COMPLEX MULTIDIMENSIONAL STRUCTURES OF THE COMBUSTION OF A NON-LINEAR MEDIUM*

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The combustion process in a medium with the thermal conductivity $kT = k_0 T^o$ and volume heat source $Q(T) = q_0 T^a$ is examined. Under specific conditions combustion occurs in a regime with peaking and the heat localization phenomenon occurs, leading to the occurrence of non-stationary dissipative structures whose number and form are determined only by the properties of the medium. The structures are described using the solutions - eigenfunctions - of the problem for a quasilinear elliptic equation. An analytic method of constructing approximations to the eigenfunctions is proposed, which makes it possible to estimate their number, investigate the architecture and numerically construct a number of eigenfunctions.

Introduction.

The theory of dissipative structures and autowave processes - or synergetics - which has been set up in the last few years, examines from a single point of view the processes of selforganisation in different physical, biological and chemical systems /1-3/. It has been shown how, without contradicting the second law of thermodynamics, one can explain the tendency to complication /3/. The development of organised structures takes place due to the dissipation of energy in open systems situated far from the state of thermodynamic equilibrium /1/. The universality of the properties of the dissipative structures is explained by the fact that, despite their different nature, they are described by the same non-linear equations. As a rule these are sets of parabolic equations with non-linear sources ans sinks /1-3/. The development of self-organization in many cases is mathematically compared with the losses of stability of the homogeneous solution or with the appearance of time-periodic solutions of the wave or autowave type /4, 5/ or with spatial-inhomogeneous steady states /1-3/ or with the development of macrostochastic regimes /6-8/.

Another class of self-organization phenomena exists, however, in which the violation of the homogeneous state is accompanied by the emergence of non-stationary dissipative structures when very rapid processes develop in a medium. This paper investigates non-stationary thermal structures which occur in a medium with a non-linear thermal conductivity and a volume heat source. In /9-21/ it is shown that under specific conditions combustion processes occur in this medium in a regime with peaking, for which some quantities (for example, the temperature distribution maximum) arbitrarily increase after a finite time t_i (the peaking time) at least one point in space.

Regimes with peaking are an intermediate asymptotic form of actual very rapid processes observed in nature. Combustion in a regime with peaking has a number of interesting properties which stimulate detailed analysis. They have a bearing on the paradoxical phenomenon of heat localization, under which heat and combustion do not propagate into a cold medium. Intense energy release takes place in the area of localization and the temperature approaches infinity after a finite time t_i , and outside this area the temperature either equal zero, or is limited. The latter is determined by the character of the initial data. Non-stationary dissipative structures emerge in the medium - intense combustion processes which are localized in separate parts of the medium. Production, self-maintenance and multiplication of structures of that type is observed in plasma /22/, and the T-layer can serve as an example /23/.

Regimes with peaking and the localization phenomenon occur in a wide range of problems of mathematical physics. There is an extensive literature on them, and reviews are given in /9-12/, /18/, see also /24-27/. In nature, of course, there are no processes in which any quantity increases to infinity. But all the features of regimes with peaking, as shown in /28, 29/, appear when the quantities increase a finite number of times with respect to the law of peaking, and this number is determined by the properties of the non-linear medium considered.

The temperature distribution in the localization area can have a complex non-monotonic form, i.e. structures of varied complexity can develop in the medium /9, 10, 13, 16/. In the one-dimensional case it is shown that the eigenfunctions of a non-linear selfsimilar problem determine all types of structures and waves in a medium with specified parameters /10/.

The following problem arises: do two-dimensional and three-dimensional thermal structures which have a complex form of localization area with a special arrangement of temperature distribution maxima in it exist and, if they do, how will the emergence and evolution of these complex structures occur? To solve this problem, it was suggested in /30, 31/ that we consider a new class of selfsimilar solutions of an equation of quasilinear heat conduction with a non-linear source, namely multidimensional solutions which are angle-inhomogeneous. This formulation is a new and untraditional one for thermal problems. It is connected with the fact that

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selfsimilar solutions - and, in general, invariant-group solutions - transmit the characteristic features of the processes examined, and are not simply particular solutions of problems in mathematical physics. They are the intermediate asymptotic forms of a wide class of other solutions with arbitrary initial data /32/, and, for a number of processes, they are even the asymptotic forms of another class of equations which themselves do not permit invariant-group solutions /19/.

When a selfsimilar eigenvalue and eigenfunction problem is formulated and a non-linear elliptic equation is obtained, the problem of solving it arises. It is necessary to clarify, first of all, whether there generally are solutions which are unlike central-symmetric solutions. The following problem arises when attempting to solve this problem numerically: in order to construct different eigenfunctions using the iteration method, we need to have fairly good initial approximations to them, and consequently we first need to investigate the problem analytically.

