A NON-LINEAR BOUNDARY VALUE PROBLEM OF IGNITION BY RADIATION

F.V. BUNKIN, V.A. GALAKTIONOV, N.A. KIRICHENKO, S.P. KURDYUMOV and A.A. SAMARSKII

Existence and non-existence conditions are obtained for stationary solutions in the model problem of laser macrokinetics which describes the heating of a semi-infinite sample by a spatially inhomogeneous beam of radiation with a power approximation of the energy liberation at the surface of the sample. The modes of localization and peaking of the heat field in the substance at the high-temperature stage of heating are examined.

1. Introduction.

Advances in high-temperature physics and macrokinetics have led to the formulation and detailed study of a wide class of non-linear boundary value problems for the heat conduction equation. Problems of the heating of a substance by radiation, which is the subject matter of the interesting new field of laser macrokinetics /1, 2/, occupy an important place. Many theoretical and experimental studies of combustion under the action of radiation have now been published, see e.g. /3/. It has been found that size effects are typical for problems of combustion physics. For instance, to initiate the combustion of a material occupying a large volume, the source must have a sufficiently high temperature and large size /4/. In problems of laser heating the inhomogeneity of the radiation field is important and is due to the finite size of the actual beam and the variable intensity distribution across it. In this connection the question arises of the kinetic parameters of the radiation (beam radius and power), for which ignition of the substance becomes possible. It is interesting to study these questions both for purposes of laser technology and for the further development of the theory of non-linear dynamic systems with inhomogeneous parameters.

The non-linear boundary value problem

\[ \frac{\partial T}{\partial t} = \Delta T + \frac{I}{r} \left( \frac{\partial T}{\partial r} \right) + g(T), \quad r > 0, \quad z > 0, \]

\[ - \frac{\partial T}{\partial r} \big|_{r=0} = I(r) + g(T), \quad \theta \Omega = \{r > 0, \quad z = 0\}, \]

\[ I(r) = I_0 \exp \left( -\frac{r_0}{r} \right), \]

\[ T(0, z) = 0, \quad z \in \Omega. \]

where \( g(T) = \exp(-T/T) \), was studied in detail in /5-8/. This is the simple mathematical model of the surface heating by radiation of a large metal sample on whose surface an exothermic reaction (oxidation) occurs. Here, \( T \) is the temperature and \( I \) and \( r_0 \) are the beam intensity and effective radius. The term \( g(T) \) describes the energy liberation of the reaction. It was shown in /5-8/ that, in the plane of parameter values \( \{I, r_0\} \) there is a monotonic stability boundary \( L_0 = h(r_0) \) such that, for all \( L > h(r_0) \), the stationary boundary value problem (1), (2) is solvable, while when \( L < h(r_0) \) there are no stationary solutions. This means that, for sufficiently low initial material temperatures, no ignition occurs if \( L < h(r_0) \), while if \( L > h(r_0) \), for any initial conditions, the exothermic reaction covers the entire surface.

Sometimes, however, non-linearities other than \( g(T) = \exp(-T/T) \) need to be considered, notably because the actual chemical reaction has, in general, several stages, the different mechanisms "switch on" at different temperatures, and there are various heat loss mechanisms (e.g., radiation \(~T^4\) and convective \(~T\)), etc. Over fairly wide temperature ranges that are of practical importance, the non-linearity \( g(T) \) in the boundary condition can often be approximated by the power law

\[ g(T) = T^\alpha, \quad \alpha = \text{const} > 0. \]

by studying such non-linearities we can draw conclusions about the corresponding systems, e.g., on the basis of comparison theorems.

The present paper deals with the conditions for the existence of stationary solutions and the ways that non-stationary solutions evolve for problem (1)-(3) with a power non-linearity in the boundary condition.

