$$\left(\frac{1}{e_1}-\frac{1}{e_2}\right)\mathrm{tg}\,\alpha\,\frac{\partial u^{2N}}{\partial z}\bigg|_{x=z\,\mathrm{tg}\,\alpha}\,F(z\,\mathrm{tg}\,\alpha),$$

$$F=\Psi_m,\Phi_p,\qquad m=0,1,\dots,N,\qquad p=1,2,\dots,N-1.$$

All the remaining matrices of the system can be evaluated with an accuracy of $O(\delta)$, by putting $\delta=0$.

In view of the stability of the boundary value problem /4/, for small values of δ we can use the solution of problem (16) and (17) when $\delta=0$.

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Translated by R.C.G.

A FINITE-DIFFERENCE METHOD FOR SOLVING THE EQUATIONS OF GAS DYNAMICS USING ADAPTIVE NETS WHICH ARE DYNAMICALLY ASSOCIATED WITH THE SOLUTION[*]

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A finite-difference method for solving non-stationary, spatially one-dimensional problems of gas dynamics is proposed. The method is based on the use of grids which are adapted and obtained by a coordinate transformation which, in its turn, is determined by the required solution. An efficient numerical algorithm has been constructed on the basis of the proposed method which enables the accuracy of the calculations to be increased considerably with a simultaneous reduction in the total number of mesh points in the net by a factor of 2-5 compared with other methods.

Introduction.

The correct choice of the computational net in problems of mathematical physics is always the most important part of a numerical calculation. This choice is particularly important in problems of gas dynamics, the solutions of which differ in a wide variety of features.

Two approaches, the Lagrangian and the Euler /1/, have been used from the outset in the development of methods for the numerical solution of the equations of gas dynamics.

In the Lagrangian methods /2, 3/, the cells of the computational net are moved together with the fluid and the velocity of motion of the mesh points is therefore determined by the velocity of the hydrodynamic flow. This method of adaptation has been found to be convenient in the case of relatively smooth flows in which there are no large deformations. Its application in problems with free surfaces and surfaces of separation is the most effective. When there are significant deformations of the flow, there is a strong distortion of the cells of the mesh which leads to a reduction in the accuracy of the calculations and a reduction in the step size for the integration. When the amplitudes of the perturbing forces are too large, this may also give rise to the so-called "reversal" of some of the cells which leads to a loss of physical meaning in the results of the calculations. There are a number of mechanisms for the regularization of computational nets of the Lagrangian type (see /4/ and /5/, for example) with the aid of which it is possible to extend the domain of these methods successfully. However, it is not possible to remove the drawbacks of the Lagrangian approach completely.

In numerical methods based on the Euler approach /1, 6/, the mesh points of the computational net are fixed and do not change during the calculations, which enables one to treat flows with strong deformations. However, in problems where the behaviour of the

solution is not previously known, the arrangement of the mesh points may turn out to be less than optimal which makes it necessary to use nets with an exceedingly fine mesh size. This drawback can be partially compensated for by a preliminary non-uniform distribution of the coordinate lines, if certain a priori information is available concerning the behaviour of the solution. However, the structure of the non-uniform meshes in problems with a complex geometry for the solution domain is not a simple problem. A number of methods have been proposed for constructing the simplest nets which solely take account of the shape of the boundaries of the domain and do not change during their motion. These are either based on a geometric approach /7/ or make use of the concept of the automatic transformation of the coordinates /8, 9/.

Both approaches have been used to develop methods for constructing nets which take account of the motion of the boundaries of the computed region. Differentiation of the elliptic system of equations with respect to time is proposed for this purpose in methods using a transformation of the coordinates /10/. In methods based on the geometric approach /11/, moving nets, which are closely connected with the motion of the boundary or with one of the special features of the solution, are introduced. In particular, the separation of a bow shock wave has been achieved with their help in /11 and /12/.

Attempts to combine the advantage of the two approaches led to the development of mixed Euler-Lagrange /13-17/ and quasi-Lagrangian /18-20/ methods.

Attempts to make use of all the advantages possessed by the moving net, Lagrangian, Euler and mixed Euler-Lagrangian approaches has led in the final analysis to the development and use of adaptive net methods. These nets are dynamically linked to the solution (see the reviews /21-23/).

At the present time, methods for constructing adaptive nets are under intensive development and it is therefore still quite impossible to state a preference for any of them. The choice of the characteristic of the solution which is used as the parameter which controls the motion of the mesh points of the net is the key problem in all of the methods for constructing adaptive nets which are dynamically linked to the solution. The accuracy of the solution also depends on how the difference scheme and the method of constructing the difference net are associated. Taking account of these requirements a number of empirical approaches to the construction of difference nets with a controlled distribution of the mesh points has been proposed by various authors.

In moving finite-element methods /24, 25/, the system of difference equations obtained using the Galerkin projection method is used for these purposes. At the same time, the opinion has been expressed in /25/ that moving finite-element methods have an indisputable advantage over finite-difference methods which use adaptive nets.