The new approach developed in /9, 10, 30, 31, 33, 34/, also in /35-37/, based on linearizing the non-linear heat conduction equation around a partial spatial-homogeneous solution with the additional condition of connecting the solution with the asymptotic form of the non-linear problem, played a decisive role in the analytical investigation. This approach is similar to the search for solutions in classical quasi-equilibrium thermodynamics, when they are sought as deviations from the equilibrium state, but in the case of non-stationary thermodynamics, deviations from a background which is increasing in a regime with peaking are considered. It is essential to note that the use of this approach in one special case enabled us to reduce the selfsimilar equation to Schrödinger's stationary equation with a Coulomb potential and corresponding boundary conditions /9, 16, 34/. This suggested the existence of exotic structures of the combustion of a medium with an area of localization, for example, in the form of a dumbbell, star, etc..

Soultions of the linear problem were subsequently investigated in detail and contrasted with solutions of the non-linear selfsimilar problem. Linearization operates in the central part of the structure, describing the whole region of non-monotony of the multidimensional selfsimilar temperature distribution. At the edge of the area of localization the termpature asymptotically approaches zero. In the one-dimensional case, by "connecting" the solution of the linearized equation in a continuous way with the asymptotic form of the non-linear problem, we were able to obtain a good approximation to the eigenfunction of the non-linear problem /30/, which differs from them on average by 5%. Previously the eigenfunctions of the onedimensional problem were constructed numerically /10, 16/, at the same time their number was finite and was determined only by the parameters of the non-linear medium. The method of connection enabled us to obtain all the existing eigenfunctions. In thise sense connection with the asymptotic form plays the role of the quantum spectrum condition, equivalent to the role of the normalization condition for the Schrödinger equation or Bohr's selection principle.

In order to carry out a similar "connection" procedure in the two-dimensional and threedimensional cases, in this paper we construct two classes of eigenfunctions, which differ from each other in the maxima-arrangement principle. The analytic approximations obtained were very good and enabled us numerically to construct eigenfunctions of non-linear selfsimilar problem and thereby to confirm their existence in most cases.

Consider the properties of these multidimensional selfsimilar solutions and the laws of their development. Thus, they are localized dissipative structures of the combustion of a non-linear medium, which develop in a regime with peaking. The problem of localization is discussed in Sect.5. Structures of a different type differ in the number of maxima in the area of localization and by the shape of the area of localization. At the same time the maxima increase in the regime with peaking and move to the centre of the symmetry and, as $t \rightarrow l_{t}$ "hit" it. Combustion outside the area of localization of the structures either does not take place at all, or operates slowly (with other t_i). The number of eigenfunctions and their form are determined only by the parameters of the non-linear medium. Consider the two-dimensional case as the clearest one. Very beautiful structures exist, the shape of whose area of localization is an ellipse, star, cross, rectangle, etc.. We can roughly characterize the architecture of the complex structure using two parameters. For this we need to divide it into layers (in one case - concentric circles, in the other - rectangles) and observe the number of maxima in the layer. The more complex the structure, the more layers it has and the more maxima in it. There is a similar picture in the three-dimensional case /23/. On the whole the architecture of the complex three-dimensional structure recalls the structure of the atom, which is characterized by the number of electron shells and the number of electrons in them. (In our case, there are the layers and maxima.) There are also fundamental differences, however. Sect.4 provides a detailed description of the architecture of the eigenfunctions.

Another fundamental property of non-linear media is connected with the representation of the complex structure: the principle of amalgamating simple structures (which have one maximum in all) into complex.structures, or the principle of superposition in non-linear problems /9/. The problems of amalgamating simple structures into complex ones, the laws of constructing the organisation of a non-linear medium, and also the problems of the stability of this organisation, are considered in Sect.5. In particular, it is established that only an organization which corresponds to one of the eigenfunctions of the selfsimilar problem can exist long-term in a non-linear medium. If combustion is not induced in accordance with the profiles of the eigenfunctions, it either fades or, "collapsing", degenerates into combustion in the form of simple structures.

1. Formulation of the problem,

We consider the process of combustion in a medium with a volume heat source and thermal conductivity which depend on the temperature in the form of a power law. The temperature distribution $T(\mathbf{r}, t)$ in the space satisfies the equation

$$\rho c_{\mathbf{z}} \frac{\partial T}{\partial t} = \operatorname{div} (\varkappa_0 T^{\sigma} \operatorname{grad} T) + \gamma_0 \rho T^{\beta}, \qquad (1.1)$$

where $\varkappa_0 > 0$, $q_v > 0$, $\sigma > 0$, $\beta > 1$, $c_v > 0$ are specified parameters.

It is assumed that the density depends on the radius in the form of a power law and in a special case it can be constant: $\rho = Ar^{r-1}, A, \epsilon > 0$.

Combustion is induced by specifying some initial limited temperature distribution $T(\mathbf{r}, 0) = T_0(\mathbf{r}) < \infty$, which will subsequently be refined.

