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A NON-LINEAR BOUNDARY VALUE PROBLEM OF IGNITION BY RADIATION*

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Existence and non-existence conditions are obtained for stationary solutions in the model problem of laser macrokinetics which describes the heating of a semi-infinite sample by a spatially inhomogeneous beam of radiation with a power approximation of the energy liberation at the surface of the sample. The modes of localization and peaking of the heat field in the substance at the high-temperature stage of heating are examined.

1. Introduction.

Advances in high-temperature physics and macrokinetics have led to the formulation and detailed study of a wide class of non-linear boundary value problems for the heat conduction equation. Problems of the heating of a substance by radiation, which is the subject matter of the interesting new field of laser macrokinetics /1, 2/, occupy an important place.

Many theoretical and experimental studies of combustion under the action of radiation have now been published, see e.g. /3/. It has been found that size effects are typical for problems of combustion physics. For instance, to initiate the combustion of a material occupying a large volume, the source must have a sufficiently high temperature and large size /4/. In problems of laser heating the inhomogeneity of the radiation field is important and is due to the finite size of the actual beam and the variable intensity distribution across it. In this connection the question arises of the kinetic parameters of the radiation (beam radius and power), for which ignition of the substance becomes possible. It is interesting to study these questions both for purposes of laser technology and for the further development of the theory of non-linear dynamic systems with inhomogeneous parameters.

The non-linear boundary value problem

$$\frac{\partial T}{\partial t}=\Delta T\equiv\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2},\qquad (r,z)=x\in\Omega=\{r\geqslant 0,\ z>0\},\tag{1}$$

$$-\left.\frac{\partial T}{\partial z}\right|_{\partial\Omega}=I(r)+g(T),\qquad \partial\Omega=\{r\geqslant 0,\ z=0\},\tag{2}$$

$$I(r)=I_0\exp\left(-\frac{r^2}{r_0^2}\right),$$

$$T(0,x)=0,\qquad x\in\Omega,\tag{2'}$$

where $g(T)=\exp(-1/T)$, was studied in detail in /5-8/. This is the simple mathematical model of the surface heating by radiation of a large metal sample on whose surface an exothermic reaction (oxidation) occurs. Here, T is the temperature and I_0 and r_0 are the beam intensity and effective radius. The term $g(T)$ describes the energy liberation of the reaction. It was shown in /5-8/ that, in the plane of parameter values $\{I_0, r_0\}$ there is a monotonic stability boundary $I_0=h(r_0)$ such that, for all $I_0>h(r_0)$, the stationary boundary value problem (1), (2) is solvable, while when $I_0<h(r_0)$ there are no stationary solutions. This means that, for sufficiently low initial material temperatures, no ignition occurs if $I_0<h(r_0)$, while if $I_0>h(r_0)$, for any initial conditions, the exothermic reaction covers the entire surface.

Sometimes, however, non-linearities other than $g(T)=\exp(-1/T)$ need to be considered, notably because the actual chemical reaction has, in general, several stages, the different mechanisms "switch on" at different temperatures, and there are various heat loss mechanisms (e.g., radiation $\sim T^4$ and convective $\sim T$), etc. Over fairly wide temperature ranges that are of practical importance, the non-linearity $g(T)$ in the boundary condition can often be approximated by the power law

$$g(T)=T^\alpha,\qquad \alpha=\text{const}>0.\tag{3}$$

By studying such non-linearities we can draw conclusions about the corresponding systems, e.g., on the basis of comparison theorems.

The present paper deals with the conditions for the existence of stationary solutions and the ways that non-stationary solutions evolve for problem (1)-(3) with a power non-linearity in the boundary condition.

2. On the conditions for the non-existence of stationary solutions.

In the stationary case problem (1)-(3) is equivalent to the non-linear integral equation /9/

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$$T(r, z)=S\left(I_0 \exp\left(-\frac{\xi^2}{r_0^2}\right)\right)(r, z)+S\left(T^\alpha(\xi, 0)\right)(r, z), \tag{4}$$

where S is an operator:

$$S(v(\xi))(r, z)=\int_0^{+\infty} \xi v(\xi)\, d\xi \frac{1}{2\pi}\int_0^{2\pi}(r^2+z^2+\xi^2-2r\xi\cos\varphi)^{-1/2}\, d\varphi.$$

We put $z=0$ in Eq. (4) and introduce the notation $T(r, 0)=T(r)$. Then,

$$T(r)=S\left(I_0 \exp\left(-\frac{\xi^2}{r_0^2}\right)\right)(r)+S\left(T^\alpha(\xi)\right)(r). \tag{4'}$$

Problems (4) and (4') are obviously equivalent from the point of view of solvability: problem (4) is solvable if and only if Eq. (4') is solvable. Moreover, if we know that $T(r, 0)>0$ is the solution of problem (4'), we can easily find from (4) the solution $T(r, z)$ in the entire half-space $x=(r, z)\in\Omega$.