Methods for constructing adaptive nets in finite-difference methods /26-34/ differ in an even greater variety of ways. Methods of constructing adaptive nets based on variational approaches /26, 27/, an equidistant distribution of the dependent variables /28/, by taking account of the error in the approximation /29, 30/ and the gradients of the numerical solution /31/, etc. (see /32/, for example) have become widely used in finite-difference methods.

The fact that the processes involved in determining the solution and the motion of the mesh points in them are treated as if they were separate and realized autonomously should be added to the drawbacks of the approaches which have been enumerated above. Furthermore, the reciprocal link between the difference scheme and the method of adjusting the net which are used is specified empirically and, as a rule, extremely crudely. It appears to us that it is, in fact, for these reasons that oscillations of the net or the linked vibrations of the solution and the net, which have been repeatedly pointed out in different papers, arise frequently when solving problems.

It is also not completely rational to determine the solutions in a domain where there are singularities by concentrating the mesh points in it. For example, the isolation of a shock wave using this approach /32/ requires about 10^2 mesh points.

A method for solving non-stationary one-dimensional problems of gas dynamics on an adaptive net, dynamically linked with the solution, which is free from the above-mentioned drawbacks, is proposed in the present paper. The approach which is described in an extension to gas-dynamic problems of the method for solving boundary value heat conduction problems /33/ and problems of the Stefan type /34/ on adaptive nets.

1. Differential formulation of the problem.

We shall define the finding of the numerical solution of the equations of gas dynamics with the aid of an adaptive net which is dynamically linked with the solution as a method of solution in which the finding of the net functions and the coordinates of the mesh points are continuously linked. The main differences between the proposed approach and those which have been previously considered are as follows.

1. The adaptive net is constructed using a corresponding coordinate transformation. The actual form of the transformation is specified in terms of a certain function Q, the form of which is determined by the singularities of the solution of the problem under investigation.

2. A close reciprocal link between the required solution and the method of adjusting the computational net is introduced at the level of the differential model which, in the general case, is a non-linear system of partial differential equations. In this system some of the equations directly describe the phenomenon under investigation while the others describe the dynamics of the computational net which are determined, in their turn, by the evolution of the solution. In the limiting cases of these equations, nets either in Eulerian or Lagrangian variables are obtained from these equations.

We note that the actual form of the link between the equations of gas dynamics and the

equation for the adjustment of the net is specified by analogy with the quasi-Lagrange approach /18/.

We will use the Eulerian system of coordinates (x, t) as the initial system of coordinates in which the mathematical formulation of the phenomenon under investigation is carried out. Then, with the help of a transformation of the general form $x=f(q, t)$, we make the transition from physical space to the computational space. The system of non-stationary equations of gas dynamics in the unidimensional approximation in Eulerian variables has the form

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = -\rho \frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \tag{1.1a}$$

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{u^2}{2}\right) + u \frac{\partial}{\partial x}\left(\varepsilon + \frac{u^2}{2}\right) = -\frac{\partial}{\partial x}(pu). \tag{1.1b}$$

In the new variables q and t, the system of Eqs.(1.1) is written in the form

$$\frac{\partial \psi}{\partial t} = -\frac{\partial Q}{\partial q}, \tag{1.2}$$

$$\frac{\partial}{\partial t}(\psi u) = -\frac{\partial p}{\partial q} - \frac{\partial}{\partial q}(Qu), \tag{1.3a}$$

$$\frac{\partial}{\partial t}\left[\psi\left(\varepsilon + \frac{u^2}{2}\right)\right] = -\frac{\partial}{\partial q}(pu) - \frac{\partial}{\partial q}\left[Q\left(\varepsilon + \frac{u^2}{2}\right)\right], \tag{1.3b}$$

$$\psi = \rho \partial x/\partial q, \tag{1.4}$$

$$Q = \rho(u - \partial x/\partial t). \tag{1.5}$$

The actual form of the function Q is determined by the rule selected for the coordinate transformation, that is, by the form of the function $f(q, t)$. We note that, in the Lagrangian, quasi-Lagrangian and mixed Euler-Lagrange approaches, the function $f(q, t)$ is assumed to be known. In the construction of an adaptive net which is dynamically linked with the solution, it is necessary that the rule governing the coordinate transformation (the function $f(q, t)$) should be determined by the required solution. On account of this, it is necessary to specify the function Q for the complete definition of system (1.2)-(1.5).

