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REGULARIZED DIFFERENCE SCHEMES FOR EVOLUTIONARY SECOND ORDER EQUATIONS

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The questions of approximate solution of unstable problems for evolutionary second order equations are discussed in this paper. The classical Cauchy problem for elliptic type equation is a significant example of such problem. Incorrectness of this problem (the Hadamard example) is due to instability of the solution towards small perturbations of the initial conditions.

The extension problem of the solutions of well-posed elliptic problems beyond the calculation region boundary is also discussed. The stability of corresponding difference schemes is investigated by basing on general theory of ρ-stability. The principle of the regularization of three-layer difference schemes is developed for the unstable problems. It is shown that the regularized difference schemes correspond to some modification of quasi-inversion method.

1. Introduction

Inverse problems for the mathematical physics equations, which are ill-posed in the classical sense, are of great importance in terms of applications.[22] In particular, inverse problems of heat transfer may be mentioned.[2,4] In the geophysical survey (gravitational, magnetometric and electric prospecting) one of the major problems is continuation of potential fields.[6,8,14] It leads to approximate solution of ill-posed problems for elliptic equation. As is well known[9] the Cauchy problem is ill-posed for elliptic equations. We may also note the extension of well-posed elliptic boundary value solutions into the region adjacent to the boundary. This problem is reduced to the one with initial conditions. When considering the stationary inverse boundary value problem of heat conduction we obtain such unstable problems.[2,4]

The problems under consideration belong to a class of conditionally well-posed problems. If a class of acceptable solutions is narrowly drawn (to distinguish a class of correctness) the solution becomes continuously depending on initial conditions. These problems are solved approximately by the regularization methods.

The approximate solution methods for ill-posed evolutionary problems may be divided into two categories. In the first one, perturbations are applied to the initial conditions which are given with a certain error. The second category of the methods for obtaining stable solutions of ill-posed problems for partial differential equations is concerned with perturbation of the equations (the method of quasi-inversion[12]).

The methods with perturbation of initial conditions are widely used for the extremal formulation of a problem.[3] In the problems of optimal control, where the systems are described by the partial differential equations,[13] the regularization method of Tikhonov is employed.[22,28] The class of approximate solution methods also includes the methods with non-local perturbation of initial conditions. In this case the regularization effect results from binding the solutions on initial and final times together. The regularization of ill-posed evolutionary problems based on non-local perturbation of initial conditions was proposed in Ref. 1. Some questions on this approach towards numerical solution of the Cauchy problem for elliptic equations were discussed in Ref. 27. Equivalence of the extremal formulations for ill-posed evolutionary problem and non-local problem is worth noticing (see, for example, Ref. 26).

The quasi-inversion method[12] is based on a perturbation of the given equation, when the problem becomes well-posed for the perturbed equations. Here the perturbation parameter acts as a regularization parameter. In Ref. 12 the Cauchy problem for elliptic equations was considered in a common irregular region. By restricting ourselves by cylindrical regions, we may construct modifications of the quasi-inversion method, which are similar to those available for the problems with inverse time for parabolic equations (see, for example, Refs. 7, 12 and 21). Based on coordinate transformation, common calculation regions may be transformed to the cylindrical calculation region. Such transformation was, in fact, performed in Ref. 23.

These methods, as opposed to the methods with perturbation of initial conditions or usual modifications of quasi-inversion method for the elliptic Cauchy problem,[12] allow construction of the most efficient computational algorithms. Incorrect problem is solved sequentially by passing from one time layer to the next one. In this manner, specific features of a given evolutionary problem can be most fully taken into account.

We may follow two different directions in solving approximately the applied ill-posed problems. In the first one, we construct a corresponding regularization problem for the given continuous unstable problem and then pass to the discrete problem. An alternative is connected with constructing discrete analogs of the unstable problem with subsequent regularization. For example, for well-posed problems of mathematical physics the theory of stability of difference schemes[18,19] is constructed independently of a continuous problem.

This paper is part of our studies in constructing difference schemes for unstable problems by basing on the regularization concept of difference schemes.[16] In Ref. 20 the regularization of difference schemes was made for ill-posed evolutionary first order equations on an example of the inverse time problem for a parabolic equation. The appropriate ρ-stable two-layer difference schemes were constructed. Here analogous results have been obtained for three-layer difference schemes as applicable to unstable problems of evolutionary second order equations.

2. Incorrect Problems for Evolutionary Second Order Equations

2.1. *The Cauchy problem for elliptic equations*

We considered the Cauchy problem for elliptic equations as an important class of unstable problem of extending the solution of a well-posed boundary value problem into the adjacent region. These problems are very important in the applications such as geophysical prospecting. Extension of the solution of the elliptic boundary value problem may be easily reduced to the Cauchy problem. The reduction of the Cauchy problem to that of the extension[25] may also prove useful.

We consider first the Cauchy problem for an elliptic second order equation. Let us note by Ω a restricted region of the m-dimensional space $\mathbb{R}^m$ with the smooth enough boundary $\partial\Omega$. In $\mathbb{R}^m \times \{-\infty < t < \infty\}$ we considered the limited cylinder

$$Q(0,T) = \{(x,t) | x \in \Omega, \ 0 < t < T\}, \qquad T > 0,$$

where $x = (x_1, x_2, \ldots, x_m)$, with the side surface

$$\Gamma(0,T) = \{(x,t) | x \in \partial\Omega, \ 0 < t < T\}.$$

For $x \in \Omega$ we determine the uniformly elliptic self-adjoint operator

$$Lu \equiv - \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) \tag{2.1}$$

with sufficiently smooth coefficients $a_{ij}(x) = a_{ji}(x)$, $x \in \Omega$.

We considered the ill-posed Cauchy problem for the elliptic second order equations. Let $u(x,t)$ satisfy the equation

$$\frac{\partial^2 u}{\partial t^2} - Lu = 0, \quad (x,t) \in Q(0,T). \tag{2.2}$$

For simplicity, we restrict our consideration with the boundary conditions

$$u(x,t) = 0, \quad (x,t) \in \Gamma(0,T). \tag{2.3}$$

Two initial conditions are given for $t = 0$. Let

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{2.4}$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \quad x \in \Omega. \tag{2.5}$$

The formulated problem (2.1)–(2.5) is the first example of ill-posed evolutionary problems.

