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STABILITY AND REGULARIZATION OF THREE-LEVEL DIFFERENCE SCHEMES WITH UNBOUNDED OPERATOR COEFFICIENTS IN BANACH SPACES*

ALEXANDER A. SAMARSKII†, IVAN P. GAVRILYUK‡, AND VLADIMIR L. MAKAROV§

Abstract. The problem of stability of difference schemes for second-order evolution problems is considered. Difference schemes are treated as abstract Cauchy problems for difference equations with operator coefficients in a Banach or Hilbert space. To construct stable difference schemes the regularization principle is employed, i.e., one starts from any simple scheme (possibly unstable) and derives absolutely stable schemes by perturbing the operator coefficients. The main result of this paper is the following: for the first time sufficient conditions are pointed out under which an unstable three-level difference scheme with unbounded operator coefficients in a Banach space can be regularized to a stable scheme. The principal stability condition is the strong P-positivity of the unbounded operator coefficients.

Key words. evolution equations, three-level difference schemes, unbounded operator coefficients, strongly P-positive operators, ρ-stability

AMS subject classifications. 65M06, 65M12, 65M20

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1. Introduction.

Second-order differential equations with operator coefficients are a powerful mathematical tool in the description and study of evolutionary partial differential equations arising in various fields of applications. In the numerical solution of evolution problems, the problem of stability of numerical methods with respect to initial data is of great importance. Considering these methods as difference schemes with operator coefficients provides a suitable model for stability analysis.

In this paper we consider difference schemes for the following initial value problem

$$\frac{d^2u}{dt^2} + Au = 0, \qquad t \in (0,T], \qquad u(0) = u_0,\ u'(0) = u_1, \tag{1}$$

where $u : R_+ \to X$ is a vector-valued function, A is a linear, densely defined, closed operator with domain $D(A)$ in a Banach space X with norm $\|\cdot\| \equiv \|\cdot\|_X$. In particular, (1) with the Laplace operator $A = -\Delta$ is the well-known wave equation.

Due to the presence of the second-order time derivative in (1), difference schemes for the numerical solution of this problem have at least three time levels, i.e., they involve approximate values y_n for $u(t_n)$ at three neighboring points of the time grid $\omega_\tau = \{t_i : i = 0, 1, 2, \dots, t_0 = 0, t_i - t_{i-1} = \tau\}$.

There are several approaches to the study of stability of difference schemes [6, 8, 9, 10, 11, 12, 14, 15, 16]. As a rule, they are based on some assumptions about the structure of difference operators and the stability analysis is performed using the

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†Institute of Mathematical Modeling, Russian Academy of Sciences, 4 Miusskaya Square, Moscow 125047, Russia (cmam@imamod.ru).

‡Berufsakademie Thueringen, Staatliche Studienakademie, Am Wartenberg 2, 99817 Eisenach, Germany (ipg@ba-eisenach.de).

§Institute of Mathematics, National Academy of Sciences, 3 Tereschenkivska Str, 01601 Kyiv, Ukraine (makarov@imath.kiev.ua).

Fourier method or certain energy inequalities. The most general and constructive theory of stability has been developed in [6, 9, 10, 11, 12]. In this theory a difference scheme is presented in a canonical form with operator coefficients, and stability conditions (in many cases necessary and sufficient conditions) are formulated via operator inequalities. This theory essentially uses the techniques and tools of Hilbert spaces, and stability estimates are given in Hilbert norms. The stability theory, together with the regularization principle [10], provide a powerful tool to obtain stable difference schemes. The main idea of regularization is to start from any simple scheme (even unstable) and by perturbing its coefficients (while taking into consideration the stability conditions) obtain a stable difference scheme or a scheme with other desired properties. All the main classes of difference schemes for the problems of mathematical physics have been designed and analyzed on the basis of this approach in [6, 9, 10, 11, 12].

Unfortunately, known stability theories do not include certain important classes of difference schemes. For instance, there are no results concerning the stability of three-level difference schemes with unbounded operator coefficients in a Banach space. Such results are also important for finite difference and finite element approximations of unbounded differential operators since the norms of these approximations tend to infinity if the discretization parameter tends to zero.

The aim of this paper is to obtain stability results for regularized three-level difference schemes with unbounded operator coefficients in a Banach space. For example, this class of schemes arises when approximating second-order evolution differential equations. Note that the initial difference scheme (without regularization) can be unstable.

We consider the following family of three-level difference schemes:

$$(I + \alpha A) y_{\bar{t}t,n} + \beta A y_{\overset{\circ}{t},n} + A y_n = 0, \qquad n = 1, 2, \dots, \tag{2}$$

with given y_0, y_1, where

$$y_{\overset{\circ}{t},n} = \frac{y_{n+1} - y_n}{2\tau}, \qquad y_{\bar{t}t,n} = \frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2},$$

α, β are parameters, and A is a linear, densely defined, closed operator in a Banach space X. If $\alpha = \beta = 0$, then we have the explicit difference scheme for (1) which is unstable in the case of an unbounded operator A.

The difference scheme (2) can be written down as

$$\begin{aligned} &\left[I + \left(\alpha + \frac{\beta\tau}{2}\right) A\right] y_{n+1} - 2\left[I + \left(\alpha - \frac{\tau^2}{2}\right) A\right] y_n \\ &\quad + \left[I + \left(\alpha - \frac{\beta\tau}{2}\right) A\right] y_{n-1} = 0, \qquad n = 1, 2, \dots . \end{aligned} \tag{3}$$

In order to get an explicit formula for the solution of (3) let us consider the scalar recurrence equation

$$a u_{n+1} - 2b u_n + c u_{n-1} = 0$$

with constant coefficients a, b, c. Setting $u_n = r^n \tilde{u}_n$ we get

$$\tilde{u}_{n+1} - 2\frac{b}{ar}\tilde{u}_n + \frac{c}{ar^2}\tilde{u}_{n-1} = 0$$

and with $r = \sqrt{ca^{-1}}, x = b\sqrt{ca^{-1}}$ we have

$$\tilde{u}_{n+1} - 2x\tilde{u}_n + \tilde{u}_{n-1} = 0.$$

This is the recurrence equation that is satisfied both by the Chebyshev polynomials $T_n(x)$ of the first kind and by those of the second kind, $U_n(x)$ (see [13]). Since U_{n-1} and U_{n-2} are linear independent and by definition $U_{-2}(x) = -1$, $U_{-1}(x) = 0$, we can write down u_n, $n = 0, 1, \ldots$, with initial values u_0, u_1 as follows:

$$u_n = (\sqrt{ca^{-1}})^n[-U_{n-2}(x)u_0 + (\sqrt{ca^{-1}})^{-1}U_{n-1}(x)u_1].$$

