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# ON DIFFERENCE SOLUTIONS OF A CLASS OF QUASI-LINEAR PARABOLIC EQUATIONS*

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The properties of implicit difference schemes for quasi-linear parabolic equations of non-linear heat conduction with a source are studied. The sufficient conditions are found for the global solvability and uniqueness of the solution of the difference problem. It is shown that the global difference solution converges by a passage to the limit to the generalized solution of the initial differential problem.

The present paper, which is a continuation of /1/, considers the difference solutions of the following boundary value problem for a quasi-linear parabolic equation of non-linear heat conduction with a source:

$$\frac{\partial u}{\partial t}=\frac{\partial^2}{\partial x^2}(u^{\sigma+1})+u^\beta,\qquad t>0,\quad x\in(0,l), \tag{1}$$

$$u(0,x)=u_0(x)\geqslant 0,\quad 0<x<l,\qquad u_0\in C([0,l]), \tag{2}$$

$$u(t,0)=u(t,l)=0,\qquad t\geqslant 0, \tag{3}$$

$\sigma>0$ and $\beta>1$ are fixed constants.

Problem (1)–(3) is associated with the implicit (non-linear) difference scheme /2/

$$\frac{\hat{u}-u}{\tau}=(\hat{u}^{\sigma+1})_{\bar{x}x}+\hat{u}^\beta,\qquad (t,x)\in\omega_\tau\times\omega_h, \tag{4}$$

$$u^0=u_{0h},\quad x\in\omega_h,\qquad \hat{u}\in H_h,\quad t\in\omega_\tau. \tag{5}$$

Here and throughout, we use the same notation as in /1,2/.

In /1/ we examined the conditions for scheme (4), (5) to be solvable at a fixed time layer, and we also showed that, if $\beta\geqslant\sigma+1$, the difference solution may not exist, may not be unique, and may increase without limit during a finite time.

Below we obtain the sufficient conditions for global solvability and uniqueness of the solution of problem (4), (5), and we also show that, as $\tau,h\to 0$, the difference solution converges to the generalized solution of the initial differential problem (1)–(3).

**5. Global solvability of the difference problem and passage to the limit with $\beta\leqslant\sigma+1$.** In this section we obtain the conditions for the global solvability of problem (4), (5) in the case when $\beta\leqslant\sigma+1$. (Here and in Sect.6, the mesh $\omega_\tau$ is assumed to be uniform.)

*Theorem 6.* Let $\beta<\sigma+1$. Then, with sufficiently small $\tau$, the difference problem has a global solution (i.e., for any $T>0$), the solution being unique.

If $\beta=\sigma+1$, a similar claim holds under the supplementary assumption

$$\lambda_1^h=\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2l}\right)>1. \tag{5.1}$$

Then, on the basis of Theorem 6, we can prove the convergence of the difference scheme (4), (5) as $\tau,h\to 0$, and at the same time we prove:

*Theorem 7.* Let the initial function $u_0(x)$ in (2) be such that $u_0^{\sigma+1}\in H_0^1(0,l)$. Then, with $\beta<\sigma+1$, the differential problem (1)–(3) has a solution satisfying the inclusions.

$$u\in L^\infty(0,T;L^{\sigma+2}(0,l)),\qquad u^{\sigma+1}\in L^\infty(0,T;H_0^1(0,l)), \tag{5.2}$$

$$\frac{\partial u}{\partial t}\in L^2(0,T;H^{-1}(0,l)),\qquad \frac{\partial}{\partial t}u^{1+\sigma/2}\in L^2(0,T;L^2(0,l)). \tag{5.3}$$

If $\beta=\sigma+1$, a similar claim holds under the supplementary assumption

$$\lambda_1^0=(\pi/l)^2>1. \tag{5.4}$$

To prove these theorems, we require some preliminary lemmas.

