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DIFFERENCE SOLUTIONS OF A CLASS OF QUASILINEAR PARABOLIC EQUATIONS. I.*

V.A. GALAKTIONOV and A.A. SAMARSKII

The properties of implicit difference schemes for quasilinear parabolic equations of non-linear heat conduction with a source are investigated. The sufficient conditions for the scheme to be solvable, for a difference solution to be non-unique and non-existent, and also for its unlimited increase over a finite time, are determined.

1. Introduction.

In this paper we study the properties of difference solutions of quasilinear parabolic equations of the type encountered in non-linear heat conduction with a source

$$\frac{\partial u}{\partial t}=\frac{\partial^2}{\partial x^2}(u^{\sigma+1})+u^\beta. \tag{1.1}$$

Here $\sigma>0, \beta>1$ are certain fixed constants.

Equation (1.1) was considered in /1–7/ (see also the bibliography in /8/) when investigating the occurrence and evolution of dissipative structures in non-linear media with volume energy dissipation. It was shown, in particular, that the solution of the Cauchy problem for (1.1) may be unbounded, when for a certain $T_0<+\infty$

$$\max_x u(t,x)\to+\infty, \qquad t\to T_0^-. \tag{1.2}$$

It was also established that when $\beta\geqslant\sigma+1$ the unbounded solution is localized in the sense that as $t\to T_0$ the solution increases without limit in a set of finite measure.

The investigation of the unusual properties of the solution of Eq.(1.1), carried out in /1–6/, rested largely on the results of a numerical solution of this problem.

In the present paper we investigate the degree of adequacy with which the solutions of the implicit difference schemes for (1.1) describe the properties of the solutions of the corresponding differential problem.

1. For Eq.(1.1) we will consider the boundary value problem

$$u(0,x)=u_0(x)\geqslant0,\quad 0<x<l,\qquad u_0\in C([0,l]), \tag{1.3}$$

$$u(t,0)=u(t,l)=0,\qquad t\geqslant0, \tag{1.4}$$

where $0<l<+\infty$ is a certain constant.

The sufficient conditions for the implicit difference scheme for Eq.(1.1) to be solvable (Sect.2), and for the solution to be non-unique (Sect. 3), non-existent(Sect.3), and (the difference analogue of condition (1.2)) to be unbounded (Sect. 4) are obtained.

In Sects.5 and 6 (to be published in the next number of the journal under the same title) we obtain the conditions for the system to be solvable as a whole, and for the solution of the difference problem to be unique, and we also establish the admissibility of the limit transition, as a result of which theorems on the existence of a solution of problem (1.1), (1.3), and (1.4) are again proved (see /9/). Note that unlike the case considered in Sect.5, $1<\beta\leqslant\sigma+1$ when $\beta>\sigma+1$ global a priori estimates of the difference solution do not exist. For $\beta>\sigma+1$ we establish in Sect.6 the limitations for which the solution always belongs to a certain difference set of stability $\mathscr{W}_h$ as soon as this inclusion is satisfied at the initial instant of time.

The results obtained in Sects,4–6 indicate that for fairly small but finite intervals h, τ of the space-time net, the solution of the implicit difference scheme exhibits many important properties, inherent in the solution of the corresponding differential equation (1.1).

2. We will introduce a uniform net in space ω_h with an interval $h=l/(M+1)$, $M>0$ is an integer, a system of time intervals $\{\tau_j\}$, $\tau_{j+1}\leqslant\tau_j$ and a time net ω_τ generated by it. Everywhere, with the exception of Sect.4,we assume the net ω_τ to be finite and uniform: $\tau_j=\tau=T/(N+1)$, $0\leqslant j\leqslant N$, $N>0$ is an integer, T is a positive constant (in Sect.4 we have $\tau_j\to0$ as $j\to\infty$ and the net ω_τ is non-uniform). We will denote by H_h the set of net functions $v_h=\{v_i|v_0=v_{M+1}=0, v_i\geqslant0, i=1,2,\ldots,M\}$.

We will assume that problem (1.1), (1.3), and (1.4) corresponds to the following implicit (non-linear) difference scheme /10/:

$$\frac{\hat u-u}{\tau_j}=(\hat u^{\sigma+1})_{\bar xx}+\hat u^\beta,\qquad (t,x)\in\omega_\tau\times\omega_h, \tag{1.5}$$

$$u^0=u_{0h},\quad x\in\omega_h,\qquad \hat u\in H_h,\quad t\in\omega_\tau, \tag{1.6}$$

where $\hat u=u_k^{j+1}$, $u=u_k^j$ is the required net function, $(v)_{\bar xx}=(v_{k+1}-2v_k+v_{k-1})/h^2$ is the notation for the

*Zh.vȳchisl.Mat.mat.Fiz.,23,3,646-659,1983 operator of second-order difference differentiation /10/, and u_{0h} is the projection of the function $u_0(x)$ on ω_h.

