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DIFFERENCE SCHEMES FOR THE PROBLEM OF FUSING HYPERBOLIC AND PARABOLIC EQUATIONS[†)]

A. A. Samarskiĭ, P. N. Vabishchevich, S. V. Lemeshevskiĭ, and P. P. Matus

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1. Introduction. In mathematical modeling of physical-chemical processes in composite bodies. it is often necessary to use the mathematical models that are based on equations of different type in different parts of the calculation domain. A particular attention in this event is paid to the fusion conditions on the boundaries of subdomains. The questions of unique solvability of boundary value problems for equations of mixed type are intensely discussed in the literature (see, for instance, [1]).

Concerning this class of problems, we distinguish boundary value problems for hyperbolic-parabolic equations whose unique solvability in the class of weak solutions was considered in [2, 3]. In the present article, taking as an example the simplest one-dimensional boundary value problem, we discuss the questions of numerical solution of the problems. We construct a homogeneous difference scheme [4] that belongs to the class of schemes with variable (discontinuous) weight factors [5–7]. We distinguish the classes of unconditionally stable schemes and study the convergence rate of an approximate solution to an exact solution.

2. Statement of the Problem. Consider the following initial-boundary value problem of fusing hyperbolic and parabolic equations in the rectangle $\overline{Q} = \overline{Q}_1 \cup \overline{Q}_2$, $Q_1 = \{(x,t) : 0 < x < \xi,\ 0 < t \leq T\}$. $Q_2 = \{(x,t) : \xi < x < l,\ 0 < t \leq T\}$:

$$\rho_1(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k_1(x)\frac{\partial u}{\partial x}\right) + f_1(x,t), \quad (x,t) \in Q_1, \tag{2.1}$$

$$\rho_2(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k_2(x)\frac{\partial u}{\partial x}\right) + f_2(x,t), \quad (x,t) \in Q_2, \tag{2.2}$$

where $\rho_m(x)$ is a strictly positive function in $\overline{Q}_m$ and $0 < c_1 \leq k_m(x) \leq c_2$ $(m = 1, 2)$. We supplement these equations with the following boundary and initial conditions:

$$u(0,t) = u(l,t) = 0, \quad t > 0, \tag{2.3}$$

$$u(x,0) = u_0(x),\ 0 \leq x \leq l; \quad \frac{\partial u}{\partial t}(x,0) = u_1(x),\ \xi \leq x \leq l. \tag{2.4}$$

The following fusion conditions are satisfied on the interface $x = \xi$ between the two domains:

$$[u] = 0, \quad \left[k\frac{\partial u}{\partial x}\right] = 0 \quad \text{for} \quad x = \xi, \quad t \geq 0, \tag{2.5}$$

where

$$[u] = u(\xi + 0, t) - u(\xi - 0, t), \quad k(x) = \begin{cases} k_1(x), & 0 < x < \xi, \\ k_2(x), & \xi < x < l. \end{cases}$$

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Constructing and studying difference schemes for the fusion problem under consideration, we assume that the coefficients $\rho_m(x)$, $k_m(x)$, and $f_m(x,t)$ $(m = 1,2)$ have the straight line $x = \xi$ a discontinuity of the first kind and are sufficiently smooth outside the discontinuity line. Under the above assumptions on the input data, the questions of existence and uniqueness of a strong solution were studied in the articles [2,3]. Henceforth we suppose that a solution $u(x,t)$ to (2.1)–(2.5) is piecewise smooth; i.e., it possesses all needed derivatives outside the line $x = \xi$ which are continuous and bounded, and satisfies the fusion conditions (2.5) on the straight line $x = \xi$. We note that the assumption that the coefficients ρ_m and k_m depend on a single variable x and that the boundary conditions (2.3) are homogeneous is only made to simplify exposition.

