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A NEW CLASS OF ASYMPTOTIC "EIGENFUNCTIONS" FOR CAUCHY'S PROBLEM FOR A NON-LINEAR PARABOLIC EQUATION*

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The asymptotic behaviour of the solutions of Cauchy's problem for a non-linear parabolic equation describing the diffusion of a body in a continuous medium with absorption of energy is investigated. The conditions under which the solution of the problem converges as $t\to+\infty$ to any spatially-non-uniform selfsimilar solution of a first-order equation are established. A description of a finite-dimensional set of these solutions (asymptotic "eigenfunctions") is given.

Introduction

This paper contains a description of a family of asymptotic eigenfunctions of Cauchy's problem for a semilinear parabolic equation which is new compared with /1–3/ and which describes the diffusion of a body in a medium with non-linear absorption of energy:

$$\mathbf{B}(u)\equiv u_t-\Delta u+u^\beta=0,\qquad t>0,\quad x\in\mathbf{R}^N, \tag{1}$$

$$u(0,x)=u_0(x)\geqslant0,\qquad x\in\mathbf{R}^N,\quad u_0\in C(\mathbf{R}^N),\quad \sup u_0<+\infty. \tag{2}$$

Here $\beta>1$ is a constant, and $u_0(x)\to0$ as $|x|\to+\infty$. As regards the nature of the discussion, notation and terminology, this analysis is closely connected with that of /3/, where we can find a brief review of the literature on non-stationary processes of heat conduction in continuous media with volume absorption.

Eq.(1) is one of the few non-linear parabolic equations in $\mathbf{R}^N$ with asymptotic behaviour of the solutions as $t\to+\infty$ which has been investigated in some detail. At the present time there is a fairly complete, and - in some domains of the parameters β and N - an apparently exhaustive description of the main elements of the attractor of Eq.(1) as sets of asymptotically stable states (eigenfunctions), to each of which its own set of attractions in the space of the initial functions corresponds.

2. Brief description of the attractor of the equation. Fundamental result.

The asymptotic behaviour as $t\to+\infty$ of the solution of problem (1), (2) depends, besides β and N, only in fact on the manner in which initial function $u_0(x)$ approaches zero as $|x|\to+\infty$ (see /1–3/). If we arrange the results obtained in these papers systematically, this picture emerges in the simplified discussion (we can set up an accurate formulation of the results discussed below using the references presented here).

Suppose

$$u_0(x)\sim|x|^{-\alpha},\qquad |x|\to+\infty, \tag{3}$$

where $\alpha>0$ is a constant. The following cases are possible.

Case 1: $\alpha=2/(\beta-1)$. We can call this case the resonance case, the behaviour of $u=u(t,x)$ as $t\to+\infty$ is defined by all three terms of Eq.(1) and in the final analysis $u(t,x)$ converges in a special norm to the appropriate selfsimilar solution of Eq.(1) of the form

$$u_A(t,x)=t^{-1/(\beta-1)}\theta_A(\xi),\qquad \xi=x/t^{1/2}, \tag{4}$$

where $\theta_A>0$ satisfies the elliptic equation

$$\Delta_\xi\theta_A+\frac{1}{2}\nabla_\xi\theta_A\xi+\frac{1}{\beta-1}\theta_A-\theta_A^\beta=0,\qquad \xi\in\mathbf{R}^N, \tag{5}$$
$$\theta_A(\xi)\sim|\xi|^{-2/(\beta-1)},\qquad |\xi|\to+\infty.$$

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The existence and asymptotic stability of radially-symmetric solutions (5), $\theta_A=\theta_A(|\xi|)$, for arbitrary $\beta>1$ are established in /3/, and an analysis of a wider set of solutions $u_A(t, x)$ when $\beta>1+2/N$ - which are asymmetric with respect to ξ - is made in /1/. Note that the structure of the set $\{\theta_A(\xi)\}$ is substantially different when $\beta\geq1+2/N$ and $\beta<1+2/N$ (see /3/). This superimposes certain requirements on the technique for proving the asmyptotic stability of selfsimilar solutions.

