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STABILITY AND CONVERGENCE OF TWO-LEVEL DIFFERENCE SCHEMES IN INTEGRAL WITH RESPECT TO TIME NORMS

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Nowadays the general theory of operator-difference schemes with operators acting in Hilbert spaces has been created for investigating the stability of the difference schemes that approximate linear problems of mathematical physics. In most cases *a priori* estimates which are uniform with respect to the t norms are usually considered. In the investigation of accuracy for evolutionary problems, special attention should be given to estimation of the difference solution in grid analogs of integral with respect to the time norms. In this paper *a priori* estimates in such norms have been obtained for two-level operator-difference schemes. Use of that estimates is illustrated by convergence investigation for schemes with weights for parabolic equation with the solution belonging to $W_2^{2,1}(Q_T)$.

1. Introduction

In the investigation of convergence of difference schemes, it is most important to have the stability of approximate solution with respect to the initial data and right-hand side. Nowadays the theory of stability (well-posedness) of operator-difference schemes[8,9] has been created and widely developed. In the framework of this theory for wide class of two- and three-level difference schemes in Hilbert space, exact (coinciding necessary and sufficient) conditions were obtained. Basic theoretical results of stability theory can be found in Refs. 10 and 12.

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Let us emphasize the main character of the general theory of stability of operator-difference schemes. As the most important generalization of the theory we should notice its application to ill-posed evolutionary problems[15] and projection-difference (FEM) schemes.[17]

It is a natural way to prove convergence of difference scheme in such norms in consistent with smoothness class of differential problem solutions.[4–6,14] By virtue of this, one should have a spectrum of estimates for difference solution. In case of difference scheme for nonstationary boundary value problems with generalized solutions,[1–3,18] attention should be paid to estimates of difference solution in integral with respect to time norms.[13,16] In most cases estimates, which are uniform with respect to the time norm, in the theory of stability of difference schemes[10,12] are made.

In this paper we demonstrate the possibility of obtaining stability conditions in integral with respect to time norms. *A priori* estimates for two-level difference schemes written in the canonical form are obtained. The point of principle here is the fact that the difference solution at half-integer time moments is evaluated via linear interpolation of grid function values at the moments. Convergence of difference schemes for parabolic equation with generalized solutions is investigated on the basis of *a priori* estimates in integral with respect to time norms.[3,7]

2. Difference–Differential Problem

Let $\mathcal{X}$ be a real finite-dimensional Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Denote as $\mathcal{X}_G$ a Hilbert space of elements of $\mathcal{X}$, with scalar product $(\cdot,\cdot)_G = (G\cdot,\cdot)$ and norm $\|\cdot\|_G = (\cdot,\cdot)_G^{1/2}$. Here $G = G^* > 0$ is a self-adjoint positive operator in $\mathcal{X}$. Let D and A be linear (e.g. difference operators in $\mathcal{X}$). Consider a difference–differential equation

$$D\frac{du}{dt} + Au = f(t), \quad 0 < t < T. \tag{2.1}$$

We suppose that operators in this equation are constant in $\mathcal{X}$ ($D \neq D(t)$, $A \neq A(t)$) and

$$D = D^* > 0, \quad A = A^* > 0. \tag{2.2}$$

In obtaining convergence estimates for difference solution, we shall consider the Cauchy problem for Eq. (2.1) with homogeneous initial condition

$$u(0) = 0. \tag{2.3}$$

Let us consider a scalar product in $\mathcal{X}$ between $u(t)$ and (2.1). Integrating one by time on the interval $[0,t]$ and taking into account (2.2), (2.3) we get the following *a priori* estimate

$$\|u(t)\|_D^2 + \int_0^t \|u(\theta)\|_A^2 \, d\theta \le \int_0^t \|f(\theta)\|_{A^{-1}}^2 \, d\theta. \tag{2.4}$$

In obtaining the *a priori* estimates for difference analogs of the problem (2.1)–(2.3), the simplest estimate is usually employed

$$\|u(t)\|_D^2 \le \int_0^t \|f(\theta)\|_{A^{-1}}^2 \, d\theta \,,$$

which immediately follows from (2.4). In a number of cases (e.g. in investigation of problems with generalized solutions), much more attention should be paid to the estimate

$$\int_0^t \|u(\theta)\|_A^2 \, d\theta \le \int_0^t \|f(\theta)\|_{A^{-1}}^2 \, d\theta \tag{2.5}$$

of problem solution in the integral with respect to the time norm.

Note that there is a positive and self-adjoint operator A^{-1}. Thus, in particular case, $\widetilde{D} = DA^{-1} = A^{-1}D$ from (2.1) we obtain

$$\widetilde{D}\frac{du}{dt} + \widetilde{A}u = A^{-1}f(t) \,, \quad 0 < t < T \,, \tag{2.6}$$

where $\widetilde{D} = \widetilde{D}^* > 0$, $\widetilde{A} = E$. Applying (2.5) to the problem (2.6), (2.3), we get

$$\int_0^t \|u(\theta)\|^2 d\theta \le \int_0^t \|A^{-1}f(\theta)\|^2 d\theta \,. \tag{2.7}$$

In the same way, multiplying (2.1) by du/dt, we deduce inequality

$$\|u(t)\|_A^2 + \int_0^t \left\|\frac{du(\theta)}{d\theta}\right\|_D^2 d\theta \le \int_0^t \|f(\theta)\|_{D^{-1}}^2 \, d\theta \,. \tag{2.8}$$

From (2.8) we have the standard estimate for the solution of (2.1)–(2.3):

$$\|u(t)\|_A^2 \le \int_0^t \|f(\theta)\|_{D^{-1}}^2 \, d\theta \,.$$

Besides, special attention should be paid to inequality

$$\int_0^t \left\|\frac{du(\theta)}{d\theta}\right\|_D^2 d\theta \le \int_0^t \|f(\theta)\|_{D^{-1}}^2 \, d\theta \,. \tag{2.9}$$

Let us present some results in obtaining the difference analogs of *a priori* estimates (2.5), (2.7) and (2.9).

