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Theory of stability and regularization of difference schemes and its application to ill-posed problems of mathematical physics

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This paper is dedicated to Professor Robert Vichnevetsky to honor him on the occasion of his 65th birthday

Abstract

Solving numerically evolutionary problems of mathematical physics, the problem of stability of difference schemes with respect to initial and boundary data is of great importance. Principal results of the general theory of stability for two- and three-level difference schemes written in the canonical form are presented in this paper. This theory has been developed by the author in 1967. Difference schemes for unsteady problems of mathematical physics are treated in the theory as abstract Cauchy problems for operator-difference problems in Hilbert spaces. The necessary and sufficient conditions for stability in different norms are also formulated for a wide class of difference schemes.

To construct stable difference schemes, a general methodological approach based on the regularization principle is employed. In this approach, we can start from any simple scheme (even unstable) as the initial one. By perturbing the initial scheme operators and taking into consideration the stability conditions, absolutely stable schemes are derived. Such an approach is used to obtain the difference schemes for approximate solution of the ill-posed problems for evolutionary equations.

1. Introduction

There are several approaches to study stability of grid approximations for the unsteady differential equations of mathematical physics [3,6,18]. As a rule, they are based on some assumptions about the structure of difference operators, and stability investigations are performed in them using the variable separation method or the method of energy inequalities.

The most general and constructive theory of stability has been developed in [7–12]. A difference scheme is considered in this theory as an independent object of study without any relation to the differential problem. The stability conditions of two- and three-level difference schemes considered in grid Hilbert spaces have been obtained in [8–10]. In this theory a

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difference scheme is presented in the canonical form and the necessary and sufficient conditions are formulated by operator inequalities. All main classes of difference schemes for common problems of mathematical physics have been analyzed on the basis of the theory.

A general approach to obtain difference schemes with the given quality is the regularization principle proposed and illustrated on some examples in [7]. This principle is particularly important for constructing stable difference schemes on the basis of some initial simple schemes. The quality of the initial scheme is improved via the perturbation of its operators, taking into account general stability conditions.

Inverse problems for the mathematical physics equations, which are ill-posed in the classical sense [17], are of great importance in terms of applications. In particular, the inverse problems of heat transfer can be mentioned [1,2]. Among the ill-posed problems for evolutionary equations the problem with inverse time for parabolic equations (the retrospective inverse problem of heat transfer) as well as the Cauchy problem for elliptic-type equation should be noted. The problems under consideration belong to the class of conditionally well-posed problems. If a class of acceptable solutions is narrowed down (to distinguish a class of correctness) a solution becomes continuously dependent on initial conditions. These problems are solved approximately by means of regularization methods [17].

Among the approximate solution methods for ill-posed problems the methods connected with perturbations of an equation itself can be recognized. This approach is known for differential equations as the quasi-inversion method [5]. It is based on a certain perturbation of the given equation, so that the problem becomes well-posed for the perturbed equation. Here the parameter of perturbation represents a parameter of regularization.

We can follow two different directions in solving applied ill-posed problems approximately. In the first one, we construct a corresponding regularized problem for a given continuous unstable problem and then pass to a discrete problem. An alternative, connected with the use of the difference scheme regularization concept, is the direct construction of the discrete analog of an unstable problem with subsequent regularization. The general theory of difference scheme stability constructed independently on the continuous problem is the theoretical basis of the approach. Some features of the solutions of conditionally stable evolutionary problems manifest itself in the growth of the solution norm with time, i.e., the ρ-stability of the difference schemes concept is used.

This paper presents a part of our joint studies with Professor P.N. Vabishchevich in constructing difference schemes for unstable problems on the basis of the regularization principle for difference schemes. A more detailed presentation of some particular results has been given in our previous publications [13–16].

2. The theory of stability of difference schemes

The principal results of the stability theory for difference schemes [11] are given below. The theory is constructed on the following fundamental ideas:

(i) A difference scheme is considered as an independent object of the study.
(ii) A difference scheme is written in the canonical form and considered in Hilbert grid spaces.
(iii) Conditions of stability are formulated in operator inequalities.

2.1. The stability of two-level difference schemes

To consider difference schemes for approximately solving evolutionary problems, we use the following designations. Let $u(t) \in H$, where H is a real finite-dimensional Hilbert space. For time $t = t_n = n\tau$, where $\tau > 0$ is a time-step, an approximate solution is denoted by y_n. Let (y, v) be the scalar product in H and $\| y \| = (y, y)^{1/2}$. Let operator D be selfadjoint and positive-definite in H ($D > 0$, if $(Dy, y) > 0$ for each $0 \neq y \in H$). We determine the space H_D with $\| y \|_D = (Dy, y)^{1/2}$ for this operator.