Expressions (1.2)-(1.3) are the equations of continuity for the motion and the energy, and (1.4) is the equation for the link between the variables x and q. Eq.(1.5) can be written in the form

$$\partial x/\partial t = u - Q/\rho, \tag{1.6}$$

where u is the velocity, ρ is the density of the gas and Q is an arbitrary function of the solution. As is shown below, a considerable improvement in the solution of the difference equations can be achieved by selecting the function Q in the form

$$Q = \pm(\gamma p\rho)^{1/2}, \tag{1.7}$$

where p is the pressure and γ is the constant ratio of the specific heat capacities $(\gamma > 1)$. The choice of the function Q in the form (1.7) means that each mesh point of the difference net either moves along the C_+-characteristic or along the C_--characteristic since, in the case of (1.7), Eq.(1.6) takes the form

$$\partial x/\partial t = u \pm c, \quad c = (\gamma p/\rho)^{1/2},$$

where c is the velocity of sound. In a number of cases, such as, for example, when describing the motion of a contact discontinuity, it is convenient to select the function Q in the form $Q = 0$. It is obvious that, in this case, the variable q is identical to the Lagrangian mass variable $m = \int \rho dx$. Moreover, when the function Q is chosen to be of the form $Q = \rho u$, we arrive at the Euler variables since, in this case, $\partial x/\partial t = 0$.

Eqs.(1.4) and (1.5) can be used to determine the Euler variable $x(q, t)$ and the change in the density ρ. However, from the computational aspect, it is more convenient to take another equation instead of (1.5). In order to obtain this equation, we differentiate the left- and right-hand sides of (1.5) with respect to the spatial variable q and, by using (1.4), we arrive at the equation for determining the change in ρ:

$$\frac{\partial}{\partial t}\left(\frac{\psi}{\rho}\right) = \frac{\partial u}{\partial q} - \frac{\partial}{\partial q}\left(\frac{Q}{\rho}\right). \tag{1.8}$$

The system of gas-dynamic equations in the variables q and t finally has the form (1.2)-(1.4), (1.8).

2. Selection of the function Q.

When constructing the function Q, it is necessary to take account of the singularities of the solution of the problem under consideration. For instance, the construction of an adaptive net for solving the Stefan problem /33/ with isolation of the position of the phase

boundary, in the simplest case requires a uniform distribution of the mesh points in each phase subregion. As a rule, the sizes of the spatial steps in these subregions are quite different and depend on the velocity at which the phase front moves. A mesh point distribution which is almost uniform is attained at each instant of time in the two subregions (with an invariant overall amount over the whole range of integration with respect to t) by specifying the function Q in the form of a diffusion flow $Q = -D\partial\psi/\partial q$, where D is a coefficient, the value of which is selected during the calculation.

A class of problems exists, the solution of which within the domain or on its boundary is characterized by the occurrence of large gradients which, if the required accuracy of the numerical solution is to be attained, necessitate the use of a finer mesh than is used within the remainder of the domain. Automatic compression of the mesh can be achieved by a concentration of the mesh points in it with the aid of the function Q, specified in the form

$$Q = -Q_0 \frac{\partial}{\partial q}\left(\psi\left|\frac{\partial T}{\partial q}\right|\right).$$

The removal of mesh points from a region where the solution only changes slightly can lead to a very irregular net in this region. In order to avoid this, it is necessary to use a smoothing mechanism and, for this purpose, the function Q, in problems with large gradients, has been specified in the form of the combination /33/:

$$Q = -D\frac{\partial\psi}{\partial q} - Q_0 \frac{\partial}{\partial q}\left(\psi\left|\frac{\partial T}{\partial q}\right|\right).$$

In problems which describe complex processes, the form of the function Q must, obviously, be determined by the predominant process. We note that, in evolutionary problems, the form of the equation used for readjusting the net must always be of the evolutionary type.

Gas-dynamic problems are distinguished by a wide variety of solutions and it is therefore difficult to indicate a universal form of the function Q which would satisfy all problems. We will now consider some methods of specifying the function Q in problems of gas dynamics using the solution of a number of actual problems as an example.

3. The difference scheme.

In the computed space $\Omega_{q,t}$, we introduce a computational net ω with a constant step size h with respect to the variable q and a step size τ with respect to the variable t:

$$\omega = \{(q_i, t^j), (q_{i+\frac{1}{2}}, t^j), q_{i+1} = q_i + h, q_{i+\frac{1}{2}} = q_i + 0.5h,$$

$$t^{j+1} = t^j + \tau, \quad i = 0, 1, \ldots, N-1, \quad j = 0, 1 \ldots\}.$$

On this net, the system of Eqs.(1.2)-(1.4), (1.8) is approximated by a family of difference schemes in which the functions ψ, ρ, u, p, and ε are calculated at the half-integral points $(q_{i+\frac{1}{2}}, t^j)$ while the values of x and Q are calculated at the integral points (q_i, t^j). The family of difference schemes has the form

$$\frac{\psi_{i+\frac{1}{2}}^{j+1} - \psi_{i+\frac{1}{2}}^{j}}{\tau} = -\frac{Q_{i+1}^{(\sigma)} - Q_i^{(\sigma)}}{h}, \tag{3.1a}$$

$$\frac{(\psi/\rho)_{i+\frac{1}{2}}^{j+1} - (\psi/\rho)_{i+\frac{1}{2}}^{j}}{\tau} = \frac{u_{i+1}^{(\sigma)} - u_i^{(\sigma)}}{h} - \frac{(Q/\rho)_{i+1}^{(\sigma)} - (Q/\rho)_i^{(\sigma)}}{h}, \tag{3.1b}$$

$$\frac{[\psi(\varepsilon + u^2/2)]_{i+\frac{1}{2}}^{j+1} - [\psi(\varepsilon + u^2/2)]_{i+\frac{1}{2}}^{j}}{\tau} = -\frac{(pu)_{i+1}^{(\sigma)} - (pu)_i^{(\sigma)}}{h} - \tag{3.1c}$$

$$\frac{[Q(\varepsilon + u^2/2)]_{i+1}^{(\sigma)} - [Q(\varepsilon + u^2/2)]_i^{(\sigma)}}{h},$$

$$(x_{i+1}^{j+1} - x_i^{j+1})/h = (\psi/\rho)_{i+\frac{1}{2}}^{j+1}. \tag{3.1d}$$