3. Operator–Difference Schemes

Let

$$\omega_\tau = \{t_n = n\tau \,, \quad n = 0, 1, \ldots, N_0 - 1 \,, \quad N_0\tau = T\}$$

be a uniform grid in time with step $\tau > 0$. For functions defined on ω_τ we shall use the following notation[10]: $y = y(t)$, $\widehat{y} = y(t+\tau)$, $\check{y} = y(t-\tau)$ and $y_t = (\widehat{y} - y)/\tau$.

Let us define space $\mathcal{H} = \mathcal{H}(\omega_\tau; \mathcal{X})$ as a set of grid functions given on ω_τ with values in $\mathcal{X}$ and norm

$$\|\cdot\|_{\mathcal{H}} = \left(\sum_{t\in\omega_\tau} \tau\|\cdot(t)\|^2\right)^{1/2} .$$

On grid ω_τ let us consider two-level operator-difference scheme written in the canonical form

$$By_t + Ay = \varphi(t)\,, \quad t \in \omega_\tau\,, \quad y(0) = 0\,, \tag{3.10}$$

where operator $B = B^* > 0$, $y = y(t) \in \mathcal{X}$, $\varphi(t) \in \mathcal{X}$. We shall study stability of difference scheme with respect to the right-hand side.

Theorem 1. *For difference scheme* (3.10) *with operators* $A = A^* > 0, B = B^* > 0$ *so that*

$$B \geq \frac{\tau}{2} A \tag{3.11}$$

the following a priori estimate holds

$$\sum_{t\in\omega_\tau} \tau\|y^{(0,5)}(t)\|_A^2 \leq \sum_{t\in\omega_\tau} \tau\,\|\varphi(t)\|_{A^{-1}}^2\,, \tag{3.12}$$

where $y^{(0,5)}(t) = (y(t) + y(t+\tau))/2$.

Proof. Let us rewrite operator Eq. (3.10) in the equivalent form

$$Gy_t + Ay^{(0,5)} = \varphi(t)\,, \quad G = B - \frac{1}{2}\tau A\,. \tag{3.13}$$

Considering a scalar product in $\mathcal{X}$ of (3.13) by $2\tau y^{(0,5)}$ and taking into account $G = G^* \geq 0$, we get an energy identity

$$(G\widehat{y}, \widehat{y}) + 2\tau\|y^{(0,5)}\|_A^2 = (Gy, y) + 2\tau(y^{(0,5)}, \varphi)\,. \tag{3.14}$$

Applying Cauchy's generalized inequality to the right-hand side

$$2\tau(y^{(0,5)}, \varphi) \leq \tau\|y^{(0,5)}\|_A^2 + \tau\|\varphi\|_{A^{-1}}^2$$

and summarizing over $t \in \omega_\tau$, we get from (3.14) the required estimate. □

The estimate (3.12) is a difference analog of *a priori* estimate (2.5). In addition, it was obtained for the difference solution at half-integer moments defined by expression $y(t+\tau/2) = (y(t+\tau) + y(t))/2$. Note that stability in such integral with respect to time norms was established in conditions (3.11), that are necessary and sufficient for the stability uniform with respect to the time norms.

In more strong assumptions about difference solution in integer time moments we can get *a priori* estimate of stability with respect to the right-hand side in the integral by time norms.

Theorem 2.[12] *For the difference scheme* (3.10) *with operators* $A = A^* > 0, B = B^* > 0$ *and*

$$B \geq (1+\varepsilon)\frac{\tau}{2}A, \quad 0 < \varepsilon < 2 \tag{3.15}$$

the following a priori estimate holds

$$\sum_{t\in\omega_\tau} \tau \|y(t)\|_A^2 \leq \frac{2}{\varepsilon^2(2-\varepsilon)} \sum_{t\in\omega_\tau} \tau \|\varphi(t)\|_{A^{-1}}^2 . \tag{3.16}$$

Proof. After rewriting Eq. (3.10) in the form

$$(B - \tau A)y_t + Ay = \varphi ,$$

let us consider its scalar product in $\mathfrak{X}$ with $2\tau\widehat{y}$. From self-adjointness of the operators B and A one can deduce that

$$2\tau((B-\tau A)y_t, \widehat{y}) = \tau((B-\tau A)y, y)_t + \tau^2((B-\tau A)y_t, y_t)$$

and get the following identity

$$(B\widehat{y}, \widehat{y}) + \tau(A\widehat{y}, \widehat{y}) + \tau(Ay, y) + \tau^2((B-\tau A)y_t, y_t) = 2\tau(\varphi, \widehat{y}) + (By, y) .$$

Taking into account

$$(A\widehat{y}, \widehat{y}) + (Ay, y) = \frac{1}{2}\|\widehat{y} + y\|_A^2 + \frac{\tau^2}{2}\|y_t\|_A^2 ,$$

we can see

$$\|\widehat{y}\|_B^2 + \tau^2((B - 0,5\tau A)y_t, y_t) + \frac{\tau}{2}\|y + \widehat{y}\|_A^2 = \|y\|_B^2 + 2\tau(\varphi, \widehat{y}) . \tag{3.17}$$

