140. A. A. Samarskii, "Numerical simulation and nonlinear processes in dissipative media," in: Self-Organization, Autowaves, and Structures Far from Equilibrium, V. I. Krinsky (ed.), Springer Verlag, Berlin (1984), pp. 119-129.
141. G. I. Taylor, "The air wave surrounding an expanding sphere," Proc. R. Soc., A 186, No. 100 (1946).

A QUASILINEAR HEAT EQUATION WITH A SOURCE: PEAKING, LOCALIZATION, SYMMETRY EXACT SOLUTIONS, ASYMPTOTICS, STRUCTURES

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A survey is given of results of investigating unbounded solutions (regimes with peaking) of quasilinear parabolic equations of nonlinear heat conduction with a source. Principal attention is devoted to the investigation of the property of localization of regimes with peaking. A group classification of nonlinear equations of this type is carried out, properties of a broad set of invariant (self-similar) solutions are investigated, and special methods of investigating the space–time structure of unbounded solutions are developed.

INTRODUCTION

Processes of spontaneous violation of a high degree of symmetry of a macroscopic state of a complex system are one of the surprising phenomena of the world surrounding us.

These processes lead to the appearance of so-called dissipative structures – ordered formations with characteristic space–time forms. For the occurrence of processes of spontaneous violation of symmetry with reduction of its degree the system must necessarily be open and the mathematical model of it must be nonlinear [29, 64, 80].

At the present time phenomena of structure formation are the focus of attention of investigators in various specialities. These phenomena are of interest to biologists in connection with the question of the origin of life, problems of prebiological evolution, and morphogenesis [49, 71, 75, 80, 81], to ecologists from the viewpoint of recognizing the laws of formation and stable functioning of biogenesis [75], and to physicists and chemists in connection with the possibility of creating new devices and installations which are new in principle. The interest of technicians is caused by the possibility of raising the productivity of old technologies and creating new intensive technologies [77]. These phenomena attract philosphers as examples of the nontrivial occurrence of a category of "part and whole" and the dialectics of self-movement [78].

In spite of the different nature of the systems, on passage from an unordered state to an ordered state they behave in a similar manner which bears witness to the existence of fundamental principles of their functioning. Representatives of various disciplines are occupied with the study of these principles within the framework of the synergetic approach [59, 64, 79].

It is altogether natural that one of the most powerful tools of modern science – mathematical modeling by means of a computational experiment [72] – is used in studying structures in nonlinear media. A combination of traditional methods of mathematical physics, modern numerical methods, and methods of processing information makes it possible to analyze the phenomenon considered from all sides, accumulate information regarding it, and create new concepts and methods adequate to the qualitative features of nonlinear phenomenona. The concept of symmetry–asymmetry is of considerable use in creating the basic concepts and constructing mathematical models.

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In studying any phenomenon the investigator deals with a hierarchical sequence of models which, as a rule, is formed by successive consideration of various factors. In such a hierarchy of models it is always possible to trace the hierarchy of symmetry. The model of the lowest level of descriptive detail hereby has maximum symmetry.

The presence of a rich symmetry makes it possible to formulate "atomistic" concepts which are to a considerable extent adequate to the class of phenomena being studied. In essence almost all dissipative structures known at the present time are from a mathematical point of view invariant or partially invariant solutions of phenomenological nonlinear equations, that is, the most symmetric solutions. Thus, stationary dissipative structures are a special case of invariant solutions — stationary solutions [39]. Autowave structures to good accuracy can be represented by means of another special case of invariant solutions — so-called traveling waves [38, 49, 56]. Finally, the nonstationary dissipative structures of regimes with peaking considered in this work and in other works of the authors [32, 40, 41, 45, 59, 73, 74] are directly connected with power self-similar solutions. Symmetry—asymmetry is the deep property of the matter which can be used not only for formulating the basic concepts but can also be taken as the foundation for mathematical modeling. We have in mind the creation of a hierarchy of models on the basis of a hierarchy of symmetry.

The present work is devoted to the study of dissipative structures of regimes with peaking which are formed in an active, dissipative nonlinear medium. As a substantial mathematical model of minimal dimension we choose the quasilinear heat equation

$$u_t = \operatorname{div}(K(u)\operatorname{grad} u) + Q(u). \tag{1}$$

In this equation $u \geqslant 0$ is the temperature of the medium, $K(u) \geqslant 0$ is the nonlinear coefficient of thermal conductivity, and $Q(u) \geqslant 0$ is a nonlinear heat source. It is assumed that K(u) and Q(u) are defined and smooth for all u > 0 and vanish only in an absolutely cold medium [K(0) = 0, Q(0) = 0].

Solutions of the Cauchy problem for Eq. (1) for various types of the pair of coefficients {K(u), Q(u)} form the object of investigation.

Of special interest are K(u) and Q(u) satisfying the conditions

$$\int_0^1 K(u)\, u^{-1} du < +\infty \tag{2}$$

and

$$\int_1^\infty [Q(u)]^{-1} du < +\infty. \tag{3}$$

Condition (2) in the absence of a heat source in the medium [Q(u) ≡ 0] ensures in the case of a compactly supported initial distribution a wave regime of propagation of a thermal perturbation in an absolutely cold medium with finite speed of the front [43, 68] (see Fig. 1).

Condition (3) in the absence of heat conduction in the medium [K(u) ≡ 0] leads to the nonexistence of a global solution of the Cauchy problem. In this case heating of the medium occurs in a regime with peaking: after a bounded interval of time $t \in [0, \tau]$ in some region of space the temperature becomes infinite. An essential feature is the localization of the region of high temperatures in space for an inhomogeneous, bounded, initial thermal perturbation (see Fig. 2).

Study of the solution of the problem with simultaneous action of the dissipative factor [K(u) ≢ 0] and a volumetric heat source [Q(u) ≢ 0] is of interest.

Investigations of unbounded solutions (regimes with peaking) occupy a special place in the theory of nonlinear evolution equations. In the general theory nonlinear problems admitting unbounded solutions are globally (in time) unsolvable. For a long time they were considered exotic examples indicating the degree of optimality of those conditions which ensure global solvability.

Successful attempts to derive conditions for unboundedness of solutions of nonlinear parabolic problems were made more than 20 years ago [93, 97, 108]. The methods proposed in these works were very fruitful and were further developed.

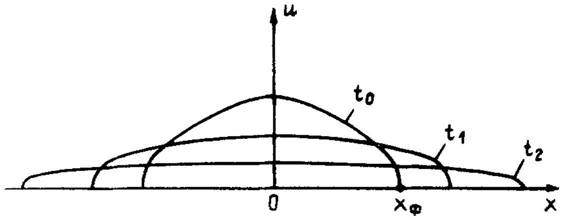


Fig. 1

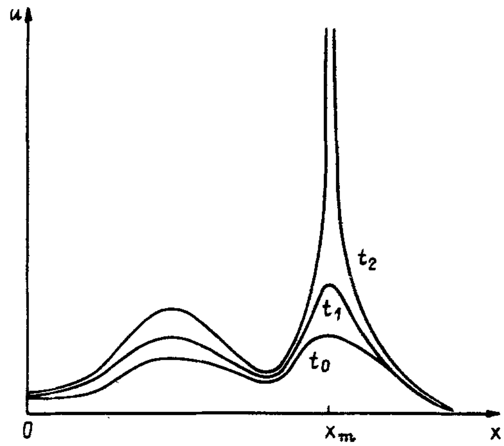


Fig. 2

Fig. 1. Evolution of an initial thermal perturbation in a medium without a source. The front of the heat wave propagates with finite speed.

Fig. 2. Evolution of an initial thermal perturbation in a medium without heat conduction. The temperature reaches an infinite value after a finite time τ at one point of space.

The development of the theory of unbounded solutions received new impetus from possibilities of practical applications, for example, problems of self-focusing of light pencils in a nonlinear media, the effect of a T-layer in a low-temperature plasma, problems of shockless compression, etc. (in this regard see [59] and other papers of the present volume). The number of mathematical publications in which unbounded solutions of nonlinear evolution equations were studied rose abruptly in the last decade and a half.

However, in mathematical investigations of unbounded solutions preference is mainly given to questions of the general theory and principally to the derivation of conditions for global insolvability of nonlinear problems. Constructive methods of investigating space–time structure of unbounded solutions has so far not advanced far enough. The main reason for this apparently is that to a wide circle of specialists in the area of nonlinear evolution equations those essentially nonstationary effects and phenomena of physical character which can arise in a nonlinear medium and stably evolve for a bounded interval of time are unknown. These effects are unusual and are of considerable theoretical and practical interest.

In particular, regimes with peaking lead to localization in space of regions of high-temperature and to the formation of nonstationary dissipative structures.

1. Unusual Effects of Regimes with Peaking

Properties of regimes with peaking can be most simply demonstrated in a one-dimensional medium with a particular pair $\{K(u), Q(u)\}$.

As such a pair we choose the power functions

$$K(u)=u^\sigma, \quad \sigma>0, \qquad Q(u)=u^\beta, \quad \beta>1. \tag{4}$$

With the restrictions $\sigma > 0$, $\beta > 1$ the functions K(u) and Q(u) satisfy conditions (2), (3). The choice of power functions is not accidental. By means of methods of group analysis it will be shown below that in the class of such functions the symmetry of the mathematical model is maximal in a particular sense. On the other hand, the powers σ and β are a convenient measure of the intensity of the heating and dissipative factors. Relations between σ and β determine the space–time order in the medium in question. An idea of the space–time structure can be obtained on the basis of a preliminary analysis of invariant solutions.

For a special choice of the initial thermal perturbation $u(x, 0) = \theta(x)$ its further evolution

$$u(x,t)=g(t,\tau)\theta(x\varphi^{-1}(t,\tau)) \tag{5}$$

is determined by the functions

$$g(t,\tau)=(1-t\tau^{-1})^{-\gamma},\quad \varphi(t,\tau)=(1-t\tau^{-1})^{\alpha},$$

where

$$\gamma=(\beta-1)^{-1},\quad \alpha=0{,}5(\beta-\sigma-1)(\beta-1)^{-1}.$$

The function $\theta(\xi)$ hereby satisfies a corresponding boundary value problem (see Sec. 3) for a second order nonlinear ordinary equation

$$(\theta^{\sigma}\theta'_{\xi})'_{\xi}-\frac{\alpha}{\tau}\xi\theta'_{\xi}+\theta^{\beta}-\frac{\gamma}{\tau}\theta=0. \tag{6}$$

Although the invariant solution (5) is a special solution of the Cauchy problem it nevertheless turns out that for practically any initial perturbations the solution of the original problem "passes out" onto this solution at an advanced stage of the heating process. Thus, the most symmetric solution describes the asymptotics of the evolution process. The basic asymptotic regimes of heating of the medium [40, 41, 45, 73, 74] are established by an analysis of Eq. (6).

The Thermal Wave. HS-Regime with Peaking. A wave regime – the so-called HS-regime with peaking – is possible in the medium in question. A wave regime of evolution of the initial perturbation is possible for $1 < \beta < \sigma + 1$. The last inequality means roughly speaking that with increase in the temperature the diffusion of heat occurs more intensively than the heating of the medium. In this case after a finite time the entire space is heated to an infinite temperature (see Fig. 3).

Localization of Heat in an S-Regime. For $\beta = \sigma + 1$ the intensity of heating and diffusion of heat equalize which leads to a parodoxical effect: heating of the medium to infinite temperature occurs over the so-called fundamental length (see Fig. 4). In spite of the presence of diffusion the heat does not propagate into cold space beyond the limits of the fundamental length. An effect of localization of the region of intense heating occurs. Moreover, a nonstationary dissipative structure is formed in which the distribution of temperature does not depend on the initial perturbation. Only the time of existence of the structure depends on the initial perturbation. Such a heating regime was called an S-regime [45, 74]. We emphasize that in the cases of the HS- and S-regimes at an advanced stage of heating the temperature perturbation has a unique structurally stable space–time form determined by the unique solution of the boundary value problem for (6) with a definite value of the time of the existence of the solution $\tau > 0$.

Dissipative Structures of the LS-Regime with Peaking. For more intense operation of the source as compared with the diffusion of heat ($\sigma + 1 < \beta < \sigma + 3$) a finite number of dissipative structures of the LS-regime with peaking are formed in the medium. The number N of qualitatively distinct structures is determined by the formula [40]:

$$N=-[-a]-1, \tag{7}$$

where $a=(\beta-1)(\beta-\sigma-1)^{-1}$.

The number of structures is connected with the number of zeros of the solution $y = y(x)$ of the following linear problem [40]:

$$y''-0{,}5(\beta-\sigma-1)xy'+(\beta-1)y=0,$$
$$y'(0)=0,\ y(0)=1.$$

The solution of this problem has the form

$$y(x)=\Phi(-(\beta-\sigma)(\beta-\sigma-1)^{-1},\ 0{,}5,\ 0{,}25(\beta-\sigma-1)x^2),$$

where $\Phi(a, b, \xi)$ is a degenerate hypergeometric function [7].

Each structure has its own space–time form which is determined by a solution of the same boundary value problem for Eq. (6) (see Fig. 5). These structures exist for the same interval of time $\tau > 0$. At the time of peaking $t = \tau$ each structure leaves a trace in the medium – the limit distribution

$$u(x,\tau)=c_i|x|^{-2(\beta-\sigma-1)^{-1}},\quad i=\overline{1,N},\quad c_i>0. \tag{8}$$

The limit distributions for different structures are distinguished by the values of the constants c_i.

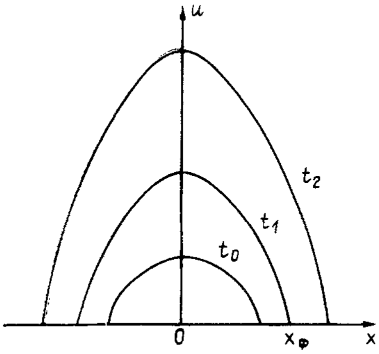


Fig. 3

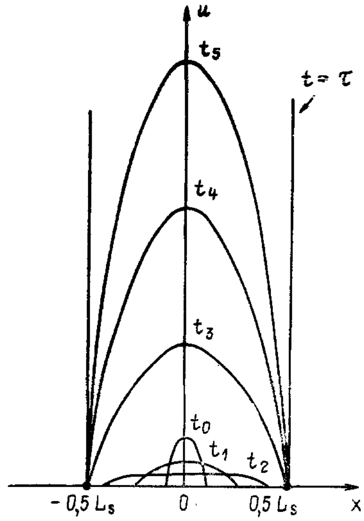


Fig. 4

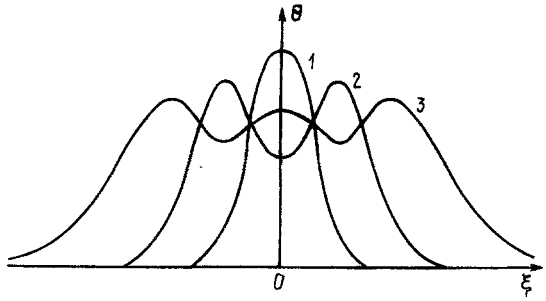
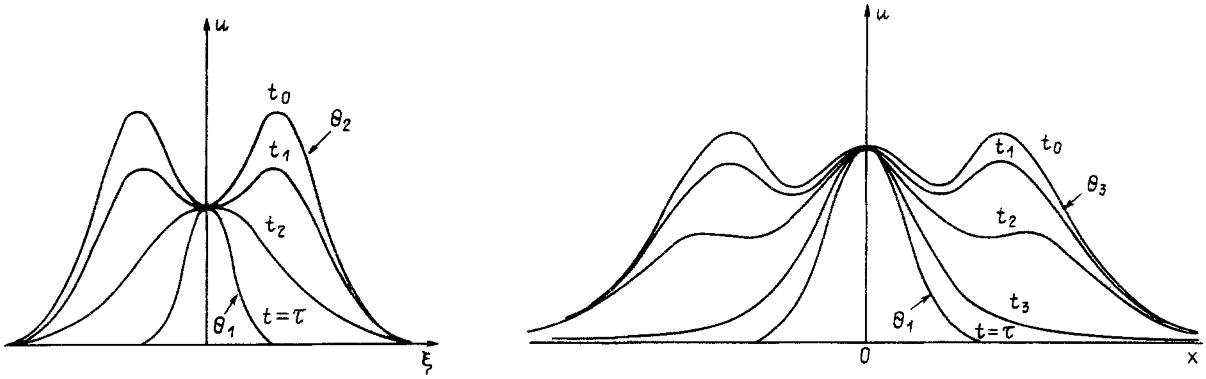


Fig. 5

Fig. 3. The thermal wave of an HS-regime with peaking.

Fig. 4. Formation of a dissipative structure of an S-regime with peaking. The structure is localized at the fundamental length L_S.

Fig. 5. Eigenfunctions of a self-similar problem in the case of an LS-regime with peaking: $\theta = \theta_i(\xi)$, $i = \overline{1, 3}$. The eigenfunctions determine the distribution of temperature in the simple (1) and complex (2, 3) dissipative structures.



a b

Fig. 6. a) Decomposition of a complex dissipative structure corresponding to the second eigenfunction; b) decomposition of a complex dissipative structure corresponding to the third eigenfunction.

The values c_i are the eigenvalues of a nonlinear boundary value problem for (6), and its solutions are eigenfunctions.

For a given time of peaking there are N isochronic structures (existing for the same interval of time) "containing" at the initial time a definite quantity of "thermal energy"

$$Q_i = \int_0^\infty \theta_i(\xi)\, d\xi, \quad Q_i > Q_{i-1}, \quad i = \overline{2, N}.$$

Thus, the eigenfunctions determine a finite number of "energy levels" existing for the same time interval τ.

On the other hand, due to profiling over space of the initial temperature, any amount of energy given at the initial time in correspondence with the finite number of eigenfunctions can exist as a finite number of isoenergy structures. The times of existence of such structures are different: the simplest structure has minimal "life time"; the most complex structure has maximal life time. The degree of complexity of the structure is determined by the character of its nonmonotonicity for $x \geqslant 0$. To the simple structure there corresponds a monotonically decreasing temperature distribution; to the most complex there corresponds a distribution with the maximum possible number of local extrema.

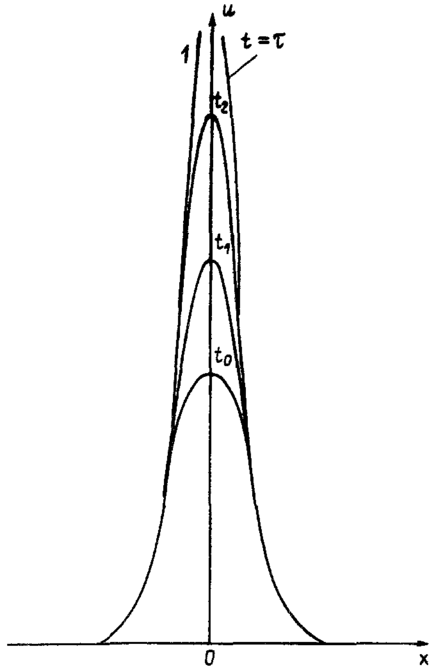


Fig. 7. Formation of a dissipative structure of an LS-regime with peaking. 1 is the limit distribution of the temperature.

Investigations have shown [41] that the simplest structure is the most stable structure. Figure 6a, b shows results of calculation of the conversion of complex structures θ_2 and θ_3 into the simple structure θ_1. The complex thermal structures are conserved during almost the entire time of peaking. Degeneration of a complex structure into the simple structure occurs during the course of a rather brief interval of time just before the time of peaking.

The existence of "needlelike" limit distributions (8) bears witness to the loalization in space of the region of intense heating (Fig. 7).

On the basis of the examples considered above the impression is formed that localization of the region of intense heating occurs for an action of the source more intense than diffusion. Indeed, for $1 < \beta < \sigma + 1$ there is no localization; for $\beta = \sigma + 1$ the region of localization is determined by the fundamental length L_S. For $\sigma + 1 < \beta < \sigma + 3$ the higher the temperature u_0, the smaller the region of space where $u \geqslant u_0$.

If for $1 < \beta < \sigma + 3$ and any initial data a regime with peaking is always realized, then for $\beta > \sigma + 3$ there exist two types of regimes – the HS-regime of cooling without peaking ($\tau < 0$) and the LS-regime with peaking ($\tau > 0$).

The invariant solution of the HS-regime without peaking is structurally instable and is the boundary between two classes of initial data. If the initial distribution dominates a temperature distribution of an invariant solution of the HS-regime of cooling, then an LS-regime with peaking occurs (see Fig. 8a). If the initial distribution is majorized with the same values of the maximal temperatures, then a damped wave is formed (see Fig. 8b). The wave exists for a rather large interval of time. In Fig. 8 the distribution of the HS-regime without peaking is denoted by a dashed curve.

It should be noted that the existence of the regimes enumerated for power dependence of K(u), Q(u) on u can be predicted on the basis of the method of averaging proposed in [42].

A preliminary analysis of the invariant solution (5) for the special case of power dependence of K(u) and Q(u) [40-42, 45, 59, 73, 74] made it possible to determine and formulate a number of concepts characteristic for the class of problems considered. These are primarily the concepts of peaking, localization, effective localization, a limit temperature distribution, complex and simple dissipative structures, and their structural stability. The precise definitions of these concepts can be found in Sec. 3.

In proving the existence of peaking and localization in the case of more general dependencies of K(u) and Q(u) an important role is played by methods of group analysis, qualitative methods of the theory of ordinary differential equations (o.d.e.), numerical methods, and methods of comparing solutions of parabolic equations, including degenerate solutions.

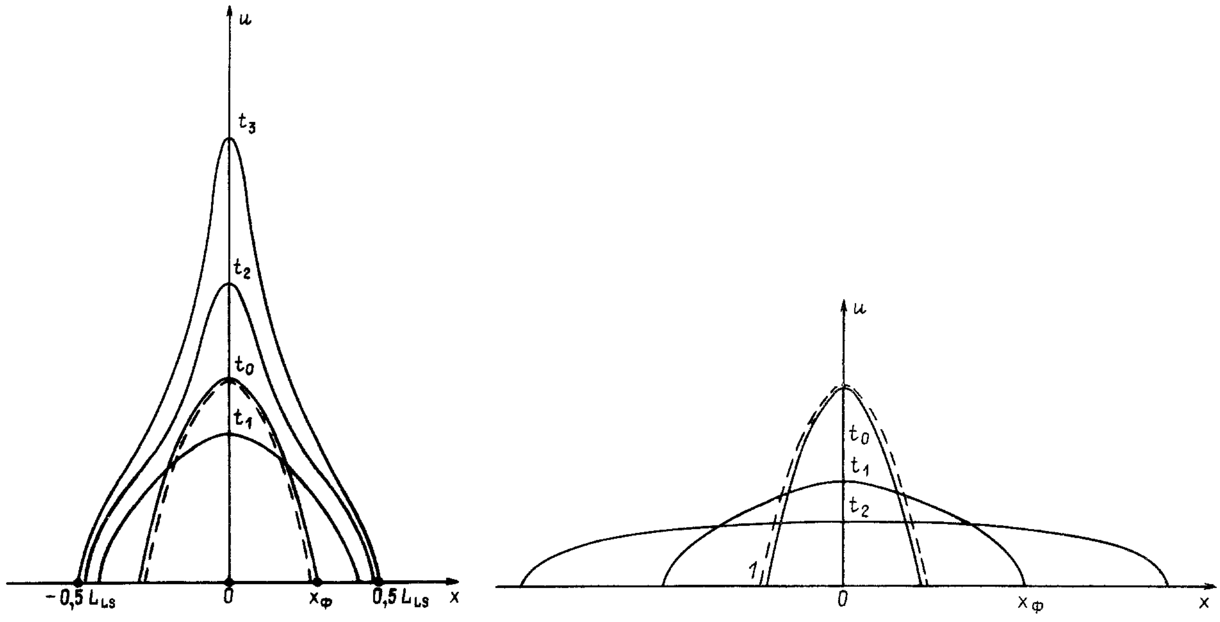


Fig. 8. a) Formation of a dissipative structure of an LS-regime with peaking for $\beta > \sigma + 3$. The structure is localized at the fundamental length. 1 is the critical temperature distribution; b) formation of a thermal wave of an HS-regime without peaking.

Methods of group analysis make it possible to determine for which pairs $\{K(u), Q(u)\}$ there is a sufficiently rich collection of invariant solutions and to determine their concrete form.

By means of the methods of the qualitative theory of o.d.e. it is possible to establish special properties of invariant solutions and to formulate the basic concepts. Numerical methods make it possible to obtain information regarding a specific invariant solution and its stability relative to perturbations.

By means of the method of approximate self-similar solutions (ASS; see [23]) it is possible to assign to the initial equations certain other basic equations. These equations may have a richer collection of invariant solutions as compared with the original equations. A remarkable circumstance is that these equations may differ from the original equations, and nevertheless the solution of the original problem at the asymptotic stage of the evolution process tends to invariant solutions of the basic equations. This circumstance makes especially important the investigation of the spectrum of invariant solutions which determine the various final forms of processes in nonlinear media.

A sufficient reserve of invariant solutions is needed for successful application of methods of operator comparison. In the method of operator comparison these solutions are used as the basic carriers of particular properties of the general solution (see [14]).

One of the interesting results of the investigation in the multidimensional case is establishing the existence of different types of group-invariant solutions on the basis of different coordinates: power self-similarity with respect to the radius and a traveling wave with respect to the angle. Such solutions determine the well-known spiral waves [49]. It is noteworthy that solutions of such type exist for only the heat equation with a source.

In the present work we do not consider the important and difficult problem of constructing and studying the architecture of multidimensional eigenfunctions of a nonlinear medium. We mention that basic constructive steps in this direction have been made in the work [60]. The problem of constructing multidimensional self-similar solutions leads to the necessity of numerical solution of nonlinear elliptic equations. The problem of the initial approximation is solved on the basis of the method of linearization in a neighborhood of a so-called homothermic solution and of a fundamental solution. This approach makes it possible to construct multidimensional structures also in trigger media.

Establishment of the asymptotics of evolution processes leading to the occurrence of a particular number of types of metastably stable dissipative structures whose space–time structure is described by group-invariant solutions plays the role of an analogue of the second law in open nonlinear systems.

2. Symmetry of the Heat Equation

2.1. What Does Group Analysis Provide? Modern group analysis is a generally recognized method of describing the symmetry of continuous mathematical models.

This direction is based on the theory of continuous groups developed by Sophius Lie at the end of the last century and is a synthesis of algebraic ideas with ideas of analysis. The concept of a continuous group developed intensively and led to the creation of an entire direction in mathematics: the theory of Lie groups and algebras, the theory of group representations, etc. Lie's approach to the problem of integration (from the beginning Lie's purpose was the creation of a theory of total integration of differential equations) was basically forgotten in the theory of differential equations. This was caused by the fact that the methods of integration developed by Lie were not a universal mathematical tool – an arbitrary system of differential equations need not admit a nontrivial group of transformations (a discussion of these questions can be found in [66]). Moreover, Lie's theory is a local theory unsuited, generally speaking, to giving directly a solution of a boundary value problem.

Nevertheless, later Lie's approach to differential equations was appreciated by applied persons, since the mathematical models used in physics and mechanics possess, as a rule, basic symmetry described by a broad group of transformations. Knowledge of this group affords the investigator considerable information for study of the mathematical model. In particular, a group property of a system of differential equations makes it possible to distinguish classes of group-invariant solutions, the finding of which is a simpler problem than finding a general solution, to generate new solutions from solutions already known, etc. This circumstance acquires special importance in studying nonlinear models where each exact solution plays an important role and where algorithms of group analysis act just as effectively as for linear models. We remark also that in contrast to traditional methods of investigation (for example, the method of a small parameter, etc.) groups methods do not use linearization of the original model. Apparently the wide investigation of group properties of sets of models of mathematical physics carried out in the sixties is connected with these circumstances. A new, independent direction called group analysis of differential equations arose after the works of Birkhoff, L. V. Ovsyannikov, L. I. Sedov, and their followers.

Contrary to popular opinion group analysis is not exhausted by methods of constructing special solutions of a system of differential equations. Already at the beginning of our century the connection of the symmetry of a mathematical model with conservation laws found a constructive formulation in the form of Noether's theorem. Since conservation laws in the majority of cases are the foundation for constructing mathematical models, Noether's theorem indicated the fundamental role of symmetry in mathematical modeling. It is not superfluous to recall that considerations of symmetry played a decisive role in the creation of quantum mechanics, the theory of elementary particles, and in other areas.

The development of group analysis led to many other ways of using the symmetry of a mathematical model. However, modern problems of mathematical physics hereby posed a number of questions which found no solution within the framework of classical Lie theory. Individual transformations not of point character were found which preserved differential equations; transformations were found which connect solutions of nonlinear equations; solutions were found which are not classical invariants (for example, the multisoliton solutions of the KdV equation). The so-called Bäcklund problem crystallized out completely, although a constructive formulation of it was given comparatively recently [48].

All this was a prerequisite of the development and generalization of Lie theory. Lie himself gave the first generalizations – his theory of first order tangential transformations. However, further advances in this direction encountered difficulties of principle character. Only recently [47] was it possible to create a satisfactory theory – the theory of Lie–Bäcklund groups – which generalizes classical Lie theory in a nontrivial manner. The new theory made it possible to resolve a number of questions which were unsolved by the classical theory and, most important, provided the possibility of finding "hidden symmetry" of mathematical models inaccessible in the classical approach. Within the framework of the theory of Lie–Bäcklund groups it is possible to obtain a number of constructive theorems in the study of conservative systems of differential equations, to formulate the Bäcklund problem, and to give effective methods of solving it.

At the present time group analysis is undoubtedly at a new stage of ascent and is becoming an effective method of investigating nonlinear differential equations.

In this part of the work we present some results of group analysis and its applications within the framework of the model of nonlinear heat conduction. The symmetry of the nonlinear heat equation with a source will here interest us mainly from the point of view of structures – the possibility of the existence of stable dissipative structures of a heat-conducting medium. The concept of structural stability, i.e., the preservation in time of a form characteristic for a given structure, rate of growth, localization in space, etc., is closely connected with the concept of invariance of a solution under transformations involving time. There is reason to suppose that precisely the invariant solutions in many cases form "attractors" of the evolution of dissipative structures of a particular type of nonlinear problems. From this position group-invariant solutions are not exotic, degenerate representatives of a manifold of thermal formations but structures which characterize important intrinsic properties of the nonlinear dissipative medium.

2.2. Group Classification of a Nonlinear Heat Equation with a Source: the Group of Point Transformations. 2.2.1. In this subsection we consider group properties of the nonlinear heat equation with a source (sink) in the one-dimensional case

$$u_t = (k(u)u_x)_x + Q(u). \tag{1}$$

For concrete functions k(u) and Q(u) the algorithms of group analysis make it possible to find groups of transformations (forming local Lie groups) admitted by Eq. (1). A natural generalization of this problem [group classification of Eq. (1)] is the enumeration of groups of transformations for all possible forms of k(u), Q(u), more precisely, the finding of an admissible group of transformations for an arbitrary pair (k, Q) (nucleus of the basic groups) and enumeration of all those special forms of (k, Q) for which extension of the group of transformations admitted by Eq. (1) occurs.

It is known (see [67]) that a criterion for the invariance of a differential equation – in our case Eq. (1) – relative to a group of point transformations of the dependent and independent variables of the form

$$\begin{aligned} t^* &= f(t, x, u; a_1, \dots, a_r), \\ x^* &= g(t, x, u; a_1, \dots, a_r), \\ u^* &= \varphi(t, x, u; a_1, \dots, a_r), \end{aligned} \tag{2}$$

where $a_1, \dots, a_r$ are the parameters of an r-parameter Lie group of transformations

$$\tilde{X}(u_t - (k(u)u_x)_x - Q(u))|_{(1)} = 0, \tag{3}$$

where the linear operator

$$\tilde{X} = \tilde{t}\frac{\partial}{\partial t} + \tilde{x}\frac{\partial}{\partial x} + \tilde{u}\frac{\partial}{\partial u} + \zeta_{u_t}\frac{\partial}{\partial u_t} + \zeta_{u_x}\frac{\partial}{\partial u_x} + \zeta_{u_{xx}}\frac{\partial}{\partial u_{xx}},$$

defines an inifinitesimal transformation of the group (2), where $\tilde{t}$, $\tilde{x}$, $\tilde{u}$ are unknown functions of the point (t, x, u), and ζ_{u_t}, ζ_{u_x}, $\zeta_{u_{xx}}$ are computed by the standard extension formulas in terms of $\tilde{t}$, $\tilde{x}$, $\tilde{u}$ (see [67]). The operator $\tilde{X}$ is connected in a one-to-one manner by the Lie equations with the group of transformations (2), and hence to find the admissible group it suffices to solve the system (3) for the functions $\tilde{t}$, $\tilde{x}$, $\tilde{u}$.

In solving the problem of group classification the system of equations (3) [we note that (3) is a system of equations, since the coefficients $\tilde{t}$, $\tilde{x}$, $\tilde{u}$ do not depend on the derivatives u_t, u_x, $u_{xx}, \dots$, and hence, expressing, for example, u_t in (3) by means of (1) in terms of u_x, u_{xx}, we obtain the possibility of "decoupling" (3) into a system of equations by equating to zero the coefficients of u_x, u_{xx} and their powers] is solved for nonspecific dependencies k = k(u), Q = Q(u). As a result classifying equations for k(u), Q(u) arise whose solutions give an extension of an admissible group of transformations.

The results of group classification can be written in more compact form if we use a group of equivalent transformations (see [67]). In our case we can use the transformations

TABLE 1

$k = e^u$

$Q(u)$	$\tilde{t}$	$\tilde{x}$	$\tilde{u}$
$\pm e^{\beta u},\ \beta \neq 0$	βt	$(\beta-1)\frac{x}{2}$	-1
$\pm e^u + \delta,\quad \delta = \pm 1$	$e^{-\delta t}$	0	$\delta e^{-\delta t}$
$\delta = \pm 1$	0	x	2
	$e^{-\delta t}$	0	$\delta e^{-\delta t}$
0	$2t$	x	0
	t	0	-1

TABLE 2

$k = u^\sigma,\quad \sigma \neq 0;\ -4/3$

$Q(u)$	$\tilde{t}$	$\tilde{x}$	$\tilde{u}$
$\pm u^{\sigma+1} + \delta u$, $\lvert\delta\rvert = 1$	$e^{-\delta\sigma t}$	0	$\delta e^{-\delta\sigma t} u$
$\pm u^n$	$2(n-1)t$	$(n-\sigma-1)x$	$-2u$
$\delta u,\ \lvert\delta\rvert = 1$	0	σx	$2u$
	$e^{-\delta\sigma t}$	0	$\delta e^{-\delta\sigma t} u$
$\delta = \pm 1$	t	$(\sigma+1)x/2$	u
0	$2t$	x	0
	0	σx	$2u$

$$\begin{gathered}\bar{k}(\bar{u}) = \gamma^2 k(\alpha u + \beta),\\ \bar{Q}(\bar{u}) = \frac{\alpha}{\delta^2} Q(\alpha u + \beta),\\ \bar{u} = \alpha u + \beta,\quad \bar{x} = \delta\gamma x,\quad \bar{t} = \delta^2 t.\end{gathered} \tag{4}$$

Two equations (1) equivalent in the sense of the transformations (4) admit similar groups and are not distinguished in the group classification.

For arbitrary $k(u)$, $Q(u)$ Eq. (1) admits (see [30, 31]) the group of translations in t and x (the nucleus of the basic groups) to which there correspond the operators $X_1 = \partial/\partial x$ and $X_2 = \partial/\partial x$. For $Q \equiv 0$ to X_1, X_2 there is added the dilation $X_3 = 2t(\partial/\partial t) + x(\partial/\partial x)$. Tables 1-4 present special forms (specializations) of the pair $\{k(u), Q(u)\}$ for which Eq. (1) admits a wider group of transformations. For each pair $\{k(u), Q(u)\}$ the coordinates $\tilde{t}$, $\tilde{x}$, $\tilde{u}$ of the basic operators of the Lie algebra are presented, but the nucleus X_1, X_2 is not listed. Tables 1-4 also contain classical results [65, 115] pertaining to the linear heat equation and the nonlinear heat equation (without a source).

2.2.2. Here we present results of the group classification of the heat equation with a source (sink) in the two-dimensional and three-dimensional cases (see [33, 35]):

$$u_t = \sum_{i=1}^{N} (k(u) u_{x_i})_{x_i} + Q(u),\quad k \geqslant 0. \tag{5}$$

Calculations according to the method of [66] lead in the case of Eq. (5) to the following equivalence group:

$$\begin{gathered}\bar{t} = b^2 t + f,\\ \bar{x}_i = ab x_i + e_i,\\ \bar{u} = cu + d,\\ \bar{k}(\bar{u}) = a^2 k(u),\\ \bar{Q}(\bar{u}) = \frac{c}{b^2} Q(u),\end{gathered} \tag{6}$$

where a, b, c, d, e_i, f are arbitrary constants, $a \cdot b \cdot c \neq 0$, $i = 1, \ldots, N$.

