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NUMERICAL METHODS.
FINITE-DIFFERENCE EQUATIONS

Finite-Difference Approximations to the Transport Equation. II

A. A. Samarskii and P. N. Vabishchevich

Institute of Mathematical Modeling, Russian Academy of Sciences, Moscow, Russia

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INTRODUCTION

Standard schemes of second-order approximation with respect to the space variables for the boundary value problem for the transport equation in a rectangle were considered in [1]. Three forms of the transport equation, namely, divergent, nondivergent, and symmetric, were extracted. The stability of two- and three-layer finite-difference schemes with standard space approximations by third-order central differences in the corresponding grid spaces were investigated in [2, 3] on the basis of the general stability theory of operator-difference schemes.

In the applied mathematical modeling of problems in continuum mechanics, great attention is paid to the monotonicity of finite-difference approximations [2], which is related to the validity of the maximum principle for a finite-difference solution. The development of monotone finite-difference approximations is traditionally aimed at the approximation of convective terms by directed differences. In the second part of the paper, we study the stability of schemes with directed differences for nonstationary transport problems with the use of convective transport operators in divergent and nondivergent forms. We derive *a priori* estimates in Banach spaces of grid functions.

THE MAXIMUM PRINCIPLE AND *a priori* ESTIMATES FOR THE DIFFERENTIAL PROBLEM

We restrict our consideration to the sample problem for a two-dimensional transport equation in a rectangle. We rewrite it in the form of the Cauchy problem for the evolution equation:

$$du/dt + \mathscr{C}u = 0, \qquad \mathscr{C} = \mathscr{C}(t), \qquad t > 0, \tag{1}$$

$$u(0) = u_0. \tag{2}$$

We subject the velocity field $\mathbf{v} = (v_1, v_2)$ to the conditions

$$(\mathbf{v} \cdot \mathbf{n}) = 0, \qquad \mathbf{x} \in \partial\Omega, \qquad t > 0, \tag{3}$$

where $\mathbf{n}$ is the normal to the boundary of the domain. Under conditions (3) on the velocity field, no boundary conditions are needed for extracting a unique solution of problem (1), (2). We consider two classes of problems, in which the convective transport is written out in the nondivergent and divergent forms. In the first case, the convective transport operator is represented in the form $\mathscr{C} = \mathscr{C}_1$, where

$$\mathscr{C}_1 u = \sum_{\alpha=1}^{2} v_\alpha(\mathbf{x}, t)\partial u/\partial x_\alpha. \tag{4}$$

For the divergent transport equation in (1), we have $\mathscr{C} = \mathscr{C}_2$, where

$$\mathscr{C}_2 u = \sum_{\alpha=1}^{2} \partial\left(v_\alpha(\mathbf{x}, t)u\right)/\partial x_\alpha. \tag{5}$$

The maximum principle [1] holds for transport equations in the divergent and nondivergent forms (4) and (5); namely, if the initial conditions are nonnegative, then the solution is nonnegative at any time $t > 0$. We also present related *a priori* estimates for the solution of problems (1), (2), (4) and (1), (2), (5).

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For the transport equation in the divergent form (1), (5), we have the following estimate for the stability with respect to initial data in $L_1(\Omega)$:

$$\|u(\mathbf{x},t)\|_1 \le \|u_0(\mathbf{x})\|_1, \tag{6}$$

where $\|g\|_1 = \int_\Omega |g(\mathbf{x})| d\mathbf{x}$ is the norm in the space $L_1(\Omega)$.

The corresponding estimate in $L_\infty(\Omega)$ for the nondivergent equation (1), (4) has the form

$$\|u(\mathbf{x},t)\|_\infty \le \|u_0(\mathbf{x})\|_\infty, \tag{7}$$

where $\|g\|_\infty = \max_{\mathbf{x}\in\Omega} |g(\mathbf{x})|$.

When constructing discrete analogs of problems (1), (2), (4) and (1), (2), (5), we keep track of the validity of the maximum principle on the finite-difference level and the validity of stability estimates like (6) and (7).

