are bounded in $L^2(\omega_T)$. These estimates are not sufficient to pass to the limit.

In this case it is not hard to prove global solvability using the standard comparison theorem. Indeed, the function $U_\alpha(x) = \alpha w_1^{1/(\sigma+1)}(x)$ is for any $\alpha > 0$ a stationary solution of the problem $\Delta U_\alpha^{\sigma+1} + U_\alpha^{\sigma+1} = 0$ in $\Omega$, $U_\alpha = 0$ on $\partial\Omega$. Therefore if $u_0 \leq U_\alpha$ in $\Omega$, then $u \leq U_\alpha$ in $\mathbf{R}_+ \times \Omega$.

It is of interest that though the boundary value problem has a continuum of stationary solutions $\{U_\alpha = \alpha w_1^{1/(\sigma+1)}, \alpha > 0\}$, only one of those is asymptotically stable, and $u^{\sigma+1} \to \alpha_*^{\sigma+1} w_1(x)$in $L^2(\Omega)$ as $t \to \infty$, where

$$\alpha_* = (u_0, w_1)\left(\int_\Omega w_1^{(\sigma+2)/(\sigma+1)}\, dx\right)^{-1}.$$

This follows from the identity $(u(t), w_1) \equiv (u_0, w_1)$ (see (23′) for $\lambda_1 = 1$) [345].

*1.3. The case $\beta > \sigma + 1$.* This is the most interesting case: both the stable set $\mathcal{W}$ and the unstable set $\mathcal{V}$ are then non-empty. Below we shall assume that $\beta > \sigma + 1$ for $N = 1, 2$ and that $\beta \in (\sigma + 1, (\sigma + 1)(N + 2)/(N - 2))$ for $N \geq 3$. First we shall prove a theorem concerning local solvability of the problem.

**Theorem 3.** *Assume that $\sigma + 1 \leq \beta < \sigma + 1 + 2(\sigma + 2)/N$. Then there exists a constant $T_* > 0$ such that on the interval $[0, T_*]$ problem (5), (6) has a solution, which satisfies the inclusions (7), (8).*

The proof relies on the following lemma [282, 283].

**Lemma 2.** *Let a function $v$ be such that $|v|^\sigma v \in H_0^1(\Omega)$. Then for any $\beta > 1$ if $N = 1, 2$, and for $1 < \beta < (\sigma + 1)(N + 2)/(N - 2)$ if $N \geq 3$, we have the estimate*

$$\|v\|_{L^{\beta+\sigma+1}(\Omega)} \leq C_2 \left\{ \sum_{i=1}^{N} \left\| \frac{\partial}{\partial x_i}(|v|^\sigma v) \right\|^2_{L^2(\Omega)} \right\}^{\nu} \|v\|^{1-2\nu(\sigma+1)}_{L^{\sigma+2}(\Omega)}.$$

*Here $\nu = N(\beta - 1)/\{(\beta + \sigma + 1)[2(\sigma + 2) + N\sigma]\}$, $C_2 = \{1 + (\sigma + 2)/[2(\sigma + 1)]\}^{2\nu}$ for $N = 1, 2$ and $C_2 = [2(N - 1)/(N - 2)]^{2\nu}$ for $N \geq 3$.*

*Proof of Theorem* 3. Let us use Lemma 2:

$$\int_\Omega \Phi(u_m(t, x))\, dx \leq C_2^{\beta+\sigma+1} \left\{ \sum_{i=1}^{N} \left\| \frac{\partial}{\partial x_i}(|u_m|^\sigma u_m) \right\|^2_{L^2(\Omega)} \right\}^{\nu_1} \|u_m\|^{\nu_2}_{L^{\sigma+2}(\Omega)},$$

$$\nu_1 = \frac{N(\beta - 1)}{2(\sigma + 2) + N\sigma}, \quad \nu_2 = \frac{\beta(2 - N) + (\sigma + 1)(N + 2)}{2(\sigma + 2) + N\sigma}(\sigma + 2) > 0, \tag{24}$$

where by the restriction $\beta < (\sigma + 1) + 2(\sigma + 2)/N$ we have that $\nu_1 < 1$. This allows us to apply Young's inequality to the right-hand side of (24), as a result of which we derive the estimate

$$\int_\Omega \Phi(u_m(t, x))dx \leq \left\{ \sum_{i=1}^{N} \left\| \frac{\partial}{\partial x_i}(|u_m|^\sigma u_m) \right\|^2_{L^2(\Omega)} \right\} + c\|u_m\|^{\nu_2/(1-\nu_1)}_{L^{\sigma+2}(\Omega)}. \tag{25}$$

Substituting this estimate into inequality (18), which is first differentiated, we obtain

$$\frac{d}{dt}\|u_m(t)\|^{\sigma+2}_{L^{\sigma+2}(\Omega)} \leq c\|u_m(t)\|^{\nu_2/(1-\nu_1)}_{L^{\sigma+2}(\Omega)}.$$

Hence it follows that $u_m$ are bounded in $L^\infty(0, T_*; L^{\sigma+2}(\Omega))$ if $T_* > 0$ is sufficiently small. Then from (17) and (25) it follows that the functions $(|u_m|^{\sigma/2} u_m)'$ are bounded in $L^2(0, T_*; L^2(\Omega))$, $|u_m|^\sigma u_m$ are bounded in $L^\infty(0, T_*; H_0^1(\Omega))$, and this concludes the proof. □

This theorem is presented to illustrate proofs of local solvability using a minimal mathematical apparatus. In the course of the proof of Theorem 3 there appears an *inessential* restriction from above on the size of $\beta$. This restriction does not arise in the construction of the local solution as a limit of a monotone sequence of classical positive solutions. Uniform boundedness of the sequence of classical solutions in a domain of the form $(0, T_*) \times \Omega$ follows from comparing each of these solutions with the spatially homogeneous solution (see § 1, 2, Ch. I).

Sequences of strictly positive classical solutions $u_\epsilon$, which converge monotonically to $u$ as $\epsilon \to 0$, can be constructed in a number of different ways. For example, as $u_\epsilon$ we can take solutions of the original equation (5) with different conditions: $u_\epsilon = \epsilon + u_0$ for $t = 0$ in $\Omega$, $u_\epsilon = \epsilon$ on $(0, T) \times \partial\Omega$. Then $u_\epsilon \geq \epsilon$ in $(0, T) \times \Omega$, and therefore on each solution $u_\epsilon$ equation (5) is uniformly parabolic. We could also do this differently: leave boundary conditions as they were, and regularize the equation by replacing in (5) the operator $\Delta u^{\sigma+1}$ by $(\sigma + 1)\nabla \cdot ((u^2 + \epsilon^2)^{\sigma/2}\nabla u)$. A priori estimates, that guarantee, in particular, convergence of $u_\epsilon$ to $u$ as $\epsilon \to 0$ are derived in practically the same way.

Let us move on now to construct the stable set $\mathcal{W}$. First let us prove certain auxiliary statements.

**Lemma 3.** *Let* $|v(x)|^\sigma v(x) \in H_0^1(\Omega)$. *Let us set*

$$J(v) = \frac{1}{2}a(v) - \frac{\sigma + 1}{\beta + \sigma + 1}b(v),$$

*where*

$$a(v) = \sum_{i=1}^{N} \left\| \frac{\partial}{\partial x_i}(|v|^\sigma v) \right\|^2_{L^2(\Omega)}, \quad b(v) = \int_\Omega \Phi(v(x))\, dx.$$

*Then we have the inequality*

$$d = \inf_{\substack{|v|^\sigma v \in H_0^1(\Omega) \\ v \neq 0}} \sup_{\lambda > 0} J(\lambda v) > 0. \tag{26}$$

*Proof.* It is not hard to see that

$$J(\lambda v) = \frac{1}{2}\lambda^{2(\sigma+1)}a(v) - \frac{\sigma + 1}{\beta + \sigma + 1}\lambda^{\beta+\sigma+1}b(v);$$

therefore, using the estimate (12), we obtain

$$\sup_{\lambda > 0} J(\lambda v) = J\left[v\left(\frac{a(v)}{b(v)}\right)^{1/[\beta-(\sigma+1)]}\right] \equiv$$

$$\equiv \frac{\beta - (\sigma + 1)}{2(\beta + \sigma + 1)}\left[\frac{(a(v))^{\beta+\sigma+1}}{(b(v))^{2(\sigma+1)}}\right]^{1/[\beta-(\sigma+1)]} \geq \frac{\beta - (\sigma + 1)}{2(\beta + \sigma + 1)}C_1^{-\frac{2(\sigma+1)(\beta+\sigma+1)}{\beta-(\sigma+1)}} > 0.$$

□

Let us introduce the set

$$\mathcal{W} = \{v : |v|^{\sigma} v \in H_0^1(\Omega);\ 0 \le J(\lambda v) < d,\ \lambda \in [0, 1]\}. \tag{27}$$

The distinguishing property of this set, which follows from the method of its construction, is reflected in

**Lemma 4.** *Equality* $\mathcal{W} = \mathcal{W}_* \cup \{0\}$ *holds. Here*

$$\mathcal{W}_* = \{v : |v|^{\sigma} v \in H_0^1(\Omega),\ a(v) - b(v) > 0,\ J(v) < d\}.$$

*Proof.* Let $v \in \mathcal{W}$, $v \not\equiv 0$. Then

$$\sup_{\lambda > 0} J(\lambda v) = J\left[v\left(\frac{a(v)}{b(v)}\right)^{1/[\beta-(\sigma+1)]}\right] \ge d.$$

and therefore $a(v)/b(v) > 1$, whence $v \in \mathcal{W}_*$. On the other hand, let $v \in \mathcal{W}_*$. Then

$$\sup_{\lambda \in (0,1)} J(\lambda v) = J(v) < d,$$

so that $v \in \mathcal{W}$. □

Some other important properties of $\mathcal{W}$ are contained in

**Lemma 5.** *The quantity $d$ in* (26) *is finite; the set $\mathcal{W}$ is bounded and contained in the ball*

$$\left\{v : |v|^{\sigma} v \in H_0^1(\Omega),\ a(v) \le \frac{2(\beta + \sigma + 1)}{\beta - (\sigma + 1)} d\right\}.$$

*Proof.* Under these assumptions the boundary value problem

$$\Delta v_\nu + \nu v_\nu^{\beta/(\sigma+1)} = 0 \text{ in } \Omega;\ v_\nu = 0 \text{ on } \partial\Omega.$$

has a positive solution in $\Omega$ for some $\nu \in \mathbf{R}_+$ (see [332]). (Moreover, such a solution exists for all $\nu > 0$; its positivity follows from the formulation of the corresponding variational problem; see [339, 96, 297].)

Let us set $v = v_\nu^{1/(\sigma+1)}$. Then $v^{\sigma+1} \in H_0^1(\Omega)$ and

$$d \le J\left\{v\left[\frac{a(v)}{b(v)}\right]^{1/[\beta-(\sigma+1)]}\right\}.$$

Hence by the equality $a(v) - \nu b(v) = 0$, we have

$$d \le J(\nu^{1/[\beta-(\sigma+1)]} v) = \frac{\beta - (\sigma + 1)}{2(\beta + \sigma + 1)} a(v) \nu^{2(\sigma+1)/[\beta-(\sigma+1)]} < \infty.$$

Take $v \in \mathcal{W}$, $v \not\equiv 0$. Then by Lemma 4 $a(v) - b(v) > 0$ and therefore

$$d > J(v) = \frac{1}{2}a(v) - \frac{\sigma+1}{\beta+\sigma+1}b(v) \geq \frac{\beta-(\sigma+1)}{2(\beta+\sigma+1)}a(v).$$

□

Let us now prove a theorem concerning global solvability of the problem.

**Theorem 4.** *Let $\beta > \sigma+1$ for $N = 1, 2$ and $\sigma+1 < \beta < (\sigma+1)(N+2)/(N-2)$ for $N \geq 3$. Assume that the initial function $u_0$ in (6) is such that $u_0 \in \mathcal{W}$. Then for any $T > 0$ there exists a generalized solution of the problem (5), (6), which satisfies the inclusions (7), (8) and belongs to $\overline{\mathcal{W}}$ for all $t \geq 0$ ($\overline{\mathcal{W}}$ is the closure of $\mathcal{W}$ in the set $\{v \mid |v|^\sigma v \in H_0^1(\Omega)\}$).*

*Proof.* As in proofs of previous theorems, the solution of the problem is constructed by Galerkin's method. Then the equivalent of (17) can he written in the form

$$\frac{4(\sigma+1)}{(\sigma+2)^2}\int_0^t \left\|\{|u_m(s)|^{\sigma/2}u_m(s)\}'\right\|^2_{L^2(\Omega)} ds = J(u_{0m}) - J(u_m(t)). \tag{28}$$

Hence it follows immediately that $u_m \in \mathcal{W}$ for all $t > 0$ and sufficiently large $m$, since $u_0 \in \mathcal{W}$, $J(u_0) < d$ and $|u_{0m}|^\sigma u_{0m} \to u_0^{\sigma+1}$ in $H_0^1(\Omega)$ as $m \to \infty$. Then by Lemma 4 $a(u_m(t)) \geq b(u_m(t))$, and therefore

$$\frac{4(\sigma+1)}{(\sigma+2)^2}\int_0^t \left\|\{|u_m(s)|^{\sigma/2}u_m(s)\}'\right\|^2_{L^2(\Omega)} ds +$$
$$+\frac{\beta-(\sigma+1)}{2(\beta+\sigma+1)}\left\{\sum_{i=1}^N \left\|\frac{\partial}{\partial x_i}(|u_m(t)|^\sigma u_m(t))\right\|^2_{L^2(\Omega)}\right\} \leq J(u_{0m}) < c.$$

Therefore the functions $\{|u_m|^{\sigma/2}u_m\}'$ are bounded in $L^2(\omega_T)$ and $|u_m|^\sigma u_m$ are bounded in $L^\infty(0, T; H_0^1(\Omega))$. From that point on the theorem is proved as the preceding ones. □

Thus for $\beta \in (\sigma+1, (\sigma+1)(N+2)/(N-2)_+)$ the problem has a global solution, if the initial function is sufficiently small. Let us note that the asymptotic behaviour of global solutions $u \in \overline{\mathcal{W}}$ here is completely different from the case $\beta \in (1, \sigma+1]$. The point is that every stationary solution $U > 0$ in $\Omega$ of the problem $\Delta U^{\sigma+1} + U^\beta = 0$ in $\Omega$, $U = 0$ on $\partial\Omega$, is unstable (concerning existence and stability of $U$ see [331, 334, 96, 200, 296]). This is easy to demonstrate, when, for example, $\Omega$ is star-shaped with respect to $x = 0$.

Indeed, in that case the function $U_\alpha = \alpha U(\alpha^m x)$, $m = |\beta-(\sigma+1)|/2 > 0$, for any $\alpha > 0$ is a stationary solution in the domain $\Omega_\alpha = \{\alpha^m x \in \Omega\}$. If $\alpha \in (0, 1)$, then clearly $\overline{\Omega} \subset \Omega_\alpha$ and $\sup_{\Omega_\alpha} U_\alpha < \sup_\Omega U$. Therefore if in the original problem

$u_0 \le U_\alpha$ in $\Omega$, then by the Maximum Principle $u \le U_\alpha$ in $\mathbf{R}_+ \times \Omega$. Therefore $u$ cannot stabilize to $U$ as $t \to \infty$, even though the difference $u_0 - U$ in $C(\overline{\Omega})$ as $\alpha \to 1^-$ can be arbitrarily small. Similarly, choosing $\alpha > 1$ we can prove instability of a stationary solution from above.

It is not hard to show, using Lemma 4, that if the condition $u_0 \in \mathcal{W}$ holds, the function $u(t, x)$ stabilizes as $t \to \infty$ to the trivial solution $U \equiv 0$. Indeed, if $u_0 \in \mathcal{W}$, then by (28) for $m \to \infty$ we have $J(u(t)) \le J(u_0) < d$. However, if $U \not\equiv 0$, then $J(U) \ge d$, and stabilization of $u$ to $U$ as $t \to \infty$ is impossible.

Let us show now that in problem (5), (6) for $\beta > \sigma + 1$ all sufficiently large initial functions $u_0$ belong to the unstable set $\mathcal{V}$ (the analysis here employs methods different from those of § 6, Ch. V).

**Theorem 5.** *Let* $\sigma + 1 < \beta < \sigma + 1 + 2(\sigma + 2)/N$ *and*

$$J(u_0) < 0, \ \|u_0\|_{L^{\sigma+2}(\Omega)} > 0. \tag{29}$$

*Then the problem* (5), (6) *does not have a global solution, and we can find* $T_0 \in (0, T_*]$, *where*

$$T_* = \frac{\beta + \sigma + 1}{(\beta - 1)[\beta - (\sigma + 1)]} (\operatorname{meas} \Omega)^{(\beta-1)/(\sigma+2)} \|u_0\|^{1-\beta}_{L^{\sigma+2}(\Omega)} < \infty,$$

*such that*

$$\lim_{t \to T_0^-} \|u(t)\|_{L^{\sigma+2}(\Omega)} = \infty.$$

*Proof.* Recall that the restriction $\beta < \sigma + 1 + 2(\sigma + 2)/N$ ensures local solvability of the problem. Let us show that $u$ cannot be defined for all $t > 0$. By passing to the limit in (28) we obtain $J(u(t)) \le J(u_0)$ a.e., and therefore

$$a(u(t)) < \frac{2(\sigma + 1)}{\beta + \sigma + 1} b(u(t)) \text{ a.e. in } \mathbf{R}_+.$$

Then for $m = \infty$ we derive from (18) the following inequality:

$$\begin{aligned} &\|u(t)\|^{\sigma+2}_{L^{\sigma+2}(\Omega)} \ge \|u_0\|^{\sigma+2}_{L^{\sigma+2}(\Omega)} + \\ &+ (\sigma + 2) \frac{\beta - (\sigma + 1)}{\beta + \sigma + 1} (\operatorname{meas} \Omega)^{(1-\beta)/(\sigma+2)} \int_0^t \|u(s)\|^{\beta+\sigma+1}_{L^{\sigma+2}(\Omega)} \, ds, \ t > 0. \end{aligned} \tag{30}$$

In deriving this inequality, we used the estimate

$$\|u(t)\|^{\beta+\sigma+1}_{L^{\beta+\sigma+1}(\Omega)} \ge (\operatorname{meas} \Omega)^{(1-\beta)/(\sigma+2)} \|u(t)\|^{\beta+\sigma+1}_{L^{\sigma+2}(\Omega)}.$$

Taking into account the fact that $\|u_0\|^{\sigma+2}_{L^{\sigma+2}(\Omega)} > 0$ and using (29), we deduce the required result from (30). □

Let us note that condition (29) of unboundedness of a solution of the problem is in a certain sense the opposite of the inclusion $u_0 \in \mathcal{W}$ (see (27)).

## 2 Equations of general form

In this subsection we consider the problem (1)–(3) with coefficients $\phi(u)$, $Q(u)$ of sufficiently general form; $Q(u)$ is taken to satisfy (4). The main task is to analyze the set $\mathcal{W}$, which is studied by two different methods. In subsection 2.3 we study the unstable set.

*1 Conditions for global solvability for arbitrary initial functions ($\mathcal{V} = \emptyset$) Derivation of global integral estimates for the solution.*

Here, as in the previous subsection, we apply the Galerkin method. The main restriction on the coefficients $\phi(u)$, $Q(u)$ consists of the following: there exist positive constants $M_1$, $M_2$, such that the inequality

$$Q(s) \le M_1 + M_2\phi(s), \; s \ge 0, \tag{31}$$

holds; moreover,

$$M_2 < \lambda_1, \tag{31'}$$

For simplicity we shall also assume that

$$\phi'(u)/|1+\phi^2(u)| \le M_3 = \text{const} > 0 \text{ for } u \ge 0. \tag{32}$$

For convenience, let us introduce the function

$$\psi(s) = \int_0^s |\phi'(\eta)|^{1/2}\, d\eta, \; s \ge 0.$$

**Theorem 6.** *Let conditions* (31), (32) *be satisfied. Then for any $T > 0$ there exists a generalized solution of problem* (1)–(3), *such that $u(t, x) \ge 0$ a.e. in $\Omega$ for any fixed $t \ge 0$ and the following inclusions hold:*

$$\psi(u) \in L^\infty(0, T; L^2(\Omega)), \; \frac{\partial}{\partial t}\psi(u) \in L^2(0, T; L^2(\Omega)), \tag{33}$$

$$\phi(u) \in L^\infty(0, T; H_0^1(\Omega)). \tag{34}$$

**Remark 1.** Condition (31), which plays a crucial part, will be necessarily satisfied if

$$Q(s)/\phi(s) \to 0, \; s \to \infty. \tag{35}$$

**Remark 2.** If in (1) we set $\phi(s) \equiv Q(s)$, then (31) holds for $M_2 = 1$ $(M_1 = 0)$. Therefore (31′) holds if $\lambda_1 > 1$. Thus in this case existence of a global solution depends only on the domain $\Omega$ (this situation occurs for equation (5) for $\beta = \sigma+1$).

The above theorem is proved by Galerkin's method, practically in the same way as Theorem 1. It is shown that the approximate solution $u_m(t)$ satisfies for all $T > 0$ the inclusions (33), (34). Condition (32) then allows us to prove that $(\phi(u_m))'$ are bounded in $L^2(0, T; L^1(\Omega))$; this estimate is needed to pass to the limit (for examples of similar analysis of quasilinear equations of general form, see [125, 294]).

In the course of proof of Theorem 6 (and Theorem 1), it is possible to derive explicit and practically useful integral estimates of the global solution in various norms. In particular, the lemmas we proved above permit us to do that at least for equation (5). More illustrative estimates in the norm of $C(\Omega)$ will be obtained, together with the global solvability condition, by a different method. Let us note that construction of the set $\mathcal{W}$ for equation (1) of general form by the method of subsection 1.3, requires quite awkward computations and a certain effort. It is much easier to do by comparison methods.

### *2 Analysis using a family of stationary solutions*

Here we present an application of the method of stationary states, which is different from the one of § 1. The method is used here to determine conditions of global solvability of the problem. Here we shall not be needing any preliminary results, as all the necessary material is contained in § 1.

Below we shall assume for simplicity that a local solution of the problem (1)–(3) exists and belongs to $C^{1,2}_{t,x}$ wherever it is positive and has for all $t > 0$ a continuous derivative $\nabla_x \phi(u(t, x))$. Then the solution obeys the Maximum Principle and depends in a continuous monotone fashion on the initial function; in particular, the comparison theorem is valid for solutions of equation (1). The proof of these assertions proceeds by constructing the generalized solution as a limit of a sequence of classical strictly positive solutions of equation (1). In the following we assume that $\phi(u_0) \in C^1(\Omega)$.

Let $\{U\}$ be a family of radially symmetric stationary solutions of equation (1) (see (1.6), (1.7)). We shall only need the estimate (1.9):

$$U(r; U_0) \geq U_-(r; U_0) = \phi^{-1}\left[\phi(U_0)(1 - r^2/r_0^2)_+\right], \quad r_0 = \left[2N\frac{\phi(U_0)}{Q(U_0)}\right]^{1/2}. \quad (36)$$

Here $U_0 > 0$ is the parameter of the family, $r = |x|$. Let us note that in certain particular cases, for example for $N = 1$ or $Q(u) = \mu\phi(u)$, exact equalities, which were derived in § 1, can be used.

**Theorem 7.** *Under the above assumptions the stable set of the problem* (1)–(3) *contains the set*

$$\mathcal{W} = \{u_0 \geq 0 \mid \exists U_0 > 0 : \Omega \subseteq \operatorname{supp} U(r; U_0), u_0(x) \leq U(r; U_0) \ in\ \Omega\}. \quad (37)$$

*Proof.* The proof is based on the Maximum Principle. Indeed, if $u_0 \in \mathcal{W}$ then by construction of $\mathcal{W}$ we shall have for all $t \geq 0$ the inequality $u(t, x) \leq U(r; U_0)$, $x \in \Omega$, since $U(r; U_0) \geq 0$ on $\partial\Omega$. The latter follows from the condition $\Omega \subseteq \operatorname{supp} U(r;\ U_0)$. □

Let us note that (37) allows us to describe the structure of initial functions in $\mathcal{W}$. In particular, the choice of the size of $U_0$ in (37) determines the maximal amplitude of the global solution, $\sup_{x\in\Omega} u(t, x) \leq U_0$, $t > 0$. The magnitude of $U_0$ depends to a large extent on the geometrical dimensions of the initial perturbation: the "wider" it is, the larger must the value of the parameter $U_0$ be, because otherwise the function $u_0(x)$ will not be majorized from above in $\Omega$ by the stationary solution $U(r; U_0)$. Inequality (36) allows us to obtain a reasonably good estimate of this dependence.

