37

Domain Decomposition Methods for Parabolic Problems

A.A.SAMARSKII [1], & P.N.VABISHCHEVICH [2]

INTRODUCTION

Parallel algorithms for multidimensional problems in mathematical physics are designed now on the basis of domain decomposition methods. The original problem is divided into a set of subproblems. Either problem is solved on its own processor (its own elementary computer) and in its own subdomain. The scope for such an approach with an approximate solution of non-stationary problems of mathematical physics is discussed. The main attention is paid for noniterative variants of the domain decomposition method with various of interchanging boundary conditions.

The present investigation is directed to a review and analysis of works on methods of domain decomposition for parabolic initial-boundary value problems. In constructing domain decomposition methods for time-dependent problems there are employed the following approaches.

- The first [Kuz88, Tal94] is based on using classic implicit schemes and involves domain decomposition methods in order to solve an elliptic grid problem at new time level.
- In the second approach peculiarities of transient problems are taken into account in more details. The corresponding region-additive schemes are investigated in various variants in works [Dry90, Lae90, Lae92, SV96,

1 Institute for Mathematical Modeling of Russian Academy of Sciences, 4 Miusskaya Square, 125047 Moscow, Russia email: samarskii@imamod.msu.su
2 Institute for Mathematical Modeling of Russian Academy of Sciences, 4 Miusskaya Square, 125047 Moscow, Russia email: vab@ibrae.ac.ru
Eleventh International Conference on Domain Decomposition Methods
Editors Choi-Hong Lai, Petter E. Bjørstad, Mark Cross and Olof B. Widlund

SV95b, Vab89, Vab94b].

- There are [DD92] constructed decomposition schemes with a special approximation of exchange boundary conditions (inhomogeneous schemes).
- Parallel variants of standard splitting schemes (for example, [JSS87]): schemes of alternating directions, factorized schemes, local one-dimensional schemes of component-wise splitting) are being designed.

The main attention is paid to iterative-free variants of a decomposition method, i.e. regional-additive schemes. These algorithms adequately account for a specific character of non-stationary problems when the transition to a new time level is connected with the solution of a set of separate problems in subdomains. Theoretical analysis of domain decomposition schemes is done with the use of standard and new splitting schemes [Abr90, Vab94a].The study is based on the modern theory of stability and convergence of operation-difference splitting schemes [Sam89, SG73, SV95a].

MODEL PROBLEM

A two-dimensional rectangular domain Ω with parallel to coordinates sides is considered. The solution of the parabolic equations is sought in the domain Ω:

$$\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha}(k_\alpha(x)\ \frac{\partial u}{\partial x_\alpha}) = f(x), \quad x = (x_1, x_2) \in \Omega, \quad t > 0. \tag{1}$$

Equation (1) is supplemented by the homogeneous boundary condition (the Dirichlet problem)

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0 \tag{2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{3}$$

Let us introduce in the domain Ω the uniform grid x_α with the uniform spacing $h_\alpha, \alpha = 1,2$. Let ϖ be the set of the internal points. An approach to more common problems in context of this paper is of editing character. Let us define on the set of functions $y \in H$ such as $y(x) \equiv 0, x \notin \varpi$, a difference operator A:

$$Ay = \sum_{\alpha=1}^{2} \Lambda_\alpha, \tag{4}$$

$$\Lambda_\alpha = -(a_\alpha(x) y_{\bar{x}})_x, \quad x \in \varpi.$$

Here the standard index-free notation of the difference scheme theory [Sam89]–[SV95a] is used, for example:

$$w_{\bar{x}} = \frac{w(x) - w(x-h)}{h}, \quad w_x = \frac{w(x+h) - w(x)}{h},$$

$$a_1(x) = 0,5(k_1(x) + k_1(x_1 - h_1, x_2)),$$

$$a_2(x) = 0,5(k_2(x) + k_2(x_1, x_2 - h_2)).$$

Let us introduce a scalar product and a norm in the Hilbert space H as follows

$$(y, v) = \sum_{x \in \varpi} y v h_1 h_2, \quad ||y|| = \sqrt{(y, y)}.$$

Note, that in H operator A is self-adjoint and positive definite [Sam89]–[SV95a], i.e. $A = A^* > 0$.