2.2. *The extension problem*

As the second example of ill-posed problems for elliptic equation (2.2) we consider the extension of the Dirichlet problem solution in the half-band

$$Q(-\infty,\, 0) = \{(x,t) | x \in \Omega,\ -\infty < t < 0\}\,.$$

Let $u(x,t)$ be determined by the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} - Lu = 0, \quad (x,t) \in Q(-\infty,\, 0)\,, \tag{2.6}$$

which is completed by the boundary conditions of the first kind

$$u(x,t) = 0, \quad (x,t) \in \Gamma(-\infty,\, 0)\,, \tag{2.7}$$

$$u(x,\, -\infty) = 0, \quad x \in \Omega\,, \tag{2.8}$$

$$u(x,0) = u_0(x) \quad x \in \Omega\,, \tag{2.9}$$

where $\Gamma(-\infty,\, 0)$ is analogous to $\Gamma(0,T)$, the side surface $Q(-\infty,\, 0)$.

The extension problem is posed in the following way. The solution of the Dirichlet problem (2.6)–(2.9) is extended into the adjacent region $Q(0,T)$, i.e., the region $Q(-\infty,\, T)$ we consider the problem for the equation

$$\frac{\partial^2 u}{\partial t^2} - Lu = 0, \quad (x,t) \in Q(-\infty,\, T) \tag{2.10}$$

with the boundary conditions

$$u(x,t) = 0, \quad (x,t) \in \Gamma(-\infty,\, T)\,. \tag{2.11}$$

The conditions (2.8) and (2.9) formulated above are used in the variable t.

The extension problem (2.8)–(2.11) is reduced to the Cauchy problem if we take into account that by solving the Dirichlet problem (2.6)–(2.9) we find the function

$$\frac{\partial u}{\partial t}(x,0) = \psi(x), \quad x \in \Omega\,. \tag{2.12}$$

After this we may consider the Cauchy problem (2.2)–(2.4), (2.12) in the region $Q(0,T)$.

2.3. *The solution instability*

The Cauchy problem (2.2)–(2.5) and the extension problem (2.8)–(2.11) belong to a class of ill-posed problems of mathematical physics. Like in the retrospective inverse problem for the parabolic equation[20] the incorrectness is due to the instability of the solution towards small perturbations of the initial conditions (the function $\phi(x)$ in (2.4) and (2.9)). The following Hadamard example[9] is well known for the Cauchy problem.

Let $m = 1$ and consider the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < a, \quad 0 < t < T \tag{2.13}$$

with the boundary conditions

$$u(0,t) = 0, \quad u(a,t) = 0\,. \tag{2.14}$$

The initial condition (2.4) is taken in the form

$$u(x,0) = k^{-s}\left(\frac{2}{a}\right)^{1/2} \sin\left(\pi k\,\frac{x}{a}\right), \quad 0 < x < a\,, \tag{2.15}$$

while (2.5) yields

$$\frac{\partial u}{\partial t}(x,0) = 0, \quad 0 < x < a\,. \tag{2.16}$$

When $s > 0$ in the norm $\mathcal{H} = L_2(0,a)$ we have

$$\|u(x,0)\|^2 = \int_0^a u^2(x,0)\,dx = k^{-2s} \to 0$$

as $k \to \infty$, i.e., the initial condition is arbitrary small. The exact solution of the problem (2.13)–(2.16) has the form

$$u(x,t) = k^{-s}\left(\frac{2}{a}\right)^{1/2} \mathrm{ch}\left(\pi\frac{k}{a}t\right) \sin\left(\pi k\frac{x}{a}\right), \qquad 0 < x < a, \quad 0 < t < T\,. \tag{2.17}$$

From the representation (2.17) it follows that

$$\|u(x,t)\| = k^{-s}\mathrm{ch}\left(\pi\frac{k}{a}t\right) \to \infty$$

as $k \to \infty$. Hence, the perturbations in the initial condition, no matter how small they are, infinitely grow when $t > 0$. An analogous example may also be given for extension problem (2.8)–(2.11).

2.4. *Conditional stability*

To solve the unstable problems (2.2)– (2.5) and (2.8)–(2.11) approximately we single out a class of *a priori* restrictions imposed upon the solution which is stable in initial data. We shall consider the restricted solution. For example, for the extension problem (2.8)–(2.11) let

$$\|u(x,t)\| \leqslant M = \text{const}, \quad -\infty < t < T. \tag{2.18}$$

Let us note by $W_k(x)$ and λ_k, $k = 1, 2, \ldots$ the eigenfunction and eigenvalues of the operator L respectively, defined by (2.1) on the set of functions satisfying the boundary conditions (2.3); $0 < \lambda_1 < \lambda_2 < \ldots$ (Ref. 11). Then obtaining the solution of the problem (2.8)–(2.11) we derive the representation

$$u(x,t) = \sum_{k=1}^{\infty} \exp(\lambda_k^{1/2} t)(u_0, W_k) W_k(x). \tag{2.19}$$

For the square norm from (2.19) we obtain

$$\|u(x,t)\|^2 = \sum_{k=1}^{\infty} (u_0, W_k)^{2(1-t/T)} ((u_0, W_k) \exp(\lambda_k^{1/2} T))^{2t/T}. \tag{2.20}$$

By basing on the Gelder inequality[10] from (2.20) we get

$$\|u(x,t)\|^2 \leqslant \left(\sum_{k=1}^{\infty} (u_0, W_k)^2 \right)^{(1-t/T)} \left(\sum_{k=1}^{\infty} (u_0, W_k) \exp(\lambda_k^{1/2} T)^2 \right)^{t/T}.$$

From this inequality we have the estimate

$$\|u(x,t)\| \leqslant \|u(x,0)\|^{1-t/T} \|u(x,T)\|^{t/T}. \tag{2.21}$$

From (2.18) and (2.21) we obtain the continuous dependence in space $L_2(D)$ for the solution $u(x,t)$ of the extension problem for the elliptic equation (2.8)–(2.11) on the initial condition in the class of functions uniformly bounded in $t \in (-\infty, T]$.