*Lemma 2.* For all $\xi,\eta\in\mathbf{R}_+^1$ we have

$$(\xi^{\sigma+1}-\eta^{\sigma+1})\xi^\beta\leqslant\frac{\sigma+1}{\beta+\sigma+1}(\xi^{\beta+\sigma+1}-\eta^{\beta+\sigma+1})+C_1[\max\{\xi,\eta\}]^{\beta-1}(\xi^{1+\sigma/2}-\eta^{1+\sigma/2})^2, \tag{5.5}$$

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where $C_1 = C_1(\sigma, \beta) > 0$ is a constant.

*Lemma 3.* Given any mesh function $v_h \in H_h$,

$$|v_h|_C \leqslant A_3 \| v_h^{\sigma+1} \|_{h,2}^{1/(\sigma+1)}, \qquad A_3 = l^{1/2(\sigma+1)}. \tag{5.6}$$

If the function $v(x) \geqslant 0$ is such that $v^{\sigma+1} \in H_0^1(0, l)$, then estimate (5.6) holds with $h=0$.

Lemma 3 is proved in /2,3/, while inequality (5.5) is proved directly.

*Proof of Theorems 6 and 7.* We fix arbitrary $T>0$. Let $u_0^{\sigma+1}(x) \in H_0^1(0, l)$.

1. We first consider the case $\beta < \sigma+1$. By Theorem 1 (see /1/), with $\beta < \sigma+1$, scheme (4) is solvable for any $\tau$. Multiplying scalarly both sides of (4) by $\hat{u}^{\sigma+1}$ and then using the estimate /4/

$$(\xi - \eta)\xi^{\sigma+1} \geqslant \frac{1}{\sigma+2}(\xi^{\sigma+2} - \eta^{\sigma+2}), \qquad \xi, \eta \in \mathbf{R}_+^1,$$

we obtain

$$\frac{1}{\sigma+2}\frac{1}{\tau}(|\hat{u}|_{h,\sigma+2}^{\sigma+2} - |u|_{h,\sigma+2}^{\sigma+2}) + \|\hat{u}^{\sigma+1}\|_{h,2}^{2} \leqslant |\hat{u}|_{h,\beta+\sigma+1}^{\beta+\sigma+1}. \tag{5.7}$$

By means of the estimate /2/

$$|v_h|_{h,\beta+\sigma+1}^{\beta+\sigma+1} \leqslant A_0 \|v_h^{\sigma+1}\|_{h,2}^{(\beta+\sigma+1)/(\sigma+1)}, \qquad v_h \in H_h, \qquad A_0 = l^{1+(\beta+\sigma+1)/2(\sigma+1)},$$

and Young's inequality /3/, we obtain

$$|\hat{u}|_{h,\beta+\sigma+1}^{\beta+\sigma+1} \leqslant \frac{1}{2}\|\hat{u}^{\sigma+1}\|_{h,2}^{2} + A_4, \tag{5.8}$$

$$A_4 = \frac{[(\sigma+1)-\beta]}{2(\beta+\sigma+1)}\left[\frac{\sigma+1}{A_0(\beta+\sigma+1)}\right]^{2(\sigma+1)/[\beta-(\sigma+1)]}.$$

In view of (5.8), inequality (5.7) takes the form

$$\frac{1}{\sigma+2}\frac{1}{\tau}|\hat{u}|_{h,\sigma+2}^{\sigma+2} + \frac{1}{2}\|\hat{u}^{\sigma+1}\|_{h,2}^{2} \leqslant A_4 + \frac{1}{\tau}\frac{1}{\sigma+2}|u|_{h,\sigma+2}. \tag{5.9}$$

From (5.9) we obtain the estimate

$$\max_{0 \leqslant j \leqslant N} |u^{j+1}|_{h,\sigma+2}^{\sigma+2} \leqslant A_4 T(\sigma+2) + |u_{0h}|_{h,\sigma+2}^{\sigma+2} \leqslant A_5 \tag{5.10}$$

and by (5.6), the inequality

$$\max_{0 \leqslant j \leqslant N} |u^{j+1}|_C \leqslant A_3\left[2A_4 + \frac{2}{\tau}\frac{1}{(\sigma+2)}A_5\right]^{1/2(\sigma+1)}. \tag{5.11}$$