The stability theory of difference schemes is based on the canonical presentation form for difference schemes. For linear two-level difference schemes this form is as follows [11]:

$$B\frac{y_{n+1} - y_n}{\tau} + Ay_n = 0, \quad n = 0, 1, \ldots, \tag{2.1}$$

for given $y_0 \in H$. In (2.1) A and B are linear operators.

Difference scheme (2.1) is call stable in space H_D if the approximate solution satisfies the estimate:

$$\| y_{n+1} \|_D \leqslant \| y_n \|_D \tag{2.2}$$

for any $y_0 \in H$. Let us cite the following basic result of the stability theory for difference schemes [11].

Theorem 2.1. *Assume in* (2.1) *that the operators A and B are steady* (*independent of n*), *that $A = A^* > 0$ and that B^{-1} does exist* ($B \neq B^*$). *Then the condition*

$$B \geqslant \tfrac{1}{2}\tau A \tag{2.3}$$

is necessary and sufficient for the stability of the scheme (2.1) *in H_A. Moreover, if $B = B^* > 0$ then condition* (2.3) *is necessary and sufficient for the stability of the scheme* (2.1) *in H_B, too.*

This result is principal. We deal with coincident necessary and sufficient conditions of stability of difference schemes. Some generalizations can be performed in different directions, such as the stability with respect to the right-hand side, consideration of the difference schemes with non-selfadjoint operators, unsteady operators, stability in more simple norms, etc.

It should be noted that if $A \neq A^* > 0$ and $B = B^* > 0$, then the following inequality for inverse operators is the necessary and sufficient condition for stability in H_B: $A^{-1} \geqslant \frac{1}{2}\tau B^{-1}$.

2.2. ρ-stability of difference schemes

The condition of stability (2.2) is not always reasonable. For example, an exact solution norm of the initial differential problem can increase or decrease in a proper manner. In particular, such a situation is typical for ill-posed problems, where the solution norm increases and the regularization allows to control the growth. It is natural that an approximate solution reproduces such a behavior of the exact one. A more general concept of ρ-stability of difference schemes is used instead of (2.2) to reach this aim.

Difference scheme (2.1) is referred to as ρ-stable [11] if

$$\| y_{n+1} \|_D \leqslant \rho \| y_n \|_D, \tag{2.4}$$

where $\rho > 0$. To specify ρ, the following two expressions are in common use

$$\rho = \exp(c\tau), \qquad \rho = 1 + c\tau,$$

where the constant c does not depend on τ. For such a ρ the evaluation of the difference solution stability with respect to the initial data follows from (2.4) in the form

$$\| y_{n+1} \|_D \leqslant \exp(ct_{n+1}) \| y_0 \|_D. \tag{2.5}$$

The general conditions for ρ-stability of two-level difference schemes are formulated in the following statement.

Theorem 2.2. *Assume in* (2.1) *that the operators A and B are stationary* (*independent of n*), *selfadjoint and that B is positive. Then the condition*

$$\frac{1-\rho}{\tau} B \leqslant A \leqslant \frac{1+\rho}{\tau} B \tag{2.6}$$

is necessary and sufficient for the ρ-stability of scheme (2.1) *in H_B.*

Proof. Since $B = B^* > 0$, operators $B^{1/2}$ and $B^{-1/2}$ exist. Let $x = B^{1/2} y$. After multiplying (2.1) by $B^{-1/2}$ we obtain

$$\frac{x_{n+1} - x_n}{\tau} + Cx_n = 0, \tag{2.7}$$

where $C = B^{1/2} A\, B^{-1/2}$. Then

$$x_{n+1} = Sx_n, \qquad S = E - \tau C, \tag{2.8}$$

where E is the unitary operator and S is the operator of transition. The condition of ρ-stability in H_B is $\| x_{n+1} \| \leqslant \rho \| x_n \|$. This inequality and the estimate $\| S \| \leqslant \rho$ are equivalent. If operator S is selfadjoint, then we can obtain the equivalent inequality $-\rho E \leqslant S \leqslant \rho E$ and further we deduce from (2.8) the following inequality

$$\frac{1-\rho}{\tau} E \leqslant C \leqslant \frac{1+\rho}{\tau} E. \tag{2.9}$$

After multiplying both inequalities (2.9) by $B^{1/2}$ we obtain inequality (2.6). □

It should be noted that the theorem has been proved without the condition $A > 0$. The constraint $A \geqslant 0$ is necessary for the scheme ρ-stability with $\rho = 1$, and condition $A > 0$ is necessary for $\rho < 1$. These conditions are contained in the left inequality (2.6). Generally, positivity of operator A is not required for the ρ-stability of scheme (2.1) with $\rho > 1$ (see (2.6)).

