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FINITE-DIFFERENCE METHODS FOR PROBLEM OF CONJUGATION OF HYPERBOLIC AND PARABOLIC EQUATIONS

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A problem of conjugation of hyperbolic and parabolic equations in domain with moving boundaries is considered. Existence and uniqueness of a strong solution of the given problem are proved. *A priori* estimate for operator-difference scheme with non-self-adjoint spatial operator is obtain. Homogeneous difference scheme with constant weights for the conjugation problem is constructed. Moreover, consistency conditions are approximated with the second-order of accuracy with respect to spatial variables. Stability and convergence of the suggested scheme are investigated.

1. Introduction

Problems of conjugation for two and more differential equations defined in different space subdomains and connected by some consistency conditions arise in the mathematical modeling of many phenomena in the media with different physical characteristics. For example, we obtain the conjugation problems of polytypic equations in the study of fluid flow in the channel surrounded by a porous medium, in phenomena of magnetic fluid dynamics etc.[7,12,13] In such case of consideration the type of equation is defined by the medium properties and the process character. Existence and uniqueness of a strong solution of the boundary value problems for such equations are considered in Refs. 1 and 2. Note that the questions related to

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numerical solution of one- and two-dimensional conjugation problems for polytypic equations in the rectangular domains are investigated in Refs. 6 and 11.

This paper concerns the two-dimensional problem of conjugation of hyperbolic and parabolic equations in domains with moving boundary. *A priori* estimate of stability for its solution is derived by means of the method of energy inequalities. A uniform three-layered difference scheme with constant weights[8] on the moving meshes is suggested for numerical solution of the problem. In this connection, consistency conditions are approximated with the second-order of accuracy with respect to spatial variables. Stability and convergence analysis for the scheme suggested is performed by the general theory of operator-difference schemes.[8]

2. Differential Problem

Let

$$Q = \{(t,\mathbf{x}):\ c_0 t < x_1 < l_1 + c_0 t, \quad 0 < x_2 < l_2, \quad 0 < t < T\}$$

be a bounded domain in the three-dimensional Euclidean space $\mathcal{R}^3$ of variables $(t,\mathbf{x}) = (t, x_1, x_2)$. Suppose Q is separated by the surface $\Gamma = \{(t,\mathbf{x}) :\ x_1 = \mathbf{x}i + c_0 t,\ 0 < \xi < l_1,\ 0 < x_2 < l_2,\ 0 < t < T\}$ into two subdomains, Q_1 and $Q_2:\ Q_1 = \{(t,\mathbf{x}):\ c_0 t < x_1 < \xi + c_0 t,\ 0 < x_2 < l_2,\ 0 < t < T\}$, $Q_2 = \{(t,\mathbf{x}) :\ \xi + c_0 t < x_1 < l_1 + c_0 t,\ 0 < x_2 < l_2,\ 0 < t < T\}$. The boundary ∂Q of Q consists of a lower base, $\Omega^0 = \{(t,\mathbf{x}) \in \partial Q:\ t = 0\}$, an upper base, $\Omega^{\mathrm{T}} = \{(t,\mathbf{x}) \in \partial Q:\ t = T\}$, and a side surface, $S = \{(t,\mathbf{x}) \in \partial Q:\ 0 < t < T\}$. The lower base $\overline{\Omega}^0$ consists of two parts: $\overline{\Omega}_1^0 = \overline{\Omega}^0 \cap \partial Q_1$ and $\overline{\Omega}_2^0 = \overline{\Omega}^0 \cap \partial Q_2$ ($\overline{\Omega}^0$ and $\overline{\Omega}_i^0$ are closures of Ω^0 and Ω_i^0, $i = 1, 2$, respectively).

In Q_1 we shall consider equation of hyperbolic type with respect to desired function $u^{(1)}(t,\mathbf{x})$

$$\frac{\partial^2 u^{(1)}}{\partial t^2} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i^{(1)}(\mathbf{x})\frac{\partial u^{(1)}}{\partial x_i}\right) + f^{(1)}(t,\mathbf{x})\,, \quad (t,\mathbf{x}) \in Q_1\,, \tag{2.1}$$

and in Q_2 we shall consider parabolic equation with respect to function $u^{(2)}(t,\mathbf{x})$

$$\frac{\partial u^{(2)}}{\partial t} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i^{(2)}(\mathbf{x})\frac{\partial u^{(2)}}{\partial x_i}\right) + f^{(2)}(t,\mathbf{x})\,, \quad (t,\mathbf{x}) \in Q_2\,, \tag{2.2}$$

where $k_i^{(m)}(\mathbf{x}) \in C^1(\overline{Q}_m)$, $0 < c_1 \le k_i^{(m)}(\mathbf{x}) \le c_2$, $i = 1, 2$, $m = 1, 2$.

In addition, assume that the coefficients of Eq. (2.1) satisfy the following condition

$$k_1^{(1)}(\mathbf{x}) - c_0^2 \ge \delta > 0\,. \tag{2.3}$$

Equations (2.1) and (2.2) are supplemented with the following boundary and initial conditions:

$$u|_S = 0\,, \quad (t,\mathbf{x}) \in S\,, \tag{2.4}$$

$$u|_{\Omega^0} = u_0(\mathbf{x}), \quad \left.\frac{\partial u^{(1)}}{\partial t}\right|_{\Omega_1^0} = u_1^{(1)}(\mathbf{x}), \tag{2.5}$$

where

$$u(t,\mathbf{x}) = \begin{cases} u^{(1)}(t,\mathbf{x}), & (t,\mathbf{x}) \in \overline{Q}_1, \\ u^{(2)}(t,\mathbf{x}), & (t,\mathbf{x}) \in \overline{Q}_2, \end{cases}$$

$$u_0(\mathbf{x}) = \begin{cases} u_0^{(1)}(\mathbf{x}), & (0,\mathbf{x}) \in \overline{\Omega}_1^0, \\ u_0^{(2)}(\mathbf{x}), & (0,\mathbf{x}) \in \overline{\Omega}_2^0. \end{cases}$$

At the interface Γ, the following consistency conditions are valid

$$\begin{gathered} u^{(1)}|_\Gamma = u^{(2)}|_\Gamma, \\ \left.\left(c_0 \frac{\partial u^{(1)}}{\partial t} + k_1^{(1)}(\mathbf{x}) \frac{\partial u^{(1)}}{\partial x_1}\right)\right|_\Gamma = \left.\left(k_1^{(2)}(\mathbf{x}) \frac{\partial u^{(2)}}{\partial x_1}\right)\right|_\Gamma. \end{gathered} \tag{2.6}$$

2.1. *Existence and uniqueness of a strong solution*

Let $\mathcal{B}$ be a Banach space, obtained by closure of a set $\{u\colon\ u^{(m)} \in C^2(\overline{Q}_m)(m=1,2),$ u satisfies the conditions (2.4) and (2.6)$\}$ with respect to the norm

$$\|u\|_{\mathcal{B}} = \left\| \frac{\partial u^{(1)}}{\partial t} \right\|_{L_2(Q_1)} + \sup_{0\le t\le T} \left(\left\| \frac{\partial u^{(2)}}{\partial t} \right\|_{L_2(\Omega^{(1)}(t))} + \sum_{m=1}^{2} \sum_{i=1}^{2} \left\| \frac{\partial u^{(m)}}{\partial x_i} \right\|_{L_2(\Omega^{(m)}(t))} \right),$$

where $\Omega^{(m)}(t)$ is a section of the subdomain $Q_m (m = 1, 2)$ by the plane $\{(t,\mathbf{x}) \in \mathcal{R}^3\colon$ $t = \text{const.}\}$, $\|\cdot\|_{L_2}$ is a norm in a space, L_2, of Lebesgue integrable functions whose squares are also Lebesgue integrable. Let $\overset{\circ}{H}{}^1(\Omega^0)$ be a Hilbert space that consists of functions $u \in L_2(\Omega^0)$ ($u = 0$ on $\overline{\Omega}^0 \cap \overline{S}$) whose first-order weak derivatives are also elements of $L_2(\Omega^0)$. The norm in $\overset{\circ}{H}{}^1(\Omega^0)$ is $\|\cdot\|_{\overset{\circ}{H}{}^1(\Omega^0)} = \|\cdot\|_{L_2(\Omega^0)} + \sum_{i=1}^2 \|\frac{\partial\cdot}{\partial x_i}\|_{L_2(\Omega^0)}$.