To obtain the other a priori estimates, we multiply (4) scalarly by $(\hat{u}^{\sigma+1} - u^{\sigma+1})/\tau$ and then use inequality (5.5), along with the inequality $(\xi^{\sigma+1} - \eta^{\sigma+1})(\xi - \eta) \geqslant C_2(\xi^{1+\sigma/2} - \eta^{1+\sigma/2})^2$, valid for all $\xi, \eta \in R_+^1$, where $C_2 = C_2(\sigma) > 0$ is a constant /4/. As a result, we obtain

$$C_2\left|\frac{\hat{u}^{1+\sigma/2} - u^{1+\sigma/2}}{\tau}\right|_{h,2}^{2} \leqslant -\frac{1}{2\tau}(\|\hat{u}^{\sigma+1}\|_{h,2}^{2} - \tag{5.12}$$

$$\|u^{\sigma+1}\|_{h,2}^{2}) + \frac{\sigma+1}{\beta+\sigma+1}\frac{1}{\tau}(|\hat{u}|_{h,\beta+\sigma+1}^{\beta+\sigma+1} - |u|_{h,\beta+\sigma+1}^{\beta+\sigma+1}) +$$

$$C_1\tau[\max\{|\hat{u}|_C, |u|_C\}]^{\beta-1}\left|\frac{\hat{u}^{1+\sigma/2} - u^{1+\sigma/2}}{\tau}\right|_{h,2}^{2}.$$

When obtaining (5.12) we have also used the fact that $((\hat{u}^{\sigma+1})_{\bar{x}x}, \hat{u}^{\sigma+1} - u^{\sigma+1})_h \geqslant -(\|\hat{u}^{\sigma+1}\|_{h,2}^{2} - \|u^{\sigma+1}\|_{h,2}^{2})/2$, by virtue of the inequality $\xi(\xi-\eta) \geqslant -(\xi^2 - \eta^2)/2$, $\xi, \eta \in \mathbf{R}_+^1$.

We now choose so large an $N$ that $C_1\tau[\max\{|\hat{u}|_C, |u|_C\}]^{\beta-1} \leqslant C_2/2$ for all $0 \leqslant j \leqslant N$, i.e., (see (5.11))

$$\tau\left[2A_4 + \frac{2}{\tau}\frac{1}{(\sigma+2)}A_5\right]^{(\beta-1)/2(\sigma+1)} \leqslant \frac{C_2}{2C_1}A_3^{1-\beta}. \tag{5.13}$$

It is easily seen that, in the case $\beta < \sigma+1$, this is always possible. Then, summing Eq. (5.12) over all $j$ from $0$ to $N$ and applying Young's inequality (see e.g., /3/), we obtain

$$\frac{C_2}{2}\sum_{j=0}^{N}\tau\left|\frac{\hat{u}^{1+\sigma/2} - u^{1+\sigma/2}}{\tau}\right|_{h,2}^{2} + \frac{1}{2}\|(u^{N+1})^{\sigma+1}\|_{h,2}^{2} \leqslant$$

$$\frac{\sigma+1}{\beta+\sigma+1}|u^{N+1}|_{h,\beta+\sigma+1}^{\beta+\sigma+1} + \frac{1}{2}\|u_{0h}^{\sigma+1}\|_{h,2}^{2} - \frac{\sigma+1}{\beta+\sigma+1}|u_{0h}|_{h,\beta+\sigma+1}^{\beta+\sigma+1} \leqslant \frac{1}{4}\|(u^{N+1})^{\sigma+1}\|_{h,2}^{2} + A_6.$$