TABLE 3

$k = u^{-4/3}$

$Q(u)$	$\tilde{t}$	$\tilde{x}$	$\tilde{u}$
$\alpha u^{-1/3}+\delta u$, $\lvert\alpha\rvert=\lvert\delta\rvert=1$	$e^{4\delta t/3}$	0	$\delta e^{4\delta t/3}u$
	0	$e^{2(\alpha/3)^{1/2}x}$	$-(3\alpha)^{1/2}\times$ $\times e^{2(\alpha/3)^{1/2}x}u$
	0	$e^{-2(\alpha/3)^{1/2}x}$	$(3\alpha)^{1/2}\times$ $e^{-2(\alpha/3)^{1/2}x}u$
$\alpha u^{-1/3}$, $\lvert\alpha\rvert=1$	$4t/3$	0	u
	0	$e^{2(\alpha/3)^{1/2}x}$	$-(3\alpha)^{1/2}\times$ $\times e^{2(\alpha/3)^{1/2}x}u$
	0	$e^{-2(\alpha/3)^{1/2}x}$	$(3\alpha)^{1/2}\times$ $e^{-2(\alpha/3)^{1/2}x}u$
$\pm u^n$, $n\neq -1/3$	$2(n-1)t$	$(n+1/3)x$	$-2u$
δu, $\lvert\delta\rvert=1$	0	$-2x/3$	u
	$e^{4\delta t/3}$	0	$\delta e^{4\delta t/3}u$
	0	$-x^2$	$3xu$
$\delta=\pm 1$	t	$-x/6$	u
0	$2t$	x	0
	0	$-2x/3$	u
	0	$-x^2$	$3xu$

TABLE 4

$k\equiv 1$

$Q(u)$	$\tilde{t}$	$\tilde{x}$	$\tilde{u}$
$\pm e^u$	t	$x/2$	-1
$\pm u^n$	$2(n-1)t$	$(n-1)x$	$-2u$
$\delta u\ln u$, $\delta=\pm 1$	0	$e^{\delta t}$	$-\delta/2\, e^{\delta t}xu$
	0	0	$e^{\delta t}u$
δu, $\delta=\pm 1$	$2t$	x	$2\delta tu$
	$4t^2$	$4xt$	$-(2t+x^2-4\delta t^2)u$
	0	$2t$	$-xu$
	0	0	u
	0	0	$b(x,t)$, $b_t=b_{xx}+\delta b$
$\delta=\pm 1$	$2t$	x	$2\delta t$
	$4t^2$	$4xt$	$\delta(6t^2+tx^2)-(x^2+2t)u$
	0	$2t$	$-xu+\delta xt$
	0	0	$u-\delta t$
	0	0	$a(x,t)$, $a_t=a_{xx}$
0	$2t$	x	0
	$4t^2$	$4xt$	$-(x^2+2t)u$
	0	$2t$	$-xu$
	0	0	u
	0	0	$a(x,t)$, $a_t=a_{xx}$

The result of solving the system of equations for (1) for N = 3 is as follows. In the case of arbitrary functions $k(u)$, $Q(u)$ Eq. (5) admits the group of translations along t, x_1, x_2, x_3 and the group of rotations about any of the spatial axes which are defined by the infinitesimal operators

$$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial x_2},\quad X_3=\frac{\partial}{\partial x_3},\quad X_4=\frac{\partial}{\partial t},$$
$$X_5=x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2},\quad X_6=x_3\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_3},$$

TABLE 5

$Q(u)$ \ $\kappa(u)$	$\kappa\equiv 1$	$\kappa=u^{-4/5}$	$\kappa=u^{6}$	$\kappa=e^{u}$	k = k(u) an arbitrary function
Q + Q(u) an arbitrary function	L_7	L_7	L_7	L_7	L_7
$Q=\pm e^{u}+\delta$, $\lvert\delta\rvert=1$	L_7	L_7	L_7	L_8	L_7
$Q=\pm e^{\alpha u}$, $\alpha\neq 0$	L_8	L_7	L_7	L_8	L_7
$Q=\pm u^{n}$, $n\neq 0;1$	L_8	L_8	L_8	L_7	L_7
$Q=\pm u^{\sigma+1}+\delta u$, $\lvert\delta\rvert=1$	L_7	L_8	L_8	L_7	L_7
$Q=\pm u\ln u$	L_{11}	L_7	L_7	L_7	L_7
$Q=\pm u$	$L_{13}\oplus L_\infty$	L_{12}	L_9	L_7	L_7
$Q=\pm 1$	$L_{13}\oplus L_\infty$	L_8	L_8	L_9	L_7
$Q=0$	$L_{13}\oplus L_\infty$	L_{12}	L_9	L_9	L_8

$$X_7=x_3\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_3}. \tag{7}$$

The group nucleus (7) can be extended only for those special forms of k(u), Q(u) listed below.

Table 5 presents an orienting "scheme" of group properties of Eq. (5). L_r denotes the space of dimension r (the Lie algebra) of operators of the form

$$X=\tilde{t}\,\frac{\partial}{\partial t}+\sum_{i=1}^{3}\tilde{x}_i\frac{\partial}{\partial x_i}+\tilde{u}\frac{\partial}{\partial u},$$

admitted by Eq. (5).

We remark that in linear cases, just as in the linear one-dimensional case (see above), the space of admissible operators is formed by a direct sum of a finite-dimensional and infinite-dimensional space.

Tables 6-10 present the coordinates of the basis operators of the Lie algebra for cases of extension of the nucleus of the basic groups (the hatched cells in Table 5), while the coordinates of the basis operators of the nucleus are not presented. In the majority of cases x_i is written in place of the three coordinates $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$.

2.2.3. In the two dimensional case (N = 2) the group nucleus is formed by the following operations:

$$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial x_2},\quad X_3=\frac{\partial}{\partial t},\quad X_4=x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2}.$$

TABLE 6

$\tilde{t}$	$\tilde{x}_i$	$\tilde{u}$	$Q(u)$
$2t$	x_i	0	$Q\equiv 0$

Note. The function k = k(u) is arbitrary.

TABLE 7

$k = e^u$:

$Q(u)$	$\tilde{t}$	$\tilde{x}_i$	$\tilde{u}$
$Q = \pm e^u + \delta$, $\|\delta\| = 1$	$e^{-\delta t}$	0	$\delta e^{-\delta t}$
$Q = \pm e^{\alpha u}$, $\alpha \neq 0$	$2\alpha t$	$(\alpha - 1) x_i$	-2
$Q = \delta = \pm 1$	0	x_i	2
	$e^{-\delta t}$	0	$\delta e^{-\delta t}$
$Q \equiv 0$	$2t$	x_i	0
	t	0	-1

TABLE 8

$k = u^\sigma$, $\sigma \neq 0$; $-4/5$:

$Q(u)$	$\tilde{t}$	$\tilde{x}_i$	$\tilde{u}$
$Q = \pm u^n$, $n \neq 0$; 1	$2(1-n)t$	$(\sigma - n + 1) x_i$	$2u$
$Q = \pm u^{\sigma+1} + \delta u$, $\|\delta\| = 1$	$e^{-\delta\sigma t}$	0	$\delta e^{-\delta\sigma t} u$
$Q = \delta u = \pm u$	0	σx_i	$2u$
	$e^{-\delta\sigma t}$	0	$\delta e^{-\delta\sigma t} u$
$Q \equiv 0$	$2t$	x_i	0
	0	σx_i	$2u$

Table 11 gives the scheme of the group characteristics of equation (5) when $N = 2$. Table 12 shows only the special case $K = u^{-1}$ not obtained from the two-dimensional case. All other cases of extension of the nucleus of the basis groups (the hatched cells in Table 11) are obtained from the corresponding cases for the three-dimensional equation (5) if we formally set $x_3 = 0$, $d/dx^3 = 0$

We note that in case of any special dimension N with coefficient of thermal conductivity

$$k(u) = u^{-\frac{4}{N+2}} \tag{8}$$

There will be significant extension of the admissible groups of transformations. At the same time the two-dimensional case $N = 2$, $k = u^{-1}$ distinguishes itself, when the groups of admissible transformations is infinite-dimensional

$$u_t = \Delta(\ln u) + \delta u, \ \delta = 0;\ \pm 1. \tag{9}$$

It is known that a considerable change of the functional properties of solutions of Eq. (5) [for example, the Cauchy problem for (5) ceases to be well posed) occurs for the coefficient of thermal conductivity

TABLE 9

$$k=u^{-4/5}:$$

$Q(u)$	$\tilde{t}$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{u}$
$Q=u^n$, $n\neq 0;\ 1$	$2(1-n)t$	$(1/5-n)x_1$	$(1/5-n)x_2$	$(1/5-n)x_3$	$2u$
$Q=\pm u^{1/5}+\delta u$, $\lvert\delta\rvert=1$	$e^{4/5\delta t}$	0	0	0	$\delta e^{4/5\delta t}u$
$Q=\delta u$, $\lvert\delta\rvert=1$	$e^{4/5\delta t}$	0	0	0	$\delta e^{4/5\delta t}u$
	0	$2x_1$	$2x_2$	$2x_3$	$-5u$
	0	$x_1^2-x_2^2-x_3^2$	$2x_1x_2$	$2x_1x_3$	$-5x_1u$
	0	$2x_1x_2$	$x_2^2-x_1^2x-{}_3^2$	$2x_2x_3$	$-5x_2u$
	0	$2x_1x_3$	$2x_2x_3$	$x_3^2-x_2^2-x_1^2$	$-5x_3u$
$Q\equiv 0$	$2t$	x_1	x_2	x_3	0
	0	$2x_1$	$2x_2$	$2x_3$	$-5u$
	0	$x_1^2-x_2^2-x_3^2$	$2x_1x_2$	$2x_1x_3$	$-5x_1u$
	0	$2x_1x_2$	$x_2^2-x_1^2-x_3^2$	$2x_2x_3$	$-5x_2u$
	0	$2x_1x_3$	$2x_2x_3$	$x_3^2-x_1^2-x_2^2$	$-5x_3u$

$$k(u)=u^{-\frac{2}{N}}. \tag{10}$$

It is interesting to note that the intersection of the regions (8) and (10) gives precisely N = 2, that is, Eq. (9) (and only it) has all the qualities enumerated.

We note that negative power exponents in the coefficient of thermal conductivity are used in describing diffusion processes in polymers, semiconductors, in porous media, in crystalline hydrogen, in some problems of chemistry, in problems of the physics of the sea, etc. In these areas modeling of the diffusion process with reduction of the dimension of the problem may distort the group properties of the problem which, in particular, may lead to the appearance of solutions which are absent in the multidimensional formulation or, conversely, to the absence of solutions present in the multidimensional case.

2.2.4. We consider the group properties of a nonlinear, anisotropic heat equation with a source (or with a sink)

$$\begin{gathered} u_t=\sum_{i=1}^{N}(k_i(u)u_{x_i})_{x_i}+Q(u), \\ k_i\geqslant 0,\quad N=2,3, \\ \sum_{i,j}\left(\frac{k_i}{k_j}\right)_u^2\neq 0. \end{gathered} \tag{11}$$

In classifying Eq. (11) on the basis of $k_i(u)$, $Q(u)$ the following equivalence group is used:

$$\begin{gathered} \bar{x}_i=a_ibx_i+e_i, \\ \bar{t}=b^2t+f, \\ \bar{u}=cu+d, \end{gathered}$$

TABLE 10

k ≡ 1:

$Q(u)$	$\tilde{t}$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{u}$
$Q=\delta u$, $\lvert\delta\rvert=1$	$2t$	x_1	x_2	x_3	$2\delta t u$
	$4t^2$	$4tx_1$	$4tx_2$	$4tx_3$	$(4\delta t^2-2Nt-x_1^2-x_2^2-x_3^2)u$, N—is the number of spatial variables
	0	0	0	0	u
	0	$2t$	0	0	$-x_1u$
	0	0	$2t$	0	$-x_2u$
	0	0	0	$2t$	$-x_3u$
	0	0	0	0	$B(t, x_1, x_2, x_3)$ is the solution of the equation $B_t=\Delta B+\delta B$
$Q=\pm 1$	$2t$	x_1	x_2	x_3	$2u$
	$4t^2$	$4tx_1$	$4tx_2$	$4tx_3$	$-(2Nt+x_1^2+x_2^2+x_3^2)u+((2N+4)t+x_1^2+x_2^2+x_3^2)t$
	0	0	0	0	$u-t$
	0	$2t$	0	0	$x_1(t-u)$
	0	0	$2t$	0	$x_2(t-u)$
	0	0	0	$2t$	$x_3(t-u)$
	0	0	0	0	$A(t, x_1, x_2, x_3)$ is any solution of the equation $A_t=\Delta A$

$Q(u)$	$\tilde{t}$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{u}$
$Q=\pm e^u$	$2t$	x_1	x_2	x_3	-2
$Q=\pm u^n$	$2(n-1)t$	$(n-1)x_1$	$(n-1)x_2$	$(n-1)x_3$	$-2u$
$Q=\delta u\ln u$, $\lvert\delta\rvert=1$	0	0	0	0	$e^{\delta t}u$
	0	$e^{\delta t}$	0	0	$-\frac{\delta}{2}x_1e^{\delta t}u$
	0	0	$e^{\delta t}$	0	$-\frac{\delta}{2}x_2e^{\delta t}u$
	0	0	0	$e^{\delta t}$	$-\frac{\delta}{2}x_3e^{\delta t}u$
$Q\equiv 0$	$2t$	x_1	x_2	x_3	0
	$4t^2$	$4tx_1$	$4tx_2$	$4tx_3$	$-(2Nt+x_1^2+x_2^2+x_3^2)u$, N - is the number of spatial variables.
	0	0	0	0	u
	0	$2t$	0	0	$-x_1u$
	0	0	$2t$	0	$-x_2u$
	0	0	0	$2t$	$-x_3u$
	0	0	0	0	$A(t, x_1, x_2, x_3)$ is any solution of the equation $A_t=\Delta A$

TABLE 11

$Q(u)$ \ $K(u)$	$K\equiv 1$	$K=u^{-1}$ $(\sigma=-1)$	$K=u^{\sigma}$ $\sigma\neq 0;-1$	$K=e^{u}$	$K=K(u)$ an arbitrary function
$Q=Q(u)$ an arbitrary function	L_4	L_4	L_4	L_4	L_4
$Q=\pm e^{u}+\delta$, $\lvert\delta\rvert=1$	L_4	L_4	L_4	L_5	L_4
$Q=\pm e^{\alpha u}$ $\alpha\neq 0$	L_5	L_4	L_4	L_5	L_4
$Q=\pm u^{n}$	L_4	L_5	L_5	L_4	L_4
$Q=\pm u^{\sigma+1}+\delta u$, $\lvert\delta\rvert=1$	L_4	L_5	L_5	L_4	L_4
$Q=\delta u \ln u$ $\lvert\delta\rvert=1$	L_7	L_4	L_4	L_4	L_4
$Q=\pm u$	$L_9\oplus L_\infty$	$L_2\oplus L_\infty$	L_6	L_4	L_4
$Q=\pm 1$	$L_9\oplus L_\infty$	L_5	L_5	L_6	L_4
$Q\equiv 0$	$L_9\oplus L_\infty$	$L_2\oplus L_\infty$	L_6	L_6	L_5

TABLE 12

$k = u^{-1}$:

$Q(u)$	$\tilde{t}$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{u}$
$Q=\pm u^{n}$ $n\neq 1$	$21(n_1-1)$	nx_1	nx_2	$-2u$
$Q=\alpha u+\delta$, $\lvert\alpha\rvert=\lvert\delta\rvert=1$	$e^{\alpha t}$	0	0	$\alpha e^{\alpha t}u$
$Q=\pm 1$	t	0	0	u
$Q=\delta u$, $\lvert\delta\rvert=1$	$e^{\delta t}$	0	0	$\delta e^{\delta t}u$
	0	$A(x_1, x_2)$	$B(x_1, x_2)$	$-2uA_{x_1}$
		A, B— any solution of the system $A_{x_1}=B_{x_2}$, $A_{x_2}=-B_{x_1}$		
$Q\equiv 0$	t	0	0	u
	0	$A(x_1, x_2)$	$B(x_1, x_2)$	$-2uA_{x_1}$

TABLE 13

$\kappa_1 \nsim \kappa_2 \nsim \kappa_3 \nsim \kappa_1$:

$Q(u)$ \ $\kappa_i(u)$	$\kappa_i = u^{\sigma_i}$, $\sigma_1 \neq \sigma_2 \neq \sigma_3 \neq \sigma_1$	$\kappa_i = e^{\alpha_i u}$, $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_1$	$\kappa_i = \kappa_i(u)$ an arbitrary function
$Q = Q(u)$ an arbitrary function	L_4	L_4	L_4
$Q = \pm e^{\alpha u}$, $\alpha \neq 0$	L_4	L_5	L_4
$Q = \pm u^n$, $n \neq 0$	L_5	L_4	L_4
$Q = \pm 1$	L_5	L_5	L_4
$Q = 0$	L_6	L_6	L_5

$$\bar{k}_i(\bar{u}) = a_i^2 k_i(u), \qquad \bar{Q}(\bar{u}) = \frac{c}{b^2} Q(u), \tag{12}$$

In the three-dimensional case it is convenient to distinguish two possibilities: 1) all three components are nonproportional $k_1 \nsim k_2$, $k_2 \nsim k_3$, $k_3 \nsim k_1$ and 2) two coefficients are proportional, $k_i \sim k_j$, $k_i \nsim k_l$, where (i, j, ℓ) is any permutation of (1, 2, 3). The second case reduces by means of the equivalent groups (12) to $k_i \equiv k_j \nsim k_l$. To be specific we assume that $k_1 \equiv k_3 \nsim k_2$ which will be convenient for reduction of the two-dimensional case to a special case of the three-dimensional case.

2.2.4.1. $k_1 \nsim k_2$, $k_2 \nsim k_3$, $k_3 \nsim k_1$. As a result of solving the definition-system for equation (11) (see [34, 35]) we obtain the group nucleus corresponding to the operators

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial x_3}, \quad X_4 = \frac{\partial}{\partial t},$$

which can be extended only for those specializations of $[k_i(u), Q(u)]$ which are listed in Table 13 and below.

The "general scheme" of group properties of Eq. (11) in the present case is presented in Table 13.

Tables 14-16 give the coordinates $\tilde{t}$, $\tilde{x}_i$, $\tilde{u}$ of the basis operators which augment the nucleus; the latter is not given.

2.2.4.2. The case $k_1 \equiv k_3 \nsim k_2$. The group nucleus in this case is determined by the operators

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial x_3},$$
$$X_4 = \frac{\partial}{\partial t}, \quad X_5 = x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}.$$

Table 17 shows the "general scheme" of the group properties of Eq. (11) in the case considered, while cases of extension of the group nucleus are listed in Tables 18-22.

In the case $k_1 = k_3 = 1$, $k_2 = u^{-4/3}$, $Q = -u^{-1/3}$, and $Q = -u^{-1/3} + \delta u$ (see Table 22) two operators augmenting the nucleus have complex-valued coefficients. In structuring real-valued invariants and invariant solutions this circumstance, however, causes no difficulties: using the linearity of the space of admissible operators it is always possible to take the real or imaginary part of a linear combination of operators relative to which invariance of the objects studied is then considered.

2.2.5. In the two-dimensional anisotropic cases $k_1 \nsim k_2$ the nucleus of the basic groups consist only of translations along the axes x_1, x_2, t:

TABLE 14

$k_i(u)$ are arbitrary functions:

$Q(u)$	$\tilde{t}$	$\tilde{x}_i$	$\tilde{u}$
$Q\equiv 0$	$2t$	x_i	0

TABLE 15

$k_i = e^{\alpha_i u}$, $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_1$:

$Q(u)$	$\tilde{t}$	$\tilde{x}_i$	$\tilde{u}$
$Q=\pm e^{\alpha u}$	$2\alpha t$	$(\alpha-\alpha_i)\, x_i$	-2
$Q=\pm 1$	0	$\alpha_i x_i$	2
$Q\equiv 0$	$2t$	x_i	0
	0	$\alpha_i x_i$	2

TABLE 16

$k_i = u^{\sigma_i}$, $\sigma_1 \neq \sigma_2 \neq \sigma_3 \neq \sigma_1$:

$Q(u)$	$\tilde{t}$	$\tilde{x}_i$	$\tilde{u}$
$Q=\pm u^n$	$2(1-n)\, t$	$(\sigma_i+1-n)\, x_i$	$2u$
$Q=\pm 1$	$2t$	$(\sigma_i+1)\, x_i$	$2u$
$Q\equiv 0$	$2t$	x_i	0
	0	$\sigma_i x_i$	$2u$

$$X_1=\frac{\partial}{\partial x_1},\quad X_2=\frac{\partial}{\partial x_2},\quad X_3=\frac{\partial}{\partial t},$$

which form a basis of the Lie algebra L_3.

Table 23 presents the "general scheme" of the group properties of the two-dimensional anisotropic equation ($k_1 \not\sim k_2$). The coordinates of the operators augmenting the nucleus L_3 (the hatched region of Table 23) are obtained from the operators of the three-dimensional case enumerated in Tables 18-22 if in them we formally set $x_3 = 0$, $d/dx_3 = 0$.

We note that in the two-dimensional anisotropic case no extensions of the group are observed. Indeed, for example, the case $k_1 = k_3 = u^{-1}$, $Q = 0$ (Table 21) on passing to the two-dimensional case $k_1 = u^{-1}$, $k_2 = 1$, $Q = 0$ gives the operator $X_1 = d/dx_1$ which is obtained from

$$\overset{\infty}{X}= A(x_1, x_3)\frac{\partial}{\partial x_1}+B(x_1, x_3)\frac{\partial}{\partial x_3}-2A_{x_1}(x_1, x_3)\,u\frac{\partial}{\partial u},$$
$$A_{x_1}=B_{x_3},\quad A_{x_3}=-B_{x_1},$$

if we set $x_3 = 0$, $\partial/\partial x_3 = 0$.

2.3. Lie–Bäcklund Groups Admitted by the Heat Equation with a Source. In this subsection we consider the group classification of the heat equation

$$u_t=\sum_{i=1}^{N}(k_i(u)\,u_{x_i})_{x_i}+Q(u),$$

TABLE 17

$K_i(u)$ \ $Q(u)$	$K_1=K_3=1$, $K_2=u^{-4/3}$	$K_1=K_3=u^{-1}$, $K_2=1$	$K_1=K_3=u^{\sigma_1}$, $K_2=u^{\sigma_2}$, $\sigma_1\neq\sigma_2$	$K_1=K_3=e^{\alpha_1 u}$, $K_2=e^{\alpha_2 u}$, $\alpha_1\neq\alpha_2$	$K_1=K_3=K$ K, K_2 is an arbitrary function
$Q=Q(u)$ arbitrary function	L_5	L_5	L_5	L_5	L_5
$Q=\pm e^{\alpha u}$, $\alpha\neq 0$	L_5	L_5	L_5	L_6	L_5
$Q=\pm u^n$, $n\neq 0;1,-1/3$	L_6	L_6	L_6	L_5	L_5
$Q=\pm u^{-1/3}$	L_8	L_6	L_6	L_5	L_5
$Q=\pm u^{-1/3}+\delta u$, $\lvert\delta\rvert=1$	L_7	L_5	L_5	L_5	L_5
$Q=\pm u$	L_7	$L_2\oplus L_\infty$	L_6	L_5	L_5
$0=\pm 1$	L_6	L_6	L_6	L_6	L_5
$Q\equiv 0$	L_8	$L_3\oplus L_\infty$	L_7	L_7	L_6

TABLE 18

$k_1=k_3=k(u)$, $k_2=k_2(u)$,
$k(u)$, $k_2(u)$ is an arbitrary function:

$Q(u)$	$\tilde{t}$	$\tilde{x}_i$	$\tilde{u}$
$Q\equiv 0$	$2t$	x_i	u

TABLE 19

$k_1=k_3=e^{\alpha_1 u}$, $k_2=e^{\alpha_2 u}$,
$\alpha_1\neq\alpha_2$:

$Q(u)$	$\tilde{t}$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{u}$
$Q=\pm e^{\alpha u}$	$2\alpha t$	$(\alpha-\alpha_1)x_1$	$(\alpha-\alpha_2)x_2$	$(\alpha-\alpha_1)x_3$	-2
$Q=\pm 1$	0	$\alpha_1 x_1$	$\alpha_2 x_2$	$\alpha_1 x_3$	2
$Q\equiv 0$	$2t$	x_1	x_2	x_3	0
	0	$\alpha_1 x_1$	$\alpha_2 x_2$	$\alpha_1 x_3$	2

$$k_i\geqslant 0,\quad N=1, 2, 3, \tag{13}$$

from the point of view of tangential Lie–Bäcklund transformations [47, 48].

2.3.1. Suppose there is given the one-dimensional evolution equation

$$u_t=F(u, u_1, u_2, \ldots, u_m),\quad u_i=\frac{\partial^i u}{\partial x^i}. \tag{14}$$

TABLE 20

$k_1 = k_3 = u^{\sigma}, \quad k_2 = u^{\sigma_2}, \quad \sigma \neq \sigma_2$:

$Q(u)$	$\tilde{t}$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{u}$
$Q = \pm u^n$	$2(1-n)t$	$(\sigma+1-n)x_1$	$(\sigma_2+1-n)x_2$	$(\sigma+1-n)x_3$	$2u$
$Q = \pm u^{-1/3}$	$8/3t$	$\left(\sigma+\frac{4}{3}\right)x_1$	$\left(\sigma+\frac{4}{3}\right)x_2$	$\left(\sigma+\frac{4}{3}\right)x_3$	$2u$
$Q = \pm u$	0	σx_1	$\sigma_2 x_2$	σx_3	$2u$
$Q = \pm 1$	$2t$	$(\sigma+1)x_1$	$(\sigma_2+1)x_2$	$(\sigma+1)x_3$	$2u$
$Q \equiv 0$	$2t$	x_1	x_2	x_3	0
	0	σx_1	$\sigma_2 x_2$	σx_3	$2u$

TABLE 21

$k_1 = k_2 = u^{-1}, \quad k_2 = 1$:

$Q(u)$	$\tilde{t}$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{u}$
$Q = \pm u^n$	$2(1-n)t$	$-nx_1$	$(1-n)x_2$	$-nx_3$	$2u$
$Q = \pm u^{-1/3}$	$8/3t$	$1/3x_1$	$4/3x_2$	$1/3x_3$	$2u$
$Q = \pm u$	0	$A(x_1, x_3)$*	0	$B(x_1, x_3)$*	$-2A_{x_1}u$
$Q = \pm 1$	$2t$	0	x_2	0	$2u$
$Q \equiv 0$	$2t$	0	x_2	0	$2u$
	0	$A(x_1, x_3)$*	0	$B(x_1 \; x_3)$*	$-2A_{x_1}u$

*Here A, B are any solution of the system: $\begin{cases} A_{x_1} = B_{x_3}, \\ A_{x_3} = -B_{x_1}. \end{cases}$

Let $\mathfrak{M}$ be the manifold defined by (14) and all its differential consequences in the space $\mathscr{Z}$ of points $z = (t, x, u, \underset{1}{u}, \underset{2}{u}, \ldots)$, where $\underset{1}{u} = \{u_1, u_t\}$, $\underset{2}{u} = \{u_2, u_{1t}, u_{tt}\}$, etc. The one-parameter Lie–Bäcklund group admitted by $\mathfrak{M}$, is given by transformations in $\mathscr{Z}$ of the form (see [47, 48])

$$
\begin{aligned}
x^* &= x + \xi(t, x, u, u_1, u_2, \ldots)a + o(a), \\
t^* &= t + \eta(t, x, u, u_1, u_2, \ldots)a + o(a), \\
u^* &= u + U(t, x, u, u_1, u_2, \ldots)a + o(a),
\end{aligned}
\tag{15}
$$

where ξ, η, $U \in \mathscr{A}$, $\mathscr{A}$ is the space of locally analytic functions of a finite number of variables from $z = (t, x, u, u_1, u_2, \ldots)$, and a is the group parameter. The symbol o(a) denotes the remainder of a formal series of special form [ensuring closedness relative to multiplication – superpositions of the transformations (15)]; if the series in (15) converge, then the symbol o(a) takes its usual meaning. The arguments do not contain derivatives of the form $\partial^{i+j}u/\partial t^i \partial x^j$, since they can be expressed in terms of u, u_1, u_2,... by means of the original equation (14).

Just as in classical Lie theory, Lie–Bäcklund groups are related in a one-to-one manner to vector fields (infinitesimal operators). Thus, the transformations (15) can be given by means of an operator of the form

$$X = \bar{U}\frac{\partial}{\partial u} + \sum_{i=1}^{\infty} \xi_i \frac{\partial}{\partial u_i} + \sum_{i=0}^{\infty} \zeta_{it} \frac{\partial}{\partial u_{it}} + \ldots,$$

TABLE 22

$$k_1=k_3=1,\quad k_2=u^{-4/3}:$$

$Q(u)$	$\tilde{t}$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{u}$
$Q=\pm u^n$	$2(1-n)t$	$(1-n)x_1$	$-(n+1/3)x_2$	$(1-n)x_3$	$2u$
$Q=\gamma u^{-1/3}$, $\gamma=\pm 1$	$2t$	x_1	0	x_3	$3/2u$
	0	0	$e^{2\sqrt{\gamma/3}x_2}$	0	$-\sqrt{3\gamma}e_2{}^{(\gamma/3)^{1/2}x_2u}$
	0	0	$e^{-2\sqrt{\gamma/3}x_2}$	0	$\sqrt{3\gamma}\,e^{2(\gamma/3)^{1/2}x_2}u$
$Q=\gamma u^{-1/3}+$ $+\delta u$, $\lvert\gamma\rvert=\lvert\delta\rvert=1$	0	0	$e^{2\sqrt{\gamma/3}x_2}$	0	$-\sqrt{3\gamma}\,e^{2(\gamma/3)^{1/2}x_2}u$
	0	0	$e^{-2\sqrt{\gamma/3}x_2}$	0	$\sqrt{3\gamma}\,e^{2(\gamma/3)^{1/2}x_2}u$
$Q=\pm u$	0	0	x_2	0	$-3/2u$
	0	0	x_2^2	0	$-3x_2u$
$Q=\pm 1$	$2t$	x_1	$-1/3x_2$	x_3	$2u$
$Q\equiv 0$	$2t$	x_1	0	x_3	$3/2u$
	0	0	x_2	0	$-3/2u$
	0	0	x_2^2	0	$-3x_2u$

where $\bar{U}=U-\xi u_1-\eta u_t$, $\zeta_i=(D_x)^i\bar{U}$, $\zeta_{it}=D_t(D_x)^i\bar{U}$, ...,

$$D_x=\frac{\partial}{\partial x}+u_1\frac{\partial}{\partial u}+u_2\frac{\partial}{\partial u_1}+\dots$$

$$D_t=\frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{1t}\frac{\partial}{\partial u_1}+\dots.$$

The criterion of invariance of (14) has the form

$$X\{u_t-F(u,u_1,\dots,u_m)\}|_{\mathfrak{M}}=0.$$

This equation can be rewritten in the form

$$(D_t-F_*)\bar{U}|_{\mathfrak{M}}=0, \tag{16}$$

where we have introduced the operator

$$F_*=\frac{\partial F}{\partial u}+\frac{\partial F}{\partial u_1}D_x+\dots+\frac{\partial F}{\partial u_m}(D_x)^m.$$

We remark that Eq. (16) always has a solution, since the evolution-Eq. (14) admits translation in x and t to which there correspond the operators

$$X_1=u_1\frac{\partial}{\partial u},\quad X_2=F\frac{\partial}{\partial u}.$$

Knowledge of one solution $\bar{U}$ of Eq. (16) often makes it possible to construct a sequence of solutions of this equation (this is precisely the situation for all Lie–Bäcklund groups admitted by evolution equations which have so far been computed. For systems of equations this is not true; see [48]).

Indeed, suppose for F_* and some differential operator L the following commutator is defined:

TABLE 23

$K_i(u)$ \ $Q(u)$	$K_1=1$, $K_2=u^{-4/3}$	$K_1=u^{\sigma_1}$, $K_2=u^{\sigma_2}$, $\sigma_1\neq\sigma_2$	$K_1=e^{\alpha_1 u}$, $K_2=e^{\alpha_2 u}$, $\alpha_1\neq\alpha_2$	$K_1(u), K_2(u)$ are arbitrary function
$Q=Q(u)$ is arbitrary function	L_3	L_3	L_3	L_3
$Q=\pm e^{\alpha u}$, $\alpha\neq 0$	L_3	L_3	L_4	L_3
$Q=\pm u^n$, $n\neq 0; 1; -1/3$	L_4	L_4	L_3	L_3
$Q=\pm u^{-1/3}$	L_6	L_4	L_3	L_3
$Q=\pm u^{-1/3}+\sigma u$, $\lvert\sigma\rvert=1$	L_5	L_3	L_3	L_3
$Q=\pm u$	L_5	L_4	L_3	L_3
$Q=\pm 1$	L_4	L_4	L_4	L_3
$Q\equiv 0$	L_6	L_5	L_5	L_4

$$[F_*, L]=F_*L-LF_*.$$

Then if on solutions of (14) the operator L satisfies the Lax equation (see [122])

$$L_t=[F_*, L], \tag{17}$$

where L_t is defined by the expression $D_tL = L_t + LD_t$, then the operator L takes any solution $\bar{U}$ of Eq. (16) again into a solution $L\bar{U}$ of this same equation. Indeed, (17) is equivalent to the equation

$$[D_t-F_*, L]=0,$$

and hence on solutions of (14)

$$(D_t-F_*)(L\bar{U})=L(D_t-F_*)\bar{U}=0.$$

In other words, the action of the operator L generates an infinite chain of solutions $L^k\bar{U}$ of Eq. (16).

We remark that solvability of Eq. (17) in a particular class, generally speaking of operators L which are not differential operators (see [48]), is a necessary condition that (16) has solutions $\bar{U}(t, x, u, u_1,\dots,u_k)$ of arbitrarily high order k, and hence that Eq. (14) admits an infinite group of Lie–Bäcklund transformations. This fact has major practical importance, since solution of the Lax equation (17) is frequently a simpler problem than solution of the defining Eq. (16).

2.3.2. The defining Eq. (16) for the one-dimensional heat Eq. (13) has the form

$$U^t=(k''u_1^2+k'u_2)U+2k'u_1U^x+kU^{xx}+Q'U,$$

where $U^t=D_tU_{|(13)}$, $U^x=D_xU$, $U^{xx}=D_x^2U$.

It is assumed that the desired function U depends on a finite number of variables: $U = U(t, x, u, u_1,\dots,u_n)$. It can be shown [36] that for $n\leqslant 2$ Eq. (18) defines the Lie group of point transformations computed in [30, 31] and presented above; therefore, it is henceforth assumed that $n > 2$.

The method of investigating the definition-Eq. (18) is based on decoupling it according to the powers of u_{n+1}, u_{n+2} contained in (18) and not contained in the number of arguments of U. It is shown in the work [36] that a nontrivial solution of (18) exists only in the following cases:

1) $k = 1$, $Q = au + b$, where a, b are arbitrary constants;

2) $k = u^{-2}$, $Q = 0$;

3) $k = u^{-2}$, $Q = au$, where a is an arbitrary constant;

4) $k = u^{-2}$, $Q = b$, where b is an arbitrary constant.

It is easy to indicate point transformations reducing the case 3) to 2) and taking case 1) into a linear equation without a source. Since point changes do not alter the structure of the Lie–Bäcklund group, it is possible to restrict attention to three variants:

a) $u_t = u_{xx}$;

b) $u_t = (u^{-2}u_x)_x$;

c) $u_t = (u^{-2}u_x)_x + b$, where b is an arbitrary constant.

a) The linear equation $u_t = u_{xx}$ admits transformations (see [105]) given by the functions

$$U_0 = u, \; U_\infty = W(t, x),$$

where W(t, x) is an arbitrary solution of the equation $w_t = w_{xx}$. The Lax equation in this case has the solution

$$L_1 = D_x, \quad L_2 = tD_x + \frac{x}{2}.$$

Any Lie–Bäcklund transformation for a linear equation is determined by a linear combination of U_0, U_∞ and solutions obtained from U_0 by means of the indicated operators L_1 and L_2. Those solutions U which depend on u_n for $n \geqslant 3$, correspond to Lie–Bäcklund transformations.

b) The equation $u_t = (u^{-2}u_x)_x$ is investigated in [89]. The admissibile Lie–Bäcklund group in this case is determined by a linear combination of the functions $U_k = L^k U_0$, $k = 1, 2, \ldots$, where $U_0 = (u^{-2}u_x)_x$, and $L = D_x^2(u^{-1})D_x^{-1}$; the operator L in this case is an integro-differential operator. It can be shown that U_k can be represented in the form

$$U_k = -[D_x^2 (u^{-1}D_x)^k](x), \quad k = 1, 2, \ldots .$$

c) The case $u_t = (u^{-2}u_x)_x + b$ is essentially new (see [36]).

In this case the Lax equation has the solution $L = D_x^2\left(u^{-1}D_x - \frac{b}{2}x\right)D_x^{-2}$, while the solution of the defining equation which "starts" the chain of Lie–Bäcklund transformations has the form

$$U_0 = -\left[D_x^2\left(u^{-1}D_x - \frac{b}{2}x\right)\right](x).$$

It can be shown (see [36]) that the series U_k can be represented in the form $U_k = -D_x^2\left[u^{-1}D_x - \frac{b}{2}x\right]^{k+1}(x)$, and any solution of the defining equation can be represented as a linear combination of the U_k.

It is shown in the work [36] that the multidimensional heat equation with a source (13) admits a nontrivial Lie–Bäcklund group only in the linear cases: $k_i = \text{const}$, $i = 1, \ldots, N$, $N = 2, 3$, $Q = au + b$, where a, b are constants.

Thus, with this the group properties of the heat equation with a source have been studied in the one-, two-, and three-dimensional cases. Below examples will be given of the use of the groups of transformations admitted by the heat equation.

2.4. Invariant Solutions of the Heat Equation with a Source in the One-Dimensional Case. 2.4.1. The presence of a nontrivial group of transformations admitted by some differential equation or system carries a particular algebraic structure into the set of its solutions. This is expressed in particular in the fact that transformations of the group take any solution again into a solution of the same system of equations. The set of solutions obtained by this process can "start" with any solution including one which is in no way connected with the admissible group of transformations. This property is especially important in investigating nonlinear equations where the "value" of each exact solution is high and where transformations taking a solution into solutions often carry a nonobvious character.

If under a transformation of the group a solution goes over into itself or remains within an invariant manifold, then such solutions are called, respectively, invariant and partially invariant solutions. The search for such solutions is alleviated in connection with the possibility of describing them in terms of invariants of the same group of transformations. Since the number of invariants is always less than the total number of dependent and independent variables in the original system, the theorem on the representation of a nonsingular invariant manifold (see [67]) guarantees a reduction of the dimension in a coordinate system connected with the invariants (the special case of this fact for groups of dilations is known as the "π-theorem" in dimension theory). The number of independent variables in the space of invariants is called the rank of the invariant solution and may assume different values. Therefore, a first step in the classification of invariant solutions of a system of differential equations whose admissible group is known is enumeration of the classes of solutions having the same rank. The next step of classification is usually the calculation of an optimal system of subgroups of the basic group to which there corresponds a collection of essentially distinct invariant solutions [67]. Such solutions cannot be obtained from one another by means of the admissible group, while all other invariant solutions are obtained in just this manner. Distinguishing a collection of essentially distinct solutions makes it possible to systematize the set of invariant solutions: altogether even for cases of a small number of variables it has a cumbersome form. In the work [31] such a classification is carried out for the one-dimensional nonlinear heat equation with a source for all forms of k(u), Q(u) for which an extension of the group of admissible transformations occurs. (We remark that for the heat equation in any dimension there may exist only invariant solutions, and there may not exist partially invariant solutions distinct from the invariant solutions.)