Denote by $\mathcal{L}$ the differential operator $\mathcal{L}u = (\mathcal{L}^{(1)}u^{(1)}, \mathcal{L}^{(2)}u^{(2)})$, where $\mathcal{L}^{(m)}\cdot = \frac{\partial^{3-m}\cdot}{\partial t^{3-m}} - \sum_{i=1}^2 \frac{\partial}{\partial x_i}(k_i^{(m)} \frac{\partial\cdot}{\partial x_i})$, $m = 1, 2$. We can consider the problem (2.1), (2.2), (2.4)–(2.6) as the following operator equation

$$Lu = F,$$

$$Lu = (\mathcal{L}u, l_0 u, l_1 u^{(1)}), \quad l_0 u = u|_{\Omega^0}, \quad l_1 u^{(1)} = \left.\frac{\partial u^{(1)}}{\partial t}\right|_{\Omega_1^0},$$

$$F = (f(t,\mathbf{x}), u_0(\mathbf{x}), u_1^{(1)}(\mathbf{x})), \quad f(t,\mathbf{x}) = \begin{cases} f^{(1)}(t,\mathbf{x}), & (t,\mathbf{x}) \in Q_1, \\ f^{(2)}(t,\mathbf{x}), & (t,\mathbf{x}) \in Q_2, \end{cases}$$

acting from $\mathcal{B}$ onto $\mathcal{H} = L_2(Q) \times \overset{\circ}{H}{}^1(\Omega^0) \times L_2(\Omega_1^0)$ and whose domain of definition is $\mathcal{D}(L) = \{u(t,\mathbf{x})\colon\ u^{(m)}(t,\mathbf{x}) \in C^2(\overline{Q}_m),\ m = 1, 2,\ u(t,\mathbf{x})$ satisfies the conditions (2.4) and (2.6)$\}$.

For the differential problem (2.1), (2.2), (2.4)–(2.6), the following theorem is valid.

Theorem 1. *Suppose that* $k_i^{(m)}(\mathbf{x}) \in C^1(\overline{Q}_m)$, $0 < c_1 \le k_i^{(m)}(\mathbf{x}) \le c_2$, $i = 1,2$, $m = 1,2$ *and assume that the condition* (2.3) *holds*; *then for the conjugation problem* (2.1), (2.2), (2.4)–(2.6), *the following estimate is valid*

$$\|u\|_{\mathcal{B}} \le c\|Lu\|_{\mathcal{H}} = c(\|\mathcal{L}u\|_{L_2(Q)} + \|l_0 u\|_{\overset{\circ}{H}{}^1(\Omega^0)} + \|l_1 u^{(1)}\|_{L_2(\Omega_1^0)} + ++), \quad c > 0\,. \tag{2.7}$$

Proof. To prove the theorem let us multiply the expressions

$$\mathcal{L}^{(m)} u^{(m)} = \frac{\partial^{3-m} u^{(m)}}{\partial t^{3-m}} - \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i^{(m)} \frac{\partial u^{(m)}}{\partial x_i}\right)$$

by $2\frac{\partial u^{(m)}}{\partial r} = \frac{\partial u^{(m)}}{\partial t} + c_0 \frac{\partial u^{(m)}}{\partial x_1}$ and integrate the product over $Q_m^{\tilde t} = \{(t,\mathbf{x}) \in Q_m : 0 < t < \tilde t \le T\}$, $m = 1,2$. Using the Ostrogradsky theorem and the identities

$$2\frac{\partial^2 u^{(1)}}{\partial t^2}\frac{\partial u^{(1)}}{\partial r} = 2\frac{\partial}{\partial t}\left(\frac{\partial u^{(1)}}{\partial t}\frac{\partial u^{(1)}}{\partial r}\right) - \frac{\partial}{\partial r}\left(\left(\frac{\partial u^{(1)}}{\partial t}\right)^2\right),$$

$$\begin{aligned} 2\frac{\partial}{\partial x_i}\left(k_i^{(m)}\frac{\partial u^{(m)}}{\partial x_i}\right)\frac{\partial u^{(m)}}{\partial r} &= 2\frac{\partial}{\partial x_i}\left(k_i^{(m)}\frac{\partial u^{(m)}}{\partial x_i}\frac{\partial u^{(m)}}{\partial r}\right) \\ &- \frac{\partial}{\partial r}\left(k_i^{(m)}\left(\frac{\partial u^{(m)}}{\partial x_i}\right)^2\right) + \frac{\partial k_i^{(m)}}{\partial r}\left(\frac{\partial u^{(m)}}{\partial x_i}\right)^2, \quad i = 1,2\,, \quad m = 1,2\,, \end{aligned}$$

we obtain the following relations:

$$\mathcal{I}_m^{\tilde t} + \int_{Q_m^{\tilde t}} \Im^{(m)}(u^{(m)})\,dt\,d\mathbf{x} = \int_{Q_m^{\tilde t}} f^{(m)}(t,\mathbf{x})\frac{\partial u^{(m)}}{\partial r}dt\,d\mathbf{x}\,, \quad m = 1,2\,. \tag{2.8}$$

Here

$$\begin{aligned} \mathcal{I}_1^{\tilde t} &= \int_{\partial Q_1^{\tilde t}} \Im_0(u^{(1)})\,ds = \int_{\partial Q_1^{\tilde t}} \left(2\frac{\partial u^{(1)}}{\partial t}\frac{\partial u^{(1)}}{\partial r}\nu_0 - \left(\frac{\partial u^{(1)}}{\partial t}\right)^2 r_\nu \right. \\ &\left. + \sum_{i=1}^{2} k_i^{(1)}\left(\frac{\partial u^{(1)}}{\partial x_i}\right)^2 r_\nu - 2\sum_{i=1}^{2} k_i^{(1)}\frac{\partial u^{(1)}}{\partial x_i}\frac{\partial u^{(1)}}{\partial r}\nu_i\right) ds\,, \end{aligned}$$

$$\mathcal{I}_2^{\tilde t} = \left\|\frac{\partial u^{(2)}}{\partial t}\right\|^2_{L_2(Q_2^{\tilde t})} + \int_{\partial Q_2^{\tilde t}} \left(\sum_{i=1}^{2} k_i^{(2)}\left(\frac{\partial u^{(2)}}{\partial x_i}\right)^2 r_\nu - 2\sum_{i=1}^{2} k_i^{(2)}\frac{\partial u^{(2)}}{\partial x_i}\frac{\partial u^{(2)}}{\partial r}\nu_i\right) ds\,,$$

and $\Im^{(m)}(u^{(m)})$ $(m = 1,2)$ are quadratic forms of the first-order derivatives of $u^{(m)}$ and $\nu(t,\mathbf{x}) = (\nu_0(t,\mathbf{x}), \nu_1(t,\mathbf{x}), \nu_2(t,\mathbf{x}))$ denotes the outward normal vector to the domain $Q_m^{\tilde t}(m = 1,2)$, $|\nu| = 1$, $r_\nu = \nu_0 + \nu_1 c_0$.

To obtain the lower bound of the integral $\int_{\partial Q_1^{\tilde t}\cap\partial Q^{\tilde t}} \Im_0(u^{(1)})\,ds$ consider it as a sum

$$\int_{\partial Q_1^{\tilde t}\cap\partial Q^{\tilde t}} \Im_0(u^{(1)})\,ds = \int_{\Omega^{(1)}(\tilde t)} \Im_0(u^{(1)})\,ds + \int_{\partial Q_1^{\tilde t}\cap S} \Im_0(u^{(1)})\,ds + \int_{\Omega_1^0} \Im_0(u^{(1)})\,ds \tag{2.9}$$

taking into account the given conditions (2.4) and (2.5). For every point $(t,\mathbf{x}) \in \partial Q_1^{\tilde t}\cap\partial Q^{\tilde t}$ we shall estimate $\Im_0(u^{(1)})$.