2.5. *Differential difference problem*

To solve the unstable problem (2.2)–(2.5) approximately we shall use different methods. we introduce the grid ω_h in the domain Ω. Without loss of generality, we shall assume that the grid ω_h is uniform along each direction. The grid step in x_i is h_i, where $i = 1, 2, \ldots, m$.

We approximate the operator L, defined by (2.1) with involvement of (2.3), by corresponding grid operator $\mathcal{A}$. A specific choice of $\mathcal{A}$ is carried out by basing on the difference scheme theory,[18] the finite elements method.[5] Let us only note the most important properties of the operator $\mathcal{A}$, which must be preserved in passing from the differential to grid operator. On the grid ω_h we determine the space of grid functions $H = L_2(\omega_h)$ with the scalar product

$$(y,z) = \sum_{x \in \omega_h} y(x) z(x) h_1 h_2 \ldots h_m.$$

The operator $\mathcal{A}$ is self-adjoint and positively defined in H.

Now we transfer from the problem (2.2)–(2.5) to the following differential difference problem. We have to find the solution of the equation

$$\frac{d^2 v}{dt^2} - Av = 0\,, \tag{2.22}$$

completed by the initial conditions

$$v(x,0) = u_0(x), \quad x \in \omega_h\,, \tag{2.23}$$

$$\frac{dv}{dt}(x,0) = 0, \quad x \in \omega_h\,. \tag{2.24}$$

For obtaining the difference solution of the problem (2.22)–(2.24) we introduce the grid in time

$$\omega_\tau = \{t | t = n\tau, \quad n = 0, 1, \ldots, N, \quad N\tau = T\}\,,$$

where $\tau > 0$ is the grid step.

3. The ρ-Stability of Three-Layer Difference Schemes

3.1. *The stability of three-layer difference schemes*

The general theory of difference scheme stability[18,19] is based on presenting difference schemes in the canonical form. For the three-layer difference schemes this form is

$$B\,\frac{y_{n+1} - y_{n-1}}{2\tau} + \tau^2 R\,\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} + Ay_n = \phi_n\,,$$
$$n = 1, 2, \ldots \tag{3.1}$$

for given $y_0(x) = u_0(x)$, $y_1(x)$, $x \in \omega_h$. In (3.1) the operators A, B and R generally depend on h_i, $i = 1, 2, \ldots, m$, τ and t. For the test problems under consideration, similar to (2.1)–(2.5), we assume that the difference operators A, B and R are stationary (do not depend on t). By basing on general theory of the difference scheme stability[19] the results were generalized in different aspects (stability towards the right-hand side, the difference schemes with nonself-adjoint operators, nonstationary operators, stability in simpler norms, etc.). Here we shall study only the stability of difference schemes in initial data; therefore in (3.1) $\phi_n = 0$.

For well-posed evolutionary problems an usual condition of stability in space H_D in initial conditions is

$$\|y_{n+1}\|_D \leqslant \|y_n\|_D, \quad \|y\|_D^2 = (Dy, y), \quad D = D^* > 0\,. \tag{3.2}$$

We have to use more complicated norms when investigating the stability of three-layer difference schemes. This question was thoroughly discussed in Ref. 19, where most important norms were considered.

For each $n = 1, 2, \ldots$ we shall determine vector

$$Y^n = \left\{ \frac{1}{2}(y_n + y_{n-1}),\ y_n - y_{n-1} \right\}.$$

A direct sum of the spaces H is designated by H^2: $H^2 = H \oplus H$. For the vectors $Y = \{y^1, y^2\}$ the summation and multiplication in H^2 are carried out coordinate by coordinate, while the scalar product

$$(Y, V) = (y^1, v^1) + (y^2, v^2).$$

For the difference scheme (3.1) with $R = R^*$, $A = A^* > 0$, $4R - A \geqslant 0$ we determine the norm in $H^2_{\mathcal{D}}$ by the expression $\|Y^n\|_{\mathcal{D}} = ((\mathcal{D}Y^n, Y^n))^{1/2}$, where

$$(\mathcal{D}Y^n, Y^n) = \frac{1}{4}\|y_n + y_{n-1}\|^2_A + \|y_n - y_{n-1}\|^2_{R-\frac{1}{4}A}, \tag{3.3}$$

while the norm in H_A is determined according to (3.2). Our study is based on the general results of the stability theory for three-layer difference schemes.[18,19]

Theorem 0 (basic).[17] Let the operators A and R in (3.1) be stationary (independent of n), self-adjoint and positive ($A = A^* > 0$, $R = R^* > 0$). Then the conditions

$$B_0 = \frac{1}{2}(B + B^*) \geqslant 0 \tag{3.4}$$

$$R - \frac{1}{4}A > 0 \tag{3.5}$$

are necessary and sufficient for the stability of the scheme (3.1) in $H^2_{\mathcal{D}}$, i.e., the bound $\|Y^{n+1}\|_{\mathcal{D}} \leqslant \|Y^n\|_{\mathcal{D}}$ is fulfilled.

This result is final and cannot be improved since we deal with the necessary and sufficient conditions.

3.2. *The ρ-stability*

In the difference schemes for ill-posed problems like (2.2)–(2.5) the stability condition $\|Y^{n+1}\|_{\mathcal{D}} \leqslant \|Y^n\|_{\mathcal{D}}$ does not fit and must be replaced by the ρ-stability condition. It is due to the fact that the solution (its norm) to the inverse problem grows (see representation of (2.19)). On the grid level it shows up in the differential difference problem (2.22)–(2.24) where the operator $\mathcal{A}$ is positively defined.

