Mathematical Models and Methods in Applied Sciences
Vol. 9, No. 1 (1999) 93–110

INVARIANT DIFFERENCE SCHEMES FOR PARABOLIC EQUATIONS WITH TRANSFORMATIONS OF INDEPENDENT VARIABLES

A. A. SAMARSKII and V. I. MAZHUKIN*
Institute of Mathematical Modelling, Russian Academy of Sciences, Miusskaya Sq. 4A, 125047 Moscow, Russia

P. P. MATUS† and V. G. RYCHAGOV‡
Institute of Mathematics, National Academy of Sciences of Belarus, Surganov Str. 11, 220072 Minsk, Belarus

I. SMUROV
Ecole Nationale d'Ingenieurs de Saint-Etienne, 58 Rue Jean Parot, 42023 Saint-Etienne Cedex 2, France

Communicated by A. A. Samarskii
Received 19 December 1997

In this paper, invariant difference schemes for nonstationary equations under independent variables transformation constructed and investigated. Under invariance of difference scheme we mean its ability to preserve basic properties (stability, approximation, convergency, etc.) in various coordinate systems. Difference schemes of the second-order approximation that satisfy the invariance property are constructed for equations of parabolic type. Stability and convergency investigation of correspondent difference problems are carried out; *a priori* estimates in various grid norms are obtained.

1. Introduction

At present together with traditional requirements of similarity, conservativity and complete conservativity for computational methods, a fulfillment of adaptivity property is also required. As a rule, adaptive methods used in computational practice are to solve problems in complex irregular domains with moving boundaries (including contact and phase), contact surfaces (shock waves), domains of big gradients and boundary layers.

In constructing adaptive grids, there are two main directions: stationary and nonstationary problems. In most cases of grid construction for multidimensional

*E-mail: mazhukin@imm.msk.ru
†Corresponding author. E-mail: matus@im.bas-net.by
‡E-mail: rychagov@im.bas-net.by

stationary problems, a method of coordinate transformation which allows mapping a complex geometric domain into rectangle or square with sides of unit length[1–3] is often used. An adaptive grid can be obtained by means of functions of inverse mapping.[4,5] An idea of coordinate transformation also has a lot of applications to adaptive methods for evolutionary problems.[6,7] However, as a result of problem's nonstationary nature such transformation is realized by means of choosing one of the moving coordinate systems. Typical samples of such transformations are Lagrange's variables in gas dynamics[8,9] and coordinate systems associated with front propagation.[10] Very fruitful idea in this direction consists of constructing grids that dynamically adapted to solution.[11–17] The use of arbitrary nonstationary coordinate system permits one to describe a grid node behavior by differential model as a component of mathematical problem. This method is proved to be effective for solution of wide range of problems of mathematical physics such as nonlinear heat conductivity,[11,12] nonlinear transfer (Burger's equation), gas dynamics,[13] one- and multidimensional Stephan problems.[16,17]

Differential problems written in different coordinate systems are equivalent from the mathematical point of view. It is intrinsic to demand a realization of similar property for difference schemes.

In Refs. 22 and 23, based on the simplest differential equation, an invariant difference scheme was introduced that such its basic properties under discrete transformations of independent variables were conserved. Some basic principles of construction of difference schemes were also formulated.

The purpose of this paper is to construct invariant difference schemes that also conserve a property of the second-order of precision by space in various coordinate systems. This circumstance has a principal meaning — because in approximating the second-order derivatives by space on ordinary nonuniform grids as a rule one can only achieve the first-order approximation. Here for invariant difference schemes on moving grids, *a priori* estimates of stability and convergency have been obtained. Construction and investigation of such schemes are based on the results of Ref. 22.

2. Model Equation

Definition 1. A difference scheme is called *invariant* by some property if it conserves the property in a given class of discrete transformations of independent variables.

For example, if we pass from one coordinate system to another, we wish to conserve such properties as approximation, stability, convergence, etc.

On the base of elementary second-order equation

$$u''(x) = -f(x)\,, \quad a < x < b\,, \quad u(a) = u(b) = 0 \tag{2.1}$$

we shall demonstrate how to construct and theoretically investigate such schemes. On the segment $[a, b]$, let us introduce an arbitrary nonuniform grid

$$\widehat{\omega}_h = \{x_i = x_{i-1} + h_i,\ i = \overline{1, N},\ x_0 = a,\ x_N = b\}\,. \tag{2.2}$$

The corresponding difference scheme of the second-order of approximation on the grid $\widehat{\omega}_h$ is[20]

$$\frac{1}{\hbar_i}\left(\frac{y_{i+1}-y_i}{h_{i+1}}-\frac{y_i-y_{i-1}}{h_i}\right)=-\bar{\varphi}_i\,, \tag{2.3}$$

$$y(a)=y(b)=0\,,$$

where $\bar{\varphi}_i = f(\bar{x}_i)$, $\bar{x}_i = x_i + (h_{i+1} - h_i)/3$, $\hbar_i = 0.5(h_i + h_{i+1})$.

Let $x = x(q)$ be a function that twice continuously differentiable and transforms closed segment $[0,1]$ in $[a,b]$ such that $x(q_i) = x_i$, where q_i is a node of the uniform grid $\omega_{h_q} = \{q_i = i/N,\ \ i = \overline{0,N}\}$, and

$$\frac{dx}{dq}=\psi(q)\geq c_0>0\,,\quad x(0)=a\,. \tag{2.4}$$

After the change of variables $x = x(q)$ in Eq. (2.1) we get

$$\frac{1}{\psi}\left(\frac{d}{dq}\left(\frac{1}{\psi}\frac{d\bar{u}}{dq}\right)\right)=-\bar{f}(q)\,,\quad 0<q<1\,,\quad \bar{u}(0)=\bar{u}(1)=0\,, \tag{2.5}$$

$$\bar{u}=\bar{u}(q)=u(x(q))=u(x)=u\,. \tag{2.6}$$

For (2.5) one can write the following difference scheme:

$$\frac{1}{\psi_{h,i}}(a\bar{y}_{\bar{q}})_{q,i}=-\bar{f}(\tilde{q}_i)\,, \tag{2.7}$$

where $\tilde{q}_i = x^{-1}(\bar{x}_i)$, $a_i = a(q_i) = [1/h_q \int_{q_{i-1}}^{q_i} \psi\, dq]^{-1} = [x_{\bar{q},i}]^{-1}$ is defined from the "best" difference scheme, and the following equality holds:

$$\bar{f}(\tilde{q}_i)=f(\bar{x}_i)\,.$$

In this case

$$\psi_{h,i}=\frac{x_{i+1/2}-x_{i-1/2}}{h_q}\,, \tag{2.8}$$

where $x_{i-1/2} = 0.5(x_{i-1} + x_i)$ is the middle of the segment $[x_{i-1}, x_i]$ and $\psi_{hi} - \psi(q_i) = x_{\overset{\circ}{q}} - x'(q_i) = O(h_q^2)$, $x_{\overset{\circ}{q}} = (x_{i+1} - x_{i-1})/2h_q$, whenever the function $x'''(q)$ is restricted for $q \in [0,1]$.