Since $\frac{\partial u^{(i)}}{\partial r} = 0$ $(m = 1,2)$ and $r_\nu = 0$ on S, it follows that $\Im_0(u^{(1)}) = 0$ at $(t,\mathbf{x}) \in S$ and, consequently,

$$\int_{\partial Q_1^{\tilde t}\cap S} \Im_0(u^{(1)})\,ds = 0\,.$$

We shall consider the integrand $\Im_0(u^{(1)})$ in the integral $\int_{\Omega^{(1)}(\tilde t)} \Im_0(u^{(1)})$ over upper base $\Omega^{(1)}(\tilde t)$ of the subdomain $Q_1^{\tilde t}$ as a quadratic form of derivatives $\partial u^{(1)}/\partial t$, $\partial u^{(1)}/\partial x_1$, $\partial u^{(1)}/\partial x_2$. Obviously we have $\nu_0(\tilde t,\mathbf{x}) = 1$, $\nu_1(\tilde t,\mathbf{x}) = \nu_2(\tilde t,\mathbf{x}) = 0$, and $r_\nu = 1$ when $(\tilde t,\mathbf{x}) \in \Omega^{(1)}(\tilde t)$. Thus,

$$\begin{aligned}\Im_0(u^{(1)}) &= 2\frac{\partial u^{(1)}}{\partial t}\frac{\partial u^{(1)}}{\partial r} - \left(\frac{\partial u^{(1)}}{\partial t}\right)^2 + \sum_{i=1}^{2} k_i^{(1)}\left(\frac{\partial u^{(1)}}{\partial x_i}\right)^2 \\ &= \left(\frac{\partial u^{(1)}}{\partial t}\right)^2 + 2c_0\frac{\partial u^{(1)}}{\partial t}\frac{\partial u^{(1)}}{\partial x_1} + \sum_{i=1}^{2} k_i^{(1)}\left(\frac{\partial u^{(1)}}{\partial x_i}\right)^2, \quad (\tilde t,\mathbf{x}) \in \Omega^{(1)}(\tilde t)\,.\end{aligned}$$

To obtain the lower bound for the quadratic form and then for the expression $\int_{\Omega^{(1)}(\tilde t)} \Im_0(u^{(1)})$, we shall use the quadratic forms positivity criterion (Sylvestr's criterion). The matrix of the form $\Im_0(u^{(1)})$ is

$$\begin{pmatrix} 1 & c_0 & 0 \\ c_0 & k_1^{(1)}(\mathbf{x}) & 0 \\ 0 & 0 & k_2^{(1)}(\mathbf{x}) \end{pmatrix}.$$

According to Sylvestr's criterion, positivity of $\Im_0(u^{(1)})$ is determined by the main minors $d_1(t,\mathbf{x})$, $d_2(t,\mathbf{x})$ and $d_2(t,\mathbf{x})$ of its matrix. We have

$$\begin{aligned} d_1(t,\mathbf{x}) &= 1 > 0\,, \\ d_2(t,\mathbf{x}) &= k_1^{(1)}(\mathbf{x}) - c_0^2 \geq \delta > 0\,, \\ d_3(t,\mathbf{x}) &= k_2^{(1)} d_2(t,\mathbf{x}) \geq c_1\delta > 0\,. \end{aligned}$$

Here we use conditions on the coefficients of Eqs. (2.1) and (2.2) and the relation (2.3). Hence,

$$\int_{\Omega^{(1)}(\tilde t)} \Im_0(u^{(1)})\,ds \geq c_3 \int_{\Omega^{(1)}(\tilde t)} \left(\left(\frac{\partial u^{(1)}}{\partial t}\right)^2 + \left(\frac{\partial u^{(1)}}{\partial x_1}\right)^2 + \left(\frac{\partial u^{(1)}}{\partial x_2}\right)^2\right) d\mathbf{x}\,.$$

Estimating the expression $\int_{\Omega_1^0} \Im_0(u^{(1)})\,ds$ from above, we obtain

$$\int_{\Omega_1^0} \Im_0(u^{(1)})\,ds \le c_4(\|u_0\|^2_{H^1(\Omega^0)} + \|u_1^{(1)}\|^2_{L_2(\Omega_1^0)})\,.$$

Thus we deduce that

$$\int_{\partial Q_1^{\tilde t}\cap\partial Q^{\tilde t}} \Im_0(u^{(1)}) \ge c_3 \int_{\Omega^{(1)}(\tilde t)} \left(\left(\frac{\partial u^{(1)}}{\partial t}\right)^2 + \sum_{i=1}\left(\frac{\partial u^{(1)}}{\partial x_i}\right)^2\right)(\tilde t, \mathbf{x})\,d\mathbf{x}$$
$$- c_4(\|u_0\|^2_{H^1(\Omega^0)} + \|u_1^{(1)}\|^2_{L_2(\Omega_1^0)})\,. \tag{2.10}$$

Now if we recall the restrictions on coefficients of Eqs. (2.1), (2.2) and the boundary conditions, we get

$$\int_{\partial Q_2^{\tilde t}\cap\partial Q^{\tilde t}} \sum_{i=1}^{2} k_i^{(2)}(\mathbf{x})\left(\left(\frac{\partial u^{(2)}}{\partial x_i}\right)^2 r_\nu - 2\frac{\partial u^{(2)}}{\partial r}\frac{\partial u^{(2)}}{\partial x_i}\nu_i\right) ds$$
$$\ge c_5 \int_{\Omega^{(2)}(\tilde t)} \sum_{i=1}^{2}\left(\frac{\partial u^{(2)}}{\partial x_i}\right) d\mathbf{x} - c_6\|u_0\|^2_{H^1(\Omega^0)}\,. \tag{2.11}$$

It follows from the first consistency condition that

$$\left.\frac{\partial u^{(1)}}{\partial r}\right|_\Gamma = \left.\frac{\partial u^{(2)}}{\partial r}\right|_\Gamma .$$

Since $\nu_0 = -\frac{c_0}{\sqrt{1+c_0^2}}$, $\nu_1 = \frac{1}{\sqrt{1+c_0^2}}$ on the interface Γ, we obtain

$$\int_{\partial Q_1^{(\tilde t)}\cap\Gamma}\left(2\frac{\partial u^{(1)}}{\partial t}\frac{\partial u^{(1)}}{\partial r}\nu_0 - \left(\frac{\partial u^{(1)}}{\partial t}\right)^2 r_\nu\right.$$
$$\left. + \sum_{i=1}^{2} k_i^{(1)}\left(\frac{\partial u^{(1)}}{\partial x_i}\right)^2 r_\nu - 2\sum_{i=1}^{2} k_i^{(1)}\frac{\partial u^{(1)}}{\partial r}\frac{\partial u^{(1)}}{\partial x_i}\nu_i\right) ds$$
$$+ \int_{\partial Q_2^{(\tilde t)}\cap\Gamma}\sum_{i=1}^{2} k_i^{(2)}\left(\left(\frac{\partial u^{(2)}}{\partial x_i}\right)^2 r_\nu - 2\frac{\partial u^{(2)}}{\partial r}\frac{\partial u^{(2)}}{\partial x_i}\nu_i\right) ds$$
$$= \frac{2}{\sqrt{1+c_0^2}}\int_{\partial Q_1^{(\tilde t)}\cap\Gamma}\frac{\partial u}{\partial r}\left(-c_0\frac{\partial u^{(1)}}{\partial t} - k_1^{(1)}\frac{\partial u^{(1)}}{\partial x_1} + k_1^{(2)}\frac{\partial u^{(2)}}{\partial x_1}\right) ds = 0\,. \tag{2.12}$$

Here we also use the second consistency condition.

Using the Cauchy–Schwarz inequality, we get the following estimates for the second term on the left-hand side and for the right-hand side of the equalities (2.8):

$$\int_{Q_m^{\tilde t}} \Im^{(m)}(u^{(m)})\,dt\,d\mathbf{x} \le \varepsilon_1\left\|\frac{\partial u^{(2)}}{\partial t}\right\|^2_{L_2(Q_2^{\tilde t})}$$

$$+ c_7(\varepsilon_1) \left(\left\| \frac{\partial u^{(1)}}{\partial t} \right\|^2_{L_2(Q_1^{\tilde t})} + \sum_{m=1}^{2} \sum_{i=1}^{2} \left\| \frac{\partial u^{(m)}}{\partial x_i} \right\|^2_{L_2(Q_m^{\tilde t})} \right),$$

$$\int_{Q_m^{\tilde t}} f^{(m)}(t,\mathbf{x}) \frac{\partial u^{(i)}}{\partial r} dt\, d\mathbf{x} \le \varepsilon_2 \left\| \frac{\partial u^{(2)}}{\partial t} \right\|^2_{L_2(Q_2^{\tilde t})}$$