Allowing a growth of the solution to (2.22)–(2.24) we shall use the ρ-stability difference schemes. The three-layer difference scheme (3.1) is called ρ-stable[18] if

$$\|Y^{n+1}\|_{\mathcal{D}} \leqslant \rho\,\|Y^n\|_{\mathcal{D}}, \tag{3.6}$$

where $\rho > 0$ is any number. Allowing a limited growth of the solution we may put

$$\rho = \exp(c\tau),$$

or

$$\rho = 1 + c\tau ,$$

where the positive constant c does not depend on the grid (on τ and h). For this ρ the stability estimate of the difference solution in the initial data may be derived from (3.6) in the form

$$\|Y^{n+1}\|_{\mathcal{D}} \leqslant \exp(ct_{n+1}) \|Y^0\|_{\mathcal{D}} . \tag{3.7}$$

General conditions for the ρ-stability of three-layer difference schemes were obtained in Ref. 17 and discussed in detail in Ref. 19.

A specific nature of the difference schemes for ill-posed problems is seen in the stability condition (3.6) where $\rho > 1$.

Theorem 1. Let the operators A, R and B in (3.1) be stationary (independent of n) and self-adjoint. Then if

$$\frac{\rho^2 - 1}{2\tau} B + (\rho^2 + 1) R > 0 \tag{3.8}$$

for the ρ-stability in $H^2_{\mathcal{D}}$ with $\rho > 0$ it is necessary and sufficient to satisfy the conditions

$$\frac{\rho^2 - 1}{2\tau} B + (\rho - 1)^2 R + \rho A \geqslant 0 , \tag{3.9}$$

$$\frac{\rho^2 - 1}{2\tau} B + (\rho + 1)^2 R - \rho A \geqslant 0 , \tag{3.10}$$

$$\frac{\rho^2 + 1}{2\tau} B + (\rho^2 - 1) R \geqslant 0 . \tag{3.11}$$

In this case the operator of the norm $\mathcal{D}$ is given by the expression

$$(\mathcal{D}Y^n, Y^n) = \frac{1}{4} \left\| \frac{1}{\rho} y_n + y_{n-1} \right\|^2_{\tilde{A}} + \left\| \frac{1}{\rho} y_n - y_{n-1} \right\|^2_{\tilde{R} - \frac{1}{4}\tilde{A}} ,$$

where

$$\tilde{A} = \frac{\rho^2 - 1}{2\tau} B + (\rho - 1)^2 R + \rho A ,$$

$$4\tilde{R} - \tilde{A} = \frac{\rho^2 - 1}{2\tau} B + (\rho + 1)^2 - \rho A .$$

We note that for the ρ-stability of the three-layer scheme (3.1) with $\rho > 1$ the condition $A > 0$ is not necessary (in the case of (2.22)–(2.24) we have $A < 0$).

4. Regularization of Three-Layer Difference Schemes

4.1. *Regularization of the explicit scheme*

We shall apply the regularization principle[16] first to well-posed evolutionary problems for the second order equations. Discussing the results for well-posed problems in parallel with those for ill-posed problems allows a more profound understanding of difference methods for unstable evolutionary problems. On the other hand, such a parallel consideration demonstrates power and generality of the unified mathematical tool used in the stability theory. Moreover, the new difference schemes proposed may prove to be of individual interest for well-posed problems.

We consider the differential difference problem (see (2.22)–(2.24)):

$$\frac{d^2v}{dt^2} + \mathcal{A}v = 0\,, \tag{4.1}$$

$$v(x,0) = u_0(x)\,, \tag{4.2}$$

$$\frac{dv}{dt}(x,0) = 0\,, \quad x \in \omega_h\,. \tag{4.3}$$

For the problem (4.1)–(4.3) we write down the usual explicit symmetrical difference scheme

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} + \mathcal{A}y_N = 0, \quad n = 0,\, 1,\, \ldots\,. \tag{4.4}$$

This scheme (see, for example, Refs. 18 and 19) is stable for sufficiently small steps in time. Really, the scheme (4.4) may be written in the canonical form (3.1) with

$$B = 0, \quad R = \frac{1}{\tau^2}E, \quad A = \mathcal{A}\,. \tag{4.5}$$

By owing to $\mathcal{A} \leqslant \|\mathcal{A}\|E$ the condition (3.5) yields

$$E - \frac{\tau^2}{4}\mathcal{A} \geqslant \left(\|\mathcal{A}\|^{-1} - \frac{\tau^2}{4}\right)\mathcal{A} \geqslant 0\,.$$

This inequality is satisfied if $\tau^2 \leqslant 4\|\mathcal{A}\|^{-1}$, i.e., when $\tau \leqslant \tau_0 = O(h)$, which is the Courant conditions.

By using the regularization principle we shall correct the scheme (4.4) and construct absolutely stable difference scheme.

Now we shall note the regularization grid operator by $\mathcal{R} = \mathcal{R}^* > 0$ and let $\alpha > 0$ be the regularization parameter (the perturbation parameter). By analogy with Ref. 16 the regularization scheme for (4.4) is written in the form

$$(E + \alpha R)\,\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} + \mathcal{A}\,y_n = 0\,. \tag{4.6}$$

We use two choices of the regularizator: $\mathcal{R} = \mathcal{A}$ and $\mathcal{R} = \mathcal{A}^2$. Let's prove the following statement.

Theorem 2. The difference scheme (4.6) with $\mathcal{A} = \mathcal{A}^* > 0$ is absolutely stable for the choice of the regularizator $\mathcal{R} = \mathcal{A}$ if

$$\alpha \geqslant \tau^2/4 - \|\mathcal{A}\|^{-1}\,, \tag{4.7}$$

and for the choice $\mathcal{R} = \mathcal{A}^2$ if $\alpha \geqslant \tau^4/64$.

The proof is based on checking the necessary and sufficient conditions (3.4) and (3.5). In our case the inequality (3.4) is always satisfied since for the scheme (4.6) we have $B = 0$. To check the condition (3.5) we should take into account that

$$R = \tau^{-2}(E + \alpha\mathcal{R}), \quad A = \mathcal{A}. \tag{4.8}$$

Proceeding from (3.5) we obtain

$$(\|\mathcal{A}\|^{-1} + \alpha)\mathcal{A} \geqslant \frac{\tau^2}{4}\mathcal{A}.$$

If the regularization parameter α is chosen in accordance with (4.7) this inequality will be satisfied.