Case 2: $\alpha>2/(\beta-1)$. a. If $\beta<1+2N$ and, for example, $u_0(x)\sim\exp(-|x|^2/4)$ as $|x|\to+\infty$ (formally this corresponds to $\alpha=+\infty$ in (3)), then the asymptotic form of $u(t, x)$ is described by a selfsimilar solution of the form (4), where $\theta_A=\theta_A(|\xi|)$ satisfies Eq.(5), whilst

$$\theta_A(\xi)\sim|\xi|^{1/(\beta-1)-N/2}\exp(-|\xi|^2/4), \qquad |\xi|\to+\infty$$

(see /3/). When $\beta\geq1+2/N$ the function $\theta_A(|\xi|)$ with a similar "exponential" asymptotic form does not exist.

b. If $\beta>1+2/N$, then when $\alpha>2/(\beta-1)$ the volume absorption is unimportant as $t\to+\infty$ and the asymptotic behaviour of the solution $u(t, x)$ is described by the space-time structure of the selfsimilar solutions of the linear heat-conduction equation

$$u_t=\Delta u, \qquad t>0, \quad x\in\mathbf{R}^N. \tag{6}$$

Namely, when $u_0\in L^1(\mathbf{R}^N)$ (this means that $\alpha>N$) the fundamental solution

$$u_a(t,x)=\frac{C_0}{(4\pi t)^{N/2}}\exp\left(-\frac{|x|^2}{4t}\right),$$

where the constant $C_0>0$ depends on $u_0(x)$ (see /2/, and also /3/, where a close result is obtained using another method) is this soluiton. When $2/(\beta-1)<\alpha<N(u_0\overline{\in}L^1(\mathbf{R}^N))$ the asymptotic nature of the process is described by other selfsimilar solutions of Eq.(6):

$$u_a(t, x)=t^{-\alpha/2}\theta_a(\xi), \qquad \xi=x/t^{1/2},$$

$$\Delta_\xi\theta_a+\frac{1}{2}\nabla_\xi\theta_a\xi+\frac{\alpha}{2}\theta_a=0, \quad \xi\in\mathbf{R}^N, \quad \theta_a(\xi)\sim|\xi|^{-\alpha}, \quad |\xi|\to+\infty.$$

The proof of the convergence $u\to u_a$, $t\to+\infty$ in this case is carried out, in /1/ and /3/, using different methods (the case $\alpha=N$ is also discussed in /3/).

c. For the critical value of the parameter $\beta=1+2/N$, as shown in /3/, the occurrence of a non-trivial approximate selfsimilar solution is possible. If, for example, $u_0=u_0(|x|)\sim\exp(-|x|^2)$, $|x|\to+\infty$ (in this case $u_0\in L^1(\mathbf{R}^N)$), then $u(t, x)$ can converge as $t\to+\infty$ only to an approximate selfsimilar solution of the following fairly unusual form:

$$u_a(t,x)=M_N(t\ln t)^{-N/2}\exp\left(-\frac{|x|^2}{4t}\right), \tag{7}$$

where $M_N=(N/2)^{N/2}(1+2/N)^{N^2/4}$ (see /3/). The appproximate selfsimilar solution (7) satisfies the equation

$$(u_a)_t=\Delta u_a-\frac{N}{2}\frac{u_a}{t\ln t}, \qquad t>1, \quad x\in\mathbf{R}^N, \tag{8}$$

which differs from both the original equation and from (6). For this case an estimate of the range of the soluition of the problem was obtained earlier in /2/: $u(t, x)\geq c(t\ln t)^{-N/2}$ for large t in each compactum from $\mathbf{R}^N$.