THE CAUCHY PROBLEM FOR A SYSTEM OF ODE

Let us firstly consider the homogeneous system of linear ordinary first-order equations

$$\frac{dw_i}{dt} + \sum_{j=1}^{N} a_{ij}(t) w_j = 0, \qquad i = 1, \dots, N. \tag{8}$$

Setting $w = w(t) = \{w_1, w_2, \dots, w_n\}$ and $A = [a_{ij}]$, we rewrite this system in the matrix (operator) form

$$dw/dt + A(t)w = 0. \tag{9}$$

We construct finite-difference schemes for the approximate solution of the Cauchy problem in which relation (9) is considered for $t > 0$ under the initial conditions

$$w(0) = u_0. \tag{10}$$

We are interested in the stability of the finite-difference solution of problem (9), (10) in L_∞ and L_1. For the norm of a vector and the induced matrix norm in L_∞, we have [4]

$$\|w\|_\infty = \max_{1\le i\le N} |w_i|, \qquad \|A\|_\infty = \max_{1\le i\le N} \sum_{j=1}^{n} |a_{ij}|. \tag{11}$$

Likewise, in L_1, we obtain

$$\|w\|_1 = \sum_{i=1}^{N} |w_i|, \qquad \|A\|_1 = \max_{1\le j\le N} \sum_{i=1}^{n} |a_{ij}|. \tag{12}$$

Problem (9), (10) is considered under the following restriction. We assume that the diagonal entries of A are nonnegative and either the nonstrict diagonal dominance takes place in rows, i.e.,

$$a_{ii} \ge \sum_{i\ne j=1}^{N} |a_{ij}|, \qquad i = 1, \dots, N, \tag{13}$$

or the nonstrict diagonal dominance takes place in columns, i.e.,

$$a_{jj} \ge \sum_{j\ne i=1}^{N} |a_{ij}|, \qquad j = 1, \dots, N. \tag{14}$$

In the theory of finite-difference schemes [2, 3], stability estimates in L_∞ are often derived from the maximum principle and related comparison theorems. Here we obtain sufficient stability conditions with the use of the notion of the logarithmic norm [5, 6] of an operator. This permits one to consider stability in L_∞ and L_1 from a common viewpoint (see also [7, 8]).

The logarithmic norm of a matrix A is defined as the number

$$\mu[A] = \lim_{\delta\to 0+} \frac{\|E+\delta A\| - 1}{\delta}.$$

For the logarithmic matrix norm in L_∞ [coordinated with (11)] and in L_1 [coordinated with (12)], we have the expressions

$$\mu_\infty[A] = \max_{1\le i\le N}\left(a_{ii} + \sum_{i\ne j=1}^{N} |a_{ij}|\right), \qquad \mu_1[A] = \max_{1\le j\le N}\left(a_{jj} + \sum_{j\ne i=1}^{N} |a_{ij}|\right).$$

By virtue of (13) and (14), in the Cauchy problem (9), (10) for the logarithmic norm of the matrix $-A$, we have

$$\mu[-A] \le 0 \tag{15}$$

in the corresponding space [namely, in L_∞ in case (13) and in L_1 in case (14)].

Of the properties of the logarithmic norm (see [5, 9]), we note the following:

(1) $\mu[cA] = c\mu[A]$, $c = \text{const} \ge 0$;

(2) $\mu[cE + A] = c + \mu[A]$, $c = \text{const}$;

(3) $\|Aw\| \ge \max\{-\mu[-A], -\mu[A]\}\|w\|$.

Of special interest is property (3), which permits one to derive a lower bound for $\|Aw\|$ readily evaluated on the basis of the matrix entries. This estimate can be combined with the ordinary upper bound for the norm of Aw: $\|Aw\| \le \|A\|\|w\|$.

Applying the logarithmic norm to (9), we obtain [5, 6] $d\|w\|/dt \le \mu[-A]\,\|w\|$, which, together with property (15), implies the stability estimate

$$\|w(t)\| \le \|u_0\| \tag{16}$$

for the solution of the Cauchy problem (9), (10) in L_∞ or L_1.

WEIGHTED SCHEMES

Let us investigate the stability of finite-difference schemes for problem (9), (10). Let us consider the simplest weighted two-layer finite-difference scheme

$$(y_{n+1} - y_n)/\tau + A(\sigma y_{n+1} + (1-\sigma)y_n) = 0, \tag{17}$$

where, for example, $A = A(\sigma t_{n+1} + (1-\sigma)t_n)$, under the initial condition

$$y_0 = u_0. \tag{18}$$

Let us state the stability conditions for the finite-difference schemes (17), (18). Let us show that, for the Cauchy problem (9), (10) with matrix A satisfying condition (13) [respectively, (14)], the weighted finite-difference scheme (17), (18) is unconditionally stable for $\sigma = 1$ and conditionally stable for $0 \le \sigma < 1$ in L_∞ (respectively, L_1) provided that

$$\tau \le (1-\sigma)^{-1}\left(\max_{1\le i\le N} a_{ii}\right)^{-1}. \tag{19}$$

Furthermore, the finite-difference solution satisfies the *a priori* estimate

$$\|y_{n+1}\| \le \|u_0\|. \tag{20}$$

It follows from (17) that $(E + \sigma\tau A)y_{n+1} = (E - (1-\sigma)\tau A)y_n$, and consequently,

$$\|(E + \sigma\tau A)y_{n+1}\| \le \|(E - (1-\sigma)\tau A)y_n\|. \tag{21}$$

By virtue of the above-mentioned properties of the logarithmic norm and relation (15), for the left-hand side of inequality (21), we have

$$\|(E+\sigma\tau A)y_{n+1}\| \geq -\mu[-E-\sigma\tau A]\,\|y_{n+1}\| = (1+\sigma\tau\mu[-A])\,\|y_{n+1}\| \geq \|y_{n+1}\|.$$

For the right-hand side of (21), we obtain $\|(E-(1-\sigma)\tau A)y_n\| \leq \|E-(1-\sigma)\tau A\|\,\|y_n\|$.