2.4.2. Linear Heat Equation with a Source of Alternating Sign. Here we present only a fragment of the classification corresponding to the special case

$$u_t = u_{xx} + \delta u \ln u,\quad \delta = \pm 1. \tag{19}$$

Equation (19) admits (see Table 4) the Lie algebra L_4 of infinitesimal operators with basis

$$X_1 = \frac{\partial}{\partial t},\quad X_2 = \frac{\partial}{\partial x},\quad X_3 = e^{\delta t}\frac{\partial}{\partial x} - \frac{\delta}{2} e^{\delta t} x u \frac{\partial}{\partial u},$$
$$X_4 = e^{\delta t} u \frac{\partial}{\partial u}, \tag{20}$$

and the optimal system of one-parameter subgroups in this case is

$$\{X_1;\ \alpha X_1 + X_2;\ X_3;\ X_2 \pm X_3;\ X_4\},$$

where α is an arbitrary constant.

The invariant solutions corresponding to this collection can be represented in the corresponding form:

$$\begin{gathered} u(t,x) = \tilde{u}(x), \\ u(t,x) = \tilde{u}(\alpha x - t), \\ u(t,x) = \tilde{u}(t) e^{-\frac{\delta x^2}{4}}, \\ u(t,x) = \tilde{u}(t) e^{-\frac{\delta x^2}{4(1 \pm e^{-\delta t})}}. \end{gathered} \tag{21}$$

There exists no invariant solution corresponding to X_4, since for it the corresponding necessary conditions are not satisfied (see [67]).

Substitution of the representations (21) into Eq. (19) leads to an ordinary differential equation for $\tilde{u}(\lambda)$, where λ is the independent variable in the given representation.

It is necessary to note that the classification of invariant solutions from the viewpoint of similarity (that is, distinguishing a collection of essentially distinct solutions) may not have great value from the viewpoint of applications. In applications the collection of solutions found is usually adapted to a particular boundary value problem. There the interest is in global properties of such solutions, while the classification is carried out by means of the group of admissible transformations, that is, by means of local analysis. Proceeding from the collection of essentially invariant solutions, it is not always easy to perceive a "suitable" solution, since passage to a similar solution may be realized by rather complex transformations of the group. In the work [30] an example is presented where passage to a solution similar to one of the invariant solutions of the optimal collection leads to a regime with peaking, while none of the solutions of the optimal system of subgroups possess this property.

Below we shall consider in more detail the family of invariant solutions of Eq. (19) corresponding to the operator (see [32])

$$X=\varepsilon X_2+X_3.$$

In this case the invariant solution can be represented in the form

$$u(t,x)=\tilde{u}(t)\,e^{-\frac{\delta x^2}{4}\left(\frac{1}{1+\varepsilon e^{-\delta t}}\right)}. \tag{22}$$

Substitution of (22) into (19) leads to an ordinary differential equation for $\tilde{u}(t)$:

$$\tilde{u}_t=-\frac{\delta}{2}\,\frac{\tilde{u}}{1+\varepsilon e^{-\delta t}}+\delta\tilde{u}\ln\tilde{u}.$$

This equation can be integrated and gives a two-parameter family of invariant solutions

$$u(t,x)=e^{-\frac{\delta x^2}{4(1+\varepsilon e^{-\delta t})}}\,e^{e^{\delta t}\ln\left[u_0\left(\frac{1+\varepsilon e^{-\delta t}}{1+\varepsilon}\right)^{1/2\varepsilon}\right]}. \tag{23}$$

We shall consider only those solutions of the Cauchy problem for Eq. (19) in which the temperature perturbation is bounded in space: $u(t, x) \to 0$ as $|x| \to \infty$. The family (23) is a solution of the Cauchy problem for the two-parameter family of initial data

$$u(0,x)=u_0 e^{-\frac{\delta x^2}{4(1+\varepsilon)}}, \tag{24}$$

where u_0, ε are parameters $u_0\geqslant 0$, $\varepsilon<-1$ for $\delta < 0$, $-1 < \varepsilon < +\infty$ for $\delta > 0$. We shall consider in more detail possible variants of the evolution of the initial perturbation (24).

a) The solution (23) for $\varepsilon = 0$ goes over into

$$u(t,x)=\sqrt{e}\,e^{-\frac{\delta x^2}{4}}\,e^{e^{\delta t}\left(\ln u_0-\frac{1}{2}\right)}, \tag{25}$$
$$u_0>0,\ \delta>0.$$

The half width of the temperature distribution (25) does not depend on time and is equal to $2\sqrt{\ln 2/\delta}$.

For $u_0 < \sqrt{e}$ at each fixed point x the temperature $u(t, x)$ decreases monotonically to zero, $\lim\limits_{t\to+\infty} u(t,x)=0$, for $u_0 = \sqrt{e}$ the solution (25) is stationary, and for $u_0 > \sqrt{e}$ the temperature increases monotonically and unboundedly at each fixed point x as $t \to +\infty$.

Figures 9 and 10 show the numerical realization of a solution with constant half width.

b) We consider the solution (23) for $\varepsilon > 0$, $\delta > 0$. In this case the half width of the initial distribution (24) is greater than the half width of the solution (25). The half width of the solution (23) is a monotonically decreasing function and as $t \to +\infty$ tends to the half width of the distribution (25). There are three possibilities for the amplitude of the solution.

If $u_0<u_2(\varepsilon)=e^{1/(2+2\varepsilon)}$, then the amplitude decreases monotonically and $\lim\limits_{t\to+\infty} u_{\max}(t)=0$.

If $u_0>u_1(\varepsilon)=(1+\varepsilon)^{1/2\varepsilon}$, then the amplitude increases monotonically and unboundedly as $t \to +\infty$.

If $u_2(\varepsilon)<u_0<u_1(\varepsilon)$, then the amplitude is a nonmonotone function of time: it first increases (but does not reach the magnitude $\sqrt{e}$ – the amplitude of the stationary solution), and it then decreases to zero as $t \to +\infty$.

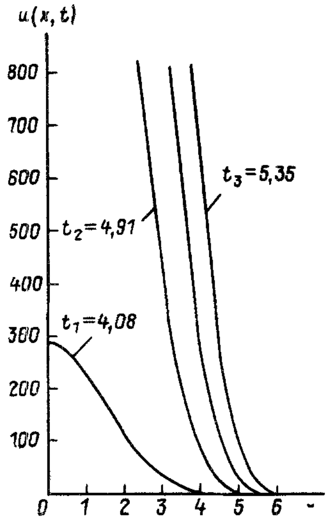


Fig. 9

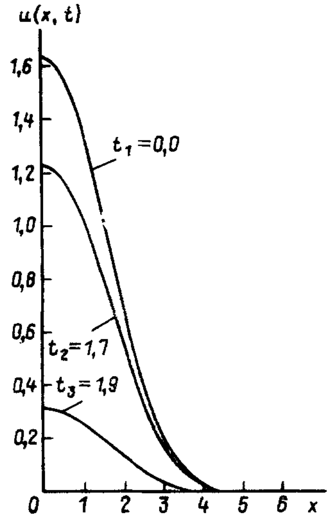


Fig. 10

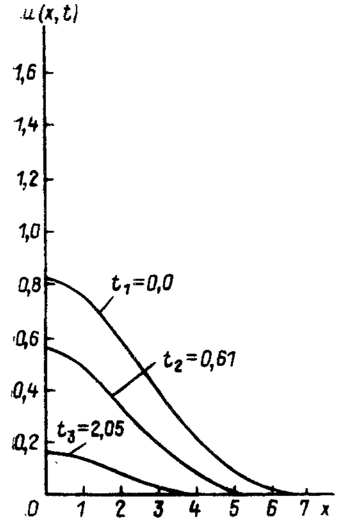


Fig. 11

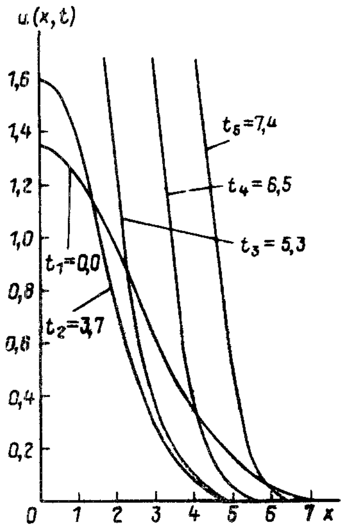


Fig. 12

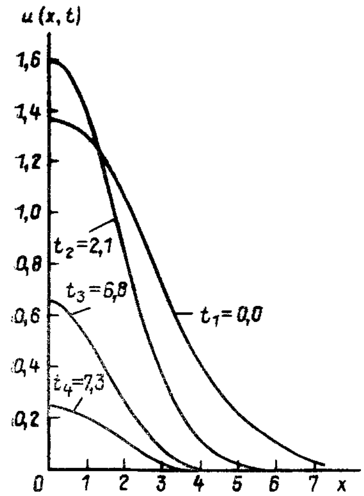


Fig. 13

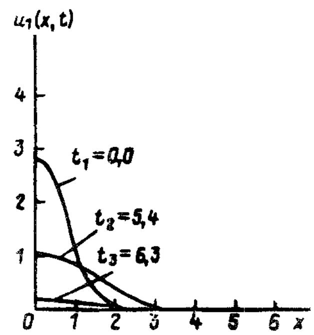


Fig. 14

The solution (23) for $\varepsilon > 0$, $\delta > 0$ is a solution with a monotonically decreasing half width.

Figures 11, 12, 13 show a numerical realization of a solution with a monotonically decreasing half width.

c) Solutions with a monotonically increasing half width are determined by formula (23) either for $\delta > 0$, $-1 < \varepsilon < 0$ or for $\delta < 0$, $\varepsilon < -1$.

In the first case the half width of the solution (23) is a monotonically increasing function. At the initial time $t = 0$ it is less than the half width (1.4), while as $t \to +\infty$ it tends to the value $2\sqrt{\ln 2/\delta}$.

If $u_0 < u_1(\varepsilon)$, then the amplitude decreases monotonically to zero.

If $u_0 > u_2(\varepsilon)$, then the amplitude increases monotonically and unboundedly as $t \to +\infty$.

If $u_1(\varepsilon) < u_0 < u_2(\varepsilon)$, then the amplitude is a nonmonotone function: it first decreases, it attains some maximum value at a finite time, and it then grows without bound in a monotonic way as $t \to +\infty$.

A numerical realization of the solution for $\delta > 0$, $-1 < \varepsilon < 0$ is shown in Figs. 14, 15, and 16.

In the second case ($\delta < 0$, $\varepsilon < -1$) the half width of the solution (23) grows monotonically and unboundedly as $t \to +\infty$.

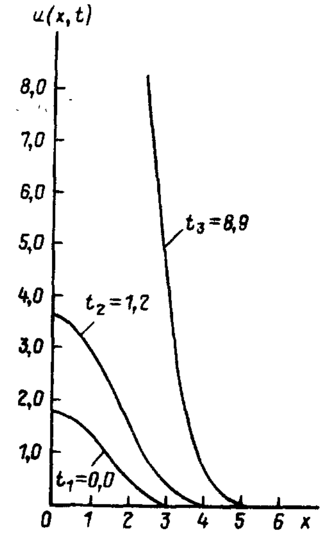


Fig. 15

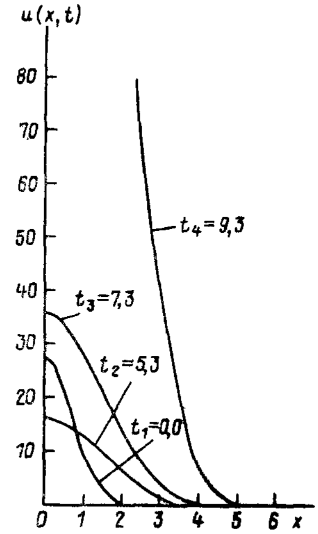


Fig. 16

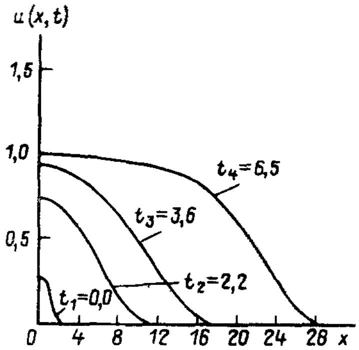


Fig. 17

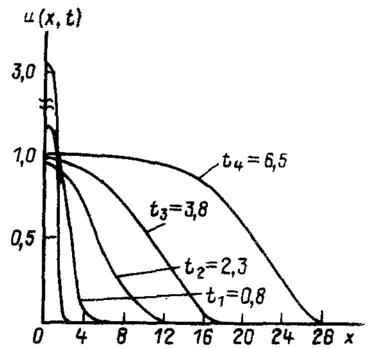


Fig. 18

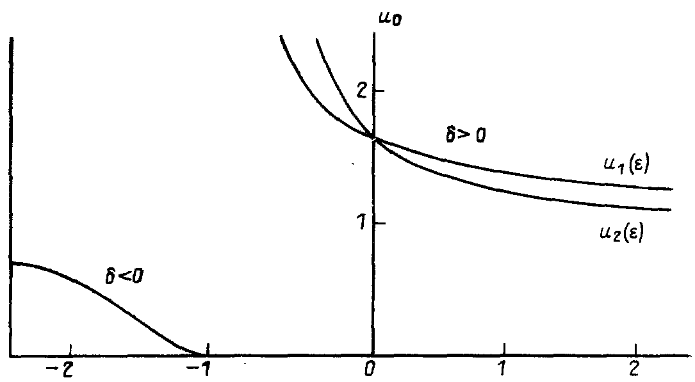


Fig. 19

If $u_0 < u_2(\varepsilon)$, then the amplitude increases monotonically to the value $u(0, \infty) = 1$.

If $u_0 > u_2(\varepsilon)$, then the amplitude first decreases monotonically and then, while growing monotonically, tends to the value $u(0, \infty) = 1$ as $t \to +\infty$.

Such a solution (see Figs. 17, 18) is analogous to the solution of a traveling wave considered in the work [54].

The behavior of the temperature distribution (23) in space and time is determined by the pair of parameters (ε, u_0). Figure 19 shows the plane of parameters of the initial data (ε, u_0) which is separated by the curves $u_1(\varepsilon)$ and $u_2(\varepsilon)$ into zones corresponding to the regimes described above.

d) In order to graphically represent the behavior of the solution (23) in time, we go over to variables of the amplitude and half width of the temperature perturbation:

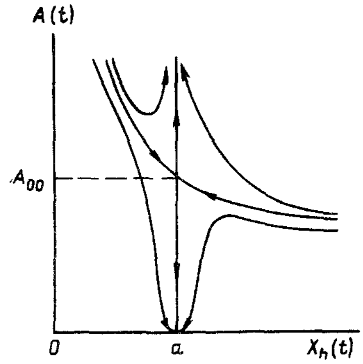


Fig. 20

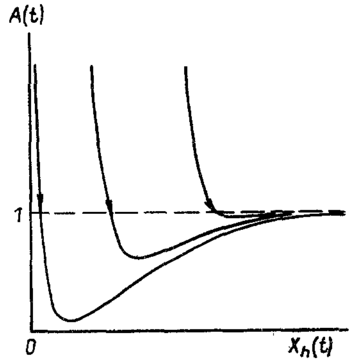


Fig. 21

$$A(t)=u(t,0)=e^{e^{\delta t}\ln\left[u_0\left(\frac{1+\varepsilon e^{-\delta t}}{1+\varepsilon}\right)^{1/2\varepsilon}\right]},$$
$$X_h(t)=2\sqrt{\frac{\ln 2}{|\delta|}}\sqrt{\operatorname{sign}(\delta)(1+\varepsilon e^{-\delta t})}. \tag{26}$$

The connection of the amplitude with the half width is given by the equation

$$\frac{dA}{dX_h}=\frac{Aa^2\operatorname{sign}(\delta)(1-2\operatorname{sign}(\delta)X_h^2a^{-2}\ln A)}{X_h\left(X_h^2-a^2\operatorname{sign}(\delta)\right)}, \tag{27}$$

where $a=2\sqrt{\ln 2/|\delta|}$ is the half width of the stationary solution (25). The integral of Eq. (27) has the form

$$A^{\left(X_h^2-a^2\operatorname{sign}(\delta)\right)}X_L^{-a^2\operatorname{sign}(\delta)}=A^{\left(X_{h_0}^2-a^2\operatorname{sign}(\delta)\right)}X_{h_0}^{-a^2\operatorname{sign}(\delta)}, \tag{28}$$

where $X_{h_0}=X_h(0)$, $A_0=A(0)=u_0(0)$.

In the (X_h, A) plane Eq. (27) has a zero isocline

$$A=e^{\frac{2\ln 2\operatorname{sign}(\delta)}{|\delta|X_h^2}},$$

and an infinite isocline $(dX_h/dA=0)$:

$$X_h^2=4\ln 2\,\frac{\operatorname{sign}(\delta)}{|\delta|},$$

which exists only for $\delta>0$.

A singular point of Eq. (27) for finite $A\neq 0$ and $X_h>0$ for $\delta>0$ is the point with coordinates $X_h=a$, $A_{00}=\sqrt{e}$. The singular point is a saddle point with separatrices (see Fig. 20)

$$X_h=a,\quad A=\left(\frac{X_h}{a}\right)^{\left(\left(\frac{X_h}{a}\right)^2-1\right)^{-1}}.$$

In the case $\delta<0$ the point (0, 0) is a saddle point with separatrices $X_h\equiv 0$ and $A\equiv 0$ (see Fig. 21).

The course of the integral curves of Eq. (27) is represented in Figs. 22, 23.

We note that all results enumerated in this section carry over easily to the case where the volumetric source of heat has a lienar "increment" $Q=\alpha u\ln u+\beta u$ by means of a transformation of the equivalence group $u\to\gamma u$, $\gamma=\text{const}$.

2.4.3. Nonlinear Heat Equation with a Source of Variable Sign. We consider the heat equation in the case where the source and the coefficient of thermal conductivity have power dependence on the temperature

$$u_t=(u^\sigma u_x)_x+\alpha u^{\sigma+1}+\delta u,\quad \sigma>0. \tag{29}$$

One of the invariant solutions (see [31]) in this case has the form

$$u(t,x)=\tilde{u}(x)(\lambda+(1-\lambda)e^{-\sigma\delta t})^{-1/\sigma}. \tag{30}$$

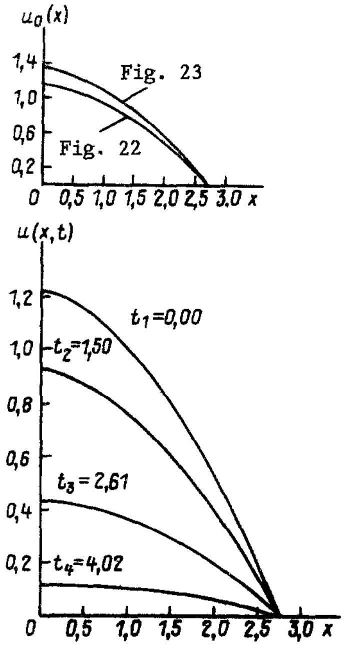


Fig. 22

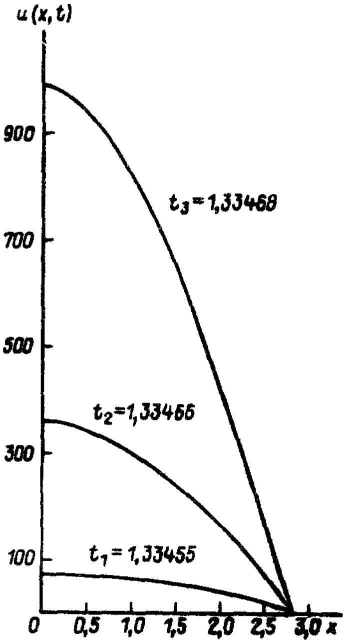


Fig. 23

The representation (30) gives a solution of the Cauchy problem if the following conditions are satisfied: $\alpha > 0$, $\lambda\delta < 0$. [These conditions ensure that the heat flux $k(u)u_x$ vanishes if the temperature vanishes.]

The function $\tilde{u}(x)$ is a solution of the equation

$$(\tilde{u}^{\sigma}\tilde{u}_x)_x+\alpha\tilde{u}^{\sigma+1}+\lambda\delta\tilde{u}=0,$$

and has the form

$$\tilde{u}(x)=\begin{cases}\left(-2\lambda\delta(\sigma+1)\dfrac{1}{\alpha(\sigma+2)}\right)^{1/\sigma}\cos^{2/\sigma}\left(\dfrac{\sigma}{2}\sqrt{\dfrac{\alpha}{\sigma+1}}\,x\right), & \text{for } |x|\leqslant L_s/2,\\ 0, & \text{for } |x|>L_s/2,\end{cases}$$

where $L_s = \sqrt{(\sigma + 1)\alpha^{-1}}/\sigma$ is the fundamental length. We remark that L_s is determined only by the parameters of the medium σ and α and does not depend on δ, that is, the presence of a linear source (sink) does not affect the spatial structure of the invariant solution.

For $\delta > 0$ for any $u_0=(-2\lambda\delta(\sigma+1)\alpha^{-1}(\sigma+2)^{-1})^{1/\sigma}>0$ the time characteristic of the solution (30) $g(t)=(\lambda+(1-\lambda)e^{-\sigma\delta t})^{-1/\sigma}\to+\infty$ as $t\to\tau=-\ln[\lambda(\lambda-1)^{-1}]/\sigma\delta$.

For $\delta < 0$ there exists a critical value of the amplitude of the initial distribution $u_*=(2|\delta|(\sigma+1)^{1/\sigma}/\alpha(\sigma+2))$ (the amplitude of the stationary solution) such that

(a) if $u_0 > u_*$, then $g(t) \to +\infty$ for $t \to \tau$;

(b) if $u_0 < u_*$, then $g(t) \to 0$ for $t \to +\infty$;

(c) if $u_0 = u_*$, then $g(t) \equiv 1$, where

$$\tau=\frac{\ln\left[u_0^{\sigma}(u_0^{\sigma}-u_*^{\sigma})^{-1}\right]}{\sigma|\delta|}.$$

The solutions considered convey the qualitative features of the Cauchy problem for the heat equation with a source of variable sign and nonlinear type: the phenomenon of localization and the presence of superfast regimes of growth of thermal perturbations.

2.5. Two-Dimensional Heat Conduction: Directed Propagation of Heat in an Anisotropic, Nonlinear Medium. 2.5.1. Here we consider an example of the construction of an invariant solution describing the propagation of heat with different types of localization in different spatial directions which occurs in a regime with peaking [4].

As shown in part 2.4, the nonlinear, anisotropic heat equation with a source

$$u_t=(u^{\sigma_1}u_x)_x+(u^{\sigma_2}u_y)_y+u^{\beta},\quad \sigma_1>0,\quad \sigma_2>0,\quad \beta>0, \tag{31}$$

is invariant relative to transformations defined by the Lie algebra of infinitesimal operators of translation in t, x, y and dilation

$$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x},\quad X_3=\frac{\partial}{\partial y},$$

$$X_4=2(1-\beta)t\frac{\partial}{\partial t}+(\sigma_1+1-\beta)x\frac{\partial}{\partial x}+(\sigma_2+1-\beta)y\frac{\partial}{\partial y}+2u\frac{\partial}{\partial u}.$$

We shall consider self-similar solutions invariant relative to the subgroup of dilations defined by the operator X_4 which we write in the form

$$X_4=2(1-\beta)(t_0-t)\frac{\partial}{\partial(t_0-t)}+(\sigma_1+1-\beta)x\frac{\partial}{\partial x}+(\sigma_2+1-\beta)y\frac{\partial}{\partial y}+2u\frac{\partial}{\partial u}. \tag{32}$$

The parameter $t_0 > 0$ introduced determines the time interval $[0, t_0)$ on which the solution exists (a solution does not exist globally). To the operator (32) there corresponds the invariant solution

$$u(t, x, y)=(t_0-t)^{1/1-\beta}\tilde{u}(\xi, \eta), \tag{33}$$

where

$$\xi=x(t_0-t)^{\frac{\sigma_1+1-\beta}{2(\beta-1)}},\quad \eta=y(t_0-t)^{\frac{\sigma_2+1-\beta}{2(\beta-1)}}.$$

Substitution of (33) into Eq. (31) leads to an equation for the function $\tilde{u}(\xi, \eta)$:

$$(\tilde{u}^{\sigma_1}\tilde{u}_\xi)_\xi+(\tilde{u}^{\sigma_2}\tilde{u}_\eta)_\eta+\frac{\sigma_1+1-\beta}{2(\beta-1)}\xi\tilde{u}_\xi+\frac{\sigma_2+1-\beta}{2(\beta-1)}\eta\tilde{u}_\eta+\tilde{u}^\beta-\frac{1}{\beta-1}\tilde{u}=0. \tag{34}$$

Regimes with peaking corresponding to the case $\beta > 1$ are considered below. The dependence of the direction of heat propagation on the relations among the parameters β, σ_1, σ_2 follows easily from (33).

Equation (34) was solved numerically by two methods: the method of finite elements and a finite-difference method (see [4]). It is assumed that the temperature distribution at the initial and subsequent times is symmetric relative to the axes Ox, Oy. This makes it possible to consider the problem only in one quadrant of the (x, y) plane.

Let

$$\Omega=\{(x, y),\ 0<x<a_1,\ 0<y<a_2\},$$
$$\Omega_1=\{(x, y),\ 0\leqslant x\leqslant b_1,\ 0\leqslant y\leqslant b_2\},$$

where a_α, b_α, $\alpha=1,2$ are positive constants,

A function u(t, x, y) is sought which satisfies Eq. (31) in the cylinder $Q_T = \Omega \times (0, T]$, where T = const, $0 < T < t_0$, the initial condition at t = 0

$$u(0, x, y)=u_0(x, y)=\begin{cases}1, & (x, y)\in\Omega_1,\\ 0, & (x, y)\bar{\in}\Omega_1\end{cases} \tag{35}$$

and the boundary conditions for t > 0

$$\begin{aligned}&u=0, \text{ if } x=a_1 \text{ or } y=a_2,\\ &\frac{\partial u}{\partial n}=0, \text{ if } x=0 \text{ or } y=0,\end{aligned} \tag{36}$$

(n is the normal to the boundary of the domain Ω).

The relation among the quantities β, $(\sigma_1 + 1)$, and $(\sigma_2 + 1)$ affects the character of heat propagation as is evident from the representation of the invariant solution (33). Below we consider two types of solutions which in analogy with one-dimensional regimes [45, 74] we call HS-S- and HS-LS-regimes. The results presented also make it possible to judge the qualitative features of self-similar regimes of heat propagation with other parameters of the medium, in particular, in an LS-S-regime which we therefore do not consider. We remark also that the formulations of the problems can easily be carried over also to a three-dimensional, anisotropic medium.

HS-S-Regime. Problem (35), (36) was solved with the following parameters:

$$\sigma_1=3,\ \sigma_2=2,\ \beta=\sigma_2+1=3.$$

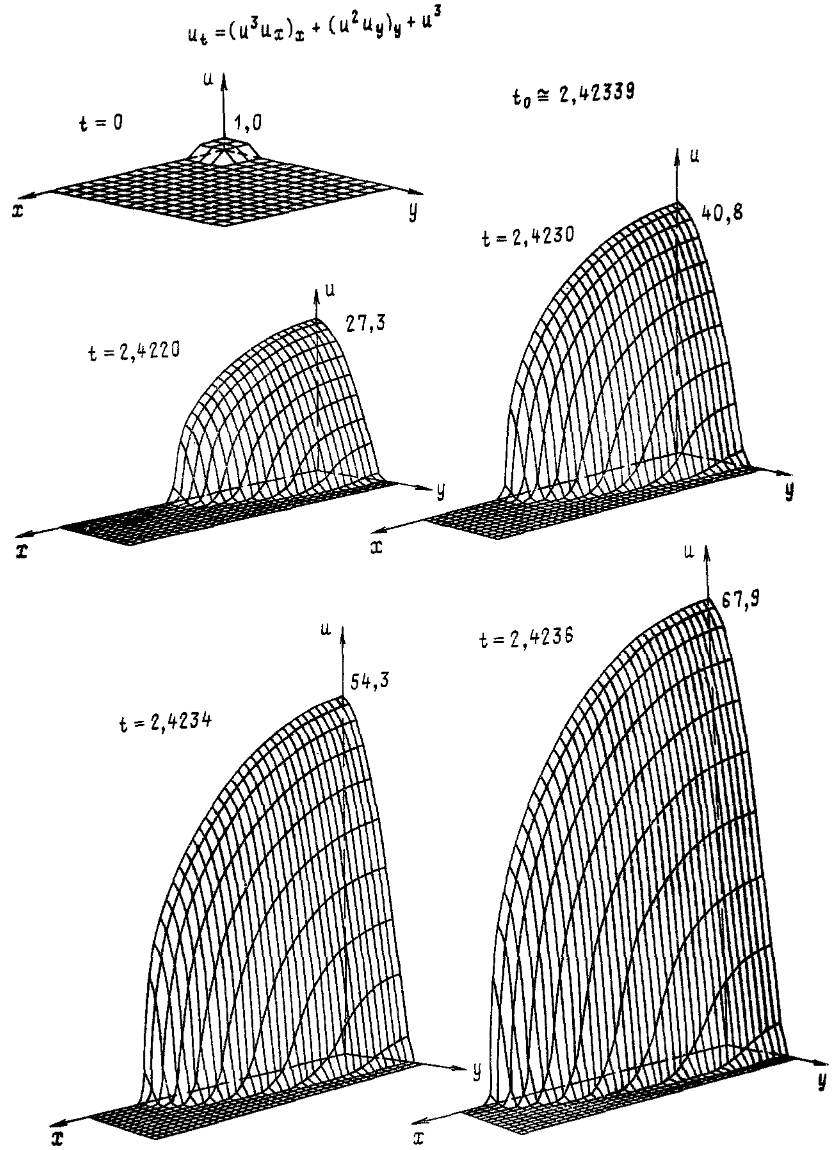


Fig. 24

In this case no propagation of heat occurs in the y direction, while in the x direction after a finite time t_0 propagation of heat to infinity occurs (an HS-regime).

Figure 24 shows the numerical solution u(t, x, y) at different times; Fig. 25 shows its level lines. It is evident that the support of the solution is a convex region. The time of peaking obtained in calculations was t_0 = 2.42389, while the discrepancy of the procedure is less than 4%.

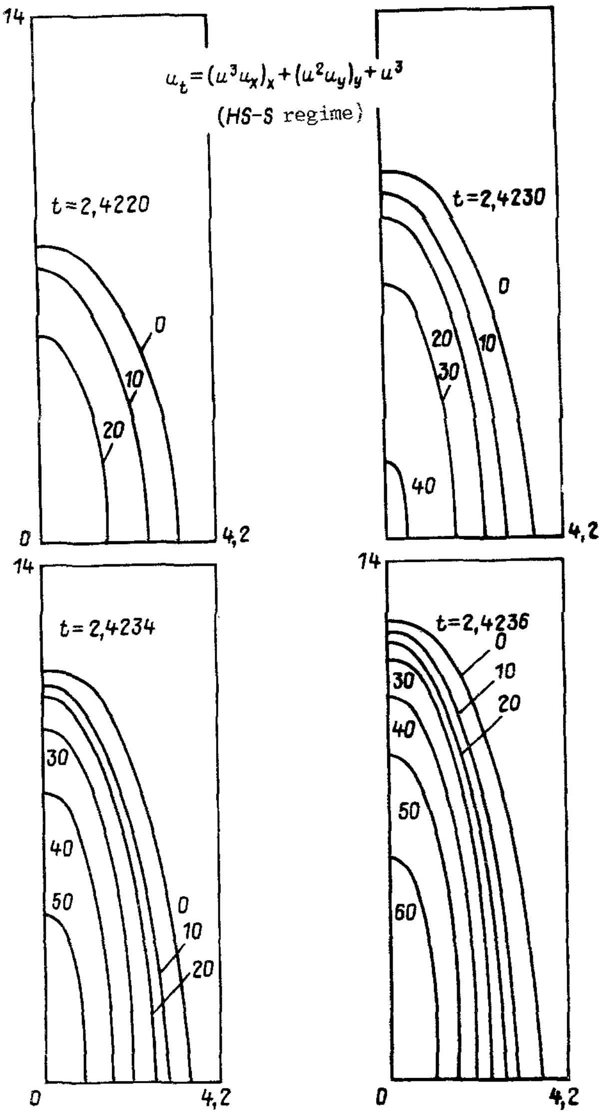
Figure 26a, b presents results of self-similar processing of the numerical solution u(t, x, y). Namely, the quantity [see (33)]

$$\tilde{u}(\xi, \eta) = (t_0 - t)^{-1/2} u(t, x, y),$$
$$x = (t_0 - t)^{-1/4} \xi, \quad y \equiv \eta,$$

was computed which in the course of time $t \to t_0$ approaches a solution of the self-similar problem. Beginning at a certain time the curves $\tilde{u}(\xi, \eta)$ practically coincide which bears witness to the passage to a self-similar regime. A solution of the elliptic equation (34) was thus obtained numerically.

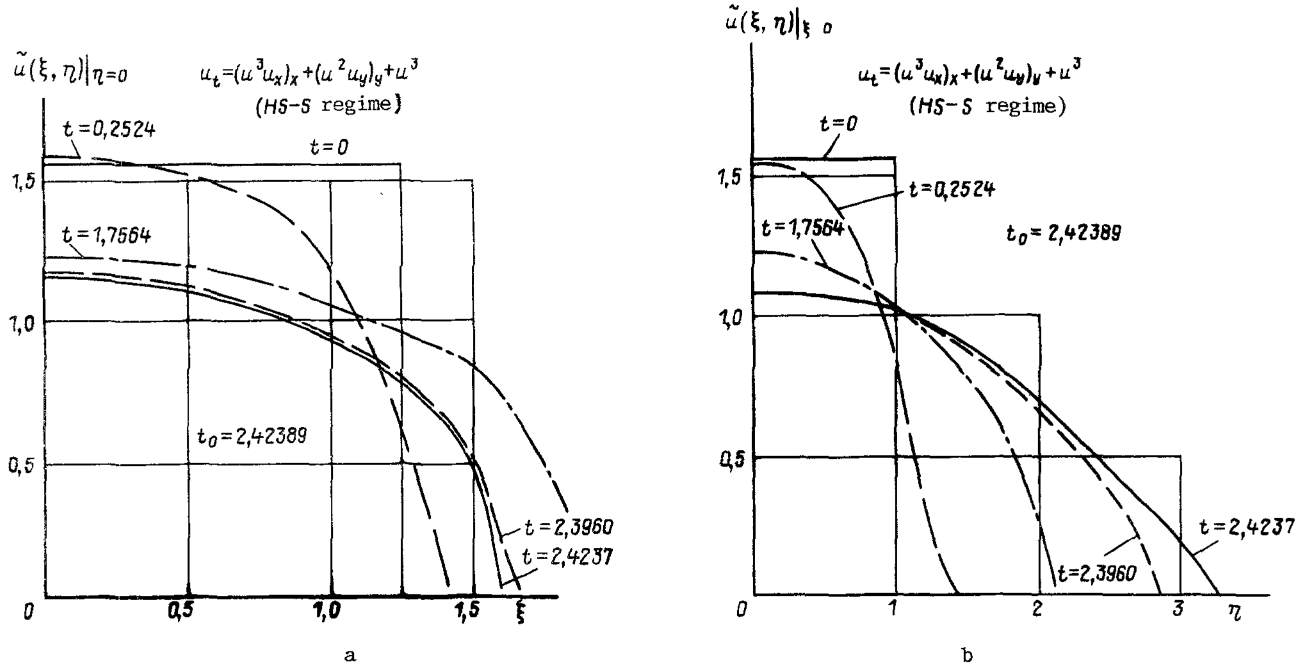
HS-LS-Regime. This regime is the most paradoxical manner of heating an anisotropic medium.