$$+ c_8(\varepsilon_2) \left(\left\| \frac{\partial u^{(1)}}{\partial t} \right\|^2_{L_2(Q_1^{\tilde t})} + \sum_{m=1}^{2} \sum_{i=1}^{2} \left\| \frac{\partial u^{(m)}}{\partial x_i} \right\|^2_{L_2(Q_m^{\tilde t})} \right)$$

$$+ c_9 \|f\|^2_{L_2(Q^{\tilde t})}, \quad m = 1, 2\,.$$

Taking into account the estimates (2.10), (2.11), the last inequalities and (2.12) we sum the equalities (2.8) for $m = 1, 2$. As a result, selecting appropriate values of ε_1 and ε_2, we obtain

$$\left\| \frac{\partial u^{(2)}}{\partial t} \right\|^2_{L_2(Q_2^{\tilde t})} + \left\| \frac{\partial u^{(1)}}{\partial t} \right\|^2_{L_2(\Omega^{(1)}(\tilde t))} + \sum_{m=1}^{2} \sum_{i=1}^{2} \left\| \frac{\partial u^{(m)}}{\partial x_i} \right\|^2_{L_2(\Omega^{(i)}(\tilde t))}$$

$$\le c_{10} \left(\left\| \frac{\partial u^{(1)}}{\partial t} \right\|^2_{L_2(Q_1^{\tilde t})} + \sum_{m=1}^{2} \sum_{i=1}^{2} \left\| \frac{\partial u^{(m)}}{\partial x_i} \right\|^2_{L_2(Q_i^{\tilde t})} \right.$$

$$\left. + \|f\|^2_{L_2(Q^{\tilde t})} + \|u_0\|^2_{H^1(\Omega^0)} + \left\| \frac{\partial u_1^{(1)}}{\partial r} \right\|^2_{L_2(\Omega_1^0)} \right).$$

The application of Gronwall's lemma yields the required estimate (2.7). □

Operator $L\colon \mathcal{B} \to \mathcal{H}$ admits a closure $\overline{L}$.[4] The solution of the operator equation $\overline{L}u = F$ is a *strong solution* of problem (2.1), (2.2), (2.4)–(2.6).

Theorem 2. *Under the conditions of Theorem* 1 *for arbitrary* $F \in \mathcal{H}$ *there exists a unique strong solution* $u \in \mathcal{B}$ *of problem* (2.1), (2.2), (2.4)–(2.6). *In addition,*

$$\|u\|_{\mathcal{B}} \le c\|F\|_{\mathcal{H}}, \quad c > 0\,. \tag{2.13}$$

The estimate (2.13) and uniqueness of the solution follow from the energy inequality (2.7). To prove the existence of the strong solution of problem (2.1), (2.2), (2.4)–(2.6) for arbitrary $F \in \mathcal{H}$, it is sufficient to prove that a set of values of the operator $\mathcal{L}$ is a compact set in $\mathcal{H}$.[4] We can do it by means of averaging operators with variable step[3] following the plan of similar proofs in Refs. 3 and 5 and using a technique for obtaining of the energy inequality (2.7).

3. Auxiliary Results

In this section we shall obtain a stability estimate for the operator-difference scheme of the form

$$Dy_{\bar t t} + By_t + Ay = \varphi\,, \quad y(0) = y_0\,, \quad y_t(0) = y_1\,, \tag{3.1}$$

where $y \in H$, H is a real finite-dimensional Hilbert space with an inner product $(\cdot,\cdot)$ and a norm $\|\cdot\|$. The operators A, B, $D:\ H \to H$ are linear ones in H, moreover

$$B \geq 0\,, \quad D = D^* > 0\,.$$

Here we use the standard notation of the difference schemes theory[8]

$$y_{\bar{t}t} = \frac{1}{\tau}(y_t - y_{\bar{t}})\,, \quad y_t = \frac{\hat{y} - y}{\tau}\,, \quad y_t = \frac{y - \check{y}}{\tau}\,,$$

$$y = y(t)\,, \quad \hat{y} = y(t+\tau)\,, \quad \check{y} = y(t-\tau)\,.$$

Let $R = R^* > 0$. Denote by H_R a Hilbert space of elements of H which is equipped with the inner product $(y,v)_R = (Ry,v)$ and the norm $\|y\|_R^2 = (y,y)_R$.

Using the method of energy inequalities, we can deduce sufficient conditions of stability of the scheme (3.1) with non-self-adjoint operator A:

$$A = A_0 + A_1\,, \quad A_0 = A_0^* > 0\,,$$

provided that the following subordination condition is valid

$$\|A_1 y\|^2 \leq \alpha_0 (A_0 y, y)\,, \quad y \in H\,. \tag{3.2}$$

Here constant $\alpha_0 > 0$ does not depend on τ.

Theorem 3. *Let the operators D and A_0 of the scheme* (3.1) *be positive and self-adjoint. Assume that*

$$B_0 = 0.5(B + B^*) \geq \varepsilon U + 0.5\tau A_0\,, \quad \varepsilon = \text{const.} > 0\,, \quad D > L\,, \tag{3.3}$$

where $L = L^* \geq 0$, $U = U^* \geq 0$, $L + U = E$, and assume that the subordination condition (3.2) *holds; then for the solution of the operator-difference scheme* (3.1) *the following a priori estimate is valid*

$$\|y_{n+1}\|_{A_0}^2 \leq M \left(\|y_0\|_{A_0}^2 + \|y_{t,0}\|_{D+2\tau^2 A_0}^2 + M_1 \sum_{k=0}^{n} \tau \|\varphi_k\|^2 \right)\,. \tag{3.4}$$

Here $M = 2e^{\alpha_1 t_{n+1}}$, $\alpha_1 = \max\{1, 2\varepsilon, 2\alpha_0(1/\varepsilon + T)\}$, $M_1 = \max\{2, 2/\varepsilon\}$.

Proof. To prove the theorem, rewrite the scheme (3.1) in the form

$$D y_{\bar{t}t} + B y_t + A_0 y = \Phi\,, \quad \Phi = \varphi - A_1 y\,. \tag{3.5}$$

Consider an inner product of Eq. (3.5) onto term $2\tau y_t$. Taking into account the relations

$$2\tau(D y_{\bar{t}t}, y_t) = (D y_t, y_t) - (D y_{\bar{t}}, y_{\bar{t}}) + \tau^2 (D y_{\bar{t}t}, y_{\bar{t}t})\,,$$

$$2\tau(A_0 y, y_t) = (A_0 \widehat{y}, \widehat{y}) - (A_0 y, y) - \tau^2 (A_0 y_t, y_t)\,,$$

$$2\tau(B y_t, y_t) = 2\tau(B_0 y_t, y_t)\,,$$

we obtain the following energy identity

$$\|y_{t,n}\|_D^2 + \|y_{n+1}\|_{A_0}^2 + \tau^2\|y_{\bar{t}t,n}\|_D^2 + 2\tau((B_0 - 0.5\tau A_0)y_t, y_t)$$
$$= \|y_{t,n-1}\|_D^2 + \|y_n\|_{A_0}^2 + 2\tau(\Phi_n, y_{t,n})\,. \tag{3.6}$$

Since

$$2\tau(\Phi, y_t) = 2\tau((L+U)\Phi, y_t)\,,$$

estimating the right-hand side by the Cauchy inequality and ε-inequality

$$2\tau(U\Phi, y_t) \le 2\tau\varepsilon_1(Uy_t, y_t) + \frac{\tau}{2\varepsilon_1}\|\Phi\|^2\,,$$

$$\begin{aligned} 2\tau(L\Phi, y_t) &= 2\tau(L\Phi, y_{\bar{t}}) + 2\tau^2(L\Phi, y_{\bar{t}t}) \\ &\le 2\tau\varepsilon_2(Ly_{\bar{t}}, y_{\bar{t}}) + \frac{\tau}{2\varepsilon_2}(L\Phi, \Phi) + 2\tau^2\varepsilon_3(Ly_{\bar{t}t}, y_{\bar{t}t}) + \frac{\tau^2}{2\varepsilon_3}(L\Phi, \Phi) \\ &\le 2\tau\varepsilon_2\|y_{\bar{t}}\|_D^2 + \frac{\tau}{2\varepsilon_2}\|\Phi\|^2 + \frac{\tau^2}{2\varepsilon_3}\|\Phi\|^2\,, \end{aligned}$$

we get the following bound

$$\|y_{t,n}\|_D^2 + \|y_{n+1}\|_{A_0}^2 + \tau^2(1-2\varepsilon_3)\|y_{\bar{t}t,n}\|_D^2 + 2\tau((B_0 - \varepsilon_1 U - 0.5\tau A_0)y_{t,n}, y_{t,n})$$
$$\le \|y_{t,n-1}\|_D^2 + \|y_n\|_{A_0}^2 + 2\tau\varepsilon_2\|y_{t,n-1}\|_D^2 + 0.5\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)$$
$$\times\,\tau\|\Phi_n\|^2 + 2\varepsilon_3\tau^2\|\Phi_n\|^2\,.$$