Thus, denoting

$$\chi(\alpha,\beta,A) \equiv \chi(A) = \left[I + \left(\alpha - \frac{\tau^2}{2}\right)A\right]\left\{\left[I + \left(\alpha + \frac{\beta\tau}{2}\right)A\right]\left[I + \left(\alpha - \frac{\beta\tau}{2}\right)A\right]\right\}^{-1/2},$$

$$Q(\alpha,\beta,A) \equiv Q(A) = \left\{\left[I + \left(\alpha + \frac{\beta\tau}{2}\right)A\right]^{-1}\left[I + \left(\alpha - \frac{\beta\tau}{2}\right)A\right]\right\}^{1/2},$$

the solution of (3) can be obviously represented by

$$y_n = Q^n(A)[-U_{n-2}(\chi)y_0 + Q^{-1}(A)U_{n-1}(\chi)y_1]. \tag{4}$$

Next, we introduce two definitions which we will use in our analysis.

DEFINITION 1.1. *Given a function $\rho = \rho(\tau)$ and a real $\sigma \geq 0$ we say that the scheme* (2) *is ρ-stable with respect to initial data in the domain $D(A^\sigma)$ of the operator A^σ if there exists a constant M independent of n such that the estimate*

$$\|y_n\| \leq M\rho^n\left(\left\|\frac{y_0 + y_1}{2}\right\|_\sigma + \left\|\frac{y_1 - y_0}{\tau}\right\|_\sigma\right) \tag{5}$$

holds for any $y_0, y_1 \in D(A^\sigma)$ with $\|u\|_\sigma = \|A^\sigma u\|$.

Here and below we denote by $M, M_1, \ldots, C, C_1, \ldots, c, c_1, \ldots,$ various positive constants.

The strongly P-positive operators that were introduced in [4] will play a principal role in our stability analysis. Let Γ be a counterclockwise oriented path consisting of two arcs, Γ_+ and Γ_-, of a parabola $y^2 = \frac{c_0^2}{2}x$, $c_0 > 0$ connected by a segment Γ_* of the line $x = \gamma > 0$ (see Figure 1). We denote by Ω_Γ the domain lying inside of Γ. Now we are in the position to give the definition of the strong P-positivity.

DEFINITION 1.2 (see [4]). *We say that an operator $A : D(A) \subset X \to X$ is strongly P-positive if its spectrum $\Sigma(A)$ lies in the domain Ω_Γ and on Γ, and outside of Γ the estimate*

$$\|(z - A)^{-1}\|_{X\to X} \leq \frac{M}{1 + \sqrt{|z|}} \tag{6}$$

holds with a positive constant M.

Remark 1. The form of the path $\Gamma \equiv \Gamma(z)$ for a bounded z is not essential for our analysis. What is important is the behavior of the resolvent and of $\Sigma(A)$ at infinity, i.e., that Γ is a parabola and the estimate (6) holds for $|z| \to \infty$.

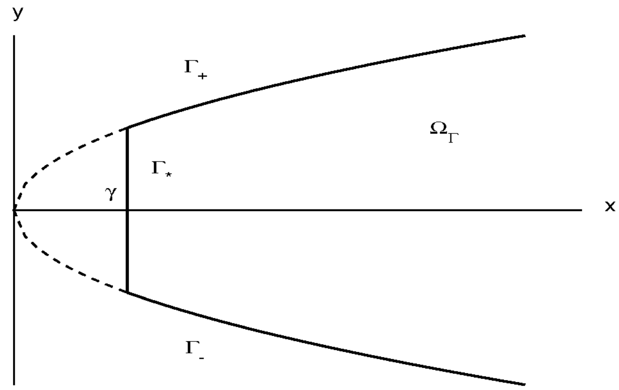

FIG. 1. *The path* Γ.

Example 1. Let us consider the one-dimensional operator $A : L_1(0,1) \to L_1(0,1)$ with the domain $D(A) = \{u | u \in H_0^2(0,1)\}$ in the Sobolev space $H_0^2(0,1)$ defined by

$$Au = -u'' \quad \forall\, u \in D(A).$$

The eigenvalues $\lambda_k = k^2\pi^2, k = 1, 2, \ldots,$ of A lie on the real axis inside the path

$$\Gamma = \begin{cases} z = \eta^2 \pm i\eta, & \eta \geq 1, \\ z = 1 \pm i\eta^2, & |\eta| \leq 1. \end{cases}$$

The Green function for the problem

$$(zI - Au) \equiv u''(x) + zu(x) = -f(x),\ x \in (0,1); u(0) = u(1) = 0$$

is

$$G(x,\xi) = \frac{1}{\sqrt{z}\sin\sqrt{z}} \begin{cases} \sin\sqrt{z}x \sin\sqrt{z}(1-\xi), & x \leq \xi, \\ \sin\sqrt{z}\xi \sin\sqrt{z}(1-x), & x \geq \xi, \end{cases}$$

i.e., we have

$$u(x) = (zI - A)^{-1} f = \int_0^1 G(x,\xi) f(\xi) d\xi.$$

In order to show that the estimate (6) holds true, it is sufficient to estimate the Green function on the parabola $z = \eta^2 \pm i\eta = \sqrt{\eta^4 + \eta^2}(\cos\phi \pm i \sin\phi)$, where

$$\cos\phi = \frac{\eta}{\sqrt{\eta^2+1}}, \quad \sin\phi = \frac{1}{\sqrt{\eta^2+1}}.$$

Actually, we have $\sqrt{z} = \sqrt[4]{\eta^4+\eta^2}(\cos\frac{\phi}{2} \pm i \sin\frac{\phi}{2}) = a \pm ib$ with

$$\cos\frac{\phi}{2} = \frac{\sqrt{\eta^2 + \sqrt{\eta^4+\eta^2}}}{\sqrt{2}\sqrt[4]{\eta^4+\eta^2}}, \quad \sin\frac{\phi}{2} = \frac{\sqrt{\sqrt{\eta^4+\eta^2} - \eta^2}}{\sqrt{2}\sqrt[4]{\eta^4+\eta^2}},$$

$$a = \frac{\sqrt{\eta^2 + \sqrt{\eta^4+\eta^2}}}{\sqrt{2}}, \quad b = \frac{\sqrt{\sqrt{\eta^4+\eta^2} - \eta^2}}{\sqrt{2}}.$$

The following estimates hold for $x \le \xi$ and for η large enough:

$$\left|\frac{\sin\sqrt{z}x\sin\sqrt{z}(1-\xi)}{\sqrt{z}\sin\sqrt{z}}\right| = \frac{[\sin^2 ax+\sinh^2 bx]^{\frac{1}{2}}[\sin^2 a(1-\xi)+\sinh^2 b(1-\xi)]^{\frac{1}{2}}}{\sqrt[4]{\eta^4+\eta^2}[\sin^2 a+\sinh^2 b]^{\frac{1}{2}}}$$
$$\le \frac{c}{\eta}$$

with an absolute constant c.
The case $\xi \le x$ can be considered analogously. The last estimate implies that $\|(zI - A)^{-1}f\|_{L_1} \le \frac{M}{1+\sqrt{|z|}}\|f\|_{L_1}$, i.e., the operator A is strongly P-positive in $X \equiv L_1(0,1)$. The same estimates for the Green function imply the strong P-positivity of A also in $L_\infty(0,1)$.