For the regularizator $\mathcal{R} = \mathcal{A}^2$ and (2.8) the inequality (3.5) is transformed in the following form:

$$\begin{aligned}\tau^2(R - 1/4A) &= E + \alpha\mathcal{A}^2 - \tau^2/4\,\mathcal{A} \\ &= (\alpha^{1/2}\mathcal{A} - \tau^2/(8\alpha^{1/2})E)^2 + (1 - \tau^4/(64\alpha))E \geqslant 0\,.\end{aligned}$$

From this it follows that when $\mathcal{R} = \mathcal{A}^2$ and $\alpha \geqslant \tau^4/64$ the regularization difference scheme (4.6) is absolutely stable.

For the scheme (4.6) the choice $\alpha = \sigma\tau^2$, $\sigma_1 = \sigma_2 = \sigma$ at $\mathcal{R} = \mathcal{A}$ corresponds to the usual scheme[18,19] with the weights

$$\begin{aligned}&\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} \\ &\qquad + \mathcal{A}(\sigma_1 y_{n+1} + (1 - \sigma_1 - \sigma_2)y_n + \sigma_2 y_{n-1}) = 0\,.\end{aligned}$$

An analysis of such regularization schemes was made in Ref. 16. It is worth to consider separately the regularization scheme (3.6), (3.8) with the regularizator $\mathcal{R} = \mathcal{A}^2$.

4.2. *Factorization schemes*

By analogy with the above case for the parabolic equation[20] we may construct efficient difference schemes by solving approximately the multidimensional problems (2.2)–(2.5). We shall consider only some simplest results obtained in this field. The difference schemes of total approximation, the split schemes are discussed in detail from different point of view in Refs. 15, 18 and 19. Here we show the possibility of constructing the factorization difference schemes by basing the regularization principle.

We consider the case when the grid operator $\mathcal{A}$ may be presented as a a sum of the commutative operators

$$\begin{aligned}&\mathcal{A} = \sum_{k=1}^{p} \mathcal{A}_k, \quad \mathcal{A}_k = \mathcal{A}_k^* > 0\,, \\ &\mathcal{A}_k\mathcal{A}_s = \mathcal{A}_s\mathcal{A}_k, \quad k, s = 1, 2, \ldots, p\,.\end{aligned} \tag{4.9}$$

Theorem 3. The factorization difference scheme (3.1) with

$$B = 0, \quad R = \prod_{k=1}^{p} (E + \alpha \mathcal{R}_k), \quad A = \mathcal{A}, \tag{4.10}$$

where $\mathcal{A}$ satisfies the conditions (4.9) is stable at $\mathcal{R}_k = \mathcal{A}_k$ if $\alpha \geqslant \tau^2/4 - \|\mathcal{A}\|^{-1}$, while at $\mathcal{R}_k = \mathcal{A}_k^2$ it is stable if $\alpha \geqslant p\tau^4/64$.

Choosing the operator R in accordance with (4.9) and (4.10) we have for $\mathcal{R}_k = \mathcal{A}_k$ that

$$\prod_{k=1}^{p} (E + \alpha \mathcal{A}_k) \geqslant E + \alpha \mathcal{A} .$$

Hence, similar to Theorem 2, the factorization scheme (3.1), (4.9) and (4.10) is stable if (4.7) is satisfied.

In the case $\mathcal{R}_k = \mathcal{A}_k^2$ the theorem is proved by using the inequality

$$\prod_{k=1}^{p} (E + \alpha \mathcal{A}_k^2) \geqslant E + \alpha \sum_{k=1}^{p} \mathcal{A}_k^2 \geqslant E + \frac{\alpha}{p} \left(\sum_{k=1}^{p} \mathcal{A}_k \right)^2$$
$$= E + \frac{\alpha}{p} \mathcal{A}^2 . \tag{4.11}$$

By involving Theorem 2 we obtain that for $\alpha \geqslant p\tau^2/64$ the factorization scheme (3.1), (3.9) and (4.10) is stable.

Based on the regularization principle we may also obtain other efficient schemes for solving the differential difference problem (4.1)–(4.3). In particular (see Refs. 15, 18 and 19), it concerns the case when the operator is split into two ($p = 2$).

5. Regularization of Three-Layer Difference Schemes for Unstable Problems

5.1. *Regularization of the explicit scheme for unstable problems*

We shall turn now to ill-posed problems with initial date for the elliptic second order equation. Let $u(x, t)$ be defined from Eq. (2.2) completed by the conditions (2.3)–(2.5). We see our task in constructing the ρ-stable difference scheme by basing on the regularization principle and in extending the results obtained for well-posed problems for evolutionary second order equations onto ill-posed problems.

We first consider the explicit symmetric difference scheme

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} - \mathcal{A} y_n = 0, \quad n = 0, 1, \dots . \tag{5.1}$$

The scheme (5.1) may be written in the canonical form (3.1) with

$$B = 0, \quad R = \frac{1}{\tau^2} E, \quad A = -\mathcal{A}, \tag{5.2}$$

i.e., $A = A^* < 0$.

Theorem 4. The explicit scheme (5.1) is ρ-stable with

$$\rho = \exp\left(\|\mathcal{A}\|^{1/2}\tau\right). \tag{5.3}$$

The proof is based on checking the ρ-stability conditions for the scheme (5.1). For the three-layer difference scheme (2.1) they have the form (3.8)–(3.11). When $B \geqslant 0$, $A \leqslant 0$, $R \geqslant 0$ and $\rho > 1$ the inequalities (3.8), (3.10) and (3.11) for all $\tau > 0$ are obviously satisfied. The inequality (3.9) with involvement of (5.2) may be transform into

$$(\rho-1)^2 E - \tau^2\rho\mathcal{A} \geqslant \left((\rho-1)^2\|\mathcal{A}\|^{-1} - \tau^2\rho\right)\mathcal{A} \geqslant 0.$$

Let us prove the following auxiliary results.