Case 3: $\alpha<2/(\beta-1)$. This paper analyses this case. The following /2/ has been well-known until now: for any $\beta>1$ we have $t^{1/(\beta-1)}u(t, x)\to\theta_H=(\beta-1)^{-1/(\beta-1)}$ uniformly in each set of the form $P_a(t)=\{x\in\mathbf{R}^N|\ |x|\leq at^{1/2}\}$, $a>0$. However, this result in fact gives only an estimate of the range of the solution $(u(t, x)\sim\theta_H t^{-1/(\beta-1)}$ in $P_a(t)$ for large t) and in no way reflects the space-time evolution of the initial perturbation $u_0(x)$.

This paper shows that when $\alpha<2/(\beta-1)$ in (3) and for some additional limitations a peculiar "degeneration" of Eq.(1) occurs at the asymptotic stage of the process and as $t\to+\infty$ the diffusion term is unimportant. As a result the asymptotic feature $u(t, x)$ is described by the selfsimilar solutions of the first-order equation

$$u_t=-u^\beta, \qquad t>0, \quad x\in\mathbf{R}^N, \tag{9}$$

which have the form ($T=\text{const}\geq0$)

$$u_a(t,x)=(T+t)^{-1/(\beta-1)}f_a(\xi), \qquad \xi=\frac{x}{(T+t)^{1/\alpha(\beta-1)}}. \tag{10}$$

Substitution of Eq.(10) into (9) gives the following equation for the function $f_a>0$

$$\frac{1}{\alpha(\beta-1)}\nabla_\xi f_a\xi+\frac{1}{\beta-1}f_a-f_a^\beta=0, \ \xi\in\mathbf{R}^N, \ f_a(\xi)\sim|\xi|^{-\alpha}, \ |\xi|\to+\infty. \tag{11}$$

Eq.(11) is easily integrated:

$$f_a(\xi)=[(\beta-1)+C^{1-\beta}(\xi/|\xi|)|\xi|^{\alpha(\beta-1)}]^{-1/(\beta-1)}, \qquad \xi\in\mathbf{R}^N, \tag{12}$$

where $C(\omega)>0$ is, generally speaking, an arbitrary fairly smooth function determined on the unit sphere $S=\{\xi\in\mathbf{R}^N|\ |\xi|=1\}$. The non-empty set of attraction in the space of the initial functions $\{u_0(x)\}$ satisfies each solution (10), (12), which is an approximate selfsimilar solution in relation to the initial Eq.(1), and the solution of Cauchy's problem $u(t, x)$

converges to an approximate selfsimilar solution in the following sense:

$$f_T(t,\xi)\equiv(T+t)^{1/(\beta-1)}u(t,\xi(T+t)^{1/\alpha(\beta-1)})\to f_a(\xi),\qquad t\to+\infty, \tag{13}$$

where the selfsimilar representation $u(t,x)$, determined in accordance with the space-time structure of the approximate selfsimilar solution (10), is denoted by $f_T(t,\xi)$. The set of asymptotically stable approximate selfsimilar solutions (10), which are part of the attractor of Eq.(1), is infinite-dimensional when $N>1$ due to the arbitrariness in the choice of the function $C(\omega)$ in (12). In the one-dimensional case the set of functions (12) is two-dimensional. Any function $f_a(\xi)$ from (12), with respect to each direction $\xi=\omega|\xi|$, has, generally speaking, different asymptotic features:

$$f_a(\xi)\cong C(\omega)|\xi|^{-\alpha},\qquad |\xi|\to+\infty. \tag{14}$$

The function

$$f_a(\xi)=\left[(\beta-1)+\left(\sum_{i=1}^{N}k_i\xi_i^2\right)^{\alpha(\beta-1)/2}\right]^{-1/(\beta-1)} \tag{15}$$

where $k_i>0$, $i=1,2,\ldots,N$ are arbitrary constants, is a characteristic representation of the family $\{f_a\}$; it corresponds to

$$C\left(\frac{\xi}{|\xi|}\right)=\left(\sum_{i=1}^{N}k_i\xi_i^2/|\xi|^2\right)^{-\alpha/2}.$$

The functions (15) from an N-dimensional set.