Let us estimate the term $2\tau(\varphi, \widehat{y})$. From

$$\widehat{y} = \frac{1}{2}(y + \widehat{y}) + \frac{1}{2}\tau y_t ,$$

it follows that

$$2\tau(\varphi, \widehat{y}) = \tau(\varphi, y + \widehat{y}) + \tau^2(\varphi, y_t) \leq \frac{\tau\varepsilon_1}{2}\|y + \widehat{y}\|_A^2 + \frac{\tau}{2\varepsilon_1}\|\varphi\|_{A^{-1}}^2$$
$$+ \frac{\tau^3\varepsilon_2}{2}\|y_t\|_A^2 + \frac{\tau}{2\varepsilon_2}\|\varphi\|_{A^{-1}}^2 .$$

So, from identity (3.17) we get

$$\|\widehat{y}\|_B^2 + \tau^2\left(\left(B - \frac{(1+\varepsilon_2)\tau}{2}A\right)y_t, y_t\right) + \frac{\tau}{2}(1-\varepsilon_1)\|y + \widehat{y}\|_A^2$$
$$\leq \|y\|_B^2 + \frac{\tau}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\|\varphi\|_{A^{-1}}^2 . \tag{3.18}$$

Since under the conditions $\varepsilon_1 = 1 - \varepsilon/2$, $\varepsilon_2 = \varepsilon/2 > 0$ and $B \geq ((1+\varepsilon)/2)\tau A$

$$\tau^2 \left(\left(B - \frac{1+\varepsilon_2}{2}\tau A \right) y_t, y_t \right) + \frac{1-\varepsilon_1}{2}\tau \|y + \widehat{y}\|_A^2 \geq \frac{\tau^3 \varepsilon}{4}\|y_t\|_A^2 + \frac{\tau\varepsilon}{4}\|y + \widehat{y}\|_A^2$$

$$= \frac{\tau\varepsilon}{4}\left(\|y + \widehat{y}\|_A^2 + \tau^2 \|y_t\|_A^2 \right) = \frac{\tau\varepsilon}{2}\left(\|y\|_A^2 + \|\widehat{y}\|_A^2 \right),$$

from (3.18) we find that

$$\|y_{k+1}\|_B^2 + \frac{\tau\varepsilon}{2}\left(\|y_k\|_A^2 + \|y_{k+1}\|_A^2 \right) \leq \|y_k\|_B^2 + \frac{2\tau}{\varepsilon(2-\varepsilon)}\|\varphi_k\|_{A^{-1}}^2 .$$

Summarizing the last inequalities over all $k = 0, 1, \ldots, n$, we get the estimate (3.16). The proof is complete. □

Let us apply Theorem 2 to the investigation of stability with respect to the right-hand side of the scheme with constant weights

$$y_t + Ay^{(\sigma)} = \varphi(t), \quad t \in \omega_\tau, \quad y(0) = 0 . \tag{3.19}$$

On the basis of identity $y^{(\sigma)} = y + \sigma\tau y_t$, it can be rewritten in the equivalent form

$$(E + \sigma\tau A)y_t + Ay = \varphi(t), \quad t \in \omega_\tau . \tag{3.20}$$

In (3.19) we assume $y^{(\sigma)}(t) = \sigma y(t+\tau) + (1-\sigma)y(t)$.

Multiplying A^{-1} on both sides of Eq. (3.20), we get another canonical form of two-level scheme:

$$\widetilde{B}y_t + \widetilde{A}y = \widetilde{\varphi}(t), \quad t \in \omega_\tau, \quad y(0) = 0, \tag{3.21}$$

where $\widetilde{B} = A^{-1} + \sigma\tau E$, $\widetilde{A} = E$, $\widetilde{\varphi} = A^{-1}\varphi$, $\widetilde{B} = \widetilde{B}^* > 0$. Verifying the sufficient condition (3.15) of stability with respect to the right-hand side

$$\widetilde{B} - (1+\varepsilon)\frac{\tau}{2}\widetilde{A} = \left(A^{-1} - \frac{E}{\|A\|} \right) + \tau \left(\sigma - \frac{1+\varepsilon}{2} + \frac{1}{\tau\|A\|} \right) E \geq 0$$

brings to us the restriction on weighting parameter

$$\sigma \geq \frac{1+\varepsilon}{2} - \frac{1}{\tau\|A\|}, \tag{3.22}$$

It also follows from the inequality $A^{-1} \geq (1/\|A\|)E$ which carried out for any self-adjoint operator A.

Thus, we have proved the following statement.

Theorem 3. *Suppose that in the scheme with weights* (3.19) $A = A^* > 0, A \neq A(t)$ *and condition* (3.22) *is satisfied. Then the difference scheme is stable in* $\mathcal{H}$ *and the following a priori estimate takes place*

$$\|y\|_{\mathcal{H}}^2 \leq \frac{2}{\varepsilon^2(2-\varepsilon)}\|A^{-1}\varphi\|_{\mathcal{H}}^2 . \tag{3.23}$$

Note that estimate (3.23) is a corresponding analog of the difference inequality (2.7).

Similarly, for the difference solution at half-integer moments we can formulate

Theorem 4. *Let $A = A^* > 0$ be a constant operator in the scheme with weights* (3.19). *Then in assumption*

$$\sigma \geq \frac{1}{2} - \frac{1}{\tau \|A\|} \tag{3.24}$$

the scheme is stable in $\mathcal{H}$ and a priori estimate

$$\sum_{t \in \omega_\tau} \tau \|y^{(0,5)}(t)\|^2 \leq \sum_{t \in \omega_\tau} \tau \|A^{-1}\varphi\|^2 \tag{3.25}$$

is valid.

Let us give now the difference analog of the estimate (2.9).