Lemma. The inequality

$$(\rho-1)^2\chi - \tau^2\rho \geqslant 0$$

for positive χ, τ and $\rho > 1$ is satisfied when

$$\rho \geqslant \exp(\chi^{-1/2}\tau).$$

This inequality will be fulfilled for $\rho \geqslant \rho_2$, where

$$\rho_2 = 1 + \frac{1}{2}\tau^2\chi^{-1} + \tau(\chi^{-1})^{1/2}\left(1 + \frac{1}{4}\tau^2\chi^{-1}\right)^{1/2}.$$

By virtue of

$$\left(1 + \frac{1}{4}\tau^2\chi^{-1}\right)^{1/2} < 1 + \frac{1}{8}\tau^2\chi^{-1},$$

$$\begin{aligned} \rho_2 &< 1 + \tau(\chi^{-1})^{1/2} + \tau^2\frac{1}{2}\chi^{-1} + \tau^3\frac{1}{8}\chi^{-3/2} \\ &< 1 + \tau\chi^{-1/2} + \frac{1}{2}(\tau\chi^{-1/2})^2 + \frac{1}{6}(\tau\chi^{-1/2})^3 \\ &< \exp(\chi^{-1/2}\tau). \end{aligned}$$

Thus the lemma has been proved.

In our case $\chi = \|\mathcal{A}\|^{-1}$ and therefore for ρ we obtain the estimate (5.3) for the explicit scheme (5.1).

Taking into account the restrictness of operator $\mathcal{A}$ (we have $\|A\| = O(h^{-2})$,[18,19] we may suggest that the grid step in space restricts the solution growth, i.e, acts as the regularization parameter). Passage from the continuous problem to the discrete one may, in principle, be considered as a method of combating the instability and the resulting regularization algorithm is a possible approach to obtaining the approximate solution of unstable evolutionary problem.

Introducing explicitly the regularization additions to grid operators of the difference scheme offers great potentialities. By analogy with the well-posed problem (see (4.8)), we write down the regularization scheme in the canonical form (3.1) with

$$B = 0,\ R = \frac{1}{\tau^2}(E + \alpha\mathcal{R}), \quad A = -\mathcal{A}. \tag{5.4}$$

Theorem 5. The regularization scheme (3.1) and (5.4) is stable at $\mathcal{R} = \mathcal{A}$ with

$$\rho = \exp(\alpha^{-1/2}\tau)\,, \tag{5.5}$$

while at $\mathcal{R} = \mathcal{A}^2$ with

$$\rho = \exp(2^{-1/2}\alpha^{-1/4}\tau)\,. \tag{5.6}$$

Again the proof is based on checking the satisfaction conditions for the inequality (3.9) which for (5.4) takes the form

$$(\rho - 1)^2(E + \alpha\mathcal{R}) - \tau^2\rho\mathcal{A} \geqslant 0\,. \tag{5.7}$$

At $\mathcal{R} = \mathcal{A}$, by analogy with Theorem 4 ($\chi = \alpha + \|\mathcal{A}\|^{-1}$), we obtain

$$\rho = \exp\left((\alpha + \|\mathcal{A}\|^{-1})^{-1/2}\tau\right).$$

By making ρ cruder we obtain the estimate (5.5).

At $\mathcal{R} = \mathcal{A}^2$ the inequality (5.7) may be transformed into:

$$E + \alpha\mathcal{A}^2 - \frac{\tau^2\rho}{(\rho-1)^2}\mathcal{A} = \left(\alpha^{1/2}\mathcal{A} - \frac{\tau^2\rho}{2(\rho-1)^2}\alpha^{-1/2}E\right)^2 + \left(1 - \frac{\tau^4\rho^2}{4(\rho-1)^4}\alpha^{-1}\right)E \geqslant 0\,.$$

The above inequality will be satisfied for given ρ if the regularization parameter

$$\alpha \geqslant \frac{\tau^4\rho^2}{4(\rho-1)^4}\,. \tag{5.8}$$

Now let us estimate ρ for a given α from the inequality (5.8). The latter may be rewritten as

$$(\rho - 1)^2 2\alpha^{1/2} - \tau^2\rho \geqslant 0\,.$$

Due to the lemma ($\chi = 2\alpha^{1/2}$) this inequality is satisfied for ρ defined according to (5.6).

The two possible choices of the regularizator, given above, allow a certain set of regularization algorithms along with the explicit regularization at the expense of appropriately chosen space grid. Selection of an algorithm out of this set is made proceeding from additional considerations involving specific features of an applied problem and available computational resources.

5.2. *Factorization schemes*

By analogy with the direct problem we consider regularized factorization schemes. We assign the operators of the difference scheme (3.1) in the form

$$B = 0, \quad R = \prod_{k=1}^{p} (E + \alpha \mathcal{R}_k), \quad A = -\mathcal{A}, \tag{5.9}$$

Theorem 6. The factorization scheme (3.1), (4.9) and (5.9) is ρ-stable at $\mathcal{R}_k = \mathcal{A}_k$ with

$$\rho = \exp(\alpha^{-1/2}\tau),$$

while at $\mathcal{R}_k = \mathcal{A}_k^2$ with

$$\rho = \exp\left(p^{1/4} 2^{-1/2} \alpha^{-1/4} \tau\right).$$

The proof is quite analogous to that of Theorem 5 with involvement (4.11). It is only necessary to note that again we restricted our consideration with simplest factorization scheme with self-adjoint and commutative operators $\mathcal{A}_k$, $k = 1, 2, \dots, p$.

5.3. *Perturbation of other operators*

The regularization theory of difference schemes is based on the perturbation of the grid operators A, B and R. The regularization effect may be achieved not only by means of perturbation upon the grid operator R. For example, instead of the explicit scheme (5.1), we consider the regularization scheme where the operator A is perturbed. Let

$$\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} - \mathcal{A}y_n + \alpha\mathcal{R}y_n = 0 \tag{5.10}$$

and $\mathcal{R} = \mathcal{A}^2$.