**Corollary 1.** *Let*

$$\overline{\lim_{s\to\infty}}\, |\phi(s)/Q(s)| = \infty. \tag{38}$$

*Then the problem* (1)–(3) *has global solutions for all initial functions.*

Indeed, if (38) holds, by (36) we have $U(r; U_0) \to \infty$ along some subsequence $U_0 = U_0^k \to \infty$ in $\mathbf{R}^N$. Therefore for any domain $\Omega$ and functions $u_0(x)$ there exists $U_0^k$, such that $u_0(x) \leq U(r; U_0^k)$ in $\overline{\Omega}$, which by the Maximum Principle ensures boundedness of the solution uniformly in $t$. For equation (5) this case obtains if $\beta < \sigma + 1$.

**Corollary 2.** *Let*

$$\overline{\lim_{s\to\infty}}\, [\phi(s)/Q(s)| = \mu = const > 0. \tag{39}$$

*Then if* $\overline{\Omega}$ *is contained in a ball of radius* $(2N\mu)^{1/2}$, *the problem* (1)–(3) *has a global solution for all initial functions.*

This claim follows immediately from (36). Furthermore, from (36) we obtain

**Corollary 3.** *Let*

$$\phi(s)/Q(s) \to 0,\ s \to \infty. \tag{40}$$

*Then, if* $\operatorname{supp} U(r; U_0) \to \{0\}$ *as* $U_0 \to \infty$, *the set* $\mathcal{W}$, *defined by* (37) *is bounded in* $C(\Omega)$. *If, on the other hand,* $\operatorname{supp} U(r; U_0) = \mathbf{R}^N$ *for all sufficiently large* $U_0 > 0$[3], *then* $\mathcal{W}$ *contains functions* $u_0(x)$ *of arbitrarily large norm in* $C(\Omega)$.

Let us note that it is precisely in the case (40) that we should expect the appearance of unbounded solutions for sufficiently large initial functions $u_0(x)$, which do not belong to the stable set (37).

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[3]This possibility occurs, for example, if condition (1.12′) holds.

*3 Analysis of unbounded solutions using the eigenfunctions method*

This method is similar to the one used in subsection 2, § 6, Ch. V. As well as the method based on conditions of $\phi$-criticality, it is useful in the analysis of unbounded solutions of boundary value problems in bounded domains.

Let us impose some restrictions on the functions that enter equation (1). We shall take the functions $\phi$, $Q$ to be convex in $\mathbf{R}_+$. We shall assume that the function $P(s) = Q(\phi^{-1}(s))$ is also convex, and that $s/P(s)$ is non-increasing in $\mathbf{R}_+$. These conditions can be written down in the form

$$\phi''(s) \geq 0, Q''(s) \geq 0, \tag{41}$$

$$Q''(s)\phi'(s) - Q'(s)\phi''(s) \geq 0, \tag{42}$$

$$Q'(s)\phi(s) - Q(s)\phi'(s) \geq 0, \quad s > 0. \tag{43}$$

All the inequalities are satisfied, for example, by the coefficients of equation (5) for $\beta \geq \sigma + 1$. As shown above, it is exactly for these values of the parameters that unbounded solutions can be expected.

As usual, we shall denote by $w_1(x)$ the positive in $\Omega$ eigenfunction of problem (9), which corresponds to the minimal eigenvalue $\lambda_1 > 0$. We shall choose the function $w_1$, such that $\|w_1\|_{L^1(\Omega)} = 1$.

The main result is contained in the following theorem, where we have introduced the notation

$$E_0 = \int_\Omega w_1(x)u_0(x)\,dx.$$

**Theorem 8.** *Let conditions* (41)–(43) *hold and let the function* $u_0$ *in* (2) *be such that*

$$\lambda_1\phi(E_0) < Q(E_0). \tag{44}$$

*Then problem* (1)–(3) *does not have a global solution and there exists* $T_0 \leq T_*$, *where*

$$T_* = \frac{Q(E_0)}{Q(E_0) - \lambda_1\phi(E_0)}\int_{E_0}^{\infty} \frac{d\eta}{Q(\eta)} < \infty,$$

*such that*

$$\overline{\lim_{t \to T_0}}\ \sup_{x \in \Omega} u(t, x) = \infty. \tag{45}$$

*Proof.* Let us set

$$E(t) = (u(t, x), w_1(x)) \equiv \int_\Omega u(t, x)w_1(x)\,dx.$$

Then $E(0) = E_0$. As in subsection 2, § 6, Ch. V, we obtain for $E(t)$ the equality

$$dE(t)/dt = -\lambda_1(w_1, \phi(u)) + (w_1, Q(u)), \quad t > 0. \tag{46}$$

which we shall analyze below.

By Jensen's inequality for convex functions, from (42) we have the estimate $(w_1, Q(u)) \equiv (w_1, P(\phi(u))) \geq P[(w_1, \phi(u))]$, using which we obtain from (46)

$$\frac{dE}{dt} \geq P[(w_1, \phi(u))]\left\{1 - \lambda_1 \frac{(w_1, \phi(u))}{P[(w_1, \phi(u))]}\right\}. \tag{47}$$

However, the function $s/P(s)$ is non-increasing in $\mathbf{R}_+$ (see (43)), and therefore in view of the convexity of $\phi$, and thus also Jensen's inequality $(w_1, \phi(u)) \geq \phi[(w_1, u)] = \phi(E)$, we have $(w_1, \phi(u))/P[(w_1, \phi(u))] \leq \phi(E)/Q(E)$. Then from (47) we get

$$\frac{dE}{dt} \geq P[(w_1, \phi(u))]\left[1 - \lambda_1 \frac{\phi(E)}{Q(E)}\right]. \tag{48}$$

From (48), (43), (44) we conclude that $E'(t) \geq 0$, that is, that $E(t) \geq E_0$. Then from (48) and Jensen's inequality it follows that $E'(t) \geq Q(E(t))[1 - \lambda_1 \phi(E_0)/Q(E_0)]$. Hence we have that the function $E(t)$ cannot be bounded for all $t \in (0, T_+]$ and there exists $T_1 \leq T_*$, such that $E(t) \to \infty$ as $t \to T_1^-$. Since $E(t) \leq \sup_x u(t, x)$, (45) follows immediately, which concludes the proof. □

**Remark.** The inequality

$$E' \geq \mu_0 Q(E), 0 < t < T; \ \mu_0 = 1 - \lambda_1 \frac{\phi(E_0)}{Q(E_0)} > 0, \tag{49}$$

obtianed in the course of the proof of Theorem 8, allows us in a fairly general case to derive an upper bound of $u(t, \cdot)$ in the norm of $L^\infty(\mathbf{R}^N)$. Indeed, from Theorem 3 in § 1 it is easy to obtain a condition on the functions $\phi$ and $Q$, under which $E(T) = \infty$, if $T$ is the blow-up time. Then integrating (49) over $(t, T)$ we have

$$\int_{E(t)}^{\infty} \frac{d\eta}{Q(\eta)} \geq \mu_0(T - t). \tag{50}$$

Finally, if in the computation of $E(t) = (u(t, x), w_1(x))$ in (50) we use the estimate (1.26) of the solution $u(t, x)$ in terms of $u(t, 0)$ and the structure of the function $U$ given by (1.9), then (50) leads to an upper bound for the function $u(t, 0) = \sup_x u(t, x)$. In a number of cases in terms of dependence on the nonlinear coefficients $\phi(u)$ and $Q(u)$, this bound is quite sharp, which is easily verified.

This is a typical example of the situation, in which from an ordinary differential inequality for some "energy-like" functional of the solution, using an inequality of the same sign for the "energy," an $L^\infty$ estimate of the solution is derived.

Let us make one observation. If $\phi \equiv Q$, then condition (44) assumes the form $\lambda_1 < 1$. Therefore the solution becomes unbounded for all initial functions $u_0 \not\equiv 0$.

Recall that if $\lambda_1 = \lambda_1(\Omega) > 1$, then the boundary value problem is always globally solvable (see Theorem 6).

To conclude, let us note that practically all the results of § 1 having to do with S- and LS- blow-up regimes can be used to describe asymptotic behaviour of unbounded solutions of the boundary value problem (1)–(3). In particular, for the problem with power type nonlinearities (5), (6), Theorems 5, 6 of § 1, as well as the results of § 6, Ch. IV, remain valid. All of these allow us to provide a sufficiently exact description of the singularity of the limiting profile $u(T_0^-, x)$.

## § 3 A parabolic system of quasilinear equations with a source

This section is concerned with the study of the properties of solutions of a parabolic system of two quasilinear equations of nonlinear heat conduction with sources:

$$u_t = \Delta u^{\nu+1} + v^p, \tag{1}$$

$$v_t = \Delta v^{\mu+1} + u^q. \tag{2}$$

Here $\mu > 0$, $\nu > 0$, $p \geq 1$, $q \geq 1$ are constants. This system describes processes of heat diffusion and combustion in two-component continua with nonlinear heat conductance and volumetric heat release. The functions $u$, $v$ can be interpreted as temperatures of interacting components of some combustible mixture. We shall be especially interested in the conditions for existence of unbounded solutions as well as conditions for their localization in the Cauchy problem.

Let us observe that there are four parameters in (1), (2), and therefore, even though the nonlinearities are of power type, for arbitrary $\mu$, $\nu$, $p$, $q$ the system does not admit self-similar solutions, which, as we know, provide us with a detailed description of asymptotic behaviour of unbounded solutions and, in particular, of the localization property of finite time blow-up regimes.

We shall consider initially the first boundary value problem for (1), (2):

$$u(0, x) = u_0(x) \geq 0,\ \ v(0, x) = v_0(x) \geq 0,\ \ x \in \Omega, \tag{3}$$

$$u(t, x) = v(t, x) = 0,\ \ t > 0, x \in \partial\Omega, \tag{4}$$

where $\Omega$ is a bounded domain in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$; $u_0$, $v_0$ are bounded continuous functions, $u_0^{1+\nu} \in H_0^1(\Omega)$, $v_0^{1+\mu} \in H_0^1(\Omega)$. First we shall determine the conditions of its global insolvability. Then, using an approach based on the analysis of a family of stationary solutions of the system, we shall derive restrictions on the parameters of the problem for which it is always globally solvable for arbitrary initial functions $u_0$, $v_0$, and establish a condition for the absence of localization in finite time blow-up in the Cauchy problem for (1), (2).

It turns out that much depends on the sign of only one parameter $m = pq - (1+\mu)(1+\nu)$. If $m < 0$ then the boundary value problem has only globally bounded solutions (there is no blow-up) and unbounded solutions in the Cauchy problem are not localized. This result appears to be optimal, that is, the inequality $m < 0$ is a necessary and sufficient condition for both these properties of solutions.

In this section we study only the most general properties of solutions of the system (1), (2). A more detailed numerical investigation of systems of this kind is undertaken in § 4, where we also consider self-similar solutions, which exist under some restrictions on the parameters of the problem.

## 1 Conditions for absence of global solutions of the boundary value problem for $p \geq 1+\mu$, $q \geq 1+\nu$

Below, under the indicated restrictions on the parameters, we single out an unstable set $\mathcal{V}$ in the space of initial functions, such that the inclusion $\{u_0, v_0\} \in \mathcal{V}$ implies global insolvability of the problem (1)–(4). This means that there exists a blow-up time $T_0 < \infty$ and

$$\overline{\lim_{t\to T_0}} \left( \|u^{\nu+1}(t,\cdot)\|^2_{L^2(\Omega)} + \|v^{\mu+1}(t,\cdot)\|^2_{L^2(\Omega)} \right) = \infty. \tag{5}$$

Let us note that formally in (5) it can happen that only one of the functions $u$ or $v$ grows without bound as $t \to T_0^-$. However, because of the "entangled" nature of the source terms in (1), (2) the functions $u$, $v$ have to become unbounded at the same time.

In the proof of the propositions formulated below, we shall assume that the local solution of the boundary value problem satisfies the natural inclusions

$$\begin{aligned} &\{u^{1+\nu/2}\}_t,\ \{v^{1+\mu/2}\}_t \in L^2(0,T; L^2(\Omega)),\\ &u^{\nu+1},\ v^{\mu+1} \in L^\infty(0,T; H^1_0(\Omega)),\ T < T_0. \end{aligned} \tag{6}$$

Under the assumption of boundedness of $u$, $v$ these can be easily derived by Galerkin approximations. It is also not hard to establish a weak Maximum Principle, so that if $u$, $v$ are bounded in $(0,T)\times\Omega$, then $u \geq 0$, $v \geq 0$ a.e. in $\Omega$, $0 < t < T$.

*1 Derivation of a systems of ordinary differential inequalities*

Let us denote by $w_1(x) > 0$ in $\Omega$ and $\lambda_1 > 0$, respectively, the first eigenfunction and the corresponding (smallest) eigenvalue of the problem

$$\Delta w + \lambda w = 0,\ x \in \Omega,\ w|_{\partial\Omega} = 0. \tag{7}$$

Let us choose the function $w_1$ so that

$$\|w_1\|_{L^1(\Omega)} = 1. \tag{8}$$

Taking the scalar product in $L^2(\Omega)$ with $w_1$ of both parts of the equations (1), (2) and integrating the resulting expressions in $t$, we obtain the system of equalities

$$(u(t), w_1) - (u_0, w_1) = -\lambda_1 \int_0^t (u^{\nu+1}(s), w_1)\, ds + \int_0^t (v^p(s), w_1)\, ds, \tag{9}$$

$$(v(t), w_1) - (v_0, w_1) = -\lambda_1 \int_0^t (v^{\mu+1}(s), w_1)\, ds + \int_0^t (u^q(s), w_1)\, ds. \tag{10}$$

Let us introduce the notation

$$a_0 = (u_0, w_1) \geq 0,\ b_0 = (v_0, w_1) \geq 0, \tag{11}$$

and let us set

$$P(t) = (u^{\nu+1}(t), w_1)^{1/(\nu+1)},\ R(t) = (v^{\mu+1}(t), w_1)^{1/(\mu+1)}. \tag{12}$$

From the Hölder inequality and (8) it follows that

$$\begin{aligned} (u(t), w_1) &\leq (u^{\nu+1}(t), w_1)^{1/(\nu+1)} = P(t), \\ (v(t), w_1) &\leq (v^{\mu+1}(t), w_1)^{1/(\mu+1)} = R(t). \end{aligned} \tag{13}$$

Furthermore, taking into account the fact that $p \geq 1 + \mu$, $q \geq 1 + \nu$, we have

$$(v^p(t), w_1) \geq R^p(t),\ (u^q(t), w_1) \geq P^q(t). \tag{14}$$

Using the notation of (11), (12) and the estimates (13), (14), we conclude that the solution of the problem (1)–(4) satisfies for all admissible $t > 0$ the system of inequalities

$$P(t) - a_0 \geq -\lambda_1 \int_0^t P^{\nu+1}(s)\, dx + \int_0^t R^p(s)\, ds, \tag{15}$$

$$R(t) - b_0 \geq -\lambda_1 \int_0^t R^{\mu+1}(s)\, dx + \int_0^t P^q(s)\, ds. \tag{16}$$

In conjunction with (15), (16) let us consider the following system of ordinary differential equations:

$$d\tilde{P}/dt = -\lambda_1 \tilde{P}^{\nu+1} + \tilde{R}^p, \tag{17}$$

$$d\tilde{R}/dt = -\lambda_1 \tilde{R}^{\mu+1} + \tilde{P}^q,\ t > 0. \tag{18}$$

Let the functions $\tilde{P}$, $\tilde{R}$ satisfy the conditions

$$\tilde{P}(0) = a_0 \geq 0, \quad \tilde{R}(0) = b_0 \geq 0. \tag{19}$$

A direct comparison of (15), (16) with the problem (17)–(19) shows that for all admissible $t$ we have the inequalities

$$P(t) \geq \tilde{P}(t), \quad R(t) \geq \tilde{R}(t). \tag{20}$$

Therefore the system of equations (17), (18) allows us, in view of (20), to define the conditions under which the functions $P(t)$, $R(t)$ cannot both be bounded for all $t > 0$, that is $\overline{\lim} \max\{P(t), R(t)\} = \infty$, $t \to T_0 < \infty$. In view of the inequalities

$$P(t) \leq \|u^{\nu+1}\|^{1/(\nu+1)}_{L^2(\Omega)} \|w_1\|^{1/(\nu+1)}_{L^2(\Omega)},$$

$$R(t) \leq \|v^{\mu+1}\|^{1/(\mu+1)}_{L^2(\Omega)} \|w_1\|^{1/(\mu+1)}_{L^2(\Omega)},$$

this ensures that (5) holds.

## *2 The case $p = 1 + \mu$, $q = 1 + \nu$*

**Theorem 1.** *Let $p = 1 + \mu$, $q = 1 + \nu$ and $u_0 + v_0 \not\equiv 0$ in $\Omega$. Moreover, let the domain $\Omega$ be such that*

$$\lambda_1 < 1. \tag{21}$$

*Then the problem (1)–(4) has no global solutions, and condition (5) holds for some $T_0 < \infty$.*

*Proof.* Let us set $E(t) = \tilde{P}(t) + \tilde{R}(t)$. Adding up equations (17) and (18) for $p = 1 + \mu$, $q = 1 + \nu$, we have that $E$ satisfies the equation

$$dE/dt = (1 - \lambda_1)[\tilde{P}^{1+\nu} + \tilde{R}^{1+\mu}], \quad t > 0, \tag{22}$$

where we also have by assumption that $E(0) = (u_0 + v_0, w_1) > 0$.

We have to consider two separate cases.

a) Let $\mu = \nu$. Then, using the inequality

$$1 + \xi^{1+\nu} \geq 2^{-\nu}(1 + \xi)^{1+\nu}, \quad \xi \geq 0, \tag{23}$$

and (21), we get from (22) that

$$dE/dt \geq (1 - \lambda_1)2^{-\nu}[\tilde{P} + \tilde{R}]^{1+\nu} \equiv (1 - \lambda_1)2^{-\nu}E^{1+\nu}, \quad t > 0.$$

Hence it follows that there exists a time $t = T_0 \leq T_*$, where

$$T_* = \frac{2^{\nu}}{\nu(1 - \lambda_1)} E^{-\nu}(0) < \infty.$$

such that $E(t) \to \infty$ as $t \to T_0$. Therefore the solution of the boundary value problem is unbounded in the sense of (5).

b) Let $\mu \neq \nu$ and for definiteness $\mu > \nu$. Then, using Young's inequality

$$\tilde{R}^{1+\mu} \geq \epsilon \tilde{R}^{1+\nu} - \epsilon^{(\mu+1)/(\mu-\nu)} A_0,$$

where $\epsilon > 0$ is an arbitrary constant and

$$A_0 = \frac{\mu - \nu}{\mu + 1} \left( \frac{1+\nu}{1+\mu} \right)^{(\nu+1)/(\mu-\nu)} > 0,$$

we obtain from (22)

$$dE/dt \geq (1-\lambda_1)[\tilde{P}^{1+\nu} + \epsilon \tilde{R}^{1+\nu}] - (1-\lambda_1) A_0 \epsilon^{(\mu+1)/(\mu-\nu)}. \tag{24}$$

Let $\epsilon \leq 1$. Then, using (23), we derive from (24) the following estimate:

$$dE/dt \geq (1-\lambda_1)\epsilon 2^{-\nu} E^{1+\nu} - (1-\lambda_1)\epsilon^{(\mu+1)/(\mu-\nu)} A_0. \tag{25}$$

Let us set

$$\epsilon = \epsilon_0 = \min\{1, [2^{-\nu-1} E^{1+\nu}(0)/A_0]^{(\mu-\nu)/(1+\nu)}\}.$$

Then for all $E \geq E(0)$ the right-hand side of (25) is positive and therefore $E(t) \to \infty$ as $t \to T_0$, where $T_0 \leq T_*$,

$$T_* = \frac{2^\nu}{\epsilon_0(1-\lambda_1)} \int_{E(0)}^{\infty} \frac{d\eta}{\eta^{1+\nu} - 2^\nu A_0 \epsilon_0^{(1+\nu)/(\mu-\nu)}} < \infty.$$

Therefore in the case under consideration the unstable set $\mathcal{V}$ is all the space $\{u_0, v_0 \mid u_0 + v_0 \not\equiv 0\}$. □

It is not hard to obtain the same result in a different way. In Figure 80 we have drawn in a schematic way the integral curves of the first order equation

$$\frac{d\tilde{P}}{d\tilde{R}} = \frac{\tilde{R}^{1+\mu} - \lambda_1 \tilde{P}^{1+\nu}}{\tilde{P}^{1+\nu} - \lambda_1 \tilde{R}^{1+\mu}}, \quad \tilde{P} > 0, \ \tilde{R} > 0, \tag{26}$$

which is equivalent to the system (17), (18), in the case $\lambda_1 < 1$, $\mu > \nu$. The dashed line shows the isocline of zero $\tilde{P}_0$: $\tilde{P} = [\lambda_1^{-1} \tilde{R}^{1+\mu}]^{1/(1+\nu)}$, the dashed and dotted line shows the isocline of infinity, $\tilde{P}_\infty$: $\tilde{P} = [\lambda_1 \tilde{R}^{1+\mu}]^{1/(1+\nu)}$. The thick curve denotes a special trajectory, the separatrix $\tilde{P}_S$: $\tilde{P} = P_S(\tilde{R})$. For large $\tilde{R}$

$$P_S(\tilde{R}) = \lambda_1^{1/(1+\nu)} \tilde{R}^{(1+\mu)/(1+\nu)} + \frac{1-\lambda_1^2}{\lambda_1} \frac{\tilde{R}}{1+\mu} + \dots . \tag{27}$$

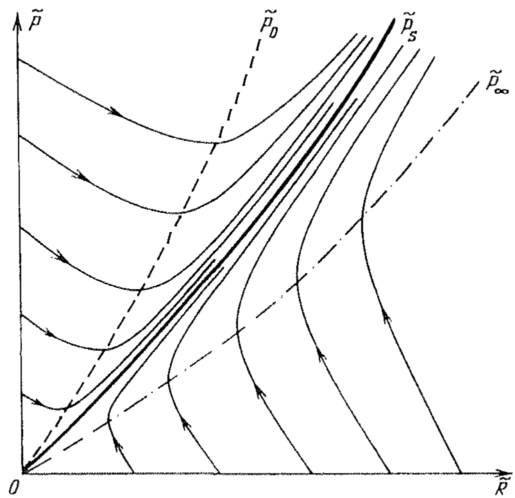



**Fig. 80.** The phase plane in the case $p = 1 + \mu$, $q = 1 + \nu$, $\lambda_1 < 1$

As time advances (the direction of evolution of the trajectories is indicated by arrows), all the integral curves bunch up into an ever narrowing neighbourhood of the separatrix $\tilde{P}_S$, which by (27) ensures that the functions $\tilde{P}(t)$, $\tilde{R}(t)$ grow to infinity in finite time.

For comparison, we have drawn in Figure 81 the integral curves of the same equation (26), but in the case $\lambda_1 > 1$. Here the functions $\tilde{P}$, $\tilde{R}$ are bounded for all $t > 0$, and, moreover $\tilde{P}(t)$, $\tilde{R}(t) \to 0$ as $t \to \infty$. In some sense this indicates global solvability of the problem (the proof of this fact is found in subsection 3).