The boundary conditions at infinity have the form $T \rightarrow 0$, $\varkappa_0 T^a \operatorname{grad} T \rightarrow 0$ as $r \rightarrow \infty$. The problem consists of obtaining all the types of thermal sturctures that can arise in a specified non-linear medium. With this aim we investigate selfsimilar solutions of problem (1.1) of the form

$$T(\mathbf{r},t) = g(t) y^{1/(n+1)}(\xi), \quad \xi = \frac{\mathbf{r}}{\psi(t)}, \quad \xi = |\xi|.$$
(1.2)

The substitution of (1.2) into Eq.(1.1) uniquely defines the form of the functions g and ψ :

$$g(t) = \left(1 - \frac{t}{\tau}\right)^{\mu}, \quad \psi(t) = \left(1 - \frac{t}{\tau}\right)^{n}, \quad \mu = -\frac{1}{\beta - 1}, \quad (1.3)$$
$$n = \frac{\beta - \sigma - 1}{s(\beta - 1)}.$$

and the form of the multidimensional selfsimilar equation with respect to the function y. It is convenient to carry out a transformation of the coordinates: $\hat{t}=t_0t$, $t_0=q_0c_0^{-1}$, $\hat{r}=r_0r$, $r_0=(\varkappa_0q_0^{-1})^{1/4}$. The function $y(\xi)$ satisfies the quasilinear elliptic equation

$$\frac{1}{\sigma+1}\Delta y - \frac{\beta-\sigma-1}{\tau s(\beta-1)}\xi^{s-1}\frac{\partial}{\partial\xi}y^{1/(\sigma+1)} + \xi^{s-2}\left(y^{\beta/(\sigma+1)} - \frac{1}{\tau(\beta-1)}y^{1/(\sigma+1)}\right) = 0,$$
(1.4)

where τ is the arbitrary parameter of the separation of variables (1.3).

In accordance with the formulation of the problem we shall seek selfsimilar solutions of Eq.(1.4), which satisfy the following boundary conditions:

$$y < \infty$$
 when $\xi = 0$, $y \to 0$, grad $y \to 0$ as $\xi \to \infty$. (1.5)

Problem (1.4), (1.5) is an problem in eigenvalues τ and eigenfunctions $y(\xi, \tau)$. We note some properties of the eigenvalues and eigenfunctions, which directly follow from an analysis of the functions $g(t), \psi(t)$ and Eq.(1.4). Since $\beta > 1$, then for the positive eigenvalue τ the eigenfunction exists for a finite time $t_i = \tau$ and develops in a regime with peaking. For the negative eigenvalue τ the eigenfunction $y(\xi, \tau)$ exists for any t>0 and describes a decaying regime. Since we are interested in regimes with peaking, we will only consider $\tau > 0$. If the problem investigated has a solution for some value of $\tau = \tau_i > 0$, it also has a solution for any other value of $\tau = \tau_i > 0$. These solutions are connected by a similarity transformation /10, 16/. The above property of the eigenfunctions and eigenvalues enables us to consider not all the possible values of τ , but to choose a convenient one. Henceforth we will assume $\tau = (\beta - 1)^{-i}$.

Thus, first we investigate the eigenfunctions of the selfsimilar solution (1.5), (1.4), and then the conditions for their implementation, i.e. the stability.

2. Selfsimilar solutions in the one-dimensional case.

1. Three types of selfsimilar solution. Consider solutions of the above problem which are functions only of the variable ξ (the plane, cylindrically- and spherically-symmetric cases). These selfsimilar solutions were examined in /lo, l3/, /l6/. It was shown in /l3, l6/ that three different types of selfsimilar solution, describing the processes of combustion of the medium, exist as a function of the relation between the indicators β and σ . When $1<3<\sigma+1$ the so-called HS-regime is implemented, which is a thermal wave, propagating with a finite front, whose amplitude and velocity increase in the regime with peaking. When $\beta=\sigma+1$ the selfsimilar problem has a unique monotonic solution. It is a non-stationary dissipative structure which is localized in the fundamental length L_T (Fig.1: the area of localization CD, the half-width AB). In the case of plane geometry and constant density the analytic solution has the form /38/ (see Fig.1)

$$T(x,t) = q_0 t_f \left(1 - \frac{t}{t_f}\right)^{-1/\sigma} \left[\frac{2(\sigma+1)}{\sigma(\sigma+2)}\cos^2\left(\frac{\pi x}{L_r}\right)\right]^{1/\sigma},$$

$$L_r = \frac{2\pi}{\sigma} \left[\frac{\varkappa_0}{q_0}(\sigma+1)\right]^{1/\epsilon}, \qquad t_f = (q_0 \sigma T_0^{\sigma})^{-1}.$$

At the edge of the area of localization and outside it the temperature and thermal flux remain equal to zero. When $\beta > \sigma + 1$ the LS-regime is implemented, which is also considered in

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this paper.