The values of the functions ψ, ρ, u, p, and ε at the integral mesh points were calculated using the interpolation formulae

$$y_i = \frac{\psi_{i+\frac{1}{2}} y_{i-\frac{1}{2}} + \psi_{i-\frac{1}{2}} y_{i+\frac{1}{2}}}{\psi_{i-\frac{1}{2}} + \psi_{i+\frac{1}{2}}}.$$

When $\sigma = 0$ and 1, the difference scheme (3.1) is of the order of approximation of $O(\tau + h^2)$ and, when $\sigma = 0.5$, the order of approximations of $O(\tau^2 + h^2)$. In the following calculations schemes, Eqs.(3.1) with $\sigma = 0.5$ and 1 were employed. The implicit schemes were solved by the method of simple iterations.

Let us now consider the solution of several non-stationary, spatially one-dimensional problems of gas dynamics. The choice of these problems was dictated by an attempt to demonstrate the possibilities of the proposed method of solution as fully as possible.

4. The problem of the formation of a shock wave.

Let us consider the problem of the formation and propagation of a shock wave using the

example of the motion of an accelerating piston. At the instant of time $t=0$, the piston occupies the position $x=0$ and then moves according to the law $x=at^2$, $a>0$ in a tube filled with a quiescent polytropic gas. During the development of the compression wave, the gradients of the solution in this zone increase and, at a certain critical instant of time, $t=t_c$, become infinite. A so-called gradient catostrophy occurs, that is, a singularity occurs in the solution. The instant of the onset of the gradient catastrophy can be determined using the formula /35/

$$t_\kappa = u_c^0/[(\gamma+1)a],$$

where u_c^0 is the velocity of sound in the quiescent gas and a is the constant in the law of motion of the piston. For $\rho_0=1$, $p_0=1$, and $a=1$, the time $t_c=0.484$, and $\gamma=\frac{5}{3}$.

Up to the instant of the onset of the gradient catastrophy, the flow is a simple com-pression wave with a weak discontinuity at the point where it joins with the constant flow. An attempt has been made to determine the instant when the discontinuity forms from the numerical solution of the system of finite difference Eqs.(3.1) with $\sigma=0.5$ and 1 on a fixed Lagrangian net $(Q=0)$ with a total number of mesh points $N=60$. Spatial profiles of the velocity u at different instants of time obtained using the implicit difference scheme $(\sigma=1)$ with first-order accuracy (Fig.1) and the scheme with second-order accuracy $(\sigma=0.5$ (Fig.2) are shown in Figs.1 and 2. The numerical solution is shown by means of broken lines and the exact solution by the solid lines. The positions of the mesh points are indicated by the small circles. The results obtained suggest that it is practically impossible to determine the instant of time at which the shock wave is formed from numerical calculations on a fixed mesh. The solutions obtained using the first order scheme (Fig.1) do not reproduce the steep fronts on account of the large approximation viscosity of the scheme. The second-order scheme yields oscillations in regions where there are large gradients (Fig.2).
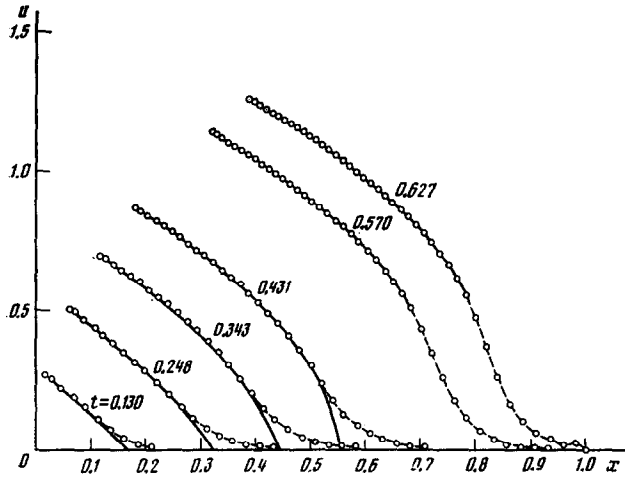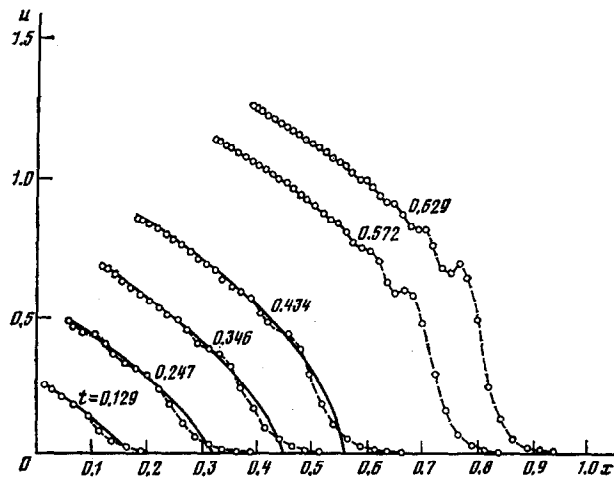