Let us consider this estimate in detail for the stability analysis in L_∞. The case of L_1 can be considered in a similar way. Taking into account relation (11) and the diagonal dominance condition (13), we obtain

$$\begin{aligned}\|E-(1-\sigma)\tau A\| &= \max_{1\leq i\leq N}\left|1-(1-\sigma)\tau\left(a_{ii}+\sum_{i\neq j=1}^{N} a_{ij}\right)\right| \\ &\leq \max_{1\leq i\leq N}\left(|1-(1-\sigma)\tau a_{ii}| + (1-\sigma)\tau\sum_{i\neq j=1}^{N}|a_{ij}|\right) \\ &\leq \max_{1\leq i\leq N}\left(|1-(1-\sigma)\tau a_{ii}| + (1-\sigma)\tau a_{ii}\right) \leq 1\end{aligned}$$

for $0\leq\sigma\leq 1$ provided that the time increment satisfies condition (19). The substitution into (21) yields $\|y_{n+1}\| \leq \|y_n\|$, whence we immediately have the desired estimate (20) for stability with respect to the initial data.

Let us apply this result to the stability analysis of finite-difference schemes for nonstationary transport equations in nondivergent and divergent forms.

SCHEMES WITH DIRECTED DIFFERENCES FOR TRANSPORT EQUATIONS

Unconditionally stable (in L_∞ and L_1) finite-difference schemes for the transport equations in nondivergent and divergent forms can be constructed on the basis of the simplest approximations of transport operators by first-order directed differences.

We use the following notation: $g(x) = g_+(x) + g_-(x)$, $g_+(x) = 0.5(g(x)+|g(x)|) \geq 0$, and $g_-(x) = 0.5(g(x)-|g(x)|) \leq 0$. Let us start from approximations on the basis of the definition of velocity fields at nodes of the grid $\bar{\omega}$. In the problems with the no-flow condition (3), approximations on the boundary are constructed with regard to the sign of the normal component of the velocity in boundary-adjacent nodes. For the finite-difference transport operator in nondivergent form, we set

$$C_1 = \sum_{\alpha=1}^{2} C_1^{(\alpha)}, \qquad C_1^{(\alpha)}y = \begin{cases} b_+^{(\alpha)}y_{\bar{x}_\alpha} + b_-^{(\alpha)}y_{x_\alpha} & \text{if } h_\alpha \leq x_\alpha \leq l_\alpha - h_\alpha, \\ 0 & \text{if } x_\alpha = 0,\ x_\alpha = l_\alpha, \end{cases} \qquad \alpha = 1,2. \tag{22}$$

Likewise (see [1]), for the approximation of the transport operator in the divergent form, we can set

$$C_2 = \sum_{\alpha=1}^{2} C_2^{(\alpha)}, \qquad C_2^{(\alpha)}y = \begin{cases} \left(b_-^{(\alpha)}y\right)_{x_\alpha} & \text{if } x_\alpha = 0, \\ \left(b_+^{(\alpha)}y\right)_{\bar{x}_\alpha} + \left(b_-^{(\alpha)}y\right)_{x_\alpha} & \text{if } h_\alpha \leq x_\alpha \leq l_\alpha - h_\alpha, \\ \left(b_+^{(\alpha)}y\right)_{\bar{x}_\alpha} & \text{if } x_\alpha = l_\alpha, \end{cases} \qquad \alpha = 1,2. \tag{23}$$

In general, the finite-difference operator C_2 does not approximate the differential transport operator in the divergent form at the boundary nodes, but the truncation error is $O(|h|)$ in the class of problems with condition (3).