3. Difference schemes. We introduce the following uniform grids of knots:

$$\bar{\omega}_h = \{x_i = ih,\ i = 0,1,\dots,N;\ hN = l\},$$

$$\omega_\tau = \{t_n = n\tau,\ n = 0,1,\dots,N_0 - 1;\ \tau N_0 = T\}.$$

We suppose that the discontinuity point of the coefficients $\xi = x_p = ph \in \omega_h$, $2 \le p \le N-2$, is a knot of the uniform space grid. Also, we consider the following grids in the domains Q_1 and Q_2:

$$\omega_1 = \omega_{1h} \times \omega_\tau, \quad \omega_2 = \omega_{2h} \times \omega_\tau,$$

where $\omega_{1h} = \{x_i = ih,\ i = 1,2,\dots,p-1\}$ and $\omega_{2h} = \{x_i = ih,\ i = p+1,p+2,\dots,N-1\}$.

We approximate the differential problem by the three-layer difference scheme

$$\rho_1 y_t = ((a y_{\bar{x}})^{(\sigma_1,\sigma_2)})_x + \varphi, \quad (x,t) \in \omega_1, \tag{3.1}$$

$$\rho_2 y_{\bar{t}t} = ((a y_{\bar{x}})^{(\sigma_1,\sigma_2)})_x + \varphi, \quad (x,t) \in \omega_2, \tag{3.2}$$

$$\hat{y}_0 = \hat{y}_N = 0, \tag{3.3}$$

$$y(x,0) = u_0(x), \quad y_t(x,0) = \bar{u}_0(x), \quad x \in \omega_{1h}^-, \qquad \omega_{1h}^- = \omega_{1h} \cup \{0\}, \tag{3.4}$$

$$y(x,0) = u_0(x), \quad y_t(x,0) = \bar{u}_1(x), \quad x \in \bar{\omega}_{2h}, \tag{3.5}$$

with constant weights σ_α, $\alpha = 1,2$.

As in the case of the third boundary value problem for the one-dimensional heat equation [4, p. 95], we approximate the fusion conditions (2.5) so as to fulfill the requirement of second-order approximation in the space variable:

$$(a y_{\bar{x}})^{(\sigma_1,\sigma_2)} + \frac{h}{2}(\rho_1 y_t + \varphi) = (a y_x)^{(\sigma_1,\sigma_2)} - \frac{h}{2}(\rho_2 y_{\bar{t}t} + \varphi), \quad x = \xi. \tag{3.6}$$

Here

$$\rho_k = \rho_k(x_i), \quad a = k(x_{i-0.5}), \quad \varphi = 0.5(f_{i-0.5} + f_{i+0.5}), \tag{3.7}$$

$$\bar{u}_0 = \rho_1^{-1}(x)(Lu_0(x) + f(x,0)), \tag{3.8}$$

$$\bar{u}_1 = u_1(x) + 0.5\tau\rho_2^{-1}(x)(Lu_0(x) + f(x,0)),$$

$$Lu = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right), \quad f(x,t) = \begin{cases} f_1(x,t), & (x,t) \in Q_1, \\ f_2(x,t), & (x,t) \in Q_2. \end{cases}$$

It is convenient to write the approximate fusion conditions (3.6) as

$$0.5(\rho_1 y_t + \rho_2 y_{\bar{t}t}) = ((a y_{\bar{x}})^{(\sigma_1,\sigma_2)})_x + \varphi, \quad x = \xi. \tag{3.9}$$

Observe also that the second initial condition $y_t(x,0) = \bar{u}_0$ for the parabolic equation is derived by the following arguments. Employing a three-layer scheme, we have to know one more initial condition.

for instance $y(x, \tau)$: therefore, it is natural to approximate this condition with accuracy $O(h^2 + \tau^2)$. The idea (see [4, p. 97]) is to search the value $y(x, \tau)$ in the form

$$y(x,\tau) = u_0(x) + \tau\mu(x)$$

and choose μ so that the error $y(x,\tau) - u(x,\tau)$ be $O(\tau^2 + h^2)$.