Hence we have the inequalities

$$\sum_{j=0}^{N} \tau \left| \frac{\hat{u}^{1+\sigma/2} - u^{1+\sigma/2}}{\tau} \right|_{h,2}^{2} \le A_7, \tag{5.14}$$

$$\max_{0\le j\le N} \|(u^{j+1})^{\sigma+1}\|_{h,2}^{2} \le A_8. \tag{5.15}$$

Thus the restriction (5.13) on the time $\tau$ step ensures global solvability of difference problem (4), (5). Notice that, using (5.6), we obtain from (5.15) the inequality

$$\max_{0\le j\le N} |u^{j+1}|_C \le A_9. \tag{5.16}$$

Let us show that, for fairly small $\tau$, the solution is unique. Let there be two solutions $\hat{u}_1$ and $\hat{u}_2$, and let $\tau$ be such that restriction (5.13) holds, and also the condition

$$\tau < \frac{1}{\beta} A_9^{1-\beta}. \tag{5.17}$$

It follows from (4) that $\hat{u}_1-\hat{u}_2=\tau(\hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1})_{\bar{x}x}+\tau(\hat{u}_1^{\beta}-\hat{u}_2^{\beta})$. Multiplying this equation scalarly by $\hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1}$, we get

$$(\hat{u}_1-\hat{u}_2, \hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1})_h = -\tau\|\hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1}\|_{h,2}^2 +$$
$$\tau(\hat{u}_1^{\beta}-\hat{u}_2^{\beta}, \hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1})_h \le$$
$$\tau\beta[\max\{|\hat{u}_1|_C, |\hat{u}_2|_C\}]^{\beta-1}(\hat{u}_1-\hat{u}_2, \hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1})_h.$$

Hence we conclude, using (5.16) and (5.17), that $(\hat{u}_1-\hat{u}_2, \hat{u}_1^{\sigma+1}-\hat{u}_2^{\sigma+1})_h=0$, i.e., $\hat{u}_1=\hat{u}_2$.

To justify the passage to the limit we need a further estimate, from (4), we have

$$\left\| \frac{\hat{u}-u}{\tau} \right\|_{h,2}^{*} \le \|(\hat{u}^{\sigma+1})_{\bar{x}x}\|_{h,2}^{*} + \|\hat{u}^{\beta}\|_{h,2}^{*}.$$

Hence, since

$$\|(\hat{u}^{\sigma+1})_{\bar{x}x}\|_{h,2}^{*} = \|\hat{u}^{\sigma+1}\|_{h,2}, \qquad \|\hat{u}^{\beta}\|_{h,2}^{*} \le (\lambda_1^h)^{-1/2}|\hat{u}|_{h,\beta}^{\beta},$$

we obtain by means of (5.15):

$$\max_{0\le j\le N} \left\| \frac{u^{j+1}-u^j}{\tau} \right\|_{h,2}^{*} \le A_{11}. \tag{5.18}$$

We introduce for simplicity the notation

$$\nabla_\tau q_\tau v_{\tau,h} = \frac{v_h^{j+1}-v_h^j}{\tau}, \qquad j\tau \le t \le (j+1)\tau.$$

From (5.10), (5.14), (5.15), (5.18), recalling the results of /4/, we obtain the following estimates:

$$q_\tau p_h (u^j)^{\sigma+1} \text{ are bounded in } L^\infty(0, T; H_0^1(0, l)), \tag{5.19a}$$
$$q_\tau p_h (u^j)^{\sigma+1} \text{ are bounded in } L^\infty(0, T; L^{(\sigma+2)/(\sigma+1)}(0, l)),$$

$$q_\tau q_h (u^j)^{\sigma+1} \text{ are bounded in } L^\infty(0, T; L^{(\sigma+2)/(\sigma+1)}(0, l)), \tag{5.19b}$$
$$q_\tau q_h (u^j)^{1+\sigma/2} \text{ are bounded in } L^\infty(0, T; L^2(0, l)),$$

$$\nabla_\tau q_\tau q_h (u^j)^{1+\sigma/2} \text{ are bounded in } L^2(0, T; L^2(0, l)), \tag{5.19c}$$
$$q_\tau p_h u^j \text{ are bounded in } L^\infty(0, T; L^{\sigma+2}(0, l)),$$

$$\nabla_\tau q_\tau p_h u^j \text{ are bounded in } L^\infty(0, T; H^{-1}(0, l)), \tag{5.19d}$$
$$[q_\tau p_h (u^j)^{\sigma+1}]_{\bar{x}x} \text{ are bounded in } L^\infty(0, T; H^{-1}(0, l)),$$

$$q_\tau q_h (u^j)^{\beta} \text{ are bounded in } L^\infty(0, T; L^{(\sigma+2)/\beta}(0, l)), \quad q_h(u^{N+1}) \text{ are bounded in } L^{\sigma+2}(0, l). \tag{5.19e}$$