Let us show that the difference schemes (2.3) and (2.7) are invariant by order of approximation. First we demonstrate that the property (2.6) holds for grid solutions of (2.3) and (2.7). It means fulfillment of equalities $y_i = \bar{y}_i$ for any $i = \overline{0,N}$ or that solutions of the difference schemes (2.3) and (2.7) are the same. Let us show that we can obtain the difference scheme (2.3) from (2.7) by algebraic transformations.

Note that the transformation $x = x(q)$ of coordinate system in accordance with (2.4) and condition $x(1) = b$ can be written in the form

$$x(q) = a + c_1 \int_0^q \psi(q)\,dq\,, \quad c_1 = (b-a)\left(\int_0^1 \psi(q)\,dq\right)^{-1}.$$

We can assume without loss of generality that $c_1 = 1$. Thus grid transformation is defined as a one-to-one correspondence $q_i \mapsto x_i$ of the form

$$x_i = a + \int_0^{q_i} \psi(q)\,dq\,, \quad \psi(q) \geq c > 0\,. \tag{2.9}$$

Taking into account the obvious equalities

$$\begin{aligned} h_i = x_i - x_{i-1} &= \int_0^{q_i} \psi(q)\,dq - \int_0^{q_{i-1}} \psi(q)\,dq \\ &= \int_{q_{i-1}}^{q_i} \psi(q)\,dq\,, \quad \hbar_i = 0.5\int_{q_{i-1}}^{q_{i+1}} \psi(q)\,dq\,, \end{aligned}$$

for any discrete function y_i we can easily obtain a sequence of difference expressions:

$$\begin{aligned} (Ly)_i &= y_{\bar{x}\hat{x},i} + f(\bar{x}_i) \\ &= \frac{1}{\hbar_i}\left(\frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right) + f(\bar{x}_i) \\ &= \frac{1}{0.5\int_{q_{i-1}}^{q_{i+1}} \psi(q)\,dq}\left(\left(\frac{1}{h_q}\int_{q_i}^{q_{i+1}} \psi(q)\,dq\right)^{-1}\frac{y_{i+1}-y_i}{h_q}\right. \\ &\quad \left. - \left(\frac{1}{h_q}\int_{q_{i-1}}^{q_i} \psi(q)\,dq\right)^{-1}\frac{y_i - y_{i-1}}{h_q}\right) + \bar{f}(\tilde{q}_i) \\ &= \frac{1}{\frac{x_{i+1}-x_{i-1}}{2h_q}}\left(a_{i+1}\frac{y_{i+1}-y_i}{h_q} - a_i\frac{y_i - y_{i-1}}{h_q}\right) + \bar{f}(\tilde{q}_i) \\ &= \frac{1}{\psi_{h,i}}(a(\psi)y_{\bar{q}})_q + \bar{f}(\tilde{q}_i) = (\bar{L}y)_i\,. \end{aligned}$$

Here $\tilde{q}_i$ is a preimage of the point $\bar{x}_i$ for the correspondence (2.9). Thus any grid function that satisfies the difference scheme (2.3) is a solution of (2.7) and vice versa.

Characteristic feature of schemes (2.3) and (2.7) is an equal order of approximation for correspondent differential operators. So, each of them is invariant by order of approximation. In addition, we have to assume that $x = x(q)$ is sufficiently smooth function.

As shown in Ref. 20:

$$\delta(x_i) = u_{\bar{x}\hat{x},i} + f(\bar{x}_i) - \left(\frac{d^2u}{dx^2} + f(x)\right)\bigg|_{x=x_i} = O(\hbar_i^2)\,.$$

Taking into account that $1/\psi_{h,i} = (\psi(q))^{-1}|_{q=q_i} + O(h_q^2)$, and $(a(\psi)\,\bar{u}_{\bar{q}})_q = d/dq(1/\psi\; d\bar{u}/dq) + O(h_q^2)$ for any function on the grid $\widehat{\omega}_h$ we obtain

$$\bar{\delta}(q_i) = \frac{1}{\psi_{h,i}}(a(\psi)\,\bar{u}_{\bar{q}})_{q,i} + \bar{f}(\tilde{q}_i) - \left(\frac{1}{\psi}\frac{d}{dq}\left(\frac{1}{\psi}\frac{d\bar{u}}{dq}\right) + \bar{f}(q)\right)\bigg|_{q=q_i} = O(h_q^2)\,. \qquad (2.10)$$

Here

$$\begin{aligned}
\bar{f}(\tilde{q}_i) &= f\left(x_i + \frac{h_{i+1} - h_i}{3}\right) \\
&= f(x_i) + \frac{h_{i+1} - h_i}{3} f'(x_i) + O(\hbar_i^2) \\
&= \bar{f}(q_i) + \frac{h_q^2 x_{\bar{q}q}}{\psi^2} \bar{f}'(q_i) + O(h_q^2) \\
&= \bar{f}(q_i) + O(h_q^2)\,, \quad h_{i+1} - h_i = x_{i+1} - 2x_i + x_{i-1} = h_q^2 x_{\bar{q}q}\,.
\end{aligned}$$

3. Consistency of Grid Norms

Usually, stability and convergence investigation of difference scheme occurred in space with some grid norm that is an analogue of the correspondent norm in space of functions of continuous argument. Let $u = u(x)$, $a \le x \le b$ be a function variable x and $\overline{u} = \overline{u}(q)$, $0 \le q \le 1$, where x and q are linked by coordinate transformation (2.4). Then norms in spaces of continuous functions C and L_2 are defined in the form $\|u\|_{C[a,b]} = \max_{a \le x \le b}|u(x)|$, $\|\overline{u}\|_{C[0,1]} = \max_{0 \le q \le 1}|\overline{u}(q)|$, $\|u\|_{C[a,b]} = \|\overline{u}\|_{C[0,1]}$, $\|u\|_{L_2[a,b]} = \{\int_a^b u^2(x)\,dx\}^{1/2} = \{\int_0^1 \psi\overline{u}^2(q)\,dq\}^{1/2}$, $\|u\|^2 = (\psi, \overline{u}^2)$. The correspondent seminorms in W_2^1, W_2^2 are:

$$\|u\|_{W_2^1[a,b]} = \left\{\int_a^b u'^2(x)\,dx\right\}^{1/2} = \left\{\int_0^1 \psi^{-1}(q)\overline{u}'^2(q)\,dq\right\}^{1/2},$$

$$\|u\|_{W_2^2[a,b]} = \left\{\int_a^b u''^2(x)\,dx\right\}^{1/2} = \left\{\int_0^1 \psi^{-1}(q)\left(\psi^{-1}(q)\overline{u}'(q)\right)'^2\,dq\right\}^{1/2}.$$

We shall see that similar consistency of norms takes place in discrete case for the invariant difference scheme (2.7) and (2.8).