2. On the conditions for the non-existence of stationary solutions.

In the stationary case problem (1)-(3) is equivalent to the non-linear integral equation

\[ T(\tau) = \int_{r_0}^{\infty} \exp\left( -\frac{r_0}{r} \right) T(r) \, dr, \]

**Note:** The text contains mathematical equations and symbols. The context suggests a detailed discussion of non-linear boundary value problems, particularly concerning ignition by radiation and the conditions for the existence or non-existence of stationary solutions. This involves a detailed examination of heat transfer, radiation effects, and the approximation of energy liberation through power laws. The discussion extends to practical implications and theoretical analyses within the realm of macrokinetics and high-temperature physics.
$T(r, z) = S \left( I_0 \exp \left( -\frac{r^2}{r_0^2} \right) \right)(r, z) + S(T_0(r, z))(r, z), \quad (4)$

where $S$ is an operator:

$S(\xi(\xi))(r, z) = \int_{\mathbb{R}_+} \xi(\xi) d\xi \frac{1}{2\pi} \int_0^{2\pi} (r^2 + z^2 - 2r \xi \cos \varphi)^{1/2} d\varphi.$

We put $\xi = 0$ in Eq. (4) and introduce the notation $T(r, 0) = T(r)$. Then,

$T(r) = S \left( I_0 \exp \left( -\frac{r^2}{r_0^2} \right) \right)(r) + S(T_0(r))(r). \quad (4')$

Problems (4) and (4') are obviously equivalent from the point of view of solvability: problem (4) is solvable if and only if Eq. (4') is solvable. Moreover, if we know that $T(r, 0) > 0$ is the solution of problem (4'), we can easily find from (4) the solution $T(r, z)$ in the entire half-space $\sigma \in r(x, z) \in \Omega$.

Theorem 1 (non-existence). Let $0 < \alpha < 2$. Then problem (1)-(3) has no stationary solutions.

For the proof we need some estimates that characterize the domain of values of the operator $S$.

Lemma 1. We have

$S \left( I_0 \exp \left( -\frac{r^2}{r_0^2} \right) \right)(r) \geq \frac{m_1}{r + r_0}, \quad r > 0, \quad (5)$

where

$m_1 = \int_{\mathbb{R}_+} \int_0^{2\pi} \eta(1 + \eta)^{-1} \exp(-\eta d\eta.$

Relation (5) follows from the inequality

$S(\xi(\xi))(r) = I_0 \int_{\mathbb{R}_+} \xi \exp \left( -\frac{r^2}{r_0^2} \right) d\xi \frac{1}{2\pi} \int_0^{2\pi} (r^2 + z^2 - 2r \xi \cos \varphi)^{1/2} d\varphi >$

$I_0 \int_{\mathbb{R}_+} \eta(r + r_0)^{-1} \exp(-\eta d\eta >$

$I_0 \int_{\mathbb{R}_+} \eta (1 + \eta)^{-1} \exp(-\eta d\eta.$

Relation (5) gives a simple estimate for the solution of Eq. (4'):

$T(r) > m_1/(r + r_0).$ We form the recurrent functional sequence

$T_0(r) = \frac{m_1}{r + r_0}, \quad T_{n+1}(r) = T_0(r) + S(T_n(r))(r), \quad n = 1, 2, \ldots. \quad (6)$

Since the operator $S(T_n)$ is monotonic with respect to $T$, we see from (4') that for all $n \geq 1$ (if $T_n(r)$ are defined for any finite $n > 1$) we must have

$T(r) > T_n(r), \quad r > 0, \quad n = 1, 2, \ldots. \quad (7)$

and in particular,

$T(r) > T_\infty(r) = \lim_{n \to \infty} T_n(r), \quad r > 0. \quad (7')$

Since $(T_n)$ is monotonic, the last limit (finite or infinite) exists.

Lemma 2. Let $0 < \alpha < 1$. Then, by (7), with $n = 2$,

$S(T_0^2(\xi))(r) = m_1 S(T_0(\xi))(r) = +\infty, \quad r > 0,$

so that Eq. (4') has no solutions.

Proof. Obviously,

$S(T_0^2(\xi))(r) > \int_{\mathbb{R}_+} \int_0^{2\pi} (r + z)^{-1} (r + z)^{-1} \eta(r + z)^{1/2} \eta(1 + \eta)^{-2} d\varphi = \int_{\mathbb{R}_+} \int_0^{2\pi} \eta(r + z)^{1/2} \eta(1 + \eta)^{-2} d\varphi,$

where $\xi = r/r_0$, when $\alpha \in (0, 1)$, these last integrals are divergent. But then, see (1), (2),