2.3. Stability of three-level difference schemes

Three-level difference schemes can be written in the following canonical form

$$B \frac{y_{n+1} - y_{n-1}}{2\tau} + \tau^2 R \frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} + Ay_n = 0, \quad n = 1, 2, \ldots, \tag{2.10}$$

for given y_0, $y_1 \epsilon H$.

We have to use more complicated norms when investigating the stability of three-level difference schemes. For every $n = 1, 2, \ldots$ we determine the vector

$$Y^n = \left\{\tfrac{1}{2}(y_n + y_{n-1}),\ y_n - y_{n-1}\right\}.$$

The direct sum of the spaces H is designated by $H^2 = H \oplus H$. For vectors $Y = \{y^1, y^2\}$ addition and multiplication in H^2 are carried out coordinate by coordinate, and the scalar product is

$$(Y, V) = (y^1, v^1) + y^2, v^2.$$

For the difference scheme (2.10) with $R = R^*$, $A = A^* > 0$, $4R - A \geqslant 0$, we introduce a norm in $H_{\mathscr{D}}^2$ as $\| Y^n \|_{\mathscr{D}} = ((\mathscr{D}Y^n, Y^n))^{1/2}$, where

$$(\mathscr{D}Y^n, Y^n) = \tfrac{1}{4}\| y_n + y_{n-1} \|_A^2 + \| y_n - y_{n-1} \|_{R-A/4}. \tag{2.11}$$

Theorem 2.3. *Let in* (2.10) *the operators A and R be stationary* (*independent of n*), *selfadjoint and positive-definite. Then*:

$$B_0 = \tfrac{1}{2}(B + B^*) \geqslant 0, \tag{2.12}$$

$$R - \tfrac{1}{4}A \geqslant 0, \tag{2.13}$$

are necessary and sufficient conditions for the stability of scheme (2.10) *in $H_{\mathscr{D}}^2$, i.e., the estimate $\| Y^{n+1} \|_{\mathscr{D}} \leqslant \| Y^n \|_{\mathscr{D}}$ is satisfied.*

Stability analysis of three-level difference schemes is based on the transition to the equivalent two-level difference schemes in H^2.

2.4. ρ- stability of three-level schemes

Allowing some increase or decrease of the solution norm of the difference problem we shall use ρ-stable difference schemes. The three-level difference scheme (2.10) is called ρ-stable if

$$\| Y^{n+1} \|_{\mathscr{D}} \leqslant \rho \| Y^n \|_{\mathscr{D}}, \tag{2.14}$$

where $\rho > 0$ is any number.

Theorem 2.4. *Let in* (2.10) *the operators A, R and B be stationary* (*independent of n*) *and selfadjoint. Then, if*

$$\frac{\rho^2 - 1}{2\tau} B + (\rho^2 + 1)R > 0, \tag{2.15}$$

then for the ρ-stability in $H_{\mathscr{D}}^2$ with $\rho > 0$ it is necessary and sufficient to satisfy the following conditions:

$$\frac{\rho^2 - 1}{2\tau} B + (\rho - 1)^2 R + \rho A \geqslant 0, \tag{2.16}$$

$$\frac{\rho^2 - 1}{2\tau} B + (\rho + 1)^2 R - \rho A \geqslant 0, \tag{2.17}$$

$$\frac{\rho^2 + 1}{2\tau} B + (\rho^2 - 1) R \geqslant 0. \tag{2.18}$$

In this case the operator of the norm $\mathscr{D}$ is given by the expression

$$(\mathscr{D}y^n, y^n) = \frac{1}{4}\left\| \frac{1}{\rho} y_n + y_{n-1} \right\|_{\tilde{A}}^2 + \left\| \frac{1}{\rho} y_n - y_{n-1} \right\|_{\tilde{R}-\tilde{A}/4}^2,$$

where

$$\tilde{A} = \frac{\rho^2-1}{2\tau} B + (\rho-1)^2 R + \rho A,$$

$$4\tilde{R} - \tilde{A} = \frac{\rho^2-1}{2\tau} B + (\rho+1)^2 R - \rho A.$$

The above conditions for the ρ-stability of three-level difference schemes have been obtained in [4] and carefully discussed in [12].