These estimates, along with the compactness theorem of /5/, justify passage to the limit as $\tau, h\to 0$ (see /4,6,7/), as a result of which we prove the existence of a global solution of problem (1)–(3), satisfying inclusions (5.2), and (5.3).

2. Now consider the case $\beta=\sigma+1$. On estimating the term on the right-hand side of (5.7) with the aid of the inequality /2/

$$|v_h|_{h,2(\sigma+1)}^{2(\sigma+1)} \le \frac{1}{\lambda_1^h}\|v_h^{\sigma+1}\|_{h,2}^2, \qquad v_h \in H_h,$$

we obtain

$$\frac{1}{\sigma+2}\frac{1}{\tau}|\hat{u}|_{h,\sigma+2}^{\sigma+2} + \left(1-\frac{1}{\lambda_1^h}\right)\|\hat{u}^{\sigma+1}\|_{h,2}^2 \le \frac{1}{\sigma+2}\frac{1}{\tau}|u|_{h,\sigma+2}^{\sigma+2}.$$

Hence (see (5.1)) it follows that (5.10) holds when $A_4=0$ and that

$$\|\hat{u}^{\sigma+1}\|^2_{h,2} \leqslant \frac{1}{\sigma+2}\frac{1}{\tau}\frac{\lambda_1^h}{\lambda_1^h-1}|u_{0h}|^{\sigma+2}_{h,\sigma+2} \leqslant \frac{A_{10}}{\tau}.$$

We choose $\tau$ so small that (see (5.13))

$$\tau^{(\sigma+2)/2(\sigma+1)} \leqslant \frac{C_2}{2C_1}A_3^{-\sigma}A_{10}^{-\sigma/2(\sigma+1)}.$$

We then have the inequality

$$\frac{C_2}{2}\sum_{j=0}^{N}\tau\left|\frac{\hat{u}^{1+\sigma/2}-u^{1+\sigma/2}}{\tau}\right|^2_{h,2}+\frac{1}{2}\left(1-\frac{1}{\lambda_1^h}\right)\|(u^{N+1})^{\sigma+1}\|^2_{h,2} \leqslant \frac{1}{2}(\|u_{0h}^{\sigma+1}\|^2_{h,2}-|u_{0h}|^{2(\sigma+1)}_{h,2(\sigma+1)}),$$

whence estimates (5.14), (5.15), (5.16). The rest of the analysis is the same as in the case when $\beta<\sigma+1$.

**6. Difference stability set and passage to the limit when $\beta>\sigma+1$** In this section we obtain the sufficient conditions for the global existence of a solution of difference problem (4), (5) in the case when $\beta>\sigma+1$, which converges as $\tau, h\to 0$ to the generalized solution of problem (1)–(3).

1. We define for all $v_h\in H_h$ the functional

$$J_h(v_h)=\frac{1}{2}a_h(v_h)-\frac{\sigma+1}{\beta+\sigma+1}b_h(v_h), \tag{6.1}$$

where $a_h(v_h)=\|v_h^{\sigma+1}\|^2_{h,2}$, $b_h(v_h)=|v_h|^{\beta+\sigma+1}_{h,\beta+\sigma+1}$.

*Lemma 4.* Let $\beta>\sigma+1$. We then have the inequality

$$d_h=\inf_{\substack{v_h\in H_h\\ v_h\neq 0}}\sup_{\lambda>0} J_h(\lambda v_h) > \frac{\beta-(\sigma+1)}{2(\beta+\sigma+1)}l^{-[\beta+3(\sigma+1)]/[\beta-(\sigma+1)]}>0. \tag{6.2}$$

The proof is similar to that of the similar proposition in /8/.