$T(r) > T_0(r) > S(T_0^2(\xi))(r) = m_1 S(T_0^2(\xi))(r) = +\infty,$

which proves the lemma.
Lemma 3. Let $\alpha > 1$. Then,
\[ S((\xi + r_0)^{-\alpha})(r) > [2(\alpha - 1)](r + r_0)^{\alpha - 1}, \ r > 0. \tag{9} \]

Proof. We use an estimate similar to (8). With $\alpha > 1$, the integrals are convergent, while
\[
\int_{r_0}^{\infty} \eta(s + \eta)^{-\alpha} (1 + \eta)^{-\alpha} \, d\eta \geq \int_{r_0}^{\infty} \frac{\eta}{s + \eta} (1 + \eta)^{-\alpha} \, d\eta, \quad s = \frac{r}{r_0} > 0.
\]
Since $\eta/(s+\eta) \geq \epsilon$, for $\eta \geq r$, we have
\[
S((\xi + r_0)^{-\alpha})(r) > \frac{1}{2} r_0^{-\alpha} \int_{r_0}^{\infty} (1 + \eta)^{-\alpha} \, d\eta = r_0^{-\alpha} \frac{(r + r_0)^{\alpha - 1}}{(\alpha - 1)^{\alpha}},
\]
which proves (9).

We will now prove Theorem 1.

1. First, let $1 < \alpha < 2$. We will show that, with $\alpha \equiv (1, 2)$ the sequence (6) is not defined for all $n$, and there is an integer $N = N(\alpha) > 1$, such that $T_{x, n}(r) = +\infty$. By (7), this ensures that problem (4') has no solution.

By Lemma 3, all the terms of sequence (6) have the lower limit
\[
T_n(r) > S[(\xi + r_0)^{-\alpha}](r) > m_1(r + r_0)^{\alpha - 1} \quad \lambda_\alpha = \alpha - 1,
\]
and finally, for all $n > 1$,
\[
T_n(r) > m_1(r + r_0)^{\alpha - 1}, \quad \lambda_\alpha = \alpha - 1, \quad m_1 = m_1^\alpha/(2\lambda_\alpha).
\tag{10}
\]
From the recurrence relation on the power $\lambda_\alpha$, in (10) we have $\lambda_\alpha = (\alpha - 1)^{n} - [1 - (2 - \alpha)^{\alpha - 1}]$, $n = 1, 2, \ldots$, so that, with $1 < \alpha < 2$, there always exists an integer $N = N(\alpha) > -\ln(2 - \alpha)/\ln\alpha > 0$ such that $\alpha - 1$, i.e.,
\[
T_{x, n}(r) > S(T_{x, n}(r)) = m_1 S((\xi + r_0)^{-\alpha} = +\infty
\]
(see Lemma 2). By (7), this implies that (4') has no solution.

2. Now consider the case $\alpha = 2$. It will be shown below that sequence (6) is now defined for all $n = 1, 2, \ldots$, and the non-existence theorem can be proved on the basis of the limiting inequality (7'). For the proof, we need two lemmas.

Lemma 4. We have
\[
S((\xi + r_0)^{-2})(r) > \frac{1}{4r} \ln \left( \frac{r}{r_0} \right), \quad r > r_0, \quad \alpha = 2.
\tag{11}
\]

Proof. From (8) with $\alpha = 2$ we have
\[
S((\xi + r_0)^{-2})(r) > r_0^{-1} \int_{r_0}^{\infty} \frac{\eta}{(1 + \eta)^2} (s + \eta)^{-3} \, d\eta, \quad s = \frac{r}{r_0}.
\]
Since $\eta/(s + \eta) \geq 1/(4\eta)$, $\eta \geq 1$, then
\[
S((\xi + r_0)^{-2})(r) > \frac{1}{4r_0} \int_{r_0}^{\infty} \eta^{-1} (s + \eta)^{-3} \, d\eta = \frac{1}{4r_0} \ln (1 + \frac{r}{r_0}),
\]
whence (11) follows.

Using (11), we obtain the inequality for the second term of sequence (6).
\[
T_n(r) > \frac{m_2}{r + r_0} + \frac{1}{4} \ln \left( \frac{r}{r_0} \right), \quad r > r_0.
\]

The operator $S(v^\alpha)$ thus changes the asymptotic behaviour of function $v$ as follows: if $v \rightarrow r^{-\alpha} \alpha \rightarrow +\infty$, then $S(v^\alpha) \geq (\ln r)/r$. We next need a lower limit for the general term of sequence $(T_n)$ with $\alpha = 2$.