Using the subordination condition (3.2) for $\|\Phi_n\|^2$, we have

$$\|\Phi_n\|^2 = \|\varphi_n - A_1 y_n\|^2 \le 2\|\varphi_n\|^2 + 2\|A_1 y_n\|^2 \le 2\|\varphi_n\|^2 + 2\alpha_0\|y_n\|_{A_0}^2\,.$$

Further, selecting $\varepsilon_3 = 0.5$, $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and recalling (3.3), we obtain

$$\|y_{t,n}\|_D^2 + \|y_{n+1}\|_{A_0}^2 \le (1+2\varepsilon\tau)\|y_{t,n-1}\|_D^2 + \left(1 + \left(\frac{2\alpha_0}{\varepsilon} + 2\alpha_0 T\right)\tau\right)\|y_n\|_{A_0}^2$$
$$+ \frac{2}{\varepsilon}(1+\tau)\tau\|\varphi_n\|^2 + 2(1+\tau)\tau\|\varphi_n\|^2\,.$$

Hence,

$$\|y_{t,k}\|_D^2 + \|y_{k+1}\|_{A_0}^2 \le (1+\alpha_1\tau)(\|y_{t,k-1}\|_D^2 + \|y_k\|_{A_0}^2 + M_1\tau\|\varphi_k\|^2)\,.$$

Summing the last inequality over $k = 1, 2, \ldots, n$, we get the following estimate

$$\|y_{n+1}\|_{A_0}^2 \le (1+\alpha_1\tau)^{n+1}(\|y_{t,0}\|_D^2 + \|y_1\|_{A_0}^2) + (1+\alpha_1\tau)^n M_1 \sum_{k=0}^{n} \tau\|\varphi_k\|^2\,.$$

Further, taking into account the relation

$$\|y_1\|_{A_0}^2 = \|y_0 + \tau y_{t,0}\|_{A_0}^2 \le 2\|y_0\|_{A_0}^2 + 2\tau^2\|y_{t,0}\|_{A_0}^2$$

and inequality $1 + \alpha_1\tau \le e^{\alpha_1\tau}$, we obtain the desired estimate (3.4). □

Note that in Ref. 10 an analogous estimate is obtained for the operator-difference scheme

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi\,, \quad y(0) = y_0\,, \quad y_t(0) = y_1\,,$$

with non-self-adjoint operator A, provided the condition (3.2) is fulfilled. Though the operator B here has satisfy stronger inequality:

$$B_0 \geq \varepsilon E\,, \quad \varepsilon > 0\,.$$

4. Difference Scheme for the Conjugation Problem

In this section we construct and investigate the numerical method for the conjugation problem (2.1), (2.2), (2.4)–(2.6).

Let

$$\omega_{h\tau} = \omega_h \times \omega_\tau$$

be a uniform moving mesh in the domain Q. Here

$$\overline{\omega}_h = \{\mathbf{x}^j_{i_1 i_2} = (x^j_{1i_1}, x_{2i_2}) :\ x^j_{1i_1} = i_1 h_1 + c_0 t_j\,; \quad x_{2i_2} = i_2 h_2\,,$$
$$0 \leq i_k \leq N_k\,, \quad h_k N_k = l_k\,, \quad k = 1, 2\}\,,$$
$$\omega_\tau = \{t_j = j\tau\,; \quad 0 < j \leq N_0 - 1\,, \quad \tau N_0 = T\}\,.$$

The set

$$\omega_h = \{\mathbf{x}^j_{i_1 i_2} :\ 0 < i_k < N_k\,, \quad k = 1, 2\}$$

is a set of interior mesh-points of $\overline{\omega}_h$, and

$$\partial\omega_h = \overline{\omega}_h \backslash \omega_h = \{\mathbf{x}^j_{i_1 i_2} :\ i_1 = 0, N_1\,, \quad 0 < i_2 < N_2 \quad 0 < i_1 < N_1\,, \quad i_2 = 0, N_2\}$$

is a set of boundary mesh-points of $\overline{\omega}_h$.

We assume that the interface Γ contains the mesh-points of $\omega_{h\tau}$, and denote this set by

$$\gamma_h = \{\mathbf{x}^j_{p_1 i_2} = (x^j_{p_1}, x^j_{i_2}) :\ x^j_{1p_1} = p_1 h_1 + c_0 t_j\,, \quad p_1 h_1 = \mathbf{x}i\,, \quad 0 < i_2 < N_2\}\,,$$

where $2 \leq p_1 \leq N_1 - 2$. In addition, in the domains Q_1 and Q_2 we shall consider the following meshes

$$\omega_1 = \omega_{1h} \times \omega_\tau\,, \quad \omega_2 = \omega_{2h} \times \omega_\tau\,.$$

Here

$$\omega_{1h} = \{\mathbf{x}^j_{i_1 i_2} :\ 0 < i_1 < p_1\,, \quad 0 < i_2 < N_2\}\,,$$
$$\omega_{2h} = \{\mathbf{x}^j_{i_1 i_2} :\ p_1 < i_1 < N_1\,, \quad 0 < i_2 < N_2 - 1\}\,.$$

On the moving mesh $\omega_{h\tau}$, we approximate the differential problem (2.1), (2.2), (2.4)–(2.6) by a three-layered scheme

$$y_{\bar{t}t} = (a_1 y_{\bar{x}_1})_{x_1}^{(\sigma_1,\sigma_2)} + (a_2 y_{\bar{x}_2})_{x_2}^{(\sigma_1,\sigma_2)} + 2c_0 y_{t\overset{\circ}{x}_1} + \varphi\,, \quad (t,\mathbf{x}) \in \omega_1\,, \tag{4.1}$$

$$y_t = (a_1 y_{\bar{x}_1})_{x_1}^{(\sigma_1,\sigma_2)} + (a_2 y_{\bar{x}_2})_{x_2}^{(\sigma_1,\sigma_2)} + c_0 y_{\overset{\circ}{x}_1} + \varphi\,, \quad (t,\mathbf{x}) \in \omega_2\,, \tag{4.2}$$

$$y|_{\partial\omega_h} = 0\,, \quad (t,\mathbf{x}) \in \partial\omega_h\,, \tag{4.3}$$

$$y(0,\mathbf{x}) = u_0(\mathbf{x})\,, \quad y_t(0,\mathbf{x}) = u_1^{(1)}(\mathbf{x})\,, \quad \mathbf{x} \in \omega_{1h}^{+}\,, \quad \omega_{1h}^{+} = \omega_{1h} \cup \gamma_h\,, \tag{4.4}$$

$$y(0,\mathbf{x}) = u_0(\mathbf{x})\,, \quad y_t(0,\mathbf{x}) = u_1^{(2)}(\mathbf{x})\,, \quad \mathbf{x} \in \omega_{2h}\,, \tag{4.5}$$

with constant weights σ_k, $k = 1,2$. Here

$$u_1^{(2)}(\mathbf{x}) = Lu_0(\mathbf{x}) + c_0 \frac{\partial u_0(\mathbf{x})}{\partial x_1} + f^{(1)}(0,\mathbf{x})\,, \quad \mathbf{x} \in \omega_{2h}\,,$$

$$Lu = \frac{\partial}{\partial x_1}\left(k_1^{(1)} \frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k_2^{(1)} \frac{\partial u}{\partial x_2}\right).$$