From equations (1)-(3) for the given $y(x, 0), x \in \varpi$ we pass to the following equation

$$\frac{dy}{dt} + Ay = 0, \quad x \in \varpi. \tag{5}$$

For the last equation difference schemes of domain decomposition schemes are constructed. Numerical implementation of these schemes is based on the solution of problems in separate subdomains of the calculation domain Ω at every time-level.

DOMAIN DECOMPOSITION

Let domain Ω consists from m separate subdomains:

$$\Omega = \Omega_1 \cup \Omega_2 \cup ... \cup \Omega_p.$$

Designing of regional-additive difference schemes is based on a special additive representation of A operator of considered equation (5) and on the application of one or another splitting schemes. The choice of a splitting operator and a splitting scheme corresponds to the choice of a definite scheme of computations in separate subdomains. In particular, to the choice of interchange boundary conditions on the boundary of sub-domains.

Let ϖ_α be the subsets of points ϖ, lying in subdomains $\Omega_\alpha, \alpha = 1, 2, ..., p$. Let us construct the decomposition difference schemes similar to presented in [Lae90] on the basis of the unit splitting for domain Ω. Let us define the following functions

$$\chi_\alpha(x) = \begin{cases} > 0, & x \in \Omega_\alpha, \\ 0, & x \notin \Omega_\alpha, \end{cases} \quad \alpha = 1, 2, \tag{6}$$

where

$$\sum_{\alpha=1}^{p} \chi_\alpha(x) = 1, \quad x \in \Omega. \tag{7}$$

We shall consider the class of decomposition schemes, where for the operator A the following additive representation takes place:

$$A = \sum_{\alpha=1}^{p} A_\alpha, \tag{8}$$

where operators $A_\alpha, \alpha = 1, 2, ..., p$ are associated with isolated subdomains and also with splitting (6),(7) and with the solution of the individual subproblems in subdomains $\Omega_\alpha, \alpha = 1, 2, ..., p$.

The simplest difference decomposition scheme is defined via the definition of operators $A_\alpha, \alpha = 1, 2, ..., p$ in the following way [Vab89, Vab94b]:

$$A_\alpha = \chi_\alpha A, \quad \alpha = 1, 2, ..., p. \tag{9}$$

The following presentation for the decomposition operators can be used:

$$A_\alpha = A\chi_\alpha, \quad \alpha = 1, 2, ..., p. \tag{10}$$

Clearly, that for such splitting operator A is not selfadjoint, i.e.$A_\alpha \neq A_\alpha^*, \alpha = 1, 2, ..., p$.

It is naturally to use in this case the symmetrical splitting (7),(8) (see, for example, [Dry90, Lae90], where

$$A_\alpha y = -\sum_{\alpha=1}^{2} (a_\beta^\alpha(x) y_{\overline{x}_\beta})_{x_\beta}, \quad x \in \varpi, \quad \alpha = 1, 2, ..., p. \tag{11}$$

Grid operators $A_\alpha, \alpha = 1, 2, ..., p.$ are approximated via difference degenerating elliptic operators

$$-\sum_{\beta=1}^{2} \frac{\partial}{\partial x_\beta}(k_\beta \chi_\alpha(x) \ \frac{\partial u}{\partial x_\beta}), \quad \alpha = 1, 2, ..., p,$$

where coefficients a_β^α are defined as a_β. With expressions (7),(8),(11) we obtain $A_\alpha = A_\alpha^* \geq 0, \quad \alpha = 1, 2, ..., p.$

REGION-ADDITIVE SCHEMES

At first, simplest case of decomposition of domain Ω by means of the straight lines $x_1 =$ const will be demonstrated. In this case while constructing of the difference schemes we can be oriented to the usage of the difference splitting schemes with $p = 2$, where Ω_1 and Ω_2 are defined as a combination of corresponding subdomains. When splitting into two operators, classical alternating direction schemes would be appropriate for transition from the previous time level to the next one.