We shall discuss in detail the properties of the selfsimilar solutions in the LS-regime. If $\beta > \sigma + 1$, the selfsimilar indicator n > 0, and since $r = \frac{1}{2}(1-t\tau^{-1})^n$, the half-width of the temperature profile narrows with time. The selfsimilar solution cannot have a finite front, since its radius would also have to shorten. It follows from the asymptotic analysis that as $\xi \rightarrow \infty$ the solutions have the asymptotic form /16/

$$y = C\xi^{-p} + \dots, \qquad p = \frac{s(\sigma+1)}{\beta - \sigma - 1}. \tag{2.1}$$



Fig.l

Fig.2

Fig.2 Cylindrically-symmetric eigenfunctions which depend only on one variable ξ : Fig.2a shows all possible eigenfunctions, the dashed line is the solution of the linearized equation $1+\alpha_{,\mu}(\xi)$, imposed on the second, third and fourth eigenfunctions; Fig.2b shows the first three eigenfunctions and the linear approximations $\tilde{y}^{i_{\mu}}(\xi)$ to them.

If we substitute (1.2) and (1.3) into (2.1), we will obtain that the principal term of the asymptotic expansion of the temperature as $r \rightarrow \infty$ does not depend on time:

$$T(\mathbf{r},t) = Cr^{-p} - B(C)\left(1 - \frac{t}{\tau}\right)r^{-q} + \dots, \qquad q = \frac{s(\sigma + \beta)}{\beta - \sigma - 1}.$$
(2.2)

This fact indicates combustion localization: the temperature increases in the regime with peaking in the shortening region close to the centre of symmetry, at the same time as outside this area it approaches the limit, time-constant temperature distribution (2.2).

In the multidimensional case three regimes of combustion of the medium with peaking also occur for the same relations between β and σ . The asymptotic form of the eigenfunctions of the LS-regime has the form (2.1), where C is the function of the direction:

$C = C(\xi / \xi) = C(\theta, \varphi).$

2. The spectrum of selfsimilar solutions in an LS-regime. We shall write Eq.(1.5) for the case when y depends on only one variable ξ :

$$\frac{1}{\sigma+1}\xi^{-\nu}\frac{\partial}{\partial\xi}\left(\xi^{\nu}\frac{\partial y}{\partial\xi}\right) - \frac{\beta-\sigma-1}{s}\xi^{s-1}\frac{\partial}{\partial\xi}y^{1/(\sigma+1)} +$$

$$\xi^{s-2}\left(y^{\beta/(\sigma+1)}-y^{1/(\sigma+1)}\right) = 0.$$
(2.3)

The numerical calculations of problem (2.3), (1.5) showed that it has a finite number N of eigenfunctions $y_i(\xi)$, i=1, 2, ..., N. The number of eigenfunctions and their qualitative behaviour depends neither on the character of the density distribution of s, nor on the geometry of the region v and are determined by the formula

$$N = [a - [[a] a^{-1}]] + 1, \qquad u = \frac{\beta - 1}{\beta - \sigma - 1}.$$

Fig.1 shows graphs of all the possible eigenfunctions (N=5) for the case $\sigma=2, \beta=3.5, s=2, \nu=1$. The first eigenfunction decreases monotonically. The following are non-monotonic, with a number of local extrema which equals their number. In the domain of its non-monotony, as follows from the calculations, the eigenfunctions perform oscillations around the homothermic (spatial-homogeneous) solution of Eq.(2.3): $y_0=1$. These oscillations are small and decrease as the number of eigenfunctions increases, and also as σ and β increase. The calculations also showed that the eigenfunctions reach the asymptotic form with good accuracy for small values of ξ immediately after leaving the domain of non-monotony. In the domain of nonmonotony the eigenfunctions are well described by the solutions of Eq.(2.3) which is linearized around y_0 :

$$\xi^2 u^* + v\xi u = \frac{\beta - \sigma - 1}{s} \xi^{s+1} u' + (\beta - 1)\xi^s u = 0$$

(it is convenient to assume that u(0) = 1, while $y = 1 + \alpha u$, $\alpha < \infty$), if the asymptotic form α of the solution of the linear problem is correctly specified. Each eigenfunction is described by the solution of the linear problem with its amplitude α_i , whilst the leading eigenfunction in the largest domain (by comparison with the domains for the lowest eigenfunctions) agrees with the solution of the linear equation. It is close to it in the whole area of non-monotony. The remaining eigenfunctions "transmit" only one, two, etc., oscillations of the solution of the linear equation in accordance with their number. Consequently, we can construct the eigenfunctions with good accuracy (besides the numerical solution), if we solve the problem of selecting a number of discrete values of α . This problem was solved using the connection method. The space id divided into two parts: $0 \le \xi \le \xi_0$ and $\xi_0 \le \xi \le \infty$. In the first domain $1 + \alpha u$ is chosen as the approximate solution of $ilde{y}$ the asymptotic form $\mathcal{C}\xi^{-p}$ is chosen in the second, and the unknowns α , C, ξ_0 are determined from the connection condition, i.e. from the equation when $\xi = \xi_0$ of both functions and their first and second derivatives. It appeared that the number of these approximations equals the number of eigenfunctions and they well describe the behaviour of the eigenfunctions qualitatively and quantitatively (we can improve the quantitative agreement if we take $\tilde{y}^{\prime h}$ instead of \tilde{y}). The main value of this procedure is the fact that we were able to extend the concepts of connection to the multidimensional case and we can numerically construct a number of fundamentally new structures.