Lemma 5. Let $\alpha = 2$. Then, for all $n \geq 2$,
\[
T_n(r) > \frac{m_2}{r + r_0} + \sum_{k=1}^{n-1} T_k(r),
\]
(13)
where \( v_k(r) = 0 \) for \( r = 0, r_1 \), and for \( r > r_1 \),

\[
v_k(r) = C_k \frac{1}{r} \left( \ln \left( \frac{r}{r_0} \right) \right)^{\nu_k} \quad \text{and} \quad C_k = \left( \frac{m_1}{2} \ln \nu_k \right)^{\nu_k} > 0.
\]

The proof is by induction. With \( n = 2 \), (13) holds (see (12)). Let it hold for \( n = N \).

Then, from (6) with \( \alpha = 2 \) we have

\[
T_{N+1}(r) = \frac{m_1}{r + r_0} + S(T_N(r))(r) \geq \frac{m_1}{r + r_0} +
\]

\[
S \left( \left( \frac{m_1}{r + r_0} + \sum_{k=1}^{N-1} v_k(r) \right)^4 \right) \geq \frac{m_1}{r + r_0} + S \left( \left( \frac{m_1}{r + r_0} + \sum_{k=1}^{N-1} v_k(r) \right)^4 \right) (r).
\]

Proceeding in the same way as when proving Lemma 4, we obtain

\[
S \left( m_4 (r + r_0)^4 \right) > \frac{m_1}{r + r_0} \ln \left( \frac{r}{r_0} \right) = v_1(r), \quad r > r_0,
\]

and in the general case (see (14))

\[
S \left( m_4 (r + r_0)^4 \right) = S \left( C_k \frac{1}{r} \left( \ln \left( \frac{r}{r_0} \right) \right)^{\nu_k} \right) \geq
\]

\[
C_k \frac{1}{r} \left( \ln \left( \frac{r}{r_0} \right) \right)^{\nu_k} \geq C_k \ln 2 - \frac{1}{r} \left( \ln \left( \frac{r}{r_0} \right) \right)^{\nu_k}.
\]

But \( C_k \ln 2 = C_{n+1} \), so that \( S \left( m_4 (r) \right) > v_{n+1}(r) \) for any \( k = 1, 2, \ldots, N-1 \). It then follows at once from (15) that (13) holds for \( n = N+1 \). This completes the proof.

We will now show that \( (4') \) has no solutions when \( \alpha = 2 \). We substitute (13) and (14) into the limiting inequality (7'), which must be satisfied by any solution \( T - T(r) > 0 \). We then obtain

\[
T(r) > T(r) > \frac{m_1}{r + r_0} + \frac{1}{r} \sum_{k=1}^{n} p_k, \quad r > r_0
\]

where

\[
p = \frac{m_1}{2} \left( \ln 2 \right)^{\nu_1} \left( \ln \left( \frac{r}{r_0} \right) \right)^{\nu_1} > 0, \quad r > r_0.
\]

It is easily seen that the series on the right-hand side of (16) is divergent if \( p > 1 \), i.e., if

\[
r > r_0, \text{exp} \left( \frac{1}{2} (m_4 \ln 2) \right) > r_0.
\]

Hence \( T(r) \) is not defined for \( r > r_0 \), so that no solution of the initial elliptic problem exists for \( \alpha = 2 \). This completes the proof of Theorem 1.

Note. Theorem 1 remains true whatever the radiation intensity distribution \( I(r) \) in the beam: instead of the exponential form \( I(r) = I_0 \exp(-r^2/ro^2) \) taken in (2), (4), we can take any function \( I(r) = I_0 \exp(-r^2/ro^2) \), \( 0 < I_0 < \infty \). The proof is then only slightly modified (see /7/).