Problem (34), (35) was solved with the following parameters: $\sigma_1 = 3$, $\sigma_2 = 1$, $\beta = 3$. In this case the invariant solution (33) approaches infinity in the x direction (an HS-regime),



14
$u_t=(u^3u_x)_x+(u^2u_y)_y+u^3$
(HS-S regime)
t=2,4220
0
10
20
0
4,2
t=2,4230
0
20
10
30
40
4,2
14
t=2,4234
30
40
50
0
10
20
0
4,2
14
t=2,4236
0
10
20
30
40
50
60
0
4,2

Fig. 25



$\tilde{u}(\xi,\eta)|_{\eta=0}$
$u_t=(u^3u_x)_x+(u^2u_y)_y+u^3$
(HS-S regime)
t=0,2524
t=0
1,5
t=1,7564
1,0
t_0=2,42389
0,5
t=2,3960
t=2,4237
0
0,5
1,0
1,5
ξ
a
$\tilde{u}(\xi,\eta)|_{\xi=0}$
$u_t=(u^3u_x)_x+(u^2u_y)_y+u^3$
(HS-S regime)
t=0
1,5
t=0,2524
t=1,7564
t_0= 2,42389
1,0
0,5
t=2,4237
t=2,3960
0
1
2
3
η
b

Fig. 26

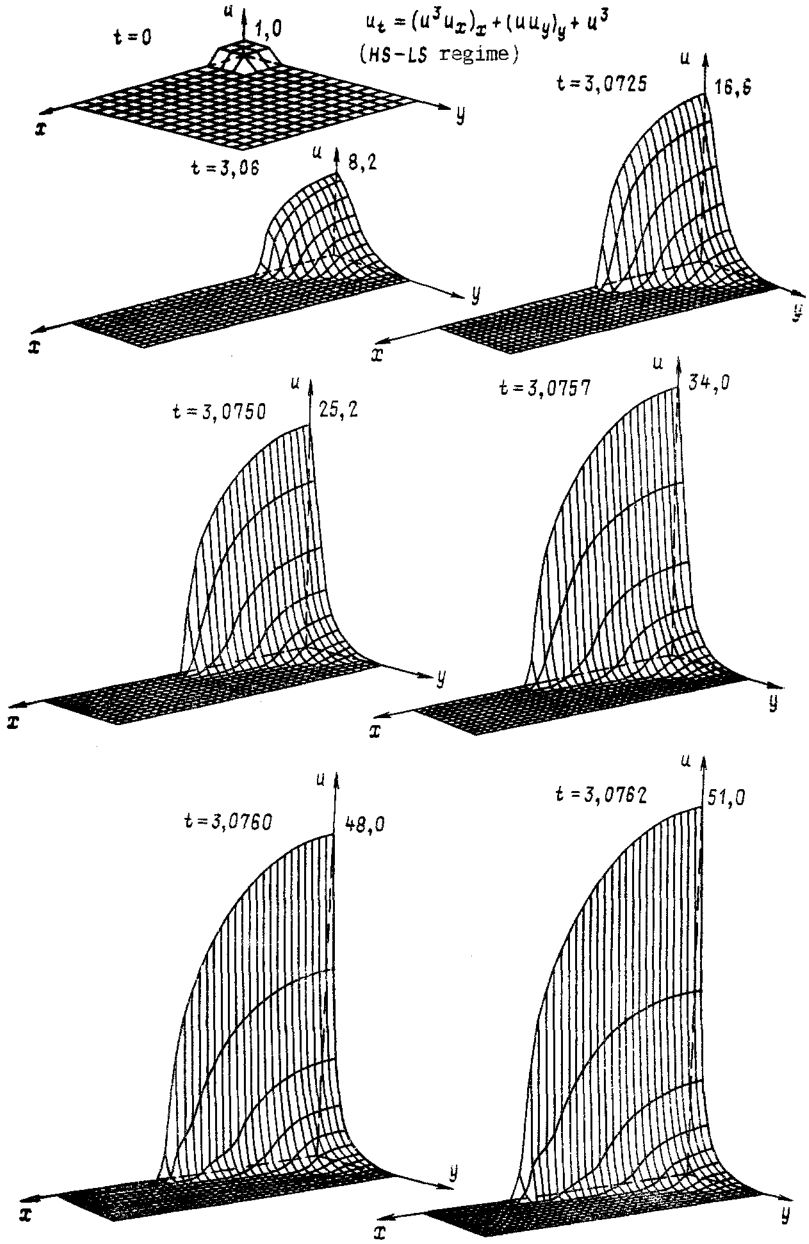


Fig. 27

while in the y direction curtailment of the effective depth of heating occurs (an LS-regime), that is, the interval along the y axis where, for example, $u \geq u_{max}/2$ tends to zero as $t \to t_0$. Thus, at the final stage the solution (see Fig. 27) is an infinitely heated line off which the temperature is finite. The level lines of the solution are shown in Fig. 28. We point out the characteristic cross-like form of the support. This form of the support is determined by two processes. Along the x axis accelerated heat propagation proceeds in an HS-regime, while along the y axis the solution tends to a self-similar LS-regime in which the heat front is located at infinity. The time of peaking obtained in the calculation was t_0 = 3.07660.

Figure 29a, b shows the results of self-similar processing of u(t, x, y) for x = 0 and y = 0, respectively. The functions $\tilde{u}(\xi, \eta)|_{x=0}$ and $\tilde{u}(\xi, \eta)|_{y=0}$ are computed from the formulas [see (33)]

$$\tilde{u}(\xi, \eta) = (t_0 - t)^{1/2} u(t, x, y),$$
$$x = \xi (t_0 - t)^{-1/4}, \quad y = \eta (t_0 - t)^{1/4}.$$

In the course of time the functions $\tilde{u}(\xi, \eta)$ tend to a solution of the elliptic equation (34),

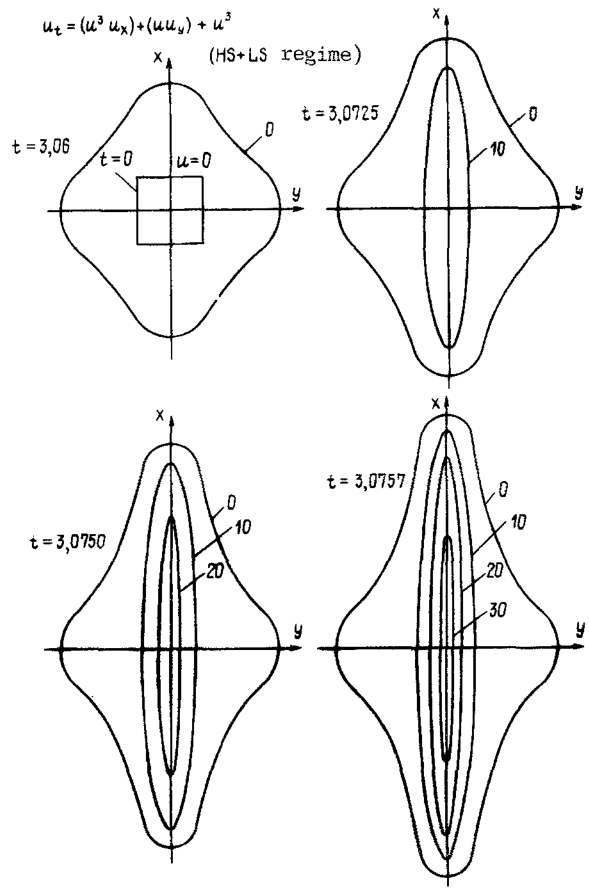


Fig. 28

We note that the heat propagation analogous to that described above will occur for any σ_1, σ_2, and β for which the relation $\sigma_1 + 1 > \beta > \sigma_2 + 1$ holds [see (33)]. Moreover, the anisotropy $\sigma_1 > \sigma_2$ may be expressed arbitrarily small. This leads only to a change of the rate of growth of the amplitude and propagation of the front; the qualitative features of the process remain the same.

We emphasize that in this subsection, on the one hand, we have presented a numerical investigation of an evolutional (structural) stability of the self-similar solution (33) of the nonlinear, anisotropic heat equation with a source (31). On the other hand, we have constructed a numerical solution $\tilde{u}(\xi, \eta)$ of the nonlinear elliptic equation (34). This is of particular interest, since so far there exist no efficient methods of integrating such equations. The construction of this solution can in principal not be done by direct solution of Eq. (34) by the method of determination, since the solution $u(\xi, \eta) \not\equiv 0$ in such a formulation is unstable (regarding theoretical methods of investigating such problems see, for example, [16, 87]).

2.5.2. An Example of the Construction of an Invariant Solution on an Infinite Group. An unlimited reserve of invariant solutions can be generated by an infinite group. We consider the two-dimensional heat equation

$$u_t = \Delta(\ln u) \equiv \left(\frac{u_x}{u}\right)_x + \left(\frac{u_y}{u}\right)_y, \tag{37}$$

which admits (see Table 12) the algebra of operators

$$X_1 = \frac{\partial}{\partial t};\quad X_2 = t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u};\quad X_\infty = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} - 2A_x u\frac{\partial}{\partial u},$$

where A(x, y), B(x, y) is an arbitrary solution of the Cauchy–Riemann system

$$A_x = B_y,\ A_y = -B_x. \tag{38}$$

A solution of the system (38) can be written in the form $A + iB = \Phi(x + iy)$, where Φ is an arbitrary differentiable function.

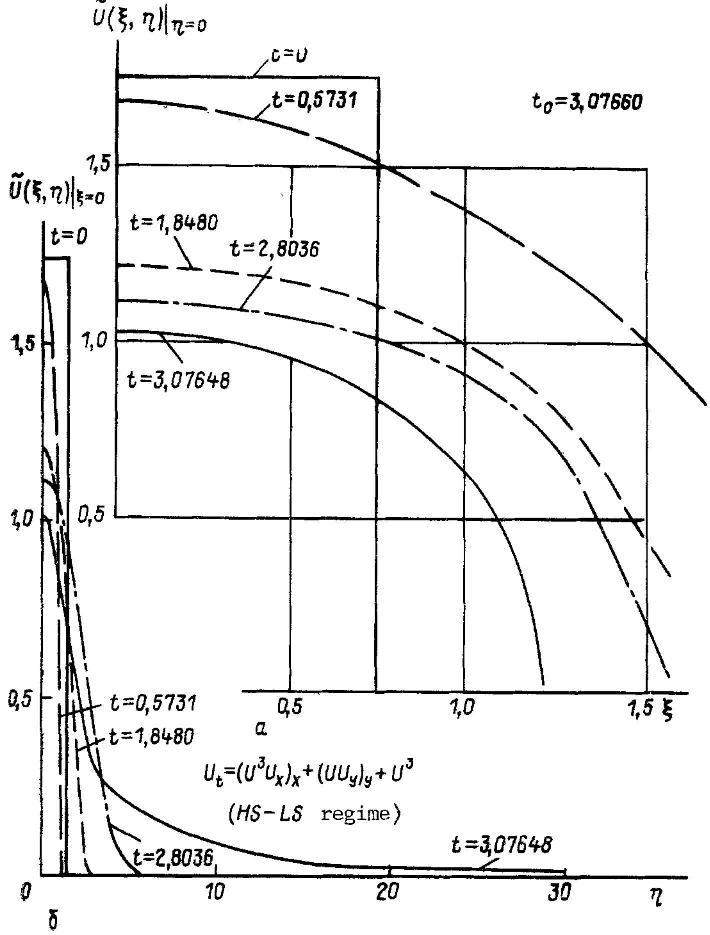


Fig. 29

As an example we consider $\Phi(x + iy) = e^{x+iy}$, where $A = e^x \cos y$, $B = e^x \sin y$, $A_x = e^x \times \cos y$, and the operator X_∞ takes the concrete form

$$\overline{X}_\infty = e^x \cos y \frac{\partial}{\partial x} + e^x \sin y \frac{\partial}{\partial y} + 2ue^x \cos y \frac{\partial}{\partial u}.$$

In order to construct an invariant solution of rank one of Eq. (37), it is necessary to consider a two-dimensional subalgebra. We consider the algebra $\langle X_2, \overline{X}_\infty\rangle$ for which a complete collection of invariants can be chosen as follows:

$$I_1 = u\frac{e^{2x}}{t}, \quad I_2 = \frac{e^x}{\sin y}.$$

We seek an invariant solution in the form

$$u(t, x, y) = t\,\frac{\tilde{u}(\lambda)}{e^{2x}}, \quad \lambda = \frac{e^x}{\sin y},$$

where the function $\tilde{u}(\lambda)$ is determined by the ordinary differential equation

$$\lambda^4(\tilde{u}\tilde{u}_{\lambda\lambda} - \tilde{u}_\lambda^2) + 2\lambda^3\tilde{u}\tilde{u}_\lambda - \tilde{u}^3 = 0. \tag{39}$$

Making in (39) the change of variables $\tilde{u}(\lambda) = e^{\hat{u}(\lambda)}$, we obtain the equation

$$\lambda\hat{u}_{\lambda\lambda} + 2\lambda^3\hat{u}_\lambda = e^{\hat{u}},$$

which by means of the substitution $\hat{u}(\lambda) = y(x)$, $\lambda = 1/x$ goes over into the equation $y'' = e^y$ whose general solution is known [53] (the method of integrating it is also based on the group of transformations which it admits). As a result the desired invariant solution can be represented in the form of the three families

$$u(x, y, t) = \frac{2t}{(\sin y + c_2 e^x)^2},$$

$$u(x, y, t) = \frac{2c_1^2 t e^{c_1(e^{-x}\sin y + c_2)}}{e^{2x}(1 - e^{c_1(e^{-x}\sin y + c_2)})^2},$$

$$u(x, y, t)=\frac{c_1^2 t}{2e^{2x}}\left[\operatorname{tg}^2\left(\frac{c_1}{2}\left(\frac{\sin y}{e^x}+c_2\right)\right)+1\right],$$

where c_1, c_2 are arbitrary constants.

2.5.3. "Spiral Waves" in a Nonlinear Thermally Conducting Medium. We consider the invariant solution describing spiral propagation of inhomogeneities in a nonlinear medium constructed in the dissertation of S. R. Svirshchevskii. The equation

$$u_t=(u^\sigma u_x)_x+(u^\sigma u_y)_y+u^\beta, \quad \beta>1, \tag{40}$$

admits (see Table 8), in particular, the operators

$$X_1=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y},$$

$$X_2=(1-\beta)t\frac{\partial}{\partial t}+\frac{\sigma+1-\beta}{2}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+u\frac{\partial}{\partial u}.$$

We write Eq. (40) and the operator $X = c_0X_1 + X_2$ in polar coordinates:

$$u_t=\frac{1}{\sigma+1}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u^{\sigma+1}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u^{\sigma+1}}{\partial \varphi^2}\right\}+u^\beta, \tag{41}$$

$$X=(1-\beta)(t_0-t)\frac{\partial}{\partial(t_0-t)}+\frac{\sigma+1-\beta}{2}\left(r\frac{\partial}{\partial r}\right)+u\frac{\partial}{\partial u}+(1-\beta)c_0\frac{\partial}{\partial \varphi}, \tag{42}$$

where c_0 and t_0 are constants, $t_0 > 0$. Proceeding from the complete collection of invariants of the operator (42)

$$I_1=r(t_0-t)^{\frac{\beta-\sigma-1}{2(1-\beta)}};\quad I_2=\varphi-c_0\ln(t_0-t);\quad I_3=u(t_0-t)^{-1/1-\beta};$$

we seek a solution of Eq. (41) invariant relative to X in the form

$$\begin{gathered} u(t, r, \varphi)=(t_0-t)^{\frac{1}{1-\beta}}\tilde{u}(R, \Phi), \\ R=r(t_0-t)^{\frac{\beta-\sigma-1}{2(1-\beta)}},\quad \Phi=\varphi-c_0\ln(t_0-t). \end{gathered} \tag{43}$$

The function $\tilde{u}(R, \Phi)$ is determined by the equation

$$\frac{1}{\sigma+1}\left\{\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \tilde{u}^{\sigma+1}}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2\tilde{u}^{\sigma+1}}{\partial \Phi^2}\right\}=\frac{\beta-\sigma-1}{2}R\frac{\partial \tilde{u}}{\partial R}-c_0\frac{\partial \tilde{u}}{\partial \Phi}+\tilde{u}-\tilde{u}^\beta. \tag{44}$$

It is easy to see that the invariant solution (43), (44) describes the propagation of inhomogeneities of the temperature field (local maxima, weak discontinuities, etc.) along a logarithmic spiral. The trajectories of such inhomogeneities are represented schematically in Fig. 30: For $\beta < \sigma + 1$ we have expanding spirals, for $\beta > \sigma + 1$ contracting spirals, and for $\beta = \sigma + 1$ we have circles. The arrows indicate the direction of motion of the inhomogeneities with increasing t ($c_0 > 0$). We note that the growth of u as well as the motion of the inhomogeneities along spirals takes place in a regime with peaking.

In the three-dimensional case it is possible to construct analogous solutions which describe the propagation of inhomogeneities u along spirals wound on the surface of cones with vertex at the origin. Numerical realization of the invariant solution (43) encounters great difficulties. The case $c_0 = 0$ of the solution (43) describing the evolution of a complex "architecture" of inhomogeneities u in a regime with peaking is realized numerically in the work [60].

2.6. Linearization and Other Questions. Heat equations admitting an infinite group of transformations occupy of special place. The presence of an infinite chain of transformations leaving the equation unchanged makes it possible to construct a corresponding chain of solutions invariant relative to the Lie–Bäcklund group. However, the chief property of such equations is related to the possibility of their linearization. The presence of an infinite Lie–Bäcklund group (more precisely, automorphicity of the equation) is a necessary condition for the presence of a (nonpoint) transformation taking it into a linear equation [48]. Although sufficient conditions for linearizability have not been obtained for all heat equations admitting a nontrivial Lie–Bäcklund group it is possible to find a linearizing transformation. The presence of linearizing transformations makes it possible to construct

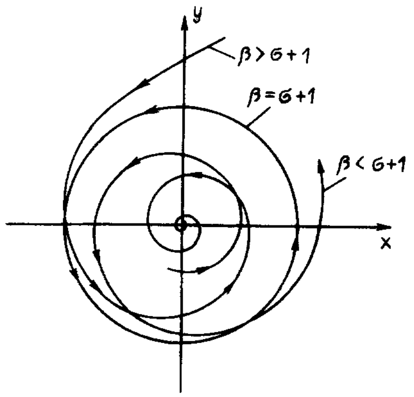


Fig. 30

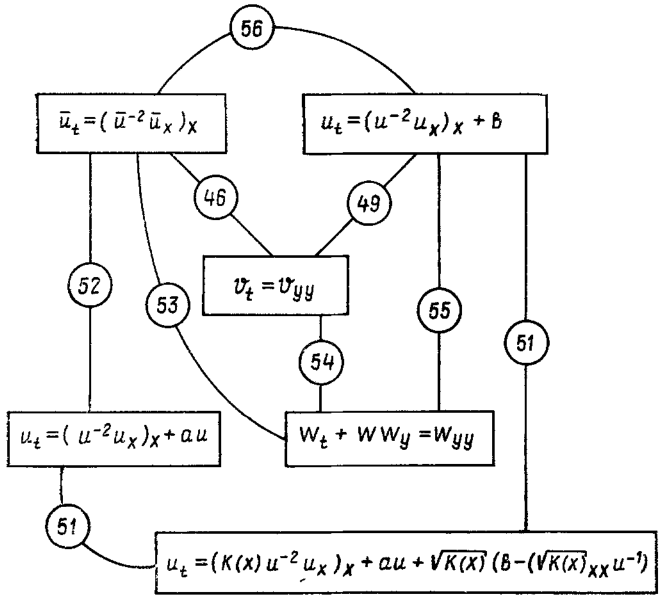


Fig. 31

solutions of the nonlinear equation by proceeding from solutions of the linear equation, to apply the principle of superposition of solutions, etc.

2.6.1. The equation

$$u_t = (u^{-2}u_x)_x, \tag{45}$$

admitting an infinite chain of Lie–Bäcklund transformations [89] can be linearized by means of the change (see [105])

$$u(t, x) = (v_y)^{-1}, \quad x = v(\bar{t}, y), \quad \bar{t} = t. \tag{46}$$

Indeed, it follows from (46) that

$$\frac{\partial}{\partial x} = (v_y)^{-1}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} - v_{\bar{t}}\,(v_y)^{1}\,\frac{\partial}{\partial y},$$

and therefore (46) takes (45) into the equation

$$v_y^{-1}\frac{\partial}{\partial y}\left(v_y\frac{\partial}{\partial y}(v_y^{-1})\right) = D_y\left[\frac{1}{v_y}(v_{\bar{t}} - v_{yy})\right] = 0,$$

or

$$v_{\bar{t}} = v_{yy} + f(\bar{t})\,v_y,$$

where f is an arbitrary function of $\bar{t}$. Hence, the transformation (46) takes any solution of the equation

$$v_{\bar{t}} = v_{yy} \tag{47}$$

into a solution of Eq. (45).

2.6.2. The equation

$$u_t = (u^{-2}u_x)_x + b, \quad b = \text{const}, \tag{48}$$

by means of the transformation (see [36])

$$u(t, x) = -\frac{b}{2}\left(\frac{v_y}{v}\right)_y^{-1}, \qquad x = -\frac{2}{b}\left(\frac{v_y}{v}\right), \quad t = \bar{t}, \tag{49}$$

goes over into the equation

$$D_y\left[\left(\frac{v_y}{v}\right)_y\right]^{-1} D_y\left[\frac{1}{v}(v_{\bar{t}} - v_{yy})\right] = 0.$$

Thus, any solution of Eq. (47) goes over under the transformation (49) into a solution of Eq. (48).

2.6.3. Heat Equation in an Inhomogeneous Medium Admitting a Nontrivial Lie–Bäcklund Group. In the work [76] a broader classification problem was solved – heat equations

$$u_t = (k(x, u)u_x)_x + Q(x, u),$$

were distinguished which admit a nontrivial Lie–Bäcklund group. It was shown that nontrivial Lie–Bäcklund transformations are admitted by a linear equation in an inhomogeneous medium ($k'_u = 0$, $Q''_{uu} = 0$) and by the nonlinear equation

$$u_t = (ku^{-2}u_x)_x + au + b\sqrt{k} - \sqrt{k}(\sqrt{k})_{xx}u^{-1}, \tag{50}$$

where $k(x)$ is an arbitrary function of x, a and b are constants, and $ab = 0$. By means of the change

$$u \to \sqrt{k(x)}\,u \tag{51}$$

Eq. (50) reduces to the equation

$$u_t = (u^{-2}u_x)_x + au + b, \quad ab = 0,$$

investigated above.

In the work [119] a special case of Eq. (50) for $a = b = 0$, $k = (\lambda x + \mu)^2$ is considered:

$$u_t = \left[\left(\frac{\lambda x + \mu}{\alpha u + \beta}\right)^2 u_x\right]_x.$$

By means of the (point) transformation

$$x \to \frac{1}{\gamma}\ln|\lambda x + \mu|, \quad t \to \frac{\lambda^2}{\gamma}t,$$

the last equation goes over into the equation

$$u_t = ((\alpha u + \beta)^{-2}u_x)_x + \gamma(\alpha u + \beta)^{-2}u_x,$$

which arises in some problems of porous media and is considered in [91].

New equations linearizable by Bäcklund transformations are obtained in [76] where a more complex model of heat conduction is considered – one of hyperbolic type.

Figure 31 shows the scheme of connections of the solutions of the equations considered where the following transformations are given in addition to those listed above:

$$\bar{u}(\bar{t}, \bar{x}) = ue^{-at}, \quad \bar{t} = -\frac{1}{2a}e^{-2at}, \quad \bar{x} = x; \tag{52}$$

$$W(y, t) = \frac{2}{\bar{u}\,\bar{x}}, \quad y = D_{\bar{x}}^{-1}\bar{u}, \quad t = \bar{t}; \tag{53}$$

$$W(y, t) = 2v_y/v; \tag{54}$$

$$u(t, x) = -bW_y^{-1}, \quad x = -\frac{1}{b}W(y, t); \tag{55}$$

$$u(t, x) = -\frac{b}{2}[\bar{u}^{-1}D_{\bar{x}}(\bar{u}\bar{x})^{-1}]^{-1}, \quad x = -\frac{2}{b\bar{x}\bar{u}}, \quad t = \bar{t}. \tag{56}$$

2.6.4. Invariant Solutions. Suppose the evolution equation

$$u_t = F(u, u_1, \ldots, u_m),$$

admits a nontrivial Lie–Bäcklund group defined by the operator

$$X = U(x, u, u_1, \ldots, u_n)\frac{\partial}{\partial u} + \cdots. \tag{57}$$

Then the solution $u = f(x, t)$ invariant relative to the corresponding transformations is defined by the relation

$$U(x, u, u_1, \ldots, u_n)|_{u=f(x,t)} = 0.$$

In other words, to find invariant solutions it is necessary to solve the system of equations

$$u_t = F(u, u_1, \ldots, u_m),$$

$$U(x, u, u_1, \ldots, u_n)=0, \tag{58}$$

of which the second is an ordinary differential equation. As n increases it is necessary to solve a differential equation of even higher order. Since this equation, as a rule, is nonlinear, attention is usually restricted to only a small number of solutions in the chain given by means of a recurrence for U (see, for example, [89]). Nevertheless, the problem of solving (58) can be simplified considerably (see [36]) if a linearizing transformation is used.

We consider the equation

$$u_t=(u^{-2}u_x)_x+b, \tag{59}$$

which admits operators (57) for which

$$U=\sum_{i=1}^{n} c_i V_i+c_0V_0+c^1V^1+c^2V^2,$$

where $n = 1, 2, \ldots, c_0, c^1, c^2, c_i$ are constants;

$$U_0=(u^{-2}u_x)_x+b,\ U^1=u_x,\ U^2=2u+xu_x-2t(u^{-2}u_x)_x-2bt$$

correspond to point transformations (found in [31]), while U_i have the form

$$U_i=-D_x^2\left(u^{-1}D_x-\frac{b}{2}x\right)^{i+1}(x),\quad i=1, 2, \ldots.$$

Solutions of Eq. (59) invariant relative to the group with $U=\sum_{i=0}^{n} c_iU_i$, are defined by the equation

$$D_x^2\sum_{i=0}^{n} c_i\left(u^{-1}D_x-\frac{b}{2}x\right)^{i+1}(x)=0. \tag{60}$$

Applying the linearizing change (49) to (60), we obtain a linear system. As a result of solving it it is found that the entire collection of solutions invariant relative to the Lie–Bäcklund transformations is given parametrically in the following form:

$$u(t, x)=-\frac{b}{2}\left(\frac{v_y}{v}\right)_y^{-1},\quad x=-\frac{2}{b}\frac{v_y}{v},$$

where

$$v=\sum_{i=1}^{r}\sum_{j=0}^{k_i} e^{\lambda_i y+\lambda_i^2 t} y^{k_i-j} A_{i(k_i-j)},$$
$$A_{i(k_i-j)}=\sum_{s=0}^{j} B_{k_i-s}\frac{(k_i-s)!}{(k_i-j)!}\sum_{l=0}^{\left[\frac{j-s}{2}\right]}(2\lambda_i)^{j-s-2l}C^{l}_{j-s-2l}\frac{t^{j-s-l}}{(j-s-l)!}, \tag{61}$$

B_{k_i-s} $(s = 0,\ldots,k_i)$ are arbitrary constants, [...] is the integral part of a number, c_m^n is the number of combinations of m things taken n at a time, and λ_i $(i = 1,\ldots,r)$ are pairwise unequal complex numbers.

We shall not pause to analyze the invariant solutions (61) of Eq. (59), since for (59) it is not possible to formulate the boundary value problems considered here.

2.6.5. For heat equations admitting an infinite series of Lie–Bäcklund transformations it is not possible to construct a corresponding chain of conservation laws (as is the case, for example, for the KdV equation). By a conservation law we mean the existence of a vector (g, h) satisfying the condition

$$D_tg+D_xh=0,$$

on solutions of the corresponding heat equation. The functions g and h depend on t, x, u, $u_1,\ldots,u_n$; g is called the density of the conservation law.

It was shown above that the equation

$$u_t=(u^{-2}u_x)_x+b \tag{62}$$

admits a Lie–Bäcklund group for which the definition equation

$$D_tU-D_x^2(u^{-2}U)=0$$

has the sequence of solutions

$$U_k = -D_x^2\left(u^{-1}D_x - \frac{b}{2}x\right)^{k+1}(x), \quad k=1, 2, \ldots \tag{63}$$

Hence, Eq. (62) formally has an infinite series of conservation laws of the form

$$D_t U_k + D_x(-D_x u^{-2} U_k) = 0, \; k=1, 2, \ldots$$

However, by (63) these conservation laws can be represented in the form

$$D_x^2\left[D_t\left(u^{-1}D_x - \frac{b}{2}x\right)^{k+1}(x) + u^{-2}U_k\right] = 0$$

and are trivial in the sense that they have a density equal to zero.

This situation has general character in considering evolution equations having even order in the leading derivative (and one of the differences from equations with an odd leading derivative consists in this).

It is known (see [54]) that if g is the density of a conservation law for an evolution equation

$$u_t = F(u, u_1, \ldots, u_m), \tag{64}$$

then the function $f = \frac{\delta}{\delta u} g \equiv \sum_{k \geqslant 0} (-D_x)^k \frac{\partial}{\partial u_k}$ satisfies the equation

$$D_t f + \sum_{k=0}^{\infty} (-D_x)^k \left(f \frac{\partial F}{\partial u_k}\right) = 0 \tag{65}$$

(formally adjoint to the defining equation of the Lie–Bäcklund group), and, conversely, if f is a solution of Eq. (65) and $f = (\delta/\delta u)g$, then g is the density of a conversion law. From this theorem (see [see [54]) it follows that in the case m = 2k Eq. (64) cannot have conservation laws whose densities depend on u_s, s > k.

Therefore, in the case of the equation

$$u_t = (k(u)u_x)_x + Q(u) \tag{66}$$

it is possible to find solutions f of the defining-Eq. (65) which depend only on t, x, u, u_1, u_2. Analysis of Eq. (65) shows that in this case $f_u = f_{u_1} = f_{u_2} = 0$, that is, f = f(t, x), and (65) has the form

$$f_t + kf_{xx} + Q'f = 0, \tag{67}$$

whereby

$$g = f(t, x)u. \tag{68}$$

It follows from Eq. (67) that (66) has nontrivial conservation laws only in two cases:

1) k = const, Q = au + b; a, b are constants. The density g in the conservation law has the form (68), and f is any solution of the equation

$$f_t + kf_{xx} + af = 0;$$

2) k ≠ const, Q' = ak + b; a, b are constants, and g has the form (68), where

$$f = \begin{cases} e^{-bt}(c_1 e^{i\sqrt{a}x} + c_2 e^{-i\sqrt{a}x}), & a \neq 0, \\ e^{-bt}(c_1 x + c_2), & a = 0, \end{cases}$$

where C_1 and C_2 are arbitrary constants.

3. Quasilinear Equation $u_t = \nabla(u^\sigma \nabla u) + u^\beta$: Unbounded Solutions, Localization, Asymptotic Behavior

The present section is devoted to the investigation of unbounded solutions of the Cauchy problem for a quasilinear equation with power nonlinearities

$$u_t = \nabla(u^\sigma \nabla u) + u^\beta, \quad t > 0, \quad x \in \mathbf{R}^N; \quad \sigma > 0, \quad \beta > 1; \tag{1}$$

$$u(0, x) = u_0(x) \geqslant 0, \quad x \in \mathbf{R}^N; \quad \sup u_0 < +\infty. \tag{2}$$

As a rule, we assume that the initial function u_0 has compact support, and $u_0^{\sigma+1}\in C^1(\mathbf{R}^N)$. Since the degenerate equation (1) describes processes with a finite speed of propagation of perturbations, for all t > 0 the solution $u = u(t, x) \geq 0$ will then also have compact support in x.

Justification of the many qualitative results of Sec. 1 is given below. We shall first introduce a number of concepts and definitions whose significance has been discussed in detail in preceding sections.

Principal attention will be devoted to the investigation of unbounded solutions of problem (1), (2) defined on a finite time interval $t\in[0, T_0)$, with

$$\overline{\lim_{t\to T_0^-}} \sup_{x\in\mathbf{R}^N} u(t, x) = +\infty. \tag{3}$$

Here a point $x=x_0\in\mathbf{R}^N$, at which unbounded growth of the solution occurs as $t \to T_0^-$ we call a point of singularity or a point of peaking if there exists a sequence $t_k\in[0, T_0)$, $t_k\to T_0$ when $k \to \infty$ such that

$$u(t_k, x_0)\to+\infty,\ k\to+\infty. \tag{4}$$

The value $T_0 = T_0(u_0)\in R_t^{+}$ which, naturally, is a function of the initial function $u_0 \not\equiv 0$, is called the time of peaking of the unbounded solution.

A considerable part of this and the next section is devoted to the study of the effect of localization of regimes with peaking in which unbounded solutions of problem (1), (2) may occur for particular values of the parameters (see Sec. 1).

Definition 1. Let $u_0(x)$ be a compactly supported function. An unbounded solution of problem (1), (2) is called localized (in the strict sense) if the set (the region of localization)

$$\Omega_L=\{x\in\mathbf{R}^N\,|\,u(T_0^-, x)>0\} \tag{5}$$

is bounded in $\mathbf{R}^N$. If Ω_L is unbounded (for example, $\Omega_L=\mathbf{R}^N$), then there is no localization.

Boundedness of Ω_L means that $u \equiv 0$ in $[0, T_0)\times\{\mathbf{R}^N\setminus\Omega_L\}$; this follows from general properties of solutions of a parabolic equation with a source. The next definition is more general; it is meaningful in the case of arbitrary equations and for any noncompactly supported initial functions.

Definition 2. An unbounded solution of problem (1), (2) is called effectively localized if the set (the region of effective localization)

$$\omega_L=\{x\in\mathbf{R}^N\,|\,u(T_0^-, x)=+\infty\} \tag{6}$$

is bounded. If ω_L is unbounded, then there is no effective localization.

In both definitions $u(T_0^-, x)=\overline{\lim\limits_{t\to T_0^-}}\, u(t,x)$.

It is obvious that $\omega_L\subseteq\Omega_L$. As a rule, we henceforth omit the word "effective" from the second definition. Introduction of the set (6) makes it possible to classify localized solutions as follows: localization in the S-regime if $\operatorname{mes}\omega_L > 0$ and in the LS-regime if $\operatorname{mes}\omega_L = 0$. In the last case the solution u(t, x) can grow to infinity as $t \to T_0^-$ only at one singular point, while at the remaining points it is bounded above uniformly with respect to $t\in(0, T_0)$.

We shall indicate the basic conditions imposed on the solution of problem (1), (2).

1) In any region of the form $(0, T_0-\tau)\times\mathbf{R}^N$ there exists a unique, bounded, nonnegative generalized solution of problem (1), (2), $u^{\sigma+1}\in L^\infty(0, T; H_0^1(\mathbf{R}^N))$, $(u^{1+\sigma/2})_t\in L^2(0, T; L^2(\mathbf{R}^N))$, $u^{1+\sigma/2}\in L^\infty(0, T; L^2(\mathbf{R}^N))$, $T=T_0-\tau$ (see the various methods of analysis of properties of generalized solutions in [9, 11, 37, 51, 55, 63, 68, 84, 114, 124, 125]).
2) The solutions satisfy the comparison theorem with respect to initial functions: if $u^{(\nu)}$, $\nu = 1, 2$ are two solutions corresponding to different initial functions $u^{(\nu)}(0, x)=u_0^{(\nu)}(x)\geq 0$ and $u_0^{(2)}\geq u_0^{(1)}$ in $\mathbf{R}^N$, then $u^{(2)}\geq u^{(1)}$ in $(0, T_0)\times\mathbf{R}^N$, $T_0=\min\{T_0^{(1)}, T_0^{(2)}\}$ (see, for example, [51, 55, 68, 84, 88]).
3) The function $\nabla u^{\sigma+1}(t, x)$ is continuous in x in $\mathbf{R}^N$ for each fixed $t\in(0, T_0)$; see [50, 55, 82, 90, 110].

4) The solution $u=u(t,x)\geqslant 0$ in the case of a compactly supported initial function u_0 is compactly supported in x for any $t\in(0,T_0)$ (this is established by comparison of u(t, x) in the sense of 2) with compactly supported, self-similar solutions constructed, for example, in [12].

5) In deriving individual results we shall use the fact that a generalized solution in $(0,T_0-\tau)\times\mathbf{R}^N$ can be obtained as the limit of a monotonically decreasing sequence of classical, strictly positive solutions of the same equation on each of which it is uniformly parabolic; see [51, 55, 68, 114] [we note that this implies the validity of 2-4)].

6) In a number of cases we shall also assume with no loss of generality that everywhere in $P[u]=\{(t,x)\in(0,T_0)\times\mathbf{R}^N | u(t,x)>0\}$ the solution is classical, $u\in C^{1,2}(P[u])$, and fails to have the required smoothness only on a surface of degeneracy $S[u]=\bar{P}[u]\setminus P[u]\setminus\{t=0,\ x\in\mathbf{R}^N\}$ [51, 55, 68].

Other natural restrictions will be formulated during the course of the exposition.

3.1. Conditions for the Occurrence of Unbounded Solutions. A very complete analysis of conditions for unboundedness of a solution of problem (1), (2) can be carried out by constructing unbounded lower solutions of Eq. (1). For this we require the following assertion where $A(v)$ denotes the operator $A(v)=v_t-\nabla(v^\sigma\nabla v)-v^\beta$, $v=v(t,x)\geqslant 0$ and D_T denotes the region $(0,T)\times\{x\in\mathbf{R}^N|\ |x|<\zeta(t)\}$, where $\zeta\in C([0,T))$ is some nonnegative function; $Q_T=(0,T)\times\mathbf{R}^N$.

Proposition. Let $u:Q_T\to[0,+\infty)$ be a solution of the Cauchy problem (1), (2), and suppose the functions $u_\mp:Q_T\to[0,+\infty)$, $u_\mp\in C^{1,2}(D_T)\cap C(\bar{D}_T)$, $u_\mp=0$ in $Q_T\setminus D_T$ satisfy in D_T the inequalities $A(u_-)\leqslant 0$, $A(u_+)\geqslant 0$. Suppose $\nabla u_\pm^{\sigma+1}\in C(\mathbf{R}^N)$ for all $t\in(0,T)$, and, moreover $u_-(0,x)\leqslant u_0(x)\leqslant u_+(0,x)$ in $\mathbf{R}^N$. Then $u_-\leqslant u\leqslant u_+$ in $(0,T)\times\mathbf{R}^N$.

The functions u_- and u_+ are called, respectively, lower and upper solutions of problem (1), (2). For a proof see, for example, [51, 52, 55, 88].

3.1.1. Construction of Unbounded Lower Solutions. We seek a lower solution of problem (1), (2) in the self-similar form

$$u_-(t,x)=(T-t)^{-1/(\beta-1)}\theta_-(\xi),\quad \xi=|x|/\zeta^*(t), \tag{7}$$

where $\zeta^*(t)=(T-t)^m$, $m=[\beta-(\sigma+1)]/2(\beta-1)$. The inequality $A(u_-)\leqslant 0$ in D_T is then equivalent to the ordinary differential inequality

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}\theta_-^\sigma\theta_-')'-m\theta_-'\xi-\frac{1}{\beta-1}\theta_-+\theta_-^\beta\geqslant 0. \tag{8}$$

Setting $\theta_-(\xi)=A(1-\xi^2/a^2)_+^{1/\sigma}$, where A, a are positive constants [we note that $\nabla_\xi\theta_-^{\sigma+1}\in C(\mathbf{R}^N)$], we reduce (8) to the following form:

$$m-n\Delta+A^{\beta-1}\Delta^{\frac{\beta-1}{\sigma}+1}\geqslant 0,\quad \Delta=(1-\xi^2/a^2)_+\in[0,1], \tag{9}$$

where

$$m=\frac{4A^\sigma}{\sigma^2a^2}+\frac{\beta-(\sigma+1)}{(\beta-1)\sigma},\quad n=\frac{1}{\sigma}\left[1+\frac{2A^\sigma}{a^2}\left(N+\frac{2}{\sigma}\right)\right]. \tag{10}$$

Conditions for the validity of (9) and thus the validity of the following assertion [19] can be derived without difficulty.