Theorem 5. *For difference scheme* (3.10) *with operators $A = A^* > 0$, $B = B^*$ and*

$$G = B - \frac{\tau}{2} A > 0 \tag{3.26}$$

the following a priori estimate holds

$$\sum_{t \in \omega_\tau} \tau \|y_t\|_G^2 \leq \sum_{t \in \omega_\tau} \tau \|\varphi(t)\|_{G^{-1}}^2 \, . \tag{3.27}$$

Proof. Consider a scalar product in $\mathcal{X}$ of Eq. (3.13) and $2\tau y_t$. From the self-adjointness of operator A, we get an energy identity

$$\|y(t+\tau)\|_A^2 + 2\tau \|y_t\|_G^2 = \|y(t)\|_A^2 + 2\tau \, (y_t, \varphi) \, . \tag{3.28}$$

Using Cauchy's generalized inequality, we get the following estimate

$$2\tau (y_t, \varphi) \leq \tau \|y_t\|_G^2 + \tau \|\varphi\|_{G^{-1}}^2 \, .$$

Substituting it into (3.23) and summarizing over all $t \in \omega_\tau$, we obtain the inequality (3.22). The proof is complete. □

Note that in this case we only demand a non-negativity for operator A. But restrictions on the operator B are amplified (compare with (3.11), (3.15) and (3.26)). From the estimate (3.27) with regard to condition $y(0) = 0$ it is also easy to get estimates uniform with respect to the t norm. One should use for that the following inequalities.

Lemma 1.[10] *For any function $v(t)$ given on uniform grid*

$$\overline{\omega}_\tau = \{t_n = n\tau \, , \quad n = 0, 1, \ldots, N_0 \, , \quad N_0 \tau = T\}$$

and vanishing at $t = 0 (v(0) = 0)$, *the following a priori estimates are fulfilled*

$$\|v\|_{C(\omega_\tau)} \le \sqrt{T}\|v_t\|_{\mathcal{H}}\,,$$

$$\|v\|_{\mathcal{H}} \le T\|v_t\|_{\mathcal{H}}\,,$$

where

$$\|v\|_{C(\omega_\tau)} = \max_{t\in\omega_\tau}|v(t)|\,, \quad \|v\|_{\mathcal{H}} = \left(\sum_{t\in\omega_\tau}^{T} \tau v^2(t)\right)^{1/2}.$$

The proof of the Lemma follows from the obvious relations

$$v^2(t) = \left(\sum_{t=0}^{t-\tau} \tau v_t\right)^2 \le t\sum_{t=0}^{t-\tau} \tau v_t^2\,, \tag{3.29}$$

$$\|v\|_{\mathcal{H}}^2 \le T\|v\|_{C(\omega_\tau)}^2\,. \tag{3.30}$$

Lemma 2. *Let* $G = G^* > 0$ *be a self-adjoint positive operator. Then for any function* $y(t) \in \mathcal{X}$, $t \in \omega_\tau$, $y(0) = 0$ *inequalities*

$$\|y(t)\|_G^2 \le t\sum_{t=\tau}^{t-\tau} \tau\|y_t\|_G^2\,,$$

$$\sum_{t\in\omega_\tau} \tau\|y(t)\|_G^2 \le T^2 \sum_{t\in\omega_\tau} \tau\|y_t\|_G^2$$

are satisfied.

Proof. Let $x = G^{1/2}y$. For the function $v(t) = \|y(t)\|_G$ let us use Lemma 1 and inequality (3.29)

$$v^2(t) \le t\sum_{t=0}^{t-\tau} \tau v_t^2\,, \tag{3.31}$$

$$\sum_{t\in\omega_\tau} \tau v^2(t) \le T^2 \sum_{t\in\omega_\tau} \tau v_t^2\,. \tag{3.32}$$

In accordance with the triangle inequality

$$v_t^2 = (\|G^{1/2}y\|_t)^2 = \frac{1}{\tau^2}\left(\|\widehat{x}\| - \|x\|\right)^2 \le \frac{1}{\tau^2}\|\widehat{x} - x\|^2 = \|x_t\|^2\,.$$

Thus, $v_t^2 \le \|y_t\|_G^2$. Substituting the last inequality into (3.31), (3.32), we get Lemma's statement.

Let's apply now the formulated conditions of stability for the scheme with weights

$$Dy_t + Ay^{(\sigma)} = \varphi(t)\,, \quad t \in \omega_\tau\,, \quad y(0) = 0\,, \tag{3.33}$$

that approximate Cauchy abstract problem (2.1)–(2.3). The scheme (3.33) can be written in canonical form (3.10) with

$$B = D + \sigma\tau A\,. \tag{3.34}$$

By virtue of (3.11), the estimate (3.12) for the difference scheme (3.10), (3.34) will be valid when $A \le \Delta D$ if

$$\sigma \ge \frac{1}{2} - \frac{1}{\Delta\tau}\,.$$

In the same way, from (3.15), (3.34) it follows that for scheme (3.10), (3.34) the estimate (3.16) holds whenever

$$\sigma \ge \frac{1+\varepsilon}{2} - \frac{1}{\Delta\tau}\,.$$

On the basis of Theorem 5 under ordinary restriction $\sigma \ge 1/2$ from (3.27) one can obtain the inequality

$$\sum_{t\in\omega_\tau} \tau\|y_t\|_D^2 \le \sum_{t\in\omega_\tau} \tau\|\varphi(t)\|_{D^{-1}}^2\,. \tag{3.35}$$

This estimate is a difference analog of the estimate (2.9) for the difference problem.