Also we use the standard notation of the theory of difference schemes[8–10]:

$$y_{x_\alpha} = \frac{y^{(+1_\alpha)} - y}{h_\alpha}\,, \quad y_{\bar{x}_\alpha} = \frac{y - y^{(-1_\alpha)}}{h_\alpha}\,, \quad y_{\overset{\circ}{x}_\alpha} = \frac{y^{(+1_\alpha)} - y^{(-1_\alpha)}}{2h_\alpha}\,, \quad \alpha = 1,2\,,$$

$$y = y_{i_1 i_2} = y(t,\mathbf{x}(t))\,, \quad y^{(\pm 1_1)} = y_{i_1 \pm 1 i_2} = y(t, x_1(t) \pm h_1, x_2)\,,$$

$$y^{(\pm 1_2)} = y_{i_1 i_2 \pm 1} = y(t, x_1(t), x_2 \pm h_2)\,,$$

$$y_{\bar{t}t} = \frac{y_t - y_{\bar{t}}}{\tau},\; y_t = \frac{\hat{y} - y}{\tau},\; y_{\bar{t}} = \frac{y - \check{y}}{\tau},\; y^{(\sigma_1,\sigma_2)} = \sigma_1 \hat{y} + (1 - \sigma_1 - \sigma_2) y + \sigma_2 \check{y}\,,$$

$$\hat{y} = y(t+\tau, \mathbf{x}(t+\tau))\,, \quad \check{y} = y(t-\tau, \mathbf{x}(t-\tau))\,.$$

Stencil functionals φ and $a_m(\mathbf{x})$ $(m = 1,2)$ are defined by formulas

$$\varphi(\mathbf{x}) = 0.5(f(t, x_1 - 0.5h_1, x_2) + f(t, x_1 - 0.5h_1, x_2))\,,$$

$$a_1(\mathbf{x}) = \begin{cases} k_1^{(1)}(x_1 - 0.5h_1, x_2) - c_0^2\,, & \mathbf{x} \in \omega_{1h}^{+}\,, \\ k_1^{(2)}(x_1 - 0.5h_1, x_2)\,, & \mathbf{x} \in \omega_{2h}\,, \end{cases}$$

$$a_2(\mathbf{x}) = \begin{cases} k_2^{(1)}(x_1, x_2 - 0.5h_2)\,, & \mathbf{x} \in \omega_{1h}^{+}\,, \\ k_2^{(2)}(x_1, x_2 - 0.5h_2)\,, & \mathbf{x} \in \omega_{2h}\,, \end{cases}$$

respectively.

Similarly as in Ref. 9, we approximate the second consistency conditions (2.6) with the second-order of accuracy with respect to spatial variables and write its approximation in the following form

$$\frac{c_0}{h_1} y_t + 0.5(y_t + y_{\bar{t}t}) = (a_1 y_{\bar{x}_1})_{x_1}^{(\sigma_1,\sigma_2)} + (a_2 y_{\bar{x}_2})_{x_2}^{(\sigma_1,\sigma_2)} + c_0 y_{t\bar{x}_1} + 0.5 c_0 y_{x_1} + \varphi\,. \tag{4.6}$$

Note that the second initial condition $y_t(0, \mathbf{x}) = u_1^{(2)}(\mathbf{x})$ for the parabolic equation is obtained from the condition of the second order of accuracy of value $y(\tau, \mathbf{x})$.[8]

4.1. *Stability of the difference scheme*

To investigate the stability of the constructed difference scheme (4.1)–(4.6) we shall use the results obtained in Sec. 3. Thus, the three-layered scheme (4.1)–(4.6) must be reduced to the canonical form (3.1).

Let H be a space of mesh functions $y = y(\mathbf{x})$ that are given on $\overline{\omega}_h$ and equal to zero on the boundary $\partial\omega_h$. In the space H we introduce the inner product

$$(y, v) = \sum_{\mathbf{x}\in\omega_h} y(\mathbf{x})v(\mathbf{x})h_1h_2\,, \quad y,\, v \in H\,,$$

and the norm

$$\|y\| = \sqrt{(y, y)}\,, \quad y \in H\,.$$

For functions $y \in H$ let us introduce an operator A as follows:

$$A = A_0 + A_1\,. \tag{4.7}$$

Here $A_0 = A_0^*$ and $A_1 \neq A_1^*$ are determined by the following expressions

$$A_0 y = \begin{cases} -\displaystyle\sum_{k=1}^{2}(a_k(\mathbf{x})y_{\bar{x}_k})_{x_k}\,, & \mathbf{x} \in \omega_h\,,\\ 0\,, & \mathbf{x} \in \partial\omega_h\,,\end{cases}$$

$$A_1 y = \begin{cases} 0\,, & \mathbf{x} \in \omega_{1h}\,,\\ -0.5c_0 y_{x_1}\,, & \mathbf{x} \in \gamma_h\,,\\ -c_0 y_{\overset{\circ}{x}_1}\,, & \mathbf{x} \in \omega_{2h}\,,\\ 0\,, & \mathbf{x} \in \partial\omega_h\,.\end{cases} \tag{4.8}$$

Properties of the operator $A_0\colon H \to H$ are well known.[8,9] In particular, it is self-adjoint and positive operator. Note that for A_1 and A_0 the subordination condition (3.2) is true.

Let us define the other operators:

$$B = G + 2A_2 + (\sigma_1 - \sigma_2)\tau A_0\,, \tag{4.9}$$

$$D = C + \sigma_2\tau^2 A_0\,. \tag{4.10}$$

Here

$$Gy = \begin{cases} 0\,, & \mathbf{x} \in \omega_{1h}^-\,,\\ \left(\dfrac{c_0}{h_1} + 0.5\right)y\,, & \mathbf{x} \in \gamma_h\,,\\ y\,, & \mathbf{x} \in \omega_{2h}^+\,,\end{cases}$$

$$Cy = \begin{cases} y\,, & \mathbf{x} \in \omega_{1h}^{-}\,, \\ 0.5y\,, & \mathbf{x} \in \gamma_h\,, \\ 0\,, & \mathbf{x} \in \omega_{2h}^{+}\,, \end{cases}$$

$$A_2 y = \begin{cases} 0\,, & \mathbf{x} \in \partial\omega_h\,, \\ -c_0 y_{\overset{\circ}{x}_1}\,, & \mathbf{x} \in \omega_{1h}\,, \\ -0.5 c_0 y_{\bar{x}_1}\,, & \mathbf{x} \in \gamma_h\,, \\ 0\,, & \mathbf{x} \in \omega_{2h}\,, \end{cases}$$

$$\omega_{1h}^{-} = \overline{\omega}_{1h} \setminus \gamma_h\,, \quad \omega_{2h}^{+} = \overline{\omega}_{2h} \setminus \gamma_h\,.$$

Thus, the scheme (4.1)–(4.6) is reduced to the canonical form (3.1) with operators A, B and D specified by formulas (4.7)–(4.10) and

$$\begin{aligned} & y(0) = y_0\,, \quad y_0 = u_0\,, \\ & y_t(0) = y_1\,, \quad y_1 = \begin{cases} u_1^{(1)}(\mathbf{x})\,, & \mathbf{x} \in \omega_{1h}\,, \\ u_1^{(2)}(\mathbf{x})\,, & \mathbf{x} \in \omega_{2h}\,. \end{cases} \end{aligned} \tag{4.11}$$

In order to use Theorem 3, we have to verify conditions (3.3). Since the operators G and A_0 are self-adjoint, we have $B_0 = G + A_{20} + (\sigma_1 - \sigma_2)\tau A_0$. Here $A_{20} = 0.5(A_2 + A_2^*)$.

To find the operator A_2^*, we consider the inner product

$$\begin{aligned} (A_2 y, v) &= -c_0 \sum_{i_2=1}^{N_2-1} h_2 \sum_{i_1=1}^{p_1-1} h_1 y_{\overset{\circ}{x}_1, i_1 i_2} v_{i_1 i_2} - 0.5 c_0 \sum_{i_2=1}^{N_2-1} h_2 h_1 y_{\bar{x}_1, p_1 i_2} v_{p_1 i_2} \\ &= -0.5 c_0 \sum_{i_2=1}^{N_2-1} h_2 \left(\sum_{i_1=1}^{p_1-1} (y_{i_1+1 i_2} - y_{i_1-1 i_2}) v_{i_1 i_2} + (y_{p_1 i_2} - y_{p_1-1 i_2}) v_{p_1 i_2} \right) \\ &= -0.5 c_0 \sum_{i_2=1}^{N_2-1} h_2 \left(\sum_{i_1=1}^{p_1-1} (y_{i_1 i_2} v_{i_1-1 i_2} - y_{i_1 i_2} v_{i_1+1 i_2}) + y_{p_1 i_2} v_{p_1-1 i_2} \right. \\ &\qquad \left. + y_{p_1-1 i_2} v_{p_1 i_2} + y_{p_1 i_2} v_{p_1 i_2} - y_{p_1-1 i_2} v_{p_1 i_2} \right) \\ &= c_0 \sum_{i_2=1}^{N_2-1} h_2 \sum_{i_1=1}^{p_1-1} h_1 y_{i_1 i_2} v_{\overset{\circ}{x}_1, i_1 i_2} - 0.5 c_0 \sum_{i_2=1}^{N_2-1} h_2 h_1 \frac{y_{p_1 i_2}(v_{p_1-1 i_2} + v_{p_1 i_2})}{h_1}\,. \end{aligned}$$