Theorem 2 (existence). Let \( \alpha > 2 \). Then, there is a monotonous boundary of stability \( r = h(r_0) > 0, \ r_0 > 0, \ h(0)^+ = +\infty, \ h(\infty)^+ = 0 \), in the plane of the parameters \( \{ I, r_0 \} \). For all \( r > 0, h(r_0) \) the stationary problem is unsolvable, while for all \( I < h(r_0) \) there exists a minimal solution \( T = T_{min}(x) > 0 \) in \( \Omega \), which is stable from below; then, \( T(t, x) \leq T_{min}(x), \ t > 0, x \in \Omega \), and \( T(t, x) \to T_{min}(x) \) as \( t \to +\infty \) uniformly in any compactum of \( \Omega \).

This theorem can be proved in the same way as in /6, 7/. However, for \( \alpha > 3 \) the lower limit of \( T(r) \) can be obtained quite easily. Let \( \alpha > 3 \). We can construct an upper solution of problem (1)-(3) as follows:

\[
T_+(x) - C \left( x + a \right)^{\nu_1}, \quad x = (r, z) \in \Omega
\]

where \( C \) and \( a \) are positive constants. The function \( T_+ \) satisfies the non-linear elliptic problem

\[
\Delta T_+ = 0 \ln \Omega, \quad -\partial T_+ / \partial z|_{\partial \Omega} = -a C \Delta T_+ + \Delta T_+, \quad a > 0
\]

and is obviously an upper solution of the initial problem if

\[
-a C \Delta T_+ + \Delta T_+ \geq I_0 \exp(-r^2/ro^2) T_+, \quad a > 0
\]

or what amounts to the same thing,

\[
a C \left( r^2 + a^2 \right)^{\nu_1} \geq I_0 \exp(-r^2/ro^2) + C \left( r^2 + a^2 \right)^{\nu_1}
\]

for any \( r > 0 \). Using the inequality \( (r^2+a^2)^{\nu_1} \exp(-r^2/ro^2) \), we see that (17) certainly holds if

\[
a C \left( 1 - C^2 \right)^{\nu_1} \left( r^2 + a^2 \right)^{\nu_1} \geq I_0 \exp(-r^2/ro^2), \quad r > 0.
\]
This last inequality holds e.g., for $3a'=2a'$, $I_0 \leq aC(1-C^{-1-\varepsilon'})/a'$, i.e., in the case $I_0 \leq \frac{1}{2} a^{-1} C \left[1-C^{-1-\varepsilon'} \right]/a'$.

The right-hand side reaches its maximum when $C = \frac{\varepsilon}{\varepsilon'}(1-1/a)\left[1-C^{-1-\varepsilon'} \right]/a'$ so that the upper solution $T_+$ exists under the constraint on the beam parameters $I_0 \leq h_1(r_0) = (\varepsilon-2\varepsilon')/(\varepsilon+2\varepsilon')/2\varepsilon$.

Let us show that, under condition (18), problem (1)-(3) with $a > 3$ has a stationary solution. In fact, the solution of the non-stationary problem then satisfies the inequality $T(t, r) < T_+(r)$ in $R_+ \times \Omega$, and moreover, $T_+ > 0$ (see Lemma 5 on criticality in /7/). Hence, for any $r \in \Omega$, there exists the finite limit

$$\lim_{t \to \infty} T(t, r) = T_+(r)$$

and it can be shown in the same way as in /7/, sect.3, that $T_+(r) > 0$ is the required stationary solution.

In short, when $a > 3$, when inequality (18) holds, problem (1)-(3) has a stationary solution, so that the function $h$ in Theorem 2 satisfies the inequality $h(r) > h_1(r_0)$ for all $r > 0$.

We shall not dwell on the non-existence of solutions when $I_0 > h(r_0)$, since it is proved in the same way as in /7/.


Let us examine the asymptotic behaviour of the solutions of the evolutionary problem (1)-(3) in cases when no stationary solutions exist.

1. Unbounded solutions with $1 \leq a < 2$. We will first show that, when $1 \leq a < 2$, development of the thermal process always occurs in a mode with peaking, i.e., the boundary value problem has no global solution.

Theorem 3. Let $a \in (1, 2]$. Problem (1)-(3) then has no global solution, and there exists $t_0 \in (0, +\infty)$ such that

$$\sup_{t \in [0, +\infty)} T(t, r, 0) = T(t, 0) \to +\infty \text{ as } t \to t_0.$$  \hspace{1cm} (19)

For the proof, we require some auxiliary lemmas.