Note that it follows directly from (13) that at each point $x\in\mathbf{R}^N$ a stabilization to the spatially-homogenous solution $t^{1/(\beta-1)}u(t,x)\to\theta_H$, $t\to+\infty$ occurs, which agrees with the conclusion in /2/. Besides this, however, the limiting Eq.(13) together with (12) gives a clear representation of the nature of the asymptotic evolution of not only the "range" of the solution of Cauchy's problem, but also its effective spatial width. It follows from (13), (10) that the effective width of the thermal structure for fairly large t can be estimated using the formula $x_{eff}(t)\sim t^{1/\alpha(\beta-1)}$, whilst it differs with respect to each direction $\xi=\omega|\xi|$ as a function of the form of the function $C(\omega)$ in (12). Eq.(13) thereby describes the law of formation as $t\to+\infty$ of the asymptotic eigenfunction in all the space $\mathbf{R}^N$ (unlike the "local" result /2/).

In conclusion we turn our attention, once more, to the curious "transformations" which the parabolic Eq.(1) can undergo at the asymptotic stage. Depending on the quantities β, N and the initial function $u_0(x)$, it can be transformed into an equation of three types: into a linear equation without a sink (6), into a first-order equation with diffusion (9) and - in the critical case $\beta=1+2/N$ - into Eq.(8) with a linear sink.

3. Proof of the asymptotic stability of degenerate approximate selfsimilar solutions.

The main aim of this section is to determine the conditions under which the limiting Eq.(13) holds. For this, without aiming for maximum generality, we shall use an extremely simple method of proof which is close to that used in /3, Sect.3/.

Theorem. Suppose

$$\frac{(2-N)_+}{\beta-1}<\alpha<\min\left\{\frac{2}{\beta-1},N\right\} \tag{16}$$

and, in addition, in (12)

$$C(\omega)>0,\quad \omega\in S,\qquad C(\omega)\in C^2(S). \tag{17}$$

Then $p>0$ exists, such that for any initial functions $u_0(x)$, conditionally satisfying, for some $T>0$,

$$|u_0(\cdot)-u_a(0,\cdot)|^{(p+1)/2}\in H^1(\mathbf{R}^N) \tag{18}$$

(u_a is determined in accordance with (10), (12)), the limiting equation

$$f_T(t,\xi)\equiv(T+t)^{1/(\beta-1)}u(t,\xi(T+t)^{1/\alpha(\beta-1)})\to f_a(\xi) \tag{19}$$

holds in $L^{p+1}(\mathbf{R}^N)$ as $t\to+\infty$.

An estimate of the rate of convergence will be obtained when proving the theorem. For convenience we shall first formulate the following lemma:

Lemma. Suppose (17) and the following conditions hold:

$$N-(\alpha+2)(p+1)<0,\qquad N+[\alpha(\beta-1)-2](p+1)>0,\qquad p>0. \tag{20}$$

Then $\Delta_\xi f_a\in L^{p+1}(\mathbf{R}^N)$.

Proof. By virtue of (17), the singularities of the integrand in $\|\Delta_\xi f_a\|^{p+1}_{L^{p+1}(\mathbf{R}^N)}$ only emerge at the points $|\xi|=\infty$ and $\xi=0$. Two conditions (20) guarantee their integrability (it is sufficient to look at the asmyptotic features: as $|\xi|\to+\infty$ it has the form (14), if $\xi\to0$, then $f_a(\xi)\cong\theta_H-\theta_H(\beta-1)^{-2}C^{1-\beta}(\xi/|\xi|)|\xi|^{\alpha(\beta-1)}$).

Proof of the theorem. The function $z=u-u_a$ satisfies in $\mathbf{R}_+\times\mathbf{R}^N$ the equation

$$z_t=\Delta z-zg(t,x)+h(t,x), \tag{21}$$

where

$$g(t,x)=\beta\int_0^1 [\eta u(t,x)+(1-\eta)u_a(t,x)]^{\beta-1}\,d\eta,\quad h(t,x)=\Delta u_a(t,x).$$

Eq.(21) is a "linear" equation in z. Note that $g(t,x)>0$ in $\mathbf{R}_+^1\times\mathbf{R}^N$ and when conditions (20) hold we will have $h(t,\cdot)\in L^{p+1}(\mathbf{R}^N)$ for each $t\geq 0$. We shall confine ourselves, below, to a formal proof of the theorem, using the natural assumptions concerning the regularity of the generalized solution of the linear Eq.(21) (see /1, 2/).