3. Regularization of difference schemes

In [7] the regularization concept was suggested and illustrated by many examples. For two- and three-level difference schemes there were formulated some recipes to improve the quality (stability, accuracy, efficiency) of difference schemes. The regularization concept was suggested and applied to the difference schemes for well-posed evolutionary problems of mathematical physics. So, this approach is especially justified for a construction of difference schemes for ill-posed problems.

The construction of stable difference schemes via the regularization concept is implemented in the following way:

(i) An initial difference scheme that does not have necessary properties is constructed for a given problem.
(ii) The difference scheme is presented in the canonical form.
(iii) Quality of the difference scheme is improved in the required aspects by the perturbation of the difference scheme operators.

The theory of regularization for the difference schemes is studied here as the principle of improvement of scheme properties by means of involving new supplementary items (regularizers) to operators of the initial difference scheme.

3.1. Evolutionary first-order equation

The following problem will be used to illustrate the main regularization ideas

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \mathscr{A}u = 0, \quad 0 < t \leqslant T, \qquad u(0) = u_0, \tag{3.1}$$

where $\mathscr{A} = \mathscr{A}^* > 0$ in H.

According to the regularization concept we write the simplest explicit difference scheme for this problem

$$\frac{y_{n+1}-y_n}{\tau}+\mathscr{A}y_n=0, \quad n=0, 1, \ldots. \tag{3.2}$$

It could be transformed to the canonical form (2.1) for

$$B=E, \qquad A=\mathscr{A}. \tag{3.3}$$

Since $\mathscr{A} \leqslant \| \mathscr{A} \| E$, the stability condition (2.3) is satisfied in H and H_A for the scheme (2.1) with (3.3), when $\tau \leqslant 2/\| A \|$. So we obtain the usual conditions for the conditional stability of the explicit scheme.

The regularization is based on the transition from the initial scheme to another scheme (perturbed). Let $\mathscr{R}=\mathscr{R}^* > 0$ be the regularization grid operator and let $\alpha > 0$ be the regularization parameter (the parameter of perturbation). Taking into account the necessary and sufficient conditions of stability (2.3), it is natural to construct the regularized scheme by the perturbation of operator B. We write the perturbed scheme in the form

$$(E+\alpha\mathscr{R})\frac{y_{n+1}-y_n}{\tau}+\mathscr{A}y_n=0, \quad n=0, 1, \ldots. \tag{3.4}$$

When the natural choices $\mathscr{R}=\mathscr{A}$ and $\mathscr{R}=\mathscr{A}^2$ are considered, the conditions of the absolute stability are given by the following theorem.

Theorem 3.1. *Difference scheme* (3.4) *with* $\mathscr{A}=\mathscr{A}^* > 0$ *is absolutely stable in* $H_{\mathscr{A}}$ *for regularizer* $\mathscr{R}=\mathscr{A}$ *if* $\alpha \geqslant \tau/2 - \| \mathscr{A} \|^{-1}$, *and for* $\mathscr{R}=\mathscr{A}^2$ *if* $\alpha \geqslant \tau^2/16$.

Proof. The scheme (3.4) is presented in canonical form (2.1) with

$$B=E+\alpha\mathscr{R}, \qquad A=\mathscr{A}. \tag{3.5}$$

Substituting (3.5) into (2.3) with $\mathscr{R}=\mathscr{A}$, we obtain

$$\left(\| \mathscr{A} \|^{-1}+\alpha-\tfrac{1}{2}\tau\right)\mathscr{A} \geqslant 0.$$

The theorem statement results from this inequality.

For the regularizer $\mathscr{R}=\mathscr{A}^2$ and (3.5) the inequality (2.3) is transformed by the following manner

$$\begin{aligned} B-\frac{\tau}{2}A &= E+\alpha\mathscr{A}^2-\frac{\tau}{2}\mathscr{A} \\ &= \left(\alpha^{1/2}\mathscr{A}-\tfrac{1}{4}\tau\alpha^{-1/2}E\right)^2+\left(1-\tfrac{1}{16}\tau^2\alpha^{-1}\right)E \geqslant 0. \end{aligned}$$

So, the regularized difference scheme (3.4) is absolutely stable for $\mathscr{R}=\mathscr{A}^2$ and $\alpha \geqslant \tau^2/16$. □

The set $\mathscr{R}=\mathscr{A}$ and $\alpha=\sigma\tau$ for the regularized scheme (3.4) corresponds to the usual scheme with weights. We can increase the stability reserve in the class of explicit schemes via perturbing operator $\mathscr{A}$ of the explicit scheme (3.2).

3.2. Second-order equation

Instead of problem (3.4) we consider now the Cauchy problem for the second-order equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \mathscr{A} u = 0, \quad 0 < t \leqslant T, \qquad u(0) = u_0, \quad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = u_1, \tag{3.6}$$

and construct some absolutely stable schemes using the regularization principle.