On the basis of Lemma 2 from (3.35), the estimate of the solution follows in both uniform metrics

$$\|y(t)\|_D^2 \le T \sum_{t\in\omega_\tau} \tau\|\varphi(t)\|_{D^{-1}}^2$$

and integral norm by t

$$\sum_{t\in\omega_\tau} \tau\|y(t)\|_D^2 \le T^2 \sum_{t\in\omega_\tau} \tau\|\varphi(t)\|_{D^{-1}}^2\,.$$

4. Convergence of Difference Schemes for Problems with Solutions in Distributions

In the rectangle

$$Q_T = \{(x,t):\ x \in \Omega = (0,1)\,,\ 0 < t < T\}$$

let us consider heat transfer equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t)\,, \quad (x,t) \in Q_T\,, \tag{4.36}$$

with homogeneous initial and boundary conditions

$$u(x,0) = u_0(x)\,, \quad x \in \Omega\,; \quad u(0,t) = u(1,t) = 0\,, \quad 0 < t < T\,. \tag{4.37}$$

For functions $u = u(x)$ defined on Ω let us introduce Sobolev space $W_2^1(\Omega)$ with norm

$$\|u\|^2_{W_2^1(\Omega)} = \|u\|^2_{L_2(\Omega)} + \left\|\frac{du}{dx}\right\|^2_{L_2(\Omega)}, \quad \|u\|^2_{L_2(\Omega)} = \int_\Omega u^2(x)\,dx\,,$$

and for functions $u = u(x,t)$ given on Q_T let us define space $W_2^{\alpha,\beta}(Q_T)$ with norm

$$\|u\|^2_{W_2^{\alpha,\beta}(Q_T)} = \|u\|^2_{L_2(Q_T)} + \sum_{s=1}^{\alpha}\left\|\frac{\partial^s u}{\partial x^s}\right\|^2_{L_2(Q_T)} + \sum_{s=1}^{\beta}\left\|\frac{\partial^s u}{\partial t^s}\right\|^2_{L_2(Q_T)},$$

$$\|u\|^2_{L_2(Q_T)} = \int_0^T \|u(x,t)\|^2_{L_2(\Omega)}\,dt\,.$$

We shall denote as $\overset{\circ}{W}{}_2^1(\Omega)$ and $W_{2,0}^{1,1}(Q_T)$ subspaces of $W_2^1(\Omega)$ and $W_2^{1,1}(Q_T)$ such that the dense sets in them are smooth functions that equal to zero near $x = 0$ and $x = 1$.

Definition 3. An element u of space $W_{2,0}^{2,1} \equiv W_2^{2,1}(Q_T) \cap W_{2,0}^{1,1}(Q_T)$ is called a generalized solution of the problem (4.36), (4.37) in space $W_2^{2,1}(Q_T)$ if Eq. (4.36) is satisfied almost everywhere in Q_T and equals to $u_0(x)$ when $t = 0$.

Further we shall use

Lemma 3.[3,7] *The problem* (4.36), (4.37) *is uniquely solvable in* $W_{2,0}^{2,1}(Q_T)$ *if* $f \in L_2(Q_T)$, $u_0 \in \overset{\circ}{W}{}_2^1(0,1)$ *and the following inequality holds*

$$\|u\|_{W_2^{2,1}(Q_T)} \le M(\|u_0\|_{W_2^1(0,1)} + \|f\|_{L_2(Q_T)})\,,$$

where positive constant M *does not depend on* u_0 *and* f.

In the domain Q_T, let us introduce uniform grids ω_τ,

$$\omega_h = \{x_i = ih\,, \quad i = 1, \ldots, N-1\,, \quad Nh = 1\}\,, \quad \omega_{h\tau} = \omega_h \times \omega_\tau\,.$$

Below we shall use Steklov's averaging operators.[14] First let us define one-dimensional averaging operators acting in each direction x and t

$$S_x v(x,t) = \frac{1}{h}\int_{x-0,5h}^{x+0,5h} v(\xi,t)\,d\xi\,,$$

$$S_t v(x,t) = \frac{1}{\tau}\int_t^{t+\tau} v(x,t')\,dt' = \int_0^1 v(x,t+\theta\tau)\,d\theta\,.$$

Let us introduce an operator of repeated averaging in direction x

$$S_x^2 v(x,t) = \frac{1}{h^2}\int_x^{x+h}\int_{\xi-h}^{\xi} v(\eta,t)\,d\eta\,d\xi\,.$$

Using the formula of integrating by parts one can obtain the following representation p. 57 of Ref. 14

$$S_x^2 v(x,t) = \frac{1}{h}\int_{x-h}^{x+h}\left(1-\frac{|x'-x|}{h}\right)v(x',t)\,dx' = \int_{-1}^{1}(1-|s|)v(x+sh,t)\,ds\,.$$

For averaging operators, the following properties take place

$$S_x^2\frac{\partial^2 u}{\partial x^2} = u_{\bar{x}x}\,,\qquad S_t\frac{\partial u}{\partial t} = u_t\,,$$

where

$$u_x \equiv \frac{u(x+h)-u(x)}{h}\,,\qquad u_{\bar{x}} \equiv \frac{u(x)-u(x-h)}{h}\,.$$

Let us approximate the problem (4.36), (4.37) by difference scheme with weights

$$y_t = y_{\bar{x}x}^{(\sigma)} + S_x^2 S_t f\,,\quad (x,t)\in\omega_{h\tau}\,, \tag{4.38}$$

$$y(x,0) = S_x^2 u_0(x)\,,\quad x\in\omega_h\,,\quad y(0,t)=y(1,t)=0\,,\quad t\in\omega_\tau\,. \tag{4.39}$$

Let us define $\mathcal{H} = L_2(\omega_{h\tau})$ as a grid analog of $L_2(Q_T)$ with integral with respect to time norm

$$\|v\|_{\mathcal{H}} = \left(\sum_{t\in\omega_\tau}\tau\|v(t)\|^2\right)^{1/2}\,,\quad \|v(t)\|^2 = \sum_{x\in\omega_h} hv^2(x,t)\,. \tag{4.40}$$