Unconditionally stable difference schemes for the solution of equation (6) with the corresponding initial condition are easily constructed via splitting (7),(8) with selfadjoint and positive definite operators $A_\alpha, \alpha = 1, 2$ (with decomposition operator (11)). The accuracy problem of the approximate solution, its dependence on the width of subdomain overlapping and also on functions $\chi_\alpha(x), \alpha = 1, 2$ is of special interest. Usage of the schemes with the asymmetric decomposition operators requires individual investigation.

Let us select among the unconditionally steady factorized schemes the scheme of the stabilizing correction similar to the classical Douglas-Rechford scheme. Let y^n be the difference solution at the time moment $t^n = n\tau$, where $\tau > 0$ is the time-step. The transition from the previous time-level to the next one is performed in accordance with the following expressions

$$\frac{y^{n+1/2} - y^n}{\tau} + A_1 y^{n+1/2} + A_2 y^n = 0,$$

$$\frac{y^{n+1} - y^{n+1/2}}{\tau} + A_2(y^{n+1} - y^n) = 0. \tag{12}$$

For the difference decomposition scheme (8),(12) with the selection of operators $A_\alpha, \alpha = 1, 2$ according to (9), (10) and (11) the following estimate of stability in respect to the initial data is valid:

$$\left\|(E + \tau A_2)y^{n+1}\right\|_D \le \left\|(E + \tau A_2)y^0\right\|_D,$$

where $D = A, A^{-1}$ and E respectively.

Investigation of convergence of decomposition method leads to the following estimates

$$\left\|(E + \tau A_2)z^{n+1}\right\|_D \le M((1 + \|\chi_2\|_A)\tau + |h|^2). \tag{13}$$

Moreover, accuracy of decomposition schemes depends on the width of subdomains overlapping Ω_1 and Ω_2 (see term at $\|\chi_2\|_A$ in the estimate (13)).

While constructing difference schemes for the parallel computer we should be oriented to the decomposition with minimum overlapping of domains, i.e. on the minimization of exchange between individual processors. At minimal overlapping of subdomains (width of subdomains overlapping equals to $O(|h|)$) we have from the estimate (13) that the convergence rate is $O(|h|^{-1/2}\tau + |h|^2)$.

Under a more general domain decomposition grid operator A in equation (5) has the form (8) with a number of operators $p > 2$. For such problems difference schemes of summarized approximation [Sam89, SV95a] have some advantages. Investigation of difference schemes of summarized approximation shows that these schemes have low accuracy of spatial approximation.

For instance, for the following fully implicit scheme of multicomponent splitting

$$\frac{y^{n+\alpha/p} - y^{n+(\alpha-1)/p}}{\tau} + A_\alpha y^{n+\alpha/p} = 0, \quad \alpha = 1, 2, ..., p \tag{14}$$

the error estimate has the form

$$\left\|z^{n+1}\right\| \le M((1 + \sum_{\alpha=1}^{p} \|D\chi_\alpha\|)\tau + |h|^2). \tag{15}$$

As for parallel implementation, additive-averaged schemes of domain decomposition [18] should be mentioned separately. For example, the implicit scheme has the following form (compare with (14))

$$\frac{\tilde{y}^{n+\alpha/p} - y^n}{\tau} + \chi_\alpha A\tilde{y}^{n+\alpha/p} = 0, \quad \alpha = 1, 2, ..., p,$$

$$y^{n+1} = \frac{1}{p}\sum_{\alpha=1}^{p} \tilde{y}^{n+\alpha/p}$$

and for the error we have estimate (15).

The principal moment here is concerned with the possibility to calculate $\tilde{y}^{n+\alpha/p}, \alpha = 1, 2, ..., p$ independently (asynchronously).

To construct parallel numerical methods, it seems to be more suitable to use vector additive difference schemes with full approximation, besides these numerical schemes are unconditionally stable at any p [Abr90, Vab94a]. Let's define vector $Y = y_1, y_2, ..., y_p$, the following set of equations for calculation of each component of this vector should be solved:

$$\frac{dy_\alpha}{dt} + \sum_{\beta=1}^{p} A_\beta y_\beta = 0, \tag{16}$$

$$y_\alpha(0) = y_0, \quad \alpha = 1, 2, ..., p. \tag{17}$$

From equations (16),(17) it follows that $y_\alpha(t) = y(t)$, then arbitrary component of vector $Y(t)$ may be chosen as a solution of the main problem for equation (5).