*3 The case $p > 1 + \mu$, $q > 1 + \nu$*

Let as before $E(t) = \tilde{P}(t) + \tilde{R}(t)$. From (17), (18) it follows that

$$dE/dt = -\lambda_1[\tilde{P}^{1+\nu} + \tilde{R}^{1+\mu}] + \tilde{R}^p + \tilde{P}^q, \quad t > 0. \tag{28}$$

Let us estimate $\tilde{P}^q$, $\tilde{R}^p$ using Young's inequality:

$$\tilde{P}^q \geq (1 + \lambda_1)\tilde{P}^{1+\nu} - A_0, \quad \tilde{R}^p \geq (1 + \lambda_1)\tilde{R}^{1+\mu} - B_0,$$

where $A_0$ and $B_0$ are constants:

$$A_0 = \frac{q - (\nu + 1)}{\nu + 1}\left[\frac{(\nu + 1)(1 + \lambda_1)}{q}\right]^{q/[q-(\nu+1)]},$$

$$B_0 = \frac{p - (\mu + 1)}{\mu + 1}\left[\frac{(\mu + 1)(1 + \lambda_1)}{p}\right]^{p/[p-(\mu+1)]}.$$

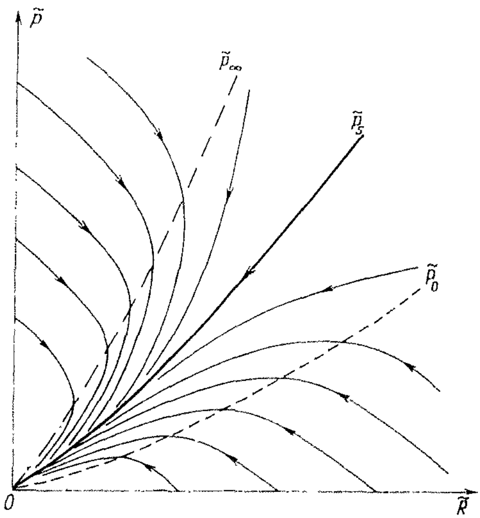



**Fig. 81.** The phase plane in the case $p = 1 + \mu$, $q = 1 + \nu$, $\lambda_1 > 1$

Then we obtain from (28) that

$$dE/dt \geq \tilde{P}^{1+\nu} + \tilde{R}^{1+\mu} - C_0, \ t > 0; \ C_0 = A_0 + B_0. \tag{29}$$

If $\mu = \nu$, using (23), we derive from (29) the inequality

$$dE/dt \geq 2^{-\nu} E^{\nu+1} - C_0, \ t > 0. \tag{30}$$

If, on the other hand, $\mu \neq \nu$, then, setting for definiteness $\mu > \nu$, and using the estimate

$$\tilde{R}^{1+\mu} \geq \tilde{R}^{1+\nu} - D_0, D_0 = \frac{\mu - \nu}{\mu + 1}\left(\frac{\nu + 1}{\mu + 1}\right)^{(\nu+1)/(\mu-\nu)},$$

we obtain from (29)

$$dE/dt \geq 2^{-\nu} E^{\nu+1} - (C_0 + D_0), \ t > 0. \tag{31}$$

It is easy to see that the right-hand side of this inequality admits passing to the limit $\mu \to \nu^+$, that is, for $\mu = \nu$ it is the same as (30). Therefore we have proved

**Theorem 2.** *Let $p > 1 + \mu$, $p > 1 + \nu$ and for definiteness $\mu \geq \nu$. Let the initial functions $u_0$, $v_0$ be such that*

$$E(0) = (u_0 + v_0, w_1) > 2^{\nu/(\nu+1)}(C_0 + D_0)^{1/(1+\nu)}. \tag{32}$$

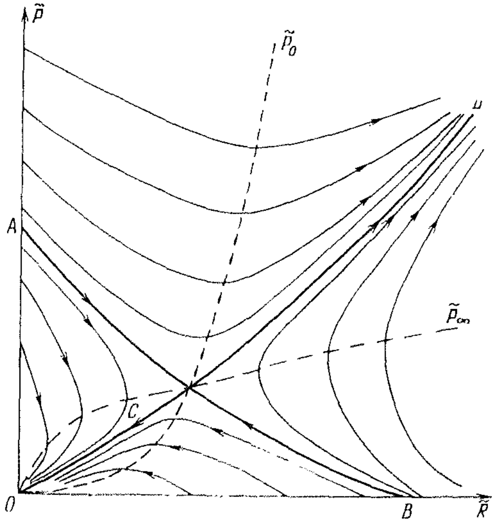



**Fig. 82.** The phase plane in the case $p > 1 + \mu$, $q > 1 + \nu$

*Then the problem* (1)–(4) *does not have a global solutions, and for some* $T_0 \leq T_*$, *where*

$$T_* = 2^\nu \int_{E(0)}^{\infty} \frac{d\eta}{\eta^{1+\nu} - 2^\nu(C_0 + D_0)} < \infty,$$

*condition* (5) *holds.*

Of course, the functions $u_0$, $v_0$ satisfying condition (32) do not exhaust all the unstable set $\mathcal{V}$. Even though Theorem 2 gives an upper bound for $T_0$, it does not use all the information contained in the system (17), (18). A more detailed and explicit description of the set $\mathcal{V}$ is provided by the analysis of the integral curves of the equation

$$\frac{d\tilde{P}}{d\tilde{R}} = \frac{\tilde{R}^p - \lambda_1 \tilde{P}^{1+\nu}}{\tilde{P}^q - \lambda_1 \tilde{R}^{1+\mu}}, \quad \tilde{P} > 0, \ \tilde{R} > 0, \tag{33}$$

which is equivalent to that system.

These are given schematically in Figure 82, where we have distinguished the isoclines of zero $\tilde{P}_0$: $\tilde{P} = |\lambda_1^{-1}\tilde{R}^p|^{1/(1+\nu)}$, of infinity, $\tilde{P}_\infty$: $\tilde{P} = |\lambda_1 \tilde{R}^{1+\mu}|^{1/q}$. The thick curves show the separatrices $A - B$ and $C - D$. The set $\mathcal{V}$ contains all the points $\{\tilde{P}, \tilde{R}\} = \{(u_0, w_1), (v_0, w_1)\}$ lying above the separatrix $A - B$. Trajectories through points in this region converge as $\tilde{P}, \tilde{R} \to \infty$ to the separatrix $C - D$, which ensures unbounded growth of the functions $\tilde{P}(t)$, $\tilde{R}(t)$ in finite time.

Let us note that the behaviour of the integral curves lying under the separatrix $A - B$ indicates the existence in the problem of a stable set $\mathcal{W}$. Here it is charac-

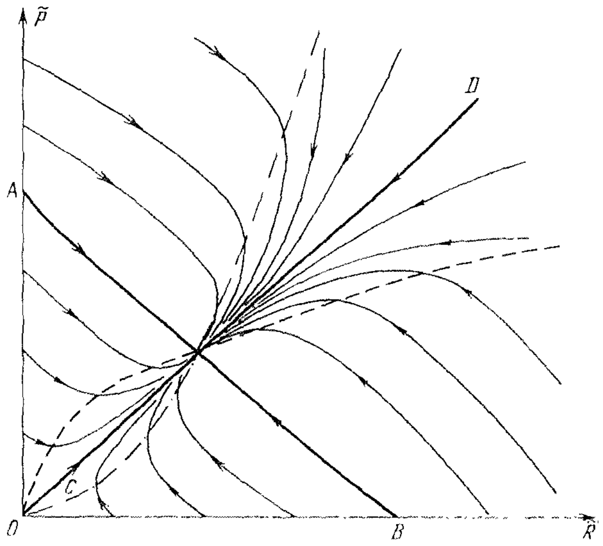



**Fig. 83.** The phase plane in the case $pq < (1+\mu)(1+\nu)$

terized by $\tilde{P}, \tilde{R} \to 0$ as $t \to \infty$. A rigorous construction of the set $\mathcal{W}$ is done in subsection 2.

**Remark.** In the derivation of the system (17), (18) we used the restrictions $p \geq 1+\mu$, $q \geq 1+\nu$ (see (14)). Nonetheless, the phase plane of the system (17), (18) correctly reflects typical behaviour of solutions of the original problem (1)–(4) for arbitrary values of the parameters. It is not hard to see that the phase plane picture depends on the sign of the one parameter $m = pq - (1+\mu)(1+\nu)$, which agrees well with results obtained below.

a) If $m = 0$, that is, $pq = (1+\mu)(1+\nu)$, then global insolvability and global solvability of the problem for arbitrary $u_0$, $v_0$ hold for $\lambda_1 < 1$ and $\lambda_1 > 1$, respectively; phase portraits in these two cases are the same as those in Figures 80, 81 (see subsection 3).

b) If $m > 0$, that is, $pq > (1+\mu)(1+\nu)$, then it follows from the analysis of the system (17), (18) that there are non-empty stable and unstable sets (see Figure 82 and subsection 2).

c) In the case $m = pq-(1+\mu)(1+\nu) < 0$ the phase plane has the appearance as in Figure 83. Here there are no trajectories to which there correspond unbounded solutions, so that the stable set can be the whole space of initial functions.

We should note that the Galerkin method, which was used in § 2 to study parabolic equations, cannot provide us with all the above results. Roughly speaking, in the process of using this technique to derive *a priori* bounds for solutions, the "control" parameter $m = pq - (1+\mu)(1+\nu)$ does not arise. For example, the global solvability condition for the system for arbitrary $u_0$, $v_0$ is given by the two inequalities $p < 1+\mu$, $q < 1+\nu$. Clearly, this domain in parameter space is much smaller than the (optimal) set $pq < (1+\mu)(1+\nu)$.

## 2 The stable set for $pq > (1+\mu)(1+\nu)$

Below we shall assume that the functions $u$, $v$ are in $C^{1,2}_{t,x}$ wherever they are positive and have in $\Omega$ continuous derivatives $\nabla u^{1+\nu}$, $\nabla v^{1+\mu}$ (these assumptions are quite natural; see § 3, Ch. I). Then the solution of the problem satisfies the Maximum Principle and depends monotonically on initial functions.

The stable set can be defined in two ways. In the course of applying one of the methods, some (later shown to be insignificant) restrictions on the parameters of the problem have to be imposed; this analysis allows us to consider quite interesting properties of the stationary solutions of the problem (1), (2), (4).

### *1 The stationary solution*

Let $pq > (1+\mu)(1+\nu)$. Let us consider the functions $U, V$, which solve the stationary system of equations (1), (2), which we write for convenience in the form

$$\Delta(|U|^{\nu}U) + |V|^{p-1}V = 0, \tag{34}$$

$$\Delta(|V|^{\mu}V) + |U|^{q-1}U = 0, \quad x \in \Omega \tag{35}$$

(obviously, for positive $U, V$ this system coincides with the original one.) The functions $U, V$ satisfy the conditions

$$U(x) = 0,\ V(x) = 0, \quad x \in \partial\Omega. \tag{36}$$

Let us make the change of variables $|U|^{\nu}U \to U$, $|V|^{\mu}V \to V$. Then for the new functions $U, V$ we obtain the problem

$$\Delta U + |V|^{\alpha-1}V = 0,\ \ \Delta V + |U|^{\beta-1}U = 0, \quad x \in \Omega, \tag{37}$$

$$U = V = 0, \quad x \in \partial\Omega. \tag{38}$$

Here we have introduced the notation $\alpha = p/(\mu+1)$, $\beta = q/(\nu+1)$ (let us note that $\alpha\beta > 1$). Solving the first equation for $V$, $V = -|\Delta U|^{1/\alpha-1}\Delta U$ and substituting into the second one, we obtain the following problem for the function $U$:

$$-\Delta(|\Delta U|^{1/\alpha-1}\Delta U) + |U|^{\beta-1}U = 0, \quad x \in \Omega, \tag{39}$$

$$U = \Delta U = 0, \quad x \in \partial\Omega. \tag{40}$$

Clearly, using the comparison theorem, the stable set can be defined in terms of the solution of the stationary problem (39), (40). However, such a construction would not be optimal. We start with a negative result,

**Proposition 1.** *Let the domain $\Omega$ be star-shaped with respect to the point $x = 0$. Assume that $N > 2(1 + 1/\alpha)$ and*

$$\beta > [N/\alpha + 2(1 + 1/\alpha)][N - 2(1 + 1/\alpha)]^{-1}, \tag{41}$$

*Then the problem* (39), (40) *has no non-trivial non-negative solutions.*

*Proof.* Following [332, 333], let us take the scalar product in $L^2(\Omega)$ of both sides of equation (39) with the function $w(x) = \sum_{i=1}^{N} x_i U_{x_i}$. Then, taking into account the boundary conditions, using Green's formula, we obtain

$$\begin{aligned} 0 = -(\Delta(|\Delta U|^{1/\alpha-1}\Delta U), w) + (U^\beta, w) = -(|\Delta U|^{1/\alpha-1}\Delta U, \Delta w) - \\ - \int_{\partial\Omega} w \frac{\partial}{\partial n}(|\Delta U|^{1/\alpha-1}\Delta U)\, ds - \frac{N}{\beta+1}\|U\|^{\beta+1}_{L^{\beta+1}(\Omega)}, \end{aligned} \tag{42}$$

where $\partial/\partial n$ stands for derivative in the direction of the outward normal to $\partial\Omega$.

It is not hard to see that

$$w(x) \le 0, \quad \frac{\partial}{\partial n}(|\Delta U|^{1/\alpha-1}\Delta U)(x) \ge 0, \quad x \in \partial\Omega,$$

This follows from the fact that $\Omega$ is star-shaped with respect to $x = 0$, the boundary conditions (40), as well as the assumptions $U \ge 0$, $\Delta U \le 0$ in $\Omega$. Therefore

$$- \int_{\partial\Omega} w \frac{\partial}{\partial n}(|\Delta U|^{1/\alpha-1}\Delta U)\, ds \ge 0, \tag{43}$$

and then from (42), using the easily verified identity

$$-(|\Delta U|^{1/\alpha-1}\Delta U, \Delta w) \equiv \left(\frac{N}{1+1/\alpha} - 2\right)\|\Delta U\|^{1+1/\alpha}_{L^{1+1/\alpha}(\Omega)},$$

we obtain the inequality

$$\left(\frac{N}{1+1/\alpha} - 2\right)\|\Delta U\|^{1+1/\alpha}_{L^{1+1/\alpha}(\Omega)} - \frac{N}{\beta+1}\|U\|^{\beta+1}_{L^{\beta+1}(\Omega)} \le 0. \tag{44}$$

On the other hand, by taking the scalar product with $U$ and integrating by parts, we have from (39) that

$$\|U\|^{\beta+1}_{L^{\beta+1}(\Omega)} = \|\Delta U\|^{1+1/\alpha}_{L^{1+1/\alpha}(\Omega)},$$

and therefore (44) means that we must have the inequality

$$\left(\frac{N}{1+1/\alpha} - 2 - \frac{N}{\beta+1}\right) \|\Delta U\|^{1+1/\alpha}_{L^{1+1/\alpha}(\Omega)} \le 0. \tag{45}$$

However, if (41) holds the left-hand side of (45) is non-negative, and therefore the only admissible solution is the function $U \equiv 0$. □

Let us note that the Strong Maximum Principle in this case gives us a strict inequality in (43), and therefore in (45), so that this proposition is valid if we have equality in (41).

*2 A family of stationary solutions*

Let $U$ be a positive in $\Omega$ solution of the problem (39), (40). Returning to the original notation, we have that the functions

$$U_1(x) = U^{1/(1+\nu)}(x),\ V_1(x) = V^{1/(1+\mu)}(x)$$

are a solution of the problem (34)–(36).

Let the domain $\Omega$ be star-shaped with respect to the point $x = 0$ (that is, from the condition $x \in \Omega$ it follows that $ax \in \Omega$ for all $a \in (0, 1)$). It is not hard to see that the family of the functions

$$\begin{aligned} U_a(x) &= a^{2(p+\mu+1)/[pq-(\nu+1)(\mu+1)]}U_1(ax), \\ V_a(x) &= a^{2(q+\nu+1)/[pq-(\nu+1)(\mu+1)]}V_1(ax), \end{aligned} \tag{46}$$

where $a \in (0, 1)$ is an arbitrary constant (the parameter of the family), satisfies equations (34), (35) and is strictly positive in domains $\Omega_a$ with boundaries $\partial\Omega_a = \{x \mid ax \in \partial\Omega\}$, and that $U_a = V_a = 0$ on $\partial\Omega_a$. Let us note that $\Omega \subset \Omega_a$ for any $a \in (0, 1)$, so that $U_a > 0$, $V_a > 0$ on $\partial\Omega$.

Using the family (46) we can now determine the stable set $W$ of the problem (1)–(4).

*3 The stable set for $pq > (1+\mu)(1+\nu)$*

**Theorem 3.** *Let $p > 1$, $q > 1$, and assume that there exists a non-trivial stationary solution $U_1, V_1$ of (34)–(36). Then there exists a non-empty stable set $\mathcal{W} = \{(u_0, v_0) \mid u_0 \ge 0, v_0 \ge 0;\ \exists\, a \in (0, 1)\ :\ u_0 \le U_a, v_0 \le V_a \text{ in } \Omega\}$, such that if $(u_0, v_0) \in \mathcal{W}$, the problem (1)–(4) has a global (bounded) solution.*

*Proof.* The theorem follows from the Maximum Principle. Under the above assumptions concerning smoothness of the solution everywhere in $\mathbf{R}_+ \times \overline{\Omega}$, the inequalities $u \le U_a$, $v \le V_a$ hold. Here the value of the parameter $a \in (0, 1)$ is chosen so that $u_0 \le U_a$, $v_0 \le V_a$ in $\Omega$. □

**Remark.** The conditions $p > 1$, $q > 1$ are more or less related to the requirement that the solution be unique. There are reasons to expect that uniqueness holds even if just the one condition $pq \geq 1$ is satisfied (at least if $pq \geq 1$ the spatially homogeneous problem $u' = v^p$, $v' = u^q$, $t > 0$; $u(0) = v(0) = 0$ has only the trivial solution; if, on the other hand, $pq < 1$, then non-trivial solutions exist).

*4 A different method of constructing the stable set*

Let us show now that a non-empty stable set $\mathcal{W}$ exists under the condition $pq > (1+\mu)(1+\nu)$ even if by Proposition 1 there is no stationary solution.

Let us consider first an example that gives a partial explanation of the reason why absence of a positive stationary solution of the problem is not that important.

**Example.** Let $N > 4$, $p = 1 + \mu$, $q = (1+\nu)(N+4)/(N-4)$, that is, condition (41) is satisfied. For these values of parameters $\alpha = 1$, $\beta = (N+4)/(N-4) > 1$, and therefore the boundary value problem (39), (40) does not necessarily have a solution. However, in this case equation (39),

$$-\Delta^2 U + |U|^{(N+4)/(N-4)-1} U = 0, \quad x \in \mathbf{R}^N, \ N > 4,$$

has a strictly positive solution

$$U(x) = C_N(a)/(a^2 + |x|^2)^{(N-4)/2} > 0, \quad x \in \mathbf{R}^N,$$

where $C_N(a) = [a^4 N(N-4)(N^2-4)]^{(N-4)/8}$, $a > 0$ is an arbitrary constant.

It is clear that the family of positive stationary solutions of the original system (34), (35),

$$U_a(x) = [C_N(a)]^{1/(1+\nu)}/(a^2 + |x|^2)^{(N-4)/[2(1+\nu)]},$$

$$V_a(x) \equiv [-\Delta U(x)]^{1/(1+\mu)} = \frac{[2C_N(a)(N-4)]^{1/(1+\mu)}(Na^2/2 + |x|^2)^{1/(1+\mu)}}{(a^2+|x|^2)^{N/[2(1+\mu)]}}, \quad (47)$$

constructed using this positive solution, can be used to construct a set $\mathcal{W}$, since $U_a > 0$, $V_a > 0$ in $\Omega$. It will have almost the same form: $\mathcal{W} = \{(u_0, v_0) \mid u_0 \geq 0, v_0 \geq 0;\ \exists a > 0 : u_0 \leq U_a, v_0 \leq V_a \text{ in } \Omega\}$.

Let us note that the family (47) of positive in $\mathbf{R}^N$ functions defines in a similar way the stable set for the Cauchy problem for the system (1), (2).

It is not hard to show that for all values of $\beta$ that satisfy condition (41) ($\alpha = p/(\mu+1)$, $\beta = q/(\nu+1)$), the system (34), (35) has a family of solutions that are strictly positive in $\mathbf{R}^N$. For example, from Proposition 1 (subsection 2.1) it follows that in this case there exists a radially symmetric solution which is everywhere positive (see subsection 4.1 of § 3, Ch. IV). Using such a family of stationary solutions in $\mathbf{R}^N$, the stable set is easily determined. However, we shall proceed in a different manner.

Indeed, to determine $\mathcal{W}$, it is sufficient to establish local solvability of the system (34), (35), for example, for sufficiently small $|x|$. The following simple assertion, which will be frequently used in the sequel, is true:

**Lemma 1.** *For arbitrary $p, q, \mu, \nu > 0$ the system (34), (35) has in the ball $\omega_1 = \{|x| < \sqrt{2N}\}$ a strictly positive, radially symmetric, monotone decreasing solution $U_1(r)$, $V_1(r)$, $r = |x|$, which satisfies the conditions*

$$U(0) = V(0) = 1, \; U'(0) = V'(0) = 0. \tag{48}$$

*In $\omega_1$ we have the estimates*

$$U_1(r) \geq \left(1 - \frac{r^2}{2N}\right)^{1/(\nu+1)}, \; V_1(r) \geq \left(1 - \frac{r^2}{2N}\right)^{1/(\mu+1)}. \tag{49}$$

Local solvability and the indicated properties of the functions $U_1$, $V_1$ follow from applying the Banach contraction mapping theorem to the integral equation

$$U^{\nu+1}(r) = 1 - \int_0^r \frac{d\rho}{\rho^{N-1}} \int_0^\rho \zeta^{N-1} d\zeta \left[1 - \int_0^\zeta \frac{d\eta}{\eta^{N-1}} \int_0^\eta \xi^{N-1} U^q(\xi) d\xi\right]^{p/(\mu+1)},$$

which is equivalent to the problem.

By (46) the functions $U_1(r)$, $V_1(r)$ define a one-parameter family of stationary solutions of the system (1), (2). Let us set $\omega_a = \{x \mid ax \in \omega_1\}$. Then for any $a > 0$ $U_a$, $V_a$ are defined and strictly positive in $\omega_a$. Now we can determine the stable set without any additional restrictions on the parameters of the problem.

**Theorem 4.** *Let $pq > (1+\mu)(1+\nu)$. Then the problem (1)–(4) has a non-empty stable set*

$$\mathcal{W} = \{(u_0, v_0) \mid u_0 \geq 0, v_0 \geq 0; \; \exists\, a > 0 : \Omega \subseteq \omega_a; \; u_0 \leq U_a, v_0 \leq V_a \text{ in } \Omega\}. \tag{50}$$

Using the estimates (49), we can also distinguish another stable set, which is smaller than (50), but illustrative:

$$\begin{aligned} \tilde{\mathcal{W}} = \Big\{ (u_0, v_0) \mid u_0 \geq 0, v_0 \geq 0; \; \exists\, a > 0 : \Omega \subseteq \{|x| < \sqrt{2N}/a\}; \\ u_0(x) \leq a^{2(p+\mu+1)/m} \left(1 - \frac{a^2|x|^2}{2N}\right)^{1/(\nu+1)}, \\ v_0(x) \leq a^{2(q+\nu+1)/m} \left(1 - \frac{a^2|x|^2}{2N}\right)^{1/(\mu+1)} \text{ in } \Omega; \\ m = pq - (1+\mu)(1+\nu) > 0 \Big\}. \end{aligned} \tag{51}$$

## 3 Conditions of global solvability of the boundary value problem for $pq \le (1+\mu)(1+\nu)$

As before, the main tool is the analysis of the family of stationary solutions.

### *1 The case $pq < (1+\mu)(1+\nu)$*

**Theorem 5.** *Let $pq < (1+\mu)(1+\nu)$, $p > 1$, $q > 1$. Then the boundary value problem (1)–(4) has a global (bounded uniformly in $t$) solution for arbitrary initial functions $u_0$, $v_0$.*

*Proof.* We shall be needing only one property of the functions $U_a$, $V_a$ of the family (46) ($U_1$, $V_1$ were defined in Lemma 1): $U_a \to \infty$, $V_a \to \infty$ as $a \to 0^+$ in $\mathbf{R}^N$. This means that the stable set (50) (or (51)) covers the space of all initial functions. In other words, in the case of any bounded domain $\Omega$ for arbitrary $u_0$, $v_0 \in C(\overline{\Omega})$ we can always find $a > 0$, so that, first of all, $\overline{\Omega} \subset \omega_a = \{x \mid ax \in \omega_1\}$ and, secondly, $u_0(x) \le U_a(x)$, $v_0(x) \le V_a(x)$ in $\Omega$. Then, since $U_a > 0$, $V_a > 0$ on $\partial\Omega$ (that is, we always have $u_0(x) < U_a(x)$, $v_0(x) < V_a(x)$ on $\partial\Omega$), using the Maximum Principle we conclude that $u \le U_a$, $v \le V_a$ in $\mathbf{R}_+ \times \overline{\Omega}$, and therefore the solution is bounded from above uniformly in $t$. □

### *2 The case $pq = (1+\mu)(1+\nu)$*

Here the situation is completely different: existence of the global solution of the problem (1)–(4) depends on solvability of the system of stationary equations (34), (35) with boundary conditions (36). Below we prove the following simple (but not optimal with respect to the admissible domains $\Omega$) result.

**Theorem 6.** *Let $pq = (1+\mu)(1+\nu)$, $p > 1$, $q > 1$, and let the diameter $D_\Omega$ of the domain $\Omega$ satisfy the condition*

$$D_\Omega < \sqrt{2N}. \tag{52}$$

*Then the problem (1)–(4) has a global solution for any initial functions $u_0$, $v_0$.*

*Proof.* For $pq = (1+\mu)(1+\nu)$ the functions (46) are not defined. In this case there is a family of stationary solutions

$$U_a(x) = aU_1(r) > 0, \quad V_a(x) = a^{q/(\mu+1)}V_1(r) > 0, \quad x \in \omega_1, \tag{53}$$

where $a > 0$ is a parameter and $U_1$, $V_1$ are defined in Lemma 1. Hence we immediately have that

$$U_a(x) \to \infty, \quad V_a(x) \to \infty \text{ in } \omega_1 \text{ as } a \to \infty. \tag{53'}$$

Condition (52) means that the domain $\Omega$ can be placed in a ball of radius $\sqrt{2N}$. Without loss of generality, we shall assume that $\overline{\Omega} \subset \omega_1$. But then by (53′) for any $u_0, v_0 \in C(\overline{\Omega})$ there exists a sufficiently large $a > 0$, such that $u \leq U_a$, $v \leq V_a$ in $\Omega$ for $t = 0$, and, since $\overline{\Omega} \subset \omega_1$ (that is, $U_a, V_a > 0$ on $\partial\Omega$) these inequalities hold for all $t > 0$. □

**Remark.** It appears that a necessary and sufficient condition of solvability of the problem (37), (38) for $\alpha\beta = 1$ is the inequality $\lambda_1 > 1$. As shown by the proof of Theorem 6, existence of a positive solution of the problem (37), (38) implies global solvability of the boundary value problem (1)–(4) for $pq = (1+\mu)(1+\nu)$ for practically all $u_0$, $v_0$. Let us note that the same condition $\lambda_1 > 1$ was obtained non-rigorously in subsection 1.