The above finite-difference operators of convective transport are approximated on the standard five-point "cross" stencil. Let us introduce the diagonal dominance conditions in this case. We set

$$\begin{aligned} Dy = \gamma(\mathbf{x},t)y(\mathbf{x},t) - \alpha_1(\mathbf{x},t)y\left(x_1-h_1,x_2\right) - \beta_1(\mathbf{x},t)y\left(x_1+h_1,x_2\right) \\ - \alpha_2(\mathbf{x},t)y\left(x_1,x_2-h_2\right) - \beta_2(\mathbf{x},t)y\left(x_1,x_2+h_2\right), \qquad \mathbf{x}\in\bar{\omega}. \end{aligned} \tag{24}$$

We assume that the corresponding coefficients α_1, α_2, β_1, and β_2 of $y(\mathbf{x})$ in (24) vanish for $\mathbf{x}\notin\bar{\omega}$.

For the natural numbering of grid points $(((i_1, i_2), i_1 = 0, \ldots, N_1), i_2 = 0, \ldots, N_2)$, the diagonal dominance condition (13) in rows for the two-dimensional five-point operator D given by (24) has the form

$$\gamma(\mathbf{x}, t) \geq |\alpha_1(\mathbf{x}, t)| + |\alpha_2(\mathbf{x}, t)| + |\beta_1(\mathbf{x}, t)| + |\beta_2(\mathbf{x}, t)|, \qquad \mathbf{x} \in \bar{\omega}. \tag{25}$$

The similar column diagonal dominance condition (14) acquires the form

$$\begin{aligned}\gamma(\mathbf{x}, t) \geq & |\alpha_1(x_1 + h_1, x_2, t)| + |\beta_1(x_1 - h_1, x_2, t)| \\ & + |\alpha_2(x_1, x_2 + h_2, t)| + |\beta_2(x_1, x_2 - h_2, t)|, \qquad \mathbf{x} \in \bar{\omega}.\end{aligned} \tag{26}$$

The grid operator of convective transport in nondivergent form (22) can be rewritten in the form (24) with

$$\begin{aligned}&\alpha_k(\mathbf{x}, t) = b_+^{(k)}(\mathbf{x}, t)/h_k, \qquad \beta_1(\mathbf{x}, t) = -b_-^{(k)}(\mathbf{x}, t)/h_k, \qquad k = 1, 2,\\ &\gamma(\mathbf{x}, t) = \alpha_1(\mathbf{x}, t) + \alpha_2(\mathbf{x}, t) + \beta_1(\mathbf{x}, t) + \beta_2(\mathbf{x}, t), \qquad \mathbf{x} \in \omega.\end{aligned}$$

For the finite-difference operator (23), we have

$$\begin{aligned}&\alpha_1(\mathbf{x}, t) = b_+^{(1)}(x_1 - h_1, x_2, t)/h_1, \qquad \alpha_2(\mathbf{x}, t) = b_+^{(2)}(x_1, x_2 - h_2, t)/h_2,\\ &\beta_1(\mathbf{x}, t) = -b_-^{(1)}(x_1 + h_1, x_2, t)/h_1, \qquad \beta_2(\mathbf{x}, t) = -b_-^{(2)}(x_1, x_2 + h_2, t)/h_2,\\ &\gamma(\mathbf{x}, t) = \alpha_1(\mathbf{x}, t) + \alpha_2(\mathbf{x}, t) + \beta_1(\mathbf{x}, t) + \beta_2(\mathbf{x}, t), \qquad \mathbf{x} \in \bar{\omega}.\end{aligned}$$

Therefore, the row diagonal dominance conditions (25) are unconditionally valid for the grid operator C_1 given by (22), and the column diagonal dominance conditions (26) hold for the grid operator C_2 given by (23).

Now we can state the stability conditions for the two-layer finite-difference scheme

$$(y_{n+1} - y_n)/\tau + C_\alpha\left(t_{n+1}^{(\sigma)}\right)(\sigma y_{n+1} + (1 - \sigma) y_n) = 0, \quad n = 0, 1, \ldots, \quad \alpha = 1, 2, \quad y_0 = u_0 \tag{27}$$

with the use of the difference transport operators (22) and (23) with directed differences. Conditions (19) on the time increment acquire the form $\tau \leq (1 - \sigma)^{-1}\left(\max_{\mathbf{x} \in \bar{\omega}} \gamma(\mathbf{x}, t)\right)^{-1}$. Hence if

$$\tau \leq \frac{1}{1 - \sigma}\left(\sum_{\alpha=1}^{2} \max_{\mathbf{x} \in \bar{\omega}} \frac{|v_\alpha(\mathbf{x}, t)|}{h_\alpha}\right)^{-1}, \tag{28}$$

then the finite-difference scheme (27), (22) is stable in $L_\infty(\bar{\omega})$, and its solution satisfies the *a priori* estimate

$$\|y_{n+1}\|_\infty \leq \|u_0\|_\infty, \qquad n = 0, 1, \ldots \tag{29}$$

The finite-difference scheme (27), (23) is also stable under the same conditions (28); moreover, in $L_1(\bar{\omega})$, we have the stability estimate

$$\|y_{n+1}\|_1 \leq \|u_0\|_1, \qquad n = 0, 1, \ldots \tag{30}$$

The estimates (29) and (30) are coordinated with the corresponding estimates [(7) and (6), respectively]. Unconditional stability takes place [see (28)] for completely implicit schemes ($\sigma = 1$).