Suppose $z(0,x)\in L^{p+1}(\mathbf{R}^N)$, $\nabla|z(0,x)|^{(p+1)/2}\in L^2(\mathbf{R}^N)$ for some $p>0$ (see (18)). We scalarly multiply Eq.(21) in $L^2(\mathbf{R}^N)$ by $|z|^{p-1}z$. Then, integrating by parts on the right-hand side, we obtain

$$\frac{1}{p+1}\frac{d}{dt}\|z\|^{p+1}_{L^{p+1}(\mathbf{R}^N)}=-\frac{4p}{(p+1)^2}\|\nabla|z|^{(p+1)/2}\|^2_{L^2(\mathbf{R}^N)}+$$

$$(|z|^{p-1}z,\Delta u_a)-(|z|^{p+1},g(t,x))\leq(|z|^{p-1}z,\Delta u_a),\quad t>0.$$

Suppose coniditions (20) hold. Using Hölder's inequality

$$(|z|^{p-1}z,\Delta u_a)\leq\|z\|^p_{L^{p+1}(\mathbf{R}^N)}\|\Delta u_a\|_{L^{p+1}(\mathbf{R}^N)},$$

and also the specific form of the approximate selfsimilar solution u_a (see (10)), by virtue of which

$$\|\Delta u_a(t)\|_{L^{p+1}(\mathbf{R}^N)}=(T+t)^\delta\|\Delta_\xi f_a\|_{L^{p+1}(\mathbf{R}^N)},\quad \delta=\frac{N-(\alpha+2)(p+1)}{\alpha(\beta-1)(p+1)}>0,$$

we arrive at the estimate:

$$\frac{d}{dt}\|z(t)\|_{L^{p+1}(\mathbf{R}^N)}\leq(T+t)^\delta m_a,\quad t>0,\quad m_a=\|\Delta_\xi f_a\|_{L^{p+1}(\mathbf{R}^N)}<+\infty.$$

Hence it directly follows that

$$\|z(t)\|_{L^{p+1}(\mathbf{R}^N)}=\begin{cases}O(t^{\delta+1}), & \delta>-1,\\ O(\ln t), & \delta=-1,\\ O(1), & \delta<-1,\end{cases}$$

as $t\to+\infty$. Finally, bearing in mind that, by virtue of (10) and the definition of the selfsimilar representation f_T,

$$\|z(t)\|_{L^{p+1}(\mathbf{R}^N)}=(T+t)^\varepsilon\|f_T(t,\cdot)-f_a(\cdot)\|_{L^{p+1}(\mathbf{R}^N)},$$

$$\varepsilon=\frac{1}{\beta-1}\left[\frac{N}{\alpha(p+1)}-1\right],$$

we obtain a final estimate of the rate of convergence in (19):

$$\|f_T(t,\cdot)-f_a(\cdot)\|_{L^{p+1}(\mathbf{R}^N)}=\begin{cases}O(t^{\delta+1-\varepsilon}), & \delta>-1,\\ O(t^{-\varepsilon}\ln t), & \delta=-1,\\ O(t^{-\varepsilon}), & \delta<-1,\end{cases}\tag{22}$$

as $t\to+\infty$. We shall require that $\varepsilon>0$, i.e.

$$N-\alpha(p+1)>0.\tag{23}$$

Then, as follows from (22), when $\delta\leq-1$ convergence always occurs. If $\delta>-1$, then

$$\delta+1-\varepsilon=\frac{1}{\alpha}\left[\alpha-\frac{2}{\beta-1}\right]<0,$$

which also guarantees the necessary result.