## 4 On localization of unbounded solutions of the Cauchy problem

It turns out that the spatial structure of the family of stationary solutions contains information about quite a subtle property of unbounded solutions of the Cauchy problem for the system (1), (2), namely, localization.

An unbounded solution of the Cauchy problem for (1), (2) with initial functions with compact support,

$$u(0,x) = u_0(x) \geq 0, \quad v(0,x) = v_0(x) \geq 0, \quad x \in \mathbf{R}^N; \tag{54}$$

$$u_0, v_0 \in C(\mathbf{R}^N), \quad u_0^{\nu+1} \in H_0^1(\mathbf{R}^N), \quad v_0^{\mu+1} \in H_0^1(\mathbf{R}^N),$$

will be called *localized* if for all $t \in [0, T_0)$, where $T_0 < \infty$ is the time for which the solution exists, the functions $u(t,x)$, $v(t,x)$ are non-zero inside some ball $\{|x| < L < \infty\}$ ($L$ does not depend on $t$), and are identically equal to zero for $|x| \geq L$. If, on the other hand, perturbations penetrate arbitrarily far from the point $x = 0$ as $t \to T_0$ (that is, as the solution grows to infinity), then we say that *there is no localization* in the Cauchy problem.

Here we shall not be establishing the conditions for a solution of the Cauchy problem to be unbounded, since our main aim is different. That could be done in a relatively simple way, for example, by constructing unbounded subsolutions as in § 3, Ch. IV. Let us note that all the results of subsection 1 (Theorems 1, 2) extend also to the case of the Cauchy problem, since every unbounded solution of the boundary value problem in an arbitrary domain $\Omega \in \mathbf{R}^N$ is a subsolution of the Cauchy problem.

The localization phenomenon in the Cauchy problem for systems of equations is conveniently studied by constructing self-similar solutions (see § 4). However, such solutions exist only for some values of parameters. Thus, for the system (1),

(2) self-similar solutions are possible for $\nu(p+1) = \mu(q+1)$. Furthermore, the questions of existence, and even more so, of stability of unbounded self-similar solutions for systems of equations remain largely open.

*1 The main result (conditions for the appearance of the HS blow-up regime)*

By analyzing a family of stationary solutions we can obtain a sufficient condition for absence of localization in the Cauchy problem. It appears also to be necessary.

**Theorem 7.** *Let $pq < (1+\mu)(1+\nu)$, $p > 1$, $q > 1$. Then all unbounded solutions of the Cauchy problem* (1), (2), (54) *are not localized.*

**Remark.** By comparing with spatially homogeneous solutions of the system (1), (2), which satisfy the equations

$$u' = v^p, \quad v' = u^q, \quad t > 0, \tag{55}$$

we see that a necessary condition for existence of unbounded solutions of the Cauchy problem is the inequality $pq > 1$.

*Proof.* Without loss of generality, we shall assume that $\overline{\lim}\, \max\{u(t,0), v(t,0)\} = \infty$, $t \to T_0 < \infty$. Let us consider the family of stationary solutions (46), where $U_1$, $V_1$ are defined in Lemma 1. By the condition $pq < (1+\mu)(1+\nu)$ the functions $U_a, V_a \to \infty$ in $\mathbf{R}^N$ as $a \to 0^+$. Therefore we can find $a_0 \in (0, 1)$, such that for all $a \in (0, a_0]$ we have that $\operatorname{supp}(u_0 + v_0) \subset \omega_a$, $u_0 \le U_a$, $v_0 \le V_a$ in $\omega_a$.

Let us fix an arbitrary $a \in (0, a_0]$. Then it follows from the Maximum Principle that the solution $u(t,x)$, $v(t,x)$ cannot be larger than the function $U_a$, $V_a$ in $\omega_a$ as long as $u \le U_a$, $v \le V_a$ on $\partial\omega_a$ (and thus as long as $\operatorname{supp}(u+v) \subset \omega_a$). Therefore by unboundedness of the solution $u, v$, for any $a \in (0, a_0]$ there exists $t_a < T_0$, such that $\operatorname{supp}[u(t_a, x) + v(t_a, x)] \not\subset \omega_a$. Hence, by passing to the limit $a \to 0^+$, we obtain the required result. □

In a similar fashion we can establish the following assertion:

**Theorem 8.** *Let $pq < (1+\mu)(1+\nu)$, $p > 1$, $q > 1$, and let the initial functions $u_0$, $v_0$ be radially symmetric and non-increasing in $r = |x|$. Then, if the solution of the Cauchy problem* (1), (2), (54) *is unbounded, for any fixed $x \in \mathbf{R}^N$*

$$\overline{\lim_{t \to T_0}}\, \max\{u(t,x), v(t,x)\} = \infty,$$

*i.e., at least one of the functions $u$ or $v$ becomes infinite as $t \to T_0^-$ on the whole space.*

Therefore for $m = pq - (1+\mu)(1+\nu) < 0$ unbounded solutions express features of combustion in the HS blow-up regime.

We are able to provide specific bounds for the amplitude and the size of the support of the generalized solution of the problem for $m < 0$. Briefly, their derivation is as follows.

Let us write down a simple homothermic solution, which satisfies equations (55). For $pq > 1$ it blows up in finite time:

$$u_m(t) = C_1(T_0 - t)^{-(p+1)/(pq-1)}, \quad v_m(t) = C_2(T_0 - t)^{-(q+1)/(pq-1)},$$
$$0 < t < T_0 < \infty, \tag{56}$$

where

$$C_1 = [(q+1)^p(p+1)/(pq-1)^{p+1}]^{1/(pq-1)},$$
$$C_2 = [(q+1)(p+1)^q/(pq-1)^{q+1}]^{1/(pq-1)}.$$

Let us use the fact that the system of equations (1), (2), which obeys the Maximum Principle, has to have an "intersection" property at least for one component of unbounded solutions having the same blow-up time (see § 3, 4, 5, Ch. IV). On that basis, let us compare the solution of the Cauchy problem and the strictly positive unbounded solution (56).

**Lemma 2.** *Let $T_0 < \infty$ be the blow-up time for an unbounded solution of the problem* (1), (2), (54). *Then for any $t \in [0, T_0)$ either*

$$\max_x u(t,x) \geq C_1(T_0 - t)^{-(p+1)/(pq-1)}, \tag{57}$$

*or*

$$\max_x v(t,x) \geq C_2(T_0 - t)^{-(q+1)/(pq-1)}. \tag{58}$$

Comparing now $u$, $v$ at each moment of time with the family (46) of stationary solutions ($U_1$, $V_1$ are taken from Lemma 1), as was done in the proof of Theorem 7, we arrive at an estimate of the support of the unbounded solution.

**Theorem 9.** *Let $pq < (1+\mu)(1+\nu)$ and let $T_0$ be the blow-up time of an unbounded solution of the Cauchy problem* (1), (2), (54), *where the initial functions have compact support and are radially symmetric, $u_0 = u_0(r)$, $v_0 = v_0(r)$, $r = |x|$. Then for each $t$ sufficiently close to $T_0$ either*

$$\text{meas supp}_r u(t,r) > \sqrt{2N}C_1^{\frac{m}{2(p+\mu+1)}}(T_0 - t)^{\frac{m(p+1)}{2(p+\mu+1)(pq-1)}}, \tag{59}$$

*or*

$$\text{meas supp}_r v(t,r) > \sqrt{2N}C_2^{\frac{m}{2(q+\nu+1)}}(T_0 - t)^{\frac{m(q+1)}{2(q+\nu+1)(pq-1)}}. \tag{60}$$

Obviously, for $m = pq - (1+\mu)(1+\nu) < 0$, $pq > 1$, estimates (59), (60) guarantee, in accordance with Theorem 7, the appearance of the HS blow-up regime as $t \to T_0^-$; meas $\text{supp}_r(u+v)$ becomes infinite as $t \to T_0^-$.

*2 S- and LS-regimes*

In the case $m \geqslant 0$ the condition $U_a, V_a \to \infty$ in $\mathbf{R}^N$ as $a \to 0^+$ is not satisfied and therefore Theorems 7, 8 are invalid. We can expect unbounded solutions to be localized for $m \geq 0$.

Here for $m > 0$, when solutions of the family (46) become infinite as $a \to \infty$ only at the one point $x = 0$ (which is characteristic of the LS-regime), the functions $U_a$, $V_a$ allow us to bound from below the singularity of the unbounded solution as $t \to T_0^-$ (see § 1).

On the other hand, if $m = 0$, then the family of stationary solutions (53) becomes infinite as $a \to \infty$ at least for all $x \in \omega_1$ (S-regime). In this case we have, for example, the following statement, which estimates the fundamental localization domain of the S-regime. It is proved as Theorem 7.

**Theorem 10.** *Let $pq = (1+\mu)(1+\nu)$, $p > 1$, $q > 1$. Let the initial functions $u_0$, $v_0$ be radially symmetric, non-increasing in $r = |x|$, and assume furthermore that $\operatorname{supp}(u_0 + v_0) \subset \omega_1$. If the solution of the Cauchy problem (1), (2), (54) is unbounded, then*

$$\varlimsup_{t \to T_0} \max\{u(t,x), v(t,x)\} = \infty$$

*everywhere in $\omega_1$.*

In other words for $pq = (1+\mu)(1+\nu)$ an unbounded solution of the Cauchy problem cannot be localized in a ball with a radius less than $\sqrt{2N}$.

*3 Comparison with self-similar results*

It is of interest to compare the results with qualitative derivations, obtained by using unbounded self-similar solutions of the Cauchy problem for (1), (2).

Let

$$\nu(p+1) = \mu(q+1), \ pq > 1; \tag{61}$$

the latter condition being equivalent to the inequality $q > \nu/\mu$ if (61) is satisfied. Then, as can be easily seen, equations (1), (2) admit unbounded self-similar solutions of the form

$$u_S(t,x) = (T_0 - t)^{-(p+1)/(pq-1)}\theta(\xi), \tag{62}$$

$$v_S(t,x) = (T_0 - t)^{-(q+1)/(pq-1)} f(\xi), \tag{63}$$

$$\xi = \frac{x}{(T_0 - t)^n}, \ n = \frac{q\mu - \nu(\mu+1)}{2(q\mu - \nu)} \equiv \frac{1}{2}\left[1 - \frac{\nu(p+1)}{pq-1}\right], \tag{64}$$

$$0 < t < T_0 < \infty, \ x \in \mathbf{R}^N.$$

The functions $\theta \geq 0$, $f \geq 0$ satisfy the following elliptic system of equations obtained by substituting the expressions (62), (63) into (1), (2) and taking (61) into account:

$$\Delta_\xi \theta^{\nu+1} - n\nabla_\xi \theta \cdot \xi - \frac{\mu}{q\mu - \nu}\theta + f^p = 0,$$
$$\Delta_\xi f^{\mu+1} - n\nabla_\xi f \cdot \xi - \frac{\nu}{q\mu - \nu} f + \theta^q = 0, \quad \xi \in \mathbf{R}^N. \tag{65}$$

Let this system have a non-trivial solution in $\mathbf{R}^N$, satisfying the conditions $\theta(\xi) \to 0$, $f(\xi) \to 0$ as $|\xi| \to \infty$. Then it follows from the form of the similarity coordinate $\xi$ in (64) that localization of the solution $u_S$, $v_S$, or its absence, depends on the sign of the parameter $l = q\mu - \nu(\mu + 1)$. The critical value $l = 0$ (that is, $q = \nu(\mu + 1)/\mu$; S-regime, $\xi = x$ and (62), (63) is the solution in separated variables) divides the space of parameters of the problem into two regions. For $l \geq 0$ the self-similar solution is localized ( $l > 0$, that is, $q > \nu(\mu + 1)/\mu$, LS-regime), while for $l < 0$ there is no localization, and $u_S(t, x)$, $v_S(t, x) \to \infty$ in $\mathbf{R}^N$ as $t \to T_0^-$ (HS-regime).

Let us compare the self-similar critical value

$$q_* = (\mu + 1)\nu/\mu \tag{66}$$

with the critical value

$$(pq)_* = (1 + \mu)(1 + \nu), \tag{67}$$

derived by the method of stationary states. By (66) we have from (61) that $p_* = \mu(q_* + 1)/\nu - 1 \equiv \mu(\nu + 1)/\nu$, and therefore $p_* q_* = (1 + \mu)(1 + \nu)$, which is exactly the same as (67).

Therefore in the "self-similar region" of parameter values, the condition of Theorems 7, 8 concerning absence of localization for $pq < (1 + \mu)(1 + \nu)$ is not only sufficient, but also necessary. It appears that this conclusion is valid also without the self-similar condition (61).

Let us observe that the rule for growth of the amplitude of the unbounded self-similar solution (62), (63) is the same as in the right-hand sides of the estimates (57), (58) as far as the form of dependence on $t$ is concerned. Those estimates were obtained by comparison with the homothermic solution (56).

It is important to note (and this fact again underlines a certain optimality of the results that follow from the method of stationary states), that sharper estimates of the size of the support of the solution (59), (60) in the HS-regime coincide with those of the self-similar solution. It can be easily checked that if condition (61) is satisfied, then we have the equalities

$$\frac{m(p+1)}{2(p + \mu + 1)(pq - 1)} = \frac{m(q+1)}{2(q + \nu + 1)(pq - 1)} = n.$$

where $n$ is the exponent in the expression (64) for the similarity coordinate $\xi$, which determines the law of motion of the fronts of self-similar thermal waves formed in each component in the HS-regime.

# § 4 The combustion localization phenomenon in multi-component media

This section can be considered as the continuation of the previous one. It deals with a qualitative and numerical analysis of combustion with finite time blow-up in multi-component media. Most properties of unbounded solutions of quasilinear parabolic systems of equations presented here have not as yet been rigorously justified.

Below we consider two different systems of equations. We shall concentrate on properties of solutions, for which there are no analogues in the theory of finite time blow-up developed earlier for a single quasilinear parabolic equation. We shall also discuss questions related to the efficacy of similarity methods.

It turns out that unbounded self-similar solutions that can be constructed for systems with power law nonlinearities, are not always "responsible" for the asymptotic stage of the blow-up process. It can happen that the asymptotic stage of the combustion process is described by self-similar solutions of completely different equations, that is, a.s.s. appear here.

## 1 A system of equations with a source

Here we consider a parabolic system of quasilinear equations, which is a generalization of the system studied in § 3. It is conveniently written down in the following form:

$$
\begin{aligned}
u_t &= k_1(u^{\sigma_1}u_x)_x + q_1 u^{\beta_1} v^{\gamma_2},\\
v_t &= k_2(v^{\sigma_2}v_x)_x + q_2 v^{\beta_2} u^{\gamma_1}, \quad t > 0,\ x \in \mathbf{R}.
\end{aligned} \tag{1}
$$

Here $\sigma_i > 0$, $\beta_i \geq 1$, $\gamma_i \geq 1$ $(i = 1, 2)$ are fixed dimensionless parameters. The number of dimensional positive constants $k_i$, $q_i$ can be reduced by rescaling $t \to t_0 t$, $x \to x_0 x$, $u \to U_0 u$, $v \to V_0 v$.

If $\delta = \sigma_2|\beta_1 - (\gamma_1 + 1)| - \sigma_1|\beta_2 - (\gamma_2 + 1)| \neq 0$, then this method can be used to get rid of all dimensional constants of the system, by setting

$$
t_0 = q_1^{-1} U_0^{1-\beta_1} V_0^{-\gamma_2}, \quad x_0 = k_1^{1/2} U_0^{\sigma_1/2} t_0.
$$

$$U_0 = \left(\frac{q_2}{q_1}\right)^{\sigma_2/\delta} \left(\frac{k_2}{k_1}\right)^{(\gamma_2+1-\beta_2)/\delta}, \quad V_0 = \left(\frac{q_2}{q_1}\right)^{\sigma_1/\delta} \left(\frac{k_2}{k_1}\right)^{(\gamma_1+1-\beta_1)/\delta}.$$

If, on the other hand, $\delta = 0$ then any three constants can be taken to be equal to one; only one dimensionless parameter remains in the system, for example,

$$k = \frac{k_2}{k_1}\left(\frac{q_2}{q_1}\right)^{\sigma_1/(\gamma_1+1-\beta_1)}, \gamma_1 + 1 - \beta_1 \neq 0.$$

Therefore instead of (1) we shall be considering the equivalent system

$$u_t = (u^{\sigma_1} u_x)_x + u^{\beta_1} v^{\gamma_2}, \tag{2}$$

$$v_t = k(v^{\sigma_2} v_x)_x + v^{\beta_2} u^{\gamma_1}, \; t > 0, x \in \mathbf{R}, \tag{3}$$

where $k = 1$ if $\delta \neq 0$, and $k > 0$ is arbitrary if $\delta = 0$. For (2), (3) we formulate the Cauchy problem

$$u(0, x) = u_0(x) \geq 0, \;\; v(0, x) = v_0(x) \geq 0, \;\; x \in \mathbf{R}, \tag{4}$$

where $u_0$, $v_0$ are bounded functions with compact support.

*1 Analysis of spatially homogeneous solutions*

First of all we have to work out under what conditions unbounded solutions are possible. This can be done by considering spatially homogeneous solutions of the problem, which do not depend on $x$ and satisfy the equations

$$u'(t) = u^{\beta_1}(t) v^{\gamma_2}(t), \;\; v'(t) = v^{\beta_2}(t) u^{\gamma_1}(t), \;\; t > 0, \tag{5}$$

$$u(0) = u_0 > 0, \; v(0) = v_0 > 0. \tag{6}$$

Let $\alpha_i = \gamma_i + 1 - \beta_i \neq 0$ $(i = 1, 2)$. Then the system has the first integral

$$\frac{u^{\alpha_1}}{\alpha_1} - \frac{v^{\alpha_2}}{\alpha_2} = C_0 \equiv \frac{u_0^{\alpha_1}}{\alpha_1} - \frac{v_0^{\alpha_2}}{\alpha_2}, \;\; t > 0, \tag{7}$$

using which it reduces to the two (uncoupled) autonomous equations

$$u'(t) = u^{\beta_1}\left(\frac{\alpha_2}{\alpha_1}u^{\alpha_1} - \alpha_2 C_0\right)^{\gamma_2/\alpha_2}, \tag{8}$$

$$v'(t) = v^{\beta_2}\left(\frac{\alpha_1}{\alpha_2}v^{\alpha_2} + \alpha_1 C_0\right)^{\gamma_1/\alpha_1}, \;\; t > 0. \tag{9}$$

From this we derive conditions for occurrence of finite time blow-up in each component. If, for example, $\alpha_1 > 0$, $\alpha_2 > 0$, then for $u$ to blow up, it suffices to have $\beta_1 + \alpha_1\gamma_2/\alpha_2 > 1$. This inequality is equivalent to the condition $-\gamma_1\gamma_2 + (\beta_1 - 1)(\beta_2 - 1) < 0$, which is satisfied in this case (since $\beta_i < 1 + \gamma_i$). Similarly, it can be checked that if $\alpha_1 > 0$, $\alpha_2 > 0$, the second component $v$ also blows up in finite time. From the identity (7) it follows that the blow-up times of $u(t)$ and $v(t)$ are the same.

An interesting situation arises if $\alpha_1$ and $\alpha_2$ have different signs, for example, if $\alpha_1 > 0$, $\alpha_2 < 0$. Here, as can be seen from (7), $C_0 > 0$, and since $\beta_2 > 1 + \gamma_2 > 1$, $v$ blows up in finite time: $v(t) \to \infty$ as $t \to T_0^- < \infty$. The function $u(t)$ in this case remains bounded: $u(t) \to (\alpha_1 C_0)^{1/\alpha_1}$, $t \to T_0^-$. The nature of homothermic combustion is still more varied if $\alpha_1 < 0$, $\alpha_2 < 0$.

The constant $C_0$ in (7) can be of either sign. For $C_0 = 0$ equations (8), (9) lead to finite time blow-up in both the components. If $C_0 < 0$, then $u(t)$ blows up, while $v(t)$ remains bounded; if $C_0 > 0$ it is the other way around.

Therefore if $\alpha_1 < 0$ (or $\alpha_2 < 0$), evolution of the components $u$, $v$ can differ: one can blow up in finite time, while the other remains bounded.

## 2 *Self-similar solutions*

Let us introduce the notation

$$m_i = \alpha_i/p, \; p = (\beta_1 - 1)(\beta_2 - 1) - \gamma_1\gamma_2.$$

If the conditions

$$\delta = \sigma_1(\gamma_2 + 1 - \beta_2) - \sigma_2(\gamma_1 + 1 - \beta_1) = 0, \tag{10}$$

$$m_1 < 0, \; m_2 < 0, \tag{11}$$

hold, the system (2), (3) admits unbounded self-similar solutions of the following form:

$$\begin{gathered} u_S(t,x) = (T_0 - t)^{m_1}\theta(\xi), \; v_S(t,x) = (T_0 - t)^{m_2} f(\xi), \\ \xi = x/(T_0 - t)^n, \; n = (m_1\sigma_1 + m_2\sigma_2 + 2)/4, \end{gathered} \tag{12}$$

where the functions $\theta \geq 0$, $f \geq 0$ satisfy the system of ordinary differential equations

$$\begin{gathered} (\theta^{\sigma_1}\theta')' - n\theta'\xi + m_1\theta + \theta^{\beta_1} f^{\gamma_2} = 0, \\ k\left(f^{\sigma_2} f'\right)' - n f'\xi + m_2 f + f^{\beta_2}\theta^{\gamma_1} = 0, \; \xi \in \mathbf{R}, \end{gathered} \tag{13}$$

and the usual conditions: $\theta(\xi)$, $f(\xi) \to 0$ as $|\xi| \to \infty$.

As can be seen from (12), many properties of self-similar solutions, which are expressed as $t \to T_0^- < \infty$, depend on the sign of the parameter $n$.

If $n < 0$ then both components evolve in the HS blow-up regime, which is not localized, $u_S, v_S \to \infty$ in $\mathbf{R}$, $t \to T_0^-$.

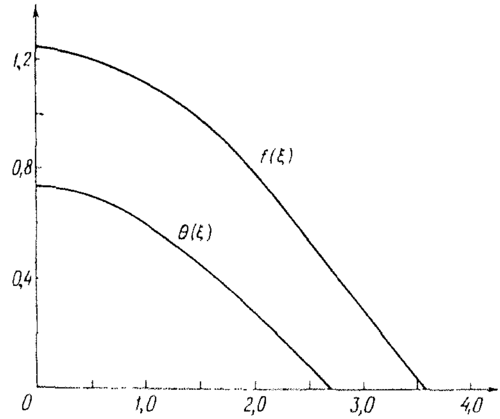



**Fig. 84.** Similarity functions of the S-regime ($n = 0$); $\sigma_1 = 1.5$, $\sigma_2 = 2$, $\beta_1 = 1.9$, $\beta_2 = 1.6$, $\gamma_1 = 1.05$, $\gamma_2 = 0.8$, $k = 1$

On the other hand, in the case $n \geq 0$ unbounded solutions are localized: if $n > 0$ (LS-regime), then $u_S, v_S$ grow without bound as $t \to T_0^-$ only at the point $x = 0$.

For $n = 0$ the S blow-up regime develops; the functions $u_S, v_S$ are a solution in separated variables, and therefore grow without bound as $t \to T_0$ on respective fundamental lengths $L_\theta = \text{meas supp}\,\theta$, $L_f = \text{meas supp}\,f$. Localization domains of each of the components are, in general, different. As an example, we show in Figure 84 the spatial profiles of the functions $\theta(\xi)$, $f(\xi)$ in the case of the S-regime. Here $L_\theta < L_f$.

Numerical studies show that for $\alpha_1 > 0$, $\alpha_2 > 0$ the self-similar solution of the S-regime is unique and stable (in the norm of the special similarity transformation, see § 2, 5, Ch. IV). For sufficiently general initial conditions, the system evolves to a stable dissipative structure on a bounded domain, with each component effectively localized on its respective fundamental length $L_\theta$ or $L_f$.

An example of such evolution to a self-similar S blow-up regime can be seen in Figure 85. The initial perturbations $u_0(x)$, $v_0(x)$ are not symmetric. Therefore initially two thermal waves in $u$, $v$ ($t = t_1$, $t = t_2$) appear. These collide at time $t = t_3$ and generate a thermal structure ($t = t_4$), which evolves in a self-similar manner in the S-regime.