Note that, when using central-difference approximations for convective terms (see [1]), we cannot hope that the two-layer finite-difference schemes (27) are stable in $L_\infty(\bar{\omega})$ or $L_1(\bar{\omega})$. In this case, we cannot ensure the diagonal dominance for any values of the grid parameters with respect to time and space.

The estimate (30) in $L_1(\bar{\omega})$ is natural for the finite-difference scheme (27), (23). In this case, no condition is imposed on the velocity field. It is often desirable to obtain an estimate in a stronger norm [in $L_\infty(\bar{\omega})$] by imposing some constraints on the properties of the medium. In this case, the situation is restricted to ϱ-stability.

In the case of the differential problem (1), (2) with the transport operator in the divergent form (5), instead of (6), we can use the ϱ-stability estimate

$$\|u(t)\|_\infty \le \exp(\mathscr{K} t)\,\|u_0\|_\infty, \qquad \mathscr{K} = \max_{\mathbf{x}\in\Omega} |\operatorname{div}\mathbf{v}|. \tag{31}$$

It can be proved by analogy with (7) with the help of the representation

$$\mathscr{C}_2 u = \mathscr{C}_1 u + \operatorname{div}\mathbf{v}\, u. \tag{32}$$

Such estimates were obtained in [10] for problem (1), (2), (5). Using (31), we obtain the corresponding ϱ-stability estimates for the finite-difference schemes (27), (23).

We can readily see that there is no analog of (32) for the finite-difference convective transport operators (22) and (23). A similar analysis was performed in [1] for the approximations of convective transport by central differences. For example,

$$C_2 y = C_1 y + \sum_{\alpha=1}^{2}\left(\left(b_+^{(\alpha)}\right)_{\bar{x}_\alpha} + \left(b_-^{(\alpha)}\right)_{x_\alpha}\right) y + \sum_{\alpha=1}^{2} h_\alpha \left(y_{x_\alpha}\left(b_-^{(\alpha)}\right)_{x_\alpha} - y_{\bar{x}_\alpha}\left(b_+^{(\alpha)}\right)_{\bar{x}_\alpha}\right)$$

for internal nodes. Hence the approximations (23) are not convenient for the finite-difference transport operator in divergent form from the viewpoint of deriving estimates in the uniform norm.

When using first-order approximations, for transport operators, we can expect a better situation from the viewpoint of the definition of transport coefficients on a displaced grid. For the nondivergent transport operator, we use the approximation

$$C_1 = \sum_{\alpha=1}^{2} C_1^{(\alpha)},$$

$$C_1^{(1)} y = \begin{cases} 2b_-^{(1)}(x_1 + 0.5h_1, x_2)\, y_{x_1} & \text{if } x_1 = 0, \\ b_+^{(1)}(x_1 - 0.5h_1, x_2)\, y_{\bar{x}_1} + b_-^{(1)}(x_1 + 0.5h_1, x_2)\, y_{x_1} & \text{if } h_1 \le x_1 \le l_1 - h_1, \\ 2b_+^{(1)}(x_1 - 0.5h_1, x_2)\, y_{\bar{x}_1} & \text{if } x_1 = l_1, \end{cases} \tag{33}$$

$$C_1^{(2)} y = \begin{cases} 2b_-^{(2)}(x_1, x_2 + 0.5h_2)\, y_{x_2} & \text{if } x_2 = 0, \\ b_+^{(2)}(x_1, x_2 - 0.5h_2)\, y_{\bar{x}_2} + b_-^{(2)}(x_1, x_2 + 0.5h_2)\, y_{x_2} & \text{if } h_2 \le x_2 \le l_2 - h_2, \\ 2b_+^{(2)}(x_1, x_2 - 0.5h_2)\, y_{\bar{x}_2} & \text{if } x_2 = l_2. \end{cases}$$

Only a specific part of the normal velocity component is taken near the boundary; it is convenient to use the doubled quantity. In this case, by (3), the truncation error preserves the first order.