Fig.1

Fig.2

The qualitative effect of the approximation viscosity on the solutions which are obtained was investigated using the method of differential approximation /36/. As an example, let us take the first differential approximation for the difference equation of motion from system (3.1) with $\sigma=0.5$:

$$\frac{\partial}{\partial t}(\psi u)+\frac{\partial p}{\partial q}+\frac{\partial}{\partial q}(Qu)=-\frac{1}{3!}\left[\frac{\tau^2}{4}\frac{\partial^3(p+Qu)}{\partial t^2\,\partial q}+\frac{h^2}{8}\frac{\partial^3(p+Qu)}{\partial q^3}\right]. \tag{4.1}$$
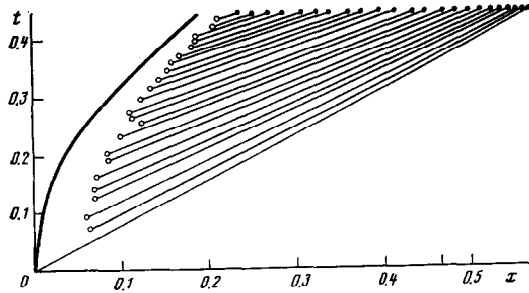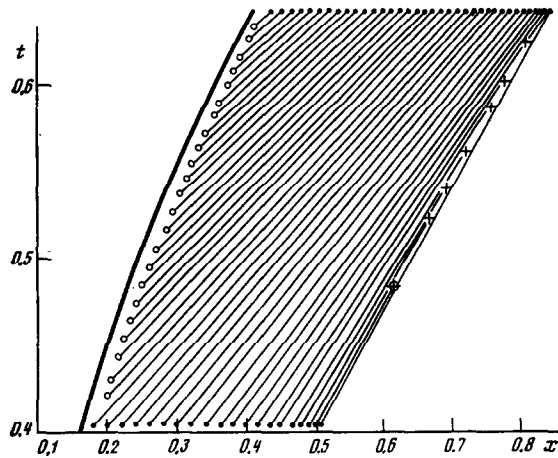
Fig.3.   (x-t)-diagram

Fig.4

We note the following: the oscillatory nature of the behaviour of u when $Q=0$ is obviously determined by the first term on the right-hand side of (4.1). Both of the terms on the right-hand side are functions of the derivatives of the quantities $p+Qu$ with respect to the spatial coordinate q. It is therefore desirable, when solving a problem using an adaptive net for the mesh points, to specify a law of motion such that the derivatives of the quantity $p+Qu$, close to large gradients in the solution, are small. It is well-known /37/ that, in the solutions of gas dynamic problems which are represented in the form of a simple wave, all the quantities are functions of just a single quantity which keeps a constant value on the characteristics. It is natural to select the function Q such that the coordinates of the mesh points in the physical space should move along the corresponding family of characteristics. In the problem under consideration, this must be the family of C_+-characteristics, the equation of which is written in the form

$$dx/dt=u+u_c,\tag{4.2}$$

where u_c is the velocity of sound. A comparison of (4.2) with Eq.(1.5) showed that the Eulerian coordinate of a meshpoint x moves along a C_+-characteristic, if

$$Q=-\rho u_c\tag{4.3}$$

is chosen.

The instant when the compression wave becomes a shock wave can be determined by the following considerations. It is known from the method of characteristics /37/ that a gradient catastrophy in a simple compression wave implies the intersection of two characteristics of the corresponding family. In the problem under consideration, when the flow Q is selected in the form (4.3), each mesh point of the net must move along a C_+-characteristic. The position and the instant of formation will be determined when the quantity ψ vanishes in the cell between the two intersecting characteristics. The calculations confirmed the hypotheses which have been put forward. The results of these calculations are presented in Figs.3-7.

There are a number of important aspects associated with the formation of a shock wave and its subsequent propagation.