Let H_h be a space of grid functions defined on irregular grid $\hat{\bar{\omega}}_h$. As in Ref. 5 we define the following norms and scalar products:

$$\|v\|_* = (v,v)_*^{1/2} = \left(\sum_{i=1}^{N-1} \hbar_i v_i^2\right)^{1/2} \quad \text{— grid norm } L_2(\hat{\omega}_h)\,,$$

$$\|v_{\bar{x}}]|^2 = (v_{\bar{x}}, v_{\bar{x}}] = \sum_{i=1}^{N} h_i v_{\bar{x},i}^2 \text{ — grid seminorm } W_2^1(\hat{\omega}_h^+),\ \hat{\omega}_h^+ = \hat{\omega}_h \cup \{x_N = b\}\,,$$

$$\|v_{\bar{x}\hat{x}}\|_*^2 = (v_{\bar{x}\hat{x}}, v_{\bar{x}\hat{x}})_* = \sum_{i=1}^{N-1} \hbar_i v_{\bar{x}\hat{x},i}^2 \text{ — grid seminorm } W_2^2(\hat{\omega}_h)\,,$$

$$\|v\|_C = \max_{x\in\hat{\omega}_h} |v(x)| \text{ — uniform norm}\,.$$

Now suppose that H_{h_q} is a space of grid functions defined on the uniform grid $\overline{\omega}_{h_q}$. For any grid function $\overline{v} \in H_{h_q}$ let us define correspondent grid norms with weight function ψ_h given by (2.8):

$$(\psi_h, \overline{v}^2) = \sum_{i=1}^{N-1} h_q \psi_{hi} \overline{v}_i^2\,, \qquad (a, \overline{v}_{\bar{q}}^2] = \sum_{i=1}^{N} h_q a_i \overline{v}_{\bar{q},i}^2\,,$$

$$(\psi_h^{-1}, (a\overline{v}_{\bar{q}})_q^2) = \sum_{i=1}^{N-1} h_q \psi_{hi}^{-1} (a_i \overline{v}_{\bar{q},i})_{q,i}^2\,, \qquad \|\overline{v}\|_C = \max_{q\in\omega_{h_q}} |\overline{v}(q)|\,,$$

where $a = a(\psi) = [x_{\bar{q}}]^{-1}$.

It is easy to prove that for invariant difference scheme, the consistency of grid norms occurred, i.e.

$$\|y\|_{C(\hat{\omega}_h)} = \|\overline{y}\|_{C(\omega_{h_q})}\,, \qquad (\psi_h, \overline{y}^2) = \|y\|_*^2\,, \tag{3.1}$$

$$(a, \overline{y}_{\bar{q}}^2] = \|y_{\bar{x}}]|^2\,, \quad (\psi_h^{-1}, (a\overline{y}_{\bar{q}})_q^2) = \|y_{\bar{x}\hat{x}}\|_*^2\,. \tag{3.2}$$

Indeed, as a result of invariance and Jacobian approximation (2.8) we get

$$(\psi_h, \overline{y}^2) = \sum_{i=1}^{N-1} (x_{i+1/2} - x_{i-1/2}) \overline{y}^2(q_i) = \sum_{i=1}^{N-1} \hbar_i y^2(x_i) = \|y\|_*^2\,.$$

In the same way, we can prove (3.2). For example,

$$(a, \overline{y}_{\bar{q}}^2] = \sum_{i=1}^{N} h_q \frac{h_q}{h_i} \frac{(\overline{y}_i - \overline{y}_{i-1})^2}{h_q^2} = \sum_{i=1}^{N} h_i y_{\bar{x},i}^2 = \|y_{\bar{x}}]|^2\,.$$

4. *A Priori* Estimates

Let us demonstrate an application of the method of energy inequalities to stability investigation of the difference scheme (2.7) and (2.8) in seminorm (3.2). First we shall prove the following statement.

Lemma 1. *For any grid function $\overline{y}(q_i)$ defined on the uniform grid ω_{h_q} such that $\bar{y}(0) = \bar{y}(1) = 0$, it follows that*

$$(\psi_h, \overline{y}^2) \leq \frac{l^2}{4}(a, \overline{y}_{\bar{q}}^2] , \quad l = b - a . \tag{4.1}$$

Proof. From (3.1) and embedding[19] $\|y\|_* \leq (l/2)\|y_{\bar{x}}]|$ it follows that $(\psi_h, \overline{y}^2) \leq (l^2/4)\|y_{\bar{x}}]|^2$. Our estimate follows from (3.2) and the last inequality. □

Now we pay our attention to the problem of stability. Let us consider scalar product of (2.7) onto $\psi_h\overline{y}$. From Green's first difference formula, the energy identity follows:

$$(a(\psi_h), \overline{y}_{\bar{q}}^2] = (\psi_h\overline{\varphi}, \overline{y}) . \tag{4.2}$$

Using the Cauchy inequality and embedding (4.1) for scalar product $|(\psi_h\overline{\varphi}, \overline{y})| \leq \|\psi_h^{1/2}\overline{\varphi}\| \, \|\psi_h^{1/2}\overline{y}\| \leq (l/2)\|\psi_h^{1/2}\overline{\varphi}\| \, \|a^{1/2}\overline{y}_{\bar{q}}]|$ we obtain by means of (4.2) the following energy inequality:

$$(a(\psi_h), \overline{y}_{\bar{q}}^2] \leq \frac{l^2}{4}(\psi_h, \overline{\varphi}^2) . \tag{4.3}$$

The last estimate conveys the stability of difference scheme by the right side in the grid seminorm $W_2^1(\omega_{h_q})$ with a weight function $a(\psi_h)$.

Note that *a priori* estimate for solution of difference problem (2.1) can be obtained because of invariance of difference schemes and grid norms in (4.3). In original coordinate system for (2.3) we get $\|y_{\bar{x}}]| \leq (l/2)\|\varphi\|_*$.

By using embedding[18]

$$\|y\|_C \leq \frac{\sqrt{l}}{2}\|y_{\bar{x}}]| ,$$

which holds for any grid function $y(x)$ that defined on arbitrary nonuniform grid $\widehat{\omega}_h$ such that $y(0) = y(l) = 0$ we get the following estimate of stability:

$$\|y\|_C \leq \frac{l^{3/2}}{4}\|\varphi\|_* .$$

5. Difference Schemes for Parabolic Equations

In rectangular domain $Q_T = \{(x,t) : a \leq x \leq b, \ 0 \leq t \leq T\}$, let us consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t) , \tag{5.1}$$

$$u|_{x=a} = 0 , \quad u|_{x=b} = 0 , \quad u(x,0) = u_0(x) . \tag{5.2}$$

Assume that the nonuniform grid $\hat{\bar{\omega}}_{h\tau} = \hat{\bar{\omega}}_h \times \omega_\tau$ is defined in Q_T. Here $\hat{\bar{\omega}}_h$ was introduced in (2.2) and $\omega_\tau = \{t_j = j\tau,\ j = \overline{0, j_0},\ \tau = T/j_0\}$. Let us suppose that $x = x(q)$ is a change of space variables such that nodes of the grid $\hat{\bar{\omega}}_h$ turns into nodes of uniform grid $\hat{\bar{\omega}}_{h_q}$.