The following scheme is an example of unconditionally stable schemes for a set of equations (16),(17)

$$(E + \sigma\tau A_\alpha)\frac{y_\alpha^{n+1} - y_\alpha^n}{\tau} + \sum_{\beta=1}^{p} A_\beta y_\beta^n = 0,$$

$$\alpha = 1, 2, ..., p$$

for case $\sigma \geq p/2$. Realization of this scheme is connected with the inversion of operators $E + \sigma\tau A_\alpha, \quad \alpha = 1, 2, ..., p$ at every time-level in just the same way as in case with general (scalar) difference additive schemes. These schemes may be considered like difference schemes with weights where the weight σ is larger than unit ($\sigma > 1$).

For the accuracy of these vector schemes of domain decomposition there are the same estimates [SV95b] as for standard schemes of summarized approximation. However, for this class of schemes it is easy to construct schemes with the second order in time.

ACKNOWLEDGEMENT

This work was supported by the Russian Foundation for Fundamental Research, through grant N 98-01-00335.

REFERENCES

[Abr90] Abrashin V. N. (1990) On a variant of alternating direction method for the numerical solution of multidimensional problems of mathematical physics.i. *Differentsial'nye Uravneniya* 26: 314–323. In Russian.

[DD92] Dawson C. N. and Dupont T. F. (1992) Explicit/implicit, conservative, galerkin domain decomposition procedures for parabolic problems. *Math. Comp* 58: 21–34.

[Dry90] Dryja M. (1990) Substructuring methods for parabolic problems. In *IV Intern. Simp. on Domain Decomposition Methods for Partial Differential Equations*, pages 264–271. SIAM.

[JSS87] Johnson S. L., Saad V., and Schultz M. H. (1987) Alternating direction methods on multiprocessors. *SIAM J.Sci. Stat. Comput.* 8: 686–700.

[Kuz88] Kuznetsov Y. A. (1988) New algorithms for approximate realization of implicit difference schemes. *Sov. J. Numer. Anal. and Math. Model.* 3: 99–114.

[Lae90] Laevsky Y. M. (1990) Quadratic elements in splitting methods. *Sov. J. Numer. Anal. and Math. Model* 5: 244–249.

[Lae92] Laevsky Y. M. (1992) On one algorithm of domain decomposition without the overlapping of subdomains for the solution of parabolic equations. *Zh. Vychisl. Mat. and Mat. Fiz.* 32: 1744–1755. In Russian.

[Sam89] Samarskii A. A. (1989) *Theory of Difference Schemes.* Nauka, Moscow, 3 edition. In Russian.

[SG73] Samarskii A. A. and Gulin A. V. (1973) *Stability of Difference Schemes.* Nauka, Moscow. In Russian.

[SV95a] Samarskii A. A. and Vabishchevich P. N. (1995) *Computational Heat Transfer*, volume 1,2. Wiley, Chicheste.

[SV95b] Samarskii A. A. and Vabishchevich P. N. (1995) Vector additive schemes of domain decomposition for parabolic problems. *Differentsial'nye Uravnenija* 31: 1563–1569. In Russian.

[SV96] Samarskii A. A. and Vabishchevich P. N. (1996) Factorized region-additive schemes for problems of convection-diffusiion. *Doklady Akademii Nauk* 336: 742–745. In Russian.

[Tal94] Tallec P. L. (1994) Domain decomposition methods in computational mechanics. *Computational Mechanics Advances* 1: 121–220.

[Vab89] Vabishchevich P. N. (1989) Finite difference schemes with decomposition of the computational domain for the solution of nonstationary problems. *Zh. Vychisl. Mat. and Mat. Fiz* 29: 1822–1829. In Russian.

[Vab94a] Vabishchevich P. N. (1994) On a class of vector additive difference schemes. *Izvestija Vuzov. Matematika* (9): 11–15. In Russian.

[Vab94b] Vabishchevich P. N. (1994) Parallel domain decomposition algorithms for time-dependent problems of mathematical physics. In *Advances in Numerical Methods and Applications*, pages 293–299. World Scientific.