1. It is known /35/ that a gradient catastrophy occurs near the base of a compression wave when the characteristics interest. One of these characteristics emerges from the piston at the instant of time $t=0$, that is, from the origin of coordinates. When a net which is uniform with respect to q and the initial values $\psi_i=1$ are used, such a characteristic does not occur in the calculations and this may lead to an appreciable error in determining the value of t_c. In our calculations, when a uniform mesh was used to approximate the characteristic emerging from the piston, the initial value of ψ_i in the first cell was chosen to be

much small than in the remaining cell: $\psi_i=10^{-4}$, $\psi_i=1$, $i=2, 3, \ldots, N-1$. Satisfaction of the inequality $\psi_i \leqslant 0.1$ in a certain cell l was considered to be the condition for the formation of a shock wave. In our calculations, using the selected values of u_c, a and γ, it was satisfied at the instant of time $t=0.475$, which is close to the calculated $t_c =0.484$, $\Delta t \approx 0.02 t_c$.

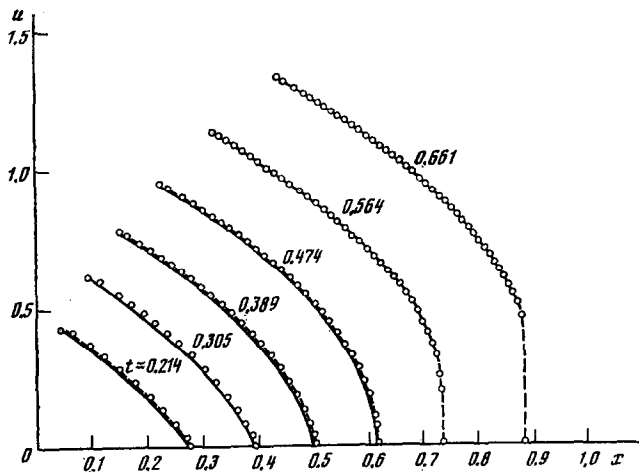

Fig.5
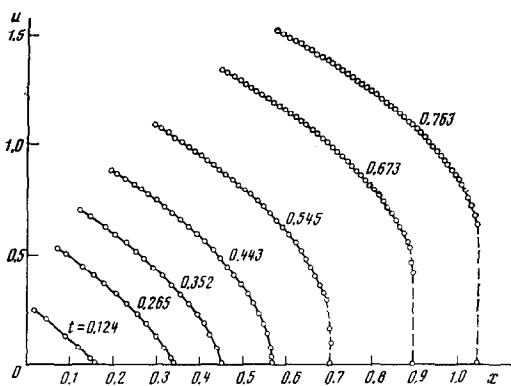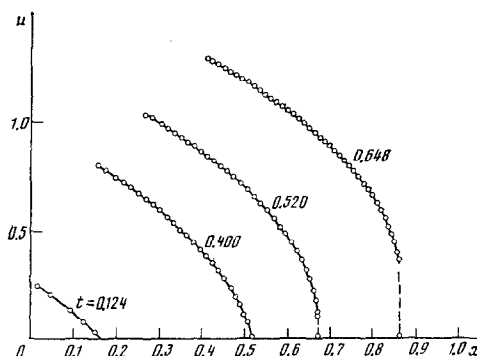

Fig.6

Fig.7

2. The boundary conditions

$$Q(q_0, t) = Q(q_R, t) = 0.$$

must be satisfied on the boundaries $q=q_0$ and $q=q_R$ of the computed region $q_0 \leqslant q \leqslant q_R$.

Hence, when the mass overcurrent is chosen in the form of (4.3), the extreme left-hand cell will increase without limit while the extreme right-hand cell collapses to a point after a finite time. In order to prevent the above-mentioned phenomena from occurring, a mechanism for generating new cells close to the piston and annihilating the fine cells on the right-hand end of the region was incorporated in the numerical algorithm. The value of the quantity ψ for the corresponding cell served as the criterion for annihilation or creation. In the new cells, the values of p and u were determined by interpolation. The density ρ was found from the requirement that the entropy should be conserved, and the function ψ was determined from the law of conservation of mass. Trajectories of the motion of the mesh points of the net are shown in Figs.3 and 4. The mesh points used in the calculations are indicated by dots, the generated mesh points are indicated by small circles while those which are annihilated are indicated by small crosses. The instant when the shock wave is formed is indicated by a small circle with a cross inside it (Fig.4). The initial net consisted of the minimum number of cells, equal to two. The initial values of the function ψ for these cells differed by several orders of magnitude. At the end of the calculations the number of mesh points had reached 40 (Fig.4).

3. After a strong discontinuity has been formed, the difficult question concerning the method of calculating the propagation of a non-stationary shock wave arises. There are two possibilities for picking out domains where there are singularities in the solution in calculations in adaptive nets. The space-time profile of a discontinuity can be reproduced with a sufficient degree of accuracy by concentrating a large number of mesh points in this region and, at the same time, not allowing any cells to collapse. However, this route is not entirely rational for two reasons. Firstly, a large overall number of mesh points is required and, secondly, excessive densification of the net with respect to the spatial variable may

bring about a substantial reduction in the integration step size with respect to time. As a result, the question arises of the efficiency of the method.

Another route is more promising when the singularities in the solution are isolated in an explicit form according to the type of solution of the Stefan problem in /34/ in which the discontinuity is located on a mesh point of the net. The boundary conditions in the form of conservation laws are written out on the discontinuity and one of these conservation laws allows one to establish the mass flux through the discontinuity. The value of this flux is then used in order to construct the net which is adapted.