Theorem 7. The regularization scheme (5.10) is ρ-stable at $\mathcal{R} = \mathcal{A}^2$ with

$$\rho = \exp(2^{-1}\alpha^{-1/2}\tau), \tag{5.11}$$

for any τ if $\alpha\|\mathcal{A}\| \leqslant 1$, and for $\tau \leqslant \left(2/(\|\mathcal{A}\|(\alpha\|\mathcal{A}\| - 1))\right)^{1/2}$ if $\alpha\|\mathcal{A}\| > 1$.

In the explicit scheme (5.10) the restriction on the time step follows from the necessity of satisfaction of the inequality (3.10). For the scheme (5.10) we have

$$B = 0, \; R = \frac{1}{\tau^2}E, \; A = -\mathcal{A} + \alpha\mathcal{A}^2. \tag{5.12}$$

The estimate (5.11) for ρ we obtain from the inequality (3.9). By involving (5.12) we obtain from (3.9)

$$\frac{(\rho-1)^2}{\tau^2} E - \rho\mathcal{A} + \rho\alpha\mathcal{A}^2$$
$$= \rho(\alpha^{1/2}\mathcal{A} - 2^{-1}\alpha^{-1/2}E)^2 + \left(\frac{(\rho-1)^2}{\tau^2} - \frac{\rho}{4\alpha}\right)E \geqslant 0 .$$

This inequality is satisfied for

$$(\rho-1)^2 4\alpha - \tau^2\rho \geqslant 0 .$$

Hence, by virtue of the lemma we obtain the estimate (5.11) for ρ.

Taking into account (3.9) we have

$$\frac{(\rho+1)^2}{\tau^2} E = \frac{(\rho-1)^2}{\tau^2} E + \frac{4}{\tau^2}\rho E$$
$$\geqslant \frac{4}{\tau^2}\rho E + \rho\mathcal{A} - \rho\alpha\mathcal{A}^2 .$$

With involvement of this inequality and inequalities $E \geqslant \|\mathcal{A}\|^{-1}\mathcal{A}$, $\mathcal{A}^2 \leqslant \|\mathcal{A}\|\,\mathcal{A}$ the relation (3.10) may be transformed into

$$\frac{(\rho+1)^2}{\tau^2} E + \rho\mathcal{A} - \rho\alpha\mathcal{A}^2$$
$$\geqslant 2\rho\left(\frac{2}{\tau^2} E + \mathcal{A} - \alpha\mathcal{A}^2\right)$$
$$\geqslant 2\rho\left(\frac{2}{\tau^2}\|\mathcal{A}\|^{-1} + 1 - \alpha\|\mathcal{A}\|\right)\mathcal{A} \geqslant 0 .$$

The last inequality is satisfied for all τ if $\alpha\|\mathcal{A}\| \leqslant 1$. However, if $\alpha\|\mathcal{A}\| > 1$ all the above restrictions upon the time step will take place.

Let us present the results about the absolute ρ-stability of the combined regularized difference scheme

$$(E + \alpha_1\mathcal{R}_1)\,\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} - \mathcal{A}y_n + \alpha_2\mathcal{R}_2 y_n = 0 ,$$
$$n = 1, 2, \dots , \qquad (5.13)$$

which was obtained by perturbation of operator R and operator A. We restrict our consideration with a particular case of fixed constrain between perturbation parameters α_1 and α_2.

Theorem 8. The regularization scheme (5.13) at $\alpha_1 = \alpha_2\tau^2/4$ and $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{A}^2$ ρ-stable for any $\tau > 0$ with

$$\rho = \exp(2^{-1}\alpha_2^{-1/2}\tau) . \qquad (5.14)$$

In the case of the scheme (5.13) at $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{A}^2$ we have

$$B = 0, \; R = \frac{1}{\tau^2}\,(E + \alpha_1 \mathcal{A}^2), \quad A = -\mathcal{A} + \alpha_2 \mathcal{A}^2 . \tag{5.15}$$

We first check the satisfaction of inequality (3.10) for any ρ and $\alpha_1 = \alpha_2 \tau^2/4$. With involvement of (5.15) and the positively of operator A we immediately obtain

$$\frac{(\rho+1)^2}{\tau^2}\,(E + \alpha_1 \mathcal{A}^2) + \rho \mathcal{A} - \rho \alpha_2 \mathcal{A}^2$$
$$> \rho \left(\frac{(\rho^{1/2} + \rho^{-1/2})^2}{\tau^2}\,\alpha_1 - \alpha_2 \right) \mathcal{A}^2 \geqslant 0 .$$

This inequality is fulfilled for the chosen α_1 and α_2.

To estimate ρ we substitute (5.15) into (3.9)

$$\frac{(\rho-1)^2}{\tau^2}\,(E + \alpha_1 \mathcal{A}^2) - \rho \mathcal{A} + \rho \alpha_2 \mathcal{A}^2$$
$$= \alpha_2\,\frac{(\rho+1)^2}{4}\,\mathcal{A}^2 - \rho \mathcal{A} + \frac{(\rho-1)^2}{\tau^2}\,E$$
$$= \left(\alpha_2^{1/2}\,\frac{\rho+1}{2}\,\mathcal{A} - \alpha_2^{-1/2}\,\frac{\rho}{\rho+1}\,E \right)^2$$
$$+ \left(\frac{(\rho-1)^2}{\tau^2} - \frac{\rho^2}{\alpha_2(\rho+1)^2} \right) E \geqslant 0 .$$

This inequality is satisfied for

$$\frac{\rho - 1}{\tau} \geqslant \alpha_2^{-1/2}\,\frac{\rho}{\rho+1} . \tag{5.16}$$

This inequality (4.16) allows us to obtain

$$\rho \geqslant 1 + \frac{1}{2}\,\alpha_2^{-1/2} \tau .$$

Hence, we obtain the estimate (5.14) for ρ.

We note also that it is possible to construct regularization difference schemes by using the perturbation of operator B in the difference scheme (3.1) when solving the Ill-posed problem by means of the explicit scheme (5.1). The corresponding difference scheme may be written in the canonical form (3.1) with the operators

$$B = \alpha \mathcal{R}, \; R = \frac{1}{\tau^2}\,E, \quad A = -\mathcal{A} . \tag{5.17}$$

For example, we shall prove the following statement.