Consider a problem of convergence of difference scheme in the grid norm of $\mathcal{H}$. Denote as $\overline{u} = S_x^2 u$ an averaging of the exact solution of problem (4.36), (4.37). Here we extend the solution by oddness over the lines $x=0$, $x=1$:

$$\widetilde{u}(x,t) = \begin{cases} -u(-x,t)\,, & x\in(-1,0]\,,\\ u(x,t)\,, & x\in[0,1]\,,\\ -u(2-x,t)\,, & x\in[1,2)\,.\end{cases}$$

It is not difficult to show that

$$\|\widetilde{u}\|^2_{W_2^{2,1}(\widetilde{Q}_T)} = 3\|u\|^2_{W_2^{2,1}(Q_T)}\,,\quad \widetilde{Q}_T = \{(x,t):\ x\in(-1,2),\ t\in(0,T)\}\,.$$

Hence, $\widetilde{u}(x,t)\in W_2^{2,1}$.

We shall compare approximate solution y with the averaging $\overline{u}$. Substituting $y = z+\overline{u}$ in (4.38), we obtain the following difference equation

$$z_t = z_{\bar{x}x}^{(\sigma)} + \psi(x,t)\,,\quad (x,t)\in\omega_{h\tau}\,, \tag{4.41}$$

$$z(x,0)=0\,,\quad x\in\overline{\omega}_h\,,\quad z(0,\hat{t}) = z(1,\hat{t}) = 0\,,\quad t\in\omega_\tau\,, \tag{4.42}$$

where

$$\psi = S_x^2 S_t f - \overline{u}_t + \overline{u}_{\bar{x}x}^{(\sigma)} \tag{4.43}$$

is the approximate error. Let us transform it to the divergent form. To this end let us apply the operator $S_x^2 S_t$ to the differential equation. We obtain

$$\overline{u}_t = S_t u_{\bar{x}x} + S_x^2 S_t f\,.$$

Let us substitute $S_x^2 S_t f$ into ψ. Then (4.43) is transformed to

$$\psi = \eta_{\bar{x}x}\,, \quad \eta = S_t u - \overline{u}^{(\sigma)}\,. \tag{4.44}$$

Let $\overset{\circ}{\mathcal{X}}$ be a set of grid functions $y(x,t)$ that are defined at every $t \in \bar{\omega}_\tau$ on $\bar{\omega}_h$ and satisfied the condition $y(0,t) = y(1,t) = 0$. For such functions let us define the operator A:

$$(Ay)(x) = -y_{\bar{x}x}\,, \quad x \in \omega_h\,.$$

Let us introduce a vector $y(t) = (y(h,t), y(2h,t), \ldots, y(1-h,t))^T$ and define space $\mathcal{X}$ as a set of vectors with scalar product $(y,v) = \sum_{x\in\omega_h} y(x)v(x)h$ and norm $\|y\| = (y,y)^{1/2}$. Then the operator A acts in $\mathcal{X}$, i.e. $A : \mathcal{X} \to \mathcal{X}$.

Properties of A are well-known.[10] In particular, A is self-adjoint and positive $A = A^* > 0$ so that for every $y, v \in \overset{\circ}{\mathcal{X}}$ on the basis of formula of summing by parts

$$(u_{\bar{x}x}, v) = -\sum_{x=h}^{1} u_{\bar{x}} v_{\bar{x}} h = (u, v_{\bar{x}x})\,,$$

i.e. $(Ay, v) = (y, Av)$. Besides, it is positively defined

$$(Ay, y) \geq 8\|y\|^2\,, \tag{4.45}$$

which follows from the grid analog of the Friedrichs inequality.[10] In particular, from (4.45) it follows that $\|A^{-1}\| \leq 1/8$ since operator A^{-1} is self-adjoint and positive. Let us define a vector $\eta(t) = (\eta(h,t), \ldots, \eta(1-h,t))$, $\eta(x,t) \in \overset{\circ}{\mathcal{X}}$.

Thus, the problem (4.41), (4.42) for approximation error can be written in the form of two-level operator difference scheme

$$z_t + Az^{(\sigma)} = A\eta\,, \quad t \in \omega_\tau\,, \quad z(0) = 0\,, \tag{4.46}$$

where $z = z(t) = (z(h,t), \ldots, z(1-h,t)) \in \mathcal{X}$, $z(x,t) \in \overset{\circ}{\mathcal{X}}$. Applying now Theorem 3 to (4.46) and taking into account that $\|A\| < 4/h^2$ whenever

$$\sigma \geq \frac{1+\varepsilon}{2} - \frac{h^2}{4\tau}$$

we get an estimate

$$\|z\|_{\mathcal{H}} \leq \frac{\sqrt{2}}{\varepsilon\sqrt{2-\varepsilon}}\|\eta\|_{\mathcal{H}}\,, \quad 0 < \varepsilon < 2\,. \tag{4.47}$$

It remains to estimate the error η in norm $\|\cdot\|_{\mathcal{H}}$. We use the following

Lemma 4.[13] *For grid function* (4.44) *the following estimate holds*

$$|\eta(x,t)| \leq \frac{M}{\sqrt{h\tau}}\left(\tau\left\|\frac{\partial u}{\partial t}\right\|_{L_2(e)} + h^2\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L_2(e)}\right)\,, \tag{4.48}$$

where $e = (x-h, x+h) \times (t, t+\tau)$, $M > 0$ *does not depend on* τ, h *and* u.