Theorem 1 (non-existence). Let $0<\alpha\leqslant 2$. Then problem (1)-(3) has no stationary solutions.

For the proof we need some estimates that characterize the domain of values of the operator S.

Lemma 1. We have

$$S\left(I_0 \exp\left(-\frac{\xi^2}{r_0^2}\right)\right)(r)\geqslant\frac{m_1}{r+r_0}, \qquad r\geqslant 0, \tag{5}$$

where

$$m_1=I_0 r_0^2\int_0^{+\infty}\eta(1+\eta)^{-1}\exp(-\eta^2)\, d\eta.$$

Relation (5) follows from the inequality

$$S(I(\xi))(r)=I_0\int_0^{+\infty}\xi\exp\left(-\frac{\xi^2}{r_0^2}\right)d\xi\,\frac{1}{2\pi}\int_0^{2\pi}(r^2+\xi^2-2r\xi\cos\varphi)^{-1/2}d\varphi\geqslant$$
$$I_0 r_0^2\int_0^{+\infty}\eta(r+r_0\eta)^{-1}\exp(-\eta^2)\, d\eta\geqslant$$
$$\frac{I_0 r_0^2}{r+r_0}\int_0^{+\infty}\eta(1+\eta)^{-1}\exp(-\eta^2)\, d\eta.$$

Relation (5) gives a simple estimate for the solution of Eq. (4'): $T(r)\geqslant m_1/(r+r_0)$. We form the recurrent functional sequence

$$T_1(r)=\frac{m_1}{r+r_0}, \quad T_{n+1}(r)=T_1(r)+S(T_n^\alpha(\xi))(r), \quad n=1, 2, \ldots . \tag{6}$$

Since the operator $S(T^\alpha)$ is monotonic with respect to T, we see from (4') that for all $n\geqslant 1$ (if $T_n(r)$ are defined for any finite $n\geqslant 1$) we must have

$$T(r)\geqslant T_n(r), \qquad r\geqslant 0, \quad n=1, 2, \ldots, \tag{7}$$

and in particular,

$$T(r)\geqslant T_\infty(r)=\lim_{n\to\infty} T_n(r), \qquad r\geqslant 0. \tag{7'}$$

Since $\{T_n\}$ is monotonic, the last limit (finite or infinite) exists.

Lemma 2. Let $0<\alpha\leqslant 1$. Then, by (7), with $n=2$,

$$S(T_1^\alpha(\xi))(r)=m_1^\alpha S((\xi+r_0)^{-\alpha})(r)=+\infty, \qquad r\geqslant 0,$$

so that Eq. (4') has no solutions.

Proof. Obviously,

$$S((\xi+r_0)^{-\alpha})(r)\geqslant\int_0^{+\infty}\xi(r+\xi)^{-1}(r_0+\xi)^{-\alpha}\, d\xi=r_0^{1-\alpha}\int_0^{+\infty}\eta(s+\eta)^{-1}(1+\eta)^{-\alpha}\, d\eta, \tag{8}$$

where $s=r/r_0$, when $\alpha\in(0, 1]$, these last integrals are divergent. But then, see (7), (6),

$$T(r)\geqslant T_2(r)>S(T_1^\alpha(\xi))(r)>m_1^\alpha S((\xi+r_0)^{-\alpha})=+\infty,$$

which proves the lemma.

Lemma 3. Let $\alpha>1$. Then,

$$S((\xi+r_0)^{-\alpha})(r)\geqslant[2(\alpha-1)(r+r_0)^{\alpha-1}]^{-1},\quad r\geqslant 0. \tag{9}$$

Proof. We use an estimate similar to (8). With $\alpha>1$, the integrals are convergent, while

$$\int_0^{+\infty}\eta(s+\eta)^{-1}(1+\eta)^{-\alpha}\,d\eta\geqslant\int_s^{+\infty}\frac{\eta}{s+\eta}(1+\eta)^{-\alpha}\,d\eta,\qquad s=\frac{r}{r_0}>0.$$

Since $\eta/(s+\eta)\geqslant{}^1/_2$ for $\eta\geqslant s$, we have

$$S((\xi+r_0)^{-\alpha})(r)\geqslant\frac{1}{2}r_0^{1-\alpha}\int_s^{+\infty}(1+\eta)^{-\alpha}\,d\eta=r_0^{1-\alpha}\frac{(s+1)^{1-\alpha}}{2(\alpha-1)},$$

which proves (9).