Example 2. This example shows that there exist important classes of the partial differential operators which are strongly P-positive.

Let $V \subset X \subset V^*$ be a triple of Hilbert spaces and let $a(.,.)$ be a sesquilinear form on V. Assume that

$$|a(u,v)| \le C\|u\|_V\|v\|_V, \qquad |\Im a(u,u)| \le c\|u\|_V\|u\|_X, \;\; u,v \in V, \tag{7}$$

and there exist constants $\delta_0 > 0$ and $\delta_1 \ge 0$ such that

$$\Re a(u,u) \ge \delta_0\|u\|_V^2 - \delta_1\|u\|_X^2 \qquad \forall u \in V, \tag{8}$$

where $\|\cdot\|_V$, $\|\cdot\|_X$ denote the norms in V and X, respectively. The boundedness of $a(.,.)$ implies that one can define a bounded operator $A : V \to V^*$ through the identity

$$a(u,v) = {}_{V^*}\!< Au, v >_V, \qquad u,v \in V,$$

where ${}_{V^*}< \cdot >_V$ denotes the duality relation between V and its adjoint space V^* of linear functionals on V. As an example one can consider the following sesquilinear form [1]:

$$a(u,v) = \int_\Omega \left(\sum_{p,q=1}^d a_{pq}(x)\frac{\partial u}{\partial x}\frac{\partial \overline{v}}{\partial x} + \sum_{p=1}^d a_p(x)\frac{\partial u}{\partial x}\overline{v} + a(x)u\overline{v}\right) dx$$

with sufficiently smooth real coefficients $a_{pq}(x)$ defined in a bounded Lipschitz domain $\Omega \subset R^d$, which corresponds to the elliptic partial differential operator A defined by

$$D(A) = \left\{ u \;|u \in H^2(\Omega) \cap \overset{\circ}{H}{}^1(\Omega)\right\},$$

$$Au = \sum_{p,q=1}^d \frac{\partial}{\partial x_p}\left(a_{pq}(x)\frac{\partial u}{\partial x_q}\right) + \sum_{p=1}^d a_p(x)\frac{\partial u}{\partial x_p} + a(x)u.$$

This operator, considered from $L_2(\Omega)$ into $L_2(\Omega)$, is unbounded. Supposing that

$$\sum_{p,q=1}^d a_{pq}\eta_p\eta_q$$

is positive definite and $a_p(x)$ are bounded, one can show that the inequalities (7), (8) hold with $V = H^1(\Omega)$, $X = L^2(\Omega)$ (see [1]). For strongly elliptic operators (8) is the well-known Gårding's inequality.

The assumptions (7), (8) guarantee that the numerical range $\{a(u,u)|\ \|u\|_X = 1\}$ of A (and the spectrum) lies inside of a parabola determined by the constants δ_0, δ_1, c. Actually, if $a(u,u) = \xi + i\eta$, then we get

$$\begin{aligned} \xi &= \Re a(u,u) \geq \delta_0 N_V - \delta_1 \geq \delta_0 c_e^{-2} - \delta_1, \\ |\eta| &= |\Im a(u,u)| \leq c\sqrt{N_V}, \end{aligned} \tag{9}$$

where $N_V = \|u\|_V^2 \geq c_e^{-2}\|u\|_X^2$ and c_e is the imbedding constant. It implies

$$\begin{aligned} &\xi > \delta_0 c_e^{-2} - \delta_1 \equiv \delta_2,\ N_V \leq \frac{1}{\delta_0}(\xi + \delta_1), \\ &|\eta| \leq c\sqrt{\frac{\xi + \delta_1}{\delta_0}}, \end{aligned} \tag{10}$$

i.e., the numerical range (and the spectrum) lies inside of the parabola $\xi = \frac{\delta_0}{c^2}\eta^2 - \delta_1$. It is easy to see that the assumption $\Re\,\Sigma(A) > \gamma_1 > \gamma_0 > 0$ provides the existence of other parabola $\Gamma = \{z = (\xi, \eta) : \xi = a\eta^2 + b\}$ with the parameters $a = \frac{(\gamma_1 - \gamma_0)\delta_0}{(\gamma_1 + \delta_1)c^2} > 0, b \in (0, \gamma_0)$ which envelop the numerical range of A (compare with [1]). The proof of the estimate (6) is completely analogous to that in [4]. Note that the condition $\delta_2 > 0$ is sufficient for $\Re\,\Sigma(A) > 0$.

The strongly elliptic partial differential operators with $\Re\,\Sigma(A) > 0$ are important examples of both strongly P-positive and strongly positive operators (also sectorial operators or infinitesimal generators of holomorphic semigroups). The framework of the strong P-positivity is important for studying cosine families of operators related to (1) (see, e.g., [5, 3]). It was shown in [4] that the strong positiveness of the operator A provides some algorithmic representations of a cosine family generated by A as well as the existence, stability, and approximation results for (1) in the case when the initial data belong to the domain of some fractional power of the operator A. Contrary to the known necessary and sufficient conditions under which an operator A generates a cosine family [5, 7] our condition is easier to prove.

In the next section we will show that the strong P-positivity of the unbounded operator A is one of the sufficient conditions for the ρ-stability of the regularized scheme (2), whereas the explicit scheme (2) with $\alpha = \beta = 0$ is unstable.

2. Stability of three-level difference schemes with strongly P-positive operator coefficients. For the sake of simplicity we set in (2)

$$\beta = \tau, \qquad \alpha = \frac{\tau^2}{2}.$$

In this case the scheme (2) takes the form

$$\left(I + \frac{\tau^2}{2}A\right) y_{\bar{t}t} + \tau A y_{\overset{\circ}{t}} + Ay = 0 \tag{11}$$

or

$$[I + \tau^2 A]y_{n+1} - 2y_n + y_{n-1} = 0, \qquad n = 1, 2, \ldots, \tag{12}$$

and the operators χ, Q are given by $\chi(A) = Q(A) = [I + \tau^2 A]^{-1/2}$.