Proof. By means of transformation

$$s = (x' - x)/h\,, \quad \theta = (t' - t)/\tau\,, \quad (x', t') \in e$$

let us map the cell e into the domain

$$\widetilde{e} = \{(s, \theta) : \ -1 < s < 1,\ 0 < \theta < 1\}\,.$$

Let us denote $v(s, \theta) = u(x + sh, t + \theta\tau)$ and rewrite the grid function η in the form

$$\begin{aligned}
\eta &= S_t u - \overline{u}^{(\sigma)} = S_t u - \overline{u} - \sigma\tau\overline{u}_t \\
&= \int_0^1 u(x, t + \theta\tau)\, d\theta - \int_{-1}^1 (1 - |s|) u(x + sh, t)\, ds - \sigma\tau \int_{-1}^1 (1 - |s|) u_t(x + sh, t)\, ds \\
&= \int_0^1 v(0, \theta)\, d\theta - \int_{-1}^1 (1 - |s|) v(s, 0)\, ds - \sigma \int_{-1}^1 (1 - |s|) \left(\int_0^1 \frac{\partial v(s, \eta)}{\partial \eta} d\eta \right) ds\,.
\end{aligned}$$

By virtue of $\int_0^1 d\theta = 1$, $\int_{-1}^1 (1 - |s|)\, ds = 1$, the last expression can be rewritten in the form:

$$\begin{aligned}
\eta = &\int_0^1 \int_{-1}^1 (1 - |s|) \left[v(0, \theta) - v(s, \theta) + v(s, \theta) - v(s, 0) \right] ds\, d\theta \\
&- \sigma\tau \int_{-1}^1 (1 - |s|) v_t(s, 0)\, ds = I_1 + I_2 + I_3\,.
\end{aligned} \tag{4.49}$$

Here

$$\begin{aligned}
I_1 &= \int_0^1 \int_{-1}^1 (1 - |s|) \left[v(0, \theta) - v(s, \theta) \right] ds\, d\theta\,, \\
I_2 &= \int_0^1 \int_{-1}^1 (1 - |s|) \left[v(s, \theta) - v(s, 0) \right] ds\, d\theta\,, \\
I_3 &= -\sigma \int_{-1}^1 (1 - |s|) \left(\int_0^1 \frac{\partial v(s, \eta)}{\partial \eta} d\eta \right) ds\,.
\end{aligned}$$

Let us use the Taylor series with the remainder term in the integral form

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \cdots + \frac{(x - x_0)^m}{m!} f^{(m)}(x_0) + R_{m+1}(x)\,, \tag{4.50}$$

$$\begin{aligned}
R_{m+1}(x) &= \frac{1}{m!} \int_{x_0}^x (x - \xi)^m f^{(m+1)}(\xi)\, d\xi \\
&= \frac{(x - x_0)^{m+1}}{m!} \int_0^1 (1 - \eta)^m f^{(m+1)}(x_0 + \eta(x - x_0))\, d\eta\,.
\end{aligned}$$

Using the expansion (4.50), we get

$$v(s,\theta) = v(0,\theta) + s\frac{\partial v}{\partial s}(0,\theta) + \int_0^s (s-\xi)\frac{\partial^2 v}{\partial \xi^2}(\xi,\theta)\,d\xi\,. \tag{4.51}$$

Applying the Cauchy inequality and identities

$$\int_{-1}^{1}(1-|s|)s\,ds = 0\,, \quad \left|\int_s^0 (s-\xi)^2\,d\xi\right|^{1/2} = \frac{1}{\sqrt{3}}|s|^{3/2}\,,$$

$$\int_{-1}^{1}\frac{1}{\sqrt{3}}(1-|s|)|s|^{3/2}\,ds = \frac{4}{5\sqrt{3}}\,,$$

and relation (4.51) let us estimate

$$\begin{aligned}
I_1 &= \int_0^1\int_{-1}^{1}(1-|s|)\left(-s\frac{\partial v}{\partial s}(0,\theta) + \int_s^0 (s-\xi)\frac{\partial^2 v(\xi,\theta)}{\partial \xi^2}\right)d\xi\,d\theta \\
&\le \int_0^1\int_{-1}^{1}(1-|s|)\left|\int_s^0 (s-\xi)^2\,d\xi\right|^{1/2}\left(\int_{-1}^{1}\left(\frac{\partial^2 v(\xi,\theta)}{\partial \xi^2}\right)^2 d\xi\right)^{1/2} ds\,d\theta \\
&= \frac{4}{5\sqrt{3}}\int_0^1\left(\int_{-1}^{1}\left(\frac{\partial^2 v}{\partial \xi^2}\right)^2 d\xi\right)^{1/2} d\theta \le \frac{4}{5\sqrt{3}}\left[\int_0^1\int_{-1}^{1}\left(\frac{\partial^2 v(\xi,\theta)}{\partial \xi^2}\right)^2 d\xi\,d\theta\right]^{1/2} \\
&= \frac{4}{5\sqrt{3}}\left\|\frac{\partial^2 v(s,\theta)}{\partial s^2}\right\|_{L_2(\tilde{e})} = \frac{4}{5\sqrt{3}}\frac{h^2}{\sqrt{h\tau}}\left\|\frac{\partial^2 u(x,t)}{\partial x^2}\right\|_{L_2(e)}\,.
\end{aligned} \tag{4.52}$$

Functionals I_2 and I_3 can also be estimated by the Cauchy inequality and Taylor expansion (4.50).