For $\alpha_1 < 0$ (or $\alpha_2 < 0$), when the nature of homothermic combustion of the two components can be significantly different, self-similar solutions do not appear. As a rule, in numerical computations one of the components blows up in finite time as $t \to T_0^-$, while the other remains bounded as $t \to T_0$. Therefore self-similar solutions with coordinated combustion of the components comprise a sort of

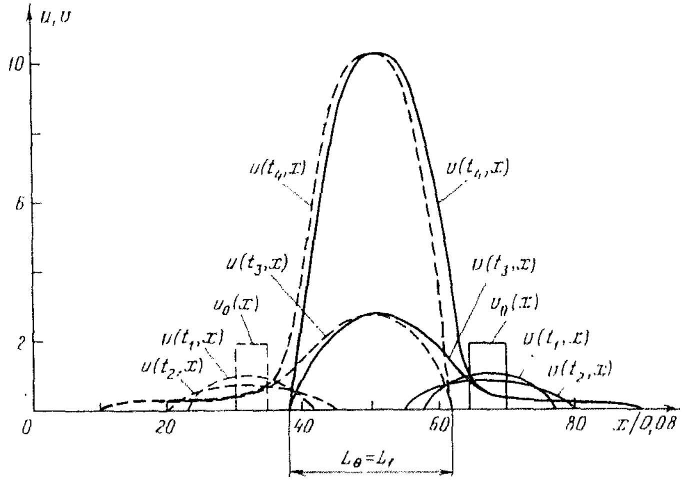



**Fig. 85.** Formation of the self-similar S-regime structure after interaction of concentration waves in the medium: $\beta_1 = \beta_2 = 1$, $\gamma_1 = \gamma_2 = 1$, $\sigma_1 = \sigma_2 = 2$, $k = 1$; $t_1 = 0.272$, $t_2 = 0.672$, $t_3 = 0.7812$, $t_4 = 0.7823$

unstable boundary, which separates sets of solutions with different (uncoordinated) patterns of evolution of $u$ and $v$.

### *3 The general case*

A natural question arises: what happens in the general case when condition (10) for the existence of self-similar solutions is not satisfied? First, numerical computations show that for $\alpha_1 > 0$, $\alpha_2 > 0$ (that is, when $\beta_1 < 1 + \gamma_1$, $\beta_2 < 1 + \gamma_2$) both components always blow up in finite time.

Secondly, if the similarity condition (10) is not satisfied, the evolution of the two components proceeds, in general, in an uncoordinated fashion, and can be markedly different. For example, it is possible for the first component to blow up in finite time in the LS-regime (unbounded growth on a set of measure zero; localization), while the second component evolves in the HS-regime, and its blow-up set covers the whole space as $t \to T_0^-$. A numerical computation of such behaviour can be seen in Figure 86. Other situations are also possible. For example, $u$ can evolve in an S-regime, while $v$ evolves in an HS-regime. Alternatively, both components can evolve in the HS-regime, but with different speeds of propagation of thermal waves.

To conclude, let us write down for the general case sufficient (and apparently necessary) conditions for the absence of localization of unbounded solutions of the problem, that is, for the occurrence of the HS-regime at least for one component.

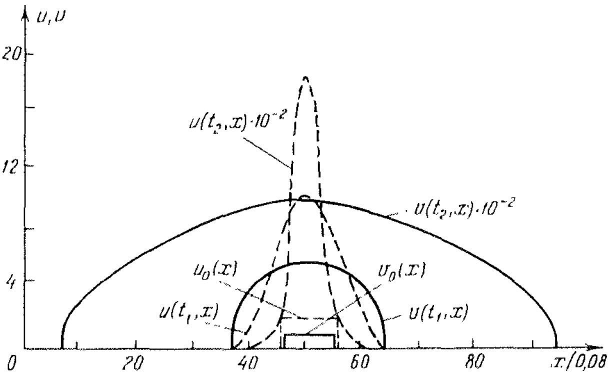



**Fig. 86.** Mixed combustion regime with finite time blow-up (LS in the first component, HS in the second): $\beta_1 = \beta_2 = 1.5$, $\gamma_1 = \gamma_2 = 1.5$, $\sigma_1 = 1$, $\sigma_2 = 3$, $k = 1$; $t_1 = 0.0745$, $t_2 = 0.0771$

It is not hard to check that equations (2), (3) admit the following one-parameter family of stationary solutions:

$$U_a(x) = a^{2\epsilon_1/m} U_1(ax), \quad V_a(x) = a^{2\epsilon_2/m} V_1(ax). \tag{14}$$

Here $a > 0$ is an arbitrary constant, $U_1$, $V_1$ is some stationary solution, for example, one similar to that constructed in Lemma 1, § 3;

$$m = \gamma_1\gamma_2 - [\beta_1 - (\sigma_1 + 1)][\beta_2 - (\sigma_2 + 1)],$$

$$\epsilon_i = 1 + \gamma_i + \sigma_i - \beta_i, \quad i = 1, 2.$$

It has to be expected that it is precisely the signs of these parameters that determine much of the asymptotic behaviour of unbounded solutions. In particular, an elementary analysis (which uses the method of stationary states) of the family (14) leads for small $a > 0$ to the following result:

**Proposition 1.** *Let* $\beta_i > 1$, $\gamma_i > 1$ *and* $\epsilon_1/m < 0$, $\epsilon_2/m < 0$. *Then every unbounded solution of the problem* (2), (3), (4) *is not localized.*

## 2 A system of equations with depletion

Below we consider the Cauchy problem for the system

$$u_t = (k_0 u^\sigma u_x)_x + q_0 v^\gamma u^\beta,$$

$$v_t = -p_0 v^\nu u^\epsilon,\ t > 0,\ x \in \mathbf{R}.$$

It describes the process of combustion in a nonlinear medium, with volumetric energy production, which takes into account decrease in the density of the substance due to its depletion as a result of burning ($v = v(t, x) \geq 0$ is the density of the combustible substance). Here $\beta > 1$, $\nu > 0$, $\epsilon > 0$ are dimensionless parameters; $k_0$, $q_0$, $p_0$ are positive dimensional constants. Diffusion of the second component is not taken into account; in fact this has no bearing on the final results.

It is easy to get rid of the constants $k_0$, $q_0$, $p_0$ by rescaling the dependent and independent variables $t, x, u, v$. Therefore in the following we shall consider the dimensionless system

$$\begin{aligned} u_t &= (u^\sigma u_x)_x + v^\nu u^\beta, \\ v_t &= -v^\nu u^\epsilon,\ t > 0,\ x \in \mathbf{R}. \end{aligned} \tag{15}$$

Let the combustion process be initiated by giving an initial temperature profile and some initial concentration of the combustible substance:

$$u(0, x) = u_0(x) \geq 0,\ v(0, x) = v_0(x) \geq 0,\ \ x \in \mathbf{R}. \tag{16}$$

Observe that for $\nu = 0$ ($\nu$ is the order of the chemical reaction of combustion) the two equations of (15) are uncoupled and the temperature distribution $u(t, x)$ satisfies an equation considered in preceding chapters.

We have to find out how depletion influences the asymptotic stage of evolution of finite time blow-up regimes in a nonlinear medium, and under which conditions localization is possible. It is of interest to note that in this case, in general, unbounded self-similar solutions do not describe correctly the asymptotic stage of the blow-up process. We shall discuss reasons for that later.

*1 Unbounded self-similar solutions*

A great advantage of the system (15) is that essentially for all values of the parameters it admits self-similar solutions:

$$u_S(t, x) = (T_0 - t)^{-1/\alpha}\theta(\xi), \tag{17}$$

$$v_S(t, x) = (T_0 - t)^{[\beta-(\epsilon+1)]/\alpha} f(\xi),\ \alpha = \epsilon\nu + (1-\nu)(\beta - 1) \neq 0, \tag{18}$$

where $\xi$ is the similarity coordinate,

$$\xi = x/(T_0 - t)^{(\alpha-\sigma)/(2\alpha)}. \tag{19}$$

If diffusion of the second component is taken into account, then another dimensional constant appears in the system, and self-similar solutions will exist under an additional restriction on the parameters.

Similarity representations of the temperature $\theta(\xi) \geq 0$ and of the density $f(\xi) \geq 0$ are determined from the system of ordinary differential equations

$$\left(\theta^{\sigma}\theta'\right)' - \frac{\alpha - \sigma}{2\alpha}\theta'\xi - \frac{1}{\alpha}\theta + f^{\nu}\theta^{\beta} = 0, \tag{20}$$

$$-\frac{\alpha - \sigma}{2\alpha}f'\xi + \frac{\beta - (\epsilon + 1)}{\alpha}f - f^{\nu}\theta^{\epsilon} = 0, \quad \xi \in \mathbf{R}, \tag{21}$$

and the boundary conditions

$$\theta(\infty) = 0, \; f(\infty) < \infty, \tag{22}$$

which have a simple physical interpretation.

From (17) we obtain the conditions for existence of finite time blow-up as $t \to T_0 < \infty$:

$$\alpha = \epsilon\nu + (1 - \nu)(\beta - 1) > 0. \tag{23}$$

In (18) the concentration $v_S$ cannot increase with time. Therefore we need another restriction on the parameters,

$$\beta > \epsilon + 1. \tag{24}$$

As usual, we classify self-similar solutions according to how the domain (half-width) of intensive combustion depends on time. It follows from (19) that its size depends on time according to

$$x_{ef}(t) \sim (T_0 - t)^{(\alpha - \sigma)/(2\alpha)}, \; 0 < t < T_0. \tag{25}$$

Therefore three cases are possible: a) if $\alpha < \sigma$, then $x_{ef}(t) \to \infty$ as $t \to T_0^-$, and by the blow-up time the combustion wave covers the whole space (HS blow-up regime); b) if $\alpha = \sigma$, then $x_{ef}(t) \equiv \text{const} > 0$, and the intensive combustion domain is constant in time (S-regime); c) if $\alpha > \sigma$, then $x_{ef}(t) \to 0$, $t \to T_0^-$, the intensive combustion domain shrinks, and unbounded growth of the temperature is observed only at the one point $x = 0$ (LS-regime).

We shall not consider in any detail the analysis of the system of ordinary differential equations (20), (21); there is no need. Let us only note that in the S-regime ($\alpha = \sigma$) it simplifies drastically; for $\nu < 1$ (21) becomes the equality

$$f(\xi) = \left[\frac{\sigma(1 - \nu)}{\sigma - \epsilon}\right]^{1/(1-\nu)} [\theta(\xi)]^{\epsilon/(1-\nu)}, \tag{26}$$

while the first equation of (20) takes the form

$$\left(\theta^{\sigma}\theta'\right)' - \frac{1}{\sigma}\theta + a^2\theta^{(\sigma+1-\nu)/(1-\nu)} = 0, \tag{26'}$$

where

$$a^2 = \left[\frac{\sigma(1 - \nu)}{\sigma - \epsilon}\right]^{\nu/(1-\nu)},$$

and is easily integrated. The function $\theta(\xi)$ is found from the quadrature

$$\int_0^{\theta''^2} \frac{dz}{[1 - b^2 z^{2/(1-\nu)}]^{1/2}} = \left[\frac{\sigma}{2(\sigma+2)}\right]^{1/2} (\xi_0 - \xi)_+, \xi > 0, \tag{27}$$

$$b^2 = a^2 \frac{\sigma(\sigma+2)(1-\nu)}{\sigma + (1-\nu)(\sigma+2)}.$$

Therefore $\theta(\xi)$, $f(\xi)$ are functions with compact support; the size of the support determines the fundamental length of the S-regime:

$$\begin{aligned} L_S \equiv 2\xi_0 = \left[\frac{2\pi(\sigma+2)}{\sigma}\right]^{1/2} \frac{\Gamma(1/2 - \nu/2)}{\Gamma(1-\nu/2)} (1-\nu) \times \\ \times \left[\frac{\sigma - \epsilon}{\sigma(1-\nu)}\right]^{\nu/2} \left[\frac{\sigma + (1-\nu)(\sigma+2)}{\sigma(\sigma+2)(1-\nu)}\right]^{(1-\nu)/2}. \end{aligned} \tag{28}$$

Naturally, for $\nu = 0$, $\epsilon = 0$ this equality defines the fundamental length in a medium without depletion: $L_S = 2\pi(\sigma+1)^{1/2}/\sigma$ (see § 1, Ch. IV).

## 2 *Asymptotic behaviour of blow-up regimes*

The above self-similar solutions are structurally unstable, that is, their (coordinated in both components) spatio-temporal structure is not observed in numerical computations.

Figure 87 presents the results of numerical simulation of equations (15) for values of parameters, which formally correspond to the S-regime ($\alpha = \sigma$). The initial functions are non-zero on an interval of length $2L_S$, where (see (28))

$$L_S = \sqrt{\pi} \frac{\Gamma(1/4)}{\Gamma(3/4)} \simeq 5.205.$$

Maxima of the initial distributions correspond to the self-similar solution (26), (27), which blows up at $T_0^S = 1$. However, the spatial profiles of $u_0(x)$ and $v_0(x)$ are not the self-similar ones. As a result, since $v_0(x)$ is too large (as compared with the self-similar one), only a part of the energy of the initial distribution $v_0(x)$ is needed to cause finite time blow-up in temperature $u$. This can be clearly seen in Figure 87,*b*. Therefore the process of substance depletion stabilizes at the asymptotic stage of the finite time blow-up process in $u$, and as $t \to T_0^-$ (the real value of the blow-up time is $T_0 = 0.525 < 1$), the density $v(t, x)$ does not change much in the intensive combustion domain.

Therefore as $t \to T_0^-$, the equation for the concentration falls away, and asymptotics of temperature blow-up is described by the single equation

$$u_t = (u^\sigma u_x)_x + Q_0 u^\beta, \tag{29}$$

where $Q_0$ is a constant equal to $(\bar{v}(T_0^-, x))^{\epsilon}$, the average value of the limiting density in the intensive combustion domain. But for these parameter values $\beta > \sigma + 1$, which ensures development of the usual LS-regime, which can be clearly seen in Figure 87, *a*. And in general, an S-regime perturbed in temperature and density from above, degenerates as $t \to T_0^-$ into a self-similar (with respect to (29)) LS-regime, since from the inequality $\sigma > \epsilon$, which always holds for the S-regime ($\alpha = \sigma$), it follows that $\beta > \sigma + 1$ in (29).

If, on the other hand, in the S-regime ($\alpha = \sigma$) the initial profiles are below those of the self-similar solution, the result most often is complete depletion of the combustible substance in the whole space and energy is no longer produced by the medium. The consequence is that the temperature does not blow up in finite time, but satisfies instead an equation without a source,

$$u_t = (u^{\sigma} u_x)_x. \tag{30}$$

Thus the self-similar solutions (17), (18) comprise an unstable boundary between two large classes of "degenerate" equations (29) and (30).

In the LS-regime (with respect to (15), $\alpha > \sigma$) initial functions lying above the self-similar ones lead to the development of the LS-regime, which corresponds to $\beta > \sigma + 1$ in (29). Figure 88 shows the result of numerical computation of the LS-regime with initial data of the S-regime, as in Figure 87. As $t \to T_0^- = 0.291$ the density stabilizes, while the temperature grows in the LS-regime ($\beta = 6 > \sigma + 1 = 3$ in (29)).

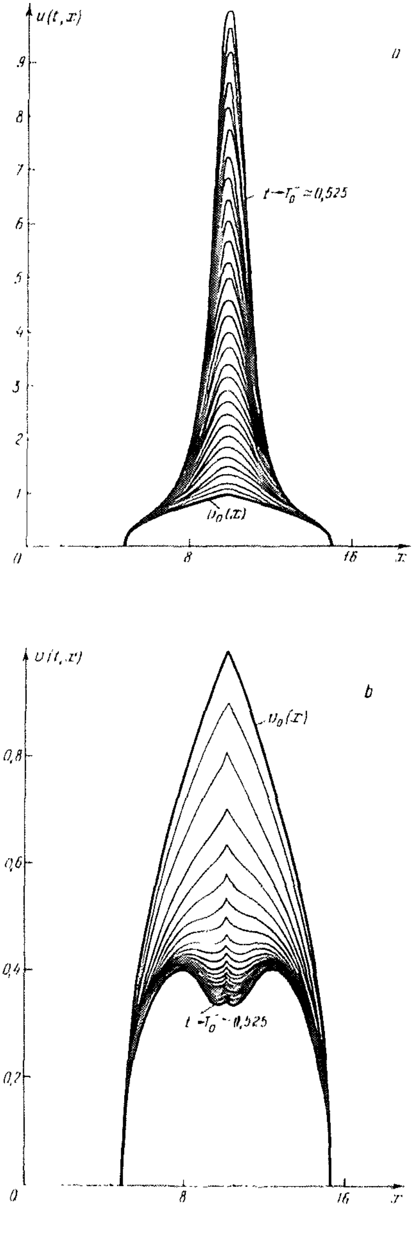
For the HS-regime ($\alpha < \sigma$), these initial data are lower than the self-similar ones. It can be seen from Figure 89 that the constant density profile $v_0(x) \equiv 1$ is also too low. Therefore initially there is fast depletion of combustible substance in areas where the initial temperature is non-zero, and then two thermal waves propagate into the surrounding space which has high density of combustible material. Due to the higher rate of energy production at high temperatures, these waves blow up in finite time ($T_0 = 1.72$).

## § 5 Finite difference schemes for quasilinear parabolic equations admitting finite time blow-up

An important place in this study of blow-up regimes is occupied by results of numerical computations on the non-stationary problems being considered. In this section we analyse properties of difference schemes for a quasilinear parabolic equation with power type nonlinearity in one space variable:

$$u_t = (u^{\sigma+1})_{xx} + u^{\beta}. \tag{1}$$

Here, as usual, $\sigma > 0$ and $\beta > 1$ are constants.




. Simulation of equations (15) in the S-regime with $\sigma = 2$, $\beta = 4$, $\epsilon = 1$, $\nu = 0.5$ ): a: the first component, b: the second component

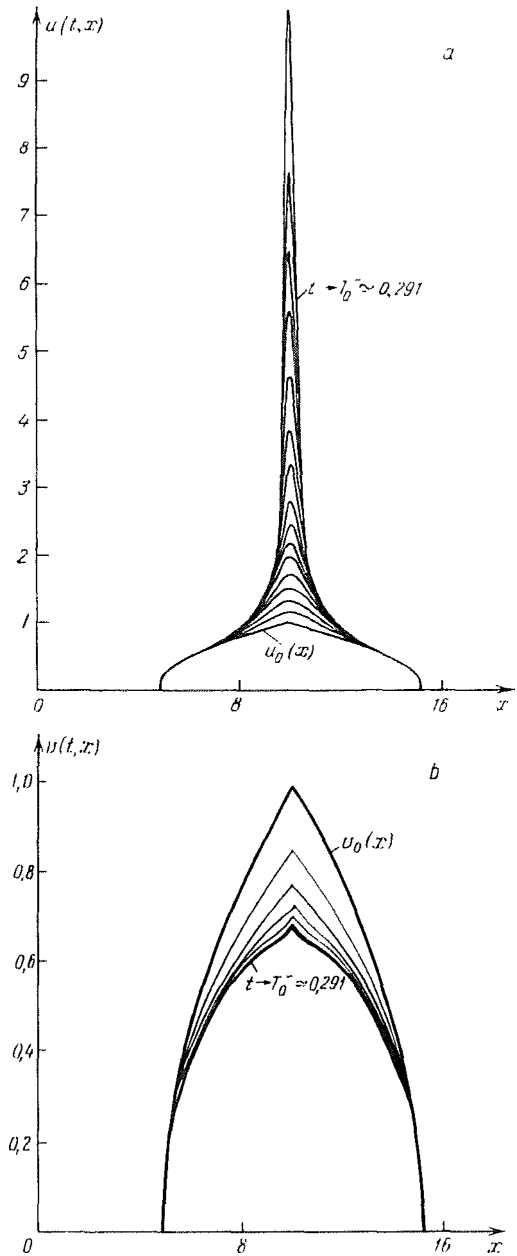



**Fig. 88.** Simulation of equations (15) in the LS-regime with $\sigma = 2$, $\beta = 6$, $\epsilon = 1$, $\nu = 0$ ($\alpha > \sigma$); a: the first component, b: the second component

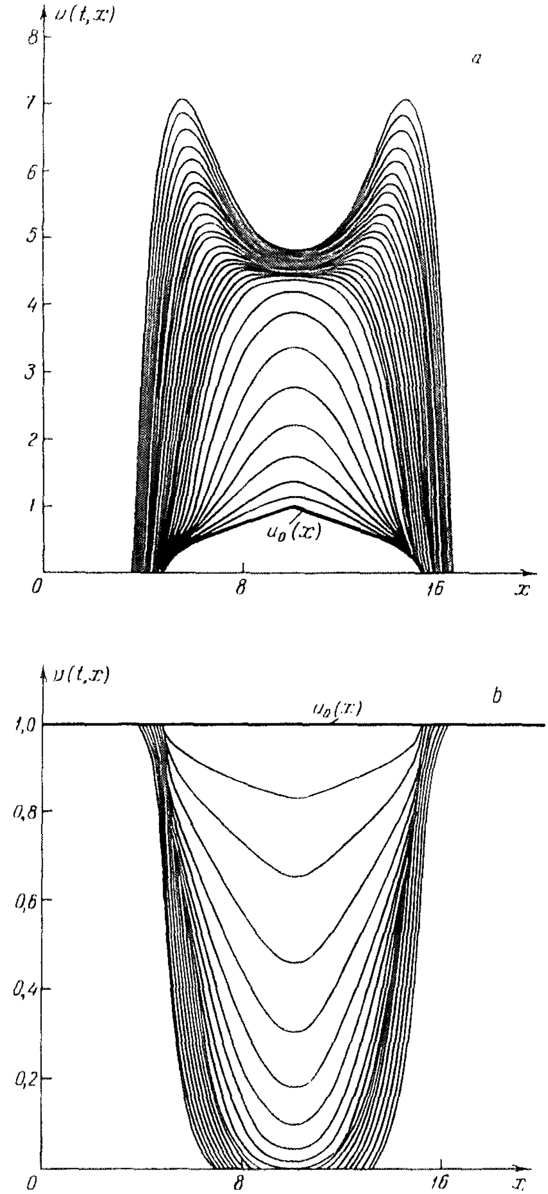



**Fig. 89.** Simulation of equations (15) in the HS-regime with $\sigma = 2$, $\beta = 3$, $\epsilon =$ $(\alpha < \sigma)$; a: the first component, b: the second component

We consider for (1) the boundary value problem in the domain $\{t > 0,\ x \in (0, l)\}$, $l = \text{const} > 0$, with the conditions

$$u(0, x) = u_0(x) \geq 0,\ 0 < x < l;\ u_0^{\sigma+1} \in H_0^1((0, l)), \tag{2}$$

$$u(t, 0) = u(t, l) = 0,\ t \geq 0. \tag{3}$$

The results we obtain here give an indication of difficulties that arise in numerical computation of blow-up regimes. The main emphasis is on the study of implicit (nonlinear) difference schemes, which, in view of their many advantages [346] as compared with explicit difference schemes, were used in all the numerical computations. Below we consider such classical questions of the theory of difference schemes as conditions of solvability of the discretized problem on a time level, conditions for global solvability of the discretized problem, and conditions of convergence of the finite difference solution to the solution of the original differential problem.

We treat in detail the question of conditions under which there is no global solution, that is, to the appearance of finite difference blow-up regimes. Related to that are two comparatively unusual properties of the implicit scheme: if the solution grows at a fast enough rate, it can happen that at a certain time level either the solution is non-unique, or it does not exist at all, that is, the scheme is no longer solvable.

All these properties (unboundedness, non-uniqueness, and non-existence of solutions) are possible in the case $\beta \geq \sigma + 1$, when the difference operator corresponding to the right-hand side of (1) is not coercive. For $1 < \beta < \sigma + 1$ and sufficiently small steps in time, a global solution always exists; moreover, it is unique.

Many of the obtained results also hold for difference schemes for equations of the type (1) with sufficiently general nonlinearities.

At the end of the section we briefly consider explicit (linear) schemes; the weak Maximum Principle is analyzed, and conditions for unboundedness of the difference solution are established.