We will now prove Theorem 1.

1. First, let $1<\alpha<2$. We will show that, with $\alpha\in(1,2)$ the sequence (6) is not defined for all n, and there is an integer $N=N(\alpha)>1$, such that $T_{N+1}(r)=+\infty$. By (7), this ensures that problem (4') has no solution.

By Lemma 3, all the terms of sequence (6) have the lower limit

$$\begin{aligned}&T_2(r)>S([m_1(\xi+r_0)^{-1}]^{\alpha})>m_2(r+r_0)^{-\lambda_2},\qquad \lambda_2=\alpha-1,\\ &m_2=\frac{m_1^{\alpha}}{2\lambda_2};\\ &T_3(r)>S([m_2(\xi+r_0)^{-\lambda_2}]^{\alpha})>m_3(r+r_0)^{-\lambda_3},\qquad \lambda_3=\alpha\lambda_2-1,\\ &m_3=\frac{m_2^{\alpha}}{2\lambda_3},\end{aligned}$$

and finally, for all $n>1$,

$$T_n(r)>m_n(r+r_0)^{-\lambda_n},\qquad \lambda_n=\alpha\lambda_{n-1}-1,\quad m_n=m_{n-1}^{\alpha}/(2\lambda_n). \tag{10}$$

From the recurrence relation on the power λ_n in (10) we have $\lambda_n=(\alpha-1)^{-1}[1-(2-\alpha)\alpha^{n-1}]$, $n=1,2,\ldots$, so that, with $1<\alpha<2$, there always exists an integer $N=N(\alpha)>-\ln(2-\alpha)/\ln\alpha>0$ such that $\alpha\lambda_N<1$, i.e.,

$$T_{N+1}(r)>S(T_N^{\alpha}(\xi))(r)>m_N^{\alpha}S((\xi+r_0)^{-\alpha\lambda_N})=+\infty$$

(see Lemma 2). By (7), this implies that (4') has no solution.

2. Now consider the case $\alpha=2$. It will be shown below that sequence (6) is now defined for all $n=1,2,\ldots$, and the non-existence theorem can be proved on the basis of the limiting inequality (7'). For the proof, we need two lemmas.

Lemma 4. We have

$$S((\xi+r_0)^{-2})(r)\geqslant\frac{1}{4r}\ln\left(\frac{r}{r_0}\right),\qquad r\geqslant r_0,\quad \alpha=2. \tag{11}$$

Proof. From (8) with $\alpha=2$ we have

$$S((\xi+r_0)^{-2})(r)\geqslant r_0^{-1}\int_1^{+\infty}\frac{\eta}{(1+\eta)^2}(s+\eta)^{-1}\,d\eta,\qquad s=\frac{r}{r_0}.$$

Since $\eta/(1+\eta)^2\geqslant 1/(4\eta)$, $\eta\geqslant 1$, then

$$S((\xi+r_0)^{-2})(r)\geqslant\frac{1}{4r_0}\int_1^{+\infty}\eta^{-1}(s+\eta)^{-1}\,d\eta=\frac{1}{4r}\ln\left(1+\frac{r}{r_0}\right),$$

whence (11) follows.

Using (11), we obtain the inequality for the second term of sequence (6):

$$T_2(r)\geqslant\frac{m_1}{r+r_0}+\frac{m_1^2}{4}\frac{1}{r}\ln\left(\frac{r}{r_0}\right),\qquad r\geqslant r_0. \tag{12}$$

The operator $S(v^2)$ thus changes the asymptotic behaviour of function v as follows: if $v\sim r^{-1}$ as $r\to+\infty$, then $S(v^2)\geqslant(\ln r)/r$. We next need a lower limit for the general term of sequence $\{T_n\}$ with $\alpha=2$.

Lemma 5. Let $\alpha=2$. Then, for all $n\geqslant 2$,

$$T_n(r)\geqslant\frac{m_1}{r+r_0}+\sum_{k=1}^{n-1}v_k(r), \tag{13}$$

where $v_k(r)=0$ for $r\in(0,r_0]$, and for $r>r_0$,

$$v_k(r)=C_k\frac{1}{r}\left[\ln\left(\frac{r}{r_0}\right)\right]^{2^{k-1}},\qquad C_k=\left[\frac{m_1}{2}\ln^{1/2}2\right]^{2^k}\ln^{-1}2>0. \tag{14}$$