For the divergent convective transport operator, we use the representation

$$C_2 = \sum_{\alpha=1}^{2} C_2^{(\alpha)}, \tag{34}$$

$$C_2^{(1)} y = \begin{cases} 2h_1^{-1}\Big(b_-^{(1)}(x_1 + 0.5h_1, x_2)\, y(x_1 + h_1, x_2) \\ \qquad + b_+^{(1)}(x_1 + 0.5h_1, x_2)\, y(x_1, x_2)\Big) & \text{if } x_1 = 0, \\ h_1^{-1}\Big(b_-^{(1)}(x_1 + 0.5h_1, x_2)\, y(x_1 + h_1, x_2) \\ \qquad - b_-^{(1)}(x_1 - 0.5h_1, x_2)\, y(x_1, x_2)\Big) \\ \qquad + h_1^{-1}\Big(b_+^{(1)}(x_1 + 0.5h_1, x_2)\, y(x_1, x_2) \\ \qquad - b_+^{(1)}(x_1 - 0.5h_1, x_2)\, y(x_1 - h_1, x_2)\Big) & \text{if } h_1 \le x_1 \le l_1 - h_1, \\ -2h_1^{-1}\Big(b_+^{(1)}(x_1 - 0.5h_1, x_2)\, y(x_1 - h_1, x_2) \\ \qquad + b_-^{(1)}(x_1 - 0.5h_1, x_2)\, y(x_1, x_2)\Big) & \text{if } x_1 = l_1, \end{cases}$$

$$C_2^{(2)}y = \begin{cases} 2h_1^{-1}\Big(b_-^{(2)}(x_1,x_2+0.5h_2)\,y(x_1,x_2+h_2) \\ \qquad +b_+^{(2)}(x_1,x_2+0.5h_2)\,y(x_1,x_2)\Big) & \text{if } x_2=0, \\ h_2^{-1}\Big(b_-^{(1)}(x_1,x_2+0.5h_2)\,y(x_1,x_2+h_2) \\ \qquad -b_-^{(2)}(x_1,x_2-0.5h_2)\,y(x_1,x_2)\Big) \\ \qquad +h_2^{-1}\Big(b_+^{(2)}(x_1,x_2+0.5h_2)\,y(x_1,x_2) \\ \qquad -b_+^{(1)}(x_1,x_2-0.5h_2)\,y(x_1,x_2-h_2)\Big) & \text{if } h_2\le x_2\le l_2-h_2, \\ -2h_2^{-1}\Big(b_+^{(2)}(x_1,x_2-0.5h_2)\,y(x_1,x_2-h_2) \\ \qquad +b_-^{(2)}(x_1,x_2-0.5h_2)\,y(x_1,x_2)\Big) & \text{if } x_2=l_2. \end{cases}$$

In this case, we have the following finite-difference analog of (32):

$$C_2y = C_1y + \operatorname{div}_h \mathbf{b}\, y. \tag{35}$$

In case (33), (34), the finite-difference divergence operator is represented as

$$\operatorname{div}_h \mathbf{b} = \sum_{\alpha=1}^{2} \operatorname{div}_h \mathbf{b}_\alpha, \tag{36}$$

$$\operatorname{div}_h \mathbf{b}_1 = \begin{cases} 2h_1^{-1}\left(b^{(1)}(x_1+0.5h_1,x_2) - b^{(1)}(x_1,x_2)\right) & \text{if } x_1=0, \\ h_1^{-1}\left(b^{(1)}(x_1+0.5h_1,x_2) - b^{(1)}(x_1-0.5h_1,x_2)\right) & \text{if } h_1\le x_1\le l_1-h_1, \\ -2h_1^{-1}\left(b^{(1)}(x_1,x_2) - b^{(1)}(x_1-0.5h_1,x_2)\right) & \text{if } x_1=l_1, \end{cases}$$

$$\operatorname{div}_h \mathbf{b}_2 = \begin{cases} 2h_2^{-1}\left(b^{(2)}(x_1,x_2+0.5h_2) - b^{(2)}(x_1,x_2)\right) & \text{if } x_2=0, \\ h_2^{-1}\left(b^{(2)}(x_1,x_2+0.5h_2) - b^{(2)}(x_1,x_2-0.5h_2)\right) & \text{if } h_2\le x_2\le l_2-h_2, \\ -2h_2^{-1}\left(b^{(2)}(x_1,x_2) - b^{(2)}(x_1,x_2-0.5h_2)\right) & \text{if } x_2=l_2. \end{cases}$$

By (35), we rewrite the two-layer scheme for Eq. (1) in the form

$$(y_{n+1}-y_n)/\tau + (C_1 + \operatorname{div}_h \mathbf{b})(\sigma y_{n+1} + (1-\sigma)y_n) = 0. \tag{37}$$

Just as in the analysis of the scheme (17), (18), we obtain [see (21)]

$$\|(E+\sigma\tau(C_1+\operatorname{div}_h \mathbf{b}))\,y_{n+1}\|_\infty \le \|(E-(1-\sigma)\tau(C_1+\operatorname{div}_h \mathbf{b}))\,y_n\|_\infty. \tag{38}$$