It remains to verify that the set of inequalities (23), (20) for the values α from (16) always has the solution $p>0$. We shall rewrite then in the equivalent form

$$p<\frac{N-\alpha}{\alpha},\quad p>\frac{N-(\alpha+2)}{(\alpha+2)},\quad p<\frac{N-[2-\alpha(\beta-1)]}{2-\alpha(\beta-1)}.\tag{24}$$

From the first inequality there directly follows the necessity for the limitation $\alpha<N$ and, consequently, of the right-hand inequality (16). The third inequality (24) signifies that we must have $2-\alpha(\beta-1)<N$, i.e. $\alpha>(2-N)_+(\beta-1)^{-1}$. It is easy to see that the second and third inequalities (24) when $N>\alpha+2$ do not contradict each other, such that the required value $p>0$ exists. This completes the proof.

Note 1. The limitation $C(\omega)>0$ in (17) is, generally speaking, unimportant. For example in the uniform case the theorem also determines the conditions of stabilization to the approximate selfsimilar solution (10), where $f_a(\xi)$ can have, in particular, the "exotic" form:

$$f_a(\xi)=\begin{cases}(\beta-1)^{-1/(\beta-1)}, & \xi\leq 0,\\ [(\beta-1)+\xi^{\alpha(\beta-1)}]^{-1/(\beta-1)}, & \xi>0.\end{cases}$$

Here $C(+1)=1$, $C(-1)=0$, $S=\{|x|=1\}$. The limitations (16) on the quantity α when $N=1$ are as follows if $1<\beta\leq 3$, then $1/(\beta-1)<\alpha<2/(\beta-1)$, and if $\beta>3$, then $1/(\beta-1)<\alpha<1$.

Note 2. When proving the thoerem the linearity of the operator $\partial/\partial t-\Delta$ in (1) is not used. Therefore, using the same method, we can carry out an analogous investigation of the asymptotic behaviour of the solutions of the quasilinear equation $u_t=\Delta u^{\sigma+1}-u^\beta$, where $\sigma>0, \beta>\sigma+1$ are constants.

We shall make one more observation. The attractor of the heat conduction equation with the runoff (1) in $\mathbf{R}^N$ when $N>1$ is infinite-dimensional (when $N=1$ it is at least two-dimensional). Moreover, the set of selfsimilar solutions of the form (4) is also infinite-dimensional. Coniser, now, the heat conduction equation with the heat source $u_t=\Delta u^{\sigma+1}+u^\beta$, $t>0$, $x\in\mathbf{R}^N$, where $\sigma>0$, $\beta>\sigma+1$ are constants. It also permits selfsimilar solutions which evolve in time in the mode with peaking:

$$u_A(t,x)=(T_0-t)^{-1/(\beta-1)}\theta(\xi), \qquad \xi=\frac{x}{(T_0-t)^m}, \tag{25}$$

$m=[\beta-(\sigma+1)]/2(\beta-1)$, $T_0>0$ is a constant (the peaking time). This is the eigenfunction of the combustion of a non-linear dissipative medium (see the review in /4/). The function $\theta(\xi)\geqslant 0$, $\theta\to 0$ as $|\xi|\to+\infty$ satisfies in $\mathbf{R}^N$ the elliptic equation

$$\Delta_\xi\theta^{\sigma+1}-m\nabla_\xi\theta\xi-\frac{1}{\beta-1}\theta+\theta^\beta=0. \tag{26}$$

When $\beta>\sigma+1$, i.e. when $m>0$ it differs from a quasilinear analogue of Eq.(5) only by the signs for the last three terms. But nevertheless the set of solutions of Eq.(26) is, to all appearances, discrete. For $N=1$ this is established in /5–7/, when $N>1$ the functions $\theta(\xi)$ are constructed numerically in /8/ (see also the list of publications in /4, 3/). They can have an extremely diverse spatial structure.

Thus, the principles for organizing the attractor of the quasilinear parabolic equations of heat conduction with a sink and source are essentially different. The asymptotic stability of the unbounded sefsimilar solutions (25) when $\beta\in(1,\sigma+1]$ is proved in /9/.

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