The proof is by induction. With $n=2$, (13) holds (see (12)). Let it hold for $n=N$. Then, from (6) with $\alpha=2$ we have

$$T_{N+1}(r)=\frac{m_1}{r+r_0}+S(T_N{}^2(\xi))(r)\geqslant\frac{m_1}{r+r_0}+ \tag{15}$$

$$S\left(\left[\frac{m_1}{r+r_0}+\sum_{k=1}^{N-1}v_k(\xi)\right]^2\right)\geqslant\frac{m_1}{r+r_0}+\ S\left(\frac{m_1{}^2}{(\xi+r_0)^2}+\sum_{k=1}^{N-1}v_k{}^2(\xi)\right)(r).$$

Proceeding in the same way as when proving Lemma 4, we obtain

$$S(m_1{}^2(\xi+r_0)^{-2})\geqslant\frac{m_1{}^2}{4}\frac{1}{r}\ln\left(\frac{r}{r_0}\right)\equiv v_1(r),\qquad r>r_0,$$

and in the general case (see (14))

$$S(v_k{}^2(\xi))=S\left(C_k{}^2\frac{1}{\xi^2}\left[\ln\left(\frac{\xi}{r_0}\right)\right]^{2^k}\right)\geqslant$$

$$C_k{}^2\int_r^{+\infty}\left[\ln\left(\frac{\xi}{r_0}\right)\right]^{2^k}\xi^{-1}(r+\xi)^{-1}d\xi\geqslant$$

$$C_k{}^2\left[\ln\left(\frac{r}{r_0}\right)\right]^{2^k}\int_r^{+\infty}\frac{d\xi}{\xi(r+\xi)}=C_k{}^2\ln 2\,\frac{1}{r}\left[\ln\left(\frac{r}{r_0}\right)\right]^{2^k}.$$

But $C_k{}^2\ln 2=C_{k+1}$, so that $S(v_k{}^2(\xi))(r)\geqslant v_{k+1}(r)$ for any $k=1,2,\dots,N-1$. It then follows at once from (15) that (13) holds for $n=N+1$. This completes the proof.

We will now show that (4′) has no solutions when $\alpha=2$. We substitute (13) and (14) into the limiting inequality (7′), which must be satisfied by any solution $T=T(r)>0$. We then obtain

$$T(r)\geqslant T_\infty(r)\geqslant\frac{m_1}{r+r_0}+\frac{1}{r\ln 2}\sum_{k=1}^{\infty}p^{2^k},\qquad r>r_0, \tag{16}$$

where

$$p=\frac{m_1}{2}(\ln 2)^{1/2}\left[\ln\left(\frac{r}{r_0}\right)\right]^{1/2}>0,\qquad r>r_0.$$

It is easily seen that the series on the right-hand side of (16) is divergent if $p\geqslant 1$, i.e., if

$$r\geqslant r_*=r_0\exp[4/(m_1{}^2\ln 2)]>r_0.$$

Hence $T(r)$ is not defined for $r\geqslant r_*$, so that no solution of the initial elliptic problem exists for $\alpha=2$. This completes the proof of Theorem 1.

Note. Theorem 1 remains true whatever the radiation intensity distribution $I(r)$ in the beam: instead of the exponential form $I(r)=I_0\exp(-r^2/r_0^2)$ taken in (2), (4), we can take any function $I(r)=I_0\theta(r/r_0)$, $\theta(\xi)\geqslant 0$, $\theta\not\equiv 0$. The proof is then only slightly modified (see /7/).

Theorem 2 (existence). Let $\alpha>2$. Then, there is a monotonic boundary of stability $I_0=h(r_0)>0$, $r_0>0$; $h(0^+)=+\infty$, $h(+\infty)=0$, in the plane of the parameters $\{I_0,r_0\}$. For all $I_0>h(r_0)$ the stationary problem is unsolvable, while for all $I_0<h(r_0)$ there exists a minimal solution $T=T_{\min}(x)>0$ in Ω, which is stable from below; then, $T(t,x)<T_{\min}(x)$, $t>0$, $x\in\Omega$, and $T(t,x)\to T_{\min}(x)$ as $t\to+\infty$ uniformly in any compactum of Ω.

This theorem can be proved in the same way as in /6, 7/. However, for $\alpha\geqslant 3$ the lower limit of $h(r_0)$ can be obtained quite easily. Let $\alpha\geqslant 3$. We can construct an upper solution of problem (1)-(3) as follows:

$$T_+(x)=C[r^2+(z+a)^2]^{-1/2},\qquad x=(r,z)\in\Omega,$$

where C and a are positive constants. The function T_+ satisfies the non-linear elliptic problem

$$\Delta T_+=0\text{ in }\Omega,\qquad -\partial T_+/\partial z|_{\partial\Omega}=aC^{-2}T_+{}^3$$

and is obviously an upper solution of the initial problem if

$$-\partial T_+/\partial z|_{\partial\Omega}\geqslant I_0\exp(-r^2/r_0{}^2)+T_+{}^\alpha,$$

or what amounts to the same thing,

$$aC(r^2+a^2)^{-3/2}\geqslant I_0\exp(-r^2/r_0{}^2)+C^\alpha(r^2+a^2)^{-\alpha/2} \tag{17}$$

for any $r\geqslant 0$. Using the inequality $(r^2+a^2)^{-\alpha/2}\leqslant a^{3-\alpha}(r^2+a^2)^{-3/2}$, we see that (17) certainly holds if

$$aC(1-C^{\alpha-1}a^{2-\alpha})(r^2+a^2)^{-3/2}\geqslant I_0\exp(-r^2/r_0{}^2),\qquad r\geqslant 0.$$