The next theorem represents the first main result of this paper.

THEOREM 2.1. *Let A be a strongly P-positive operator with the domain $D(A)$ having a spectrum $\Sigma(A)$ placed inside of a parabola $y^2 = \frac{c_0^2}{2}x$, $c_0 \equiv \text{const} > 0$, $\Re\Sigma(A) > \gamma$, $\tau < \sqrt{2}c_0^{-1}$. Then the difference scheme (2) with $\beta = \tau$, $\alpha = \frac{\tau^2}{2}$, is ρ-stable with respect to initial data in $D(A^\sigma)$ with $\rho = (1 - \frac{c_0\tau}{\sqrt{2}})^{-1/2}$.*

Proof. Using the Dunford–Cauchy integral we represent the solution of (12) as follows:

$$y_n = \frac{1}{2\pi i}\int_\Gamma -Q^n(z)U_{n-2}(\chi(z))(z-A)^{-1}dzy_0 \tag{13}$$

$$+\frac{1}{2\pi i}\int_\Gamma Q^{n-1}(z)U_{n-1}(\chi(z))(z-A)^{-1}dzy_1$$

or, in view of the elementary relations

$$y_0 = \frac{y_0+y_1}{2} + \frac{y_0-y_1}{2}, \qquad y_1 = \frac{y_0+y_1}{2} - \frac{y_0-y_1}{2},$$

we get

$$y_n = \frac{1}{4\pi i}\int_\Gamma f_n^{(+)}(z)(z-A)^{-1}dz(y_0+y_1) \tag{14}$$

$$-\frac{1}{4\pi i}\int_\Gamma f_n^{(-)}(z)(z-A)^{-1}dz(y_0-y_1),$$

where

$$f_n^{(+)}(z) = Q^{n-1}(z)[U_{n-1}(\chi) - Q(z)U_{n-2}(\chi)], \tag{15}$$

$$f_n^{(-)}(z) = Q^{n-1}(z)[U_{n-1}(\chi) + Q(z)U_{n-2}(\chi)].$$

Taking into account the form of the path Γ we can transform the integrals in (14) as follows (we use the notations $z = x + iy = x + ic_0\sqrt{\frac{x}{2}}$, $\bar{z} = x - ic_0\sqrt{\frac{x}{2}}$, $dz = (1 + ic_0/(2\sqrt{2x}))dx$, $d\bar{z} = (1 - ic_0/(2\sqrt{2x}))dx$):

$$I_n^{(\pm)} \equiv \frac{1}{4\pi i}\int_\Gamma f_n^{(\pm)}(z)(z-A)^{-1}dz$$

$$= \frac{1}{4\pi i}\int_\gamma^\infty [f_n^{(\pm)}(\bar{z})(\bar{z}-A)^{-1} - f_n^{(\pm)}(t)(z-A)^{-1}]dx$$

$$-\frac{c_0}{8\sqrt{2}\pi}\int_\gamma^\infty [f_n^{(\pm)}(\bar{z})(\bar{z}-A)^{-1} - f_n^{(\pm)}(z-A)^{-1}]\frac{dx}{\sqrt{x}}$$

$$-\frac{1}{4\pi}\int_0^{c_0\sqrt{\frac{\gamma}{2}}} f_n^{(\pm)}(\gamma+iy)(\gamma+iy-A)^{-1}dy - \frac{1}{4\pi}\int_0^{c_0\sqrt{\frac{\gamma}{2}}} f_n^{(\pm)}(\gamma-iy)(\gamma-iy-A)^{-1}dy$$

$$= -\frac{1}{2\pi}\int_\gamma^\infty \mathrm{Im} f_n^{(\pm)}(z)(\bar{z}-A)^{-1}dx + \frac{c_0}{2\pi\sqrt{2}}\int_\gamma^\infty f_n^{(\pm)}(z)\sqrt{x}(\bar{z}-A)^{-1}(z-A)^{-1}dx$$

$$+\frac{c_0 i}{4\sqrt{2}\pi}\int_\gamma^\infty \mathrm{Im} f_n^{(\pm)}(z)(\bar z - A)^{-1}\frac{dx}{\sqrt{x}} - \frac{c_0^2 i}{8\pi}\int_\gamma^\infty f_n^{(\pm)}(z-A)^{-1}(\bar z - A)^{-1}dx$$

$$+\frac{i}{2\pi}\int_0^{c_0\sqrt{\frac{\gamma}{2}}} f_n^{(\pm)}(\gamma+iy)y(\gamma+iy-A)^{-1}(\gamma-iy-A)^{-1}dy$$

$$-\frac{1}{2\pi}\int_0^{c_0\sqrt{\frac{\gamma}{2}}}\mathrm{Re} f_n^{(\pm)}(\gamma+iy)(\gamma-iy-A)^{-1}dy. \tag{16}$$

In what follows we will use the relations

$$\begin{aligned}
&U_{n-1}(\chi) - Q(\chi)U_{n-2}(\chi)\\
&= U_{n-1}(\chi) - \chi U_{n-2}(\chi) = T_{n-1}(\chi)\\
&= \frac{1}{2}[(\chi+\sqrt{\chi^2-1})^{n-1} + (\chi - \sqrt{\chi^2-1})^{n-1}],\\
&U_{n-1}(\chi) + Q(\chi)U_{n-2}(\chi)\\
&= U_{n-1}(\chi) + \chi U_{n-2}(\chi) = \frac{1}{\sqrt{\chi^2-1}}\\
&\times [(2\chi+\sqrt{\chi^2-1})(\chi+\sqrt{\chi^2-1})^{n-1} - (2\chi-\sqrt{\chi^2-1})(\chi-\sqrt{\chi^2-1})^{n-1}],
\end{aligned}$$

$$|U_{n-1}(\chi) - \chi U_{n-2}(\chi)| \le [\Phi(\chi)]^{n-1}, \tag{17}$$

$$|U_{n-1}(\chi) + \chi U_{n-2}(\chi)| \le [\Phi(\chi)]^{n-1}\frac{|2\chi+\sqrt{\chi^2-1}| + |2\chi - \sqrt{\chi^2-1}|}{|\sqrt{\chi^2-1}|}$$

with

$$\Phi(\chi) = \max\{|\chi+\sqrt{\chi^2-1}|, |\chi-\sqrt{\chi^2-1}|\}, \tag{18}$$

where $T_n(x)$ are Chebyshev polynomials of the first kind [13].