Let the following explicit symmetrical difference scheme be taken as initial:

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} + \mathscr{A} y_n = 0, \quad n = 1, 2, \ldots. \tag{3.7}$$

The scheme (3.7) is presented in the canonical form of three-level difference schemes (2.10) with

$$B = 0, \qquad R = \frac{1}{\tau^2} E, \qquad A = \mathscr{A}. \tag{3.8}$$

Since $\mathscr{A} \leqslant \| \mathscr{A} \| E$, the conditions of stability (2.12)–(2.13) give

$$E - \tfrac{1}{4}\tau^2 \mathscr{A} \geqslant \left(\| \mathscr{A} \|^{-1} - \tfrac{1}{4}\tau^2 \right) \mathscr{A} \geqslant 0.$$

This inequality holds if $\tau^2 \leqslant 4 \| \mathscr{A} \|^{-1}$.

Analogously to (3.4) we present the regularized difference scheme for (3.7) in the form

$$(E + \alpha \mathscr{R}) \frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} + \mathscr{A} y_n = 0. \tag{3.9}$$

The following statement is correct for the regularizer choices noted above.

Theorem 3.2. *The difference scheme* (3.9) *with* $\mathscr{A} = \mathscr{A}^* > 0$ *is absolutely stable for regularizer* $\mathscr{R} = \mathscr{A}$ *if*

$$\alpha \geqslant \tfrac{1}{4}\tau^2 - \| \mathscr{A} \|^{-1}, \tag{3.10}$$

and for $\mathscr{R} = \mathscr{A}^2$ *if* $\alpha \geqslant \tau^4/64$.

Proof. Let us check the necessary and sufficient conditions for stability (2.12)–(2.13). Evidently, inequality (2.12) is always satisfied since the regularized scheme (3.9) is presented in the canonical form (2.10) with

$$B = 0, \qquad R = \tau^{-2}(E + \alpha \mathscr{R}), \qquad A = \mathscr{A}. \tag{3.11}$$

If $\mathscr{R} = \mathscr{A}$ the following inequality is obtained from (2.13) and (3.11):

$$\left(\| \mathscr{A} \|^{-1} + \alpha \right) \mathscr{A} \geqslant \tfrac{1}{4}\tau^2 \mathscr{A},$$

which is satisfied for parameter α according to (3.10).

If $\mathscr{R} = \mathscr{A}^2$, inequality (2.13) is transformed to the form

$$\tau^2\left(R - \tfrac{1}{4}A\right) = E + \alpha\mathscr{A}^2 - \tfrac{1}{4}\tau^2\mathscr{A}$$
$$= \left(\alpha^{1/2}\mathscr{A} - \tfrac{1}{8}\tau^2\alpha^{-1/2}\right)E\Big)^2 + \left(1 - \tfrac{1}{64}\tau^4\alpha^{-1}\right)E \geqslant 0.$$

Therefore the regularized difference scheme is absolutely stable when $\mathscr{R} = \mathscr{A}^2$ and $\alpha \geqslant \tau^4/64$. □

It should be noted that the choice $\alpha = \sigma\tau^2$, $\sigma_1 = \sigma_2 = \sigma$ in $\mathscr{R} = \mathscr{A}$ corresponds to the usual scheme with weights:

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} - \mathscr{A}(\sigma_1 y_{n+1} + (1 - \sigma_1 - \sigma_2)y_n + \sigma_2 y_{n-1}) = 0.$$

The properties of this scheme have been well studied in [11,12].

4. Regularized difference schemes for ill-posed problems

The principle of regularization of difference schemes is applied for the construction ρ-stable difference schemes. The simplest Cauchy problems for the first- and second-order operator equations with selfadjoint operators will be considered.

4.1. Conditionally well-posed problem

Let us consider the following problem in H:

$$\frac{\mathrm{d}u}{\mathrm{d}t} - \mathscr{A}u = 0, \quad 0 < t \leqslant T, \qquad u(0) = u_0, \tag{4.1}$$

with selfadjoint and positive-definite operator $\mathscr{A}$.