THEOREM 1. Suppose in problem (1), (2) $u_0(x)$ is such that

$$u_0(x)\geqslant u_-(0,x)\equiv T^{-1/(\beta-1)}\theta_-\left(\frac{|x|}{T^m}\right),\quad x\in\mathbf{R}^N, \tag{11}$$

where $\theta_-(\xi)=A(1-\xi^2/a^2)_+^{1/\sigma}$, and T, a, A are positive constants the last two of which are connected by the inequalities

$$m>0,\quad A^{\beta-1}\geqslant(n-m)(m/n)^{-[\beta+\sigma-1]/\sigma}. \tag{12}$$

Then a solution of problem (1), (2) exists for a finite time $T_0=T_0(u_0)\leqslant T$.

It is easy to see that the system (12) is solvable for a, A for any values $\sigma>0$ and $\beta>1$. An elementary analysis of the lower solution (7) shows that for $\beta\in(1,\sigma+1)$ any solution of problem (1), (2) for $u_0\not\equiv 0$ is unbounded. A stronger result will be proved below.

3.1.2. Absence of Nontrivial Global Solutions for $1 < \beta < \sigma + 1 + 2/N$. THEOREM 2. Let $\beta \in (1, \sigma+1+2/N)$, $u_0 \not\equiv 0$. Then the solution of the Cauchy problem is unbounded.

The proof of the theorem is based on comparing u(t, x) with a known self-similar solution of the equation without a source

$$v_t = \nabla (v^\sigma \nabla v), \quad t>0, \quad x \in \mathbf{R}^N. \tag{13}$$

It has the following form (see [5, 43, 44]):

$$v_A(t, x) = (T_1+t)^{-N/(2+N\sigma)} f(\eta), \quad \eta = |x| (T_1+t)^{-1/(2+N\sigma)},$$
$$f(\eta) = B(\eta_0^2 - \eta^2)_+^{1/\sigma}; \quad B = [\sigma/2(2+N\sigma)]^{1/\sigma}. \tag{14}$$

Here T_1 and η_0 are arbitrary positive constants.

Suppose without loss of generality $u_0(x) > 0$ in some neighborhood of the point $x = 0$. We fix an arbitrary $T_1 > 0$ and choose $\eta_0 > 0$ in (14) so small that $u_0(x) \geqslant v_A(0, x)$ in $\mathbf{R}^N$. Then obviously $u(t,x) \geqslant v_A(t,x)$ in $\mathbf{R}^N$ for all admissible $t > 0$. We shall show that for $\beta < \sigma + 1 + 2/N$ there exists $t_1 \geqslant 0$, such that $v(t_1, x)$ satisfies inequality (11) [$v_A(t_1, x) \geqslant u_-(0, x)$ in $\mathbf{R}^N$] for some $T > 0$ and hence because of the estimate $u(t_1, x) \geqslant v_A(t_1, x)$ a solution of the Cauchy problem exists for a time no greater than $t_1 + T$.

The inequality $v_A(t_1, x) \geqslant u_-(0, x)$ in $\mathbf{R}^N$ is equivalent to the following:

$$(T_1+t_1)^{-N/(2+N\sigma)} B\eta_0^{2/\sigma} \geqslant AT^{-1/(\beta-1)},$$
$$\eta_0 (T_1+t_1)^{1/(2+N\sigma)} \geqslant T^{[\beta-(\sigma+1)]/2(\beta-1)} a. \tag{15}$$

Suppose in the first of them equality holds; it then remains to verify that the second is satisfied for large $T > 0$. It can be reduced to the form

$$T^{\frac{1}{2(\beta-1)}\left[\beta - \left(\sigma+1+\frac{2}{N}\right)\right]} \leqslant a^{-1} \left(\frac{B}{A}\right)^{1/N} \eta_0^{\frac{2}{N\sigma}+1} \tag{15'}$$

and is valid for all sufficiently large T if $\beta < \sigma + 1 + 2/N$ which completes the proof.

We note that conditions (15), (15') make it possible to analyze also the "critical" case $\beta = \sigma + 1 + 2/N$, but the result obtained at the qualitative level [40, 41] regarding the absence of global solutions for $\beta = \sigma + 1 + 2/N$ cannot be proved in this manner. For the case $\sigma = 0$ Theorem 2 has been known for a relatively long time (see [63, 97]); an analysis of the critical value $\beta = 1 + 2/N$ is carried out in [85, 102, 111]. Of course the technique of investigating a semilinear equation using the possibility of inverting the linear operator $(\partial/\partial t - \Delta)$ is not applicable in the quasilinear case.

3.2. Global Solutions for $\beta > \sigma + 1 + 2/N$. Optimality of Theorem 2 is established below.

Conditions for global solvability of the problem will be obtained by means of the construction of upper solutions u_+ of the following form:

$$u_+(t, x) = (T+t)^{-1/(\beta-1)} \theta_+(\xi), \quad \xi = |x|/(T+t)^m,$$
$$\theta_+(\xi) = A(1-\xi^2/a^2)_+^{1/\sigma}, \quad T>0, \quad A>0, \quad a>0. \tag{16}$$

The inequality $\mathbf{A}(u_+) \geqslant 0$ [we note that $\nabla u_+^{\sigma+1} \in C(\mathbf{R}^N)$] is equivalent to

$$m_* + n_*\Delta + A^{\beta-1}\Delta^{\frac{\beta+\sigma-1}{\sigma}} \leqslant 0, \quad \Delta = (1-\xi^2/a^2)_+ \in (0, 1);$$
$$m_* = \frac{4A^\sigma}{\sigma^2 a^2} - \frac{\beta-(\sigma+1)}{(\beta-1)\sigma}, \quad n_* = \frac{1}{\sigma}\left[1 - \frac{2A^\sigma}{a^2}\left(N+\frac{2}{\sigma}\right)\right]. \tag{17}$$

For the validity of (17), for example, it suffices that $m_* \leqslant 0$, $m_* + n_* + A^{\beta-1} \leqslant 0$, if

$$\frac{4}{\sigma^2}\frac{A^\sigma}{a^2} \leqslant \frac{\beta-(\sigma+1)}{(\beta-1)\sigma}, \quad A^{\beta-1} \leqslant \frac{2N}{\sigma}\frac{A^\sigma}{a^2} - \frac{1}{\beta-1}. \tag{18}$$

It is not hard to obtain a condition for solvability of (18). From the second inequality we obtain the necessity of the restriction $2NA^\sigma/\sigma a^2 > 1/(\beta - 1)$ which together with the first gives

$$\frac{2}{N\sigma(\beta-1)}<\frac{4}{\sigma^2}\frac{A^\sigma}{a^2}\leqslant\frac{\beta-(\sigma+1)}{(\beta-1)\sigma}.$$

Therefore, (18) has a solution if

$$\frac{2}{N\sigma(\beta-1)}<\frac{\beta-(\sigma+1)}{(\beta-1)\sigma}$$

or, equivalently, $[\beta-(\sigma+1+2/N)]/\sigma(\beta-1)>0$, i.e., $\beta > \sigma + 1 + 2/N$. We have thus proved the following theorem [19].

THEOREM 3. Suppose $\beta > \sigma + 1 + 2/N$ and for some $T > 0$

$$u_0(x)\leqslant u_+(0, x)\equiv T^{-1/(\beta-1)}\theta_+(|x|/T^m), \quad x\in\mathbf{R}^N, \tag{19}$$

where $\theta_+(\xi)=A(1-\xi^2/a^2)_+^{1/\sigma}$ and the constants A, $a > 0$ satisfy (18). Then the Cauchy problem (1), (2) has a global (bounded) solution, and

$$u(t, x)\leqslant(T+t)^{-1/(\beta-1)}\theta_+(|x|/(T+t)^m) \text{ в } \mathbf{R}_+^1\times\mathbf{R}^N. \tag{19'}$$

Summarizing the results formulated in Theorems 1 and 3, we arrive at the following assertion: for $\beta > \sigma + 1 + 2/N$ for all "large" initial functions problem (1), (2) is not globally solvable, while for sufficiently "small" u_0 there exists a global solution.

Remark. It is hard to show that global solutions for $\beta > \sigma + 1 + 2/N$ are not necessarily finite as in (19). To construct a "nonfinite" set of stability it suffices to take for θ_+ in the upper solution (16), for example, the function $\theta_+(\xi)=A(a^2+\xi^2)^{-1/(\beta-1)}>0$, and to choose the magnitudes of the constants $A > 0$ and $a > 0$ so that $\mathbf{A}(u_+)\geqslant 0$ in $\mathbf{R}_+^1\mathbf{R}^N$.

3.3. Three Types of Self-Similar Regimes of Combustion with Peaking. It is convenient to begin the investigation of concrete properties of solutions of the rather complex nonlinear problem (1), (2) from an analysis of special self-similar solutions which were investigated in a preliminary way in Sec. 1. Although these special solutions are realized for a certain special choice of $u_0(x)$, analysis of their very simple space–time structure nevertheless makes it possible to basically judge the character of the course of the combustion process with peaking in the general case. The passage from special solutions to investigation of solutions of very general form is realized on the basis of special comparison theorems formulated in Subsec. 4. A detailed qualitative and numerical investigation of various regimes of combustion with peaking in the present problem was carried out in [40, 45, 59, 74]. The first investigations of complex self-similar thermal structures and their stabilities were carried out in [40, 41, 59, 73, 74] (see Sec. 1).

The investigation of self-similar solutions carried out below uses individual results of [1, 2, 101] and is based on some of the approaches developed there.

3.3.1. Formulation of Self-Similar Problems. As shown in Sec. 1, for any* $\sigma > 0$ and $\beta > 1$ Eq. (1) has an unbounded self-similar solution

$$u_A(t, x)=(T_0-t)^{-\frac{1}{\beta-1}}\theta_A(\xi), \quad \xi=\frac{x}{(T_0-t)^m}; \quad m=\frac{\beta-(\sigma+1)}{2(\beta-1)}, \tag{20}$$

where $T_0 > 0$ is the time of existence of the solution. The function $\theta_A(\xi)\geqslant 0$ satisfies in $\mathbf{R}^n$ the following elliptic equation which is obtained after substitution of the expression (20) into (1):

$$\nabla(\theta_A^\sigma\nabla\theta_A)-m\nabla\theta_A\xi-\frac{1}{\beta-1}\theta_A+\theta_A^\beta=0, \ \xi\in\mathbf{R}^N. \tag{21}$$

Equation (21) always has the trivial solution $\theta_A \equiv 0$ and also the spatially homogeneous solution $\theta_A(\xi)\equiv\theta_H=(\beta-1)^{-1/(\beta-1)}$. We shall be interested in nontrivial solutions $\theta_A \not\equiv 0$ such that $\theta_A(\xi) \to 0$ as $|\xi| \to +\infty$. Radially nonsymmetric solutions of the elliptic equation (21) have so far been studied only qualitatively and numerically [60] (we note that (21) apparently admits an entire spectrum of solutions with extremely varied spatial structure; examples are given in [60]).

We henceforth restrict our analysis to symmetric solutions:

$$\theta_A=\theta_A(\xi), \ \xi=|x|/(T_0-t)^m\geqslant 0. \tag{20'}$$

Then (21) becomes the ordinary differential equation

*The case $\sigma = 0$ is considered in Sec. 4.

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}\theta_A^{\sigma}\theta_A')'-m\theta_A'\xi-\frac{1}{\beta-1}\theta_A+\theta_A^{\beta}=0,\ \xi>0, \tag{22}$$

$$\theta_A'(0)=0,\ \theta_A(+\infty)=0\ (\theta_A(0)>0). \tag{23}$$

Equation (22) degenerates at $\theta_A = 0$, and hence problem (22), (23) admits, generally speaking, a generalized solution not having the necessary smoothness at points of degeneracy. However, in all cases the solution θ_A must be such that $\theta_A^{\sigma}\theta_A'$ is continuous for all $\xi \geqslant 0$ (and, of course, $\theta_A \in C^2$ wherever $\theta_A > 0$). In particular, this means that $\theta_A^{\sigma}\theta_A' = 0$, where $\theta_A = 0$.

3.3.2. Localization of Combustion in an S-Regime, $\beta = \sigma + 1$. For $\beta = \sigma + 1$ Eq. (22) takes the especially simple form

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}\theta_A^{\sigma}\theta_A')'-\frac{1}{\sigma}\theta_A+\theta_A^{\sigma+1}=0,\ \xi>0. \tag{24}$$

In the one-dimensional case (N = 1) it can be easily integrated, and one of its solutions has the form (we recall that $\xi \equiv |x|$ for $\beta = \sigma + 1$)

$$\theta_A=\theta_s(x)=\begin{cases}\left[\dfrac{2(\sigma+1)}{\sigma(\sigma+2)}\cos^2\left(\dfrac{\pi|x|}{L_s}\right)\right]^{1/\sigma}, & |x|<L_s/2,\\ 0, & |x|>L_s/2,\end{cases} \tag{25}$$

where $L_s=2\pi(\sigma+1)^{1/2}/\sigma$. This solution was first constructed in [45, 74]. Its distinguishing feature is that the corresponding self-similar generalized solution (20)

$$u_A(t,x)=(T_0-t)^{-1/\sigma}\theta_s(x)\left(u_A\equiv 0 \text{ in } [0,T_0)\times\left\{|x|\geqslant\frac{L_s}{2}\right\}\right) \tag{26}$$

exhibits the effect of localization of the unbounded solution in the region $\Omega_L=\omega_L=\{|x|<L_s/2\}$. In spite of the unbounded growth of the solution as $t \to T_0^-$ at all points of the region of localization, thermal perturbations do not penetrate into the surround cold space. The width of the region of localization $L_s = \operatorname{mes}\Omega_L$ is called the fundamental length.

It will be shown below that L_s is a fundamental characteristic of the nonlinear medium in question. For N = 1 Eq. (24) admits a countable set of distinct solutions formed from an arbitrary number of elementary solutions (26) [by periodic continuation of the function (26) to both sides with period L_s] which because of the condition of thermal isolation generate in accordance with (20) combustion independent of one another at the fundamental lengths L_s. The equation $u_t=(u^{\sigma+1})_{xx}+u^{\sigma+1}$ also admits another class of unbounded, localized solutions of the following form: $u(t,x)=\varphi(t)[1-(A+\cos(\lambda x))/g(t)]_+^{1/\sigma}$, where $\lambda = \sigma/(\sigma + 1)$, A is a constant. Substitution of this expression into the equation gives a system of ordinary differential equations for the nonnegative functions $\varphi(t)$, $g(t)$ (the idea of constructing such "noninvariant" solutions for a parabolic equation with a sink $u^{\sigma+1}$ in place of a source was advanced in [88]).

The multidimensional case N > 1 is considered in detail below. Self-similar solutions of the S-regime exist in spaces of arbitrary dimension, but here, in contrast to the one-dimensional case, there are no nonmonotone solutions.

THEOREM 4. For any N > 1, $\beta = \sigma + 1$ there exists a compactly supported solution $\theta_A \geqslant 0$ of problem (22), (23). The function θ_A decreases monotonically wherever it is positive. The problem has no nonmonotone solutions.

We note first of all that the fact that the solution of problem (22), (23) has compact support follows from a local analysis of the equation in the region of sufficiently small $\theta_A > 0$. This gives the following asymptotics of a compactly supported solution:

$$\theta_A(\xi)=\left\{\frac{\sigma}{2(\sigma+2)}(\xi_0-\xi)^2\right\}^{1/\sigma}(1+\varepsilon(\xi)),\ \xi<\xi_0, \tag{27}$$

where $\xi_0=\operatorname{mes}\operatorname{supp}\theta_A<+\infty$ ($\theta_A=0$ for $\xi\geqslant\xi_0$) and $\theta(\xi) \to 0$ as $\xi \to \xi_0^-$.

To prove existence it is convenient to consider, together with (22), (23), the family of Cauchy problems

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}|\theta|^{\sigma}\theta')'-\frac{1}{\sigma}\theta+|\theta|^{\sigma}\theta=0,\ \xi>0, \tag{28}$$

$$\theta(0)=\mu>0,\ \theta'(0)=0 \tag{29}$$

and to choose the quantity $\mu > 0$ so that $\theta = \theta(\xi;\ \mu) \geqslant 0$ satisfies the condition $\theta(+\infty;\ \mu) = 0$; $\theta(\xi;\ \mu)$ will then obviously be the desired function $\theta_A(\xi)$.

The proof of Theorem 4 is based on the following lemmas.

LEMMA 1. Suppose

$$0 < \mu < \mu_* = \left[\frac{2(\sigma+1)}{\sigma(\sigma+2)}\right]^{1/\sigma}. \tag{30}$$

Then $\theta(\xi;\ \mu) > 0$ for $\xi \geqslant 0$. Moreover, $|\theta(\xi;\mu)|$ is uniformly bounded for $\xi \geqslant 0$ for any $\mu > \theta_H$: $|\theta(\xi;\ \mu)| \leqslant \mu$.

Proof. Multiplying (28) by $|\theta|^\sigma\theta'$ and integrating the equality obtained over $(0,\ \xi)$, we arrive at the identity

$$\frac{1}{2}(|\theta|^\sigma\theta')^2(\xi) + (N-1)\int_0^\xi (|\theta|^\sigma\theta')^2(\eta)\frac{d\eta}{\eta} + \Phi(\theta(\xi)) = \Phi(\mu), \tag{31}$$

where $\Phi(\mu)$ denotes the function

$$\Phi(\mu) = \frac{|\mu|^{2\sigma+2}}{2(\sigma+1)} - \frac{|\mu|^{\sigma+2}}{\sigma(\sigma+2)}.$$

From (31) it follows immediately that $\Phi(\theta(\xi;\ \mu)) \leqslant \Phi(\mu)$ for all $\xi > 0$. It is not hard to see that this ensures the validity of the assertions of the lemma.

LEMMA 2. There exists $\mu = \mu^* > \theta_H$ such that $\theta(\xi;\ \mu^*)$ vanishes.

Proof. We suppose otherwise: suppose $\theta(\xi;\ \mu) > 0$ in $\mathbf{R}^1_+$ for any $\mu > \theta_H$. Setting $\varphi = |\theta|^\sigma\theta$, $\varphi(0) = \mu^{\sigma+1}$, we obtain the following integral equation for the new function φ:

$$\varphi'(\xi) = (\sigma+1)\,\xi^{1-N}\int_0^\xi \eta^{N-1}\left[\frac{1}{\sigma}|\varphi|^{-\frac{\sigma}{\sigma+1}}\varphi - \varphi\right]d\eta, \quad \xi > 0.$$

Setting now $\psi_\mu(\xi) = \varphi(\xi)/\varphi(0) \equiv \varphi(\xi)/\mu^{\sigma+1}$, we have

$$\psi'_\mu(\xi) = (\sigma+1)\,\xi^{1-N}\int_0^\xi \eta^{N-1}\left[\frac{1}{\sigma}\mu^{-\sigma}|\psi_\mu|^{-\frac{\sigma}{\sigma+1}}\psi_\mu - \psi_\mu\right]d\eta, \quad \xi > 0. \tag{32}$$

It follows from Lemma 1 that $|\psi_\mu(\xi)| \leqslant 1$ in $\mathbf{R}^1_+$ if $\mu > \theta_H$, while from (32) we find that $|\psi'_\mu(\xi)| \leqslant$ const on any compact set $[0,\ \xi_m]$. From the Arzela–Ascoli compactness theorem it follows that for some sequence $\mu_k \to +\infty$, $\psi_{\mu_k}(\xi) \to w(\xi) \geqslant 0$ uniformly on each compact set $[0,\ \xi_m]$. An equation for w is obtained from an integral equation equivalent to (32) by passing to the limit $\mu = \mu_k \to +\infty$ and has the form

$$w'(\xi) = -(\sigma+1)\,\xi^{1-N}\int_0^\xi \eta^{N-1}w(\eta)\,d\eta, \quad \xi > 0; \quad w(0) = 1, \tag{33}$$

with $w(\xi) \geqslant 0$. Because of (33) $w' < 0$, and hence $w > 0$ in $\mathbf{R}^1_+$. Now (33) is equivalent to the following problem:

$$w'' + \frac{N-1}{\xi}w' + (\sigma+1)w = 0, \quad \xi > 0; \quad w'(0) = 0, \quad w(0) = 1, \tag{34}$$

whose solution vanishes [at the point $\xi_1 = z_N^{(1)}/(\sigma+1)^{1/2}$, where $z_N^{(1)} > 0$ is the smallest root of the Bessel function $J_{(N-2)/2}$].

To complete the proof we need the assertion regarding conditions for continuous dependence of the solution of problem (28), (29) on $\mu > 0$. We note that in the general case continuous dependence, generally speaking, is lacking, since Eq. (28) is degenerate.

LEMMA 3. Suppose the function $\theta(\xi;\ \mu_1)$, $\mu_1 > 0$ is such that $(|\theta|^\sigma\theta)' \neq 0$ at those points $\xi \in (0,\ \xi_m]$, where $\theta = 0$. Then $(|\theta|^\sigma\theta)(\xi;\ \mu)$ and $(|\theta|^\sigma\theta)'(\xi;\ \mu)$ depend continuously on μ on the compact set $[0,\ \xi_m]$ in a neighborhood of the value of the parameter $\mu = \mu_1$.

Regarding the proof, see Lemma 5 in [20] which is closely related in meaning.

Proof of Theorem 4. We construct the set $\mathscr{M} = \{\mu_0 > 0 \mid \text{for all } 0 < \mu < \mu^0 : \theta-(\xi;\ \mu) > 0 \text{ in } \mathbf{R}_+^1\}$. Then $\mathscr{M} \neq \{\mu^0 \leqslant \theta_H\}$ (Lemma 1) and $\mathscr{M}$ is bounded above (Lemma 2). Therefore, there exists

$\theta_0 = \sup\mathscr{M} > \theta_H$. It then follows from Lemma 3 that the function $\theta(\xi;\ \theta_0)$ is the desired function θ_A which satisfies conditions (23) with the asymptotics (27). Monotonicity of θ_A follows from the identity (31).

3.3.3. The Lemma on Stationary Solutions. To investigate the solvability of problem (22), (23) for $\beta \neq \sigma + 1$ we need some properties of radially symmetric solutions of the stationary equation

$$\nabla(U^\sigma \nabla U) + U^\beta = 0, \quad U = U(x) \geqslant 0. \tag{35}$$

Setting $U^{\sigma+1} = V$ and transforming the independent variable $x \to x(\sigma + 1)^{1/2}$, we obtain for the function $V = V_\lambda(|x|)$ the following problem:

$$\frac{1}{r^{N-1}}(r^{N-1}V_\lambda')' + V_\lambda^\alpha = 0, \quad r = |x| > 0, \tag{36}$$

$$V_\lambda(0) = \lambda, \quad V_\lambda'(0) = 0, \tag{37}$$

where $\lambda > 0$ is a constant (the parameter of the family $\{V_\lambda\}$), $\alpha = \beta/(\sigma+1) > 0$. Because of the invariance of Eq. (36), for any $\lambda > 0$ we have the identity

$$V_\lambda(r) \equiv \lambda V_1\left(\lambda^{\frac{\alpha-1}{2}} r\right). \tag{38}$$

LEMMA 4. Suppose $\alpha > 0$. Then

(a) for $\alpha < (N+2)/(N-2)_+$ [i.e., $0 < \alpha < +\infty$ for $N = 1$ or $N = 2$ and $0 < \alpha < (N + 2)/(N - 2)$ for $N > 2$] the solution of problem (36), (37) vanishes at some point $r = r_0(\lambda) \equiv r_0(1)\lambda^{(1-\alpha)/2}$, and $V_\lambda'(r_0) \neq 0$, so that the problem has no solution $V_\lambda \geqslant 0$ in $\mathbf{R}_+^1$;

(b) if $\alpha \geqslant (N+2)/(N-2)_+$, then $V_\lambda(r) > 0$ in $\mathbf{R}_+^1$ and $V_\lambda \to 0$ as $r \to +\infty$.

For the proof, see, for example, [10, 107, 101]. At the "critical" value $\alpha = (N + 2)/(N - 2)_+$ the solution can be represented in explicit form:

$$V_\lambda(r) = \lambda\left[\frac{N(N-2)}{N(N-2) + \lambda^{4/(N-2)}r^2}\right]^{(N-2)/2}, \quad r = |x| \geqslant 0 \tag{39}$$

(this family was considered in [116]).

Returning to Eq. (35) we find that for $0 < \beta < (\sigma + 1)(N + 2)/(N - 2)_+$ no stationary solutions $U = U(|x|) \geqslant 0$ in $\mathbf{R}_+^1$ exist (moreover, there are even no nonsymmetric solutions $U = U(x) \not\equiv 0$; see [98]). On the other hand, for $\beta \geqslant (\sigma+1)(N+2)/(N-2)_+$ all symmetric solutions of it are strictly positive [69, 101, 107].

The basis of the proof of Lemma 4 is a known integral identity [69]. The investigation carried out in [69] and for more general quasilinear elliptic equations in [70] shows that the "critical" value of the parameter, on passing through which the properties of solutions change abruptly, can be found from a condition for imbedding function spaces corresponding to various terms of the equation. For example, for Eq. (36) for $\alpha \geqslant (N+2)/(N-2)_+$ compactness of the imbedding $H_0^1(B_1) \subset L^{\alpha+1}(B_1)$, B_1 is violated; B_1 isthe ball of unit radius in $\mathbf{R}^N$ [62]. In the critical case $\alpha = (N + 2)/(N - 2)_+$ Eq. (36) is invariant relative to a certain conformal mapping [46, 48] which in final analysis made it possible to find the family of solutions (39) [116].

Proceeding from these propositions, it is possible to analyze other problems. For example, for the equation

$$\nabla(|\nabla U|^\sigma \nabla U) + U^\beta = 0, \quad \sigma > 0, \quad \beta > 1$$

$\beta = [N(\sigma+1) + (\sigma+2)]/[N-(\sigma+2)]_+$ is "critical," and for this β there exists the family of solutions (see [20])

$$U(x) = \left\{\left[\frac{a^2 N(N-(\sigma+2))}{(\sigma+1)}\right]^{1/(\sigma+2)} \Big/ (a^2 + |x|^{(\sigma+2)/(\sigma+1)})\right\}^{[N-(\sigma+2)]/(\sigma+2)}.$$

In the case of the equation of fourth order

$$-\Delta^2 U + U^\beta = 0, \quad \beta > 1,$$

at the "critical" value $\beta = (N + 4)/(N - 4)_+$ there exists the following family of strictly positive solutions $\mathbf{R}^N$ [25]:

$$U(x) = \{a^4 N(N-4)(N^2-4)\}^{(N-4)/8}/(a^2 + |x|^2)^{(N-4)/2}.$$

3.3.4. Nonlocalized Self-Similar Solutions of the HS-Regime, $\beta < \sigma + 1$. It can be established by local analysis that a solution of problem (22), (23) for $\beta \in (1, \sigma+1)$ can only be a compactly supported function with the following asymptotics near a point of the front $\xi = \xi_0 = \operatorname{mes}\operatorname{supp}\theta_A$:

$$\theta_A(\xi) = \left[\frac{(\sigma+1-\beta)\sigma}{2(\beta-1)}\xi_0(\xi_0-\xi)\right]^{1/\sigma}(1+\omega(\xi)), \tag{40}$$

where $\omega(\xi) \to 0$ as $\xi \to \xi_0^-$.

THEOREM 5. For any $1 < \beta < \sigma + 1$ and $N \geqslant 1$ there exists a compactly supported solution θ_A of problem (22), (23) with $\theta_A' < 0$ wherever $\theta_A > 0$. The problem has nonmonotone solutions. For $n = 1$ a solution $\theta_A \not\equiv 0$ is unique.

The proof of the first part of the theorem is practically no different from the analysis of the S-regime presented above. Uniqueness of the solution for $N = 1$ is established in Subsec. 3.6 on the basis of an investigation of the original parabolic partial differential equation.

We shall say a few works regarding properties of the self-similar solution (20) for $\beta \in (1, \sigma+1)$. It represents an HS-regime of combustion with peaking, and there is no localization. The front of the thermal wave moves according to the law

$$|x_\Phi(t)| = \xi_0 (T_0 - t)^{[\beta-(\sigma+1)]/2(\beta-1)} \to +\infty, \quad t \to T_0^-,$$

i.e., after finite time the wave encompasses all space, and $u_A \to +\infty$ as $t \to T_0^-$ everywhere in $\mathbf{R}^N$.

3.3.5. Localization in a Self-Similar LS-Regime with Peaking, $\beta > \sigma + 1$. The self-similar solutions (20) for $\beta > \sigma + 1$ still more clearly than in the case of an S-regime convey the property of localization of processes of diffusion of heat and combustion. Here it is a question of effective localization, since the self-similar problem (22), (23) for $\beta > \sigma + 1$ has no compactly supported solutions (in contrast to the case $\beta \leqslant \sigma+1$), and, as local analysis shows, any function $\theta_A(\xi)$ has the following asymptotics:

$$\theta_A(\xi) = C_A \xi^{-2/[\beta-(\sigma+1)]}(1+\nu(\xi)), \quad \nu(\xi) \to 0, \quad \xi \to +\infty, \tag{41}$$

where $C_A = C_A(\sigma, \beta, N)$ is a constant.

THEOREM 6. Suppose $\beta > \sigma + 1$. Then

(a) if $\beta < (\sigma + 1)(N + 2)/(N - 2)_+$, then problem (22), (23) has at least one solution $\theta_A = \theta_A(\xi) > 0$ in $\mathbf{R}_+^1$, which decreases in a strictly monotonic fashion in ξ and has the asymptotics (41), whereby

$$C_A \geqslant \left\{\frac{2N}{[\beta-(\sigma+1)]}\left[\frac{\beta-(\sigma+1)}{\beta}\right]^{\frac{\beta}{\sigma+1}}\right\}^{\frac{1}{\beta-(\sigma+1)}} > 0; \tag{41'}$$

(b) for $N = 1$ problem (22), (23) has no fewer than

$$K_0 = -[-a] - 1, \quad a = \frac{\beta-1}{\beta-(\sigma+1)} > 1,$$

distinct solutions differing in the number of extremal points for $\xi \in [0, +\infty)$ (the first is a monotone solution which has only a maximum point at $\mu = 0$; if $K_0 > 1$ the second solution has a minimum at $\xi = 0$ and exactly one maximum for $\xi > 0$, etc.)

The main part in the proof of Theorem 6 is the method proposed in [40, 73] of describing the local behavior of nonmonotone solutions near a homogeneous $\theta \equiv \theta_H$ by "linearization" of the equation. Together with (22), (23) we consider the family of Cauchy problem

$$\frac{1}{\xi^{N-1}}(\xi^{N-1}|\theta|^\sigma\theta')' - m\theta'\xi - \frac{1}{\beta-1}\theta + |\theta|^{\beta-1}\theta = 0, \quad \xi > 0, \tag{42}$$

$$\theta(0) = \mu > 0, \quad \theta'(0) = 0; \quad m = \frac{\beta-(\sigma+1)}{2(\beta-1)}. \tag{43}$$

We set $\theta \equiv \theta(\xi; \mu) = \theta_H + \varepsilon v(\xi)$, where $\varepsilon > 0$ is a constant which plays the role of a small parameter below. For $v(\xi)$ we then obtain the problem

$$\theta_H^\sigma \frac{1}{\xi^{N-1}}(\xi^{N-1}v')' - mv'\xi + v = \varepsilon\Phi_\varepsilon(v), \quad \xi > 0, \tag{44}$$

$$v(0)=\nu\equiv(\mu-\theta_H)/\varepsilon,\ v'(0)=0, \tag{45}$$

where $\Phi_\varepsilon: C^2 \to C$ is a bounded, quasilinear operator of second order. From (44) it follows that because of the continuous dependence of a solution of the equation on the parameter ε in a neighborhood of $\varepsilon = 0$ on any compact set $[0, \xi_m]$ the solution $v(\xi)$ for a particular choice of $|\varepsilon| \ll 1$ and $|\mu - \theta_H| \ll 1$ is close to a solution of the corresponding linear problem

$$\theta_H^\sigma \frac{1}{\xi^{N-1}}(\xi^{N-1}y')' - my'\xi + y = 0, \quad \xi > 0, \qquad y(0)=\nu\neq 0, \quad y'(0)=0. \tag{46}$$

By the change $\xi=\eta^{1/2}(2\theta_H^\sigma/m)^{1/2}$ the equation for y reduces to a degenerate hypergeometry equation $\eta y''+y'(c-\eta)-ay=0$, $y(0)=\nu$, where $c=N/2$, $a=-1/2m=-(\beta-1)/[\beta-(\sigma+1)]$. Solutions of this problem are nonmonotone and have for $\eta > 0$ exactly $K = -[-a]$ roots (see, for example, [7]). Returning to problem (44), (45) and then to the original problem (42), (43) we find that for sufficiently small $|\mu-\theta_H|$ any function $\theta(\xi;\mu)$ in the region of its positivity has no fewer than

$$K_0=-[-a]-1\geqslant 1 \quad \text{for} \quad a=\frac{\beta-1}{\beta-(\sigma+1)}>1 \tag{47}$$

extremal points for $\xi > 0$. This fact is of principle significance. We note that for $\sigma = 0$ from (47) we have $K_0 \equiv 0$ which, as will be explained below, bears witness to the absence of nontrivial self-similar solutions θ_A for $\sigma = 0$ [and $1<\beta\leqslant(N+2)/(N-2)_+$].

We shall briefly describe the main features of the proof of proposition (a) of Theorem 6. We set $\mathcal{N}=\{\mu>\theta_H \mid \theta(\xi;\mu)>0$ on $(0, \xi_\mu)$ and has on $(0, \xi_\mu)$ at least one minimum point$\}$. From the foregoing analysis $\mathcal{N}\neq\varnothing$. If $\mathcal{N}$ is bounded above, then the function $\theta(\xi;\theta_0)$, $\theta_0=\sup\mathcal{N}$ will obviously be the desired solution of problem (22), (23) with asymptotics (41).

We shall thus prove that $\sup\mathcal{N}<+\infty$. In problem (42), (43) we set $\psi_\mu(\xi)=\mu^{-(\sigma+1)}\varphi(\xi/\mu^{m(\beta-1)})$, where $\varphi=|\theta|^\sigma\theta$. For ψ_μ we then obtain the following integral equation:

$$\psi_\mu'(\xi)=-(\sigma+1)\xi^{1-N}\int_0^\xi \eta^{N-1}|\psi_\mu|^{\frac{\beta}{\sigma+1}-1}\psi_\mu d\eta+\mu^{1-\beta}G(\psi_\mu), \tag{48}$$

where $\psi_\mu(0)=1$, $\psi_\mu'(0)=0$ and $G(\psi_\mu)$ is an integral operator bounded in C,

$$G(\psi_\mu)=m(\sigma+1)|\psi_\mu|^{\frac{1}{\sigma+1}-1}\psi_\mu+(\sigma+1)\xi^{1-N}\int_0^\xi \eta^{N-1}\left[\frac{1}{\beta-1}-mN\right]|\psi_\mu|^{\frac{1}{\sigma+1}-1}\psi_\mu d\eta,$$

which is not contractive in a neighborhood of $\psi_\mu \equiv 0$. In contrast to the case $\beta\leqslant\sigma+1$ (see Subsec. 3.2) for $\beta > \sigma + 1$ it is a priori not possible to say anything regarding the boundedness of $|\psi_\mu|$ and $|\psi_\mu'|$ on a compact set. Therefore, the continuous dependence of $\psi_\mu(\xi)$ on μ in a neighborhood of $\mu = +\infty$ will be used. For $\mu = +\infty$ (48) formally becomes the equation

$$\psi_\infty'(\xi)=-(\sigma+1)\xi^{1-N}\int_0^\xi \eta^{N-1}|\psi_\infty|^{\frac{\beta}{\sigma+1}-1}\psi_\infty d\eta, \quad \psi_\infty(0)=1$$

or, equivalently, $\xi^{1-N}(\xi^{N-1}\psi_\infty')'+(\sigma+1)|\psi_\infty|^{\beta/(\sigma+1)-1}\psi_\infty=0$, $\xi>0$; $\psi_\infty(0)=1$, $\psi_\infty'(0)=0$. Therefore, as follows from Lemma 4 in Subsec. 3.3, for $\beta < (\sigma + 1)(N + 2)/(N - 2)_+$ the function $\psi_\infty(\xi)$ vanishes at some point $\xi = \xi_*$ [$\psi_\infty > 0$ on $(0, \xi_*)$], and $\psi_\mu'(\xi_*)<0$. We consider the compact set $K_\varepsilon=[0,\xi_*-\varepsilon]$, $\varepsilon>0$. Then $\psi_\infty(\xi)\geqslant\psi_\infty(\xi_*-\varepsilon)>0$ on K_ε, and hence on K_ε there is continuous dependence of ψ_μ and ψ_μ' on μ in a neighborhood of $\mu = +\infty$, i.e., in particular, $\psi_\mu(\xi_*-\varepsilon)\to\psi_\infty(\xi_*-\varepsilon)$, $\psi_\mu'(\xi_*-\varepsilon)\to\psi_\infty'(\xi_*-\varepsilon)$ as $\mu \to +\infty$. Now $G(\psi_\mu)=O(\|\psi_\mu\|_C^{1/(\sigma+1)})\to 0$ as $\|\psi_\mu\|_C\to 0$. Therefore, it is possible to use Schauder's theorem to extend $\psi_\mu(\xi)$ from the point $\xi=\xi_*-\varepsilon$ to a neighborhood of $\xi = \xi_*$; moreover, due to the "smallness" of $G(\psi_\mu)$ the derivative $\psi_\mu'(\xi)$ on $\{|\xi-\xi_*|<\varepsilon\}$ will change inappreciably, and as a result $\psi_\mu(\xi)$ vanishes for all sufficiently large $\mu > 0$. This completes the proof of proposition (1). The estimate (41) will be proved in Subsec. 3.8.