It follows from (14)–(18) that

$$\|y_n\| \le C\left\{\left[\int_\gamma^\infty \frac{\Phi_n(z_\Gamma)}{|z_\Gamma|^\sigma}dx + \int_0^{c_0\sqrt{\frac{\gamma}{2}}}\Phi_n(z_\gamma)dy\right]\left\|A^\sigma\frac{y_0+y_1}{2}\right\|\right.$$

$$\left. + \left[\int_\gamma^\infty \frac{\Phi_n(z_\Gamma)\Phi_{(1)}(z_\Gamma)}{|z_\Gamma|^\sigma}dx + \int_0^{c_0\sqrt{\frac{\gamma}{2}}}\Phi_n(z_\gamma)\Phi_{(1)}(z_\Gamma)dy\right]\left\|A^\sigma\frac{y_1-y_0}{2}\right\|\right\}, \tag{19}$$

where

$$z_\Gamma = x + ic_0\sqrt{\frac{x}{2}}, \qquad z_\gamma = \gamma + iy,$$

$$\Phi_n(z) = [|\chi(z)|\Phi(\chi(z))]^{n-1}, \tag{20}$$

$$\Phi_{(1)}(z) = \frac{\max\{|2\chi(z) + \sqrt{\chi^2(z)-1}|, |2\chi(z) - \sqrt{\chi^2(z)-1}|\}}{|\sqrt{\chi^2(z)-1}|}.$$

First of all we have to estimate

$$|\chi(z)|\Phi(\chi(z)) = \max\left\{|1+i\tau\sqrt{z}|^{-1}, |1-i\tau\sqrt{z}|^{-1}\right\}. \tag{21}$$

Let $z = x+iy = \rho e^{i\theta}$, $x>0, y\in(-\infty,\infty)$, $\rho=\sqrt{x^2+y^2}$, $\cos\theta = \frac{x}{\rho}$, $\sin\theta=\frac{y}{\rho}$; then

$$|q_\pm(z)|^{-2} \equiv |1\pm i\tau\sqrt{z}|^2 = |1\pm i\tau(x^2+y^2)^{1/4}e^{i\theta/2}|^2$$

$$= [1\mp\tau(x^2+y^2)^{1/4}\sin\frac{\theta}{2}]^2 + \tau^2(x^2+y^2)^{1/2}\cos^2\frac{\theta}{2}$$

$$= [1\mp 2\tau(x^2+y^2)^{1/4}\sin\frac{\theta}{2}] + \tau^2(x^2+y^2)^{1/2}.$$

Since the angle θ lies in the first or fourth quadrant, we have

$$\sin\frac{\theta}{2} = \frac{y}{\sqrt{\sqrt{x^2+y^2}+x}}\frac{1}{\sqrt{2}(x^2+y^2)^{1/2}}$$

and

$$|q_\pm(z)|^{-2} = 1\pm\sqrt{2}\tau\frac{y}{\sqrt{\sqrt{x^2+y^2}+x}} + \tau^2(x^2+y^2)^{1/2}.$$

We see that for $z=z_\Gamma$

$$|q_\pm(z)|^{-2} = 1\pm\sqrt{2}\tau\frac{c_0\sqrt{x/2}}{\sqrt{x+\sqrt{x^2+c_0^2x/2}}} + \tau^2(x^2+c_0^2x/2)^{1/2} \ge 1-\tau\frac{c_0}{\sqrt{2}} \tag{22}$$

and for $z=z_\Gamma$

$$|q_\pm(z)|^{-2} = 1\pm\sqrt{2}\tau\frac{y}{\sqrt{\gamma+\sqrt{\gamma^2+y^2}}} + \tau^2(\gamma^2+y^2)^{1/2} \ge 1-\sqrt{2}\tau\frac{c_0\sqrt{\gamma/2}}{\sqrt{\gamma+\sqrt{\gamma^2+c_0^2\gamma/2}}}$$

$$\ge 1-\tau\frac{c_0}{\sqrt{2}}, \tag{23}$$

$$0\le y\le c_0\sqrt{\gamma/2}, \qquad \tau\le\sqrt{2}/c_0.$$

Next, we consider $\Phi_{(1)}(z)$ for $z=z_\Gamma$ and $z=z_\gamma$. It is easy to see that

$$\Phi_{(1)}(z) = \frac{1}{\tau}\max\left\{\left|\frac{2}{\sqrt{z}}+i\tau\right|, \left|\frac{2}{\sqrt{z}}-i\tau\right|\right\}, \tag{24}$$

$$\Phi_{(1)}(z_\Gamma) \le \frac{1}{\tau}\left[\frac{2}{\sqrt{x^2+c_0^2x/2}}+\tau\right] \le \frac{1}{\tau}\left[\frac{2}{\gamma}+\tau\right], \tag{25}$$

$$\Phi_{(1)}(z_\gamma) \le \frac{1}{\tau}\left[\frac{2}{\sqrt{\gamma^2+y^2}}+\tau\right] \le \frac{1}{\tau}\left[\frac{2}{\gamma}+\tau\right]. \tag{26}$$

Now we are in the position to estimate $\|y_n\|$. Taking into account (19)–(25) we get

$$\|y_n\| \le c\left(1-\frac{c_0\tau}{\sqrt{2}}\right)^{-n/2}\left(\left\|\frac{y_0+y_1}{2}\right\|_\sigma+\left\|\frac{y_1-y_0}{\tau}\right\|_\sigma\right), \tag{27}$$

where $\sigma > 0$, $\|u\|_\sigma = \|A^\sigma u\|$. The proof is complete. ∎

Remark 2. From the stability estimate one gets

$$\|y_n\| \le Ce^{\frac{c_0\tau n}{2\sqrt{2}}}\left(\|\frac{y_0+y_1}{2}\|_\sigma+\|\frac{y_1-y_0}{\tau}\|_\sigma\right) \equiv Ce^{\frac{c_0T}{2\sqrt{2}}}\left(\|\frac{y_0+y_1}{2}\|_\sigma+\|\frac{y_1-y_0}{\tau}\|_\sigma\right),$$

i.e., the stability constant increases exponentially with respect to the length of the time interval $T = n\tau$.

Remark 3. The explicit scheme (2) with $\alpha = \beta = 0$, i.e.,

$$y_{\bar{t}t} + Ay = 0$$

or in the index form

$$y_{n+1} - 2\left[I - \frac{\tau^2}{2}A\right]y_n + y_{n-1} = 0,$$

is unstable if A is an unbounded operator in a Banach space E.