Let us derive an estimation for the problem solution. Conditional correctness of the problem will follow from the estimate. We introduce

$$\Phi(t) = \| u \|^2 = (u, u). \tag{4.2}$$

After differentiation of expression (4.2), taking into account (4.1), we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 2\left(u, \frac{\partial u}{\partial t}\right) = -2(u, \mathscr{A}u). \tag{4.3}$$

Due to the fact that operator $\mathscr{A}$ is selfadjoint, the second differentiation leads to:

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} = -4\left(\mathscr{A}u, \frac{\partial u}{\partial t}\right) = 4\left\|\frac{\partial u}{\partial t}\right\|^2. \tag{4.4}$$

The inequality below follows from (4.2)–(4.4) and the Cauchy–Bunjakovsky inequality:

$$\Phi \frac{\mathrm{d}^2\Phi}{\mathrm{d}t^2} - \left(\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right)^2 = 4\left(\| u \|^2 \left\| \frac{\partial u}{\partial t} \right\|^2 - \left(u, \frac{\partial u}{\partial t}\right)^2 \right) \geqslant 0. \tag{4.5}$$

This is equivalent to the inequality:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln \Phi(t) \geqslant 0. \tag{4.6}$$

It means that the function $\ln \Phi(t)$ is convex. From (4.6) we have

$$\ln \Phi(t) \leqslant \frac{t}{T} \ln \Phi(T) + \left(1 - \frac{t}{T}\right) \ln \Phi(0).$$

Hence we get

$$\phi(t) \leqslant (\Phi(T))^{t/T} (\Phi(0))^{1-t/T}.$$

Taking (4.2) into consideration we obtain the required estimate for the solution of problem (4.1)

$$\| u(t) \| \leqslant \| u(T) \|^{t/T} \| u(0) \|^{1-t/T}. \tag{4.7}$$

Assume that the solution of problem (4.1) is now considered in a class of limited in H solutions, that is

$$\| u(t) \| \leqslant M. \tag{4.8}$$

We receive the following estimate from (4.7) in the class of appropriate limitations (4.8):

$$\| u(t) \| \leqslant M^{t/T} \| u_0 \|^{1-t/T}. \tag{4.9}$$

It means that the continuous dependence on initial conditions takes place for problem (4.1) in the class of limited solutions when $0 < t < T$. Based on this fact, the algorithms to solve approximately the ill-posed problem (4.1) must separate the class of limited solutions in some way. Besides, the estimate of type (4.9) allowing growth of the solution norm with time must be typical for an approximate solution.

4.2. Regularized schemes

The regularization principle that has been illustrated above for the well-posed problem (3.1) will be used now for the construction of difference schemes for the ill-posed problem (4.1). First we consider the explicit difference scheme

$$\frac{y_{n+1} - y_n}{\tau} - \mathscr{A} y_n = 0, \quad n = 0, 1, \ldots. \tag{4.10}$$

According to the regularization principle we rewrite the scheme in canonical form with

$$B = E, \qquad A = -\mathscr{A}, \tag{4.11}$$

i.e., $A = A^* < 0$.

Theorem 4.1. *The explicit scheme* (4.10) *is* ρ*-stable in* H *and*

$$\rho = 1 + \|\mathscr{A}\|\tau. \tag{4.12}$$

Proof. Let us check the conditions of ρ-stability (2.6) for the scheme (2.1) with (4.12). As our assumptions we have $B > 0$ and $A < 0$ for $\rho > 1$. Then the right-hand side of the two-side operator inequality (2.6) is evidently satisfied for all $\tau > 0$. The left-hand side acquires the form $(\rho - 1)/\tau E \geqslant A$, when ρ is chosen in accordance with (4.12). □

Theorem 4.1 underlines a regularizing effect of the utilization of usual difference schemes for approximately solving ill-posed problems for parabolic equations. This is directly connected with the limitation of the corresponding grid elliptic operator $\mathscr{A}$. For example, $\|\mathscr{A}\| = O(h^{-2})$ for parabolic second-order equations, where h is a step of a spatial grid [11], and so this grid step limits the growth of a solution, i.e., it acts as the parameter of regularization.

When ill-posed problems are solving approximately [17], the choice of the regularization parameter must be adjusted to the level of error in the initial data. We have only constructed here some stable numerical algorithms for ill-posed evolutionary problems and studied the influence of the regularization parameter only for stability of the corresponding difference scheme. When the regularization parameter is given, then parameter α indicates the minimum of ρ provided by the theory of stability.

Basing on the explicit scheme (4.10) for problem (4.1) we present the regularized scheme in the canonical form (2.1) with

$$B = E + \alpha\mathscr{R}, \qquad A = -\mathscr{A}. \tag{4.13}$$

Theorem 4.2. *The regularized scheme* (2.1) *with* (4.13) *is* ρ*-stable in* H *with*

$$\rho = 1 + \tau/\alpha \tag{4.14}$$

when $\mathscr{R} = \mathscr{A}$ *and*

$$\rho = 1 + \tfrac{1}{2}\alpha^{-1/2}\tau \tag{4.15}$$

when $\mathscr{R} = \mathscr{A}^2$.