Let us introduce a uniform grid in space $\omega_h$, with step size $h = l/(M+1)$, where $M > 0$ is an integer; a system of time intervals $\{\tau_j\}$, $\tau_{j+1} \leq \tau_j$ and the corresponding grid in time, $\omega_\tau$. Everywhere, apart from subsection 1.4, we shall take the grid $\omega_\tau$ to be finite and uniform: $\tau_j \equiv \tau = T/(N+1)$, $0 \leq j \leq N$, $N > 0$ is an integer, $T > 0$ is a constant (in subsection 1.4 $\tau_j \to 0$ as $j \to \infty$, and $\omega_\tau$ is a non-uniform grid). Let us denote by $H_h$ the set of grid functions

$$v_h = \{v_i \mid v_0 = v_{M+1} = 0,\ v_i \geq 0,\ i = 1, 2, \ldots, M\}.$$

## 1 Implicit (nonlinear) difference scheme

Following [346], we write down the implicit difference scheme corresponding to the problem (1)–(3):

$$(\hat{u} - u)/\tau_j = (\hat{u}^{\sigma+1})_{xx} + \hat{u}^{\beta}, \quad (t, x) \in \omega_\tau \times \omega_h, \tag{4}$$

$$u^0 = u_{0h} \geq 0, \quad x \in \omega_h; \quad \hat{u} \in H_h, t \in \omega_\tau, \tag{5}$$

where we have introduced the usual notation $\hat{u} = u_k^{j+1}$, $u = u_k^j$ is the desired grid function defined on the discrete set $\omega_{\tau,h} = \omega_\tau \times \omega_h$, $v_{xx} = (v_{k-1} - 2v_k + v_{k+1})/h^2$ denotes the second difference operator, and $u_{0h}$ is the projection of $u_0(x)$ onto $\omega_h$.

In the formulation of the problem (4), (5) and all the subsequent results, we assume that the difference solution $\hat{u}$ is non-negative (otherwise the operation of taking an arbitrary power is not defined). This is one of the properties of implicit schemes for parabolic equations. It is easily verified that the scheme

$$(\hat{u} - u)/\tau_j = (|\hat{u}|^{\sigma}\hat{u})_{xx} + (\max\{0, \hat{u}\})^{\beta}, \quad (t, x) \in \omega_{\tau,h}, \tag{6}$$

which is identical to (4) for $\hat{u} \geq 0$, cannot admit for any $\tau_j$, $h$ negative values of $\hat{u}$ if $u \geq 0$ in $\omega_h$ (furthermore, $\hat{u} > 0$ in $\omega_h$ if $u \not\equiv 0$). This follows from an analysis of (6) at a point of negative minimum in $x$ of the function $\hat{u}$ (see § 7, Ch. V). An analogous weak Maximum Principle holds for the differential problem (see § 2).

Let us introduce the necessary finite difference functional spaces. The space of grid functions $V_h = \{v_i \mid i = 0, 1, \ldots, M + 1, v_0 = v_{M+1} = 0\}$ is equipped with the scalar product and the norm

$$(v_h, w_h)_h = h \sum_{i=1}^{M} v_i w_i, \quad |v_h|_{h,2} = (v_h, v_h)_h^{1/2}. \tag{7}$$

Norms in the grid analogues of the spaces $L^q(0, l)$, $q \geq 1$, and $H_0^1(0, l)$ have, respectively, the form

$$|v_h|_{h,q} = \left( h \sum_{i=1}^{M} |v_i|^q \right)^{1/q},$$

$$\|v_h\|_{h,2} = \left( h \sum_{i=0}^{M} \left| \frac{v_{i+1} - v_i}{h} \right|^2 \right)^{1/2}.$$

We denote by $\|\cdot\|_{h,2}^*$ the norm dual to $\|\cdot\|_{h,2}$ with respect to the scalar product (7):

$$\|v_h\|_{h,2}^* = \sup_{w_h \in V_h, w_h \neq 0} \frac{|(v_h, w_h)_h|}{\|w_h\|_{h,2}}.$$

We have the equality

$$\|(v_h)_{\bar{x}x}\|^*_{h,2} = \|v_h\|_{h,2}, \quad v_h \in V_h. \tag{8}$$

In the grid analogue of the space $C(0, l)$, the norm takes the form

$$|v_h|_C = \max_{1 \le i \le M} |v_i|, \quad v_h \in V_h.$$

Let us introduce the extension operators $p_h$, $q_h$ by assuming that $p_h v_h$ is a continuous function, which is linear on each interval $(ih, (i+1)h)$, such that $p_h v_h(ih) = v_i$ $(i = 0, 1, \ldots, M+1)$; $q_h v_h$ is a piecewise constant extension of the grid function $v_h \in V_h$, which is equal to $v_i$ for all $ih < x < (i+1)h$. It is clear that $p_h v_h \in H^1_0(0, l)$, $q_h v_h \in L^q(0, l)$, and that

$$\|q_h v_h\|_{L^q(0,l)} = |v_h|_{h,q}, \quad \|p_h v_h\|_{H^1_0(0,l)} = \|v_h\|_{h,2}.$$

In the same way for grid functions $v_{\tau,h}$, defined on the nodes of the grid $\omega_{\tau,h}$, we introduce the extension operator $q_\tau$, defined by $q_\tau p_h v_{\tau,h} = p_h v_h^{j+1}$, $q_\tau q_h v_{\tau,h} = q_h v_h^{j+1}$ for all $j\tau < t < (j+1)\tau$, $j = 0, 1, \ldots, N$ (the grid $\omega_\tau$ is assumed here to be uniform).

Let us denote [346] by

$$\lambda_1^h = \frac{4}{h^2} \sin^2 \frac{\pi h}{2l}, \tag{9}$$

$$\psi_h(x) = \frac{\tan(\pi h/(2l))}{h} \sin \frac{\pi x}{l}, 0 < x < l, \tag{10}$$

respectively, the first (smallest) eigenvalue and the first eigenfunction of the difference problem

$$(\psi_h)_{\bar{x}x} + \lambda_1 \psi_h = 0, \quad x \in \omega_h; \ \psi_h \in V_h. \tag{11}$$

The function $\psi_h$ in (10) is chosen so that $|\psi_h|_{h,1} = 1$. We observe that $\psi_h(x) > 0$ in $\omega_h$.

We start by considering the question of solvability of the scheme on a time level, that is, the question of existence and properties of the transition operator [346] from one time level to the next. Below we denote by $A_0, A_1, \ldots$ various constants independent of $\tau$, $h$.

### *1 Sufficient conditions for solvability of the difference scheme at a fixed time level*

1. We shall show first that for $\beta < \sigma + 1$ and also in the case $\beta = \sigma + 1$, $\lambda_1^h > 1$ (this imposes an upper bound on the length of the interval $[0, l]$; see (9)) the scheme (4) is solvable with respect to the grid function $\hat{u}$ for any magnitude of $\tau$. For this purpose we shall need the following assertion, which is the finite difference analogue of Lemma 1 of § 2 (see [296, 346]).

**Lemma 1.** *Any function $v_h \in H_h$ satisfies the estimates*

$$|v_h|^{2(\sigma+1)}_{h,2(\sigma+1)} \leq \frac{1}{\lambda_1^h}\|v_h^{\sigma+1}\|^2_{h,2}, \tag{12}$$

$$|v_h|^{\beta+\sigma+1}_{h,\beta+\sigma+1} \leq A_0\|v_h^{\sigma+1}\|_{h,2}^{(\beta+\sigma+1)/(\sigma+1)}, \; A_0 = l^{1+(\beta+\sigma+1)/[2(\sigma+1)]}. \tag{13}$$

Let us consider the continuous operator $P_h: \mathbf{R}^M \to \mathbf{R}^M$:

$$P_h(\hat{u}) = \{(\hat{u}_k - u_k)/\tau - (\hat{u}_k^{\sigma+1})_{\bar{x}x} - \hat{u}_k^\beta, \; k = 1, 2, \ldots, M\}. \tag{14}$$

Existence of a root of the equation $P_h(\hat{u}) = 0$ means that the scheme (4) is solvable.

Let initially $1 < \beta < \sigma + 1$. Then

$$(P_h(\hat{u}), \hat{u}^{\sigma+1})_h = \frac{1}{\tau}(\hat{u} - u, \hat{u}^{\sigma+1})_h + \|\hat{u}^{\sigma+1}\|^2_{h,2} - |\hat{u}|^{\beta+\sigma+1}_{h,\beta+\sigma+1}. \tag{15}$$

Using the inequality (13) as well as the easily checked estimate

$$(\xi - \eta)\xi^{\sigma+1} \geq \frac{1}{\sigma+2}(\xi^{\sigma+2} - \eta^{\sigma+2}), \quad \xi, \eta \in \mathbf{R}_+, \tag{16}$$

we obtain from (15)

$$(P_h(\hat{u}), \hat{u}^{\sigma+1})_h > -\frac{1}{\sigma+2}\frac{1}{\tau}|u|^{\sigma+2}_{h,\sigma+2} - |\hat{u}|^{\beta+\sigma+1}_{h,\beta+\sigma+1} + A_1|\hat{u}|^{2(\sigma+1)}_{h,\beta+\sigma+1},$$

$$A_1 = l^{-(\beta+3\sigma+3)/(\beta+\sigma+1)}.$$

Let us estimate the second term using Young's inequality. As a result we have

$$|\hat{u}|^{\beta+\sigma+1}_{h,\beta+\sigma+1} \leq \frac{A_1}{2}|\hat{u}|^{2(\sigma+1)}_{h,\beta+\sigma+1} + A_2,$$

$$A_2 = \frac{\sigma+1-\beta}{2(\sigma+1)}\left[\frac{\beta+\sigma+1}{A_1(\sigma+1)}\right]^{(\beta+\sigma+1)/(\sigma+1-\beta)},$$

and then the final estimate has the form

$$(P_h(\hat{u}), \hat{u}^{\sigma+1})_h > \frac{A_1}{2}|\hat{u}|^{2(\sigma+1)}_{h,\beta+\sigma+1} - \left(A_2 + \frac{1}{\sigma+2}\frac{1}{\tau}|u|^{\sigma+2}_{h,\sigma+2}\right). \tag{17}$$

From this, by the Brouwer fixed point theorem for continuous operators in a finite-dimensional space (see, for example [296]), we conclude that the equation $P_h(\hat{u}) = 0$ has at least one solution in the ball

$$|\hat{u}|^{2(\sigma+1)}_{h,\beta+\sigma+1} < \frac{2}{A_1}\left(A_2 + \frac{1}{\sigma+2}\frac{1}{\tau}|u|^{\sigma+2}_{h,\sigma+2}\right). \tag{18}$$

Outside this ball there are no solutions, since there, as follows from (17), $(P_h(\hat{u}), \hat{u}^{\sigma+1})_h > 0$.

Let us move on now to analyse the case $\beta = \sigma + 1$. Then from (15), (12) we have that

$$(P_h(\hat{u}), \hat{u}^{\sigma+1})_h > (\lambda_1^h - 1)|\hat{u}|_{h,2(\sigma+1)}^{2(\sigma+1)} - \frac{1}{\sigma+2}\frac{1}{\tau}|u|_{h,\sigma+2}^{\sigma+2}.$$

Therefore for $\lambda_1^h > 1$ the equation $P_h(\hat{u}) = 0$ has at least one solution $\hat{u}$, such that

$$|\hat{u}|_{h,2(\sigma+1)}^{2(\sigma+1)} < \frac{1}{(\lambda_1^h - 1)(\sigma+2)\tau}|u|_{h,\sigma+2}^{\sigma+2}. \tag{19}$$

Thus we have proved

**Theorem 1.** *Let $\beta < \sigma + 1$ or $\beta = \sigma + 1$, $\lambda_1^h > 1$. Then for any $\tau > 0$ there exists at least one solution $\hat{u} \in H_h$ of the scheme* (4), *which belongs to one of the sets* (18) *or* (19), *while there are no solutions outside these sets.*

As the estimates obtained in subsection 1.5 show, under the conditions of Theorem 1, the difference scheme (4) has a unique solution for sufficiently small $\tau > 0$.

2. For $\beta > \sigma + 1$ or $\beta = \sigma + 1$, $\lambda_1^h \le 1$ the parabolic operator of the scheme (4) is no longer coercive, so that the Brouwer theorem, which uses coercivity of the operator, is not applicable and Theorem 1 is invalid. Therefore we shall seek for sufficiently small $\tau$ a solution $\hat{u}$ close to $u$.

Let us set $\hat{u} - u = \hat{z}$ and let us introduce the continuous operator $F_h : \mathbf{R}^M \to \mathbf{R}^M$:

$$F_h(\hat{z}) = \{\tau[(\hat{z}_k + u_k)^{\sigma+1}]_{xx} + \tau(\hat{z}_k + u_k)^\beta, k = 1, 2, \ldots, M\}.$$

Existence of a fixed point of the operator $F_h$ implies solvability of the scheme (4). We have the obvious estimate

$$|F_h(\hat{z})|_C \le (|u|_C + |\hat{z}|_C)^\beta\tau + \frac{2}{h^2}(|u|_C + |\hat{z}|_C)^{\sigma+1}\tau.$$

Hence it follows that $F_h$ maps the set $X_{C_0} = \{\hat{z} \mid |\hat{z}|_C \le C_0\}$ into itself (here $C_0 > 0$ is an arbitrary constant), if

$$\tau \le \frac{C_0}{(|u|_C + C_0)^\beta + 2h^{-2}(|u|_C + C_0)^{\sigma+1}}. \tag{20}$$

Therefore by the Schauder fixed point theorem [101] we have

**Theorem 2.** *Let condition* (20) *hold. Then the scheme* (4) *has a solution $\hat{u} \in H_h$, such that, moreover, $|\hat{u} - u|_C \le C_0$.*

**Remark.** Setting $C_0 = |u|_C$ in (20), we obtain the following estimate of the practically maximal possible time step $\tau_{sol}$, for which the scheme is solvable at a fixed time level:

$$\tau_{sol} \simeq (2^\beta |u|_C^{\beta-1} + 2^{\sigma+2}|u|_C^{\sigma} h^{-2})^{-1}. \tag{21}$$

Below we shall show that as far as the dependence of $\tau_{sol}$ on $|u|_C$ is concerned, this estimate is optimal. It is important for large $|u|_C$, when the solution exhibits finite time blow-up. In the class of uniformly bounded $|u|_C$ we obtain the usual for parabolic equations estimate $\tau_{sol} = O(h^2)$ for $h \ll 1$ (see [346]).

*2 Conditions for non-uniqueness of the difference solution*

Let us show that for $\beta \geq \sigma+1$ and sufficiently small $\tau$ the implicit scheme (4) has in addition to the small solution constructed in Theorem 2 another, large solution, which is close to the root $\hat{U} = \tau^{-1/(\beta-1)}$ of the difference equation

$$\hat{U}/\tau = \hat{U}^\beta. \tag{22}$$

This equation is the same as the original one if we neglect the term $(\hat{u}^{\sigma+1})_{\bar{x}x}$ and set $u \equiv 0$. The second solution has the property that $|\hat{U}|_C \to \infty$ as $\tau \to 0$.

Let us set $\hat{z} = \hat{u} - \tau^{-1/(\beta-1)}$ and define the continuous operator $G_h : \mathbf{R}^M \to \mathbf{R}^M$:

$$G_h(\hat{z}) = \{\tau(\hat{z}_k + \tau^{-1/(\beta-1)})^\beta + \tau[(\hat{z}_k + \tau^{-1/(\beta-1)})^{\sigma+1}]_{\bar{x}x} -$$
$$- \tau^{-1/(\beta-1)} + u - \hat{z}_k,\ k = 1, 2, \ldots, M\}.$$

Existence of a root of the equation $G_h(\hat{z}) = 0$ implies solvability of the scheme (4).

Let us consider the expression

$$(G_h(\hat{z}), \hat{z})_h = ((\hat{z} + \tau^{-1/(\beta-1)})^\beta - \tau^{-\beta/(\beta-1)}, \hat{z})_h \tau +$$
$$+ \tau([(\hat{z} + \tau^{-1/(\beta-1)})^{\sigma+1}]_{\bar{x}x}, \hat{z})_h + (u, \hat{z})_h - |\hat{z}|_{h,2}^2 = I_1 + I_2 + I_3 - |\hat{z}|_{h,2}^2$$

on the sphere $|\hat{z}|_{h,2} = a_0 > 0$. Obviously $|\hat{z}|_C \leq a_0 h^{-1/2}$, and therefore, by setting

$$\eta_0 = a_0 h^{-1/2} \tau^{1/(\beta-1)} \tag{23}$$

we obtain

$$I_2 \geq -|\hat{z}|_{h,2} |(\hat{z} + \tau^{-1/(\beta-1)})^{\sigma+1}_{\bar{x}x}|_{h,2}\tau \geq -\tau^{[\beta-(\sigma+2)]/(\beta-1)} \frac{2a_0}{h^2}(1+\eta_0)^{\sigma+1} l^{1/2},$$

$$I_3 \geq -|\hat{z}|_{h,2}|u|_{h,2} = -a_0|u|_{h,2}.$$

To estimate the term $I_1$, we use the inequality

$$\eta[(1+\eta)^\beta - 1] \geq \frac{\beta+1}{2}\eta^2, \tag{24}$$

which holds for all $\beta > 1$, $|\eta| \le C_*$, where $C_* = C_*(\beta) > 0$ is some constant. Then, choosing $\tau$ so small that

$$\eta_0 = a_0 h^{-1/2} \tau^{1/(\beta-1)} \le C_*(\beta), \tag{25}$$

taking (24) into consideration, we obtain

$$I_1 \ge \frac{\beta+1}{2} \|z\|_{h,2}^2 = \frac{\beta+1}{2} a_0^2.$$

Thus we have the inequality

$$(G_h(z), z)_h \ge$$
$$\ge \frac{\beta-1}{2} a_0 \left\{ a_0 - \frac{2}{\beta-1} \left[ \|u\|_{h,2} + \frac{2}{h^2} \tau^{[\beta-(\sigma+2)]/(\beta-1)} (1 + C_*(\beta))^{\sigma+1} l^{1/2} \right] \right\}.$$

Hence $(G_h(z), z)_h \ge 0$ for all

$$\|z\|_{h,2} = a_0 = \frac{2}{\beta-1} \left[ \|u\|_{h,2} + \tau^{[\beta-(\sigma+2)]/(\beta-1)} \frac{2}{h^2} (1 + C_*(\beta))^{\sigma+1} l^{1/2} \right]. \tag{26}$$

It remains to check that for small $\tau$ conditions (25), (26) are compatible. Substituting into (25) the value of $a_0$ from (26), we have

$$\eta_0 = \frac{2}{\beta-1} h^{-1/2} \|u\|_{h,2} \tau^{1/(\beta-1)} + \frac{4}{\beta-1} h^{-5/2} [1 + C_*(\beta)]^{\sigma+1} l^{1/2} \tau^{[\beta-(\sigma+1)]/(\beta-1)}, \tag{27}$$

whence $\eta_0 \to 0$ as $\tau \to 0$ if $\beta > \sigma + 1$, so that condition (25) does not contradict (26) for small $\tau$. Thus we have proved

**Theorem 3.** *Let $\beta > \sigma + 1$. Then for sufficiently small $\tau$ the difference scheme (4) has in addition to the solution constructed in Theorem 2, another solution. If $\beta = \sigma + 1$ then this conclusion is still valid, provided that* (see (27))

$$\frac{4}{\sigma} h^{-5/2} [1 + C_*(\sigma+1)]^{\sigma+1} l^{1/2} < C_*(\sigma+1).$$

Fortunately the second (large) solution, which has no physical interpretation, is unstable, and a correct solution algorithm for the implicit scheme [346] converges only to the required solution. In this context, we might observe that in any neighbourhood of the solution $\hat{U} = \tau^{-1/(\beta-1)}$ the operator

$$F_h(z) = \{\tau(z_k + \tau^{-1/(\beta-1)})^\beta - \tau^{-1/(\beta-1)}, k = 1, 2, \cdots, M\}$$

is not contracting, and therefore the solution $\hat{U}$ cannot be obtained by the method of successive approximations. This testifies to its instability.

*3 Conditions of non-existence of the difference solution*

A second manifestation of non-coercivity of the operator of the scheme (4) for $\beta \geqslant \sigma + 1$ is the fact that it can actually be locally insolvable (on a given time-level).

To establish the conditions for non-existence of the solution, let us use the estimate

$$\left(\hat{u}^{\sigma+1}\right)_{\bar{x}x} \geq -\frac{2}{h^2}\hat{u}^{\sigma+1},\ x \in \omega_h,$$

in view of which we deduce from (4) the inequality

$$\hat{u} \geq u + \tau\hat{u}^{\sigma+1}(\hat{u}^{\beta-(\sigma+1)} - 2/h^2), \tag{28}$$

which the function $\hat{u}$ must satisfy everywhere in $\omega_h$.

It is clear that it is sufficient to verify this inequality at the point at which max $u$ is achieved, that is, to determine the conditions under which the inequality

$$\xi \geq |u|_C + \tau\xi^{\sigma+1}(\xi^{\beta-(\sigma+1)} - 2/h^2) \tag{29}$$

has no solutions in $\mathbf{R}_+$.

Let us consider first the case $\beta = \sigma + 1$. Then (29) assumes the form

$$\xi \geq |u|_C + \tau\xi^{\sigma+1}(1 - 2/h^2),\ \xi \in \mathbf{R}_+,$$

and, as is easily seen, has no solution if

$$h^2 > 2,\ \tau > \tau^* = \frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}}|u|_C^{-\sigma}\left(1 - \frac{2}{h^2}\right)^{-1}. \tag{30}$$

Now let $\beta > \sigma + 1$. Using Young's inequality

$$\xi^{\sigma+1} \leq \frac{h^2}{4}\xi^\beta + \epsilon,\ \xi \in \mathbf{R}_+,$$

$$\epsilon = \frac{\beta - (\sigma+1)}{\beta}\left[\frac{4(\sigma+1)}{\beta h^2}\right]^{(\sigma+1)/[\beta-(\sigma+1)]},$$

we see that (29) has no solutions if everywhere in $\mathbf{R}_+$

$$\xi < |u|_C - \frac{2\tau}{h^2}\epsilon + \frac{\tau}{2}\xi^\beta.$$

Hence we obtain conditions for insolvability of the scheme for $\beta > \sigma + 1$:

$$|u|_C > \frac{2\epsilon}{h^2}\tau + \frac{\beta-1}{\beta}\left(\frac{2}{\beta\tau}\right)^{1/(\beta-1)}. \tag{31}$$

**Theorem 4.** *Let $\beta = \sigma + 1$. Then, if condition* (30) *holds, the scheme* (4) *is insolvable. In the case $\beta > \sigma + 1$ solutions do not exist if* (31) *is satisfied.*

Inequalities (30), (31) provide us with quite sharp estimates of the size of the time step, for which no iterative process of solving the implicit scheme (4) will converge (for the simple reason that the difference solution does not exist). These estimates can be utilized in real numerical computations. Therefore let us consider (31) more carefully.

Let us set

$$a_0 = \frac{2[\beta - (\sigma + 1)]}{\beta}\left[\frac{4(\sigma + 1)}{\beta}\right]^{(\beta+1)/[\beta-(\sigma+1)]}, \quad b_0 = \frac{\beta - 1}{\beta}\left(\frac{2}{\beta}\right)^{1/(\beta-1)}.$$

Then (31) has the form

$$|u|_C \geq a_0\tau(h^2)^{-\beta/[\beta-(\sigma+1)]} + b_0\tau^{-1/(\beta-1)}$$

and is satisfied, for example, in the case

$$|u|_C = \beta a_0 d_0 (h^2)^{-1/[\beta-(\sigma+1)]}, \; d_0 = \left[\frac{b_0}{a_0(\beta - 1)}\right]^{\beta/(\beta-1)}, \tag{32}$$

$$\tau = \tau_{insol} = d_0(h^2)^{(\beta-1)/[\beta-(\sigma+1)]}. \tag{33}$$

At the same time condition (21) for the solvability of the scheme with $|u|_C$ taken from (32), has the form

$$\begin{gathered} \tau_{sol} = f_0(h^2)^{(\beta-1)/[\beta-(\sigma+1)]}, \\ f_0 = [2^\beta(\beta a_0 d_0)^{\beta-1} + 2^{\sigma+2}(\beta a_0 d_0)^\sigma]^{-1}, \end{gathered} \tag{33'}$$

and has the same dependence on the size of the spatial grid as (33). Hence we conclude that condition (21) for the solvability of the scheme for $\beta > \sigma + 1$ is optimal for large $|u|_C$ (for example, when the difference solution is unbounded).

## *4 Unbounded difference solutions*

Let us move on now to determine the conditions for global insolvability of the difference problem (4), (5) for $\beta \geq \sigma + 1$[4]. Recall that the time grid here is not uniform: $\tau_j \to 0$ as $j \to \infty$ and $\sum_{j=0}^{\infty}\tau_j = T_0 < \infty$, where $T_0$ is the time of existence of the solution.

The proof of unboundedness of the solution will utilize the method used earlier in § 2, as well as in § 6, Ch. V.

[4]It will be shown in subsection 1.5 that, just as in the differential case (§ 2), there are no unbounded solutions if $\beta < \sigma + 1$.