Insert the value $\frac{\partial u}{\partial t}\Big|_{t=0}$ in the formula

$$u(x,\tau) - u_0(x) = \tau \frac{\partial u}{\partial t}\Big|_{t=0} + \frac{\tau^2}{2}\frac{\partial^2 u}{\partial t^2}\Big|_{t=0} + O(\tau^3)$$

and use the differential equation

$$\rho_1 \frac{\partial u}{\partial t}\Big|_{t=0} = Lu_0 + f_1(x,0), \quad 0 < x < \xi.$$

Then

$$\mu = (Lu_0 + f_1(x,0))/\rho_1(x).$$

Hence, we obtain (3.4).

Also, we indicate another way of specifying $y(x,\tau)$ with accuracy $O(h^2 + \tau^2)$. We carry out the first step in the domain $(x,t) \in \omega_1$ by using the two-layer scheme

$$\rho_1 \frac{y^1 - y^0}{\tau} = ((ay_{\bar{x}})^{(1/2,0)})_x + \varphi^0, \quad 0 < x < \xi, \tag{3.10}$$

$$y(x,0) = u_0(x), \quad y(0,\tau) = 0, \quad y(x_p,\tau) = u_0(x_p) + \tau\bar{u}_1(x_p). \tag{3.11}$$

Observe that, for finding the second boundary condition $y(x_p,\tau)$ at the fusion point, we have used the given initial condition for the wave equation $y_t(x,0) = \bar{u}_1$ in (3.5) at $\xi = x_p$.

For our approximation to the parabolic equation in the domain ω_1, the scheme (3.1), (3.10), (3.11) leads to a scheme with variable weight factors [5, 6]. We will discuss this question in more detail below. Implementation of the difference schemes is standard for $\sigma_1 \neq 0$ and bases on the sweep method (as in the case of nonhomogeneous boundary conditions).

4. Stability of difference schemes with constant weights. To study the stability questions, we use the canonical form of three-layer operator difference schemes like [4]

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad 0 < t \in \omega_\tau, \tag{4.1}$$

$$y(0) = y_0, \quad y_t(0) = \bar{y}_0, \tag{4.2}$$

where $y \in H$, H is a real finite-dimensional Hilbert space, and $A, B, D : H \to H$ are linear operators in H.

Let $\overset{\circ}{\Omega}{}_h^n$ stand for the set of grid functions $y_i^n = y(x_i, t^n)$ that are given on $\bar{\omega}_h$ and vanish on the boundary. For these functions, we define the operator A as follows:

$$(Ay)_i^n = -(ay_{\bar{x}})_{x,i}, \quad i = 1, 2, \ldots, N-1, \quad y_0^n = y_N^n = 0. \tag{4.3}$$

Introducing the vector $y_n = \left(y_1^n, y_2^n, \ldots, y_{N-1}^n\right)^T$, we define the space H to be the set of these vectors y with the inner product and the norm

$$(y, v) = (y_n, v_n) = \sum_{i=1}^{N-1} y_i^n v_i^n h, \quad \|y\| = \sqrt{(y,y)}. \tag{4.4}$$

Then the operator A acts from H into H. Given $D = D^* > 0$, we denote by H_D the Hilbert space with the inner product $(y,v)_D = (Dy,v)$ and the norm $\|y\|_D^2 = (y,y)_D$.