Consequently,

$$A_2^* y = \begin{cases} 0\,, & \mathbf{x} \in \partial\omega_h\,, \\ c_0 y_{\overset{\circ}{x}_1}\,, & \mathbf{x} \in \omega_{1h}\,, \\ -0.5 c_0 \dfrac{y^{(-1_1)} + y}{h_1}\,, & \mathbf{x} \in \gamma_h\,, \\ 0\,, & \mathbf{x} \in \omega_{2h}\,. \end{cases}$$

Thus, for the operator A_{20}, we have

$$A_{20} y = \begin{cases} 0\,, & \mathbf{x} \in \omega_{1h}^-\,, \\ -0.5 \dfrac{c_0}{h_1} y\,, & \mathbf{x} \in \gamma_h\,, \\ 0\,, & \mathbf{x} \in \omega_{2h}^+\,. \end{cases}$$

Since $A_0 > 0$ and $G \geq 0$ then $B_0 \geq 0$, provided $\sigma_1 \geq \sigma_2$.

If $L = C$ and

$$Uy = \begin{cases} 0\,, & \mathbf{x} \in \omega_{1h}^-\,, \\ 0.5y\,, & \mathbf{x} \in \gamma_h\,, \\ y\,, & \mathbf{x} \in \omega_{2h}^+\,, \end{cases}$$

then the condition (3.3) is obviously fulfilled for

$$\sigma_1 \geq \sigma_2 + 0.5\,, \quad \sigma_2 > 0\,. \tag{4.12}$$

In that way, if the conditions (4.12) are satisfied then for the solution of the difference scheme (4.1)–(4.6) *a priori* estimate (3.4) is valid.

4.2. *Convergence of the difference scheme*

Here we shall investigate an accuracy of the proposed difference scheme (4.1)–(4.6) for the conjugation problem (2.1), (2.2), (2.4)–(2.6). Let $y \in H$ be a solution of the problem (3.1), (4.7)–(4.11) and $u(t, \mathbf{x})$ be a solution of the differential problem (2.1), (2.2), (2.4)–(2.6). Write the equation for an error $z = y - u$. Substituting $y = z + u$ in (3.1), we get

$$Dz_{\bar{t}t} + Bz_t + Az = \psi\,, \quad z(0) = 0\,, \quad z_t(0) = \nu(\mathbf{x})\,. \tag{4.13}$$

Here z, ψ, $\nu \in H$,

$$\psi(\mathbf{x}) = \begin{cases} (a_1 u_{\bar{x}_1})_{x_1}^{(\sigma_1,\sigma_2)} + (a_2 u_{\bar{x}_2})_{x_2}^{(\sigma_1,\sigma_2)} + 2c_0 u_{t\overset{\circ}{x}_1} + \varphi - u_{\bar{t}t}\,, & \mathbf{x} \in \omega_{1h}\,, \\ (a_1 u_{\bar{x}_1})_{x_1}^{(\sigma_1,\sigma_2)} + (a_2 u_{\bar{x}_2})_{x_2}^{(\sigma_1,\sigma_2)} + c_0 u_{t\overset{\circ}{x}_1} & \\ \qquad + 0.5 u_{\overset{\circ}{x}_1} + \varphi - \dfrac{c_0}{h_1} u_t - 0.5(u_t + u_{\bar{t}t})\,, & \mathbf{x} \in \gamma_h\,, \\ (a_1 u_{\bar{x}_1})_{x_1}^{(\sigma_1,\sigma_2)} + (a_2 u_{\bar{x}_2})_{x_2}^{(\sigma_1,\sigma_2)} + c_0 u_{\overset{\circ}{x}_1} + \varphi - u_t\,, & \mathbf{x} \in \omega_{2h}\,, \end{cases}$$

is a truncation error of Eqs. (2.1) and (2.2) and the consistency condition (2.6). The term

$$\nu(\mathbf{x}) = \mathcal{O}(\tau + h_1^2 + h_2^2)$$

defines a truncation error of the second initial condition.

To estimate the accuracy of the difference scheme (4.1)–(4.6), we shall assume that

$$\begin{gathered} k_i^{(m)} \in C^3(Q_m) \cap C^2(\Gamma)\ (i=1,2)\,, \quad u^{(m)} \in C^4(Q_m) \cap C^3(\Gamma),\ m=1,2\,, \\ k_2^{(1)}(\mathbf{x}) = k_2^{(2)}(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma\,. \end{gathered} \tag{4.14}$$

Employing Taylor series expansions we have

$$\begin{aligned} u_{x_\alpha} &= \frac{\partial u}{\partial x_\alpha} + \frac{h_1}{2}\frac{\partial^2 u}{\partial x_\alpha^2} + \frac{h_1^2}{6}\frac{\partial^3 u}{\partial x_\alpha^3} + \mathcal{O}(h_\alpha^3)\,, \\ u_{\bar{x}_\alpha} &= \frac{\partial u}{\partial x_\alpha} - \frac{h_1}{2}\frac{\partial^2 u}{\partial x_\alpha^2} + \frac{h_1^2}{6}\frac{\partial^3 u}{\partial x_\alpha^3} + \mathcal{O}(h_\alpha^3)\,, \quad \alpha = 1,2\,, \\ u_{\overset{\circ}{x}_1} &= \frac{\partial u}{\partial x_1} + \mathcal{O}(h_1^2)\,, \quad u_{t\overset{\circ}{x}_1} = \frac{\partial^2 u}{\partial r \partial x_1} + \mathcal{O}(\tau + h_1^2)\,, \\ u_t &= \frac{\partial u}{\partial r} + \mathcal{O}(\tau)\,, \quad u_{\bar{t}t} = \frac{\partial^2 u}{\partial r^2} + \mathcal{O}(\tau^2)\,, \quad u^{(\sigma_1,\sigma_2)} = u + \mathcal{O}(\tau)\,. \end{aligned}$$

On the mesh ω_{1h} for the coefficient a_1 the following expansions are valid

$$\begin{aligned} a_1 &= k_1^{(1)}(\mathbf{x}) - c_0^2 - \frac{h_1}{2}\frac{\partial k_1^{(1)}}{\partial x_1}(\mathbf{x}) + \frac{h_1^2}{8}\frac{\partial k_1^{(1)}}{\partial x_1^2}(\mathbf{x}) + \mathcal{O}(h_1^3)\,, \quad \mathbf{x} \in \omega_{1h}\,, \\ a_1^{(+1_1)} &= k_1^{(1)}(\mathbf{x}) - c_0^2 + \frac{h_1}{2}\frac{\partial k_1^{(1)}}{\partial x_1}(\mathbf{x}) + \frac{h_1^2}{8}\frac{\partial k_1^{(1)}}{\partial x_1^2}(\mathbf{x}) + \mathcal{O}(h_1^3)\,, \quad \mathbf{x} \in \omega_{1h}\,. \end{aligned}$$

Hence,

$$\begin{aligned} a_1^{(+1_1)} u_{x_1} &= (k_1^{(1)} - c_0^2)\frac{\partial u^{(1)}}{\partial x_1} + \frac{h_1}{2}\frac{\partial}{\partial x_1}\left((k_1^{(1)} - c_0^2)\frac{\partial u^{(1)}}{\partial x_1}\right) \\ &\quad + \frac{h_1^2}{2}\left(\frac{1}{3}(k_1^{(1)} - c_0^2)\frac{\partial^3 u^{(1)}}{\partial x_1^3} + \frac{1}{2}\frac{\partial k_1^{(1)}}{\partial x_1}\frac{\partial^2 u^{(1)}}{\partial x_1^2} + \frac{1}{4}\frac{\partial^2 k_1^{(1)}}{\partial x_1^2}\frac{\partial u^{(1)}}{\partial x_1}\right) + \mathcal{O}(h_1^3)\,, \\ a_1 u_{\bar{x}_1} &= (k_1^{(1)} - c_0^2)\frac{\partial u^{(1)}}{\partial x_1} - \frac{h_1}{2}\frac{\partial}{\partial x_1}\left((k_1^{(1)} - c_0^2)\frac{\partial u^{(1)}}{\partial x_1}\right) \\ &\quad + \frac{h_1^2}{2}\left(\frac{1}{3}(k_1^{(1)} - c_0^2)\frac{\partial^3 u^{(1)}}{\partial x_1^3} + \frac{1}{2}\frac{\partial k_1^{(1)}}{\partial x_1}\frac{\partial^2 u^{(1)}}{\partial x_1^2} + \frac{1}{4}\frac{\partial^2 k_1^{(1)}}{\partial x_1^2}\frac{\partial u^{(1)}}{\partial x_1}\right) + \mathcal{O}(h_1^3)\,. \end{aligned}$$