3. Two-dimensional eigenfunctions. Construction of approximations.

We shall now construct two-dimensional eigenfunctions. We shall linearize Eq.(1.4) around the homothermic solution: y=1+u, $|u| \leq 1$. We will then obtain a linear equation with respect to the function $u(\xi, \varphi)$:

$$\Delta u = \frac{\beta - \sigma - 1}{s} \xi^{s-1} \frac{\partial u}{\partial \xi} + (\beta - 1) \xi^{s-2} u = 0, \quad |u| < \infty, \quad \xi = 0.$$
(3.1)

The direct transference of the connection procedure to two-dimensional and three-dimensional problems presents great difficulties and does not essentially simplify the problem: we need to solve the problem with a free boundary $\xi_0(\varphi)$, and α and C are transformed from numbers into infinite sets of Fourier coefficients. However if we solve the problem in coordinates, where the variables in Eq.(3.1) are separated, and we assume that the eigenfunctions have some internal symmetry, i.e., they are fairly accurately described using one harmonic, the connection procedure is considerably simplified. There are two such sets of coordinates – polar and (for integral s) Cartesian – for the two-dimensional problem. In accordance with this, two sets of approximations to the eigenfunctions were also constructed.

1. Approximation of the E/Mm class. We shall solve Eq.(3.1) in polar coordinates using the method of separation of variables ξ and φ . Then the general solution has the form

$$u = \sum_{m \ge 0} \alpha_m R_m(\xi) \cos(m\varphi + \varphi_{0m}), \qquad (3.2)$$

where

$$R_m(\xi) = \xi^m \mathcal{M}\left(-a_m, b_m, \frac{\beta - \sigma - 1}{s^2} \xi^s\right),$$

 $a_m = a - m/s$, $b_m = 1 + 2m/s$. $a = (\beta - 1) / (\beta - \sigma - 1)$, M(a, b, z) is a degenerate hypergeometric function, and α_m are arbitrary constants.

We will assume that the *m*-th harmonic predominates in the eigenfunctions with respect to φ . Then in series (3.2) we can confine ourselves only to the term $\alpha_m R_m(\xi) \cos(m\varphi)$, and the function $C(\varphi)$ in the asymptotic form has the form $C(\varphi) = C_y + C_m \cos(m\varphi)$ and, consequently, we can use the connection method. Two functions are constructed for this:

$$f_{j}(\xi) = \begin{cases} 1 + \alpha_{j} R_{m}(\xi), & \xi < \xi_{j}, \\ C_{j} \xi^{-p}, & \xi > \xi_{j}, \end{cases}$$
(3.3a)

$$f_{j+1}^{*}(\xi) = \begin{cases} 1 - \alpha_{j} R_{m}(\xi), & \xi < \xi_{j+1}^{*}, \\ C_{j-1}^{*} \xi^{-p}, & \xi > \xi_{j+1}^{*}. \end{cases}$$
(3.3b)

The quantities $\alpha_{j_1} C_{j_1} C_{j_{j+1}}^* \xi_{j_{j+1}}^*$ are determined from the following conditions: f_{j_1} and $f_{j_{j+1}}^*$ are positive, the function f_{j_1} has two continuous derivatives, $f_{j_{j+1}}^*$ has one, and the number of extrema of the functions equals their number. The half-sum of these functions is taken:

$$\begin{aligned} G(\xi) &= 0.5 (f_j + f_{j+1}^*) = 1, & \xi < \xi_j, \\ G(\xi) &= C_0 \xi^{-\nu} = 0.5 (C_j + C_{j+1}^*) \xi^{-\nu}, & \xi > \xi_{j+1}^* \end{aligned}$$

and their half-difference:

$$\begin{split} R(\xi) &= 0.5 \left(f_j - f_{j+1}^* \right) = \alpha_j R_m(\xi), & \xi < \xi_j, \\ R(\xi) &= 0.5 \left(C_j - C_{j+1}^* \right) \xi^{-p} = C_m \xi^{-p}, & \xi > \xi_{j+1}^*, \end{split}$$

and the following function is taken as the approximation:

$$\widetilde{y}_{mj}(\xi,\varphi) = G(\xi) + R(\xi)\cos(m\varphi),$$

which equals 1+u for small ξ and $C(\varphi)\xi^{-p}$ for large ξ , i.e. it satisfies the boundary

conditions. It is convenient to denote these approximations and the corresponding eigenfunctions by E/Mm. It follows from Eqs.(3.2) and (3.3) that for each *m* the number of approximations which we can construct equals the number of real positive zeros of the function $R_m(\xi)$:

$$N_m = j_{\max} = [a_m - [[a_m]a_m^{-1}]] + 1.$$
(3.4)