By [4, p. 392, Theorem 6; 4, p. 399, Lemma 3], if

$$A^* = A > 0, \quad D^* = D \geq \frac{(1+\varepsilon)\tau^2}{4}A, \quad B \geq 0, \quad \varepsilon > 0, \tag{4.5}$$

then we have the following a priori estimate for a solution to the difference scheme (4.1), (4.2):

$$\|y_{n+1}\|_A \leq \sqrt{\frac{1+\varepsilon}{\varepsilon}}(\|y(0)\|_A + \|y_t(0)\|_D + \max_{0\leq k\leq n}(\|\varphi_k\|_{A^{-1}} + \|\varphi_{t,k}\|_{A^{-1}})). \tag{4.6}$$

To reduce the scheme (3.1)–(3.5), (3.9) to the canonical form (4.1), (4.2), we moreover define the operator A by (4.3) and the operators B and D as follows:

$$D = S + \frac{\tau^2}{2}(\sigma_1+\sigma_2)A, \quad S = \operatorname{diag}\{s_1, s_2, \dots, s_{N-1}\}, \tag{4.7}$$

$$s(x) = \begin{cases} 0.5\tau\rho_1(x), & x \in \omega_{1h}, \\ 0.25\tau\rho_1(x) + 0.5\rho_2(x), & x = \xi, \\ \rho_2(x), & x \in \omega_{2h}; \end{cases} \tag{4.8}$$

$$B = G + (\sigma_1 - \sigma_2)\tau A, \quad G = \operatorname{diag}\{g_1, g_2, \dots, g_{N-1}\}, \tag{4.9}$$

$$g(x) = \begin{cases} \rho_1(x), & x \in \omega_{1h}, \\ 0.5\rho_1(x), & x = \xi, \\ 0, & x \in \omega_{2h}. \end{cases}$$

We also pose the initial conditions

$$y_0(x) = u_0(x), \quad x \in \omega_h, \tag{4.10}$$

$$y_t(0) = \bar{y}_0, \quad \bar{y}_0(x) = \begin{cases} \bar{u}_0(x), & x \in \omega_{1h}, \\ \bar{u}_1(x), & x \in \omega_{2h}^+, \end{cases} \quad \omega_{2h}^+ = \omega_{2h} \cup \{\xi\}. \tag{4.11}$$

Then the identity

$$v^{(\sigma_1,\sigma_2)} = v + \tau(\sigma_1 - \sigma_2)v_{\overset{\circ}{t}} + \frac{\tau^2}{2}(\sigma_1+\sigma_2)v_{\bar{t}t} \tag{4.12}$$

implies that the difference scheme (3.1)–(3.5), (3.9) reduces to canonical form for three-layer operator difference schemes with the operators A, B, and D, the right-hand side φ, and the initial conditions y_0 and $\bar{y}_0$ defined by the corresponding formulas (4.3)–(4.11).

Theorem 1. *Suppose that the parameters of the difference scheme* (4.1), (4.2), (4.7)–(4.11) *satisfy the conditions*

$$\sigma_1 \geq \sigma_2, \quad \sigma_1 + \sigma_2 \geq \frac{1+\varepsilon}{2}. \tag{4.13}$$

Then the scheme (4.1), (4.2), (4.7)–(4.11) *is stable with respect to initial data and the right-hand side; moreover, a solution to the problem satisfies estimate* (4.6).

PROOF. We have $A = A^* > 0$, $D = D^* > 0$, and $B \geq 0$ whenever $\sigma_1 \geq \sigma_2$. Therefore, to prove the theorem, it suffices to validate the inequality

$$D - \frac{1+\varepsilon}{4}\tau^2 A = S + \left(\frac{\sigma_1+\sigma_2}{2} - \frac{1+\varepsilon}{4}\right)\tau^2 A \geq 0. \tag{4.14}$$

Once this inequality is obvious under the assumptions (4.13), the theorem is proven.

5. Difference schemes with variable weight factors. We have already mentioned above that difference schemes with nonconstant weights for the fusion problem for equations of different types arise in the case of the three-layer scheme (3.1) for a parabolic equation with the initial conditions (3.10) and (3.11). In practice, it is the two-layer difference scheme that is usually used (in this case $\sigma_2 = 0$) for numerical approximation of a parabolic equation and the three-layer scheme is used for approximation of a hyperbolic equation.