When considering the left-hand side of (38), we use the inequality $\mu_\infty[-C_1]\le 0$ and the property $\mu[A+B]\ge\mu[A]-\mu[-B]$ of the logarithmic norm. Hence

$$\begin{aligned}\|(E+\sigma\tau(C_1+\operatorname{div}_h \mathbf{b}))\,y_{n+1}\| &\ge -\mu_\infty[-E-\sigma\tau(C_1+\operatorname{div}_h \mathbf{b})]\,\|y_{n+1}\| \\ &\ge (1+\sigma\tau(\mu_\infty[-C_1]-\mu_\infty[-\operatorname{div}_h \mathbf{b}]))\,\|y_{n+1}\| \ge (1-\sigma\tau K)\,\|y_{n+1}\|,\end{aligned}$$

where, by (31), $K=\max_{\mathbf{x}\in\bar{\omega}}|\operatorname{div}_h \mathbf{b}|$. Under the above constraints (28) on the time increment, for the right-hand side of (38), we obtain

$$\begin{aligned}\|(E-(1-\sigma)\tau(C_1+\operatorname{div}_h \mathbf{b}))y_n\| &\le \|E-(1-\sigma)\tau C_1\|\,\|y_n\| + (1-\sigma)\tau\,\|\operatorname{div}_h \mathbf{b}\|\,\|y_n\| \\ &\le (1+(1-\sigma)\tau K)\,\|y_n\|.\end{aligned}$$

The substitution into (38) yields the layerwise estimate $(1-\sigma\tau K)\,\|y_{n+1}\|\le(1+(1-\sigma)\tau K)\,\|y_n\|$. Hence the ϱ-stability can be estimated as

$$\|y_{n+1}\|_\infty \le \varrho\,\|y_n\|_\infty, \qquad n=0,1,\ldots, \tag{39}$$

with $\varrho=\exp((1+\sigma)\tau K)$ under the additional [apart from (28)] constraints $\tau\le 3/(4\sigma K)$ for the time increment. The estimate (39) can be treated as a finite-difference analog of the estimate (31) for the differential problem (1), (2), (5).

Using approximations of the divergent convective transport in the form (23), we can obtain the ϱ-stability estimate (39) with the constant

$$K = \sum_{\alpha=1}^{2} \max_{\mathbf{x} \in \bar{\omega}} \left(\left| \left(b_{+}^{(\alpha)} \right)_{\bar{x}_\alpha} \right| + \left| \left(b_{-}^{(\alpha)} \right)_{x_\alpha} \right| \right),$$

which substantially differs from the constant $\mathscr{K}$ of the original differential problem. Thus, the use of approximations (34) with the definition of transport coefficients on displaced grids is preferable to the approximations (23) with coefficients defined at nodes.

MONOTONE AND CONSERVATIVE FINITE-DIFFERENCE SCHEMES

Schemes for which the maximum principle is valid are said to be *monotone.* We consider homogeneous two-layer finite-difference schemes that can be represented in the form

$$D(t_n)\, y_{n+1} = G(t_n)\, y_n, \qquad n = 0, 1, \ldots \tag{40}$$

This scheme is monotone if for nonnegative initial conditions ($y_0 \geq 0$), the solution is nonnegative for any other discrete value of time ($y_n \geq 0$, $n = 0, 1, \ldots$). Let us give sufficient conditions for the monotonicity of the finite-difference scheme (40) on the basis of the simplest results of the theory of nonnegative matrices in linear algebra [4, 11].

We restrict our consideration to related separate properties of the matrices $D(t_n)$ and $G(t_n)$. If $y_n \geq 0$, then $g_n = G(t_n)\, y_n$ is nonnegative provided that $G(t_n)$ is a nonnegative matrix, that is, a matrix with nonnegative entries. If $g_n \geq 0$, then the solution y_{n+1} of the equation $D(t_n)\, y_{n+1} = g_n$ is nonnegative provided that $D(t_n)$ is a monotone matrix. Therefore, the two-layer finite-difference scheme (40) is monotone if $G(t_n)$ is a nonnegative matrix and $D(t_n)$ is a monotone matrix.

The finite-difference scheme (17) can be rewritten in the form (40) with $D(t_n) = E + \sigma\tau A$ and $G(t_n) = E - (1-\sigma)\tau A$. Under standard conditions imposed on the weight, namely, $0 \leq \sigma \leq 1$, $G(t_n)$ is a nonnegative matrix if all of its offdiagonal elements are nonpositive:

$$a_{ij} \leq 0, \qquad i \neq j, \tag{41}$$

and the time increment satisfies condition (19).