It follows from (3.5) that as *m* increases the number of approimxations N_m decreases. When $m \ge as$ the function R_m becomes monotonic, $R_m(\xi) \ne 0$ when $\xi > 0$ and it is impossible to carry out the connection and construct approximations. Therefore, the overall number of approximations E/Mm, by which the number of eigenfunctions is estimated, comprises

$$N_{\rm EM} = \sum_{0 < m < as} N_m.$$

2. Approximation of the Ei/j class. Consider the constant density case s=2. Since $\xi\partial/\partial\xi = \xi_1\partial/\partial\xi_1 + \xi_2\partial/\partial\xi_2$, where ξ_1 , ξ_2 are Cartesian coordinates, the linearized equation allows of separation of the variables ξ_1 , and ξ_2 . If the parameter $s\neq 2$, but is integral, separation is carried out with respect to the variables

$$\xi_1 = \frac{2}{s} \xi^{s/2} \cos \frac{s\varphi}{2}, \quad \xi_2 = \frac{2}{s} \xi^{s/2} \sin \frac{s\varphi}{2}.$$

The solution of Eq.(3.1) is sought in the form

$$u = \alpha_1 G(\xi_1) + \alpha_2 G(\xi_2), \qquad G(x) = M\left(-a, 0.5, \frac{\beta - \sigma - 1}{4}x^2\right),$$

where α_i, α_z are constants. Matching with the asymptotic form is carried out separately with respect to ξ_i and ξ_z . For this, by analogy with the previous case, we construct the functions

$$f_j(x) = \begin{cases} 1 + \alpha_j G(x), & |x| < \xi_{0j}, \\ C |x|^{-p}, & |x| > \xi_{0j}. \end{cases}$$

They agree in their structure with the linear approximations to the one-dimensional eigenfunctions in the plane case (one spatial variable), which are obtained in Sect.2 and depend only on one coordinate ξ_i . Their number is determined by the formula $N_i = j_{max} = [a - [[a]a^{-1}]] + 1.$

The following function is taken as the required two-dimensional approximation:

 $\tilde{y}_{i/j}(\xi_1, \xi_2) = f_i(\xi_1) f_j(\xi_2).$

At the same time for $|\xi_1| < \xi_{0i}$, $|\xi_2| < \xi_{0j}$

$$\tilde{y}_{i/j} = 1 + u + \alpha_1 \alpha_2 G(\xi_1) G(\xi_2) \approx 1 + u.$$

We shall denote these approximations and the corresponding eigenfunctions by Ei/j. The functions Ei/j and Ej/i are identical apart from an angle of rotation π/s , and we can therefore assume that $i \le j$. The total number of approximations Ei/j is estimated from the formula N=N, (N,+1)/2.

4. Architecture of the two-dimensional eigenfunctions.

The presence of approximate solutions of the selfsimilar problem enabled us to obtain many eigenfunctions numerically and thereby to confirm their existence. At the same time the functions \mathfrak{g} served as the initial approximation guaranteeing the convergence of the numerical iteration process to the corresponding eigenfunction. More than 20 eigenfunctions were obtained for different values of β , σ and s. In most cases the predictions of the linear analysis hold well in the arrangement of the maxima and the ratio of their heights.

1. Architecture of eigenfunctions of the EjMm class. All the EjMm eigenfunctions have the form of a set of maxima ("hills"), arranged near the point $\xi=0$. It is convenient to classify them by the number and arrangement of the maxima according to the values of m and j.

When j=1 the eigenfunction has *m* maxima situated on the circumference at the corners of a right *m*-gon. In the case j=2 the maxima are arranged in two layers on two concentric circumferences with *m* pieces in each at the corners of the right *m*-gons, which are rotated by an angle of π/m with respect to each other. The other E/Mm eigenfunctions are organised in a similar way: they have jm maxima situated in j layers on concentric circles at the vertices of right *m*-gons. Thus, complication of the organization occurs in two directions: an increase in the number of maxima in the layer and an increase in the number of layers. This is illustrated schematically in Fig.3 ($\beta=2.5$, $\sigma=1$, s=2). The figure also shows the shapes of the effective areas of localization of the corresponding approximations (lines of the level $\tilde{y}=0.1$). The heights of the maxima in each given eigenfunction increase monotonically as one move away from the centre.

2. Architecture of the eigenfunctions of the Ei/j class. As in the case of the EjMm eigenfunctions, we shall describe the form of the linear approximations Ei/j, which indicates the number and arrangement of the maxima and minima. We shall first consider the case s=2.



Fig.3

Fig.4

When $-\infty < x < \infty$ the function $f_i(x)$, $1 \le i \le N_i$, and *i* maxima and *i*-1 minima. The function $\tilde{y}_{i/l}(\xi_1, \xi_2)$ will have a maximum at the points at which $f_i(\xi_1)$ and $f_j(\xi_2)$ simultaneously have a maximum (the same holds for the minima). Consequently, the approximation E_i/j consists of *i* series of maxima, with *j* pieces and *i*-1 series from *j*-1 minima in each. Fig.4 illustrates schematically all the eigenfunctions E_i/j given by the linear approximation for the medium $\beta = 2.5, \sigma = 1, s = 2$.