We define the variable weights σ_1 and σ_2 by the formulas

$$\sigma_1(x) = \begin{cases} \sigma, & x \in \omega_{1h}, \\ 0.5(\sigma + \sigma_1^*), & x = \xi, \\ \sigma_1^*, & x \in \omega_{2h}, \end{cases} \qquad \sigma_2(x) = \begin{cases} 0, & x \in \omega_{1h}, \\ 0.5\sigma_2^*, & x = \xi, \\ \sigma_2^*, & x \in \omega_{2h}. \end{cases}$$

Here σ, σ_1^*, and σ_2^* are constants. Then we may rewrite (3.1)–(3.5), (4.10) as

$$\tilde{\rho}_1 y_t + \tilde{\rho}_2 y_{\bar{t}t} = ((a y_{\bar{x}})^{(\sigma_1(x), \sigma_2(x))})_x + \varphi, \quad (x,t) \in \omega, \tag{5.1}$$

$$y(0,\hat{t}) = y(l,\hat{t}) = 0, \quad t \in \omega_\tau, \tag{5.2}$$

$$y(x,0) = u_0(x), \quad x \in \omega_h, \quad y_t(x,0) = \bar{u}_1(x), \quad x \in \omega_{2h}^+, \tag{5.3}$$

where $\omega = \omega_h \times \omega_\tau$ and the coefficients $\tilde{\rho}_k$ are defined by the formulas

$$\tilde{\rho}_1(x) = \begin{cases} \rho_1(x), & x \in \omega_{1h}, \\ 0.5\rho_1(\xi), & x = \xi, \\ 0, & x \in \omega_{2h}, \end{cases} \tag{5.4}$$

$$\tilde{\rho}_2(x) = \begin{cases} 0, & x \in \omega_{1h}, \\ 0{,}5\rho_2(\xi), & x = \xi, \\ \rho_2(x), & x \in \omega_{2h}. \end{cases} \tag{5.5}$$

We study stability of the scheme along the lines of [6]. Let $\overset{1}{\Omega}_h$ be the set of grid functions $v_i = v(x_i)$ given on the grid $\bar{\omega}_h$ and satisfying the condition $v_0 = 0$. Alongside the Hilbert space H, we introduce the space H_1 that is the set of vectors of the form $v_n = (v_1^n, v_2^n, \ldots, v_N^n)^T$. We endow H_1 with the inner product

$$(v_n, w_n] = \sum_{i=1}^{N} v_i^n w_i^n h. \tag{5.6}$$

Let us show that the operator $A : H \to H$ defined by (4.3) admits the representation $A = T^*T$. Define the linear operators $T : H \to H_1$ and $T^* : H_1 \to H$ by the formulas

$$(Ty)_i = \sqrt{a_i} y_{\bar{x},i}, \quad i = 1, 2, \ldots, N, \quad y_0 = y_N = 0, \tag{5.7}$$

$$(T^* v)_i = -(\sqrt{a} v)_{x,i}, \quad i = 1, 2, \ldots, N-1. \tag{5.8}$$

The operators T and T^* are adjoint to each other with respect to the inner product (5.6). Indeed, by the summation by parts formula, for arbitrary $y \in H$ and $v \in H_1$ we have

$$(v, Ty) = \sum_{i=1}^{N} v_i \sqrt{a_i} y_{\bar{x},i} h = \sum_{i=1}^{N-1} -(\sqrt{a} v)_{x,i} y_i h = (T^* v, y). \tag{5.9}$$

Now, we define the operators B and D as follows:

$$B = G + \tau T^*(\Sigma_1 - \Sigma_2)T, \tag{5.10}$$

$$G = \operatorname{diag}\{\tilde{\rho}_{11}, \tilde{\rho}_{12}, \dots, \tilde{\rho}_{1N-1}\}, \tag{5.11}$$

$$D = C + 0.5\tau^2 T^*(\Sigma_1 + \Sigma_2)T, \tag{5.12}$$

$$C = \operatorname{diag}\{c_1, c_2, \dots, c_{N-1}\}, \quad c_i = 0.5\tilde{\rho}_{1i}\tau + \tilde{\rho}_{2i}.$$

$$\Sigma_k : H_1 \to H_1, \quad \Sigma_k = \operatorname{diag}\{\sigma_{k1}, \sigma_{k2}, \dots, \sigma_{kN-1}\}. \tag{5.13}$$