Consequently,

$$(a_1 u_{\bar{x}_1})_{x_1} = \frac{\partial}{\partial x_1}\left((k_1^{(1)}(\mathbf{x}) - c_0^2)\frac{\partial u^{(1)}}{\partial x_1}\right)(t,\mathbf{x}) + \mathcal{O}(h_1^2)\,, \quad (t,\mathbf{x}) \in \omega_1\,.$$

Similarly, we obtain

$$(a_1 u_{\bar{x}_1})_{x_1} = \frac{\partial}{\partial x_1}\left(k_1^{(2)}(\mathbf{x})\frac{\partial u^{(2)}}{\partial x_1}\right)(t,\mathbf{x}) + \mathcal{O}(h_1^2)\,, \quad (t,\mathbf{x}) \in \omega_2\,,$$

$$(a_2 u_{\bar{x}_2})_{x_2} = \frac{\partial}{\partial x_2}\left(k_2^{(m)}(\mathbf{x})\frac{\partial u^{(m)}}{\partial x_2}\right)(t,\mathbf{x}) + \mathcal{O}(h_2^2)\,, \quad (t,\mathbf{x}) \in \omega_m,\ m = 1,2\,.$$

Since $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial t} + c_0 \frac{\partial u}{\partial x_1}$, then for the truncation error we have

$$\psi(t,\mathbf{x}) = \mathcal{O}(\tau + h_1^2 + h_2^2)\,, \quad (t,\mathbf{x}) \in \omega_1 \cup \omega_2\,.$$

Now we consider the truncation error on the interface Γ. Using expansions

$$a_1^{(+1_1)} u_{x_1} = k_1^{(2)}\frac{\partial u^{(2)}}{\partial x_1} + \frac{h_1}{2}\frac{\partial}{\partial x_1}\left((k_1^{(2)} - c_0^2)\frac{\partial u^{(2)}}{\partial x_1}\right) + \mathcal{O}(h_1^2)\,, \quad \mathbf{x} \in \gamma_h\,,$$

$$a_1 u_{\bar{x}_1} = (k_1^{(1)} - c_0^2)\frac{\partial u^{(1)}}{\partial x_1} - \frac{h_1}{2}\frac{\partial}{\partial x_1}\left((k_1^{(1)} - c_0^2)\frac{\partial u^{(1)}}{\partial x_1}\right) + \mathcal{O}(h_1^2)\,, \quad \mathbf{x} \in \gamma_h\,,$$

and the second consistency condition (2.6), we obtain

$$(a_1 u_{\bar{x}_1})_{x_1} = \frac{c_0}{h_1}\frac{\partial u^{(1)}}{\partial r}(t,\mathbf{x}) + 0.5\left(\frac{\partial}{\partial x_1}\left(k_1^{(2)}(\mathbf{x})\frac{\partial u^{(2)}}{\partial x_1}\right)(t,\mathbf{x}) + \frac{\partial}{\partial x_1}\left(k_1^{(1)}(\mathbf{x})\frac{\partial u^{(1)}}{\partial x_1}\right)(t,\mathbf{x})\right) + \mathcal{O}(h_1)\,, \quad (t,\mathbf{x}) \in \gamma_h \times \omega_\tau\,.$$

Since the conditions (4.14) are valid and $\left.\frac{\partial u^{(1)}}{\partial x_2}\right|_\Gamma = \left.\frac{\partial u^{(2)}}{\partial x_2}\right|_\Gamma$, it follows that

$$(a_2 u_{\bar{x}_2})_{x_2} = 0.5\left(\frac{\partial}{\partial x_2}\left(k_2^{(2)}(\mathbf{x})\frac{\partial u^{(2)}}{\partial x_2}\right)(t,\mathbf{x}) + \frac{\partial}{\partial x_1}\left(k_2^{(1)}(\mathbf{x})\frac{\partial u^{(1)}}{\partial x_2}\right)(t,\mathbf{x})\right) + \mathcal{O}(h_2^2)\,, \quad (t,\mathbf{x}) \in \gamma_h \times \omega_\tau\,.$$

Thus on the interface Γ we have

$$\psi(t,\mathbf{x}) = \mathcal{O}(\tau + h_1 + h_2^2)\,.$$

Recalling Theorem 3 and conditions (4.12), for the solution of the problem (4.13), we obtain

$$\|z_{n+1}\|_{A_0} \le \sqrt{M}\left(\|z_t(0)\|_{D+2\tau^2 A_0} + \sum_{k=0}^{n} \tau\|\psi_k\|\right)\,.$$

Further,

$$\|\nu(\mathbf{x})\|_{D+2\tau^2 A_0} = \mathcal{O}(\tau + h_1^2 + h_2^2)\,.$$

Thus, we have proved the following statement.

Theorem 4. *Suppose that the conditions* (2.3) *and* (4.14) *are valid. Then under conditions* (4.12) *the solution of the difference scheme* (4.1)–(4.6) *converges to the solution of the differential problem* (2.1), (2.2), (2.4)–(2.6), *and for the error* $z(t)$ *the following estimate holds*

$$\max_{t\in\omega_\tau} \|z(t)\|_{A_0} \leq M_1(\tau + h_1^{3/2} + h_2^2)\,,$$

where $M_1 = \text{const.} > 0$.

References

1. V. I. Korzyuk, *Problem about conjugation of equations of hyperbolic and parabolic types*, *Differents Uravneniya* **4** (1968) 1855–1866, in Russian.
2. V. I. Korzyuk, *Mixed problem for some non-stationary equations with discontinuous coefficients*, *Differents Uravneniya* **6** (1970) 343–357, in Russian.
3. V. I. Korzyuk, *Friederichs lemma for everaging operators with variable step*, *Vestnik Belgosuniversiteta. Ser. 1. Fizika, Mat. Infor.* **2** (1996) 32–36, in Russian.
4. V. I. Korzyuk, *The method of energy inequalities and everaging operators*, *Vestnik Belgosuniversiteta. Ser. 1. Fizika, Mat. Infor.* **3** (1996) 55–71, in Russian.
5. V. I. Korzyuk, *Boundary value problem for the third order Mangeron's equation*, *Differents Uravneniya* **33** (1997) 1683–1690, in Russian.
6. S. V. Lemeshevsky, *Computational methods for the conjugation problem of hyperbolic and parabolic equations*, in **Proc. of the Second Int. Conf. on Finite-Difference Methods: Theory and Application**, Minsk, July 6–9, 1998, Vol. 2, pp. 99–102.
7. E. G. Sakhanovsky and Ya. S. Uflyand, *Influence of conductivity anisotropy on unsteady motion of conducting gas in plane channel*, *Prikl. Mat. Mekh.* **26** (1962) 542–547, in Russian.
8. A. A. Samarskii, **Theory of Difference Schemes** (Nauka, 1977), in Russian.
9. A. A. Samarskii and V. B. Andreev, **Difference Methods for Elliptic Equations** (Nauka, 1976), in Russian.
10. A. A. Samarskii and A. V. Goolin, **Stability of Difference Schemes** (Nauka, 1973), in Russian.
11. A. A. Samarskii, P. N. Vabishchevich, S. V. Lemeshevsky and P. P. Matus, *Difference schemes for conjugation problem of hyperbolic and parabolic equations*, *Sib. Mat. Z.* **39** (1998) 954–962, in Russian.
12. Ya. S. Uflyand, *To reference about oscillations spreading in composite electric lines*, *Inzh.-Fiz. Z.* **7** (1961) 89–92, in Russian.
13. Ya. S. Uflyand and I. B. Chekmariov, *Investigation of unsteady flow of conducting fluid in plain channel with moving boundary*, *Z. Tekh. Fiz.* **30** (1960) 465–471, in Russian.