Proof. We only have to check again the fulfillment of the left-hand side of the two-side inequality (2.6), which takes the following form for (4.13)

$$\frac{\rho - 1}{\tau}(E + \alpha\mathscr{R}) \leqslant \mathscr{A}. \tag{4.16}$$

When $\mathscr{R} = \mathscr{A}$ and ρ is chosen in the form (4.14), inequality (4.16) is satisfied.

When $\mathscr{R} = \mathscr{A}^2$, inequality (4.16) is transformed in the following manner:

$$E + \alpha\mathscr{A}^2 - \frac{\tau}{\rho - 1}\mathscr{A} = \left(\alpha^{1/2}\mathscr{A} - \frac{\tau}{2(\rho - 1)}\alpha^{-1/2}E\right)^2 + \left(1 - \frac{\tau^2}{4(\rho - 1)^2}\alpha^{-1}\right)E \leqslant 0.$$

The last inequality is satisfied when ρ is chosen in the form (4.15). □

4.3. Second-order equation

We should discuss the possibility of construction of regularized schemes for approximately solving the ill-posed Cauchy problem for an evolutionary second-order equation. Let us consider the following problem:

$$\begin{aligned} &\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - \mathscr{A}u = 0, \quad 0 < t \leqslant T, \\ &u(0) = u_0, \qquad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = u_1, \end{aligned} \tag{4.17}$$

with $\mathscr{A} = \mathscr{A}^* > 0$ in H.

We take the explicit symmetrical difference scheme as the initial one:

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} - \mathscr{A}y_n = 0, \quad n = 1, 2, \ldots. \tag{4.18}$$

The scheme (4.18) is presented in the canonical form of the three-level difference scheme (2.10) with

$$B = 0, \qquad R = \frac{1}{\tau^2}E, \qquad A = -\mathscr{A}, \tag{4.19}$$

i.e., when $A = A^* < 0$.

Theorem 4.3. *The explicit scheme* (4.18) *is* ρ*-stable with*

$$\rho = \exp(\|\mathscr{A}\|^{1/2}\tau). \tag{4.20}$$

Proof. Let us check the fulfillment of conditions (2.15)–(2.18) for ρ-stability of the scheme (4.18). When $B \geqslant 0$, $A \leqslant 0$, $R \geqslant 0$ and $\rho > 1$ (see (4.19)) the inequalities (2.15), (2.17) and (2.18) are evidently satisfied for every $\tau > 0$. Taking into account (4.19), inequality (2.16) is transformed to

$$(\rho - 1)^2E - \tau^2\rho\mathscr{A} \geqslant \left((\rho - 1)^2\|\mathscr{A}\|^{-1} - \tau^2\rho\right)\mathscr{A} \geqslant 0.$$

The estimates are obtained using the following useful result [15]. □

Lemma. *The inequality*

$$(\rho - 1)^2\chi - \tau^2\rho \geqslant 0$$

for positive χ, τ *and* $\rho > 1$ *is satisfied when*

$$\rho \geqslant \exp(\chi^{-1/2}\tau).$$

In our case $\chi = \|\mathscr{A}\|^{-1}$ and therefore we obtain for ρ the required estimate (4.3) for the explicit scheme (4.18).

Taking into account the limitation of the grid operator $\mathscr{A}$ (when the Cauchy problem for elliptic equations is under consideration), we can conclude that the step of a spatial grid limits the solution growth, i.e., it acts as the regularization parameter.

Let us construct absolutely stable difference schemes for the approximate solution of the problem by introducing some regularizing additions to the grid operators of the difference scheme. In just the same way as it was done for the direct problem (4.17), we can start from the explicit scheme (4.18) and derive the regularized scheme in the canonical form (2.10) with

$$B=0, \qquad R=\frac{1}{\tau^2}(E+\alpha\mathscr{R}), \qquad A=-\mathscr{A}. \tag{4.21}$$

Theorem 4.4. *The regularized scheme* (2.10) *with* (4.21) *is ρ-stable when $\mathscr{R}=\mathscr{A}$ with*

$$\rho=\exp(\alpha^{-1/2}\tau), \tag{4.22}$$

and when $\mathscr{R}=\mathscr{A}^2$ with

$$\rho=\exp(2^{-1/2}\alpha^{-1/4}\tau). \tag{4.23}$$

Proof. We transfer from (2.16) for (4.21) to the inequality

$$(\rho-1)^2(E+\alpha\mathscr{R})-\tau^2\rho\mathscr{A}\geqslant 0. \tag{4.24}$$

When $\mathscr{R}=\mathscr{A}$, similar to the proof of Theorem 4.3 ($\xi=\alpha+\|\mathscr{A}\|^{-1}$), we receive the following expression for ρ

$$\rho=\exp\left(\left(\alpha+\|\mathscr{A}\|^{-1}\right)^{-1/2}\tau\right).$$

Putting ρ more coarse we obtain the estimate (4.22).