It is obvious that the difference scheme (5.1)–(5.3) with the variable weight factors reduces to the canonical form (4.1), (4.2) with the operators $A = T^*T$, B, and D defined by (5.7)–(5.13) and with

$$y_t(0) = \bar{y}_0, \quad \bar{y}_0 = \begin{cases} \rho_1^{-1}(Ay^{(\sigma)} + \varphi), & x \in \omega_{1h}, \\ \bar{u}_1(x), & x \in \omega_{2h}^+. \end{cases} \tag{5.14}$$

It is obvious that the operator A is positive and self-adjoint. Since σ, σ_1^*, and σ_2^* are positive constants, the operators Σ_1 and Σ_2 are nonnegative and $D = D^* > 0$. By construction, $B \geq 0$ for $\Sigma_1 \geq \Sigma_2$. In order to use the a priori estimate (4.6), we check the condition (4.5) which is sufficient for stability. Observe that

$$D - \frac{1+\varepsilon}{4}\tau^2 A = C + \frac{\tau^2}{2}T^*\left(\Sigma_1 + \Sigma_2 - \frac{1+\varepsilon}{2}E\right)T \geq 0$$

for $\Sigma_1 + \Sigma_2 \geq \frac{1+\varepsilon}{2}E$. This inequality and the condition $\Sigma_1 \geq \Sigma_2$ are satisfied for $\sigma_1(x) + \sigma_2(x) \geq 0.5(1+\varepsilon)$ and $\sigma_1(x) \geq \sigma_2(x)$ respectively.

Theorem 2. *Suppose that the parameters of the difference scheme* (5.1)–(5.3) *satisfy the conditions*

$$\sigma_1(x) \geq \sigma_2(x), \quad \sigma_1(x) + \sigma_2(x) \geq \frac{1+\varepsilon}{2}, \quad x \in \omega_h.$$

Then the scheme is stable with respect to initial data and the right-hand side; moreover, a solution to the problem meets the estimate (4.6).

By the embeddings [4]

$$\|y_{\bar{x}}]| \leq \frac{1}{\sqrt{c_1}}\|y\|_A, \quad \|y\|_C \leq \frac{\sqrt{l}}{2\sqrt{c_1}}\|y\|_A, \tag{5.15}$$

where $\|y_{\bar{x}}]| = \sqrt{(y_{\bar{x}}, y_{\bar{x}}]}$ and $\|y\|_C = \max_{x \in \omega_h}|y(x)|$, (4.6) implies the corresponding a priori estimates in the seminorm of W_2^1 as well as in the uniform metric.

6. Approximation error and convergence. Consider the question of convergence of the difference scheme (5.1) with variable weight factors. By analogy with [8], we state the following conditions to be used below.

CONDITION A. The functions k_m', k_m'', ρ_m', f_m', u_m'', and $(k_m u')''$ ($m = 1, 2$, $v' = \partial v/\partial x$) satisfy the Lipschitz condition in x over each subdomain Q_m, $\frac{\partial u}{\partial t}$ satisfies the Lipschitz condition in t over Q_1, and $\frac{\partial^2 u}{\partial t^2}$ satisfies the Lipschitz condition in t over Q_2.

CONDITION B. The left and right limit values of the functions f_m, f_m', f_m'' ($m = 1, 2$), u', u'', and u''' satisfy the Lipschitz condition in t over the line $x = \xi$ for $0 \leq t \leq T$.