Actually, choosing $y_0 = 0$ we get

$$y_2 = 2\left[I - \frac{\tau^2}{2}A\right]y_1.$$

Since A is unbounded there exists $y_1^{(k)}$ with $\|y_1^{(k)}\| = 1$ such that $\|(I - \frac{\tau^2}{2}A)y_1^{(k)}\| \ge k$ for any arbitrarily large k. Thus, the estimate (5) cannot be valid for all y_0, y_1, i.e., the scheme is not stable with respect to the initial data.

Note that the scheme (12) related to the differential equation (1) is of the first order of approximation with respect to τ, whereas the unstable explicit scheme is of the second order. The following regularized difference scheme,

$$\left(I + \frac{\tau^2}{2}A\right)y_{\bar{t}t,n} + Ay_n = 0, \qquad n = 1, 2, \ldots, \tag{28}$$

with given y_0, y_1 is a special case of (2) for $\beta = 0$, $\alpha = \frac{\tau^2}{2}$. This scheme has the second order of approximation with respect to τ. It can also be interpreted as the regularized explicit scheme. The next main result of the paper deals with the stability of this scheme.

THEOREM 2.2. *Let A be strongly P-positive with a spectrum $\Sigma(A)$ inside the parabola $y^2 = c_0^2x/2$ and $\beta = 0$, $\alpha = \frac{\tau^2}{2}$, $\tau \le 2\sqrt{2}c_0^{-1}$, $\Re\Sigma(A) > \gamma$. Then the difference scheme (2) is ρ-stable with respect to initial data in $D(A^\sigma)$, $\sigma > 0$ with $\rho = (1 + \frac{c_0\tau}{\sqrt{2}} + \frac{c_0^2\tau^2}{4})^{\frac{1}{2}}$.*

Proof. In the case under consideration we have

$$Q(A) \equiv I, \qquad \chi(A) = \left[I + \frac{\tau^2}{2}A\right]^{-1}.$$

Similar to the proof of Theorem 2.1 we have to estimate

$$|Q^n\Phi(\chi(z))| \equiv |\Phi(\chi(z))|, \qquad \chi \pm \sqrt{\chi^2-1} = \left(1 \pm i\tau\sqrt{\frac{z}{2}}\sqrt{2+\frac{\tau^2}{2}z}\right) \Big/ \left(1+\frac{\tau^2}{2}z\right),$$

$$|U_{n-1}(\chi) \mp U_{n-2}(\chi)| = \frac{|(\chi+\sqrt{\chi^2-1})^{n-\frac{1}{2}} \pm (\chi-\sqrt{\chi^2-1})^{n-\frac{1}{2}}|}{\sqrt{2}|\sqrt{\chi \pm 1}|}.$$

Assume that $z=\rho e^{i\phi}$ lies on the positive arc of the parabola. Then $\rho^2 = x^2 + \frac{c_0^2}{2}x$, $\cos\phi = \frac{x}{\rho} > 0$, $\sin\phi = \frac{c_0\sqrt{x}}{\sqrt{2}\rho} > 0$. Setting $2+\frac{\tau^2}{2}z = \rho_1 e^{i\theta_1}$ we have

$$\rho_1 = \left[4 + 2\tau^2\rho\cos\phi + \frac{\tau^4\rho^2}{4}\right]^{1/2} = \left[4+2\tau^2 x + \frac{\tau^4}{4}\left(x^2+\frac{c_0^2}{2}x\right)\right]^{1/2},$$

$$\cos\theta_1 = \frac{2+\frac{\tau^2}{2}\rho\cos\phi}{\rho_1} > 0, \qquad \sin\theta_1 = \frac{\tau^2\rho\sin\phi}{2\rho_1} > 0,$$

$$|\chi+\sqrt{\chi^2-1}|^2 = \frac{1+\sqrt{2}\tau(\rho\rho_1)^{1/2}\sin\frac{\phi+\theta_1}{2} + \frac{\tau^2}{2}\rho\rho_1}{1+\tau^2\rho\cos\phi + \frac{\tau^4\rho^2}{4}}.$$

It is easy to see that $\frac{\phi+\theta_1}{2} \in [0, \frac{\pi}{2}]$ and

$$\sin\frac{\phi+\theta_1}{2} = \sqrt{\frac{1-\cos(\phi+\theta_1)}{2}}$$

$$= \frac{1}{\sqrt{2}}\left[1 - \frac{x}{\rho}\frac{2+\frac{\tau^2}{2}\rho\cos\phi}{\rho_1} + \frac{c_0\sqrt{x}}{\sqrt{2}\rho}\frac{\tau^2\rho\sin\phi}{2\rho_1}\right]^{1/2},$$

$$|\chi+\sqrt{\chi^2-1}|^2 = \left\{1+\tau\left[\rho\rho_1 - x\left(2+\frac{\tau^2}{2}x\right) + \frac{c_0^2 x\tau^2}{4}\right]^{1/2} + \frac{\tau^2}{2}\rho\rho_1\right\} \Big/$$

$$\left[1+\tau^2 x + \frac{\tau^4\rho^2}{4}\right] = 1 + \tau\frac{[\rho\rho_1 - x(2+\frac{\tau^2}{2}x) + \frac{c_0^2 x}{4}\tau^2]^{1/2}}{1+\tau^2 x + \frac{\tau^4\rho^2}{4}}$$

$$+\,\tau^2\frac{\frac{1}{2}\rho\rho_1 - x - \frac{\tau^2\rho^2}{4}}{1+\tau^2 x+\frac{\tau^4\rho^2}{4}} \equiv 1 + \tau\mu_1(x) + \tau^2\mu_2(x). \tag{29}$$