Lemma 6. Let $T(t, r, 0)$ be defined in $(0, t_0) \times \Omega$. Then,

$$\partial T/\partial t \geq 0,$$  \hspace{1cm} (20)

$$\partial T/\partial r \leq 0,$$  \hspace{1cm} (21)

For the proof, see Lemma 5 in /7/.

Lemma 7. With $a \in (1, 2]$ the function $T(t, r, 0)$ has no upper limit with respect to $t$ for any fixed $r > 0$.

Proof. Assume the contrary. Then, three cases are possible.

1. There is a constant $M > 0$ such that $T(t, r, 0) < M$ for all $r \not= 0$ for any $r > 0$. Then, by inequality (21), $T(t, r, 0) < M$ in $R_+ \times \Omega$. From (20) we find in turn that, for any $r \not= 0$,

there exists the finite limit

$$T_+(r) = \lim_{t \to +\infty} T(t, r).$$

and it is easily seen that $T_+(r)$ must be a stationary solution of the problem, see /7/, which by Theorem 1, does not exist. The stabilization of any solution, uniformly bounded with respect to $t$, to the stationary solution, also follows from the existence of the Lyapunov function

$$V(T(t)) = \frac{1}{2} \int_0^{r_0} \left[ \exp \left( -\frac{t}{r_0} \right) T + \frac{t}{\alpha + 1} T^{2\alpha} \right] \rho(r) dr,$$

which is monotonic on the evolutionary trajectories:

$$\frac{d}{dt} V(T(t)) = -\frac{2}{(2\alpha + 1)} T(t) \rho(t) < 0 \hspace{1cm} t > 0.$$  \hspace{1cm} (22)

2. There exists $r > 0$ such that $T(t, r, 0) \to +\infty$ as $t \to +\infty$ in $(0, r)$ and $T(t, r, 0)$ is uniformly bounded for all $r > r_0$. Then, by (20), $T(t, r, 0) \to +\infty$ as $t \to +\infty$ uniformly in $(0, r_0]$. In this case, from the integral equation

$$T(t, r, 0) = \frac{2}{(2\alpha + 1)} \left[ \left( \frac{t}{r_0} \right)^{2\alpha} \exp \left( -\frac{t}{r_0} \right) T + \frac{t}{\alpha + 1} T^{2\alpha} \right] \rho(r),$$  \hspace{1cm} (22)

$\Sigma$R 28-2-8
equivalent to problem (1)-(3), with a=0 we find that $T(t,r,0) \to \pm \infty$ as $t \to \pm \infty$ for all $r > 0$.

2. The function $T(t,r,0) \to \pm \infty$ as $t \to \pm \infty$ only at the point $r=0$. By inequality (20) (see Sect.1), this case is equivalent to $T(t,r,0)$ stabilizing as $t \to \pm \infty$ to the "singular" stationary solution $T(r)$ of Eq.(4'), which is defined for any $r > 0$, where $T(0)=\infty$. But it was shown during the proof of Theorem 1 that no such stationary solutions, defined in the neighbourhood of $r=\infty$, exist when $1 < a < 2$.

This contradiction proves the lemma.

Proof of Theorem 3. Assume that (19) does not hold and that the function $T(t,r,s)$ is defined everywhere in $R \times \Omega$. By Lemmas 6 and 7, we then conclude that $T(t,r,0) \to \pm \infty$ as $t \to \pm \infty$ uniformly in any compactum $\{0 \leq r \leq r_0\}$, and it is easily seen, using (22), that

$$T(t,r,0) \to \pm \infty \quad \text{as} \quad t \to \pm \infty$$

uniformly in any compactum $\{0 \leq r \leq r_0, 0 \leq s \leq s_0\}, s > 0$.

Let us show that, in these conditions, no solution can be global. Consider the function $T(t,s)$ in the domain $\infty \leq t \leq \infty, 0 < s < 1$, where a sufficiently large $s > 0$ is taken. By (23), for any $M > 0$ there is a $t_0 > 0$ such that $T(t,s) > M$ in $\{t \geq t_0\}$ uniformly in any compactum $\{0 < s \leq s_0\}$, $s > 0$.