$$\begin{aligned}
I_2 &= \int_0^1\int_{-1}^{1}(1-|s|)\left(\int_0^\theta \frac{\partial v(s,\eta)}{\partial \eta}d\eta\right) ds\,d\theta \\
&\le \int_0^1 \sqrt{\theta}\int_{-1}^{1}(1-|s|)\left(\int_0^1\left(\frac{\partial v(s,\eta)}{\partial \eta}\right)^2 d\eta\right)^{1/2} ds\,d\theta \\
&= \frac{2}{3}\left\|\frac{\partial v}{\partial \theta}\right\|_{L_2(\tilde{e})} = \frac{2}{3}\frac{\tau}{\sqrt{h\tau}}\left\|\frac{\partial u}{\partial t}\right\|_{L_2(e)}\,,
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
I_3 &\le \sigma\int_{-1}^{1}(1-|s|)\left(\int_0^1\left(\frac{\partial v(s,\eta)}{\partial \eta}\right)^2 d\eta\right)^{1/2} ds \\
&\le \frac{\sqrt{2}}{\sqrt{3}}\sigma\left\|\frac{\partial v}{\partial \eta}\right\|_{L_2(\tilde{e})} = \frac{\sqrt{2}}{\sqrt{3}}\frac{\sigma\tau}{\sqrt{h\tau}}\left\|\frac{\partial u(x,t)}{\partial t}\right\|_{L_2(e)}\,.
\end{aligned} \tag{4.54}$$

Combining all the estimates (4.53), (4.54), we arrive at the statement of the lemma.

□

Let us show that

$$\|\eta\|_{\mathcal{H}} \le \sqrt{2}M\left(\tau\left\|\frac{\partial u}{\partial t}\right\|_{L_2(Q_T)} + h^2\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L_2(Q_T)}\right). \tag{4.55}$$

In fact, in accordance with the definition

$$\|v\|^2_{L_2(e)} = \int_t^{t+\tau}\int_{x-h}^{x+h} v^2(x,t)\,dx\,dt$$

from (4.48), we get

$$|\eta|^2 \le \frac{2M^2}{h\tau}\left(\tau^2\int_t^{t+\tau}\int_{x-h}^{x+h}\left(\frac{\partial u}{\partial t}\right)^2 dx\,dt + h^4\int_t^{t+\tau}\int_{x-h}^{x+h}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx\,dt\right).$$

Summarizing the last inequality over all grid nodes $\omega_{h\tau}$, we obtain

$$\sum_{t\in\omega_\tau}\sum_{x\in\omega_h} h\tau\eta^2(x,t) \le 2M^2\left(\tau^2\left\|\frac{\partial u}{\partial t}\right\|^2_{L_2(Q_T)} + h^4\left\|\frac{\partial^2 u}{\partial x^2}\right\|^2_{L_2(Q_T)}\right).$$

Hence, inequality (4.55) is satisfied. Substituting now (4.55) into error estimate (4.47), we get

$$\|z\|_{\mathcal{H}} = \|y - S_x^2 u\|_{\mathcal{H}} \le \frac{2M}{\varepsilon\sqrt{2-\varepsilon}}\left(\tau\left\|\frac{\partial u}{\partial t}\right\|_{L_2(Q_T)} + h^2\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L_2(Q_T)}\right). \tag{4.56}$$

Hence, we have proved the following statement.

Theorem 6. *Let* $u(x,t) \in W^{2,1}_{2,0}$, $f \in L_2(Q_T)$, $u_0 \in \overset{\circ}{W}{}^1_2(0,1)$. *If*

$$\sigma \ge \frac{1+\varepsilon}{2} - \frac{h^2}{4\tau}, \quad 0 < \varepsilon < 2,$$

then the difference scheme with weights (4.38), (4.39) *converges to the generalized solution of problem* (4.36), (4.37) *and for any* τ, h *estimate* (4.56) *holds.*

References

1. L. D. Ivanovich, B. S. Jovanovich and E. E. Shili, *Mat. Vestnik* **36** (1984) 206–212.
2. B. S. Jovanovich, *Mat. Vesnik* **34** (1982) 279–292.
3. O. A. Ladyzhenskaya, **Boundary Value Problems of Mathematical Physics** (Nauka, 1973), in Russian.
4. R. D. Lasarov, *Dokl. Bolg. Akad. Nauk* **35** (1982) 7–10, in Russian.
5. R. D. Lasarov and V. L. Makarov, *Dokl. Akad. Nauk SSSR* **259** (1981) 282–286, in Russian.
6. R. D. Lasarov and A. A. Samarskii, *Zh. Vychisl. Mat. Mat. Fiz.* **20** (1980) 371–387, in Russian.
7. V. P. Mikhailov, **Differential Equations in Partial Derivatives** (Nauka, 1983), in Russian.

8. A. A. Samarskii, *Zh. Vychisl. Mat. Mat. Fiz.* **7** (1967) 1096–1133, in Russian.
9. A. A. Samarskii, *Dokl. Akad. Nauk* **181** (1968) 808–811, in Russian.
10. A. A. Samarskii, **Theory of Difference Schemes** (Nauka, 1977), in Russian.
11. A. A. Samarskii, *Aktual. Problemy Mat. Fis. Vychisl. Mat.* (1984) 174–183.
12. A. A. Samarskii and A. V. Gulin, **Stability of Difference Schemes** (Nauka, 1973), in Russian.
13. A. A. Samarskii, B. S. Jovanovich, P. P. Matus and V. S. Shcheglik, *Differents. Uravn.* **33** (1997).
14. A. A. Samarskii, R. D. Lasarov and V. L. Makarov, **Difference Schemes for Differential Equations with Generalized Solutions** (Nauka, 1987), in Russian.
15. A. A. Samarskii and P. N. Vabishchevich, *Mat. Mode.* **2** (1990) 89–98, in Russian.
16. A. A. Samarskii, P. N. Vabishchevich and P. P. Matus, *Dokl. Akad. Nauk* **354** (1997) 745–747, in Russian.
17. P. N. Vabishchevich and A. A. Samarskii, *Zh. Vychisl. Mat. Mat. Fiz.* **35** (1995) 1011–1021, in Russian.
18. A. A. Zlotnik, *Zh. Vychisl. Mat. Mat. Fiz.* **18** (1978) 1454–1465, in Russian.