The question of the existence of nontrivial solutions of problem (22), (23) for $\beta\geqslant(\sigma+1)(N+2)/(N-2)_+$ remains open.

Proposition (b) of Theorem 6 is proved in detail in [1, 2]. A discrete spectrum of solutions of problem (22), (23) can be constructed in this manner for particular values of β, σ also in the multidimensional case $N > 1$. The idea of linearization of the equation relative to the homogeneous solution $\theta \equiv \theta_H$ makes it possible to investigate the structure of nonsymmetric solutions of the elliptic equation (21) for $\beta > \sigma + 1$. So far interesting qualitative and numerical results have been obtained in this direction [60].

In conclusion we present an important result below which follows from a simple analysis of Eq. (22).

THEOREM 7. Suppose $\sigma+1<\beta\leqslant(\sigma+1)N/(N-2)_+$. Then any self-similar solution (20) is critical, that is, everywhere in $(0, T_0)\times\mathbf{R}^N$

$$\frac{\partial u_A}{\partial t}\equiv(T_0-t)^{-\frac{1}{\beta-1}-1}\left[m\theta'_A\xi+\frac{1}{\beta-1}\theta_A\right]>0 \tag{49}$$

and hence

$$u_A(t, x)<u_A(T_0^-, x)\equiv C_A|x|^{-\frac{2}{\beta-(\sigma+1)}},\quad x\in\mathbf{R}^N\setminus\{0\}. \tag{50}$$

The estimate (50) graphically indicates the effective localization of combustion in an LS-regime — the temperature grows without bound as $t \to T_0^-$ only at the one point $x = 0$; in $\mathbf{R}^N\setminus\{0\}$ it is bounded above uniformly with respect to $t\in(0, T_0)$ by the limit distribution $u_A \times (T_0^-, x)$. Strict localization in the Cauchy problem (1), (2) for $\beta > \sigma + 1$ with compactly supported $u_0(x)$, $N = 1$ will be proved in Subsec. 3.5.

We shall now proceed to an investigation of general properties of rather arbitrary unbounded solutions of the Cauchy problem. Here an important role is played by a special approach to comparing different unbounded solutions of Eq. (1) having the same time of peaking.

3.4. Main Comparison Theorem. Application of the traditional apparatus of the theory of parabolic equations — comparison theorems based on the boundary conditions — is difficult for investigation of the asymptotic properties of unbounded solutions. The situation is that majorizing one solution of the Cauchy problem by another, $u\geqslant v$, as a rule, means that the solutions $u \not\equiv v$ have different times of peaking; therefore, beginning from some time, one of them ceases to exist, and the comparison loses meaning. In the theorem formulated below solutions with the same times of existence are compared, while the number of spatial intersections of the profiles of the solutions is taken as the main "comparison criterion." As will be shown below, this approach makes it possible to describe the space—time structure of unbounded solutions of general type.

Below $u\geqslant 0$ and $v\geqslant 0$ denote different solutions of the one-dimensional equation

$$u_t=(u^\sigma u_x)_x+u^\beta. \tag{1'}$$

We say that two solutions u and v for fixed $t=t_0\geqslant 0$ "intersect" with respect to x on a (bounded) interval $I=[a, b]$, $a\leqslant b$, if $w(t_0, x)\equiv u(t_0, x)-v(t_0, x)=0$ on I, and w assumes values of different signs in any ε-neighborhood of it $\{a-\varepsilon<x<b+\varepsilon\}$, $\varepsilon>0$.

This definition is specially oriented toward investigation of generalized solutions nonanalytic in x. If u and v, $u \not\equiv v$, are analytic in x (which is natural for solutions of semilinear parabolic equations with analytic coefficients; see [92]), then obviously any intersection for $t = t_0 > 0$ occurs at a point, and they are isolated with respect to x.

We introduce the following notation: $N(t_0)$ is the number of intersections of $u(t_0, x)$ or $v(t_0, x)$ or, equivalently, the number of changes of sign of the difference $w(t_0, x)$ in the x-region considered.

THEOREM 8 (the basic comparison theorem). Suppose $u(t, x)\geqslant 0$ and $v(t, x)\geqslant 0$ are two distinct, unbounded solutions of Eq. (1') having the same time of peaking $t = T_0 < +\infty$. Suppose that $N(0) < +\infty$. The following assertions hold:

(I) Suppose u and v are defined in a region $\omega_T=(0, T]\times(\eta_1(t), \eta_2(t))$, $T<T_0$, which is not necessarily bounded, where η_i are either continuous functions or separately or together are equal to $\pm\infty$. Then $N(T)$ in $[\eta_1(T), \eta_2(T)]$ does not exceed the number of changes of sign of the difference $w = u - v$ on the parabolic boundary $\partial\omega_T$.

(II) Suppose u, v are solutions of the Cauchy problem for (1') with $v_0(x) \equiv v(0, x) > 0$ in $\mathbf{R}^1$, and $u_0(x) \equiv u(0, x)$ is compactly supported. Then $N(t)\geqslant 2$, and N(t) is nonincreasing in $t\in(0, T_0)$.

Suppose u_0 and v_0 are compactly supported functions in $\mathbf{R}^1$ with connected supports. Then

(III) $\{t \in [0, T_0) \mid u(t, x) > v(t, x)$ for all $x \in \overline{\operatorname{supp}} \, v(t, x)\} = \varnothing$;

(IV) if $\operatorname{supp} u_0 \cap \operatorname{supp} v_0 = \varnothing$, then $N(t) \leqslant 1$ for all $t \in (0, T_0)$.

For uniformly parabolic equations assertion (I) is a natural consequence of the strong maximum principle, and in a somewhat different formulation it has been known for a relatively long time [126] (see also [117, 121]). In the proof the fact is used that the difference $w = u - v$ satisfies in ω_T a "linear" parabolic equation. Its extension to the case of unbounded $|\eta_i| = +\infty$, $i = 1$ or 2, in the case where the behavior of u and v at infinitely distant points in x causes no special difficulties; see, for example, [26, 27]. In the case of a degenerate equation (1') in the proof the possibility is used of approximation (with representation of the number of intersections on $\partial\omega_T$) of generalized solutions u and v by sequences of classical, positive solutions on each of which the equation is uniformly parabolic [15-17].

The condition of equality of the times of peaking of the solutions u and v is used in an essential manner in (II) and (III). It is not difficult to verify that violation of the conclusions of each of these assertions, because of sufficient regularity of the solutions, leads to contradiction of the condition of coincidence of the times of existence of u and v. Comparison theorems of this type were broadly used in the works [15-17, 26-28] where additional facts can be found.

To conclude this subsection we note that the concept of intersection is defined in a natural manner for symmetric solutions of Eq. (1) depending on $r = |x|$. We shall not specially formulate the corresponding comparison theorem (which is altogether analogous to the foregoing), and below in analyzing unbounded solutions $u = u(t, |x|)$ for $N > 1$ we shall appeal to the corresponding one-dimensional assertions.

3.5. Localization of Unbounded Solutions for $\beta \geqslant \sigma + 1$. We restrict ourselves below to the analysis of the one-dimensional Cauchy problem

$$u_t = (u^\sigma u_x)_x + u^\beta, \ t > 0, \ x \in \mathbf{R}^1; \ \sigma > 0, \ \beta > 1; \tag{51}$$

$$u(0, x) = u_0(x) \geqslant 0, \ x \in \mathbf{R}^1; \ \sup u_0 < +\infty, \tag{52}$$

where $u_0 \not\equiv 0$ is a compactly supported function with connected support, $\omega(0) = \operatorname{supp} u_0 \equiv \{x \in \mathbf{R}^1 \mid u_0(x) > 0\} = (h_-(0), h_+(0)) \subset \mathbf{R}^1$, and $u_0^{\sigma+1}$ is uniformly Lipschitz continuous in $\mathbf{R}^1$. Then the support of an unbounded, generalized solution u(t, x) for each $t \in (0, T_0)$ is also bounded and connected: $\omega(t) \equiv \operatorname{supp} u(t, x) = (h_-(t), h_+(t)) \subset \mathbf{R}^1$. Moreover, because of well-known properties of solutions of an equation with a source $\omega(t) \subseteq \omega(t + \tau)$ for all $t, t+\tau \in [0, T_0)$, $\tau > 0$, so that mes $\times$ $\omega(t) \equiv h_+(t) - h_-(t)$ does not decrease with time L_0, and $h_\pm(t) \in C([0, T_0))$.

At the present time differential and other local properties of the functions $h_\pm(t)$ have been studied in detail (see [83, 109, 123]; interesting results have been obtained for the multidimensional equation $u_t = \Delta u^{\sigma+1}$, $\sigma > 0$ [90, 94, 123]). We shall primarily be interested in the behavior of $h_\pm(t)$ for $t \to T_0^-(u_0) < +\infty$. The conditions $|h_\pm(T_0^-)| < +\infty$ are equivalent to localization (in the strict sense) of the unbounded solution. As will be shown below, this occurs in S- ($\beta = \sigma + 1$) and LS- ($\beta > \sigma + 1$) regimes. For $1 < \beta < \sigma + 1$ (an HS-regime; see Subsec. 3.6) $h_\pm(t) \to \pm\infty$ as $t \to T_0^-$, and there is no localization.

The main result of this subsection is formulated in the following general assertions [15].

THEOREM 9. Let $\beta = \sigma + 1$. Then an unbounded solution of the Cauchy problem (51), (52) is localized, and

$$h_+(T_0^-) \leqslant h_+(0) + L_s, \quad h_-(T_0^-) \geqslant h_-(0) - L_s, \tag{53}$$

where $L_S = 2\pi(\sigma + 1)^{1/2}/\sigma$ is the fundamental length of the S-regime, and, in particular, $\operatorname{mes} \omega(T_0^-) \leqslant \operatorname{mes} \omega(0) + 2L_S$.

The estimates (53) show that the quantity L_S is actually a fundamental characteristic (not depending on the initial perturbation) of the nonlinear medium. In correspondence with (53) each front of the thermal structure independent of the form of the initial perturbation u_0 and the time of existence $T_0 = T_0(u_0) < +\infty$ can travel a distance no greater than the fundamental length L_S.

THEOREM 10. Let $\beta > \sigma + 1$ and let $T_0 < +\infty$ be the time of existence of an unbounded solution (51), (52). Then it is localized, and

$$h_+(T_0^-) \leqslant h_+(0) + \xi^* T_0^m, \quad h_-(T_0^-) \geqslant h_-(0) - \xi^* T_0^m, \tag{54}$$

where $m=[\beta-(\sigma+1)]/2(\beta-1)$ and $\xi^*=\xi^*(\sigma,\beta)>0$ is a constant.

We emphasize that here, in contrast to the S-regime, the distance which a front of the solution can travel depends through the quantity $T_0 = T_0(u_0)$ on the initial function u_0. Upper bounds for $T_0 = T_0(u_0)$ are obtained in Subsec. 1; a lower bound can easily be obtained by comparison with a spatially homogeneous solution.

Other theorems which specify for a special form of $u_0(x)$ the depth of penetration of the thermal wave will be proved below. We mention that many results carry over to the multidimensional case without principle changes.

3.5.1. Proof of Theorem 9. Investigation of the effect of localization for $\beta = \sigma + 1$ is carried out by comparison with the localized self-similar solution $u_A=(T_0-t)^{-1/\sigma}\theta_s(x)$, where (see part 3.3.2)

$$\theta_s(x) = \begin{cases} \left[\dfrac{2(\sigma+1)}{\sigma(\sigma+2)}\cos^2\left(\dfrac{\pi x}{L_s}\right)\right]^{1/\sigma}, & |x|<L_s/2, \\ 0, \quad |x| \geqslant L_s/2, \end{cases} \tag{55}$$

so that $\operatorname{mes}\operatorname{supp} u_A = L_s$.

We shall prove the first estimate of (53); the second is established in a similar way. For convenience we denote by $u_s(t, x; x_0, T_0)$ the function $u_A=(T_0-t)^{-1/\sigma}\theta_s(x-x_0)$, localized in the region $|x-x_0|<L_s/2$ and having the same time of peaking $t = T_0$ as $u(t, x)$. We set $x_0=h_+(0)+L_s/2$, $x_1=x_0+L_s/2$.

It is then obvious that $u_0(x)$ and $u_s(0, x; x_0, T_0)$ have only one intersection at the point $x = h_+(0)$, so that $N(0) = 1$, and hence $N(t) \leqq 1$ for all $t\in(0, T_0)$. We shall show that $h_+(t)\leqslant h_+(0)+L_s$. We suppose this is not so and for some $t=t_1\in(0, T_0)$, $h_+(t_1)>h_+(0)+L_s$. But then $u(t_1, x_1)>u_s(t_1, x_1; x_0, T_0)=0$ and $u(t_1, h_+(0))>u_s(t_1, h_+(0); x_0, T_0)=0$. Therefore, either $N(t_1)\geqslant 2$, which contradicts assertion (IV) in the comparison theorem or $N(t_1) = 0$ and $\operatorname{supp} u_s(t_1, x; x_0, T_0)\subset\operatorname{supp} u(t_1, x)$, which contradicts assertion (III) (i.e., the assumption of the equality of the times of peaking of the solutions u and u_s). This completes the proof.

Comparison with the self-similar solution u_A for $\beta = \sigma + 1$ makes it possible to establish other finer properties of S-regimes with peaking.

3.5.2. Condition of Invariance of the Support of an Unbounded Solution for $\beta = \sigma + 1$. The support of the self-similar solution (26), (25) does not change during all the time of its existence $t\in(0, T_0)$. It will be shown below that this property is possessed by a broad set of order (non-self-similar) solutions which are localized in the region $\operatorname{supp} u_0$ where they are initially prescribed. In the formulation of the theorem $u_s(t, x; x_0, T_0)$ denotes the solution introduced in part 3.5.1.

THEOREM 11. Let $\beta=\sigma+1$, $\operatorname{mes}\operatorname{supp} u_0>L_s$ and suppose $T_0 < +\infty$ is the time of existence of a solution of problem (51), (52). Suppose there exists $\lambda_0 > 0$ such that $u_s(0, x; x_0, \lambda_0)\leqslant u_0(x)$ in $\mathbf{R}^1$, where $x_0=h_+(0)-L_s/2$, and the functions $u_0(x)$, $u_s(0, x; x_0, \lambda)$ intersect only once for all $0 < \lambda < \lambda_0$. Then $h_+(t) = h_+(0)$ for any $t\in(0, T_0)$.

It is curious that for the immobility of a point of the right front during the entire time of peaking a "nonlocal" condition is needed on the character of the behavior of u_0 only in an L_s-neighborhood of it $[h_+(0) - L_s, h_+(0)]$; the behavior of $u_0(x)$ in the rest of space, i.e., for all $x\leqslant h_+(0)-L_s$, is not reflected in the mobility of the front. This underscores the universality of a characteristic nonlinear medium such as the fundamental length L_s of an S-regime which now emerges as a type of radius of the effective influence of thermal perturbations.

Combining Theorem 11 with an analogous assertion regarding the immobility of the left front, $h_-(t) \equiv h_-(0)$, we obtain the set which unbounded solutions with unchanged support generate [15].

Proof of Theorem 11. It is obvious that $T_0=T_0(u_0)\leqslant\lambda_0$. If $T_0 = \lambda_0$, the the assumption regarding the motion of the right front because of assertion (III) of Theorem 8 leads to a contradiction. Thus, $T_0 < \lambda_0$. If then $N(t)$ is the number of intersections of $u(t, x)$ and $u_s(t, x; x_0, T_0)$, $x_0=h_+(0)-L_s/2$, then under the conditions of the theorem $N(0) = 1$, and hence $N(t)\leqslant 1$. The arguments used in completing the proof of Theorem 9 are now repeated.

3.5.3. The Condition of Localization at the Fundamental Length L_S. Below conditions are obtained under which an unbounded solution during the time of peaking is localized in the region $\{|x|<L_S/2\}$, i.e., both fronts of the solution during the time of its existence travel a total distance not exceeding L_S − mes supp u_0 < L_S. It is hereby assumed that

$$u_0(-x)=u_0(x) \text{ in } \mathbf{R}^1; \quad u_0(x) \text{ does not increase for } x>0. \tag{56}$$

Under these conditions $u(t, -x)\equiv u(t, x)$, $u_x(t, x)\leqslant 0$ for $x\in(0, h_+(t))$ and $\sup_x u(t, x)\equiv u(t, 0)$.

THEOREM 12. Suppose $\beta=\sigma+1$, $\overline{\operatorname{supp} u_0}\subset\{|x|<L_S/2\}$ and (56) is satisfied. Suppose, moreover, that there exists $\lambda_0 > 0$ such that $u_s(0, x; 0, \lambda_0)\geqslant u_0(x)$ in $\mathbf{R}^1$, and $u_0(x)$ and $u_0(0, x; 0, \lambda)$ have exactly two intersections for all $\lambda > \lambda_0$. Then $|h_\pm(t)|\leqslant L_s/2$ for any $t\in(0, T_0(u_0))$.

The theorem is proved in analogy to the previous theorem. Under the assumptions made $T_0 < \lambda_0$, and hence $u_S(0, x; 0, T_0)$ and $u_0(x)$ have exactly two intersections: $N(0) = 2$, i.e., $N(t)\leqslant 2$. But then either $N(t) \equiv 2$ (which proves the theorem) or $N(t_1) = 0$ for some $t_1\in(0, T_0)$, which contradicts the equality of the times of peaking of the two distinct solutions.

3.5.4. Proof of Theorem 10 (Localization in an LS-Regime, $\beta > \sigma + 1$). In Subsec. 3.3.5 it is shown that for $\beta > \sigma + 1$ Eq. (51) has no self-similar solution capable of conveying the property of strict localization as in an S-regime. However, it is possible to construct a lower self-similar solution with a localized point of the front:

$$u_A^-(t, x)=(T_0-t)^{-1/(\beta-1)}\theta_-(\xi), \quad \xi=x/(T_0-t)^m, \tag{57}$$

$$(\theta_-^\sigma\theta_-')'-m\theta_-'\xi-\frac{1}{\beta-1}\theta_-+\theta_-^\beta=0, \tag{58}$$

where $m=[\beta-(\sigma+1)]/2(\beta-1)>0$.

LEMMA 5. Let $\beta > \sigma + 1$. Then there exists a solution $\theta_-(\xi) > 0$ on $(-\xi^*, 0)$ of Eq. (58) such that

$$\theta_-(0)=0, \quad (\theta_-^\sigma\theta_-')(0)=0, \tag{59}$$

and $\theta_-(-\xi^*) = 0$.

Local solvability of problem (58), (59) for $\xi < 0$ is established by reducing it to an equivalent integral equation and analyzing the latter with the help of Schauder's theorem. The fact that $\theta_-(\xi)$ vanishes at some point $\xi = -\xi^* < 0$ follows from the identity obtained after multiplication of (58) by $(\theta_-^\sigma\theta_-')(\xi)$ and integrating over the admissible interval $(\xi, 0)$; see [15].

We now set $\theta_-(\xi) = 0$ for $\xi > 0$. Then $\theta_-(\xi)$ is a generalized solution of Eq. (58) on $(-\xi^*, +\infty)$, and hence $u_A^-(t, x)$ in (57) is a generalized solution of Eq. (51) in the region $(0, T_0)\times(x_*(t), +\infty)$ with a mobile left boundary $x_*(t)\equiv-\xi^*(T_0-t)^m\to 0^-$, $t\to T_0^-$, on which $u_A(t, x_*(t)) = 0$. Therefore, (57) exhibits localization in the strict sense: the point of the right front $x_\Phi(t)\equiv 0$ is fixed in spite of the unbounded growth of u_A^- in a left neighborhood of it. If we additionally set $\theta_-(\xi) = 0$ for $\xi < -\xi^*$, then u_A^- is obviously an unbounded lower solution of Eq. (51) in $(0, T_0)\times\mathbf{R}^1$ [we note that $(\theta_-^\sigma\theta_-')(-\xi^*)>0$].

We denote by $u_{LS}^-(t, x; x_0, T_0)$ the function $u_A^-(t, x - x_0)$, and we set $x_0=h_+(0)+\xi^*T_0^m$, $T_0 = T_0(u_0)$ − the time of peaking of $u(t, x)$. In the region $(0, T_0)\times\{x_0-\xi^*(T_0-t)^m<x<+\infty\}$ intersections of $u(t, x)$ and u_{LS}^- can occur only due to their occurrence on the lateral boundary $(0, T_0)\times\{x=x_0-\xi^*(T_0-t)^m\}$, since for $t = 0$ $u_0(x)$ and $u_{LS}^-(0, x; x_0, T_0)$ do not intersect in this region. But then by construction $N(t)\leqslant 1$ for all $t\in(0, T_0)$, which, as we have seen earlier, proves the theorem.

With the help of the lower solution u_A^- in exactly the same way as in the case of an S-regime it is possible to find conditions for the immobility of the front of the solution during the entire time of peaking and also other properties (see [15]).

In conclusion we mention that $\xi^*(\sigma, \sigma+1)=L_s$, so that Theorem 9 is a special case of Theorem 10 for $\beta = \sigma + 1$ (when $m\equiv[\beta-(\sigma+1)]/2(\beta-1)=0$).

3.6. Nonlocalized, Unbounded Solutions of the HS-Regime, $1 < \beta < \sigma + 1$. In the next assertion with the help of the comparison theorem sharp upper and lower bounds are obtained for the length of the support of an arbitrary, unbounded solution of problem (51), (52) for $\beta\in(1, \sigma+1)$ which is compactly supported in x.

THEOREM 13. Let $1 < \beta < \sigma + 1$. Then a solution of the Cauchy problem (51), (52) is not localized, and if $t = T_0(u_0) < +\infty$ is the time of peaking, then there are the "lower" bounds

$$h_+(t) \geqslant h_-(0) + \xi_0 T_0^m [(1 - t/T_0)^m - 1], \tag{60}$$

$$h_-(t) \leqslant h_+(0) - \xi_0 T_0^m [(1 - t/T_0)^m - 1], \quad t \in (0, T_0), \tag{61}$$

where $m = [\beta - (\sigma+1)]/2(\beta - 1) < 0$ [and hence $h_\pm(t) \to \pm\infty$ as $t \to T_0^-$]. Moreover, there are the "upper" bounds

$$h_+(t) \leqslant h_+(0) + \xi_0 T_0^m [(1 - t/T_0)^m + 1], \tag{62}$$

$$h_-(t) \geqslant h_-(0) - \xi_0 T_0^m [(1 - t/T_0)^m + 1], \quad t \in (0, T_0). \tag{63}$$

Here $\xi_0 = \xi_0(\sigma, \beta) > 0$ is the length of the support of the compactly supported self-similar function $\theta_A(\xi)$ ($\xi_0 = \mathrm{mes\,supp}\,\theta_A < +\infty$) whose existence is proved in Theorem 5, Subsec. 3.3.4.

The estimates formulated in the theorem imply that

$$h_\pm(t) = \pm \xi_0 (T_0 - t)^m + O(1), \quad t \to T_0^-, \tag{64}$$

for $\beta \in (1, \sigma+1)$ and any solution of problem (51), (52) compactly supported in x. From (64), in particular, it follows immediately that the self-similar function θ_A for $N = 1$ is unique (see Theorem 5 in Subsec. 3.3.4) [16].

In method the proof of Theorem 13 is practically no different from previous ones. Having "at our disposal" a self-similar solution u_A with time of peaking $T_0 = T_0(u_0) < +\infty$ first on one side and then on the other side of $\mathrm{supp}\, u_0$, so that in each of the cases $N(0) = 1$, from the comparison theorem we find that $N(t) \leqslant 1$ for all $t \in (0, T_0)$. This leads to the lower and upper bounds for $h_\pm(t)$. For the details we refer to the works [15, 16].

In conclusion we mention that absence of localization for $\beta \in (1, \sigma+1)$ is also established by the method of stationary states in part 8.1 where, for example, it is shown that in an HS-regime $u(t, x) \to +\infty$ as $t \to T_0^-$ everywhere in R^1.

3.7. On Asymptotic Stability of Unbounded, Self-Similar Solutions. The main purpose of this subsection is to prove "structural" stability of the self-similar solutions constructed, i.e., to establish conditions under which the asymptotic behavior of $u(t, x)$ as $t \to T_0^-$ is described by $u_A(t, x)$. Earlier we discussed some principle difficulties of the analysis of the space-time structure of unbounded solutions which arise in the absence of stability of solutions with respect to perturbations of the initial function. Therefore, in this subsection we shall not strive for maximum generality which requires considerable efforts directly mainly in overcoming nonprinciple difficulties. The key features of the proof are presented for the example of the one-dimensional Cauchy problem using some restrictions on the form of the initial perturbation u_0.

Everywhere below we consider problem (51), (52), and we assume that the conditions on u_0 formulated in Subsec. 3.5 and also conditions (56) are satisfied. Then $u(t, x)$ is a solution even in x, and $\sup_x u \equiv u(t, 0)$.

We first give estimates of the amplitude of unbounded solutions of problem (51), (52).

LEMMA 6. Let $\sigma > 0$, $\beta > 1$. If $T_0 = T_0(u_0) < +\infty$, then

$$\sup_{x \in R^1} u(t, x) > \theta_H (T_0 - t)^{-1/(\beta-1)}, \quad t \in [0, T_0); \quad \theta_H = (\beta - 1)^{-1/(\beta-1)}. \tag{65}$$

The proof is based on comparison of a solution $u(t, x)$ compactly supported in x with the homothermic solution $v(t) = \theta_H(T_0 - t)^{-1/(\beta-1)}$ (they must intersect, i.e., $N(t) \geqslant 2$ for all $t \in [0, T_0)$, whence (65) follows).

LEMMA 7. Suppose $\sigma > 0$, $\beta > 1$ and (56) is satisfied. If $T_0 = T_0(u_0) < +\infty$, then there exists a constant $\theta_* > \theta_H$ such that

$$\sup_{x \in R^1} u(t, x) \equiv u(t, 0) < \theta_* (T_0 - t)^{-1/(\beta-1)}, \quad t \in [0, T_0). \tag{66}$$

A proof of Lemma 7 can be found in [15, 16] (see also the assertions in [26-28] having analogous proofs). In all three cases $\beta < \sigma+1$, $\beta = \sigma+1$, $\beta > \sigma+1$ it is carried out by comparing $u(t, x)$ with a lower self-similar solution u_A^- of the form

$$u_A^-(t, x) = (T_0 - t)^{-1/(\beta-1)} \theta(|\xi|; \mu), \quad \xi = x/(T_0 - t)^m, \tag{67}$$

where the function $\theta(\xi; \mu)$ wherever it is positive is a solution of the Cauchy problem (42), (43).

As an example we present the proof of (66) in the case $\beta > \sigma + 1$ (for $\beta \leqslant \sigma+1$ it is still simpler). As follows from the proof of proposition (a) of Theorem 6, for all sufficiently large $\mu > 0$ the function $\theta(\xi;\ \mu)$ vanishes at some point $\xi = \xi_\mu > 0$ ($\theta(\xi;\ \mu) > 0$ on $[0,\ \xi_\mu)$). Moreover, by the continuous dependence of $\theta^{\sigma+1}$ and $(\sigma^{\sigma+1})'_\xi$ on μ in a neighborhood of $\mu = +\infty$ we have the estimates

$$\xi_\mu \sim \mu^{\frac{\sigma+1-\beta}{2}} \to 0,\ |[\theta^{\sigma+1}(\xi_\mu;\ \mu)]'| \sim \mu^{\frac{\beta+\sigma+1}{2}},\ \mu \to +\infty$$

[this follows from the condition $\theta(\xi;\mu) \sim V_\lambda^{1/(\sigma+1)}((\sigma+1)^{1/2}|\xi|)$, $\lambda = \mu^{\sigma+1}$ as $\mu \to +\infty$; see Subsec. 3.3.5]. Therefore, it is always possible to choose a sufficiently large $\mu = \theta_*$ so that, first of all, $\overline{\operatorname{supp} u_A^-(0,x)} \subset \operatorname{supp} u_0$ [then obviously $\overline{\operatorname{supp} u_A^-(t,\ x)} \subset \operatorname{supp} u_A^-(0,\ x) \subset \operatorname{supp} u_0$ for all $t \in (0,\ T_0)$] and, secondly, $u_0(x)$ and $u_A^-(0,\ x)$ have exactly two intersections. Then by the comparison theorem $N(t) \equiv 2$ in the region $(0,\ T_0) \times \operatorname{supp} u_A^-(t,\ x)$, which leads to (66).

Remark. Following the procedure indicated, it is not hard to show that for $1 < \beta < \sigma + 1$ the estimate (66) holds for arbitrary compactly supported $u_0(x)$. Indeed, in this case $\xi_\mu \to +\infty$ as $\mu \to +\infty$. Therefore, by Theorem 13 [see also (64)] it is possible to find a $\mu = \theta_*$ such that, first of all $\operatorname{supp} u(t,\ x) \subset \{|x| < \xi_\mu (T_0-t)^m\}$ for all $t \in [0,\ T_0)$ and, secondly, $u_0(x) \leqslant u_A^-(0,\ x)$ in $\{|x| < \xi_\mu T_0^m\}$. Then obviously $u(t,\ x) \leqslant u_A^-(t,\ x)$ in $\{|x| < \xi_\mu (T_0-t)^m\}$, which gives (66).

3.7.1. Asymptotic Stability of the Self-Similar Solution of the S-Regime ($\beta = \sigma + 1$). If conditions (56) are satisfied the only self-similar solution which can possibly be asymptotically stable is

$$u_A(t,\ x) = (T_0-t)^{-1/\sigma}\theta_s(x),\ t \in (0,\ T_0),\ x \in \mathbf{R}^1, \tag{68}$$

where θ_s is the function (55). It satisfies the ordinary differential equation (see Subsec. 3.3.2)

$$(\theta_s^\sigma \theta_s')' - \frac{1}{\sigma}\theta_s + \theta_s^{\sigma+1} = 0,\ x \in \mathbf{R}^1. \tag{69}$$

We specially emphasize that $\theta = \theta_s$ is the unique nontrivial solution of it which is even, compactly supported, and monotone in x for $x > 0$.

In accordance with the structure of the self-similar solution (68) we introduce a self-similar treatment of the unbounded solution $u(t,\ x)$ of problem (51), (52), $\beta = \sigma + 1$, with time of peaking $T_0 = T_0(u_0) < +\infty$:

$$\theta(t,\ x) = (T_0-t)^{1/\sigma} u(t,\ x) \text{ in } (0,\ T_0) \times \mathbf{R}^1. \tag{70}$$

The asymptotic (structural) stability of the self-similar solution (68) implies the following theorem (see [16]).

THEOREM 14. Suppose $\beta = \sigma + 1$, and conditions (56) hold. Then uniformly in $\mathbf{R}^1$

$$\theta(t,\ x) \equiv (T_0-t)^{1/\sigma} u(t,\ x) \to \theta_s(x),\ t \to T_0^-. \tag{71}$$

It can be verified without difficulty that the function $\theta = \theta(\tau,\ x)$ at the new time $\tau = -\ln(1-t/T_0): [0,\ T_0) \to [0,\ +\infty)$ satisfies the Cauchy problem

$$\theta_\tau = (\theta^\sigma \theta_x)_x - \frac{1}{\sigma}\theta + \theta^{\sigma+1},\ \tau > 0,\ x \in \mathbf{R}^1, \tag{72}$$

$$\theta(0,\ x) = \theta_0(x) \equiv T_0^{1/\sigma} u_0(x),\ x \in \mathbf{R}^1. \tag{73}$$

Comparison of Eqs. (72) and (69) shows that (71) is equivalent to the proof of stabilization as $\tau \to +\infty$ of the solution of problem (72), (73) to the stationary solution $\theta \equiv \theta_s(x)$. The main difficulty here is that $\theta \equiv \theta_s$ is an unstable stationary solution of Eq. (72); $\theta \equiv 0$ and $\theta = +\infty$, roughly speaking, are stable here ($\theta \equiv \theta_s$ lies between them and is hence unstable). Corresponding examples are presented in [16, 32]. Stabilization to other "stationary solutions" is forbidden by the following:

LEMMA 8. Under the conditions of Theorem 14 for all $\tau > 0$

$$\sup_{x \in \mathbf{R}^1} \theta(\tau,\ x) > \sigma^{-1/\sigma},\ \sup_{x \in \mathbf{R}^1} \theta(\tau,\ x) < \theta_*; \tag{74}$$

$$\operatorname{supp} \theta(\tau,\ x) \subset [-l_0 - L_S,\ l_0 + L_S],\ 2l_0 = \operatorname{mes} \operatorname{supp} u_0. \tag{75}$$

The estimates (74) were proved in Lemmas 6 and 7; the inclusion (75) was proved in Theorem 9. By (75) the Cauchy problem (72), (73) is equivalent to a boundary value problem in an arbitrary region $\Omega \supset (-l_0 - L_s, l_0 + L_s)$ with the condition $\theta(\tau, x) = 0$ on $\mathbf{R}_+^1 \setminus \partial\Omega$. The following estimates can be derived without difficulty (see, for example, [11, 125]):

$$\theta^{1+\sigma/2} \in L^\infty(\mathbf{R}_+^1; L^2(\Omega)), \quad (\theta^{1+\sigma/2})_\tau \in L^2(\mathbf{R}_+^1 \times \Omega), \\ \theta^{\sigma+1} \in L^\infty(\mathbf{R}_+^1; H_0^1(\Omega)). \tag{76}$$

With the presence of estimates (74)-(76) stabilization of $\theta(\tau, x)$ in the weak sense to some stationary solution of Eq. (72) as $\tau = \tau_i \to +\infty$ follows from general results of [8, 84] with use of the fact that the equivalent boundary value problem has the Lyapunov function

$$V(\theta)(\tau) = \int_\Omega \left\{ \frac{(\theta^{\sigma+1})_x^2}{2(\sigma+1)} + \frac{(\sigma+1)}{\sigma(\sigma+2)} \theta^{\sigma+2} - \frac{1}{2} \theta^{2(\sigma+1)} \right\} dx,$$

which does not increase in τ on any evolutional trajectories. Formal computations give

$$\frac{d}{d\tau} V(\theta)(\tau) = -\frac{4(\sigma+1)}{(\sigma+2)^2} \int_\Omega (\theta^{1+\sigma/2})_\tau^2 dx \leqslant 0.$$

Here the independence of the limit function of the choice of sequence $\{\tau_i\}$ follows from the uniqueness of the corresponding nontrivial stationary solution $\theta \equiv \theta_s(x)$. Stabilization in $C(\Omega)$ follows from stronger estimates; by the method of S. N. Bernshtein in the form [82] it is not hard to show that $|(\theta^{\sigma+1})_x| \leqslant \mathrm{const}$ in $\mathbf{R}_+^1 \times \mathbf{R}^1$. From this it follows that $\theta^{\sigma+1}(\tau, x)$ is Hölder continuous also in the variable τ in $\mathbf{R}_+^1 \times \mathbf{R}^1$ [57, 100]. For details we refer to [16] (see other examples of investigating stabilization of solutions of degenerate equations in [84, 95, 118, 127]).

Remark. In the process of proving Theorem 14 we actually constructed the set of attraction $\mathscr{W}$ of the unstable stationary solution $\theta = \theta_s(x)$ in the Cauchy problem for (72) [if $\theta_0 \in \mathscr{W}$, then $\theta(\tau, \cdot) \to \theta_s(\cdot)$ as $\tau \to +\infty$ everywhere in $\mathbf{R}^1$]. It has the form $\mathscr{W} = \{\theta_0 = T_0^{1/\sigma} u_0(x) \mid u_0 \geqslant 0$ satisfies (56) and $T_0 = T_0(u_0) < +\infty\}$. It is obvious that $\mathscr{W}$ is unbounded, for example, in $C(\mathbf{R}^1)$, is infinite-dimensional, and, of source, is not dense in C. We emphasize that such an unbounded set of attraction of an unstable solution can in principle not be determined by "linear" analysis of solutions near a stationary solution (by this method it can be proved only that there exist functions $\theta_0 \in \mathscr{W}$, lying in a small neighborhood of $\theta = \theta_s$; see, for example, [106]). A somewhat different approach to the construction of $\mathscr{W}$ was used in [120].

As a corollary of (71) we present the following result.

COROLLARY. Under the conditions of Theorem 14 $u(t, x) \to +\infty$ as $t \to T_0^-$ at any point of the region of localization $\{|x| < L_s/2\}$.

Here condition (71), generally speaking, does not forbid unbounded growth of the solution outside the region of localization which, however, must proceed with speed $o((T_0 - t)^{-1/\sigma})$, i.e., more slowly than according to the self-similar law. One of the optimal results from this point of view can be obtained by combining Theorem 14 and Theorem 12 of Subsec. 3.5.3; this gives the following assertion.

THEOREM 15. Under the conditions of Theorem 12 $u(t, x) \to +\infty$ as $t \to T_0^-$ everywhere in $\{|x| < L_s/2\}$, and $u \equiv 0$ in $[0, T_0) \times \{|x| \geqslant L_s/2\}$. Combustion at all points $x \in \mathbf{R}^1$ hereby asymptotically approaches self-similar combustion: $(T_0 - t)^{1/\sigma} u(t, x) \to \theta_s(x)$, $t \to T_0^-$.

3.7.2. Asymptotic Stability of the Self-Similar HS-Regime. As shown earlier, for $1 < \beta < \sigma + 1$ Eq. (51) has a unique, self-similar, unbounded solution

$$u_A = (T_0 - t)^{-1/(\beta-1)} \theta_A(\xi), \quad \xi = x/(T_0 - t)^m; \quad m = \frac{\beta - (\sigma+1)}{2(\beta-1)}, \tag{77}$$

$$(\theta_A^\sigma \theta_A')' - m\theta_A' \xi - \frac{1}{\beta-1} \theta_A + \theta_A^\beta = 0, \quad \xi \in \mathbf{R}^1, \tag{78}$$

where $\theta_A \not\equiv 0$ is an even function which decreases for $\xi \in (0, \xi_0)$ [$\xi_0 = \mathrm{mes}\{\xi > 0 \mid \theta_A(\xi) > 0\}$]. We shall briefly characterize the main problems which arise in proving its asymptotic stability.