Hence, by the maximum principle,

$$T(t,s) \geq N(t,s)$$

where $N$ is the solution of the boundary value problem

$$
\begin{align*}
0 < r < r_0, & \quad \frac{\partial N}{\partial r} = N, & t > t_0, & \quad x \geq \partial r_0, \\
0 < s \leq s_0, & \quad N = 0, & t > t_0, & \quad x \geq \partial s_0, \\
N(t,s) = N_0(t), & \quad s > 0.
\end{align*}
$$

Here, $N_0(t)$ is any sufficiently smooth function which satisfies the matching conditions, with $N_0(t) \in \Omega$.

It is easily shown that, as a result of increasing the constants $M$ and $r_0$, we can choose $N_0(t)$ in such a way that

$$\int_0^{r_0} r^2 N_0^2(t) dr > \frac{\alpha + 1}{2} \int_0^{r_0} |N_0|^2 dx.$$  

Such a $N_0$ can in general be written in explicit form. Moreover, it is easily shown that, if $N_0$ is linear in $x$, we have $\int_0^{r_0} |N_0|^2 dx \sim M^2 r_0^2$, $\int_0^{r_0} |N_0|^2 dx \sim M r_0^2$, and for sufficiently large $M$ we have $\int_0^{r_0} |N_0|^2 dx \sim \int_0^{r_0} |N_0|^2 dx$ (it is easily seen that, for large $r_0$, we can arrange for the matching conditions of the boundary functions to be satisfied by a "small" disturbance of the function $N_0$, which does not violate inequality (26) with $M > 1$).

When condition (26) holds the solution of problem (25) exists during a finite time interval (see [10, Theorem 11.1]) and there exists $t_0 > t_0$ such that

$$\lim \sup \ T(t,s) = \pm \infty$$

We then obtain (19) from (24), and for the existence time of the solution we have the upper limit $t_0 < t_0$.

2. Nodes with peaking with $2 < a < 3$. Theorem 2 shows that, with $a > 2$, the solution of problem (1)-(3) in the domain $L < h(r)$ is globally unbounded and tends to the minimal stationary solution. Hence, in order to initiate combustion, we have to supply a sufficiently large non-trivial initial disturbance. We consider below the boundary value problem with the initial condition

$$T(0,z) = T_0(z) \geq 0, \quad z = (r, s) \in \Omega, \sup T_0 < \infty$$

and find the conditions under which the solution is unbounded.

Theorem 4. Let $\alpha = (1, 3)$ and let $t > 0$ exist such that

$$T(t,s) = T_0(s) \geq 0, \quad z = (r, s) \in \Omega, \sup T_0 < \infty,$$

and find the conditions under which the solution is unbounded.

Theorem 4. Let $\alpha = (1, 3)$ and let $t > 0$ exist such that

$$T(t,s) = T_0(s) \geq 0, \quad z = (r, s) \in \Omega, \sup T_0 < \infty,$$

where $\alpha = 2 - (2 - a)/(a + 1)^2$, $C_1 = (a + 1)^3/(a - 1)^2$, $a = (a + 1)/(a - 1)$. A solution of problem (1), (2), (3), (27) then exists in a finite time interval, and for some $t_0 < t_0$ we have

$$\lim \sup_{t \to t_0} T(t,s) = \pm \infty.$$
\[ T_\cdot(t,z) = (t-t)^{-1/(\alpha-1)}/(\xi, \eta), \]
\[ \xi = r(t-t)^{-\alpha}, \quad \eta = (t-t)^{-\alpha}, \]

where
\[ f(t, \eta) - C [\eta + (\eta + a)^{-1}]^{1/(\alpha-1)}, \]
and \( C \) and \( a > 0 \) are constants which are found below. The function \( T_\cdot \) is certainly a lower solution if
\[ T_\cdot(t,0) \leq \Delta T_\cdot \in (0, t) \times \Omega, \quad -\frac{\partial T_\cdot}{\partial t} < T_\cdot < 0 \in (0, t) \times \partial \Omega, \]
\[ (31) \]

or what amounts to the same thing,
\[ \xi^{-1}(\xi/\alpha)^{\alpha-2} - f(\xi, \eta)/(2(\alpha-1)) > 0 \text{ in } \Omega, \]

\[ f(t, \eta) = 0 \quad \text{in } \partial \Omega, \quad \xi > 0, \quad \eta > 0, \]

\[ (32) \]