We use the simplest criterion of the matrix monotonicity: a matrix with strict diagonal dominance and nonpositive offdiagonal entries is monotone. Consequently, the matrix $D(t_n)$ is monotone for any $\tau > 0$ if condition (41) is satisfied and the nonstrict diagonal dominance in rows [see (13)] or columns [see (14)] takes place for the matrix A.

Therefore, under the constraints (13) [or (14)] and (41), the two-layer weighted scheme (17), (18) is monotone provided that the time increment satisfies condition (19). In particular, the purely implicit scheme ($\sigma = 1$) is absolutely monotone (for any τ). Note that on the basis of the logarithmic norm, we have proved the stability in the corresponding spaces under weaker assumptions, namely, without the requirement (41). This permits one to prove the monotonicity of the above-considered two-layer finite-difference schemes with directed differences for the transport equation in divergent and nondivergent forms provided that the time increment satisfies condition (28). The additional condition (41) that offdiagonal entries are nonpositive is necessarily satisfied for schemes with directed differences.

Let us discuss the conservativeness of finite-difference schemes with directed differences for the transport equation (1), (5): the conservativeness of a finite-difference scheme is treated [1] in the sense of the layerwise relation $(y_{n+1}, 1) = (y_0, 1)$, $n = 0, 1, \ldots$, and is provided by the property

$$(C_2 y, 1) = 0 \tag{42}$$

of the finite-difference convective transport operator in divergent form. Since property (42) is valid for the finite-difference operator (34), it follows that the use of displaced grids for the definition of coefficients of the convective transport [the approximation (34)] yields a conservative scheme. The situation with the finite-difference operator (23), for which condition (42) fails, is more complicated. The requirement (3) preserves some freedom in the approximation of the normal transport component on the boundary. One of such possibilities is realized in (23). The second possibility is similar to (33) and is related to doubling the convective transport velocity on the boundary. In this case, instead of (23), we have

$$C_2 = \sum_{\alpha=1}^{2} C_2^{(\alpha)}, \qquad C_2^{(\alpha)} y = \begin{cases} 2\left(b_-^{(\alpha)} y\right)_{x_\alpha} & \text{if } x_\alpha = 0, \\ \left(b_+^{(\alpha)} y\right)_{\bar{x}_\alpha} + \left(b_-^{(\alpha)} y\right)_{x_\alpha} & \text{if } h_\alpha \le x_\alpha \le l_\alpha - h_\alpha, \\ 2\left(b_+^{(\alpha)} y\right)_{\bar{x}_\alpha} & \text{if } x_\alpha = l_\alpha, \end{cases} \qquad \alpha = 1, 2.$$

Property (42) is already valid for this convective transport operator; i.e., the corresponding schemes are conservative.

The above analysis allows one to extract the class of basic approximations for convective transport operators. Using central-difference approximations as well as directed approximations, one must aim the definition of convective transport coefficients at displaced grids. This provides the coordination of approximations of convective transport operators in various forms and, therefore, the stability of finite-difference schemes in related norms; the constants of ϱ-stability prove to be coordinated with the differential problem. In addition, finite-difference schemes are conservative for such approximations.

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REFERENCES

1. Samarskii, A.A. and Vabishchevich, P.N., *Differents. Uravn.*, 1998, vol. 34, no. 12, pp. 1675–1685.
2. Samarskii, A.A., *Teoriya raznostnykh skhem* (Theory of Finite-Difference Approximations), Moscow, 1989.
3. Samarskii, A.A. and Gulin, A.V., *Ustoichivost' raznostnykh skhem* (Stability of Finite-Difference Approximations), Moscow, 1973.
4. Gantmakher, F.R., *Teoriya matrits* (Theory of Matrices), Moscow, 1988.
5. Dekker, K. and Verwer, J.G., *Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations*, Amsterdam, 1984.
6. Hairer, E., Norsett, S.P., and Wanner, G., *Solving Ordinary Differential Equations. I. Nonstiff Problems*, Berlin, 1987.
7. Vabishchevich, P.N. and Samarskii, A.A., *Zh. Vychislit. Mat. Mat. Fiz.*, 1997, vol. 37, no. 2, pp. 182–186.
8. Vabishchevich, P.N. and Samarskii, A.A., *Chislennye metody resheniya zadach konvektsii–diffuzii* (Numerical Methods for Convection–Diffusion Problems), Moscow, 1999.
9. Desoer, C. and Haneda, H., *IEEE Trans. Circuit Theory*, 1972, vol. 19, pp. 480–486.
10. Maslennikova, V.N. and Bogovskii, M.E., *Dokl. RAN*, 1994, vol. 339, pp. 446–450.
11. Horn, R.A. and Johnson, C.R., *Matrix Analysis*, Cambridge, 1986.