Introducing the self-similar representation of the solution $u(t, x)$ of problem (51), (52) in the usual manner,

$$\theta(t, \xi) = (T_0 - t)^{1/(\beta-1)} u(t, \xi(T_0 - t)^m),$$

where $T_0 = T_0(u_0) < +\infty$, we obtain in the new time $\tau = -\ln(1 - t/T_0)$ the Cauchy problem for $\theta = \theta(\tau, \xi)$:

$$\theta_\tau=(\theta^\sigma\theta_\xi)_\xi-m\theta_\xi\xi-\frac{1}{\beta-1}\theta+\theta^\beta,\quad \tau>0,\quad \xi\in\mathbf{R}^1, \tag{79}$$

$$\theta(0,\xi)=\theta_0(\xi)\equiv T_0^{1/(\beta-1)}u_0(\xi T_0^m),\quad \xi\in\mathbf{R}^1. \tag{80}$$

To prove stabilization of $\theta(\tau, \xi)$ to the stationary solution $\theta \equiv \theta_A$ of Eq. (79) special estimates of $\theta(\tau, \xi)$ are required. Suppose conditions (56) are satisfied. Then by Lemmas 6 and 7 (Subsec. 3.7) and the estimates of Theorem 13 (Subsec. 3.6) we immediately obtain for all $\tau\geqslant 0$

$$\sup_{\xi\in\mathbf{R}^1}\theta(\tau,\xi)>(\beta-1)^{-1/(\beta-1)},\quad \sup_{\xi\in\mathbf{R}^1}\theta(\tau,\xi)<\theta_*; \tag{81}$$

$$\operatorname{supp}\theta(\tau,\xi)\subset[-\xi_0-(l_0+\xi_0T_0^m)T_0^{-m}\exp(m\tau),\ \xi_0+(l_0+\xi_0T_0^m)T_0^{-m}\exp(m\tau)]\to[-\xi_0,\xi_0],\quad \tau\to+\infty. \tag{82}$$

As in the case $\beta = \sigma + 1$, the last inclusion implies that the Cauchy problem is equivalent to a boundary value problem in a bounded region $\Omega\subset\mathbf{R}^1$ with the condition $\theta = 0$ on $\mathbf{R}_+^1\times\partial\Omega$. By the method of S. N. Bernshtein uniform Hölder continuity of the function $\theta^{\sigma+1}(\tau,\xi)$, in $\mathbf{R}_+^1\times\mathbf{R}^1$ can then be established; therefore, stabilization of $\theta^{\sigma+1}(\tau, \xi)$ as $\tau \to +\infty$ to the unique (see Theorem 5 of Subsec. 3.3.4) stationary solution $\theta = \theta_A(\xi)$ follows from the existence of a Lyapunov function $V(\theta)(\tau)$ with suitable properties. The function $V(\theta)$ is constructured according to the general approach [8] (it cannot be written out in explicit form). Known properties of the two-parameter family of solutions of the stationary equation (78) are used in constructing V. The proof of stabilization uses in an essential way the uniform boundedness of the support of a generalized solution of the Cauchy problem (79), (80).

<u>3.7.3. Stability of the Self-Similar LS-Regime, $\beta > \sigma + 1$.</u> Leaving in place all computations of Subsec. 3.7.2, we characterize the basic problems which arise in the proof of asymptotic stability of the self-similar solution (77) for $\beta > \sigma + 1$.

First of all, $\operatorname{supp}\theta(\tau,\xi)\to\mathbf{R}^1$ as $\tau \to +\infty$, i.e., the possible limit function $\bar\theta(\xi)$ [$\theta(\tau, \xi) \to \bar\theta(\xi)$ as $\tau \to +\infty$ in $\mathbf{R}^1$] is not compactly supported. Therefore, in contrast to S- and HS-regimes, here nothing so far forbids stabilization of $\theta(\tau, \xi)$ as $\tau \to +\infty$ to the spatially homogeneous solution† $\bar\theta\equiv(\beta-1)^{-1/(\beta-1)}$. This difficulty, by the way, can be avoided; in Subsec. 3.8.2 conditions are obtained on $u_0(x)$ under which $u(t,x)\leqslant C_A|x|^{-2/[\beta-(\sigma+1)]}$ in $(0,T_0)\times(\mathbf{R}^1\setminus\{0\})$, which is equivalent to the inequality $\theta(\tau,\xi)\leqslant C_A|\xi|^{-2/[\beta-(\sigma+1)]}$ in $\mathbf{R}_+^1\times(\mathbf{R}^1\setminus\{0\})$ [this obviously forbids the stabilization $\theta(\tau,\cdot)\to(\beta-1)^{-1/(\beta-1)}$ as $\tau \to +\infty$ uniformly on each compact set of $\mathbf{R}^1$]. However, additional complications arise in constructing the Lyapunov function by the method of [8] and in the derivation of the corresponding integral estimates.

Secondly, the question of uniqueness of a self-similar function $\theta_A(\xi)$ of simplest form has so far not been resolved (this is important for the independence of the limit function of the choice of sequence $\tau_i \to +\infty$). Therefore, the question of asymptotic stability of the self-similar LS-regime remains open.

<u>3.8. Asymptotics of Unbounded Solutions of the LS-Regime near a "Singular" Point.</u> Here, for example, an analysis of the multidimensional Cauchy problem is carried out for $\beta > \sigma + 1$ under the assumption that $u_0=u_0(|x|)$ is a bounded function on $\mathbf{R}^N$, and $u_0^{\sigma+1}$ is uniformly Lipschitz continuous in $\mathbf{R}^N$; u_0 is not necessarily compactly supported; $u_0(0) > 0$. It is assumed that $x = 0$ is a point of singularity of the unbounded solution, i.e., there exists a sequence $t_k\to T_0^-(u_0)<+\infty$, such that $u(t_k, 0)\to+\infty$.

The main problem is in deriving estimates of $u(T_0^-, x)$ in a neighborhood of $x = 0$ [the sense in which $u(T_0^-, x)$ is to be understood will be specified in each concrete case]. It will be shown that the limit distribution which the self-similar solution $u_A(T_0^-,x)=C_A\times|x|^{-2/[\beta-(\sigma+1)]}$ forms is characteristic for a broad class of non-self-similar solutions of the LS-regime.

<u>3.8.1. A Lower Bound.</u> This problem can be solved quite precisely on the basis of the method of stationary states [13, 14, 17, 26] as possibilities of a type of "approximation" of evolution properties of solutions on a "field" of stationary solutions continuous with respect to a parameter. A general characterization of this method of investigating various nonlinear evolution parabolic problems is given in [24]; application to parabolic systems of quasilinear equations can be found in [24, 25].

†This occurs, for example, in the case $\sigma = 0$; see Subsec. 4, Sec. 4.

We denote by $\{U(|x|;\ U_0)\}$ the family of symmetric stationary solutions of Eq. (1):

$$\frac{1}{r^{N-1}}(r^{N-1}U^\sigma U_r')_r' + U^\beta = 0,\quad r=|x|>0;$$
$$U_r'(0;\ U_0)=0,\quad U(0;\ U_0)=U_0, \tag{83}$$

where $U_0 > 0$ is the parameter of the family. The basic properties of the functions {U} are indicated in Lemma 4 (Subsec. 3.3.3). In particular, if $\beta \in (1, (\sigma+1)(N+2)/(N-2)_+)$, then each of them vanishes at the point $r_0(U_0) \equiv r_0(1)\, U_0^{[\sigma+1-\beta]/2} < +\infty$. We also present the identity

$$U(|x|;\ U_0) \equiv U_0 U(|x|\, U_0^{[\beta-(\sigma+1)]/2};\ 1). \tag{84}$$

We extend $U(|x|;\ U_0)$ to the region $r > r_0(U_0)$ by zero. Since $U(|x|;\ U_0)$ are not stationary solutions in $\mathbf{R}^N$, it will be convenient to call them stationary states. For $\beta \geqslant (\sigma+1)(N+2)/(N-2)_+$ the functions $U(|x|; U_0)>0$ are stationary solutions in $\mathbf{R}^N$.

The family of functions $\{U|x|;\ U_0)\}$ forms in the plane $\{|x|, U\}$ a "field" of stationary states which is continuous with respect to the parameter $U_0 > 0$. For $\beta > \sigma + 1$ a general characteristic of the family {U} is the envelope $L = L(r)$, $r > 0$, to the functions $U(r;\ U_0)$ on the {r, U} plane which is defined from the following condition: for any $r > 0$ there exists $U_0 > 0$ such that

$$L(r)=U(r;\ U_0),\quad L'(r)=U_r'(r;\ U_0). \tag{85}$$

As will be shown below, the form of the envelope $L(|x|)$ makes it possible to estimate the structure of the limit distribution $u(T_0^-, x)$ near the singular point $x = 0$ (see [13, 17, 26]).

<u>LEMMA 9.</u> Let $\beta > \sigma + 1$. Then problem (85) has a unique solution

$$L(|x|)=C_0|x|^{-2/[\beta-(\sigma+1)]},\quad |x|>0,$$

where the constant $C_0 = C_0(\sigma, \beta, N) > 0$ is defined from the system of transcendental equations

$$C_0=\alpha_0^{2/[\beta-(\sigma+1)]}U(\alpha_0;\ 1),$$
$$-\frac{2}{[\beta-(\sigma+1)]}C_0=\alpha_0^{1+2/[\beta-(\sigma+1)]}U_r'(\alpha_0;\ 1),\quad \alpha_0>0. \tag{86}$$

Here we have the estimate

$$C_0>C_*=\left\{\frac{2N}{\beta-(\sigma+1)}\left[\frac{\beta-(\sigma+1)}{\beta}\right]^{\frac{\beta}{\sigma+1}}\right\}^{\frac{1}{\beta-(\sigma+1)}}>0. \tag{87}$$

The first part of the lemma can be proved directly with use of the identity (84). We note only that tangency of $L(|x|)$ and $U(|x|;\ U_0)$ in accordance with equalities (85) is realized for $U_0=(\alpha_0/r)^{2/[\beta-(\sigma+1)]}$, where $\alpha_0 > 0$ is the constant in (86). The existence and uniqueness of a solution of the system (86) follows from known properties of the function $U(|x|;\ 1)$ (it can be expressed in terms of a special function of mathematical physics). We consider in more detail the derivation of (87) (see [17, 26]). Using the estimate $r^{1-N}(r^{N-1}\times U^\sigma U')'=-U^\beta \geqslant -U_0^\beta$, we find without difficulty that

$$U(|x|;\ U_0)>U_0\left(1-r^2\frac{\sigma+1}{2N}U_0^{\beta-(\sigma+1)}\right)_+^{1/(\sigma+1)} \equiv U_-(|x|;\ U_0),$$
$$0<r<(2N/(\sigma+1))^{1/2}U_0^{(\sigma+1-\beta)/2}. \tag{84'}$$

The family of functions $\{U_-\}$ is very simple, and from the same considerations it is easy to construct an envelope of the following form: $L_-(|x|)=C_*|x|^{-2/[\beta-(\sigma+1)]}$. However, by (84') in $L_*(|x|)>L_-(|x|)$ in $\mathbf{R}^N\setminus\{0\}$, which gives (87).

<u>THEOREM 16.</u> Suppose $\sigma+1<\beta<(\sigma+1)(N+2)/(N-2)_+$. Then under the assumptions made regarding $u_0 = u_0(|x|)$ there exists $\varepsilon > 0$ such that the unbounded solution of problem (1), (2) satisfies the lower bound

$$u(T_0^-,\ x)\equiv\varliminf_{t\to T_0^-}u(t,\ x)\geqslant C_0|x|^{-2/[\beta-(\sigma+1)]},\quad x\in\{0<|x|<\varepsilon\}. \tag{88}$$

<u>Proof.</u> We choose $U_0^*>u_0(0)$ so large that, first of all, $\{|x|\leqslant r_0(U_0)\}\subset \operatorname{supp} u_0$ for any $U_0\geqslant U_0^*$ and, secondly, $U(|x|;\ U_0)$ intersects $u_0(|x|)$ in $|x|$ at exactly two points for all $U_0\geqslant U_0^*$. We set $\varepsilon=r_0(U_0^*)$. Then obviously $u(t,\ x)>0=U(|x|;\ U_0)$ for $|x|=r_0(U_0)$, if $U_0\geqslant U_0^*$. In

accordance with the comparison theorem the number of intersections N(t) of the distinct solutions $U(|x|; U_0)$ and u(t, x) of Eq. (1) in $\{r=|x|<r_0(U_0)\}$ is such that $N(t)\leqslant 1$ for all $t\in(0, T_0)$. Since x = 0 is a singular point, there exists $t_k\in(0, T_0)$, such that $u(t_k, 0)>U_0$. But then by symmetry $N(t_k) = 0$, and hence $u(t_k, x)\geqslant U(|x|; U_0)$ in $\mathbf{R}^N$. By the maximum principle this will be true also for all $t\in(t_k, T_0)$. Considering now that $U_0\geqslant U_0^*$ is chosen arbitrarily, we obtain

$$u(T_0^-, x)\geqslant \sup_{U_0\geqslant U_0^*} U(|x|; U_0)\equiv L(|x|), \quad 0<|x|<\varepsilon, \tag{89}$$

which by Lemma 9 gives (88).

We note a curious corollary of the method of proof of the theorem.

COROLLARY 1. Under the conditions of Theorem 16 there exists $t_0\in[0, T_0)$, such that $u_t(t, 0) \geq 0$ for all $t_0 \in [t_0, T_0]$.

Thus, the definition of a singular point x = 0 given at the beginning of the section in the symmetric case is equivalent to the condition $u(t, 0) \to +\infty$ as $t \to T_0^-$. We note that for N = 1 this is valid for arbitrary solutions. The proof in the general case requires application of a special comparison theorems based on an analysis of the character of spatial intersections of distinct solutions, and is not considered here (in this regard, see [27, 28] and Sec. 4).

COROLLARY 2. Under the conditions of Theorem 16 for any $p\geqslant[\beta-(\sigma+1)]N/2$

$$\lim_{t\to T_0^-} \int_{\{|x|<\varepsilon\}} u^p(t, x)\, dx=+\infty. \tag{90}$$

The validity of (90) follows directly from (88). A similar assertion was known earlier for the case $\sigma = 0$; see [86, 103, 128] where (9) was not proved for the "critical" value $p = [\beta - 1]N/2$ when the integral has a weak logarithmic divergence at the point x = 0. The proof in [86, 128] used the semilinear structure of the equation for $\sigma = 0$. From the position of the method of stationary states for the derivation of integral and pointwise estimates of $u(T_0^-, x)$ the type of nonlinearity in the differential operator of the equation makes no difference; examples are presented in [13, 17, 24].

If the method of stationary states is applied to investigate the HS-regime, then the following result is obtained [19, 13]. Suppose $1<\beta<\sigma+1$, $u_0=u_0(|x|)\geqslant 0$ in $\mathbf{R}^N$, and $u_0^{\sigma+1}$ is a uniformly bounded, nonincreasing, Lipschitz continuous function. Then $u(t, x) \to +\infty$ as $t\to T_0^-(u_0)<+\infty$ everywhere in $\mathbf{R}^N$. In the proof the fact is used that for $\beta<\sigma+1$ $U(|x|; U_0)\to+\infty$ for $U_0\to+\infty$ everywhere in $\mathbf{R}^N$. Therefore, the validity of the assertion formulated follows from an estimate similar to (89)

$$u(T_0^-, x)\geqslant \sup_{U_0>U_0^*} U(|x|; U_0)=+\infty, \quad x\in\mathbf{R}^N.$$

3.8.2. An Upper Bound. We shall show that under particular conditions the lower bound (88) is sharp with respect to the character of the dependence on |x| near x = 0. One example is already known: the self-similar solution (20') which for $\sigma+1<\beta\leqslant(\sigma+1)N/(N-2)_+$, as shown in Theorem 7, is bounded above: $u_A(t, x)<u(T_0^-, x)\equiv C_A|x|^{-2/[\beta-(\sigma+1)]}$ in $\mathbf{R}^N\setminus\{0\}$. With the help of this result a similar upper bound is obtained below for a broad class of $u_0(x)$ satisfying all the conditions introduced earlier.

THEOREM 17. Suppose $\sigma + 1 < \beta \leqslant(\sigma+1)N/(N-2)_+$, $T_0=T_0(u_0)<+\infty$ and $u_0=u_0(|x|)$ intersects with respect to r = |x| the function $T_0^{-1/(\beta-1)}\theta_A(|x|/T_0^m)$ (θ_A is defined in Theorem 6) at precisely two points, while $u_0(0)>T_0^{-1/(\beta-1)}\theta_A(0)$. Suppose further that u_0 is critical, i.e., $u_t\geqslant 0$ in P[u]. If x = 0 is a singular point of the solution u(t, x), then

$$u(t, x)<C_A|x|^{-2/[\beta-(\sigma+1)]} \text{ в } (0, T_0)\times(\mathbf{R}^N\setminus\{0\}). \tag{91}$$

Before proceeding to the proof of the theorem we shall show that the conditions on u_0 in the hypotheses are not too restrictive. As concerns $u_0(x)$ being critical, this question is resolved very simply. If, for example, $u_0\in C^2(\mathbf{R}^N)$, $u_0>0$ and

$$\mathbf{A}(u_0)\equiv \nabla(u_0^\sigma\nabla u_0)+u_0^\beta\geqslant 0 \text{ in } \mathbf{R}^N, \tag{92}$$

then by the maximum principle $u_t\geqslant 0$ in $(0, T_0)\times\mathbf{R}^N$, since the function $z = u_t$ satisfies a "linear" parabolic equation. Condition (92) is satisfied by, for example, $u_0(x)=A\exp(-\alpha|x|^2)$, if $A^{\beta-(\sigma+1)}\geqslant 2\alpha\exp\{[\beta-(\sigma+1)]/2(\sigma+1)\}$, $\alpha > 0$. Another example of a critical function is $u_0(x) = U(|x|; U_0)$ for any $U_0 > 0$ (see [14, 17, 26]).

The condition of the presence of precisely two intersections of $u_0(x)$ and the self-similar function $u_A(0, x)$, where u_A has the same time of peaking, is somewhat more difficult to verify. Without considering this in detail, we mention that the compactly supported function $u_0=U(|x|; U_0)$, satisfies all the conditions of the theorem; see [17]. We remark also that the condition that u_0 be critical for N = 1, generally speaking, is superfluous; regarding this, see Subsec. 1, Sec. 4.

Proof of Theorem 17. We suppose that (91) does not hold and there exist $r_* > 0$ and $t_* \in (0, T_0)$, such that $u(t_*, r_*) \geq C_A r_*^{-2/[\beta-(\sigma+1)]}$. Then by the critical property this is satisfied in $(t_*, T_0)\times\{|x|=r_*\}$. In the region $S_*=(t_*, T_0)\{|x|<r_*\}$ we consider two solutions u and u_A with the same time of peaking $t=T_0<+\infty$. Under the assumptions made $N(t) \equiv 0$ in $[t_*, T_0)$, while since on $(t_*, T_0)\times\{|x|=r_*\}$ – the lateral boundary of S_* – there is the estimate

$$\inf_{t\in(t_*,T_0)} u(t, x) \geq \sup_{t\in(t_*,T_0)} u_A(t, x), \quad |x|=r_*,$$

it is not hard to show that there exists a $\tau > 0$ so small that $u_A(t+\tau, x) \leq u(t, x)$ in $(t_*, T_0-\tau)\times\{|x|<r_*\}$. This contradicts the equality of the times of peaking of the solutions u and u_A.

COROLLARY. Under the conditions of Theorem 17 for any $p\in(0, [\beta-(\sigma+1)]N/2)$ and $\varepsilon > 0$

$$\int_{\{|x|<\varepsilon\}} u^p(t, x)\,dx < C_A^p \int_{\{|x|<\varepsilon\}} |x|^{-2p/[\beta-(\sigma+1)]}\,dx < +\infty, \quad t\in(0, T_0).$$

4. Semilinear Parabolic Equation $u_t = u_{xx} + u^\beta$

In this section we present some results of investigating the Cauchy problem for the one-dimensional semilinear parabolic equation

$$u_t=u_{xx}+u^\beta, \quad t>0, \quad x\in\mathbf{R}^1; \quad \beta=\text{const}>1, \tag{1}$$

$$u(0, x)=u_0(x)\geq 0, \quad x\in\mathbf{R}^1. \tag{2}$$

Many of the results presented below can be carried over without special changes to the case of the multidimensional equation $u_t = \Delta u + u^\beta$.

This problem has much in common with its quasilinear analogue; therefore, our main attention below will be given to those properties of unbounded solutions which are not present in the quasilinear case. Conditions for the occurrence of unbounded solutions of the Cauchy problem (1), (2) have been studied for more than 20 years; see, for example, [93, 97, 108, 112, 113] and the bibliography in the survey [21]. Without considering this in detail, we mention only that for $1<\beta\leq 3$ any solutions $u \not\equiv 0$ is unbounded, while for $\beta > 3$ there exists a class of global solutions; see [63, 97].

The following conditions are imposed on the initial function u_0:

$$\sup u_0=M_1<+\infty; \quad |u_0(x_1)-u_0(x_2)|\leq M_2|x_1-x_2|, \quad x_1, x_2\in\mathbf{R}^1, \tag{3}$$

where M_1, M_2 are positive constants.

Everywhere below we assume that $T_0=T_0(u_0)<+\infty$.

4.1. The Critical Set. The theorem formulated below simplifies the exposition of subsequent results.

THEOREM 1. Under the assumptions made there exists a constant $M_K=M_K(M_1, M_2)>0$, such that if $u(t_0, x_0)\geq M_K$, then $u_t(t, x_0)\geq 0$ for all $t\in[t_0, T_0)$.

In particular this means that the conditions – there exists a sequence $t_k\to T_0^-<+\infty$, such that $u(t_k, x_0)\to+\infty$ as $k \to +\infty$ (the condition that $x = x_0$ is a point of singularity of the unbounded solution) and $u(t, x_0)\to+\infty$ as $t \to T_0^-$ – are equivalent.

The proof of Theorem 1 [27] uses a rather fine theorem of comparison of u(t, x) with a family of stationary solutions $\{v(x)\equiv U(x-a; U_0), U_0\in\mathbf{R}_+^1, a\in\mathbf{R}^1\}$, based on an analysis of the character of intersections of the functions u and v with respect to x. An approximate formulation of this comparison theorem is as follows: if N(0) = 2 for any small "C^1-deformations" of $u_0(x)$ and v(x), then at any point of intersection $x=x_0$, $w(t, x_0)\equiv u(t, x_0) - v(x_0) = 0$ the condition $w_x(t, x_0)\neq 0$ holds. In this connection we note that solutions of Eq. (1) are analytic in x [92]; therefore all intersections for t > 0 are isolated points (with respect to x).

4.2. An Upper Bound for the Amplitude of an Unbounded Solution. Several approaches are known to the derivation of an important upper bound for the amplitude of an unbounded solution; see [15, 16, 26-28, 96, 129, 130]. The following result is the most general [27, 28] (see also [26]).

THEOREM 2. Suppose (3) is satisfied and $u_0(x) \to 0$ as $|x| \to +\infty$. Then there exists a constant $\mu_* = \mu_*(T_0, M_1, M_2) > 0$ such that

$$u(t, x) < \mu_* (T_0 - t)^{-1/(\beta-1)} \text{ in } [0, T_0) \times \mathbf{R}^1. \tag{4}$$

The proof is based on a comparison (an analysis of the character of intersections) of u(t, x) with a family constructed in [26, 27] of unbounded, lower self-similar solutions.

4.3. A Lower Bound for the Amplitude. The next result follows immediately from Theorem 8 in Sec. 3; see assertion (II). As a comparison function v the spatially homogeneous solution $v \equiv (\beta-1)^{-1/(\beta-1)}(T_0-t)^{-1/(\beta-1)}$ is taken.

THEOREM 3. Under the conditions of Theorem 2

$$\sup_{x \in \mathbf{R}^1} u(t, x) > (\beta-1)^{-1/(\beta-1)} (T_0 - t)^{-1/(\beta-1)} \text{ in } [0, T_0). \tag{5}$$

4.4. Asymptotic Behavior of Unbounded Solutions. The asymptotics of u(t, x) for $t \to T_0^- < +\infty$ in the Cauchy problem for a semilinear equation is considerably different from what would be expected in the quasilinear case.

THEOREM 4. Suppose (3) is satisfied, and, moreover, $u_0(-x) = u_0(x)$ in $\mathbf{R}^1$, and $u_0(x)$ is nonincreasing in x in $\mathbf{R}_+^1$. Then

$$\theta(t, \xi) \equiv (T_0 - t)^{1/(\beta-1)} u(t, \xi (T_0 - t)^{1/2}) \to (\beta-1)^{-1/(\beta-1)} \text{ as } t \to T_0^- \tag{6}$$

uniformly on each compact set in ξ of $\mathbf{R}^1$.

The proof [27, 28] proceeds basically by the same scheme as that used in Subsec. 3.7, Sec. 3. The equation for the self-similar representation $\theta = \theta(\tau, \xi)$, $\tau = -\ln(1 - t/T_0)$ has the form

$$\theta_\tau = A(\theta) \equiv \theta_{\xi\xi} - \frac{1}{2} \xi \theta_\xi - \frac{1}{\beta-1} \theta + \theta^\beta, \quad \tau > 0, \ \xi \in \mathbf{R}^1. \tag{7}$$

Under the assumptions made x = 0 is a point of singularity of the solution u(t, x), and $\sup_{x \in \mathbf{R}^1} u(t, x) \equiv u(t, 0)$. By Theorem 2 the solution θ of the Cauchy problem for (7) is then uniformly bounded, $\theta(\tau, \xi) < \mu_*$ in $\mathbf{R}_+^1 \times \mathbf{R}^1$, and uniform boundedness of θ_ξ as well as boundedness of θ_τ on each set of the form $[1, +\infty) \times \{|\xi| < l\}$ can be established by the method of S. N. Bernshtein. Theorem 3 then forbids stabilization to the trivial solution of the stationary equation (7) $\theta \equiv 0$, since (5) implies that $\theta(\tau, 0) > \theta_H = (\beta-1)^{-1/(\beta-1)}$ in $\mathbf{R}_+^1$. The following result is of major significance for the proof of (6).

LEMMA 1. The function $\theta \equiv \theta_H$ is the unique nontrivial, nonnegative solution of the equation $A(\theta) = 0$ in $\mathbf{R}^1$.

For β = 3 the lemma is proved in [104]; for arbitrary β > 1 it is proved in [3].

To complete the proof of Theorem 4 it now suffices to note that (7) admits the Lyapunov function

$$V(\theta)(\tau) = \int_{-\infty}^{+\infty} \exp(-\xi^2/4) \left(\frac{1}{2} \theta_\xi^2 + \frac{1}{2(\beta-1)} \theta^2 - \frac{1}{\beta+1} \theta^{\beta+1} \right) d\xi,$$

which is monotone on the evolution trajectories:

$$\frac{d}{d\tau} V(\theta)(\tau) = - \int_{-\infty}^{+\infty} \exp(-\xi^2/4) \theta_\tau^2 d\xi \leqslant 0.$$

From the last equality we obtain the uniform estimate ($C_1 > 0$ does not depend on T > 1)

$$\int_1^T \int_{-\infty}^{+\infty} \exp(-\xi^2/4) \theta_\tau^2 d\xi d\tau \leqslant C_1.$$

Remark 1. As shown in the work [99], the assertions formulated in Theorem 4 and Lemma 1 are characteristic also for the multidimensional problem

$$u_t = \Delta u + u^\beta,\ t>0,\ x\in \mathbf{R}^N. \tag{8}$$

Here there arises the problem of investigating the solvability of the elliptic equation [the analogue of the one-dimensional equation $\mathbf{A}(\theta)=0$, see (7)]

$$\Delta_\xi \theta - \frac{1}{2}\sum_{i=1}^{N}\frac{\partial\theta}{\partial\xi_i}\xi_i - \frac{1}{\beta-1}\theta + \theta^\beta = 0 \text{ in } \mathbf{R}^N.$$

In [99] it is established that $\theta \equiv \theta_H$ is the unique, nontrivial solution for $1<\beta\leqslant(N+2)/(N-2)_+$. The basis for the proof of this interesting fact is the derivation of a special energy identity on the basis of the technique of [69, 70]. For $\beta>(N+2)/(N-2)_+$ there may exist nontrivial solutions $\theta(\xi)\to 0$, $|\xi|\to+\infty$; an example of such a solution was constructed in [22] for the case $\beta = 2$ and $6 < N < 16$ [the lower bound of the dimension of the space $N > 6$ is connected with the inequality $\beta>(N+2)/(N-2)=2$ for $N = 6$]. We note that in the presence of the estimates (4) and (5) Theorem 4 follows from a result of [99] where methods close to [26, 28] were used for the proof of stabilization.

Remark 2. Returning from the self-similar representation $\theta(\tau, \xi)$ to $u(t, x)$, we find that Theorem 4 gives an idea of the asymptotic behavior of the unbounded solution $u(t, x)$ on any compact sets of the form $P_a(t)=\{|x|\leqslant a(T_0-t)^{1/2}\}$: $u(t,x)\cong\theta_H(T_0-t)^{-1/(\beta-1)}$ in $P_a(t)$. Of course, this does not enable us to establish the space-time structure of $u(t, x)$ in a neighborhood of the point $t = T_0^-$, $x = 0$, since it determines only the amplitude of the solution and not its spatial width. In [26] (earlier for the case $\beta = 3$ in [104]) it was shown that the behavior of $u(t, x)$ as $t \to T_0^-$ in regions $\{|x|<a(T_0-t)^{1/2}|\ln(T_0-t)|^{1/2}\}$ of larger dimension is described by the degenerate approximate self-similar solution with a nontrivial spatial structure

$$u_a(t,x)=(T_0-t)^{-1(\beta-1)}f_*(\eta),\quad \eta=x/(T_0-t)^{1/2}|\ln(T_0-t)|^{1/2}. \tag{9}$$

The function u_a satisfies the equation without diffusion $u_t = u^\beta$; substitution of (9) into this equation after passing to the limit $t \to T_0^-$ leads to the following ordinary differential equation of first order for $f = f_*(\eta)$:

$$-\frac{1}{2}\eta f'_\eta - \frac{1}{\beta-1}f + f^\beta = 0,\quad \eta\in\mathrm{R}^1.$$

It has an entire family of solutions $f(\eta)=\{(\beta-1)+C\eta^2\}^{-1/(\beta-1)}$, $C > 0$. The value

$$C=C_*=(\beta-1)^2/4\beta, \tag{10}$$

to which there corresponds the function $f_*(\eta)$ in (9) is selected from the quite natural condition of analyticity of $\theta(\tau, \xi)$ at the point $\tau = +\infty$, $\eta = 0$.

The question of rigorous justification of the asymptotics (9) remains open. A numerical verification was carried out in [26, 104]. In Subsec. 4.6 a result is presented which indirectly argues in favor of the asymptotics (9).

4.5. Localization of Unbounded Solutions of the Cauchy Problem. In this subsection conditions are obtained for the effective localization of unbounded solutions, i.e., for boundedness of the set (the region of localization)

$$\omega_L=\{x\in\mathrm{R}^1\,|\,u(T_0^-,x)\equiv\overline{\lim_{t\to T_0^-}}\,u(t,x)=+\infty\}. \tag{11}$$

THEOREM 5. Suppose conditions (3) are satisfied and, moreover, there exists a constant $d > 0$ such that

$$u_0(x)\leqslant d|x|^{-2/(\beta-1)} \text{ for sufficiently large } |x|; \tag{12}$$

$$u_0(x) \text{ decreases monotonically in } |x| \text{ as } |x|\to+\infty. \tag{13}$$

Then the region of localization (11) is bounded.

The proof (see [27, 28] and also [26]) is based on a comparison of $u(t, x)$ with a special lower self-similar solution of Eq. (1) of the usual form

$$u_A(t,x)=(T_0-t)^{-1/(\beta-1)}\theta(\xi),\quad \xi=x/(T_0-t)^{1/2},$$

where the function $\theta\geqslant 0$ satisfies the equation $A(\theta)=0$, A is the operator of (7). It has no nontrivial solution $\theta \not\equiv \theta_H$ in $\mathbf{R}^1$ (Lemma 1) but admits special lower solutions exhibiting the property of localization.

LEMMA 2 [26]. For any d > 0 there exists C > d such that the equation $\boldsymbol{A}(\theta)=0$ has in $(-\xi_0, +\infty)$, $\xi_0>0$ a solution $\theta=\theta(\xi)>0$, whereby $\theta(-\xi_0)=0$, $\theta'(-\xi_0)>0$, and

$$\theta(\xi)=C\xi^{-2/(\beta-1)}+o(\xi^{-2/(\beta-1)}),\quad \xi\to+\infty. \tag{14}$$

The next lemma gives simple properties of the self-similar solution

$$v(t, x; x_0)\equiv u_A(t, x-x_0)\equiv(T_0-t)^{-1/(\beta-1)}\theta((x-x_0)/(T_0-t)^{1/2}), \tag{15}$$

which is strictly positive in the region $(0, T_0)\times(\eta_1(t), +\infty)\equiv(0, T_0)\times\Omega(t; x_0)$, $\eta_1(t)=x_0-\xi_0(T_0-t)^{1/2}$, while v = 0 on $(0, T_0)\times\partial\Omega$.

LEMMA 3 [26]. For any fixed $t_0\in[0, T_0)$

$$v(t_0, x; x_0)/x^{-2/(\beta-1)}\to C,\quad x\to+\infty; \tag{16}$$

for any fixed $x > x_0$ there exists the limit

$$v(t, x; x_0)\to C(x-x_0)^{-2/(\beta-1)},\quad t\to T_0^-; \tag{17}$$

everywhere in $(0, T_0)\times\{x>x_0\}$ the solution v is critical ($v_t > 0$), and by (17)

$$v(t, x; x_0)<v(T_0^-, x; x_0)\equiv C|x-x_0|^{-2/(\beta-1)}. \tag{18}$$

The validity of Theorem 5 follows from an assertion in which, to be specific, boundedness of ω_L on the right is established. We first recall that by Theorem 1 the solution of the Cauchy problem is critical in the region $\{x\in\mathbf{R}^1, u>M_K\}$.

LEMMA 4. Under the conditions of Theorem 5 there exists a constant C > d and $x_0\in\mathbf{R}^1$, such that

$$u(t, x)\leqslant\max\{C(x-x_0)^{-2/(\beta-1)}, M_K\},\quad t\in(0, T_0),\quad x>x_0, \tag{19}$$

i.e., $\omega_L\cap\{x>x_0\}=\varnothing$.

Proof. By Lemmas 2 and 3 in the family of self-similar solutions (15) there is a function v [$T_0=T_0(u_0)<+\infty$ and C > d in (14)] such that $u_0(x)$ and $v(0; x; x_0)$ intersect in $\Omega(0; x_0)$ at exactly one point, i.e., N(0) = 1. This is ensured by conditions (12), (13). Then by the comparison theorem (Subsec. 3.4, Sec. 3) $N(t)\leqslant1$ for $t\in(0, T_0)$. Then it is not hard to verify that violation of (19) at even one point $(t_*, x_*)\in(0, T_0)$ $\{x>x_0\}$ implies that (19) is not satisfied everywhere in $(t_*, T_0)\times\{x=x_*\}$, which contradicts the equality of the times of peaking of the unbounded solutions u and v considered in the region $(t_*, T_0)\times\{\eta_1(t)<x<x_*\}$ (a detailed proof is given in [27]; an analogous technique was used in [26, 28]).

4.6. Behavior of Unbounded Solutions near a Singular Point. The principal aim of this subsection is to show that under particular conditions unbounded growth of u(t, x) as $t\to T_0^-(u_0)<+\infty$ occurs at only one point (i.e., an LS-regime with peaking develops) and to establish estimates of the limit distribution in a neighborhood of the singular point from above and below.

4.6.1. A Lower Bound. It can be obtained by the same method as in the quasilinear case (Subsec. 3.8, Sec. 3). However, since each solution has a critical set (Theorem 1) it is possible to obtain a stronger result in the present csae.

THEOREM 6. Suppose (3) is satisfied and x = 0 is a point of singularity of the solution of the problem. Then there exists $\varepsilon_0 > 0$ such that

$$\max\{u(T_0^-, x), u(T_0^-, -x)\}\geqslant C_0|x|^{-2/(\beta-1)},\quad x\in(-\varepsilon_0, \varepsilon_0)\setminus\{0\}, \tag{20}$$

where $C_0=C_0(\beta)>0$ is a constant, and $C_0>C_*=(\beta-1)(2\beta^{-\beta})^{1/(\beta-1)}$.

As in the quasilinear case, the proof is based on comparing u(t, x) with a family $\{U(|x|; U_0)\}$ of stationary solutions of Eq. (1). The envelope $L=L(|x|)>0$ in $\mathbf{R}^1\setminus\{0\}$ of the family {U} is determined as in Subsec. 3.8.1, Sec. 3 and is indicated on the right side of inequality (20).