1. Let us set

$$E^{(j+1)} \equiv \hat{E} = (\hat{u}, \psi_h)_h, \ t \in \omega_\tau, \tag{34}$$

where $\psi_h$ is the first eigenfunction (10) of the problem (11). Taking the scalar product of the system of equations (4) with $\psi_h$, we obtain the sequence of equalities

$$\begin{aligned} \frac{\hat{E} - E}{\tau_j} &= -\lambda_1^h(\hat{u}^{\sigma+1}, \psi_h)_h + (\hat{u}^\beta, \psi_h)_h, \ t \in \omega_\tau, \\ E^{(0)} &= E_0 = (u_{0h}, \psi_h)_h, \end{aligned} \tag{35}$$

deriving which we have taken into account the fact that

$$((\hat{u}^{\sigma+1})_{\bar{x}x}, \psi_h)_h = (\hat{u}^{\sigma+1}, (\psi_h)_{\bar{x}x})_h = -\lambda_1^h(\hat{u}^{\sigma+1}, \psi_h)_h.$$

By normalization of $\psi_h > 0$ we have the Hölder inequality

$$(\hat{u}^\beta, \psi_h)_h = ((\hat{u}^{\sigma+1})^{\beta/(\sigma+1)}, \psi_h)_h \geq (\hat{u}^{\sigma+1}, \psi_h)_h^{\beta/(\sigma+1)},$$

in view of which we obtain from (35)

$$\frac{\hat{E} - E}{\tau_j} \geq (\hat{u}^{\sigma+1}, \psi_h)_h^{\beta/(\sigma+1)}[1 - \lambda_1^h(\hat{u}^{\sigma+1}, \psi_h)_h^{(\sigma+1-\beta)/(\sigma+1)}].$$

Thence, again applying the Hölder inequality $(\hat{u}^{\sigma+1}, \psi_h)_h \geq (\hat{u}, \psi_h)_h^{\sigma+1}$, we deduce the inequality

$$\frac{\hat{E} - E}{\tau_j} \geq (\hat{u}^{\sigma+1}, \psi_h)_h^{\beta/(\sigma+1)} \left(1 - \frac{\lambda_1^h}{\hat{E}^{\beta-(\sigma+1)}}\right). \tag{36}$$

Let $E_0$ be such that

$$\mu_0 = 1 - \lambda_1^h E_0^{\sigma+1-\beta} > 0. \tag{37}$$

Then from (36) we conclude that $\hat{E} > E$ in $\omega_\tau$ for sufficiently small $\tau_j$, $j = 0, 1, \ldots$, and therefore

$$\frac{\hat{E} - E}{\tau_j} \geq \hat{E}^\beta \left[1 - \frac{\lambda_1^h}{E_0^{\beta-(\sigma+1)}}\right] = \mu_0 \hat{E}^\beta, \ t \in \omega_\tau. \tag{38}$$

Since

$$\|\hat{u}\|_C = \max_{1 < k < M} \hat{u}_k \geq \hat{E}, \ t \in \omega_\tau, \tag{39}$$

in order to determine conditions for unboundedness of the solution of problem (4), (5), we have to find a system $\{\tau_j\}$ of time intervals, such that $\sum \tau_j = T_0 < \infty$ and that from (38) it would follow that $E^{(j)} \to \infty$ as $j \to \infty$. Then by (39) the difference solution will blow up in finite time, that is $\|u^j\|_C \to \infty$ as $j \to \infty$.

Let

$$\tau_j = A\rho^{-\alpha j}, \quad j = 0, 1, \ldots, \tag{40}$$

where $A > 0$, $\alpha > 0$, $\rho > 1$ are constants. Then

$$T_0 = A\rho^\alpha/(\rho^\alpha - 1) < \infty. \tag{41}$$

Let us determine the conditions for $A, \alpha, \rho$, for which

$$E^{(j)} \geq E_0\rho^j, \quad j = 0, 1, \ldots$$

For that, it is enough that

$$E_0\rho^j + \tau_j\mu_0 E_0^\beta \rho^{\beta j} \geq E_0\rho^{j+1}, \quad j = 0, 1, \ldots. \tag{42}$$

Substituting into (42) $\tau_j$ from (40) and simplifying, we obtain the condition $1 + A\mu_0 E_0^{\beta-1}\rho^{j(\beta-\alpha-1)} \geq \rho$, $j = 0, 1, \ldots$. This condition will hold if

$$\alpha = \beta - 1, \; \rho = 1 + A\mu_0 E_0^{\beta-1}. \tag{43}$$

Therefore we have proved

**Theorem 5.** *Let $\beta \geq \sigma + 1$ and let the initial function $u_{0h}$ in* (5) *be such that* (37) *holds. Let the finite difference problem* (4), (5) *be solvable on a sequence of time steps* (40), *where the constants $A, \alpha, \rho$ satisfy* (43). *Then the solution exists for time* (41), *and*

$$|u^j|_C \geq E_0\rho^j \to \infty, \quad j \to \infty.$$

**Remark.** In § 2 it is shown that in the differential (continuous) case the problem for $\beta = \sigma + 1$ has an unbounded solution if $\lambda_1^0 = (\pi/l)^2 < 1$. If, on the other hand, $\lambda_1^0 > 1$, then it is globally solvable. From condition (37), which for $\beta = \sigma + 1$ takes the form $\lambda_1^h < 1$, and the easily verified inequality $\lambda_1^h < \lambda_1^0$ (see (9)), we then easily conclude that it is possible for the finite difference problem to have an unbounded solution, while the differential problem is globally solvable. This will happen if the length of the interval $l$ is such that $\lambda_1^0 > 1$ but $\lambda_1^h < 1$.

Inequality (38), which was derived in the course of the proof of Theorem 5, can be used to analyze the problem of insolvability of the scheme on a given time-level. For example, from (38) it is easy to derive the following estimates for insolvability of the scheme at the $j$-th time step in the case $\beta = \sigma + 1$. They are sharper than the estimates of (30):

$$\lambda_1^h < 1, \; \tau \geq \tau_* = (E^{(j)})^{-\sigma}\frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}(1-\lambda_1^h)}.$$

The first one of these is optimal, since for $\lambda_1^h > 1$ the solution always exists (see Theorem 1).

Let us present an interesting corollary, which has a differential analogue (see § 3, Ch. IV).

**Corollary.** *Let $\beta \in ]\sigma + 1, \sigma + 3)$ and let us be given a function $u_0(x) \geq 0$, $x \in \mathbf{R}_+$. Let us fix an arbitrary $h > 0$. Then for sufficiently large $M$ there is a collection of time steps $\{\tau_j\}$, satisfying $T_0 = \sum \tau_j < \infty$, such that the solution of the finite difference problem* (4), (5) *for $l = (M+1)h$ and $u_{0h} \not\equiv 0$ is unbounded. If $|u_{0h}|_{h,1} > 2$ for some $M > 1$, then the same conclusion is valid also in the case $\beta = \sigma + 3$.*

*Proof.* For large $l$ $\mu_0$ in (37) satisfies the estimate

$$\mu_0 \simeq 1 - (|u_{0h}|_{h,1}/2)^{\sigma+1-\beta}(\pi/l)^{\sigma+3-\beta}.$$

Therefore under the above assumptions $\mu_0 > 0$ for sufficiently large $l$, which by Theorem 5 ensures global insolvability of the problem (4), (5). □

2. To conclude, let us give an example of an unbounded solution of the finite difference problem (4), (5), which can be written down explicitly. This example shows, in particular, that the requirement (40) of Theorem 5 concerning solvability on a sequence of time steps, is not especially burdensome.

Let $\beta = \sigma + 1$. As in the continuous case (see § 1, Ch. IV), we shall seek the difference solution of the problem in separated variables:

$$u_k^j = S^j\theta_k, \ (t, x) \in \omega_\tau \times \omega_h. \tag{44}$$

Substituting $u_k^j$ into (4) we obtain the following problems for the grid functions $S^j$, $\theta_k$

$$\frac{\hat{S} - S}{\tau_j} = \frac{1}{\sigma}\hat{S}^{\sigma+1}, \ t \in \omega_\tau; \tag{45}$$

$$(\theta^{\sigma+1})_{\bar{x}x} + \theta^{\sigma+1} = \frac{1}{\sigma}\theta, \ x \in \omega_h, \ \theta \in H_h. \tag{46}$$

Let there be given a system of time intervals (40), where $\rho > 1$, $\alpha = \sigma$. Then a solution of the problem (45) is the function

$$S^j = \rho^j, \ j = 0, 1, \ldots, A = \sigma\rho^{-(\sigma+1)}(\rho - 1). \tag{47}$$

Let us construct a solution of the problem (46) in the particular case $\sigma = 2$. Let us fix an arbitrary $M > 0$ and set $h = 2\sin\{3\pi/[2(M+1)]\}$. In this case the length of the interval is

$$l_h = \frac{3\pi h}{2}\left(\arcsin\frac{h}{2}\right)^{-1}, \ 0 < h \leq 2. \tag{48}$$

Then it can be easily seen that the solution of the problem (46) has the form

$$\theta_k = \left\{2\left[3\left(1 - \frac{4}{h^2}\sin^2\frac{a_h h}{2}\right)\right]^{-1}\right\}^{1/2} \sin(a_h kh), \ k = 0, 1, \ldots, M+1, \tag{49}$$

where $a_h = \pi/l_h$.

By (44), the functions (47), (49) define an unbounded difference solution of problem (4), (5) for $\sigma = 2$, $\beta = 3$, $l = l_h$, which grows without bound at all points of $\omega_h$, conserving its spatial structure. As $h \to 0$ the difference solution (49) converges to a solution of the corresponding ordinary differential equation (see § 1, Ch. IV):

$$\theta(x) = (3/4)^{1/2} \sin(x/3), \ 0 < x < l_0 = 3\pi. \tag{50}$$

The length of the support of this solution, $l_0$, defines the fundamental length of the nonlinear medium. The difference fundamental length (48) is close to $l_0 = 3\pi$ for $h \ll 1$. For large $h$ the difference can be significant; for example, $l_h = 9$ for $h = 1$, $l_h = 6$ for $h = 2$.

Let us note that the grid function (49) is not the projection onto $\omega_h$ of the solution (50), the differential analogue of problem (46) for $\sigma = 2$, though it has a similar structure. The dissimilarity is even more substantial in the case $\sigma = 1$, when equation (46) also has a simple solution:

$$\theta_k = A_h \sin^2(b_h kh) + B_h, \ k = 0, 1, \ldots, M+1,$$

where

$$b_h = \frac{1}{2h}\arcsin\frac{h}{2}, \ 0 < h \le 2; \ A_h = \frac{1}{\kappa_h}|2(2\kappa_h - 1)|^{1/2},$$

$$B_h = \frac{1}{2\kappa_h}\{1 - |2(2\kappa_h - 1)|^{1/2}\}, \ \kappa_h = 1 - \frac{2}{h^2}\left[1 - \left(1 - \frac{h^2}{4}\right)^{1/2}\right],$$

which, however, does not satisfy the boundary conditions and is strictly positive. However, in the limit $h \to 0$, when $\kappa_h \to 3/4$, $B_h \to 0$, the function $\theta_k$ is the solution of the differential problem for $l \ge 4\pi$.

Let us move on now to questions of global solvability of the difference problem and of convergence of the difference solution as $h \to 0$ to the generalized solution of the differential problem (1)–(3). Two separate cases have to be considered.

## *5 Global solvability and passage to the limit for $\beta \le \sigma + 1$*

Recall that below the grid $\omega_\tau$ is assumed to be uniform.

**Theorem 6.** *Let $\beta < \sigma + 1$. Then for sufficiently small $\tau$ the difference problem has a unique global solution. If $\beta = \sigma + 1$, then the same assertion is true under the additional requirement*

$$\lambda_1^h = \frac{4}{h^2} \sin^2 \frac{\pi h}{2l} > 1. \tag{51}$$

*In both cases $\beta < \sigma+1$ and $\beta = \sigma+1$, $\lambda_1 > 1$, the difference solution converges as $\tau, h \to 0$ to the generalized solution of the differential problem (1)–(3) constructed in Theorem 2 of § 2.*

We shall need two lemmas, the first of which is verified directly. The second is proved in [296, 346].

**Lemma 2.** *For all $\xi, \eta \in \mathbf{R}_+$ the following inequality holds:*

$$\begin{aligned} \left(\xi^{\sigma+1} - \eta^{\sigma+1}\right) \xi^{\beta} \le \frac{\sigma + 1}{\beta + \sigma + 1} \left(\xi^{\beta+\sigma+1} - \eta^{\beta+\sigma+1}\right) + \\ + C_1[\max\{\xi, \eta\}]^{\beta-1}(\xi^{1+\sigma/2} - \eta^{1+\sigma/2})^2, \end{aligned} \tag{52}$$

*where $C_1 = C_1(\sigma, \beta) > 0$ is a constant.*

**Lemma 3.** *For any grid function $v_h \in H_h$*

$$|v_h|_C \le A \| v_h^{\sigma+1} \|_{h,2}^{1/(\sigma+1)}, \quad A_3 = l^{1/[2(\sigma+1)]}. \tag{53}$$

*Proof of Theorem 6.* Let us fix an arbitrary $T > 0$.

1) Let us first consider the case $\beta < \sigma + 1$. By Theorem 1 the difference scheme (4) with $\beta < \sigma + 1$ is solvable for any $\tau$, that is, the function $\hat{u}$ is defined everywhere in $\omega_{\tau,h}$. We shall need estimates of the finite difference solution.

Taking the scalar product of both sides of (4) by $\hat{u}^{\sigma+1}$ and using the obvious inequality

$$(\xi - \eta)\xi^{\sigma+1} \ge \frac{1}{\sigma + 2}(\xi^{\sigma+2} - \eta^{\sigma+2}), \quad \xi, \eta \in \mathbf{R}_+,$$

we obtain

$$\frac{1}{\sigma + 2} \frac{1}{\tau}(|\hat{u}|_{h,\sigma+2}^{\sigma+2} - |u|_{h,\sigma+2}^{\sigma+2}) + \|\hat{u}^{\sigma+1}\|_{h,2}^2 \le |\hat{u}|_{h,\beta+\sigma+1}^{\beta+\sigma+1}. \tag{54}$$

We estimate the right-hand side of this inequality using (13) and Young's inequality, taking into account the fact that $\beta < \sigma + 1$. As a result we obtain the inequality

$$|\hat{u}|_{h,\beta+\sigma+1}^{\beta+\sigma+1} \le \frac{1}{2}\|\hat{u}^{\sigma+1}\|_{h,2}^2 + A_4, \tag{55}$$

$$A_4 = \frac{\sigma + 1 - \beta}{2(\beta + \sigma + 1)} \left[\frac{\sigma + 1}{A_0(\beta + \sigma + 1)}\right]^{2(\sigma+1)/[\beta-(\sigma+1)]}.$$

and then from (54) we have

$$\frac{1}{\sigma+2}\frac{1}{\tau}|\hat{u}|^{\sigma+2}_{h,\sigma+2}+\frac{1}{2}\|\hat{u}^{\sigma+1}\|^2_{h,2}\le A_4+\frac{1}{\tau}\frac{1}{\sigma+2}|u|^{\sigma+2}_{h,\sigma+2}. \tag{56}$$

Hence we obtain the estimates

$$\max_{0\le j\le N}|u^{j+1}|^{\sigma+2}_{h,\sigma+2}\le A_4T(\sigma+2)+|u_{0h}|^{\sigma+2}_{h,\sigma+2}\le A_5 \tag{57}$$

and, by (53), the inequality

$$\max_{0\le j\le N}|u^{j+1}|_C\le A_3\left(2A_4+\frac{2}{\tau}\frac{1}{\sigma+2}A_5\right)^{1/[2(\sigma+1)]}. \tag{58}$$

To derive other estimates, we take the scalar product of (4) with $(\hat{u}^{\sigma+1}-u^{\sigma+1})/\tau$ and use the inequality (52) as well as the inequality

$$(\xi^{\sigma+1}-\eta^{\sigma+1})(\xi-\eta)\ge C_2(\xi^{1+\sigma/2}-\eta^{1+\sigma/2})^2$$

for all $\xi,\eta\in\mathbf{R}_+$, where $C_2=C_2(\sigma)>0$ is a constant. As a result, we have

$$\begin{aligned} C_2\left|\frac{\hat{u}^{1+\sigma/2}-u^{1+\sigma/2}}{\tau}\right|^2_{h,2} &\le -\frac{1}{2\tau}\left(\|\hat{u}^{\sigma+1}\|^2_{h,2}-\|u^{\sigma+1}\|^2_{h,2}\right)+\\ &+\frac{\sigma+1}{\beta+\sigma+1}\frac{1}{\tau}(|\hat{u}|^{\beta+\sigma+1}_{h,\beta+\sigma+1}-|u|^{\beta+\sigma+1}_{h,\beta+\sigma+1})+\\ &+C_1\tau[\max\{|\hat{u}_C,|u|_C\}]^{\beta-1}\left|\frac{\hat{u}^{1+\sigma/2}-u^{1+\sigma/2}}{\tau}\right|^2_{h,2}. \end{aligned} \tag{59}$$

In the derivation of (59) we also used the inequality

$$((\hat{u}^{\sigma+1})_{\bar{x}x},\hat{u}^{\sigma+1}-u^{\sigma+1})_h\le-(\|\hat{u}^{\sigma+1}\|^2_{h,2}-\|u^{\sigma+1}\|^2_{h,2})/2$$

(since $\xi(\xi-\eta)\ge(\xi^2-\eta^2)/2$ for any $\xi,\eta\in\mathbf{R}_+$).

Let us choose $N$ so large (that is, $\tau=T/(N+1)$ so small), tha $C_1\tau[\max\{|\hat{u}|_C,|u|_C\}]^{\beta-1}\le C_2/2$ for all $0\le j\le N$. It follows from (58) tha for this it is sufficient that

$$\tau\left(2A_4+\frac{2}{\tau}\frac{1}{\sigma+2}A_5\right)^{(\beta-1)/[2(\sigma+1)]}\le\frac{C_2}{2C_1}A_3^{1-\beta}. \tag{60}$$

For $\beta < \sigma + 1$ that can always be done. Then, summing the inequalities (59) in $j$ from 0 to $N$, and applying Young's inequality, we obtain

$$\frac{C_2}{2}\sum_{j=0}^{N}\left|\frac{\hat{u}^{1+\sigma/2}-u^{1+\sigma/2}}{\tau}\right|_{h,2}^{2}+\frac{1}{2}\|(u^{N+1})^{\sigma+1}\|_{h,2}^{2}\le$$

$$\le\frac{\sigma+1}{\beta+\sigma+1}|u^{N+1}|_{h,\beta+\sigma+1}^{\beta+\sigma+1}+\frac{1}{2}\|u_{0h}^{\sigma+1}\|_{h,2}^{2}-$$

$$-\frac{\sigma+1}{\beta+\sigma+1}|u_{0h}|_{h,\beta+\sigma+1}^{\beta+\sigma+1}\le\frac{1}{4}\|(u^{N+1})^{\sigma+1}\|_{h,2}^{2}+A_6$$

(to derive the last inequality we used the estimate (13)). Hence we have the estimates

$$\sum_{j=0}^{N}\tau\left|\frac{\hat{u}^{1+\sigma/2}-u^{1+\sigma/2}}{\tau}\right|_{h,2}^{2}\le A_7, \tag{61}$$

$$\max_{0\le j\le N}\|(u^{j+1})^{\sigma+1}\|_{h,2}^{2}\le A_8. \tag{62}$$

Thus restriction (60) on the size of the time step $\tau$ ensures global boundedness of the difference solution of problem (4), (5) for $\beta < \sigma + 1$. Let us note that by (53), from (62) follows the estimate

$$\max_{0\le j\le N}|u^{j+1}|_C\le A_9. \tag{63}$$

Let us show now that in the case

$$\tau < A_9^{1-\beta}/\beta \tag{64}$$

the solution is unique. Let $\hat{u}_1$, $\hat{u}_2$ be two solutions of the problem. Then from (4) we have that $\hat{u}_1-\hat{u}_2=\tau(\hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1})_{\bar{x}x}+\tau(\hat{u}_1^{\beta}-\hat{u}_2^{\beta})$. Taking the scalar product of this equality with $\hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1}$, we obtain

$$(\hat{u}_1-\hat{u}_2,\hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1})_h=$$

$$=-\tau\|\hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1}\|_{h,2}^{2}+\tau(\hat{u}_1^{\beta}-\hat{u}_2^{\beta},\hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1})_h\le$$

$$\le\tau\beta[\max\{|\hat{u}_1|_C,|\hat{u}_2|_C\}]^{\beta-1}(\hat{u}_1-\hat{u}_2,\hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1})_h.$$

Hence by (63), (64) we conclude that $(\hat{u}_1-\hat{u}_2,\hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1})_h=0$, that is, $\hat{u}_1\equiv\hat{u}_2$.

To prove the validity of passing to the limit, we shall need another estimate. From (4) we have

$$\|(\hat{u}-u)/\tau\|_{h,2}^{*}\le\|(\hat{u}^{\sigma+1})_{\bar{x}x}\|_{h,2}^{*}+\|\hat{u}^{\beta}\|_{h,2}^{*}.$$

Therefore, taking into account (8), and using (62), we obtain

$$\max_{0\le j\le N} \|(u^{j+1}-u^j)/\tau\|^*_{h,2} \le A_{10}. \tag{65}$$

For convenience, let us introduce the notation

$$\nabla_\tau q_\tau v_{\tau h} = (v_h^{j+1}-v_h^j)/\tau, \quad j\tau \le t \le (j+1)\tau.$$

From (57), (61), (62), (65), using the results of [339], we obtain the following estimates; the functions $q_\tau p_h(u^j)^{\sigma+1}$ are uniformly bounded in $L^\infty(0,T;H_0^1(0,l))$ and in $L^\infty(0,T;L^{(\sigma+2)/(\sigma+1)}(0,l))$; $q_\tau q_h(u^j)^{\sigma+1}$ are uniformly bounded in $L^\infty(0,T;L^{(\sigma+2)/(\sigma+1)}(0,l))$; $q_\tau q_h(u^j)^{1+\sigma/2}$ in $L^\infty(0,T;L^2(0,l))$; $\nabla_\tau q_\tau q_h(u^j)^{1+\sigma/2}$ in $L^2(0,T;L^2(0,l))$; $q_\tau p_h u^j$ in $L^\infty(0,T;L^{\sigma+2}(0,l))$; $\nabla_\tau q_\tau p_h u^j$ in $L^\infty(0,T;H^{-1}(0,l))$; $\left[q_\tau p_h(u^j)^{\sigma+1}\right]_{\bar{x}x}$ in $L^\infty(0,T;H^{-1}(0,l))$; $q_\tau q_h(u^j)^\beta$ in $L^\infty(0,T;L^1(0,l))$; $q_h(u^{N+1})$ in $L^{\sigma+2}(0,l)$. This information is sufficient to be able to use a compactness theorem [86] to pass to the limit as $\tau, h\to 0$ (details can be found in [296, 339]).

As a result we have another proof of existence of a global generalized solution of the problem (1)–(3) for $\beta<\sigma+1$, which satisfies all the inclusions of Theorem 1, § 2. Observe that we have in addition that $u_t\in L^2(0,T;H^{-1}(0,l))$.

2) Let us consider the case $\beta=\sigma+1$. Applying the inequality (12) to estimate the right-hand side of (54) for $\beta=\sigma+1$, we have

$$\frac{1}{\sigma+2}\frac{1}{\tau}|\hat{u}|^{\sigma+2}_{h,\sigma+2} + \left(1-\frac{1}{\lambda_1^h}\right)\|\hat{u}^{\sigma+1}\|^2_{h,2} \le \frac{1}{\sigma+2}\frac{1}{\tau}|u|^{\sigma+2}_{h,\sigma+2}. \tag{66}$$

Hence (see (51)) we have the estimate (58) with $A_4=0$ and the inequalities

$$\|\hat{u}^{\sigma+1}\|^2_{h,2} \le \frac{1}{\sigma+2}\frac{1}{\tau}\frac{\lambda_1^h}{\lambda_1^h-1}|u_{0h}|^{\sigma+2}_{h,\sigma+2} \le \frac{A_{11}}{\tau}.$$

Let us take $\tau$ so small that (see (60))

$$\tau^{(\sigma+2)/[2(\sigma+1)]} \le \frac{C_2}{2C_1}A_3^{\ \sigma}A_{11}^{-\sigma/[2(\sigma+1)]}.$$

Then it is not hard to see that the following inequality is satisfied:

$$\frac{C_2}{2}\sum_{j=0}^{N}\tau\left|\frac{\hat{u}^{1+\sigma/2}-u^{1+\sigma/2}}{\tau}\right|^2_{h,2} + \frac{1}{2}\left(1-\frac{1}{\lambda_1^h}\right)\|(u^{N+1})^{\sigma+1}\|^2_{h,2} \le$$
$$\le \frac{1}{2}\left(\|u_{0h}^{\sigma+1}\|^2_{h,2} - |u_{0h}|^{2(\sigma+1)}_{h,2(\sigma+1)}\right).$$

from which the estimates (61)–(63) follow. Subsequent analysis is carried out as in the case $\beta < \sigma + 1$. □

Thus in the cases $\beta < \sigma + 1$ and $\beta = \sigma + 1$, $\lambda_1^h > 1$, no matter how fine the grid in time (which, obviously, can be taken to be non-uniform), there is no blow-up of finite difference solutions, which agrees with the conclusions of § 2 concerning the differential problem. This agreement is even more clearly seen for $\beta > \sigma + 1$.