Suppose that $y \in H$ is a solution to the problem (4.1), (4.2), (5.7)-(5.13) and $u(x,t)$ is a solution to the differential problem (2.1)-(2.5). Write down the equation for the error $z = y - u$, where $u_n = (u_1^n, u_2^n, \dots, u_{N-1}^n)^T$. Inserting $y = z + u$ in (4.1) and (4.2), we obtain

$$Dz_{\bar{t}t} + Bz_{\overset{\circ}{t}} + Az = \psi, \quad z(0) = 0, \quad z_t(0) = \nu(x). \tag{6.1}$$

Here $z, \psi, \nu \in H$,

$$\psi_i = -\tilde{\rho}_1 u_{t,i} - \tilde{\rho}_2 u_{\bar{t}t,i} + ((au_{\bar{x}})^{(\sigma_1,\sigma_2)})_{x,i} + \varphi_i, \quad i = 1, 2, \dots, N-1, \tag{6.2}$$

is the approximation error for the equations (2.1) and (2.2) and the conjugation conditions (2.5), and

$$\nu = \begin{cases} -\rho_1 u_t + (au_{\bar{x}})_x^{(\sigma_1,\sigma_2)} + \varphi, & x \in \omega_{1h}, \\ \bar{u}_1(x) - u_t(x,0), & x \in \omega_{2h}^+, \end{cases}$$

determines the approximation error of the second initial condition.

By analogy with [4, p. 421], we now transform the expression for the residual. To this end, we consider the equations (2.1) and (2.2) at time $t = t_j$ and integrate them with respect to x from $x_{i-0.5}$ to $x_{i+0.5}$:

$$W_{i+0.5} - W_{i-0.5} + \int_{x_{i-0.5}}^{x_{i+0.5}} \left(f_1(x,t) - \rho_1(x)\frac{\partial u(x,t)}{\partial t}\right) dx = 0, \quad (x,t) \in \omega_1,$$

$$\begin{aligned} W_{p+0.5} - W_{p-0.5} &+ \int_{x_{p-0.5}}^{\xi} \left(f_1(x,t) - \rho_1(x)\frac{\partial u(x,t)}{\partial t}\right) dx \\ &+ \int_{\xi}^{x_{p+0.5}} \left(f_2(x,t) - \rho_2(x)\frac{\partial^2 u(x,t)}{\partial t^2}\right) dx = 0, \quad x = \xi, \end{aligned} \tag{6.4}$$

$$W_{i+0.5} - W_{i-0.5} + \int_{x_{i-0.5}}^{x_{i+0.5}} \left(f_2(x,t) - \rho_2(x)\frac{\partial^2 u(x,t)}{\partial t^2}\right) dx = 0, \quad (x,t) \in \omega_2.$$

Here $W = ku'$.

Divide each of the identities (6.4) by h and subtract them from (6.2) to obtain

$$\psi = \eta_{1x} + \psi_1, \quad \eta_1 = (au_{\bar{x}})^{(\sigma_1,\sigma_2)} - \overline{W} = O(h^2 + \tau).$$

Here $\bar{v} = v(x_{i-0.5}, t)$ and

$$\psi_1 = \begin{cases} -\rho_1 u_t + \varphi - \dfrac{1}{h}\displaystyle\int_{x_{i-0.5}}^{x_{i+0.5}} \left(f_1(x,t) - \rho_1 \frac{\partial u(x,t)}{\partial t}\right) dx, & (x,t) \in \omega_1, \\ -0.5\,(\rho_1 u_t + \rho_2 u_{\bar{t}t}) + \varphi - \dfrac{1}{h}\displaystyle\int_{x_{n-0.5}}^{\xi} \left(f_1(x,t) - \rho_1 \frac{\partial u(x,t)}{\partial t}\right) dx & \\ \qquad - \dfrac{1}{h}\displaystyle\int_{\xi}^{x_{n+0.5}} \left(f_2(x,t) - \rho_2 \frac{\partial^2 u(x,t)}{\partial t^2}\right) dx, & x = \xi, \\ -\rho_2 u_{\bar{t}t} + \varphi - \dfrac{1}{h}\displaystyle\int_{x_{i-0.5}}^{x_{i+0.5}} \left(f_2(x,t) - \rho_2 \frac{\partial^2 u(x,t)}{\partial t^2}\right) dx, & (x,t) \in \omega_2. \end{cases}$$