Let us estimate the functions $\mu_1(x)$, $\mu_2(x)$. We represent

$$\mu_1(x) = \left\{ \left(x^2 + \frac{c_0^2}{2}x\right)\left(4 + 2\tau^2 x + \frac{\tau^4}{4}\left(x^2 + \frac{c_0^2}{2}x\right)\right) - \left[x\left(2 + \frac{\tau^2}{2}x\right) - \frac{c_0^2 x}{4}\tau^2\right]^2 \right\}^{1/2} \Big/ \Delta_1(x),$$

where

$$\Delta_1(x) = \left[1 + \tau^2 x + \frac{\tau^4}{4}\left(x^2 + \frac{c_0^2}{2}x\right)\right]\left(\left[\left(x^2 + \frac{c_0^2}{2}x\right)\left(4 + 2\tau^2 x + \frac{\tau^4}{4}\left(x^2 + \frac{c_0^2}{2}x\right)\right)\right]^{1/2} + x\left(2 + \frac{\tau^2}{2}x - \frac{c_0^2\tau^2}{4}\right)\right)^{1/2} \geq \left(1 + \tau^2 x + \frac{\tau^4 x^2}{4}\right)^{1/2}(4x)^{1/2}$$

and get

$$\mu_1(x) = \left\{ \left(x^2 + \frac{c_0^2}{2}x\right)(4 + 2\tau^2 x) - \tau^2\left(x^2 + \frac{c_0^2}{2}x\right)x\left(2 - \frac{\tau^2}{2}c_0^2\right) - x^2\left(2 - \frac{\tau^2}{2}c_0^2\right)^2 \right\}^{1/2} \Big/ \Delta_1(x) = \left\{2c_0^2 x\left(\frac{\tau^4}{4}x^2 + \tau^2 x + 1\right)\right\}^{1/2} \Big/ \Delta_1(x) \leq \frac{c_0}{\sqrt{2}}. \tag{30}$$

Setting

$$\Delta_2(x) = \left[1 + \tau^2 x + \frac{\tau^4}{4}\left(x^2 + \frac{c_0^2}{2}x\right)\right]\left\{\frac{1}{2}\left(x^2 + \frac{c_0^2}{2}x\right)^{1/2}\left[4 + 2\tau^2 x + \frac{\tau^4}{4}\left(x^2 + \frac{c_0^2}{2}x\right)\right]^{1/2} + x + \frac{\tau^2}{4}\left(x^2 + \frac{c_0^2}{2}x\right)\right\} \geq 2x$$

we have

$$\mu_2(x) = \left\{\frac{1}{4}\left(x^2 + \frac{c_0^2}{2}x\right)\left[4 + 2\tau^2 x + \frac{\tau^4}{4}\left(x^2 + \frac{c_0^2}{2}x\right)\right] - \frac{\tau^4}{16}\left(x^2 + \frac{c_0^2}{2}x\right)^2 - \frac{\tau^2}{2}x\left(x^2 + \frac{c_0^2}{2}x\right) - x^2\right\} \Big/ \Delta_2(x) = \frac{c_0^2}{2}x/\Delta_2(x) \leq \frac{c_0^2}{4}, \tag{31}$$

provided that

$$2 - \frac{\tau^2}{4}c_0^2 \geq 0. \tag{32}$$

Now, it follows from (29), (30) that

$$|\chi + \sqrt{\chi^2 - 1}|^2 \le 1 + \frac{c_0\tau}{\sqrt{2}} + \frac{c_0^2\tau^2}{4}. \tag{33}$$

Finally, we must estimate $|1 \pm \chi|^{-1/2}$ on Γ_+. It is easy to see that

$$|1+\chi|^2 = \left|\frac{2 + \frac{\tau^2}{2}z}{1 + \frac{\tau^2}{2}z}\right|^2 = \frac{(2 + \frac{\tau^2}{2}x)^2 + \frac{\tau^4}{8}c_0^2 x}{(1 + \frac{\tau^2}{2}x) + \frac{\tau^2}{8}c_0^2 x} \ge 1,$$

$$|1-\chi|^2 = \frac{\tau^4}{4}\left|\frac{z}{1 + \frac{\tau^2}{2}z}\right|^2 = \frac{\tau^4}{4}\mu_3(x), \tag{34}$$

$$\mu_3(x) = \frac{x^2 + \frac{c_0^2}{2}x}{(1 + \frac{\tau^2}{2}x)^2 + \frac{\tau^4}{8}c_0^2 x}.$$

Since

$$\mu_3'(x) = \left[\left(\tau^2 - \frac{\tau^4}{8}c_0^2\right)x^2 + 2x + c_0^2\right] \Big/ \left[\left(1 + \frac{\tau^2}{2}x\right)^2 + \frac{\tau^4}{8}c_0^2 x\right]^2 > 0$$

provided that (30) holds true, we obtain

$$|1-\chi|^2 \ge \frac{\tau^2}{4}\mu_3(\gamma). \tag{35}$$

Following the idea of the proof of Theorem 2.1 and taking into account the estimates (32)–(35) we arrive at the statement of Theorem 2.2. □

Remark 4. One can see from the conditions $\tau < \sqrt{2}c_0^{-1}$ for the scheme (12) and $\tau < 2\sqrt{2}c_0^{-1}$ for the scheme (28) that the opening of the parabola determines the upper limit of the time-step τ for which these schemes are ρ-stable.

Let us consider the following inhomogeneous difference scheme:

$$(I + \alpha A)y_{\bar{t}t,n} + \beta A y_{\overset{\circ}{t},n} + Ay_n = f_n, \qquad n = 1, 2, \ldots, \tag{36}$$

$$y_0 = y_1 = 0.$$

Below we define a type of stability that plays an important role for inhomogeneous problems.

DEFINITION 2.3. *Given a function $\rho(\tau)$ we say that the scheme (36) is ρ-stable with respect to the right-hand side in $D(A^\sigma)$ with some real $\sigma > 0$ if there exists a constant $M > 0$ independent of n such that the estimate*

$$\|y_n\| \le M\rho^n \sum_{p=0}^{n-1} \tau \|f_p\|_\sigma$$

holds for any discrete function $f_p \in D(A^\sigma)$.

The solution of (36) can be represented as

$$y_n = \left\{\left[I + \left(\alpha + \frac{\beta\tau}{2}\right)A\right]\left[I + \left(\alpha - \frac{\beta\tau}{2}\right)A\right]\right\}^{-1/2} \tau^2 \sum_{p=0}^{n-1} Q^{n-p}(A)U_{n-p-1}(\chi(A))f_p$$

$$= \frac{\tau^2}{2\pi i} \sum_{p=0}^{n-1} \left\{ \int_\Gamma \left[\left(1 + \left(\alpha + \frac{\beta\tau}{2}\right) z\right) \left(1 + \left(\alpha - \frac{\beta\tau}{2}\right) z\right)\right]^{-1/2} \right.$$

$$\left. \times\, Q^{n-p}(z) U_{n-p-1}\,(\chi(z))(z-A)^{-1} dz \right\} f_p,$$

$$n = 2, 3, \ldots; \qquad y_0 = 0,\ y_1 = 0.$$

Using the inequality

$$|Q^k(z)U_{k-1}(\chi(z))| \le \frac{|Q(z)\Phi(\chi(z))|^k}{\sqrt{|1-\chi^2(z)|}},$$

the estimates (22), (24), (25), (32), (33), (34), and following the idea of the proof of Theorems 2.1 and 2.2 we get the following statement.