After substitution of x, for (5.1) we get

$$\frac{\partial \bar{u}}{\partial t} = \frac{1}{\psi}\frac{\partial}{\partial q}\left(\frac{1}{\psi}\frac{\partial \bar{u}}{\partial q}\right) + \bar{f}(q,t)\,, \tag{5.3}$$

where $\psi = \psi(q)$ is a Jacobian of transformation. It is obvious that

$$\psi(q) = \frac{dx}{dq}\,.$$

In case of regular change of variables, the last formula coincides with (2.4). As it was shown in Ref. 20 for Eq. (5.1) there is a difference scheme of the second-order of approximation at point $\bar{x}_i$. It has the form

$$y_{(\omega_1,\omega_2)t,i} = y^{(\sigma)}_{\bar{x}\hat{x},i} + \varphi^{(\sigma)}_i\,, \quad \varphi_i = f(\bar{x}_i, t_j)\,, \quad \sigma = \text{const} > 0\,, \tag{5.4}$$

where

$$\omega_{1,i} = \frac{h_{i+1} - h_i}{6h_{i+1}} - \left|\frac{h_{i+1} - h_i}{6h_{i+1}}\right|,$$

$$\omega_{2,i} = \frac{h_i - h_{i+1}}{6h_i} - \left|\frac{h_i - h_{i+1}}{6h_i}\right|,$$

$$i = \overline{1, N-1}\,,$$

$$u_{(\omega_1,\omega_2),i} = \omega_{1i}u_{i+1} + (1 - \omega_{1i} - \omega_{2i})u_i + \omega_{2i}u_{i-1}\,,$$

$$u^{(\sigma)} = \sigma u^{j+1}_i - (1-\sigma)\,u^j_i\,.$$

In coordinates (q,t) we can associate to the differential Eq. (5.3) a difference scheme that can be obtained from scheme (5.4) by means of algebraic transformations. Taking into account the equality (2.9) we obtain

$$\bar{y}_{(\alpha_1,\alpha_2)t,i} = \frac{1}{\psi_{h,i}}(a(\psi)\bar{y}_{\bar{q}})^{(\sigma)}_q - \bar{f}^{(\sigma)}(\tilde{q}_i, t)\,, \tag{5.5}$$

$$a_i = \left(\frac{1}{h_q}\int_{q_{i-1}}^{q_i} \psi(q)\,dq\right)^{-1} = [x_{\bar{q}}]^{-1}\,, \quad \psi_{h,i} = x_{\overset{\circ}{q},i}\,,$$

$$\alpha_1 = \frac{h_q x_{\bar{q}q}}{6x_q} - \left|\frac{h_q x_{\bar{q}q}}{6x_q}\right|, \quad \alpha_2 = -\frac{h_q x_{\bar{q}q}}{6x_{\bar{q}}} - \left|\frac{h_q x_{\bar{q}q}}{6x_{\bar{q}}}\right|.$$

Let us confirm that the difference scheme (5.5) approximates Eq. (5.3) with a second order by q. Note that it is sufficient to prove the equality

$$\bar{u}_{(\alpha_1,\alpha_2)t,i} - \frac{\partial \bar{u}}{\partial t}(q_i, t_j) = O(h_q^2 + \tau)\,.$$

Here we should take into account that for operator by space we have (2.10). Since $\alpha_1 = O(h_q)$, $\alpha_2 = O(h_q)$, we obtain

$$\bar{u}_{(\alpha_1,\alpha_2)t,i} = \alpha_1\bar{u}_{t,i+1} + (1-\alpha_1-\alpha_2)\bar{u}_{t,i} + \alpha_2\bar{u}_{t,i-1}$$
$$= \bar{u}_{t,i} + h_q\alpha_1\bar{u}_{qt,i} - h_q\alpha_2\bar{u}_{\bar{q}t} = \bar{u}_t + O(h_q^2)\,.$$

Further, to prove *a priori* estimates we shall use the following difference analogues of embedding theorems.[20]

Lemma 2. *Let $y(x)$ be a grid function defined on the nonuniform grid $\hat{\bar{\omega}}_h$ and $y(a) = y(b) = 0$. Then the following inequalities hold:*

$$\|y\|_* \le \frac{l^2}{4}\|y_{\bar{x}\hat{x}}\|_*\,, \quad \|y_{\bar{x}}]| \le \frac{l}{2}\|y_{\bar{x}\hat{x}}\|_*\,, \quad \|y_{\bar{x}}\|_C \le M\|y_{\bar{x}\hat{x}}\|_*\,, \tag{5.6}$$

where $l = b-a$, $\|y_{\bar{x}}\|_C = \max_{1\le i\le N}|y_{\bar{x},i}|$, $M^2 = \varepsilon + l/4 + l^2(1+c_3)/8\varepsilon$, $\varepsilon > 0$ is an arbitrary number, and for the constant c_3 we have

$$c_3^{-1} \le \max_i\left(\frac{h_i}{h_{i+1}}\right) \le c_3\,. \tag{5.7}$$

Let us consider the case of implicit scheme (5.4) when $\sigma = 1$. Using the method of energy inequalities and Lemma 2 we shall prove the following theorem.

Theorem 1. *Consider the scheme* (5.4) *under the assumptions* (5.7) *and*

$$\tau \ge \frac{c_4}{\varepsilon/2}\max_i|h_{i+1}-h_i|^2\,, \tag{5.8}$$

where $c_4 = \max_i(\hbar_i/h_i)$, $\varepsilon > 0$ is an arbitrary number. Then the scheme is stable by initial data and right-hand side, and the following estimates hold:

$$\begin{aligned} \|y_{\bar{x}\hat{x}}^{n+1}\|_* &\le M_1\left[\|y_{0\bar{x}\hat{x}}\|_* + \max_{0\le j\le n+1}\|\varphi^j\|_* + \tau\sum_{j=0}^{n}\|\varphi_t^j\|_*^2\right]\,,\\ \|y_{\bar{x}}^{n+1}\| &\le M_2\left[\|y_{0\bar{x}\hat{x}}\|_* + \max_{0\le j\le n+1}\|\varphi^j\|_* + \tau\sum_{j=0}^{n}\|\varphi_t^j\|_*^2\right]\,, \end{aligned} \tag{5.9}$$

where $M_1 = (1+\tau)^{n+1} \le e^T$, $M_2 = M_1M$ (M as defined in Lemma 2).