We choose the quantity $U_0^*>0$ so large that $U_0^*>M_1=\sup u_0$ and, moreover, $U(|x|; U_0)$ for all $U_0>U_0^*$ intersects $u_0(x)$ at exactly two points. We set $\varepsilon_0=\max\{\varepsilon>0\,|\,L(\varepsilon)\geqslant U_0^*, L(\varepsilon)\geqslant M_K\}$. Then for any $x_*\in(-\varepsilon_0, \varepsilon_0)\setminus\{0\}$ we have the following: the function $U(|x|; U_0)$, where $U_0 > 0$ is determined from the condition $L(|x_*|)=U(|x_*|; U_0)$ [i.e., the envelope L = L(|x|) is tangent to $U(|x|; U_0)$ at the points $x=\pm|x_*|$], is such that the number of intersections of $u_0(x)$ and $U(|x|; U_0)$ is N(0) = 2. Then by the comparison theorem $N(t)\leqslant2$ for all $t\in[0, T_0)$. Because of the continuity of $u(t, 0)$ in $t \in [0, T_0)$ there exists $t_0 \in (0, T_0)$, such that $u(t_0, 0) = U_0 \equiv U(0; U_0)$. Then, since $N(t_0) \leqslant 2$, all intersections of $u_0(x)$ and $U(|x|; U_0)$ will lie either in $x \geqslant 0$, or in $x \leqslant 0$, and hence $u(t_0, x) \geqslant U(|x|; U_0)$ either for $x < 0$ or for $x > 0$. In any case we obtain $\max\{u(t_0, x_*),\ u(t_0, -x_*)\} \geqslant L(|x_*|) \equiv C_0|x_*|^{-2/(\beta-1)}$. Suppose to be specific that $u(t_0, x_*) \geqslant L(|x_*|)$. Now $L(|x_*|) \geqslant M_K$ by construction. Hence, $u_t(t, x_*) \geqslant 0$ for all $t \in [t_0, T_0)$ (Theorem 1), i.e., $u(t, x_*) \geqslant L(|x_*|)$ for all $t \in [t_0, T_0)$, which proves (20).

A somewhat different version of the proof of Theorem 6 is presented in [27]. From (20) we obtain the

<u>COROLLARY.</u> Under the conditions of Theorem 6 for any $p \geqslant (\beta-1)/2$ and $\varepsilon > 0$

$$\int_{-\varepsilon}^{\varepsilon} u^p(t, x)\,dx \to +\infty, \quad t \to T_0^-. \tag{21}$$

A similar result was obtained earlier for the interval of values $p > \max\{1,\ (\beta-1)/2\}$ (see [86, 128]) not containing the "critical" value $p = (\beta - 1)/2$. We note that it is possible to prove the validity of an inequality of the type (21) for $\varepsilon = +\infty$, $p \geqslant (\beta-1)/2$ without special separation of a concrete point of singularity [27].

<u>4.6.2. An Upper Bound.</u> The main problem of this subsection is to determine the degree of optimality of the lower bound (20) with respect to the character of the dependence of $u(T_0^-, x)$ on $|x|$. In contrast to the quasilinear case, for $\sigma = 0$ Eq. (1) has no localized self-similar solutions (Lemma 1). Therefore, an upper bound for $u(T_0^-, x)$ cannot be derived by the same method used in Subsec. 3.8.2, Sec. 3. For this purpose the approach proposed in [96] turned out to be effective. An upper bound for $u(T_0^-, x)$ which is apparently optimal (within the framework of this method) will be obtained below. For simplicity we carry out the investigation of the boundary value problem for (1) in the region $(0, T_0) \times (-R, R)$, $R =$ const > 0 with the conditions

$$u(t, -R) = u(t, R) = 0,\ t > 0;\ u(0, x) = u_0(x) \geqslant 0,\ x \in (-R, R) \tag{22}$$

(a problem of this type was considered in [96] where the investigation was carried out for the multidimensional Eq. (8)]. It is assumed that $u_0 \in C^1([-R, R])$ and $T_0 = T_0(u_0) < +\infty$.

The idea of [96] is to derive an estimate of the function

$$w(t, x) = u_x + c(x)F(u),$$

where $c(x) \geqslant 0$, $F(u) \geqslant 0$ are some smooth functions which remain to be determined. The function w satisfies a "linear" (in w) parabolic equation, and by the maximum principle it is then concluded that $w \leqslant 0$ in $(0, T_0)\,(-R, R)$, if $w(0, x) \leqslant 0$, $w_x(t, \pm R) \lesseqgtr 0$ and, moreover, in application to Eq. (1)

$$\beta u^{\beta-1}F - u^\beta F'_u - 2c'_x F'_u F \geqslant 0, \quad F'' \geqslant 0, \quad c'' \geqslant 0 \tag{23}$$

for all $u \geqslant 0$, $x \in [-R, R]$. It is not difficult to verify that integration of the equality $w \equiv u_x + c(x)F(u) \leqslant 0$ for a suitable choice of the functions c and F makes it possible to obtain an upper bound for $u(T_0^-, x)$.

In [96] for this purpose $c = \varepsilon|x|^{1+\delta}$, $F(u) = u^\gamma$, were chosen where $1 < \gamma < \beta$, $\delta > 0$ is arbitrary, and $\varepsilon > 0$ is sufficiently small. It is easy to verify that (23) is then satisfied. Without going into the procedure of choosing suitable initial functions $u_0(x)$ and $\varepsilon > 0$ [in order that $w(0, x) \leqslant 0$ and $w_x(t, \pm R) \lesseqgtr 0$], we note only that integration of the inequality $u_x + \varepsilon|x|^{1+\delta} \times u^\gamma \leqslant 0$ over $(0, x)$ leads to the estimate

$$u(t, x) \leqslant \varkappa|x|^{-2/(\nu-1)}, \quad 1 < \nu < \beta, \tag{24}$$

where the quantity ν can, generally speaking, be made arbitrarily close to the value β^-; $\varkappa > 0$ in (24) depends on ν, δ, ε and $u_0(x)$. With regard to the character of the dependence on $|x|$ the upper bound $u(T_0^-, x)$ (24) and lower bound (20) are quite close. As will now be shown, the "gap" between them can be made still smaller.

For this we first consider in place of (22) the boundary value problems with the conditions

$$u(t, \pm R) = A,\ t > 0;\ u(0, x) = u_0{}^*(x) \geqslant A,\ x \in (-R, R), \tag{25}$$

where $u_0{}^*(x) = A(2 - x^2/R^2)$, and $A > 0$ is sufficiently large. The last ensures the unboundedness of the solution of the problem, and $u(t, x) \geqslant A$ in $(0, T_0{}^*) \times (-R, R)$.

THEOREM 7. For any R > 0 there exists A > 0 such that the solution of problem (25) is unbounded, $T_0^* = T_0(u_0^*) < +\infty$, and there is the following asymptotically sharp estimate:

$$u(T_0^-, x) \leqslant \left[\frac{(\beta-1)^2}{8\beta}\right]^{-1/(\beta-1)} |x|^{-2/(\beta-1)} |\ln|x||^{1/(\beta-1)}, \ |x| \to 0. \tag{26}$$

For the proof in [28] a solution of the system of inequality (23) was found which is sharp with regard to the dependence of F(u) on u for $u \to +\infty$. It has the form $c(x) = |x|/2\beta$, $F(u) = u^\beta / \ln u$ ($F''(u) \geqslant 0$ for $u \geqslant \exp[2/(\beta-1)]$). Integrating the inequality $u_x + xu^\beta/2\beta \ln u \leqslant 0$ over (0, x), $x \in (0, R)$, and considering that by Theorem 4 $u(t, 0) = (\beta-1)^{-1/(\beta-1)}(T_0 - t)^{-1/(\beta-1)} + o((T_0 - t)^{-1/(\beta-1)})$ (because of the estimates (4) and (5) it is then also valid for the boundary value problem (25); see also the general assertion in [99]), we then obtain the following estimate which is asymptotically sharp for $t \to T_0^-$, $x \to 0$:

$$u(t, x) \leqslant [(T_0 - t)|\ln(T_0 - t)||]^{-1/(\beta-1)} \times$$
$$\left\{(\beta-1) + \frac{(\beta-1)^2}{4\beta}\left[\frac{|x|}{(T_0-t)^{1/2}|\ln(T_0-t)|^{1/2}}\right]^2\right\}^{-1/(\beta-1)} \left\{\ln\left[(T_0 - t)|\ln(T_0 - t)| + \frac{(\beta-1)}{4\beta}x^2\right]\right\}^{1/(\beta-1)}. \tag{27}$$

Setting here t = T_0^-, we arrive at (26).

The upper bound (26) and lower bound (20) (it holds also for a solution of the boundary value problem) now differ in the character of the dependence of $u(T_0^-, x)$ on |x| only by a logarithmic factor which increases weakly as $|x| \to 0$. Inequality (26) establishes, moreover, the optimality of the corollary of Theorem 6. From (26) we also obtain the following estimate: under the conditions of Theorem 7 for any $\alpha < -3/2$ and $\varepsilon > 0$ sufficiently small

$$\int_{-\varepsilon}^{\varepsilon} u^{(\beta-1)/2}(T_0^-, x)|\ln u(T_0^-, x)|^\alpha dx < +\infty.$$

Remark. From (27) it is not hard to obtain the following estimate: on the trajectories $\eta \equiv |x|/(T_0 - t)^{1/2}|\ln(T_0 - t)|^{1/2} = \text{const}$ as $t \to T_0^-$ there is the estimate

$$u(t, x) \leqslant (T_0 - t)^{-1/(\beta-1)}\left\{(\beta-1) + \frac{(\beta-1)^2}{4\beta}\eta^2\right\}^{-1/(\beta-1)}. \tag{28}$$

The approximate self-similar solution u_a [see (9) in Subsec. 4.4], which was constructed in [26] proceeding from other considerations, appears on the right here.

Passage from the special result formulated in Theorem 7 to rather general solutions of boundary problem (22) is established by means of the comparison theorem. For example, we have

THEOREM 8. Suppose x = 0 is a point of singularity of an unbounded solution of problem (22), and suppose the function $u_0(x)$ and $u_0^*(x)$ intersect at precisely two points, while $T_0(u_0) = T_0(u_0^*) = T_0 < +\infty$. Then for all sufficiently small |x| > 0 there is the asymptotically sharp estimate

$$\min\{u(T_0^-, x), u(T_0^-, -x)\} \leqslant \left[\frac{(\beta-1)^2}{8\beta}\right]^{-1/(\beta-1)} |x|^{-2/(\beta-1)} |\ln|x||^{1/(\beta-1)}.$$

Regarding the method of proof, see Subsec. 3.8.2, Sec. 3 and also [27].

CONCLUSIONS

1. The search for principles of localization and also conditions for formation, self-sustainment, and complication of structures should be carried out with substantial models of minimal dimension not overloaded with details. The relative simplicity of the model makes it possible to develop a number of new mathematical methods for studying the developed nonlinear stage of evolution processes. Computational experiment makes it possible to play through the scenario of nontrivial behavior of an open nonlinear system.

2. Investigations of the simplest model of a heat conducting medium turned out to have content. Depending on the relations of the exponents of the nonlinear medium (σ, β), its heating can take place in three regimes. The most interesting regime was the LS-regime with peaking. Heating of the medium in this regime can lead to simple and complicated dissipative structures localized at the fundamental lengths. Complex structures may be considered as a particular manner of existence of simple structures having, generally speaking, different times of peaking. The regions of localization of the simple structures within the complex

structure overlap in such a manner that the structures synchronized their rate of growth — they had one common time of peaking. Construction of eigenfunctions of the self-similar problem provides all possible types of such coordination. The criterion for the "viability" of the union of the parts into a whole in such a medium is the synchronization of the processes of heating and diffusion of heat. If synchronization is destroyed rapid degeneration of the complex structure into simple structures occurs.

3. Regimes with peaking and the phenomena of localization and formation of structures accompanying them have the character of intermediate asymptotics. For the occurrence of properties of localization growth according to a law with peaking is required only for one or two orders.

4. As shown by means of the method of approximate self-similar solutions, the asymptotic stage of processes in models not admitting group-invariant solutions can be described by group-invariant solutions of other equations.

5. From the analysis of the structures of the LS-regime with peaking it follows that processes near the center of the structures are characteristic in a particular sense for their states in the past, while processes on the periphery are connected with the future. In the HS-regime with peaking the sites of spatial occurrence of the past and future are interchanged. The connection of past and future in the spatial structure existing in the present is the paradoxical principle of construction of dissipative structures of the type indicated.

6. The development of concepts and methods of investigating the phenomena in nonlinear media established in the process of the analysis may play an essential role in understanding tendencies of evolution processes in the problem of obtaining the spectrum of elementary particles in a unified field theory and explaining and predicting the spectrum of biological, economic, and social structures as products of self-organization of processes in corresponding media.

7. In this approach the medium as a whole contains in its material characteristics in nonmanifest, potential form all types of structures which can occur in it and stably or metastably exist as the asymptotics or a type of intermediate "goals" of the evolutionary development.

LITERATURE CITED

1. M. M. Ad'yutov, Yu. A. Klokov, and A. P. Mikhailov, "Investigation of self-similar structures in a nonlinear medium," Preprint, Inst. Prikl. Mat. AN SSSR, No. 108 (1982).
2. M. M. Ad'yutov, Yu. A. Klokov, and A. P. Mikhailov, "Self-similar heat structures with contracted half width," Differents. Uravn., 19, No. 7, 1107-1114 (1983).
3. M. M. Ad'yutov and L. A. Lepin, "The absence of peaked self-similar structures in a medium with a source for constant thermal conductivity," Differents. Uravn., 20, No. 7, 1279-1281 (1984).
4. M. I. Bakirova, S. N. Borshukova, V. A. Dorodnitsyn, and S. R. Svirshchevskii, "On the directed propagation of heat in a nonlinear, anisotropic medium," Preprint, Inst. Prikl. Mat. AN SSSR, No. 182 (1985).
5. G. I. Barenblatt, "On some non-steady-state motions of a liquid and gas in a porous medium," Prikl. Mat. Mekh., 16, No. 1, 67-78 (1952).
6. G. I. Barenblatt and M. I. Vishik, "On finite propagation speed in problems of nonstationary filtration of a liquid and gas," Prikl. Mat. Mekh., 20, No. 3, 411-417 (1956).
7. H. Bateman and E. Erdelyi, Higher Transcendental Functions [in Russian], Vol. 1, Nauka, Moscow (1973).
8. V. S. Belonosov and T. I. Zelenyak, Nonlocal Problems in the Theory of Quasilinear Parabolic Equations [in Russian], Novosibirsk Univ. (1976).
9. A. I. Vol'pert and S. I. Khudyaev, "On the Cauchy problem for quasilinear, degenerate equations of second order," Mat. Sb., 78, No. 3, 374-396 (1969).
10. A. I. Vol'pert and S. I. Khudyaev, Analysis in Classes of Discontinuous Functions and Equations of Mathematical Physics [in Russian], Nauka, Moscow (1975).
11. V. A. Galaktionov, "On a boundary value problem for the nonlinear parabolic equation $u_t = \Delta u^{\sigma+1} + u^{\beta}$," Differents. Uravn., 17, No. 5, 836-842 (1981).
12. V. A. Galaktionov, "On some properties of traveling waves in a medium with nonlinear thermal conductivity and a source of heat," Zh. Vychisl. Mat. Mat. Fiz., 21, No. 4, 980-989 (1981).
13. V. A. Galaktionov, "On conditions of localization of unbounded solutions of quasilinear parabolic equations," Dokl. AN SSSR, 264, No. 5, 1035-1040 (1982).

14. V. A. Galaktionov, "On globally unsolvable Cauchy problems for quasilinear parabolic equations," Zh. Vychisl. Mat. Mat. Fiz., 23, No. 5, 1072-1087 (1983).
15. V. A. Galaktionov, "Proof of localization of unbounded solutions of the nonlinear parabolic equation $u_t = (u^\sigma u_x)_x + u^\beta$," Differents. Uravn., 21, No. 1, 15-23 (1985).
16. V. A. Galaktionov, "The asymptotic behavior of unbounded solutions of the nonlinear parabolic equation $u_t = (u^\sigma u_x)_x + u^{\sigma+1}$," Differents. Uravn., 21, No. 7, 1126-1134 (1985).
17. V. A. Galaktionov, "Asymptotics of unbounded solutions of the nonlinear equation $u_t = (u^\sigma u_x)_x + u^\beta$ near a "singular" point," Dokl. AN SSSR, 288, No. 6, 1293-1297 (1986).
18. V. A. Galaktionov, G. G. Elenin, S. P. Kurdyumov, and A. P. Mikhailov, "The effect of burning-out on localization of combustion and formation of structures in a nonlinear medium," Preprint No. 27, Inst. Prikl. Mat AN SSSR (1979).
19. V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailov, and A. A. Samarskii, "On unbounded solutions of the Cauchy problem for the parabolic equation $u_t = \nabla(u^\sigma \nabla u) + u^\beta$," Dokl. AN SSSR, 252, No. 6, 1362-1364 (1980).
20. V. A. Galaktionov, S. P. Kurdyumov, S. A. Posashkov, and A. A. Samarskii, "A quasilinear parabolic equation with a complex spectrum of unbounded self-similar solutions," in: Mathematical Modeling (Processes in Nonlinear Media) [in Russian], Nauka, Moscow (1986), pp. 142-182.
21. V. A. Galaktionov, S. P. Kurdyumov, and A. A. Samarskii, "On a parabolic system of quasilinear equations. I," Differents. Uravn., 19, No. 12, 2123-2140 (1983).
22. V. A. Galaktionov, S. P. Kurdyumov, and A. A. Samarskii, "On asymptotic stability of invariant solutions of nonlinear heat equations with a source," Differents. Uravn., 20, No. 4, 614-632 (1984).
23. V. A. Galaktionov, S. P. Kurdyumov, and A. A. Samarskii, "On approximate self-similar solutions of a class of quasilinear heat equations with a source," Mat. Sb., 124, No. 2, 163-188 (1984).
24. V. A. Galaktionov, S. P. Kurdyumov, and A. A. Samarskii, "On a method of stationary states for nonlinear evolution parabolic problems," Dokl. AN SSSR, 278, No. 6, 1296-1300 (1984).
25. V. A. Galaktionov, S. P. Kurdyumov, and A. A. Samarskii, "On a parabolic system of quasilinear equations. II," Differents. Uravn., 21, No. 9, 1544-1559 (1985).
26. V. A. Galaktionov and S. A. Posashkov, "The equation $u_t = u_{xx} + u^\beta$. Localization and asymptotic behavior of unbounded solutions," Preprint No. 97, Inst. Prikl. Mat. AN SSSR (1985).
27. V. A. Galaktionov and S. A. Posashkov, "New variants of using the strong maximum principle for parabolic equations and some of their applications," Preprint No. 167, Inst. Prikl. Mat. AN SSSR (1985).
28. V. A. Galaktionov, "Application of new comparison theorems to the investigation of unbounded solutions of nonlinear parabolic equations," Differents. Uravn., 22, No. 7, 1165-1173 (1986).
29. P. Glensdorf and I. Prigozhin, The Thermodynamic Theory of Structure, Stability, and Fluctuations [Russian translation], Mir, Moscow (1973).
30. V. A. Dorodnitsyn, "Group properties and invariant solutions of a nonlinear heat equation with a source or sink," Preprint No. 57, Inst. Prikl. Mat. AN SSSR (1979).
31. V. A. Dorodnitsyn, "On invariant solutions of a nonlinear heat equation with a source," Zh. Vychisl. Mat. Mat. Fiz., 22, No. 6, 1393-1400 (1982).
32. V. A. Dorodnitsyn, G. G. Elenin, and S. P. Kurdyumov, "On some invariant solutions of a heat equation with a source," Preprint No. 31, Inst. Prikl. Mat. AN SSSR (1980).
33. V. A. Dorodnitsyn, I. V. Knyaeva, and S. P. Svirshchevskii, "Group properties of a nonlinear heat equation with a source in two- and three-dimensional cases," Preprint No. 79, Inst. Prikl. Mat. AN SSSR (1982).
34. V. A. Dorodnitsyn, I. V. Knyazeva, and S. P. Svirchchevskii, "Group properties of an anisotropic heat equation with a source," Preprint No. 134, Inst. Prikl. Mat. AN SSSR (1982).
35. V. A. Dorodnitsyn, I. V. Knyaeva, and S. P. Svirshchevskii, "Group properties of a heat equation with a source in two- and three-dimensional cases," Differents. Uravn., 19, No. 7, 1215-1223 (1983).
36. S. P. Svirshchevskii, "On Lie–Bäcklund groups admitted by a heat equation with a source," Preprint No. 101, Inst. Prikl. Mat. AN SSSR (1983).
37. Yu. A. Dubinskii, "Weak convergence in nonlinear elliptic and parabolic equations," Mat. Sb., 67, No. 4, 609-642 (1965).
38. G. G. Elenin, "Formation of quasistationary traveling waves for unstable flows of a barotropic gas," Preprint No. 126, Inst. Prikl. Mat. AN SSSR (1977).

39. G. G. Elenin, V. V. Krylov, A. A. Polezhaev, and D. S. Chernavskii, "Features of the formation of contrast dissipative structures," Dokl. AN SSSR, 271, No. 1, 84-88 (1983).
40. G. G. Elenin and S. P. Kurdyumov, "Conditions for complication of the organization of a nonlinear dissipative medium," Preprint No. 106, Inst. Prikl. Mat. AN SSSR (1977).
41. G. G. Elenin, S. P. Kurdyumov, and A. A. Samarskii, "Nonstationary dissipative structures in a nonlinear heat-conducting medium," Zh. Vychisl. Mat. Mat. Fiz., 23, No. 2, 380-390 (1983).
42. G. G. Elenin and K. E. Plokhotnikov, "On a method of qualitative investigation of a one-dimensional quasilinear heat equation with a nonlinear source of heat," Preprint No. 91, Inst. Prikl. Mat. AN SSSR (1977).
43. Ya. B. Zel'dovich and A. S. Kompaneets, "On the theory of heat propagation for a thermal conductivity depending on the temperature," Collection Dedicated to the Seventieth Birthday of Academician A. F. Ioffe, Izd. AN SSSR, Moscow (1950), pp. 61-71.
44. Ya. B. Zel'dovich and Yu. P. Raizer, Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena [in Russian], Nauka, Moscow (1966).
45. N. V. Zmitrenko, S. P. Kurdyumov, A. P. Mikhailov, and A. A. Samarskii, "Occurrence of structures in nonlinear media and the nonstationary thermodynamics of regimes with peaking," Preprint No. 74, Inst. Prikl. Mat. AN SSSR (1976).
46. N. Kh. Ibragimov, "On the group classification of second-order differential equations," Dokl. AN SSSR, 183, No. 2, 174-177 (1968).
47. N. Kh. Ibragimov, "on the theory of groups of Lie—Bäcklund transformations," Mat. Sb., 109, No. 2, 229-253 (1979).
48. N. Kh. Ibragimov, Transformation Groups in Mathematical Physics [in Russian], Nauka, Moscow (1983).
49. G. R. Ivanitskii, I. Sh. Krinskii, and E. E. Sel'kov, Mathematical Biophysics of a Cell [in Russian], Nauka, Moscow (1978).
50. A. S. Kalashnikov, "On differential properties of generalized solutions of equation of the type of nonstationary filtration," Vestn. Mosk. Gos. Univ., Mat. Mekh., No. 1, 62-68 (1974).
51. A. S. Kalshnikov, On the character of propagation of perturbations in problems on nonlinear heat conduction with absorption," Zh. Vychisl. Mat. Mat. Fiz., 14, No. 4, 891-905 (1974).
52. A. S. Kalashnikov, "On the effect of absorption on the propagation of heat in a medium with a thermal conductivity depending on the temperature," Zh. Vychisl. Mat. Mat. Fiz., 16, No. 3, 689-697 (1976).
53. E. Kamke, Handbook on Ordinary Differential Equations [in German], Chelsea Publ.
54. O. V. Kaptson, "Classification of evolution equations according to conservation laws," Funkts. Anal., 16, No. 1, 72-73 (1982).
55. R. Kreshner, "On some properties of generalized solutions of quasilinear degenerate parabolic equations," Acta Math. Acad. Sci. Hung., 32, Nos. 3-4, 301-330 (1978).
56. A. N. Kolmogorov, I. G. Petrovskii, and N. S. Piskunov, "Investigation of a diffusion equation connected with an increase of the amount of matter and its application to a problem of biology," Byull. Mosk. Gos. Univ., 1, No. 6, 1-26 (1937).
57. S. N. Kruzhkov, "Results on the character of continuity of solutions of parabolic equations and some of their applications," Mat. Zametki, 6, No. 1, 97-108 (1969).
58. S. P. Kurdyumov, "Investigations of the localization of heat and occurrence of structures in nonlinear media," Differential Equations and Applications. Prod. 2nd Conf., Russe 29 June-2 July, 1982, Russe, 1982, pp. 393-397.
59. S. P. Kurdyumov, "Eigenfunctions of combustion of a nonlinear medium and constructive laws of constructing its organization," in: Modern Problems of mathematical Physics and Computational Mathematics [in Russian], Nauka, Moscow (1982), pp. 217-243.
60. S. P. Kurdyumov, E. S. Kurkina, A. B. Potapov, and A. A. Samarskii, "The architecture of multidimensional thermal structures," Dokl. SSSR, 284, No. 5, 1071-1075 (1984).
61. S. P. Kurdyumov, G. G. Malinetskii, Yu. A. Poveshchenko, Yu. P. Popov, and A. A. Samarskii, "The interaction of dissipative thermal structures in nonlinear media," Dokl. AN SSSR, 251, No. 4, 836-839 (1980).
62. O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc. (1965).
63. J. L. Lions, Some Methods of Solving Nonlinear Boundary Value Problems [Russian translation], Mir, Moscow (1972).
64. G. Nikolis and I. Prigozhin, Self-Organizaiton in Nonequilibrium Systems [Russian translation], Mir, Moscow (1979).

65. L. V. Ovsyannikov, "Group properties of a nonlinear heat equation," Dokl. AN SSSR, 125, No. 3, 492-495 (1959).
66. L. V. Ovshyannikov, Group Properties of Differential Equations [in Russian], Novosibirsk Univ. (1962).
67. L. V. Ovsyannikov, Group Analysis of Differential Equations [in Russian], Nauka, Moscow (1978).
68. O. A. Oleinik, A. S. Kalashnikov, and Chzhou Yui-lin', "The Cauchy problem and boundary value problems for equations of the type of nonstationary filtration," Izv. AN SSSR, Ser. Mat., No. 5, 667-704 (1958).
69. S. I. Pokhozhaev, "On eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$," Dokl. AN SSSR, 165, No. 1, 36-39 (1965).
70. S. I. Pokhozhaev, "On eigenfunctions of quasilinear elliptic problems," Mat. Sb., 82, No. 2, 192-212 (1970).
71. Yu. M. Romanovskii, N. V. Stepanova, and D. S. Chernavskii, Math. Biophyis., Nauka, Moscow (1984).
72. A. A. Samarskii, "Mathematical modeling and computational experiment," Vestn. AN SSSR, No. 5, 39-49 (1979).
73. A. A. Samarskii, G. G. Elenin, N. V. Zmitrenko, S. P. Kurdyumov, and A. P. Mikhailov, "Combustion of a nonlinear medium in the form of complex structures," Dokl. AN SSSR, 237, No. 6, 1330-1333 (1977).
74. A. A. Samarskii, N. V. Zmitrenko, S. P. Kurdyumov, and A. P. Mikhailov, "Thermal structures and the fundamental length in a medium with nonlinear thermal conductivity and volumetric heat sources," Dokl. AN SSSR, 227, No. 2, 321-324 (1976).
75. Yu. M. Svirezhev and D. O. Logofet, Stability of Biological Societies [in Russian], Nauka, Moscow (1978).
76. S. R. Svirshchevskii, "Group properties of a system of heat equations of hyperbolic type," Preprint No. 20, Inst. Prikl. Mat. AN SSSR (1986).
77. M. G. Slin'ko, "Problems of the theory of chemical reactors," Khim. Promyshlennost', No. 5, 3-7 (1984).
78. I. T. Frolov, Life and Knowledge [in Russian], Mysl', Moscow (1981).
79. G. Haken, Synergetics [Russian translation], Mir, Moscow (1980).
80. V. Ebeling, Formation of Structures for Irreversible Processes [Russian translation], Mir, Moscow (1979).
81. M. Eigen, Self-Organization of Matter and of the Evolution of Biological Macromolecules [Russian translation], Mir, Moscow (1973).
82. D. G. Aronson, "Regularity properties of flows through porous media," SIAM J. Appl. Math., 17, No. 4, 461-467 (1969).
83. D. G. Aronson, "Regularity properties of flows through porous media: the interface," Arch. Rat. Mech. Anal., 37, No. 1, 1-10 (1970).
84. D. G. Aronson, M. Crandall, and L. A. Peletier, "Stabilization of solutions of a degenerate nonlinear diffusion problem," Nonlinear Anal., Theory, Meth., Appl., 6, No. 10, 1001-1022 (1982).
85. D. G. Aronson and H. F. Weinberger, "Multidimensional nonlinear diffusion arising in popular genetics," Adv. Math., 30, No. 1, 33-76 (1978).
86. J. M. Ball, "Remarks on blow-up and nonexistence theorems for nonlinear evolution equations," Quart. J. Math., 28, No. 112, 473-486 (1977).
87. J. G. Berryman and C. J. Holland, "Stability of the separable solution for fast diffusion," Arch. Rat. Mech. Anal., 74, No. 4, 379-388 (1980).
88. M. Bertsch, R. Kersner, and L. A. Peletier, "Positivity versus localization in degenerate diffusion equations," Nonlinear Anal., Theory, Meth., Appl., 9, No. 10, 987-1008 (1985).
89. G. Bluman and S. Kumei, "On the remarkable nonlinear diffusion equation $(\partial/\partial x)[a(u + \bar{b})^{-2}(\partial u/\partial x)] - (\partial u/\partial t) = 0$," J. Math. Phys., 21, No. 5, 1019-1023 (1980).
90. L. A. Caffarelli and A. Friedman, "Regularity of the free boundary of gas flow in an n-dimensional porous medium," Indiana Univ. Math. J., 29, No. 3, 361-391 (1980).
91. A. S. Fokas and Y. C. Yortson, "On the exactly solvable equation $u_t = ((\alpha u + \beta)^{-2}u_x)_x + \gamma(\alpha u + \beta)^{-2}u_x$ occurring in two-phase flow in porous media," SIAM J. Appl. Math., 42, No. 3, 318-332 (1982).
92. A. Friedman, "On the regularity of the solutions of nonlinear elliptic and parabolic systems of partial differential equations," J. Math. Mech., 7, No. 1, 43-59 (1958).
93. A. Friedman, "Remarks on nonlinear parabolic equations," in: Applications of Nonlinear Partial Differential Equations in Mathematical Physics, Am. Math. Soc., Providence, R.I. (1965), pp. 3-23.

94. A. Friedman, "Variational principles and free-boundary problems," Wiley-Interscience, New York (1982).
95. A. Friedman and S. Kamin, "The asymptotic behavior of gas in an n-dimensional porous medium," Trans. Am. Math. Soc., 262, No. 2, 551-563 (1980).
96. A. Friedman and B. McLeod, "Blow-up of positive solutions of semilinear heat equations," Indiana Univ. Math. J., 34, No. 4, 425-447 (1985).
97. H. Fujita, "On the blowing up of solutions to the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$," J. Fac. Sci. Univ. Tokyo, 13, No. 1, Sect. IA, 109-124 (1966).
98. B. Gidas and J. Spruck, "Global and local behavior of positive solutions of nonlinear elliptic equations," Commun. Pure Appl., 34, No. 5, 525-598 (1981).
99. Y. Giga and R. V. Kohn, "Asymptotically self-similar blow-up for semilinear heat equations," Commun. Pure Appl. Math., 38, No. 2, 297-319 (1985).
100. B. H. Gilding, "Hölder continuity of solutions of parabolic equations," J. London Math. Soc., 13, No. 1, 103-106 (1976).
101. A. Haraux and F. B. Weissler, "Nonuniqueness for a semilinear initial value problem," Indiana Univ. Math. J., 31, No. 2, 167-189 (1982).
102. K. Hayakawa, "On nonexistence of global solutions of some semilinear parabolic differential equations," Proc. Jpn. Acad., 49, No. 7, 503-505 (1973).
103. D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes Math., 840 (1981).
104. L. M. Hocking, K. Stewartson, and J. T. Stuart, "A nonlinear instability burst in plane parallel flow," J. Fluid Mech., 51, 705-735 (1972).
105. N. H. Ibragimov, "Sur l'équivalence des équations d'évolution qui admettent une algèbre de Lie—Bäcklund infinie," C. R. Acad. Sci., 293, Ser. 1, No. 5, 657-660 (1981).
106. M. Ito, "The conditional stability of stationary solutions for semilinear parabolic equations," J. Fac. Sci. Univ. Tokyo, Sec. 1A, 25, No. 3, 263-275 (1978).
107. D. D. Joseph and T. S. Lundgren, "Quasilinear Dirichlet problems driven by positive sources," Arch. Rat. ech. Anal., 49, No. 4, 241-269 (1973).
108. S. Kaplan, "On the growth of solutions of quasilinear equations," Commun. Pure Appl. Math., 16, No. 3, 305-330 (1963).
109. B. F. Kerr, "The porous medium equation in one dimension," Trans. Am. Math. Soc., 234, No. 2, 381-415 (1977).
110. B. F. Kerr, "The behavior of the support of solutions of the equations of nonlinear heat conduction with absorption in one dimension," Trans. Am. Math. Soc., 249, No. 2, 381-415 (1977).
111. K. Kobayashi, T. Sirao, and H. Tanaka, "On the blowing up problems for semilinear heat equations," J. Math. Soc. Jpn., 29, No. 3, 407-424 (1977).
112. H. Levine, "Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$," Arch. Rat. Mech. Anal., 51, No. 5, 371-386 (1973).
113. H. A. Levine, "Nonexistence of global weak solutions to some properly and improperly posed problems of mathematical physics: the method of unbounded Fourier coefficients," Math. Ann., 214, No. 3, 205-220 (1975).
114. H. A. Levine and P. E. Sacks, "Some existence and nonexistence theorems for solutions of degenerate parabolic equations," J. Diff. Eqs., 52, No. 2, 135-161 (1984).
115. S. Lie, Über die Integration durch bestimmte Integral von einer Klasse linearer partiellen Differentialgleichungen," Arch. Math., 6, 328-368 (1881).
116. C. Loewner and L. Nirenberg, "Partial differential equations invariant under conformal or projective transformations," in: Contrib. Analysis, Academic Press, New York—London (1974), pp. 245-272.
117. H. Matano, "Nonincrease of lap-number of a solution for a one-dimensional semilinear parabolic equation," J. Fac. Sci. Univ. Tokyo, Sect. 1A, 29, No. 2, 401-441 (1982).
118. D. Mottoni, A. Schiaffino, and A. Tesei, "Attractivity properties of non-negative solutions for a class of nonlinear degenerate parabolic problems," Ann. Mat. Pura Appl., 136, No. 1, 35-48 (1984).
119. A. Munier, J. R. Burgan, J. Guttierrez, E. Fijalkow, and M. R. Feix, "Group transformations and the nonlinear heat diffusion equation," SIAM J. Appl. Math., 40, NO. 2, 191-207 (1981).
120. W. M. Ni, P. E. Sacks, and J. Tavantzis, "On the asymptotic behavior of solutions of certain quasilinear parabolic equations," J. Different. Equat., 54, No. 1, 97-120 (1984).
121. V. M. Ni and P. Sacks, "The number of peaks of positive solutions of semilinear parabolic equations," SIAM J. Math. Anal., 16, 460-471 (1985).

122. P. J. Olver, "Evolution equations possessing infinitely many summaries," J. Math. Phys., 18, No. 6, 1212-1215 (1977).
123. L. A. Peletier, "The porous media equation," in: Applications of Nonlinear Analysis in the Physical Sciences, Pitman, Boston–Melbourne (1981), pp. 229-241.
124. P. E. Sacks, "The initial and boundary value problem for a class of degenerate parabolic equations," Commun. Part. Diff. Eqs., 8, No. 7, 693-733 (1983).
125. P. E. Sacks, "Global behavior for a class of nonlinear evolution equations," SIAM J. Math. Anal., 16, No. 2, 233-250 (1985).
126. D. H. Sattinger, "On the total variation of solutions of parabolic equations," Math. Ann., 183, No. 1, 78-92 (1969).
127. M. Schatzman, "Stationary solutions and asymptotic behavior of a quasilinear degenerate parabolic equation," Indiana Univ. Math. J., 33, No. 1, 1-29 (1984).
128. F. B. Weissler, "Local existence and nonexistence for semilinear parabolic equations in L^p," Indiana Univ. Math. J., 29, No. 1, 79-102 (1980).
129. F. B. Weissler, "Single point blow-up for a semilinear initial value problem," J. Diff. Eqs., 55, No. 2, 204-224 (1984).
130. F. B. Weissler, "An L^∞ blow-up estimate for a nonlinear heat equation," Commun. Pure Appl. Math., 38, No. 3, 291-205 (1985).

CLASSIFICATION OF SOLUTIONS OF A SYSTEM OF NONLINEAR DIFFUSION EQUATIONS IN A NEIGHBORHOOD OF A BIFURCATION POINT

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The theory of reaction-diffusion systems in a neighborhood of a bifurcation point is considered. The basic types of space–time ordering, diffusion chaos in such systems, and sequences of bifurcations leading to complication of solutions are studied. A detailed discussion is given of a hierarchy of simplified models (one- and two-dimensional mappings, systems of ordinary differential equations, and others) which make it possible to carry out a qualitative analysis of the problem studied in the case of small regions. A number of generalizations of the equations studied and the simplest types of ordering in the two-dimensional case are described.

1. Two-Component Systems and the Classification Problem

In many systems which are studied in physics, chemistry, and biology there arise self-sustaining structures of various types [24, 32, 36, 40, 42, 62]. The question of the properties of nonlinear media where structures are formed and of the general regularities of their occurrence is one of the fundamental questions of modern science.

We shall characterize the deviation from equilibrium in the systems studied by a parameter λ ($\lambda = 0$ corresponds to the equilibrium state). It follows from classical thermodynamics that the evolution of such a system proceeds in the direction of increasing entropy; any order hereby vanishes. A necessary condition for the existence of stable structures is exchange with an external medium (the system must be open).

For small deviations ($0 < \lambda < \lambda_0$) concepts of linear nonequilibrium thermodynamics are applicable. This theory describes processes in a neighborhood of thermodynamic equilibrium and "... encompasses all cases where the flows (or velocities of irreversible processes) are linear functions of the "thermodynamic forces" (gradients of the temperature or concentrations)" [52]. It has been shown that in this range of parameters a stationary state of the system is close to the equilibrium state (for each value of λ it is unique and stable). It is therefore said that all such states lie on the thermodynamic branch.

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