*6 Finite difference stable sets and passing to the limit for $\beta > \sigma + 1$*

Here we shall show that for $\beta > \sigma + 1$ we can construct a difference stable set $\mathcal{W}_h$, which has a structure quite similar to the one for the differential problem constructed in § 2. The latter will be denoted below by $\mathcal{W}_0$, to emphasize that it is $\mathcal{W}_h$ for $h = 0^+$.

Let us define for all $v_h \in H_h$ the functional

$$J_h(v_h) = \frac{1}{2}a_h(v_h) - \frac{\sigma+1}{\beta+\sigma+1}b_h(v_h),$$

where $a_h(v_h) = \|v_h^{\sigma+1}\|_{h,2}^2$, $b_h(v_h) = |v_h|_{h,\beta+\sigma+1}^{\beta+\sigma+1}$. Using Lemma 1, it is not hard to prove (see Lemma 3 in § 2)

**Lemma 4.** *Let $\beta > \sigma + 1$. Then we have the inequality*

$$d_h = \inf_{v_h \in H_h, v_h \not\equiv 0} \sup_{\lambda > 0} J_h(\lambda v_h) > \frac{\beta - (\sigma+1)}{2(\beta+\sigma+1)} l^{-[\beta+3(\sigma+1)]/[\beta-(\sigma+1)]} > 0.$$

Now we can define the finite difference stable set $\mathcal{W}_h$ (which is non-empty by Lemma 4):

$$\mathcal{W}_h = \{v_h \mid v_h \in H_h, 0 \le J_h(\lambda v_h) < d_h, \lambda \in [0, 1]\}. \tag{67}$$

From the construction of $\mathcal{W}_h$ we obtain (see Lemma 4, § 2)

**Lemma 5.** *We have the equality $\mathcal{W}_h = \mathcal{W}_h^* \cup \{0\}$, where*

$$\mathcal{W}_h^* = \{v_h \mid v_h \in H_h, a_h(v_h) - b_h(v_h) > 0, J_h(v_h) < d_h\}.$$

Let us state the main result of this subsection.

**Theorem 7.** *Let $\beta > \sigma + 1$ and assume that $u_{0h} \in \mathcal{W}_h$. Then for sufficiently small $\tau$ the finite difference problem* (4), (5) *has a global solution which belongs to $\mathcal{W}_h$ for all $t \in \omega_\tau$. Moreover, the solution is unique. If, furthermore $u_0 \in \mathcal{W}_0$, the finite difference solution converges as $\tau, h \to 0$ to the generalized solution of the original differential problem, which was constructed in Theorem 4, § 2.*

**Remark.** In the conditions of Theorem 7, no refinement of the time step $\tau$ (which by itself has to be sufficiently small) can lead to the finite difference solution being unbounded. Therefore, as follows from Theorem 7, inequality (37) cannot hold if $u_{0h} \in \mathcal{W}_h$, that is,

$$1 - \lambda_1^h (u_{0h}, \psi_h)_h^{\sigma+1-\beta} \leq 0, \quad u_{0h} \in \mathcal{W}_h.$$

This inequality is an additonal characteristic of the difference stable set $\mathcal{W}_h$.

*Proof of Theorem 7.* This largely follows the lines of proof of the "differential" Theorem 4, § 2. If $u_{0h} \in \mathcal{W}_h$, then by Lemma 5 we have $a_h(u_{0h}) - b_h(u_{0h}) > 0$ and therefore

$$J_h(u_{0h}) = \frac{1}{2} a_h(u_{0h}) - \frac{\sigma+1}{\beta+\sigma+1} b_h(u_{0h}) \geq \frac{\beta-(\sigma+1)}{2(\beta+\sigma+1)} a_h(u_{0h}).$$

Therefore it follows from (53) that

$$|u_{0h}|_C \leq A_3 \left[ \frac{2(\beta+\sigma+1)}{\beta-(\sigma+1)} J_h(u_{0h}) \right]^{1/[2(\sigma+1)]} \leq A_{12},$$

that is, $\mathcal{W}_h$ is bounded in $C(\omega_h)$.

Let us show that the condition

$$\tau \leq \frac{\min\{1, C_2/(2C_1)\}}{(1+A_{12})^\beta + 2h^{-2}(1+A_{12})^{\sigma+1}} \tag{68}$$

ensures solvability of scheme (4) at each time level.

Let us make the first step in time. From Theorem 2 (see condition (20) for $C_0 = 1$) it follows that under the restriction (68) the scheme is solvable. Since by (68) $|u^1|_C \leq |u^0|_C + 1 \leq A_{12} + 1$,

$$\tau [\max\{|u^1|_C, |u^0|_C\}]^{\beta-1} \leq C_2/(2C_1),$$

and, consequently, we obtain from (59) the inequality

$$\frac{C_2}{2} \left| \frac{(u^1)^{1+\sigma/2} - (u^0)^{1+\sigma/2}}{\tau} \right|_{h,2}^2 \leq \frac{1}{\tau} [J_h(u^0) - J_h(u^1)]. \tag{69}$$

Let us prove that $u^1 \in \mathcal{W}_h$. Indeed, let $u^1 \notin \mathcal{W}_h$. Then since $u^1 \to u^0$ as $\tau \to 0$ (see proof of Theorem 2), and $u^0 \in \mathcal{W}_h$, there exists a $\tau = \tau_*$, such that $u^1 \in \partial\mathcal{W}_h$. By (67) this means that $J_h(u^1) = d_h$. Hence we have a contradiction with (69), since by assumption $J_h(u^0) < d_h$.

Thus, $u^1 \in \mathcal{W}_h$. Then from Lemma 5 we have that $a_h(u^1) > b_h(u^1)$. Using this inequality to estimate the second term in the right-hand side of (69), we have

$$\frac{C_2}{2}\left|\frac{(u^1)^{1+\sigma/2}-(u^0)^{1+\sigma/2}}{\tau}\right|^2_{h,2} + \frac{1}{\tau}\frac{\beta-(\sigma+1)}{2(\beta+\sigma+1)}\|(u^1)^{\sigma+1}\|^2_{h,2} \le \frac{1}{\tau}J_h(u^0).$$

Hence

$$\frac{\beta-(\sigma+1)}{2(\beta+\sigma+1)}\|(u^1)^{\sigma+1}\|^2_{h,2} \le J_h(u^0),$$

and therefore $|u^1|_C \le A_{12}$. This justifies making the next step in time under restriction (68) on the magnitude of $\tau$ and so on.

Thus, if (68) holds, the difference scheme has a global solution $\hat{u} \in \mathcal{W}_h$, $|\hat{u}|_C \le A_{12}$ for all $0 < j \le N$, and furthermore

$$\frac{C_2}{2}\sum_{j=0}^{N}\tau\left|\frac{\hat{u}^{1+\sigma/2}-u^{1+\sigma/2}}{\tau}\right|^2_{h,2} + \frac{\beta-(\sigma+1)}{2(\beta+\sigma+1)}\|\hat{u}^{\sigma+1}\|^2_{h,2} \le J_h(u_{0h}).$$

Uniqueness for $\tau < \beta^{-1}A_{12}^{1-\beta}$ of a uniformly bounded solution $u \in \mathcal{W}_h$ is proved as in Theorem 6. □

The finite difference solution constructed above satisfies estimates which allow us to pass to the limit as $\tau, h \to 0$. Here it is only necessary to observe that in the case $u_0 \in \mathcal{W}_0$ the inclusion $u_{0h} \in \mathcal{W}_h$ holds for all sufficiently small $h > 0$. As a result we have the existence of a global generalized solution of the problem (1)–(3), which is in $\overline{\mathcal{W}_0}$ for all $t > 0$.

To conclude, let us note that by the estimate (21) for global solvability of the finite difference problem in the set $\mathcal{W}_h$, it is sufficient that $\tau = O(h^2)$ for $h \ll 1$. The restriction (68) on the size of the time step $\tau$ is essential, since too large steps can sooner or later "throw" the solution out of $\mathcal{W}_h$, and it will become unbounded. The necessary shortening of the time steps $\tau_j$ will be automatically performed by the iterative algorithm we are using, when Theorem 4 (on the non-existence of solution) comes to the fore.

## 2 Explicit (linear) difference scheme

Below we shall consider briefly unbounded solutions of the explicit difference scheme for the problem (1)–(3):

$$(\hat{u}-u)/\tau_j = (u^{\sigma+1})_{\bar{x}x} + u^\beta,\ (t, x) \in \omega_\tau \times \omega_h, \tag{70}$$

$$u^0 = u_{0h} \ge 0,\ x \in \omega_h;\ \hat{u} \in H_h, t \in \omega_\tau. \tag{71}$$

The main difference between this scheme and the implicit one lies in the fact that the "elliptic" operator in the right-hand side of (70) is computed from the values of the grid function $u$ not on the next time level as in (4), but on the current time level. Therefore, obviously, the question of solvability of the scheme (70), (71) does not arise. We shall not discuss in detail the comparative advantages and disadvantages of the two schemes (this was already mentioned earlier). Such a detailed analysis can be found in [346].

Let us consider one basic question: when is the solution of the explicit scheme (70), (71) non-negative for any initial function $u_0(x) \geq 0$, $x \in \omega_h$, that is, when does the solution satisfy the weak Maximum Principle? If it does not, (70) does not necessarily make sense (it will not be possible in (70) to perform the operation of raising to an arbitrary power). It turns out that in some cases ($\beta < \sigma + 1$) the weak Maximum Principle is not satisfied at all, while for $\beta \geq \sigma + 1$ it requires very severe restrictions on $\tau_j$, $h$.

### *1 The weak Maximum Principle*

The time grid here is taken to be uniform, $\tau_j \equiv \tau$.

**Theorem 8.** *For $\beta < \sigma + 1$ the weak Maximum Principle does not hold. If $\beta = \sigma + 1$, a necessary and sufficient condition for it to hold is the inequality $h^2 \geq 2$. In the case $\beta > \sigma + 1$ such a condition is the following:*

$$\tau \leq F_0(h^2)^{(\beta-1)/[\beta-(\sigma+1)]}, \quad F_0 = \frac{\beta - 1}{2[\beta - (\sigma + 1)]} \left(\frac{\beta - 1}{2\sigma}\right)^{-\sigma/[\beta-(\sigma+1)]}. \tag{72}$$

*Proof.* We start from the fact that a necessary and sufficient condition of non-negativity of $\hat{u}$ for any $u \geq 0$ in $\omega_h$ is the requirement that the function

$$P_\lambda(\xi) = \xi + \tau\xi^\beta - 2\lambda\xi^{\sigma+1}, \quad \lambda = \tau/h^2, \tag{73}$$

be non-negative everywhere in $\mathbf{R}_+$. This follows from the form of (70) resolved with respect to $\hat{u}$: $\hat{u}_k = P_\lambda(u_k) + \lambda(u_{k-1}^{\sigma+1} + u_{k+1}^{\sigma+1})$, if we set $u_{k-1} = u_{k+1} = 0$ for some $0 < k < M$ (such a situation could obtain, for example, at the initial moment of time). An elementary analysis of the function (73) leads to the conclusions of the theorem. □

Let us note that in the dependence of $\tau$ on $h$, inequality (72) resembles condition (33′) of solvability of the implicit scheme on a time level for $\beta > \sigma+1$. However, there is a crucial difference between the two: while (33′) is needed only at a fully developed stage of blow-up ($|u|_C$ large), without (72), in general, the computation simply cannot be started. Naturally, for certain initial functions $u_{0h}$ condition (72) may not be required.

*2 Unbounded difference solutions of the explicit scheme*

Here we shall consider the appearance of finite difference finite time blow-up in the problem (70), (71) for $\beta \geq \sigma + 1$. To that end we could apply the eigenfunction method, which was used in subsection 1.4; for the explicit scheme it is much simpler.

Below we employ a different method. In the following we take the grid $\omega_\tau$ to be non-uniform, $\tau_j \to 0$ as $j \to \infty$, $T_0 = \sum_{j=0}^{\infty} \tau_j < \infty$.

Let the Maximum of an initial function $u_{0h} \not\equiv 0$, the value $|u_{0h}|_C$, be attained at some point $x \in \omega_h$. It is easily seen that at that point the value of the grid function $u^{j+1}$ is estimated in terms of $u^j$ as follows:

$$u^{j+1} \geq u^j + \tau_j \left[ (u^j)^\beta - \frac{2}{h^2} (u^j)^{\sigma+1} \right], \quad t \in \omega_\tau, \tag{74}$$

$$u^0 = |u_{0h}|_C > 0.$$

Here we are assuming that $u \geq 0$ in $\omega_\tau \times \omega_h$ (see Theorem 8).

From (74) it is not hard to derive conditions for the solution to be unbounded. Let us write (74) in the following form:

$$u^{j+1} \geq u^j + \tau_j (u^j)^\beta \left[ 1 - \frac{2}{h^2} (u^j)^{\sigma+1-\beta} \right], \quad t \in \omega_\tau, \tag{75}$$

Let the initial function $u_{0h}$ be such that

$$\mu_0 = 1 - \frac{2}{h^2} |u_{0h}|_C^{\sigma+1-\beta} > 0, \tag{76}$$

Then it immediately follows from (75) that $u^{j+1} > u^j$ for all $j = 0, 1, \ldots$, and therefore we have from (75) that

$$u^{j+1} \geq \mu_0 \tau_j (u^j)^\beta, \quad t \in \omega_\tau. \tag{77}$$

It remains to choose suitable sizes for the time steps $\tau_j$. For example, let us set

$$\tau_j = \frac{q}{\mu_0} (u^j)^{1-\beta}, \quad j = 0, 1, \ldots; \quad q = \text{const} > 1. \tag{78}$$

Then (77) assumes the form

$$u^{j+1} \geq q u^j, \quad j = 0, 1, \ldots; \quad u^0 = |u_{0h}|_C.$$

Hence we immediately have

$$u^j \geq |u_{0h}|_C \, q^j, \quad j = 0, 1, \ldots, \tag{79}$$

that is, $u^j \to \infty$ as $j \to \infty$.

Let us show that this is indeed finite difference finite time blow-up. By (78), (79) we have the following estimate for the time of existence of the difference solution:

$$
\begin{aligned}
T_0 &= \sum_{j=0}^{\infty} \tau_j = \frac{q}{\mu_0} \sum_{j=0}^{\infty} (u^j)^{1-\beta} \le \\
&\le \frac{q}{\mu_0} |u_{0h}|_C^{1-\beta} \sum_{j=0}^{\infty} q^{j(1-\beta)} = \frac{|u_{0h}|_C^{1-\beta}}{\mu_0} \frac{q^\beta}{q^{\beta-1}-1} < \infty .
\end{aligned} \tag{80}
$$

Thus, if (76) holds, $u_{0h}$ belongs to the unstable set $V_h$.

**Theorem 9.** *Let $\beta > \sigma + 1$ and assume that condition* (76) *holds. Then there exists a sequence of time steps $\{\tau_j\}$, defined by* (78), *such that the solution of the explicit scheme* (70), (71) *is unbounded and the time of existence of the solution satisfies the estimate* (80).

Let us note that to get finite time blow-up in the explicit scheme, it is necessary to choose small time steps. For example, in the framework of the above approach, we derive the following estimate for the magnitude of $\tau_0$ (the first step): if (76) holds, we have the inequality

$$
|u_{0h}|_C > (2/h^2)^{1/[\beta-(\sigma+1)]},
$$

and therefore from (78) we obtain

$$
\begin{aligned}
\tau_0 = \frac{q}{\mu_0} |u_{0h}|_C^{1-\beta} &< \frac{q}{\mu_0} 2^{(1-\beta)/[\beta-(\sigma+1)]} (h^2)^{(\beta-1)/[\beta-(\sigma+1)]} = \\
&= O[(h^2)^{(\beta-1)/[\beta-(\sigma+1)]}],\ h \ll 1 .
\end{aligned} \tag{81}
$$

Naturally, in view of (78), (79) subsequent steps will be even shorter. This estimate has the same dependence of $\tau_0$ on $h$ as the optimal inequality (33′), which ensures solvability of the implicit scheme. Thus to have blow-up in the explicit scheme, its apparent simplicity notwithstanding, we still have to compute with very small time steps. The reason for this is clear: from (70) it immediately follows that marching in time with large time steps never leads to finite time blow-up. This is true, for example, for a uniform grid $\omega_\tau$. Let us also note that a restriction of the form of (81) is needed to have the weak Maximum Principle (see Theorem 8).

## Remarks and comments on the literature

§ 1. The presentation of results of § 1 largely follows [123, 127, 131, 169, 170] ([131] contains a brief account of the results pertaining to the one-dimensional equation). Certain results of § 1 were obtained earlier in [120]. Theorem 6 for $\sigma = 0$, $p > (\beta - 1)N/2$, $p \geq 1$, and more general initial functions was proved by a different method in [379] (see also theorems 3.2 and 3.3 in [26]). Let us note that [379, 26] make significant use of the semilinearity of equation (4) for $\sigma = 0$, i.e. of the ability to invert the linear operator $\partial/\partial t - \Delta$ and reduce the problem to an equivalent integral equation. Therefore the method of [379] is not applicable in the quasilinear case when $\phi'(u) \not\equiv$ const. In the case $\sigma = 0$, $N = 1$ and arbitrary $u_0(x)$ Theorem 6 was proved in [170]; an exact form of the envelope of the family $\{U(r;\ U_0)\}$ in Theorem 5 was also established there: $u(T_0^-, x) \geq G(r) = C_0|x|^{-2/(\beta-1)}$, where $C_0 > C_*$ is a constant. Some classes of equations of the form (30), (31) were studied in [127, 150, 347] (see § 7, Ch. IV). In the most general form degenerate a.s.s. of similar quasilinear equations were considered in [160]. Many other examples of the use of the method of stationary states are contained in [164]; for other possibilities see [137, 174, 175, 180, 181, 189]. Another approach to construction of lower bounds for unbounded solutions of quasilinear equations of the form (1) has been developed by [306] (see also [223], where boundary value problems in a bounded domain are considered). In its final results, this approach is similar to the method of stationary states. For example, in [306] it is shown that for $1 < \beta < \sigma + 1$ the solution grows without bound on the whole space, while for $\beta = \sigma + 1$ a "lower bound" for the localization domain is obtained.

The main results of this section are based on intersection comparison with the given set $\mathcal{S}$ of particular stationary solutions to the quasilinear heat equation (1) with one space varable $r = |x|$. A similar comparison can be performed with respect to an arbitrary set $\mathcal{B}$ of other solutions if it is sufficiently large ("complete" in the sense of existence and uniqueness of tangent solutions in spatial variable). In this case we arrive at the notion of generalized B-convexity/concavity properties of the solutions with respect to the given functional set $\mathcal{B}$. Under certain assumptions these properties are proved to be to he preserved in time or to appear eventually in time, see general results in [194]. Observe that the "criterion" of complete blow-up for a general one-dimensional quasilinear heat equation with source [193] is a straightforward consequence fo such intersection comparison with the set of travelling wave solutions depending on the variable $\xi = x - \lambda t$, and looks like the property of eventual B-convexity.

**§ 2.** Theorems of subsection 1 are proved in [121, 125]. A similar problem was considered, using a different approach, prior to that in [372]; the results obtained there are not quite optimal. The main claims of subsection 2.1 are contained in [120, 125] (results of similar generality were established in [294]). The greater part of conclusions of subsection 2.2 can be found in [125]. A generalization of the concavity method to study unbounded solutions of parabolic equations and systems of equations with a given type of nonlinearity was undertaken in [124]. Later conditions for appearance of unbounded solutions of quasilinear parabolic equations were established in [307, 294]. [307] employs the method of eigenfunctions, which is not unlike that of [120, 125] (see Theorem 8 of § 2); [294] uses the same approach as [124]. A similar analysis of the boundary value problem for the quasilinear equation $u_t = \psi(u)u_{xx} + \phi(u)$ in the one-dimensional case appeared earlier in [225]; in particular, a version of the method of eigenfunctions was used there. A brief survey of the literature on unbounded solutions can be found in [157] and [290].

**§ 3.** The majority of results of § 3 is contained in [161] (see also [157]). Other examples of the use of the method of stationary states to derive lower bounds for unbounded solutions can be found in [159, 164, 174]. The problem of computing upper bounds and thus proof of localization for systems of equations is almost completely open. The single result of [105], obtained by the method of [108], deals with the semilinear system (1), (2) for $\mu = \nu = 0$ and $p = q$. Let us note the paper [93] (see also [290]), which considers the Cauchy problem for a semilinear system and shows that for $(\gamma + 1)/(pq - 1) \geq N/2$, $\gamma = \max\{p, q\}$, every non-trivial non-negative solution blows up in finite time, thus determining the critical (in the sense of Fujita [112]) exponent of the source.

**§ 4.** Numerical and qualitative results of subsection 1 are taken from [273, 279]. The analysis of subsection 2 comes from [142].

**§ 5.** All the main assertions of § 5 are proved in [182, 183]. Studies of unbounded solutions of explicit finite difference schemes for the semilinear ($\sigma = 0$) equation using different methods were conducted in [311, 312]. Interesting results concerning localization in the context of an explicit-implicit scheme for the equation with $\sigma = 0$, when the source term $u^\beta$ is taken not from the next, but from the current time level, are obtained in [64]. In particular, it is shown there that for $\beta = 2$ the difference solution becomes infinite at three central points, while for $\beta > 2$ it happens at a single point.

The results of § 3 and the proposition of § 4 are based on the derivations of [161, 159]. Let us note that the practically optimal result concerning global solvability of the boundary value problem for $pq < (1 + \mu)(1 + \nu)$ (Theorem 5 of § 3), established by using the method of stationary states, is hard to obtain by using the usual techniques of a priori estimates, for example, those employed in Galerkin's method. This is indicated by the analysis of [157], as well as, for example, the results of [299], where restrictions of the form $p < 1 + \mu$, $q < 1 + \nu$ are obtained. These restrictions are the natural ones for a single equation (see § 2). They characterize the easily explainable relation between intensities of processes of heat diffusion and combustion, which is necessary for the occurrence of thermal perturbations of finite amplitude. As shown in § 3, these conditions are far from being optimal for systems of equations.

# Open problems

1. (§ 1) Describe the whole class of coefficients $(k, Q)$ for which the absence of localization condition (16) (for $N = 1$) is not only sufficient, but also necessary (for the case $k(u) = u^\sigma$, $Q(u) = u^\beta$, $\beta \geq \sigma + 1$, $\sigma > 0$, this has been done in [129]).

2. (§ 1) Derive an upper bound for $u(T_0, x)$ of the form (29′) in Theorem 5 for arbitrary initial functions $u_0(x)$ (for a particular class of $u_0$ this has been done in [131, 172, 173]).

3. (§ 1) Prove localization of unbounded solutions of equation (1) with general coefficients (30) for $\alpha \geq 2$ (in the case $k(u) \equiv 1$, $Q(u) = (1+u)\ln^\beta(1+u)$, $\beta \geq 2$, this has been done in [189] ($\beta = 2$) and [177] ($\beta > 2$); see also [192] and § 7, Ch. IV).

4. (§ 2) Determine conditions for which the behaviour of unbounded solutions of the boundary value problem (5), (6) as $t \to T_0(u_0) < \infty$, $\beta \in (\sigma + 1, (\sigma + 1)(N+2)/(N-2)_+)$ is described by the self-similar solutions constructed in § 1, Ch. IV. Analyze the asymptotic behaviour of unbounded solutions of the problem for $\beta \geq (\sigma+1)(N+2)/(N-2)_+$ (let us note that in this case there is an unusual class of global solutions; see [314]).

5. (§ 3) Is it possible to construct a family of explicit solutions of equation (39) in $\mathbf{R}^N$ for the critical value of the parameter $\beta = [N/\alpha + 2(1 + 1/\alpha)]/[N - 2(1 + 1/\alpha)]$, $\alpha \neq 1$, similar to the one given in the example in subsection 2.4 for the semi-linear case $\alpha = 1$, $\beta = (N+4)/(N-4)$?

6. (§ 3) Determine conditions for localization of compactly supported unbounded solutions of the Cauchy problem for the system (1), (2). Is the condition $m = pq - (1+\mu)(1+\nu) \geq 0$ sufficient for that?

7. (§ 3) What a.s.s. describes asymptotic behaviour of unbounded solutions of the Cauchy problem for the system (1), (2) in the cases when it does not have self-similar solutions?

8. (§ 3) Determine conditions for solvability of the elliptic system (65). What is the structure of the set of its solutions for various values of parameters (some numerical results are contained in [273, 279]).

9. (§ 4) Find conditions for localization of unbounded solutions of the problem (15), (16).

10. (§ 5) Are the finite difference solutions of the explicit scheme (70), (71) localized for $\beta \geq \sigma + 1$ in the case of an initial function $u_{0h} \geq 0$ with "compact support" (that is, can it happen that $\{x \in \omega_h \mid u(T_0, x) = \infty\} \neq \omega_h$)? The implicit scheme does not have this asymptotic property.

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