Proof. Multiplying Eq. (5.4) by $-2\tau\hbar_i y_{t\bar{x}\hat{x},i}$ and summing over all internal nodes of the grid $\widehat{\omega}_h$, we get

$$\begin{aligned} 2\tau\|y_{t\bar{x}}]|^2 + \tau^2\|y_{t\bar{x}\hat{x}}\|_*^2 + \|\hat{y}_{\bar{x}\hat{x}}\|_*^2 &= \|y_{\bar{x}\hat{x}}\|_*^2 + 2\tau(h_+\omega_1 y_{tx} - h\omega_2 y_{t\bar{x}}, y_{t\bar{x}\hat{x}})_*\\ &\quad + 2\tau(\hat{\varphi}, y_{t\bar{x}\hat{x}})_*\,, \end{aligned} \tag{5.10}$$

where $h_+ = h_{i+1}$, $h = h_i$, $\hat{\varphi} = \varphi^{j+1}$. Using the Cauchy inequality and the identity $\tau(\hat{v}, y_t)_* = (\hat{v}, \hat{y})_* - (v, y)_* - \tau(v_t, y)_*$, one can see that the following relation holds:

$$-\|\hat{y}_{\bar{x}\hat{x}}\|_*^2 + \|y_{\bar{x}\hat{x}}\|_*^2 + 2\tau(\hat{\varphi}, y_{t\bar{x}\hat{x}})_* \leq -\|\hat{y}_{\bar{x}\hat{x}} - \hat{\varphi}\|_*^2 + (1+\tau)\|y_{\bar{x}\hat{x}} - \varphi\|_*^2 + \tau(1+\tau)\|\varphi_t\|_*^2 .$$

From embedding (5.6), condition (5.7), and the Cauchy inequality, we get the following:

$$\begin{aligned} 2\tau(h_+\omega_1 y_{tx}, y_{t\bar{x}\hat{x}})_* &\leq 2\tau\left(\max_i |h_+\omega_1|\right) \|y_{tx}\|_* \|y_{t\bar{x}\hat{x}}\|_* \\ &\leq 2\tau\sqrt{c_4}\left(\max_i |h_+\omega_1|\right) \|y_{t\bar{x}}]| \ \|y_{t\bar{x}\hat{x}}\|_* \\ &\leq \frac{2c_4}{\varepsilon}\left(\max_i |h_+\omega_1|\right)^2 \|y_{t\bar{x}}]|^2 + \frac{\varepsilon\tau^2}{2}\|y_{t\bar{x}\hat{x}}\|_*^2 . \end{aligned}$$

In the same way,

$$-2\tau(h\omega_2 y_{t\bar{x}}, y_{t\bar{x}\hat{x}})_* \leq \frac{2c_4}{\varepsilon}\left(\max_i |h\omega_2|\right)^2 \|y_{t\bar{x}}]|^2 + \frac{\varepsilon\tau^2}{2}\|y_{t\bar{x}\hat{x}}\|_*^2 .$$

Since $\max_i |h\omega_2| = \max_i |h_+\omega_1| = \max_i |h_+ - h|$, using the last estimates in (5.10), we have

$$\begin{aligned} &\|\hat{y}_{\bar{x}\hat{x}} - \hat{\varphi}\|_*^2 + 2\left(\tau - \frac{2c_4}{\varepsilon}\max|h_+ - h|^2\right)\|y_{t\bar{x}}]|^2 + (1-\varepsilon)\tau^2\|y_{t\bar{x}\hat{x}}\|_*^2 \\ &\quad \leq (1+\tau)\|y_{\bar{x}\hat{x}} - \varphi\|_*^2 + \tau(1+\tau)\|\varphi_t\|_*^2 . \end{aligned}$$

Taking into account the conditions of Theorem 1 we obtain

$$\|y_{\bar{x}\hat{x}}^{n+1} - \varphi^{n+1}\|_*^2 \leq (1+\tau)^{n+1}\|y_{0\bar{x}\hat{x}} - \varphi_0\|^2 + \tau\sum_{j=0}^{n}(1+\tau)^{n-j+1}\|\varphi_t^j\|_*^2 .$$

From here and (5.6), Theorem 1 is proven. □

Remark. Recall that the scheme (5.4) possesses the second-order of approximation. For the solution of the scheme (5.4) as well as for its derivative one can easily prove, by means of (5.9), that

$$\|z\|_C , \quad \|z_{\bar{x}}\|_C \leq c_0(h^2 + \tau) ,$$

where $z = y - u$, $c_0 = \text{const}$, $h = \max_i h_i$.

From invariance of the difference schemes, consistency of the grid norms, and inequalities (5.9) we immediately obtain the corresponding *a priori* estimates for the solution of the scheme (5.5) defined on the uniform grid $\overline{\omega}_{h_q}$. Moreover, the condition (5.8) on τ can be formulated in terms of functions on the net $\bar{\omega}_{h_q}$ as $\tau \geq c_5 h_q^2$, $c_5 = (2c_4/\varepsilon)\max_i |x_{\bar{q}q,i}|$. Note that in case of uniform grid $h_{i+1} - h_i =$

$h_q^2 x_{\bar{q}q} = 0$, i.e. the restriction on τ is eliminated. So the schemes are absolutely stable.

5.1. *Monotone difference schemes of second order of precision*

Let us consider an implicit difference scheme that approximate Eq. (5.1) with the second order at point $\bar{x}_i$. Moreover, we shall assume fulfillment of the monotone condition and for clearness that for steps of the grid $\widehat{\omega}_h$ we have $h_{i+1} \geq h_i$. The monotone condition means that for any three-point equation written in the canonical form

$$L[y_i] = A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i\,, \quad i = 1, 2, \ldots, N-1\,, \tag{5.11}$$

it follows that the conditions of maximum principle (see Ref. 18) hold.

Theorem (maximum principle). *Suppose that for any* $i = 1, 2, \ldots, N-1$ *coefficients of Eq.* (5.11) *satisfy property*

$$|A_i| > 0\,, \quad |B_i| > 0\,, \quad D_i = |C_i| - |A_i| - |B_i| > 0\,, \quad i = 1, 2, \ldots, N-1\,. \tag{5.12}$$

Then from conditions

$$L[y_i] \geq 0\,, \quad (L[y_i] \leq 0)$$

which fulfilled for any $i = 1, 2, \ldots, N-1$ *it follows that non-constant grid function* y_i *cannot achieve the largest positive (least negative) value at internal points, i.e. when* $i = 1, 2, \ldots, N-1$.

In this assumptions the difference scheme for Eq. (5.1) has the form

$$y_t + \frac{h_{i+1} - h_i}{3} y_{tx} = \hat{y}_{x\hat{x}} + \varphi\,, \quad y_0 = y_N = 0\,, \quad \varphi = \hat{f}(\bar{x}_i)\,. \tag{5.13}$$

Note that unlike scheme (5.4), derivative by time is approximated by flux. Due to this circumstance it is possible to use the maximum principle to investigate the scheme. Although estimates obtained by means of maximum principle are carried out in more severe constraints on time step τ it should be noted that the principle is applicable to problems of arbitrary dimension.

From the maximum principle, the following statement[18] immediately follows.