The isolation of a strong discontinuity means that it is located at a certain mesh point $i=l$, at which the values of $\rho_l, u_l, p_l,$ and ε_l are different to the left and right of this point. It is known that the Hugoniot relationships must be satisfied on this discontinuity. In the computational space they are written in a form which excludes the appearance of difference sources:

$$u_2-Q/\rho_2=u_1-Q/\rho_1, \qquad p_2+Qu_2=p_1+Qu_1,$$

$$p_2u_2+Q(\varepsilon_2+u_2{}^2/2)=p_1u_1+Q(\varepsilon_1+u_1{}^2/2).$$

Here, the indices 1 and 2 denote values in front of and behind the shock wave respectively.

The numerical calculations were found to be stable when the values from the region of constant flow $\rho_1, u_1,$ and ε_1 and one of the values, u_2, for example, from the region beind the shock wave front were extrapolated onto the surface of the discontinuity. After the formation of a shock wave where a fixed mesh point of the computational net corresponds to the position of its front, the cells to the right and the left of it were annihilated using the technique which has been described above.

4. We will now consider the question of the link between the choice of Q and the difference scheme. It is known /1-3/ that implicit schemes of the first order of accuracy on fixed nets greatly "blurs" weak discontinuities in a gas-dynamic flow. As a rule, second-order schemes lead to the appearance of parasitic oscillations in the solution.

The use of adaptive nets, which are dynamically related to the solution, substantially changes these qualities of difference schemes. Calculations of first-order accuracy using the scheme (3.1) $(\sigma=1,$ Fig.5) and schemes of second-order accuracy $(\sigma=0.5,$ Fig.6) showed that the blurring and oscillatory effects practically disappeared in both cases. The difference between the numerical solutions and the exact solutions did not exceed 1-2%. The exact solution is shown by the solid lines, the numerical solution is shown by the broken lines and the positions and number of mesh points used are indicated by the small circles. A comparison showed that the two schemes yield slightly different results. The instant at which the strong discontinuity arises was determined with an identical accuracy in the two cases. Both schemes clearly reproduce the discontinuity region when not more than 40 mesh points are used.

In order to ensure that all of this is also characteristic of other difference schemes, one of the widely used predictor-corrector type schemes, the Lax-Wendroff scheme /1/ was employed for the purpose of comparing the calculations. The results obtained using this scheme are shown in Fig.7. They hardly differ from the results obtained using (3.1) either qualitatively or quantitatively.

It may therefore be concluded that the use of the method based on the employment of adaptive nets for solving problems in gas dynamics reduces the requirements regarding the difference schemes.

Other qualities of the proposed method were investigated using the example of problems which admit of selfmodelling solutions.

5. The problem of the propagation of a shock wave on a background with an exponentially increasing density.

As an example of this problem /38/ we shall carry out an additional experimental investigation of the accuracy of the numerical solution obtained on a net constructed with the help of the mechanisms for the generation and adaptation of the meshpoints described in Sect. 4.

At the initial instant of time $t=t_0$, the values of the gas-dynamic functions have the form /16/

$$u(x,t_0)=\begin{cases} 0, & x\geqslant x_0, \\ \Delta(1-\xi_0)/t_0, & x\leqslant x_0, \end{cases}$$

$$\varepsilon(x,t_0)=\begin{cases} 0, & x\geqslant x_0, \\ 0.5(\Delta/t_0)^2(1+2\xi_0), & x\leqslant x_0, \end{cases}$$

$$\rho(x,t_0)=\begin{cases} \rho_0\exp[(x-x_0)/\Delta], & x\geqslant x_0, \\ 3\rho_0(1+2\xi_0)^{-1/2}, & x\leqslant x_0. \end{cases}$$

Here, $\xi_0=(x_0-x)/\Delta, x_0$ is the position of the discontinuity at the instant t_0, ρ_0 is the unperturbed background density when $x=x_0$ and Δ is the characteristic scale of the pulse width.

The selfmodelling solution when $t>t_0$ has the form

$$u(x,t)=\begin{cases} 0, & x\geqslant x_F(x), \\ \Delta(1-\xi)/t, & x\leqslant x_F(t), \end{cases}$$

$$\varepsilon(x,t) = \begin{cases} 0, & x \geqslant x_F(t), \\ 0.5(\Delta/t)^2(1+2\xi), & x \leqslant x_F(t), \end{cases}$$

$$\rho(x,t) = \begin{cases} \rho_0 \exp[(x-x_0)/\Delta], & x \geqslant x_F(t), \\ 3\rho(x_F)(1+2\xi)^{-1/2}, & x \leqslant x_F(t). \end{cases}$$