When $\mathscr{R}=\mathscr{A}^2$, we derive from (4.22):

$$\begin{aligned} E&+\alpha\mathscr{A}^2-\frac{\tau^2\rho}{(\rho-1)^2}\mathscr{A}\\ &=\left(\alpha^{1/2}\mathscr{A}-\frac{\tau^2\rho}{2(\rho-1)^2}\alpha^{-1/2}E\right)^2\left(1-\frac{\tau^4\rho^2}{4(\rho-1)^4}\alpha^{-1}\right)E\geqslant 0.\end{aligned}$$

This inequality is satisfied for given ρ, if the parameter of regularization is

$$\alpha\geqslant\frac{\tau^4\rho^2}{4(\rho-1)^4}. \tag{4.25}$$

Let us estimate ρ for given α from inequality (4.25) transformed to the form

$$(\rho-1)^2 2\alpha^{1/2}-\tau^2\rho\geqslant 0.$$

In accordance with the lemma ($\chi=2\alpha^{1/2}$), the inequality is fulfilled for ρ chosen from (4.23). □

The noted possibilities of construction of regularized schemes can be connected with well-known as well as new variants of the quasi-inversion method, if we consider these schemes as the approximations of the corresponding differential equations.

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References

[1] O.M. Alifanov, *Inverse Problems of Heat Exchange* (Mashinostroenie, Moscow, 1988) (in Russian).
[2] J.V. Beck, B. Blackwell and C.R. Clair Jr, *Inverse Heat Conduction. Ill-Posed Problems* (Wiley, New York, 1985).
[3] P.G. Ciarlet and J.L. Lions, eds., *Handbook of Numerical Analysis, Vol.* 1 (North-Holland, Amsterdam, 1990).
[4] A.V. Gulin, Necessary and sufficient conditions for the stability of three-level difference schemes, *Zh. Vychisl. Mat. i Mat. Fiz.* 8 (1968) 899–902 (in Russian).
[5] R. Lattes, and J.L. Lions, *Méthode de Quasi-Reversibilité et Applications* (Dunod, Paris, 1967).
[6] R.D. Richtmyer and K.W. Morton, *Difference Methods for Initial Value Problems* (Interscience, New York, 1967).
[7] A.A. Samarskii, On regularization of difference schemes, *Zh. Vychisl. Mat. i Mat. Fiz.* 7 (1967) 62–93 (in Russian).
[8] A.A. Samarskii, Classes of stable schemes, *Zh. Vychisl. Mat. i Mat. Fiz.* 7 (1967) 1096–1133 (in Russian).
[9] A.A. Samarskii, Necessary and sufficient conditions for the stability of two-level difference schemes, *Dokl. Acad. Nauk* 181 (1968) 808–811 (in Russian).
[10] A.A. Samarskii, On stability of three-level difference schemes, *Dokl. Acad. Nauk* 192 (1970) 998–1001 (in Russian).
[11] A.A. Samarskii, *Theory of Difference Schemes* (Nauka, Moscow, 1989) (in Russian).
[12] A.A. Samarskii and A.V. Gulin, *Stability of Difference Schemes* (Nauka, Moscow, 1975) (in Russian).
[13] A.A. Samarskii and P.N. Vabishchevich, Difference schemes for unstable problems, *Math. Modeling* 2 (11) (1990) 89–98 (in Russian).
[14] A.A. Samarskii and P.N. Vabishchevich, Regularized difference schemes for equations with non-selfadjoint operators, *Math. Modeling* 4 (2) (1990) 36–44 (in Russian).
[15] A.A. Samarskii and P.N. Vabishchevich, Regularized difference schemes for evolutionary second-order equations, *Math. Models Methods Appl. Sci.* 2 (1992) 295–315.
[16] A.A. Samarskii and P.N. Vabishchevich, Regularized three-level difference schemes with non-selfadjoint operators, *Differentsial'ny Uravneniya* 29 (1993) 1244–1251 (in Russian).
[17] A.N. Tikhonov and V.Y. Arsenin, *Solution of Ill-posed Problems* (Winston, Washington, DC, 1977).
[18] V. Thomee, Stability theory for partial difference operators, *SIAM Rev.*, 11 (1969) 152–195.