We define the difference stability set $\mathscr{W}_h$ by the expression

$$\mathscr{W}_h=\{v_h\,|\,v_h\in H_h,\ 0\leqslant J_h(\lambda v_h)<d_h,\ \lambda\in[0,1]\}. \tag{6.3}$$

By (6.2), $\mathscr{W}_h$ is not an empty set.

By the construction of $\mathscr{W}_h$ we have:

*Lemma 5.* Let $\beta>\sigma+1$. Then $\mathscr{W}_h=\mathscr{W}_h^*\cup\{0\}$, where

$$\mathscr{W}_h^*=\{v_h\,|\,v_h\in H_h,\ a_h(v_h)-b_h(v_h)>0,\ J_h(v_h)<d_h\}. \tag{6.4}$$

*Note 1.* If the functions $v(x)\geqslant 0$ in (6.1)–(6.4) are such that $v^{\sigma+1}\in H_0^1(0,l)$, then Lemmas 4 and 5 remain true with $h=0$ (see /8/, and also the similar assertions in /9, 10/). In this case, instead of $J_h, \mathscr{W}_h, \mathscr{W}_h^*\ldots$ we shall write $J_0, \mathscr{W}_0, \mathscr{W}_0^*\ldots$.

2. *Theorem 8.* Let $\beta>\sigma+1$ and let the function $u_{0h}$ in (5) be such that $u_{0h}\in\mathscr{W}_h$. Then, for fairly small $\tau$, the difference problem (4), (5) has a global solution, belonging to $\mathscr{W}_h$ for all $t\in\omega_\tau$, the solution being unique.

*Note 2.* In the conditions of Theorem 8, no reduction of the step $\tau$ ever leads to unboundedness of the difference solution. Hence, by Theorem 5 (see /1/), inequality (4.4) may not be satisfied if $u_{0h}\in\mathscr{W}_h$, i.e., $1-\lambda_1^h(u_{0h},\psi_h)_h^{(\sigma+1)-\beta}\leqslant 0$, $u_{0h}\in\mathscr{W}_h$. This inequality is a further characteristic of the difference stability set.

Passing to the limit at $\tau, h\to 0$, we obtain from Theorem 8:

*Theorem 9.* Let $\beta>\sigma+1$ and let the function $u_0$ in (2) be such that $u_0\in\mathscr{W}_0$. Then the differential problem (1)–(3) has a global solution, belonging to $\overline{\mathscr{W}}_0$ for all $t\in(0,T)$, and satisfying the inclusions (5.2), (5.3).

Here, $\overline{\mathscr{W}}_0$ is the closure of $\mathscr{W}_0$ in the set $\{u\,|\,u(x)\geqslant 0,\ u^{\sigma+1}\in H_0^1(0,l)\}$.

*Note 3.* It was shown in /8/ that satisfaction of the condition $J_0(u_0)<0$, which is in a sense opposite to the inclusion $u_0\in\mathscr{W}_0$ (see (6.3)), implies that the solution of problem (1)–(3) is unbounded, and for the time of existence of a solution we have the estimate

$$T_0<\frac{\beta+\sigma+1}{(\beta-1)[\beta-(\sigma+1)]}l^{(\beta-1)/(\sigma+2)}\|u_0\|^{1-\beta}_{L^{\sigma+2}(0,l)}<+\infty.$$

*Proof of Theorems 8 and 9.* We fix an arbitrary $T>0$. Given any function $u_0(x)\in\mathscr{W}_0$, $u_0\neq 0$. Then, $u_{0h}\in\mathscr{W}_h$ for fairly small $h$. From this and Lemma 5,

$$\frac{\beta-(\sigma+1)}{2(\beta+\sigma+1)}a_h(u_{0h})\leqslant J_h(u_{0h}),$$

and hence (see Lemma 3)

$$|u_{0h}|_C\leqslant A_3\left[\frac{2(\beta+\sigma+1)J_h(u_{0h})}{\beta-(\sigma+1)}\right]^{1/2(\sigma+1)}\leqslant A_{11}.$$

Let us show that the constraint

$$\tau \leqslant \frac{\min\{1, C_2/2C_1\}}{(1+A_{11})^{\beta}+2h^{-2}(1+A_{11})^{\sigma+1}} \tag{6.5}$$

ensures that scheme (4) is solvable at each step.