This last inequality holds e.g., for $3r_0^2=2a^2$, $I_0\leqslant aC(1-C^{\alpha-1}a^{2-\alpha})/a^3$, i.e., in the case

$$I_0\leqslant {}^2/_3 r_0^{-2}C[1-C^{\alpha-1}({}^3/_2)^{(2-\alpha)/2}r_0^{2-\alpha}].$$

The right-hand side reaches its maximum when

$$C=\alpha^{-1/(\alpha-1)}({}^2/_3)^{(2-\alpha)/2(\alpha-1)}r_0^{-(2-\alpha)/(\alpha-1)},$$

so that the upper solution T_+ exists under the constraint on the beam parameters

$$I_0\leqslant h_+(r_0)=(\alpha-1)\,\alpha^{-\alpha/(\alpha-1)}({}^2/_3)^{\alpha/[2(\alpha-1)]}r_0^{-\alpha/(\alpha-1)}. \tag{18}$$

Let us show that, under condition (18), problem (1)-(3) with $\alpha\geqslant3$ has a stationary solution. In fact, the solution of the non-stationary problem then satisfies the inequality $T(t,x)\leqslant T_+(x)$ in $\mathbb{R}_+^1\times\Omega$, and moreover, $T_t\geqslant0$ (see Lemma 5 on criticality in /7/). Hence, for any $x\in\Omega$, there exists the finite limit

$$\lim_{t\to+\infty}T(t,x)=T_*(x)$$

and it can be shown in the same way as in /7, Sect.3/ that $T_*(x)>0$ is the required stationary solution.

In short, when $\alpha\geqslant3$, when inequality (18) holds, problem (1)-(3) has a stationary solution, so that the function h in Theorem 2 satisfies the inequality $h(r_0)\geqslant h_+(r_0)$ for all $r_0>0$.

We shall not dwell on the non-existence of solutions when $I_0>h(r_0)$, since it is proved in the same way as in /7/.

3. On modes with peaking and the effect of reaction localization in the non-stationary problem.

Let us examine the asymptotic behaviour of the solutions of the evolutionary problem (1)-(3) in cases when no stationary solutions exist.

1. Unbounded solutions with $1<\alpha\leqslant2$. We will first show that, when $1<\alpha\leqslant2$, development of the thermal process always occurs in a mode with peaking, i.e., the boundary value problem has no global solution.

Theorem 3. Let $\alpha\in(1,2]$. Problem (1)-(3) then has no global solution, and there exists $t_0\in(0,+\infty)$ such that

$$\sup_{x\in\Omega}T(t,x)\equiv T(t,0)\to+\infty \quad\text{as}\quad t\to t_0^-. \tag{19}$$

For the proof, we require some auxiliary lemmas.

Lemma 6. Let $T(t,x)$ be defined in $(0,t_0)\times\bar\Omega$. Then,

$$\partial T/\partial t\geqslant0, \tag{20}$$

$$\partial T/\partial r\leqslant0,\qquad \partial T/\partial z\leqslant0\ \text{in}\,(0,t_0)\times\bar\Omega. \tag{21}$$

For the proof, see Lemma 5 in /7/.

Lemma 7. With $\alpha\in(1,2]$ the function $T(t,r,0)$ has no upper limit with respect to t for any fixed $r\geqslant0$.

Proof. Assume the contrary. Then, three cases are possible.