By the smoothness conditions, we have

$$\frac{1}{h}\int\limits_{x_{i-0.5}}^{x_{i+0.5}} f(x,t)\,dx = \varphi + \frac{h^2}{8}\bar{f}'_{x,i} + O(h^2), \quad x \in \omega_h,$$

$$\frac{1}{h}\int\limits_{x_{i-0.5}}^{\xi} \rho_1 \frac{\partial u}{\partial t}\,dx + \frac{1}{h}\int\limits_{\xi}^{x_{i+0.5}} \rho_2 \frac{\partial u}{\partial t}\,dx$$
$$= 0.5\left(\rho_1 \frac{\partial u(x,t)}{\partial t} + \rho_2 \frac{\partial^2 u}{\partial t^2}\right)_{x=\xi} + \frac{h^2}{8}\bar{p}'_{x,p} + O(h^2),$$

$$p(x,t) = \begin{cases} \rho_1(x)\frac{\partial u(x,t)}{\partial t}, & (x,t) \in Q_1, \\ \rho_2(x)\frac{\partial^2 u(x,t)}{\partial t^2}, & (x,t) \in Q_2. \end{cases}$$

We can consequently rewrite the approximation error as

$$\psi = \eta_x + \psi^*, \quad \eta = \eta_1 + \frac{h^2}{8}(\bar{p}' - \bar{f}'), \quad \psi^*,\, \psi_t^* = O(h^2 + \tau).$$

We turn to finding the order of the accuracy of the scheme. By (4.6), a solution to the problem (6.1) satisfies the estimate

$$\|z_{n+1}\|_A \le \sqrt{\frac{1+\varepsilon}{\varepsilon}}\left(\|z_t(0)\|_D + \max_{0\le k\le n}(\|\psi_k\|_{A^{-1}} + \|\psi_{t,k}\|_{A^{-1}})\right). \tag{6.5}$$

Immediate calculations yield

$$\|z_t(0)\|_D^2 = \|\nu(x)\|_D^2 = (C\nu,\nu) + 0.5\tau^2((\Sigma_1+\Sigma_2)T\nu, T\nu).$$

Since C, Σ_1, and Σ_2 are bounded operators, we have

$$\|\nu(x)\|_D \le c_0(\|\nu\|^2 + \tau^2\|\nu_{\bar{x}}]|^2)^{1/2} \le c(h^2+\tau), \tag{6.6}$$

where c_0 and c are constants independent of h and τ.

Similarly (see, for instance, [4, p. 442]), we have

$$\|\psi\|_{A^{-1}} \le \frac{1}{\sqrt{c_1}}\left(\|\eta]| + \frac{l}{2\sqrt{2}}\|\psi^*\|\right) \le c(h^2+\tau), \quad \|\psi_t\|_{A^{-1}} \le c(h^2+\tau).$$

Theorem 3. *Suppose that Conditions* A *and* B *are satisfied in each of the subdomains* Q_m $(m = 1, 2)$. *Then for*

$$\sigma_1(x) \ge \sigma_2(x), \quad \sigma_1(x) + \sigma_2(x) \ge \frac{1+\varepsilon}{2}, \quad x \in \omega_h,$$

a solution to the difference scheme (5.1)–(5.3) *converges to a solution to the differential problem* (2.1)–(2.5); *moreover, the following estimate holds*:

$$\max_{t\in\omega_\tau} \|z(t)\|_C \le c(h^2+\tau).$$

The proof of the theorem follows from (6.5)–(6.7) and the embedding (5.15).

We consider some more general problems similarly. Of particular interest are difference schemes for multidimensional problems with curvilinear fusion lines.

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