We take the first step. By Theorem 2 (see condition (2.9) in /1/ with $C_0=1$), scheme (4) is solvable under condition (6.5). Then, from (5.12), recalling that

$$\tau[\max\{|u^1|_C, |u^0|_C\}]^{\beta-1} \leqslant \frac{C_2}{2C_1}$$

(notice that, by (6.5), we have $|u^1|_C \leqslant |u^0|_C+1$, see Theorem 2), we obtain

$$\frac{C_2}{2}\left|\frac{(u^1)^{1+\sigma/2}-(u^0)^{1+\sigma/2}}{\tau}\right|^2_{h,2} \leqslant \frac{1}{\tau}[J_h(u^0)-J_h(u^1)]. \tag{6.6}$$

Let us show that $u^1 \in \mathscr{W}_h$. In fact, assume that $u^1 \notin \mathscr{W}_h$. Then, since $u^1 \to u^0$ as $\tau \to 0$ and $u^0 \in \mathscr{W}_h$, we can always find $\tau$, satisfying condition (6.5), such that $u^1 \in \partial\mathscr{W}_h$. By (6.3), this means that $J_h(u^1)=d_h$. Hence we obtain a contradiction with (6.6), since, by hypothesis, $J_h(u^0)<d_h$.

Thus $u^1 \in \mathscr{W}_h$. We then find by Lemma 5 that $a_h(u^1)>b_h(u^1)$. On estimating the right-hand side of (6.6) with the aid of this inequality, we have

$$\frac{C_2}{2}\left|\frac{(u^1)^{1+\sigma/2}-(u^0)^{1+\sigma/2}}{\tau}\right|^2_{h,2}+\frac{1}{\tau}\frac{\beta-(\sigma+1)}{2(\beta+\sigma+1)}\|(u^1)^{\sigma+1}\|^2_{h,2} \leqslant \frac{1}{\tau}J_h(u^0).$$

Hence

$$\frac{\beta-(\sigma+1)}{2(\beta+\sigma+1)}\|(u^1)^{\sigma+1}\|^2_{h,2} \leqslant J_h(u^0)$$

and hence $|u^1|_C \leqslant A_{11}$. This last inequality justifies taking the next time step with oondition (6.5) on $\tau$, etc.

Notice that, for the global solvability of the problem, it is sufficient that $\tau=o(h^2)$, $h \ll 1$, in the difference stability set $\mathscr{W}_h$.

In short, when condition (6.5) holds, the difference problem has a global solution, and $\hat{u} \in \mathscr{W}_h$, $|\hat{u}|_C \leqslant A_{11}$ for all $0 \leqslant j \leqslant N$, and moreover,

$$\frac{C_2}{2}\sum_{j=0}^{N}\tau\left|\frac{\hat{u}^{1+\sigma/2}-u^{1+\sigma/2}}{\tau}\right|^2_{h,2}+\frac{\beta-(\sigma+1)}{2(\beta+\sigma+1)}\|\hat{u}^{\sigma+1}\|^2_{h,2} \leqslant J_h(u_{0h}).$$

The uniqueness, for sufficiently small $\tau$, of the uniformly bounded difference solution is proved in the same way as in Theorem 6.

The difference solution satisfies estimates (5.19), which enable us to pass to the limit as $\tau$, $h \to 0$. As a result, we establish the existence of a global solution of problem (1)–(3) with $\beta>\sigma+1$, where $u \in \overline{\mathscr{W}}_0$ for any $t>0$ by virtue of the condition. $u_t \in \mathscr{W}_h$ for $t \in \omega_\tau$.

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