Corollary. *Suppose that the properties* (5.12) *are satisfied. Then for the solution of the problem*

$$L[y_i] = -F_i\,, \quad i = 1, 2, \ldots, N-1\,, \quad y_0 = 0\,, \quad y_N = 0$$

it follows that

$$\|y\|_C \leq \left\| \frac{F}{D} \right\|_C . \tag{5.14}$$

Let us represent the difference scheme (5.13) in the form (5.11). For the coefficients of three-point equation, we get

$$A_i = \frac{\tau}{\hbar h_{i+1}} - \frac{h_{i+1} - h_i}{3h_{i+1}}, \quad B_i = \frac{\tau}{\hbar_i h_i}, \quad C_i = 1 + A_i + B_i,$$
$$F_i = \frac{2h_{i+1} + h_i}{3h_{i+1}} y_i + \frac{h_{i+1} - h_i}{3h_{i+1}} y_{i+1} + \tau\varphi_i. \tag{5.15}$$

It is readily seen that positivity of the coefficients (5.15) take place when

$$\tau > \frac{h_{i+1}^2 - h_i^2}{6}.$$

From (5.14) it follows that at every time moment

$$\|y^{j+1}\|_C \le \max_i \left| \frac{2h_{i+1} + h_i}{3h_{i+1}} y_i^j + \frac{h_{i+1} - h_i}{3h_{i+1}} y_{i+1}^j + \tau\varphi_i^j \right|$$
$$\le \max_i \{|y_i^j|, |y_{i+1}^j|\} + \tau \max_i |\varphi_i^j| = \|y^j\|_C + \tau\|\varphi^j\|_C.$$

Summing over all $j = 0, 1, \ldots, n$, we get

$$\|y^{n+1}\|_C \le \|y^0\|_C + \tau \sum_{j=0}^{n} \|\varphi^j\|_C. \tag{5.16}$$

The approximation error of the scheme (5.13) at the point $\bar{x}_i$ is equal by order to $O(\hbar_i^2 + \tau)$. Using (5.16) we shall get an estimate of the order of convergency for (5.13). For $z = y - u$ we have

$$z_{t,i} + \frac{h_{i+1} - h_i}{3} z_{tx,i} = \hat{z}_{\bar{x}\hat{x},i} + \psi_i.$$

From (5.16) we obtain

$$\|z^{n+1}\|_C \le \tau \sum_{j=0}^{n} \|\psi^j\|_C = O(\hbar_i^2 + \tau).$$

The scheme that invariant to (5.13) has the form

$$\bar{y}_{(\alpha_1,\alpha_2)t,i} = \frac{1}{\psi_{h,i}} (a(\psi)\,\hat{\bar{y}}_{\bar{q}})_q - \hat{\bar{f}}(\tilde{q}_i, t), \quad \alpha_{1,i} = \frac{h_q^2 x_{\bar{q}q,i}}{3x_{q,i}}, \quad \alpha_{2,i} = 0.$$

The results obtained above and the principle of consistency for the grid norms allow us to derive estimates of stability and convergency in space of variables (q, t).

5.2. *Difference schemes for problems with moving boundaries*

Suppose that the domain $\bar{Q}_T$ is a parallelogram: $\bar{Q}_T = \{(x,t) : a + vt \le x \le b + vt,\ 0 \le t \le T\}$, $v = \text{const} > 0$, $0 \le a < b$. We shall assume the conditions on the left and right boundaries of the domain as well as initial data for Eq. (5.1) as

$$u|_{x=a+vt} = 0\,, \quad u|_{x=b+vt} = 0\,, \quad u(x,0) = u_0(x)\,.$$

The initial domain $\bar{Q}_T$ transforms to rectangular $\overline{D} = \{(q,t) : 0 \le q \le 1,\ 0 \le t \le T\}$ by a change of variables $x(q,t) = \psi q + vt' + a$ with $\psi = b - a$. If $\psi = \partial x/\partial q$ is a metric coefficient and $\partial x/\partial t = v$ is a speed of the system movement which in general should be determined, then for the partial derivatives, we get

$$\begin{aligned} &\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \frac{\partial x}{\partial t'}\frac{1}{\psi}\frac{\partial}{\partial q} = \frac{\partial}{\partial t'} - \frac{v}{\psi}\frac{\partial}{\partial q}\,, \\ &\frac{\partial}{\partial x} = \frac{\partial q}{\partial x}\frac{\partial}{\partial q} = \frac{1}{\psi}\frac{\partial}{\partial q}\,, \quad \frac{\partial^2}{\partial x^2} = \frac{1}{\psi}\frac{\partial}{\partial q}\left(\frac{1}{\psi}\frac{\partial}{\partial q}\right). \end{aligned} \tag{5.17}$$

Equation (5.1) can be written in the form

$$\frac{\partial \bar{u}}{\partial t'} - r\frac{\partial \bar{u}}{\partial q} = \frac{\partial}{\partial q}\left(k\frac{\partial \bar{u}}{\partial q}\right) + \bar{f}\,, \tag{5.18}$$

where $r = v/\psi > 0$, $k = 1/\psi^2 > 0$ are constants.

Note that any uniform grid in calculated space (q,t) generates a uniform grid with the constant step $h = \psi h_q$ in "physical" space (see Fig. 1) $\overline{\omega}_h^j = \{(x_i^j, t_j),\ x_i^j = ih + x_0^j,\ h = \psi h_q,\ x_0^j = a + vt_j,\ i = \overline{0,N}\}$, $j = \overline{0, j_0}$.

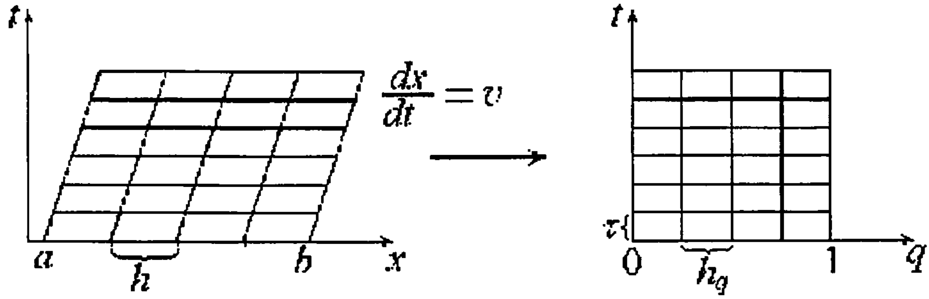

Fig. 1. Transformation of computational domains.

For Eq. (5.18) let us consider a monotone difference scheme of the second-order of approximation (see Ref. 18)

$$\begin{gathered} \overline{y}_t = \overline{\varkappa}(k\overline{y}_{\bar{q}}^{(\sigma)})_q + r\overline{y}_q^{(\sigma)} + \overline{\varphi}\,, \\ \overline{\varkappa} = \frac{1}{1+R}\,, \quad R = \frac{0.5rh_q}{k}\,, \quad \overline{\varphi} = \overline{f}^{(\sigma)}\,, \quad \overline{y}_0^{j+1} = \overline{y}_N^{j+1} = 0\,, \quad \overline{y}_i^0 = u_{0i}\,, \end{gathered} \tag{5.19}$$

where $\overline{y}_t = [\overline{y}(q_i, t'_{j+1}) - \overline{y}(q_i, t'_j)]/\tau = (\hat{\overline{y}} - \overline{y})/\tau$.