For other values of s the maxima and minima are arranged along the lines $|\xi_1| = \text{const}$ and $|\xi_2| = \text{const}$.

It is interesting that sometimes the arrangement of the maxima in the eigenfunctions Ei/j and EjMm agrees or is similar. In this case they describe one and the same actually existing eigenfunction. The eigenfunctions ELMO and El/1, ELM4 and E2/2, ELMs and El/2 are examples in Figs.3 and 4. In the series (both horizontal and vertical) the values of the maxima increase from the middle to the edge, but strict regularity in the increase of the maxima as one moves away from the centre of the eigenfunctions is sometimes violated.

3. Results of the numerical calculations of the selfsimilar problem. We were able to construct eigenfunctions of both classes which are well described by approximations. The eigenfunction E2M6 of the medium with $\beta=3.5$, $\sigma=2$, s=2.5 is represented in Fig.5 The arrangements of the maxima and their heights are given quite well by the linear approximation. But in several cases fairly considerable deviations from the approximate solutions were observed. For example, the approximation E2M3 consists of two layers, in each of which there are up to three maxima, at the centre $\tilde{y}=1$. A significant minimum is observed at the centre of the numerically constructed eigenfunction ($\beta=3.5$, $\sigma=2$, s=4), and the maxima of the external layer were split and have a smaller height in comparison with the first layer (see Fig.6). It is possible that at least three harmonics (m=0,3,6) should be used to describe this structure in the linear approximation.



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Fig.5



Fig.6

However, not all predicted eigenfunctions are constructed. In a number of cases the iteration process did not converge and it was difficult to draw any conclusions about the existence of corresponding eigenfunctions. In the family E/Mm, without considering the one-dimensional eigenfunctions for m=0, we were able to construct about 30% of the predicted eigenfunctions. For example, in Fig.3 of nine predicted eigenfunctions two are obtained (ElM2 and ElM4, which we can consider as El/2 and E2/2), and for the eigenfunction E2M2 the iteration process converged to the eigenfunction E2/3. For the family Ei/j the results are significantly better. Thus, five of the six predicted eigenfunctions (besides E3/3 are obtained in Fig.4.

It is interesting to note that if the approximations E_jMm and E_i/j are similar, then the more often the eigenfunction is implemented, the more similar it is to E_i/j (obtained in five cases), irrespective of the coordinates in which the calculations are carried out. The reverse was only sometimes observed: when $\beta=4.5$, $\sigma=3$, s=2 the ELM4 eigenfunctions $(y(0)\approx 1)$ were implemented, and not $E_2/2$ (at the centre of the minima).

Thus, the question of the number of different classes of two-dimensional eigenfunctions and of the number of eigenfunctions in each class is still open. Confirmation of the existence of complex two-dimensional structures is the main results of the investigations.

5. Localization and conditions of formation of thermal structures.

1. The numerical calculations of the Cauchy problem for the two-dimensional and threedimensional heat conduction equations /17, 39/, and also the numerous analyses of the one-dimensional problem, showed the following /10, 16/. Selfsimilar solutions are unstable with respect to the initial data in the Lyapunov sense. Small perturbations of the initial distribution lead to a small change in the peaking time. This, in turn, leads to an arbitrarily between the solutions beginning from some instant of time which is close large difference to the moment of peaking. The stability of selfsimilar solutions in a regime with peaking was first investigated in /10, 33/, where the concept of the C-stability (structural) of selfsimilar solutions is introduced, i.e. stability in the sense of arrival at a selfsimilar regime. In the LS-regime only the first eigenfunction, which has one maxima in all, is a C-stable selfsimilar solution. Complex eigenfunctions do not possess this property, therefore, in order to obtain them, we need to excite the medium to resonacne, i.e. to formulate as the initial conditions a profile which is selfsimilar or fairly close to it. For an approximation to the moment of peaking these structures as a rule decompose into separately combusting simple structures or merge into one. However the time that the complex structure exists is large: $\Delta t \sim 0.99 t_{\rm f}$, and the temperature is able to increase by a factor of 10-100.

The principle of the amalgamation of simple structures into complex ones and the principle of superposition in non-linear systems /9/ is connected with the representation of the complex structure. All the eigenfunctions of the selfsimilar problem (besides the first, which describes a simple structure with one maximum) can be considered as an amalgamation of several simple structures (each of which has its moment of peaking t_{li} , as if they burn in isolation). The complex structure has yet another moment of peaking, $t_l=\tau$, which is common for the whole system. This was first shown in /33/ for one-dimensional structures, where the interaction between two simple structures and one temperature maximum was shown. It is shown that if the distance l between them is greater than the so-called resonance length L_{τ} , they burn independently, if $l < L_{\tau}$ they fairly quickly merge into one, and if $l \approx L_{\tau}$, they consistently burn for a long time and form a complex structure with two maxima, corresponding to the second eignefunction. The estimate for L_{τ} in the plane case for the constant density s=2 is obtained in /14, 38/:

$$L_{\tau}^{\bullet} = \pi \left(\frac{2(\beta + \sigma + 1)}{\sigma(\beta - 1)} \right)^{\prime \prime} T_{\max}^{[\beta - (\sigma + 1)]/2}.$$

Stricter estimates are given in /21, 40/.