1. There is a constant $M>0$ such that $T(t,r,0)\leqslant M$ for all $t\geqslant0$ for any $r\geqslant0$. Then, by inequality (21), $T(t,r,z)\leqslant M$ in $\mathbb{R}_+^1\times\bar\Omega$. From (20) we find in turn that, for any $x\in\bar\Omega$, there exists the finite limit

$$T_*(x)=\lim_{t\to+\infty}T(t,x),$$

and it is easily seen that $T_*(x)$ must be a stationary solution of the problem, see /7/, which by Theorem 1, does not exist. The stabilization of any solution, uniformly bounded with respect to t, to the stationary solution, also follows from the existence of the Lyapunov function

$$V(T)(t)=\frac{1}{2}\|\nabla T\|^2_{L^2(\Omega)}(t)-2\pi\int_{\partial\Omega}r\left[\exp\left(-\frac{\xi^2}{r_0^2}\right)T+\frac{1}{\alpha+1}T^{\alpha+1}\right]dr,$$

which is monotonic on the evolutionary trajectories:

$$\frac{d}{dt}V(T)(t)=-\|T_t\|^2_{L^2(\Omega)}(t)\leqslant0,\qquad t>0.$$

2. There exists $r_*>0$ such that $T(t,r,0)\to+\infty$ as $t\to+\infty$ in $(0,r_*)$ and $T(t,r,0)$ is upper-bounded for all $r>r_*$. Then, by (20), $T(t,r,0)\to+\infty$ as $t\to+\infty$ uniformly in $[0,r_*/2]$. In this case, from the integral equation

$$T(t,r,z)=\frac{2}{(2\sqrt{\pi})^3}\int_0^t(t-\tau)^{-3/2}d\tau\int_0^\infty\xi\,[\exp(-\xi^2/r_0^2)+ \tag{22}$$

$$T^{\alpha}(\tau,\xi,0)]\,d\xi\,\frac{1}{2\pi}\int_0^{2\pi}\exp\left[-\frac{r^2+z^2+\xi^2-2r\xi\cos\varphi}{4(t-\tau)}\right]d\varphi,$$

equivalent to problem (1)-(3), with $z=0$ we find that $T(t,r,0)\to+\infty$ as $t\to+\infty$ for all $r\geqslant 0$.

3. The function $T(t,r,0)\to+\infty$ as $t\to+\infty$ only at the point $r=0$. By inequality (20) (see Sect.1), this case is equivalent to $T(t,r,0)$ stabilizing as $t\to+\infty$ to the "singular" stationary solution $T_*(r)$ of Eq.(4'), which is defined for any $r>0$, where $T_*(0)=\infty$. But it was shown during the proof of Theorem 1 that no such stationary solutions, defined in the neighbourhood of $r=+\infty$, exist when $1<\alpha\leqslant 2$.

This contradiction proves the lemma.

Proof of Theorem 3. Assume that (19) does not hold and that the function $T(t,r,z)$ is defined everywhere in $\mathbb{R}_+^1\times\bar{\Omega}$. By Lemmas 6 and 7, we then conclude that $T(t,r,0)\to+\infty$ as $t\to+\infty$ uniformly in any compactum $\{0\leqslant r\leqslant r_*\}$, and it is easily seen, using (22), that

$$T(t,r,z)\to+\infty \text{ as } t\to+\infty \tag{23}$$

uniformly in any compactum $\{0\leqslant r\leqslant r_*,\ 0\leqslant z\leqslant z_*\},\ z_*>0$.

Let us show that, in these conditions, no solution can be global. Consider the function $T(t,x)$ in the domain $\omega=\{0<r<r_*,\ 0<z<1\}$, where a sufficiently large $r_*>0$ is taken. By (23), for any $M_*>0$ there is a $t_*>0$ such that $T(t,x)\geqslant M_*$ in $(t_*,+\infty)\times\bar{\omega}$. Hence, by the maximum principle,

$$T(t,x)\geqslant v(t,x) \text{ in } (t_*,+\infty)\times\bar{\omega}, \tag{24}$$

where v is the solution of the boundary value problem

$$v_t=\Delta v,\quad t>t_*,\quad x\in\omega,\quad -\frac{\partial v}{\partial z}=v^{\alpha},\quad t\geqslant t_*,\quad x\in\partial\omega_1= \tag{25a}$$
$$\{0<r<r_*,\ z=0\},$$

$$v=0,\quad t\geqslant t_*;\quad x\in\partial\omega_2=\partial\omega\setminus\partial\omega_1,\quad v(t_*,x)=v_0(x),\quad x\in\bar{\omega}. \tag{25b}$$

Here, $v_0(x)\geqslant 0$ is any sufficiently smooth function which satisfies the matching conditions, wile $v_0\leqslant M_*$ in $\bar{\omega}$.