THEOREM 2.4. *Under assumptions of Theorem* 2.1 ($\alpha = \frac{\tau^2}{2}, \beta = \tau$) *or Theorem* 2.2 ($\beta = 0, \alpha = \frac{\tau^2}{2}$) *the corresponding difference schemes from the family* (36) *are ρ-stable with respect to the right-hand side in $D(A^\sigma)$ with $\rho(\tau) = (1 - \frac{c_0\tau}{\sqrt{2}})^{-1/2}$ and $\rho(\tau) = (1 + \frac{c_0\tau}{\sqrt{2}} + \frac{c_0^2\tau^2}{4})^{\frac{1}{2}}$, respectively.*

Example 3. In some sense, this example shows the sharpness of our results. Indeed, let us consider the scalar problem

$$\frac{d^2u}{dt^2} + Au = 0, \quad t \in (0, T],$$
$$u(0) = u_0, \quad u'(0) = i\sqrt{A}u_0,$$

where $A = x - ic_0\sqrt{\frac{x}{2}}$ can be viewed for $x \to \infty$ as an "unbounded" strongly P-positive operator with the spectrum inside the parabola $y^2 = (c_0 + \varepsilon)^2 x/2$ with an arbitrarily small positive ε. The solution of the problem is the function

$$u(x,t) = \exp\left\{ i\sqrt{x - ic_0\sqrt{\frac{x}{2}}}\ t \right\} u_0$$

$$= \exp\left\{ i\sqrt[4]{x^2 + c_0^2\frac{x}{2}} \left(i\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}\right) t \right\} u_0,$$

where

$$\cos\varphi = \frac{x}{\sqrt{x^2 + c_0^2\frac{x}{2}}} > 0, \quad \sin\varphi = -\frac{c_0\sqrt{\frac{x}{2}}}{\sqrt{x^2 + c_0^2\frac{x}{2}}} < 0.$$

It is easy to see that

$$|u(x,t)| = \rho_d(x,t)|u_0|$$

with

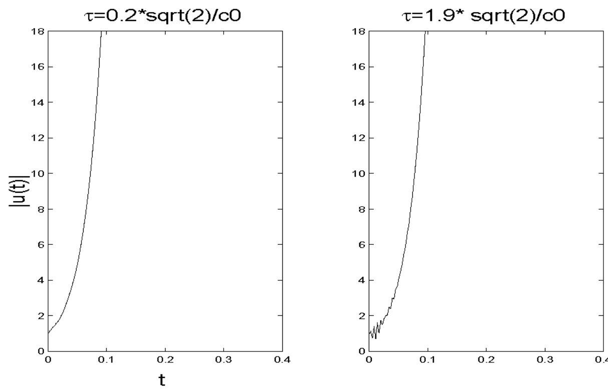

FIG. 2. *Solution of scheme* (28): *the sufficient stability condition is fulfilled.*

$$\rho_d(x,t) = \exp\left\{ \sqrt[4]{x^2 + c_0^2\frac{x}{2}}\,\frac{1}{\sqrt{2}}\sqrt{1 - \frac{x}{\sqrt{x^2 + c_0^2\frac{x}{2}}}}\; t \right\}$$

$$= \exp\left\{ \frac{t}{\sqrt{2}}\sqrt{\sqrt{x^2 + c_0^2\frac{x}{2}} - x} \right\}.$$

For the solution of the corresponding difference scheme (28) Theorem 2.2 provides the estimate

$$|y_n(x)| = |y(x, n\tau)| \le c\rho^n(\tau)|u_0|$$

with $\rho(\tau) = (1 + \frac{(c_0+\varepsilon)\tau}{\sqrt{2}} + \frac{(c_0+\varepsilon)^2\tau^2}{4})^{\frac{1}{2}}$. In particular we have for a fixed $t = n\tau$ and $x \to \infty$

$$|u(\infty, t)| = \lim_{x\to\infty} |u(x,t)| = \rho_d(\infty, t)|u_0|,$$

$$|y(t)| \le c\rho(\tau)^{\frac{t}{\tau}}|u_0|,$$

where $\rho_d(\infty, t) = \lim_{x\to\infty} \rho_d(x,t) = \exp\frac{c_0 t}{2\sqrt{2}}$. It is easy to check that

$$\lim_{\tau\to 0} \rho(t)^{\frac{t}{\tau}} = \lim_{\tau\to 0}\left[1 + \frac{(c_0+\varepsilon)\tau}{\sqrt{2}} + \frac{(c_0+\varepsilon)^2\tau^2}{4}\right]^{\frac{t}{2\tau}} = \exp\left\{\frac{(c_0+\varepsilon)t}{2\sqrt{2}}\right\},$$

i.e., the parabola containing the spectrum of A defines the behavior of the difference solution asymptotically in t in exactly the same way as the exact solution.

Example 4. Let us consider the difference scheme (28) with A as in Example 3 for $x = 10^4, c_0 = 10^2, y_0 = 1, y_1 = 1 + \tau i\sqrt{x - ic_0\sqrt{\frac{x}{2}}}$, and $n = 1, 2, \ldots, 100$. One can see that the absolute value of the solution as function of t computed by (28) is stable when the sufficient stability condition $\tau < 2\sqrt{2}/c_0$ holds (Figure 2). The next figure shows that the instability can occur if the condition $\tau < 2\sqrt{2}/c_0$ is violated (see Figure 3).

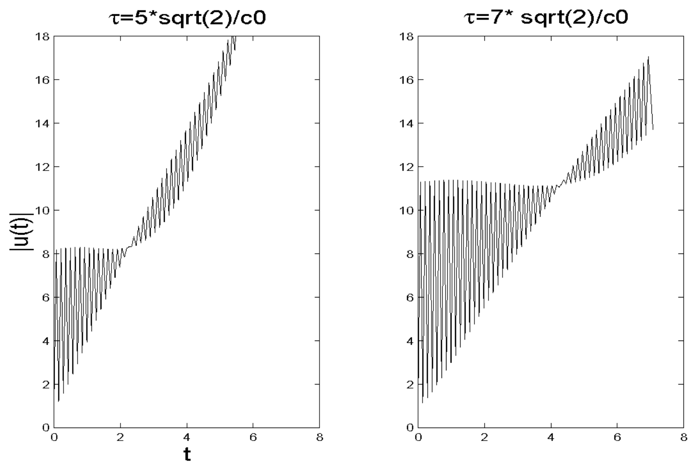

FIG. 3. *Solution of scheme* (28): *the sufficient stability condition is violated.*

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