(condition (31) is the same as (28)). Substituting for \( f \) from (30) into (32) and (33), we arrive at the inequalities
\[ \xi^{-1}[\eta - 4(3-\alpha)/(\alpha-1) - a]/2 - f(\xi, \eta)/(2(\alpha-1)) > 0 \text{ in } \Omega, \]
\[ C^{-1} \geq 2a/(\alpha-1). \]

Hence follow the bounds on the values of the parameters \( a \) and \( C \):
\[ a^* = \frac{4(3-\alpha)/(\alpha-1)}{2(\alpha-1)} = a^*, \quad C^{-1} \geq 2a/(\alpha-1). \]

In particular, these bounds are satisfied with the constants \( a^* \) and \( C^* \) indicated in the theorem.

With this choice of \( a^* \) and \( C^* \), therefore, \( T_\cdot(t,z) \) is a lower solution, so that \( T_\cdot \geq T_\cdot \text{ in } \Omega \) for all admissible \( t > 0 \). Hence (29) follows at once.

3. On the asymptotic stage of combustion. It is interesting to see how the temperature field evolves after the initiation of combustion.

It is natural to assume that, at the stage of developed combustion, the energy liberation of the reaction greatly exceeds the radiation energy. At this stage, therefore, the temperature field must be regularly described by the simplified problem
\[ T_t = \Delta T, \quad t > 0, \quad z \in \Omega, \quad -\frac{\partial T_t}{\partial t} \bigg|_{\partial \Omega} = \Gamma^a. \]

As distinct from the initial problem, the boundary condition here does not contain a term describing the absorption of radiation.

Problem (34) admits of the construction of similarity solutions which evolve into a mode with peaking:

\[ T_\ast(t,z) = (t-t)^{-1/(\alpha-1)}/(\xi, \eta), \]

\[ \xi = r(t-t)^{-\alpha}, \quad \eta = (t-t)^{-\alpha}, \]

where \( \gamma > 0 \) satisfies the non-linear elliptic problem (see (32))
\[ \Delta \gamma - (\xi/\alpha)^{\alpha-2}[2(3-\alpha)/(\alpha-1)]^{-1} = 0 \text{ in } \Omega, \]
\[ -f(t, \eta) = 0 \quad \text{in } \partial \Omega, \quad \xi > 0, \quad \eta > 0. \]

This asymptotic behaviour is supported by the fact that, with \( a^* < 3/2 \), we can construct unbounded lower solutions with the same space-time structure.

The amplitude of solution (35) increases without limit: \( T_\ast(t,0,0) \rightarrow f(t,0,0)/(t-t)^{-1/2}(\alpha-1) \rightarrow \infty, \quad t^{-k}. \) The effective width of the resulting thermal structure falls in each direction:
\[ \lim_{t \rightarrow k} \| \xi, \eta \| = 0, \quad t^{-k}, \]
and we can expect moreover that \( T_\ast(t, z) \rightarrow \infty \), \( t^{-k} \), at only the one point \( z = 0 \). Thus virtually all the energy liberated at this stage of the combustion is localized in the neighbourhood of this point.

With respect to its space-time structure, \( T_\ast \) is a typical localized similarity solution of the LS mode with peaking, see /11/. A reaction energy liberation localization effect was earlier found in another non-linear laser heating problem /12, 13/, where the mode of localization with peaking arose as an intermediate asymptotic form of the temperature field evolution and was caused by the variation of the optical characteristics of the material during the reaction.

It must be said in conclusion that a rigorous proof of the asymptotic form (35) remains an open question. Great efforts, see e.g., /14, 15/, are needed to prove the asymptotic stability of similarity solutions with a point of time singularity, even in the case of much simpler parabolic non-linear problems. It also remains an open question whether there is a non-trivial solution of the elliptic problem (36), which is more complicated than that considered in Sect.2.
REFERENCES


15. GALAKTIONOV V.A., Asymptotic behaviour of unbounded solutions of the non-linear parabolic equation \( u_t = (u^m)_x + u^n \), DIFFERENTS. Urav. 21, 7, 1126-1134, 1985.

Translated by D.E.B.