It is easily shown that, as a result of increasing the constants M_* and r_*, we can choose v_0 in such a way that

$$\int_{\partial\omega_1} v_0^{\alpha+1}(s)\,ds>\frac{\alpha+1}{2}\int_{\omega_1}|\nabla v_0|^2\,dx. \tag{26}$$

Such a v_0 can in general be written in explict form. Moreover, it is easily shown that, if v_0 is linear in z, we have $\|v_0\|^{\alpha+1}_{L^{\alpha+1}(\partial\omega_1)}\sim M_*^{\alpha+1}r_*^2$, $\|\nabla v_0\|^2_{L^2(\omega)}\sim M_*^2r_*^2$ and for sufficiently large M_* we have $\|v_0\|^{\alpha+1}_{L^{\alpha+1}(\partial\omega_1)}\gg\|\nabla v_0\|^2_{L^2(\omega)}$ (it is easily seen that, for large r_*, we can arrange for the matching conditions of the boundary functions to be satisfied by a "small" disturbance of the function v_0, which does not violate inequality (26) with $M_*\gg 1$).

When condition (26) holds the solution of problem (25) exists during a finite time interval (see /10, Therem II.1) and there exists $t_1>t_*$ such that

$$\varlimsup_{t\to t_1^-}\sup_{x\in\omega} v(t,x)=+\infty.$$

We then obtain (19) from (24), and for the existence time of the solution we have the upper limit $t_0\leqslant t_1$.

2. Modes with peaking with $2<\alpha\leq 3$. Theorem 2 shows that, with $\alpha>2$, the solution of problem (1)-(3) in the domain $I_0<h(r_0)$ is globally unbounded and tends to the minimal stationary solution. Hence, in order to initiate combustion, we have to supply a sufficiently large non-trivial initial disturbance. We consider below the boundary value problem with the initial condition

$$T(0,x)=T_0(x)\geqslant 0,\qquad x=(r,z)\in\bar{\Omega},\quad \sup T_0<+\infty, \tag{27}$$

and find the conditions under which the solution is unbounded.

Theorem 4. Let $\alpha\in(1,3)$ and let $t_*>0$ exist such that

$$T_0(x)>t_*^{-1/[2(\alpha-1)]}C_0\,[r^2t_*^{-1}+(zt_*^{-1/2}+a_0)^2]^{-1/(\alpha-1)} \text{ in } \bar{\Omega}, \tag{28}$$

where $a_0=2[(3-\alpha)/(\alpha-1)]^{1/2}$, $C_0=[16(3-\alpha)(\alpha-1)^{-2}]^{1/[2(\alpha-1)]}$. A solution of problem (1), (2), (3), (27) then exists in a finite time interval, and for some $t_0\leqslant t_*$ we have

$$\lim_{t\to t_0^-}T(t,0,0)=+\infty. \tag{29}$$

Proof. We will show that, under our assumptions, the problem has an unbounded lower solution

$$T_-(t,x)=(t_*-t)^{-1/[2(\alpha-1)]}f(\xi,\eta),$$
$$\xi=r(t_*-t)^{-1/2},\qquad \eta=z(t_*-t)^{-1/2},$$

where

$$f(\xi,\eta)=C[\xi^2+(\eta+a)^2]^{-1/(\alpha-1)},\tag{30}$$

and C and $a>0$ are constants which are found below. The function T_- is certainly a lower solution if

$$(T_-)_t\leqslant\Delta T_- \text{ in } (0,t_*)\times\Omega,\qquad -\partial T_-/\partial z\leqslant T_-^{\alpha} \text{ in } (0,t_*)\times\partial\Omega,\tag{31}$$
$$T_-(0,x)\leqslant T_0(x) \text{ in } \bar\Omega,$$

or what amounts to the same thing,

$$\xi^{-1}(\xi f_\xi)_\xi+f_{\eta\eta}-(\xi f_\xi+\eta f_\eta)/2-f/[2(\alpha-1)]\geqslant0 \text{ in } \Omega,\tag{32}$$

$$-f_\eta(\xi,0)\leqslant f^\alpha(\xi,0),\qquad \xi\geqslant0\tag{33}$$

(condition (31) is the same as (28)). Substituting for f from (30) into (32) and (33), we arrive at the inequalities

$$\xi^2+\eta^2+4(3-\alpha)/(\alpha-1)-a^2\geqslant0,\qquad \xi\geqslant0,\qquad \eta\geqslant0,$$
$$C^{\alpha-1}\geqslant2a/(\alpha-1).$$

Hence follow the bounds on the values of the parameters a and C:

$$a^2\leqslant4(3-\alpha)/(\alpha-1)=a_0^2,\qquad C^{\alpha-1}\geqslant2a/(\alpha-1).$$

In particular, these bounds are satisfied with the constants a_0 and C_0 indicated in the theorem.

With this choice of a_0 and C_0, therefore, $T_-(t,x)$ is a lower solution, so that $T\geqslant T_-$ in $\bar\Omega$ for all admissible $t>0$. Hence (29) follows at once.

3. *On the asymptotic stage of combustion.* It is interesting to see how the temperature field evolves after the initiation of combustion.