Here, $x_F(t) = x_0 + 1.5\Delta \ln(t/t_0)$ is the position of the front at the instant t and t, $\xi = [x_F(t)-x]/\Delta$. The numerical values of the parameters were as follows: $t_0 = 2$, $x_0 = 6$, $\rho_0 = 1$, $\Delta = 4$, $\gamma = 2$.
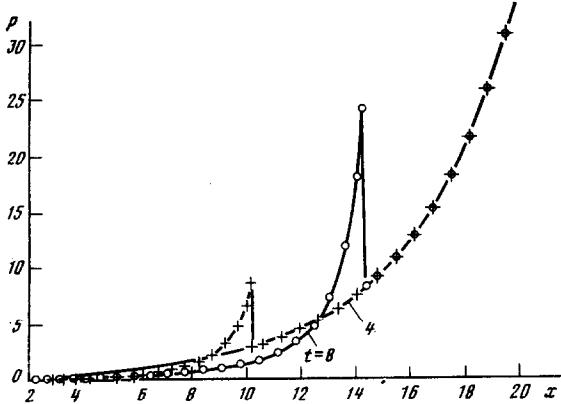

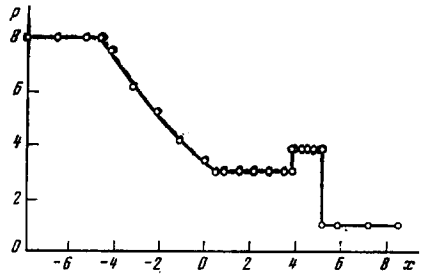Fig.8   Fig.9

Results of calculations using the above-mentioned technique with the creation and annihilation of cells both close to the boundaries of the region as well as in the neighbour-hood of the shock wave front are presented in Fig.8 for the first case: at the initial instant the net consisted of 31 mesh points and , in the region encompassed by the motion, there were 11 mesh points ($+$ is for $t=4$ and 0 is for $t=8$). In the second case the total number of mesh points at the initial instant was 61 and of these 41 lay in the domain $x \leqslant x_0$. The error in determining the density did not exceed 5% in both cases. For comparison, we note that nets with a total number of mesh points $N = 37-150$ were employed in analogous calculations in /16/.

6. The problem of the decay of a strong discontinuity.

Among the solutions of the system of gas-dynamic equations, some are encountered which either contain a shock wave (index s.w.) or a contact discontinuity (index c.d.) In the problem of the breakdown of a strong discontinuity, there are simultaneously two types of discontinuities which move with constant velocities. This problem has been widely employed in the analysis of the quality of difference schemes by different authors /16, 39, 40/.

Let us formulate the problem in the following form: when $t=0$, let us put $p_1 = 480$, $\rho_1 = 8$, $u_1 = 0$, $p_r = 1$, $\rho_r = 1$, $u_r = 0$, where values to the right and to the left of the discontinuity at the point $x = 0$ are denoted by "r" and "l". The problem being considered is a selfmodelling problem with respect to the variable $\xi = x/t$. The selfmodelling coordinates of the discontinuities have the following form: $\xi_l = -10$, $\xi_r = 1.0933$, $\xi_{c.d.} = 8.32$, $\xi_{s.w.} = 11.24$ ($\gamma = 5/3$).

The exact solution in regions I–IV is described as follows:

domain	I	II	III	IV	V
ρ	8	$(2-u/15)^3$	3,0193	3.85	1
u	0	$0.75(\xi+10)$	8.32	8.32	0
p	480	$15\rho^{5/3}$	94.52	94.52	1

In the numerical calculations, 5 cells were chosen in each of the regions I–V. In region II, the mesh points of the net were moved along the C_--characteristics, that is, the current Q was selected in the form $Q = \rho u_c$. In the regions of piecewise-constant solutions, a diffusion mechanism for pushing the mesh points apart: $Q = -D_0 \partial \psi / \partial q$ was chosen. Profiles of the density at the instant of time $t = 0.46$ are shown in Fig.9. The solid line is the exact solution and the small circles are the numerical solution. The number and position of the markers corresponds to the number and position of the mesh points of the net. The maximum error was observed in the region of the rarefaction wave and this was 0.5%.

Conclusion.

An analysis of the solutions of a number of well-known test problems (Sects.4-6) enables us to make the following assertions.

A finite-difference method has been proposed for solving non-stationary, spatially one-dimensional problems of gas dynamics. The method is based on the use of adaptive nets which are obtained by a coordinate transformation which is determined by the required solution.

An efficient computational algorithm has been constructed on the basis of the proposed method which enables one to increase the accuracy of the calculations considerably (the error with respect to the exact solution was 1-5% for the different problems) with a simultaneous decrease in the total number of mesh points of the net by a factor of two to five compared with other methods.

The method enables one to determine the site and the moment of formation of discontinuous

solutions with great accuracy.

Using this method, the fronts of strong, weak and contact discontinuities can be isolated almost precisely.

The method is economic and readily allows one to introduce a mechanism for generating mesh points.

The method reduces the requirements regarding the quality of the difference schemes. The adaptive nets used in this method, which are dynamically linked to the solution, lead to a reduction in the approximation viscosity and to the suppression of parasitic oscillations.

The efficiency of the proposed method can be increased by using different optimization procedures which have not been carried out in the present paper.

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Translated by E.L.S.