Approximation error. Let us consider the residual $\delta_{h_q} = -\overline{u}_t + v(\psi^{-1}\overline{u}_q^{(\sigma)}) + \overline{\varkappa}\psi^{-1}(\psi^{-1}\overline{u}_{\bar{q}}^{(\sigma)})_q$ of scheme (5.19).

Note that $R = O(h_q)$. Using Taylor series expansion of the function $\overline{u}(q, t')$ at the neighborhood of (q_i, t'_j) it can be proved that

$$\delta_{h_q} = \tau\sigma\left(\frac{1}{\psi}\frac{\partial}{\partial q}\left(\frac{1}{\psi}\frac{\partial^2 \overline{u}}{\partial t'\partial q}\right) + \frac{1}{\psi}\frac{\partial^2 \overline{u}}{\partial t'\partial q}\right) + O\big((h_q\psi)^2 + \tau\big)\,.$$

Suppose that $\overline{y}$ is a solution of the difference scheme (5.19) on the grid $\omega_{h_q} \times \omega_\tau$ and the function y defined at the node of the moving grid $\overline{\omega}_h^j$. To analyze invariance property let us introduce the following notation: $y = y(x(q_i, t_j), t_j) = \overline{y}(q_i, t'_j) = \overline{y}$, $\hat{y} = y(x(q_i, t_{j+1}), t_{j+1}) = \overline{y}(q_i, t'_{j+1}) = \hat{\overline{y}}$.

From (5.18) and algebraic relations

$$\frac{y_q}{\psi} = \frac{\overline{y}^j_{i+1} - \overline{y}^j_i}{\psi h_q} = \frac{y_{i+1} - y_i}{h} = y_x\,, \quad \psi^{-1}(\psi^{-1}y_{\bar{q}})_q = y_{\bar{x}x}\,,$$

$$\frac{\overline{r}}{\overline{k}} = v\psi\,, \quad R = \frac{\psi h_q v}{2} = \frac{hv}{2}\,, \tag{5.20}$$

we get the following difference scheme

$$\begin{gathered}\frac{(\hat{y} - y)}{\tau} - v y_x^{(\sigma)} = \varkappa y_{\bar{x}x}^{(\sigma)} + \varphi\,,\\ y(x_0^{j+1}, t_{j+1}) = y(x_N^{j+1}, t_{j+1}) = 0\,, \quad \varkappa = \frac{1}{1 + 0.5hv}\,.\end{gathered} \tag{5.21}$$

The last scheme approximates the initial difference Eq. (5.1) on the grid $\bar{\omega}_h \times \omega_\tau$.

Residual for (5.21) is an implication of (5.20), (5.17), and it has the form

$$\delta_{h_q} = \delta(x_i(t_j), t_j) = -u_t + v u_x^{(\sigma)} + \varkappa u_{\bar{x}x}^{(\sigma)} = \tau\sigma\left(\frac{\partial^3 u}{\partial t\partial x^2} + \frac{\partial^2 u}{\partial t\partial x}\right) + O(h^2 + \tau)\,.$$

From here it follows that the local order of approximation in initial physical space remains the same.

To investigate a stability of the scheme (5.21) on the moving grid $\overline{\omega}_h^j$ let us use the general theory of stability.[18] To do that one has to bring a scheme to the canonical form of two-layered operator difference schemes

$$By_t + Ay = \varphi\,, \quad y(0) = u_0\,, \tag{5.22}$$

where $y = (0, y_1^j, y_2^j, \ldots, y_{N-1}^j, 0)$ is a required vector and A, B are linear operators defined in finite-dimensional space $H = \omega_h^j$.

Suppose that the operators A and B are defined as

$$B = E + \tau\sigma A, \quad A = A_1^+ + A_2, \tag{5.23}$$

$$(A_1^+ y)_i = \begin{cases} 0, & i = 0, \\ -v y_{x,i}, & i = \overline{1, N-1}, \\ 0, & i = N, \end{cases} \tag{5.24}$$

$$(A_2 y)_i = \begin{cases} 0, & i = 0, \\ -\varkappa y_{\bar{x}x}, & i = \overline{1, N-1}, \\ 0, & i = N. \end{cases} \tag{5.25}$$

Let us show the positivity of A. Since $A_2 = A_2^* > 0$, it is sufficient to prove that the operator A_1^+ is non-negative. This follows from

$$(A_1^+ y, y) = -\sum_{i=1}^{N-2} hv \left(\frac{y_i + y_{i+1}}{2} \frac{y_{i+1} - y_i}{h} - \frac{h}{2} y_{x,i}^2 \right) + v y_{N-1}^2$$

$$= \frac{v}{2} y_1^2 + \frac{hv}{2} \sum_{i=1}^{N-2} h y_{x,i}^2 + \frac{v}{2} y_{N-1}^2 > 0.$$

Further we shall use the following:

Lemma 3. *For any function $y(x)$ defined on the grid*

$$\bar{\omega}_h = \{x_i = ih, \ 0 \le i \le N, \ x_0 = 0, \ x_N = l\}$$

and such that $y(0) = y(l) = 0$, the following inequality holds

$$\|y\|_A \le M_1 \|\tilde{A} y\|, \tag{5.26}$$

where $(Ay)_i = -(a y_{\bar{x}})_{x,i}$, $(\tilde{A} y)_i = -(a y_{\bar{x}})_{x,i} - v y_{x,i}$, $M_1 = l c_2 / (2\sqrt{2c_1}(c_2 + hv))$.

Proof. Multiplying $-(v y_x)_i$ by $h y_i$ and summing over all the interior nodes of the grid $\bar{\omega}_h$, we get the identity

$$(-v y_x, y) = \frac{hv}{2} \sum_{i=1}^{N-1} h y_{\bar{x},i}^2 + \frac{v}{2} y_{N-1}^2 = \frac{hv}{2} \|y_{\bar{x}}]|^2. \tag{5.27}$$

Since $a \le c_2$, we can see that for $\|y_{\bar{x}}]|^2$ the following inequality holds

$$\|y_{\bar{x}}]|^2 = \left[\frac{1}{a} a y_{\bar{x}}, y_{\bar{x}} \right] \ge \frac{1}{c_2} \|y\|_A^2.$$

If we combine this with (5.27), we obtain

$$(-v y_x, y) \ge = \frac{hv}{c_2} \|y\|_A^2.$$

On the other hand,

$$(\tilde{A}y, y) = (-(ay_{\bar{x}})_x - vy_x, y) \geq \left(1 + \frac{hv}{c_2}\right) \|y\|_A^2 .$$

Finally, using the Cauchy inequality and the embedding $\|y\| \leq 2/2\sqrt{2}c_1\|y\|_A$, we get

$$(\tilde{A}y, y) \leq \frac{l}{2\sqrt{2}c_1}\|\tilde{A}y\| \, \|y\|_A .$$

The last two inequalities proves the Lemma. □

Also, to prove *a priori* estimates we shall use (see Ref. 18):