Theorem 9. The difference scheme (5.17) is ρ-stable at $\mathcal{R} = \mathcal{A}$ with

$$\rho = \exp(\alpha^{-1} \tau) . \tag{5.18}$$

Indeed, with such a choice of operators A, B and R the inequalities (3.8), (3.10) and (3.11) are obviously satisfied, while the inequality (3.9) with involvement of (5.17) is transformed into

$$\begin{aligned}\frac{\rho^2-1}{2\tau}B + (\rho-1)^2 R + \rho A &\\ &= \frac{\rho^2-1}{2\tau}\alpha A + (\rho-1)^2\frac{E}{\tau^2} - \rho A\\ &> \left(\frac{\rho^2-1}{2\tau}\alpha - \rho\right)A \geqslant 0\,.\end{aligned}$$

The last inequality is satisfied for $\rho \geqslant \rho_2$, where

$$\begin{aligned}\rho_2 &= \frac{\tau}{\alpha} + \left(1+\frac{\tau^2}{\alpha^2}\right)^{1/2}\\ &< 1 + \frac{\tau}{\alpha} + \frac{1}{2}\frac{\tau^2}{\alpha^2} < \exp(\alpha^{-1}\tau)\,.\end{aligned}$$

We obtain from this the estimate (5.18) for ρ.

Theorems 7–9 illustrate the possibilities for obtaining the ρ-stable difference schemes under the perturbation of different grid operators in canonical form (3.1).

6. Three-Layer Regularization Schemes and the Quasi-Inversion Method

6.1. *Basic variant of the quasi-inversion method*

The regularization difference schemes given above are obtained on the basis of the regularization principle which is formulated irrespective of a specific continuous problem under consideration. Regularization of difference schemes is based on perturbation of grid operators. If ill-posed problems are solved approximately the regularization principle of difference schemes may be considered as a method of quasi-inversion for a discrete problem. The perturbation of difference scheme operators on a continuous level corresponds to a perturbation of the original differential equation. Therefore, such regularization difference schemes for ill-posed Cauchy problems may sometimes be interpreted as difference schemes of the quasi-inversion method.

For each variant of regularization of unstable schemes we write down a corresponding variant of quasi-inversion method. In this context the regularization schemes may be considered as difference schemes of quasi-inversion method.

For the Cauchy problem (2.1)–(2.5) we shall use the quasi-inversion method variant based on obtaining the solution of the equation

$$\frac{\partial^2 u_\alpha}{\partial t^2} - Lu_\alpha + \alpha L^2 u_\alpha = 0, \quad (x,t) \in Q(0,T) \tag{6.1}$$

to determine $u_\alpha(x,t)$. This variant corresponds to the quasi-inversion method modification for a retrospective inverse problem for the parabolic equation considered in the book.[12]

For the solution of the problem (6.1), (2.3)–(2.5) we have the estimate

$$\|u_\alpha(x,t)\| \leqslant \exp\left(\frac{1}{2}\alpha^{-1/2}t\right)\|u_\alpha(x,0)\|, \tag{6.2}$$

which provides the stability of the solution in initial data.

To solve the problem (6.1), (2.4) and (2.5) numerically we use the difference scheme with the weights

$$\begin{aligned} &\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} \\ &\quad - \mathcal{A}(\sigma_1 y_{n+1} + (1-\sigma_1-\sigma_2)y_n + \sigma_2 y_{n-1}) \\ &\quad + \alpha\mathcal{A}^2(\sigma_3 y_{n+1} + (1-\sigma_3-\sigma_4)y_n + \sigma_4 y_{n-1}) = 0. \end{aligned} \tag{6.3}$$

The scheme (6.3) coincides with the regularization scheme (5.13) at $\sigma_1 = \sigma_2 = 0$, $\sigma_3 = \sigma_4$, $\alpha_1 = \sigma_3\tau^2\alpha$, $\alpha_2 = \alpha$ and $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{A}^2$. The value ρ (see (5.14)) exactly agrees with an estimate of the solution stability for the continuous problem (6.2).

6.2. *Other variants of the quasi-inversion method*

Another variant of the quasi-inversion method[24] is based on obtaining the solution of the equation

$$\frac{\partial^2 u_\alpha}{\partial t^2} - Lu_\alpha + \alpha L\frac{\partial^2 u_\alpha}{\partial t^2} = 0, \quad (x,t) \in Q(0,T). \tag{6.4}$$

The corresponding estimate of stability for the problem (6.4), (2.4) and (2.5) has the form

$$\|u_\alpha(x,t)\| \leqslant \exp(\alpha^{-1/2}t)\|u_\alpha(x,0)\|. \tag{6.5}$$

The three-layer difference scheme with weights for Eq. (6.4) has the form

$$\begin{aligned} &(E + \alpha\mathcal{A})\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau^2} \\ &\quad - \mathcal{A}(\sigma_1 y_{n+1} + (1-\sigma_1-\sigma_2)y_n + \sigma_2 y_{n-1}) = 0. \end{aligned} \tag{6.6}$$

The scheme (6.6) coincides with regularization scheme (3.1) and (5.4) at $\sigma_1 = \sigma_2 = 0$ and the choice $\mathcal{R} = \mathcal{A}$, while the value ρ (see (5.5)) agrees well with the estimate (6.5).

The variant of quasi-inversion method, which corresponds to the regularization scheme (3.1) and (5.4) at the choice $\mathcal{R} = \mathcal{A}^2$ is based on solving the equation

$$\frac{\partial^2 u_\alpha}{\partial t^2} - Lu_\alpha + \alpha L^2\frac{\partial^2 u_\alpha}{\partial t^2} = 0, \quad (x,t) \in Q(0,T). \tag{6.7}$$

For the solution of problem (6.7), (2.4) and (2.5) we have the estimate

$$\|u_\alpha(x,t)\| \leqslant \exp(2^{-1/2}\alpha^{-1/4}t)\|u_\alpha(x,0)\|.$$

A similar estimate is obtained for the solution of the difference problem (see (5.6)).

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