Fig.7

Similar analyses were made for two-dimensional structures in /17, 39/. At the same time not only the distance between the simple structures is important, but also their configuration, which must have a certain symmetry. The results of this paper confirm this conclusion.

Thus, in order that the several maxima arranged in the space develop with one moment of peaking, preserving its shape, it is necessary that their configuration corresponds to the multidimensional eigenfunction. The basic organizing principle when amalgamating simple structures into complex ones is the synchronization of the processes occurring in them (the establishment of a common moment of peaking).

2. Selfsimiar solutions in the LS-regime do not have a finite front. However, as is clear from the above drawings, they quickly arrive at an asymptotic form which corresponds to the solidified temperature profile. This fact indicates localization of the combustion process in the LS-regime. From the theorems of operator comparison and the stationary state method /18-20, 40/ there follows the localization of the simple structures with one temperature maximum. In the case of complex structures the localization is confirmed using numerical calculations /10, 17, 33/. The line of level 0.05-0.01 in fact determines the shape of the area of localization. Inside this area the combustion process is described by a selfsimilar law, which is only distorted near the boundaries. The shape of the areas of localization of the two-dimensional structures are shown in Figs.3 and 4. As we see, they have an unusual form: a cross, a star, an ellipse, a rectangle, etc.. Numerical calculations of problem (1.1), (1.2) confirmed the conclusion concerning localization and the shape of the areas of localization.

An example of this calculation is shown in Fig.7. A selfsimilar profile was taken as the initial profile. The localization structure has the type E2/2, $\beta=2.5$, $\sigma=1$, s=2. The peaking time $t_t=0.6535$. The level and section lines for the instants of time $t_t=0.0$ (Fig.7a) and $t_z=0.6250$ (Fig.7b) are given. Selfmaintenance of the structure architecture occurs; however because of random perturbations in the numerical computation the profile deviates from the selfsimilar one, denoted for $t=t_2$ by crosses. It is clear that the structures are indeed localized, and outside the area of localization up to $t=t_1$ the temperature remains equal to zero. The selfsimilar profile, preserved over a long time, is reproduced in ever smaller scales. It decomposes only when the temperature increases by a factor of roughly 50. The existence time is close to the moment of peaking.

Remark. The effect of localization of the diffusion processes is theoretically strictly proved for a wide class of media in problems with boundary regimes and for the Cauchy problem in a medium without sources (see review /15, 18, 41/; possible applications are discussed in /9, 11, 28, 29, 41-46/). The localization phenomenon also occurs in hydrodynamic processes /11, 47-49/ and in other media /11, 12/.

3. A distinctive feature of the selfsimilarity $\xi = r[A(t_t-t)^n]^{-1}$ examined when n>0 in an LS-regime is the scale contraction. The area of intensive combustion with maxima contracts to the centre, and when $t=t_t$ the limit temperature distribution $T=Cr^{-p}$, remains, which corresponds to the asymptotic form. Thus, the asymptotic form as $\xi \to \infty$ gives a representation about the future structure $(t \to t_t)$. On the other hand, the behaviour of the selfsimilar solution at the centre for ξ , which is close to zero, corresponds to the past of the structure, when larger values of r correspond to small ξ and, correspondingly, the area of r-non-monotony has larger dimensions. Thus, examining the character of the processes at the present in different spatial parts of dissipative structures with a space-time invariant of the $\xi = r[A(t_t-t)^n]^{-1}$ type, we can judge their behaviour tendencies and the organization of processes in them in the past and in the future.

Conclusion.

At the present time there is great interest in studying the asymptotic forms of non-linear processes, since they exhibit the most characteristic features of processes in different media. Dissipative structures are often such asymptotic forms: for example, spectra of stationary and oscillating solutions in /1-3/, and travelling waves /3-5/, /31/. These asymptotic forms, as a rule, are invariant-group, selfsimilar solutions /7, 50, 51/.

In this paper we have investigated the emergence, development and significance of complex multidimensional thermal structures in a medium with a non-linear thermal conductivity and a volume heat source, and how their number and type depend on the parameters of the non-linear medium. A spectrum of structures was constructed for selfsimilarity of the form $\xi = axt^n$. A spectrum of shapes which are only characteristic for a given non-linear medium and are defined only by its parameters was obtained. The fundamental statement is the proof of the fact that a medium, by virtue of non-linearity, contains in potential form all the structures which can exist in it at a developed asymptotic stage and which the processes occurring in it can approach, as they approach attractors. It is interesting that these results are similar to the ideas of the ancient philosophers about the potential undeveloped shapes contained in a single originally homogeneous substance.

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