Lemma 4. *Let A be a positive and not self-adjoint operator. If $\sigma \geq 0.5$, then for the scheme* (5.22), (5.23) *one has*

$$\|y^{j+1}\| \leq \|y_0\| + \|(A^{-1}\varphi)^0\| + \|(A^{-1}\varphi)^j\| + \sum_{k=1}^{j} \tau\|(A^{-1}\varphi)_{\bar{t},k}\| . \qquad (5.28)$$

From here it follows that the difference scheme (5.21) on moving grid is stable in grid norm L_2. By means of Theorem 2 one can obtain strong estimate from Ref. 21. Here we cite this theorem as

Lemma 5. *Consider the scheme* (5.22), (5.23) *under the assumptions of Lemma* 4. *The scheme is stable by initial data, right-hand side, and the following estimate holds*:

$$\|Ay^{j+1}\| \leq \|Ay^0\| + \|\varphi^0\| + \|\varphi^j\| + \sum_{k=0}^{j-1} \tau\|\varphi_{t,k}\| .$$

This lemma allows us to get the following *a priori* estimate for the solution of (5.21)

$$\|vy_x^j + y_{\bar{x}x}^j\| \leq \|vy_x^0 + y_{\bar{x}x}^0\| + \|\varphi^0\| + \|\varphi^j\| + \sum_{k=0}^{j-1} \tau\|\varphi_{t,k}\| .$$

From the last estimate, Lemma 3, and the embedding $\|y\|_C \leq \sqrt{l}/(2\sqrt{c_1})\|y\|_A$ one can get the estimate of the scheme (5.21) in the norm C

$$\|y\|_C \leq M\left(\|vy_x^0 + y_{\bar{x}x}^0\| + \|\varphi^0\| + \|\varphi^j\| + \sum_{k=0}^{j-1} \tau\|\varphi_{t,k}\|\right), \quad M = \frac{l^{3/2}}{4\sqrt{2}c_1} .$$

Due to consistency of the grid norms $\|y\|^2 = \sum_{i=1}^{N-1} h(y_i^j)^2 = \sum_{i=1}^{N-1} h_q\psi(\overline{y}_i^j)^2 = (\psi, (\overline{y}_i^j)^2)$ one can see that the solution $\overline{y}$ of scheme (5.19) is stable in the calculated space $\omega_{h_q} \times \omega_\tau$. For example, *a priori* estimate in the grid norm $L_2(\omega_{h_q})$ has the form

$$\|\psi^{1/2}\overline{y}^{j+1}\| \leq \|\psi^{1/2}\overline{y}^0\| + \|\psi^{1/2}(A^{-1}\overline{\varphi})^0\| + \|\psi^{1/2}(A^{-1}\overline{\varphi})^j\| + \sum_{k=1}^{j} \|\psi^{1/2}(A^{-1}\overline{\varphi})_{\bar{t},k}\| ,$$

where $A = A_1^+ + A_2$ is an operator in finite-dimensional Hilbert space H_{h_q}. According to (5.20), (5.24), (5.25) this operator can be defined as follows:

$$(A_1^+\overline{y})_i = \begin{cases} 0\,, & i = 0\,, \\ -r\overline{y}_{q,i}\,, & i = \overline{1, N-1}\,, \\ 0\,, & i = N\,, \end{cases}$$

$$(A_2\overline{y})_i = \begin{cases} 0\,, & i = 0\,, \\ -\varkappa k\overline{y}_{\bar{q}q}\,, & i = \overline{1, N-1}\,, \\ 0\,, & i = N\,. \end{cases}$$

Thus the requirement of absolute stability on the moving grid ω_h^j as well as on the rectangular grid ω_{h_q} is fulfilled whenever $\sigma \geq 0.5$.

6. Conclusion

From computational practice point of view, the invariancy requirement allows us to construct difference schemes for differential problems written in different coordinate systems in such a way that relatively to a given class of transformations of independent variables, important properties such as conservativity, the order of approximation, stability, convergency, etc. are preserved.

References

1. J. E. Thompson, Z. U. A. Warsi and C. W. Mastin, *J. Comput. Phys.* **47** (1982) 1–108.
2. J. U. Brackbill and J. S. Saltzman, *J. Comput. Phys.* **46** (1982) 342–368.
3. H. Zhang and K. Moalemi, *Num. Heat Transfer. Part* **B27** (1995) 255–276.
4. M. Vinokur, *J. Comput. Phys.* **50** (1983) 215–234.
5. A. J. Wathen, *J. Comput. Phys.* **101** (1992) 51–54.
6. R. G. Hindman, P. Kutler and D. Anderson, *AIAA J.* **19** (1981) 424–431.
7. K. Nakahaschi and G. S. Deiwert, *AIAA J.* **25** (1987) 513–520.
8. R. Richtmyer and K. Morton, **Difference Methods for Initial-Value Problems** (Wiley, 1967).
9. A. A. Samarskii and U. P. Popov, **Difference Methods for Problems of Gas Dynamics** (Nauka, 1992) (in Russian).
10. S. K. Godunov, A. V. Zabrodin, M. Ya. Ivanov, A. N. Kraiko and G. P. Prokopov, **Numerical Methods for Multi-Dimensional Problems of Gas Dynamics** (Moscow, 1976).
11. N. A. Dar'in and V. I. Mazhukin, *Dokl. Akad. Nauk SSSR* **298** (1988) 64–68.
12. V. F. Vasilevskii and V. I. Mazhukin, *Different Urav.* **25** (1989) 1188–1193.
13. V. I. Mazhukin, A. A. Samarskii, O. Costelianos and A. V. Shapranov, *Math. Model.* **5** (1993) 32–56.
14. N. A. Dar'in, V. I. Mazhukin and A. A. Samarskii, *Dokl. Akad. Nauk SSSR* **302** (1988) 1078–1081.
15. P. V. Breslavskii and V. I. Mazhukin, *Math. Model.* **7** (1995) 48–78.
16. N. A. Dar'in and V. I. Mazhukin, *Different Urav.* **123** (1987) 1154–1160.
17. N. A. Dar'in and V. I. Mazhukin, *Math. Model.* **1** (1989) 29–43.

18. A. A. Samarskii, **Theory of Difference Schemes** (Nauka, 1977) (in Russian).
19. A. A. Samarskii and A. V. Gulin, **Stability of Difference Schemes** (Nauka, 1973) (in Russian).
20. A. A. Samarskii, P. N. Vabishchevich and P. P. Matus, *Different. Urav.* **32** (1996) 265–274.
21. P. N. Vabishchevich and P. P. Matus, *Dokl. Akad. Nauk Belarusi* **37** (1993) 15–17.
22. A. A. Samarskii, V. I. Mazhukin, P. P. Matus and M. M. Chuiko, *Different. Urav.* **32** (1997) 1–10.
23. A. A. Samarskii, V. I. Mazhukin and P. P. Matus, *Dokl. RAN* **352** (1997) 602–605.