It is natural to assume that, at the stage of developed combustion, the energy liberation of the reaction greatly exceeds the radiation energy. At this stage, therefore, the temperature field must be regularly described by the simplified problem

$$T_t=\Delta T,\qquad t>0,\qquad x\in\Omega,\qquad -\frac{\partial T}{\partial z}\bigg|_{\partial\Omega}=T^{\alpha}.\tag{34}$$

As distinct from the initial problem, the boundary condition here does not contain a term describing the absorption of radiation.

Problem (34) admits of the construction of similarity solutions which evolve into a mode with peaking:

$$T_A(t,x)=(t_0-t)^{-1/[2(\alpha-1)]}f(\xi,\eta),\tag{35}$$
$$\xi=r(t_0-t)^{-1/2},\qquad \eta=z(t_0-t)^{-1/2},$$

where $f>0$ satisfies the non-linear elliptic problem (see (32))

$$\Delta f-(\xi f_\xi+\eta f_\eta)/2-[2(\alpha-1)]^{-1}f=0 \text{ in } \Omega,\tag{36a}$$

$$-f_\eta(\xi,0)=f^\alpha(\xi,0),\qquad \xi\geqslant0.\tag{36b}$$

This asymptotic behaviour is supported by the fact that, with $\alpha\in(1,3)$, we can construct unbounded lower solutions with the same space-time structure.

The amplitude of solution (35) increases without limit: $T_A(t,0,0)=f(0,0)(t_0-t)^{-1/[2(\alpha-1)]}\to+\infty$, $t\to t_0^-$. The effective width of the resulting thermal structure falls in each direction:

$$\|x_{\text{eff}}\|(t)\sim(t_0-t)^{1/2}\to0,\qquad t\to t_0^-,$$

and we can expect moreover that $T_A(t,x)\to+\infty$, $t\to t_0^-$ at only the one point $x=0$. Thus virtually all the energy liberated at this stage of the combustion is localized in the neighbourhood of this point.

With respect to its space-time structure, T_A is a typical localized similarity solution of the LS mode with peaking, see /11/. A reaction energy liberation localization effect was earlier found in another non-linear laser heating problem /12, 13/, where the mode of localization with peaking arose as an intermediate asymptotic form of the temperature field evolution and was caused by the variation of the optical characteristics of the material during the reaction.

It must be said in conclusion that a rigorous proof of the asymptotic form (35) remains an open question. Great efforts, see e.g., /14, 15/, are needed to prove the asymptotic stability of similarity solutions with a point of time singularity, even in the case of much simpler parabolic non-linear problems. It also remains an open question whether there is a non-trivial solution of the elliptic problem (36), which is more complicated than that considered in Sect.2.

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THE APPROXIMATION OF THE BOLTZMANN EQUATION BY STOCHASTIC EQUATIONS*

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An algorithm for approximating the solution of the spatially non-uniform Boltzmann equation by solutions of a system of stochastic differential equations is proposed.

1. Suppose $\mathbb{R}^3$ is a three-dimensional Euclidean space, $(,)$ is the scalar product in $\mathbb{R}^3$, and $S^{(2)}$ is the unit sphere in $\mathbb{R}^3$. We will define the following functions:

$$\psi:\mathbb{R}^3\times\mathbb{R}^3\times S^{(2)}\to\mathbb{R}^3,\quad \psi(\xi,\eta,\omega)=(\eta-\xi,\omega)\omega,$$
$$\mathrm{cl}:\mathbb{R}^3\times\mathbb{R}^3\times S^{(2)}\to\mathbb{R}^3,\quad \mathrm{cl}(\xi,\eta,\omega)=\xi+\psi(\xi,\eta,\omega),$$
$$\xi,\eta\in\mathbb{R}^3,\quad \omega\in S^{(2)}.$$

The Boltzmann equation has the form

$$D_tf+(\xi,D_xf)=\int[f(x,\mathrm{cl}(\xi,\eta,\omega),t)f(x,\mathrm{cl}(\eta,\xi,\omega),t)-f(x,\xi,t)f(x,\eta,t)]h(\xi,\eta,\omega)\,d\eta\,d\omega. \tag{1}$$

Here t is the time, x is the space coordinate, ξ is the velocity coordinate, D is the differentiation operator, and f is the distritubion function of the number of particles of the gas. The region of variation of the variables is such that

$$t\in\mathbb{R}_+^1,\quad x\in\mathbb{R}^3,\quad \xi,\eta\in\mathbb{R}^3,\quad \omega\in S^{(2)},\quad d\omega=(4\pi)^{-1}\sin\theta\,d\theta\,d\varphi,\quad 0\le\theta<\pi,\quad 0\le\varphi<2\pi.$$

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