When formulating problem (1.5), (1.6) and all subsequent results, we will assume that the difference solution is non-negative. In fact, the difference scheme

$$\frac{\hat{u}-u}{\tau_j}=(|\hat{u}|^\sigma\hat{u})_{\bar{x}x}+[\max\{0,\hat{u}\}]^\beta, \qquad (t,x)\in\omega_\tau\times\omega_h \tag{1.7}$$

is identical with (1.5) when $\hat{u}\geqslant 0$. However, as can easily be seen, the solution $\hat{u}$ of scheme (1.7) cannot be negative if $u\geqslant 0$ in ω_h (moreover, $\hat{u}>0$ in ω_h so long as $u\neq 0$). Note that a similar "weak" maximum principle also occurs for the differential problem.

A detailed study of the difference scheme (1.5) without the non-linear term on the right side has been carried out in a number of papers (see, for example,/10–15/). It is shown in Sects.3–5 that the presence of a source considerably changes the properties of the solution. The most interesting case from this point of view is the one where $\beta\geqslant\sigma+1$ when a difference solution may not exist, may be non-unique, and may be unlimited. The latter indicates that (compare (1.2))

$$\max_{1\leqslant k\leqslant M} u_k{}^j\to+\infty, \qquad j\to+\infty \tag{1.8}$$

(in this case we assume that $\sum_{j=0}^{\infty}\tau_j<+\infty$). When $1<\beta<\sigma+1$ and for fairly small τ a solution as a whole will always exist, and it will be unique.

3. Many results in Sects.2–4 can be reformulated for difference schemes for parabolic equations of the type (1.1) having the general form

$$\frac{\partial u}{\partial t}=\frac{\partial^2}{\partial x^2}\varphi(u)+Q(u), \qquad t>0, \quad x\in(0,l), \tag{1.9}$$

where $\varphi, Q\in C^2(R_+{}^1)\cap C([0,+\infty))$ are specified functions, and $\varphi(u)>0, \varphi'(u)>0, Q(u)>0$ when $u>0$, $\varphi'(0)\geqslant 0$, $Q(0)=0$,

$$\int_1^{+\infty}\frac{d\eta}{Q(\eta)}<+\infty \tag{1.10}$$

(inequality (1.10) is a necessary condition for unbounded solutions of Eq.(1.9) to exist, see /16/). The implicit difference scheme for Eq.(1.9) has the form /10/

$$\frac{\hat{u}-u}{\tau_j}=[\varphi(\hat{u})]_{\bar{x}x}+Q(\hat{u}), \qquad (t,x)\in\omega_\tau\times\omega_h. \tag{1.11}$$

Without any major changes the difference schemes (1.5), (1.11) can be investigated for the case of several spatial variables.

4. We will use the notation employed in /17/. The space $V_h=\{v_i|i=0,1,\dots,M+1;\ v_0=v_{M+1}=0\}$ is provided with a scalar product and a norm, which are found from the equations

$$(v_h,w_h)_h=h\sum_{i=1}^{M}v_iw_i, \qquad |v_h|_{h,2}=(v_h,v_h)_h{}^{1/2}. \tag{1.12}$$

The norms in the net analogues of the spaces $L^q(0,l)$, $q\geqslant 1$ and $H_0{}^1(0,l)$ have the following forms, respectively,

$$|v_h|_{h,q}=\left(h\sum_{i=1}^{M}|v_i|^q\right)^{1/q}, \qquad \|v_h\|_{h,2}=\left(h\sum_{i=0}^{M}\left|\frac{v_{i+1}-v_i}{h}\right|^2\right)^{1/2}.$$

We will denote the norm, dual to $\|\cdot\|_{h,2}$ with respect to the scalar product (1.12) by $\|\cdot\|_{h,2}^*$ i.e.

$$\|v_h\|_{h,2}^*=\sup_{\substack{w_h\in V_h\\ w_h\neq 0}}\frac{|(v_h,w_h)_h|}{\|w_h\|_{h,2}}.$$

The following equation holds:

$$\|(v_h)_{\bar{x}}\|_{h,2}^*=\|v_h\|_{h,2}, \qquad v_h\in V_h. \tag{1.13}$$

In the net analogue of the space $C(0,l)$ the norm has the form

$$|v_h|_C=\max_{1\leqslant i\leqslant M}|v_i|, \qquad v_h\in V_h.$$

We will introduce operators of extension p_h and q_h, assuming that p_hv_h is a continuous function, linear in each interval $[ih,(i+1)h]$, and $p_hv_h(ih)=v_i$, $i=0,1,\dots,M+1$; q_hv_h is a piecewise constant supplementation of the net function $v_h\in V_h$, which for all $ih<x<(i+1)h$ is equal to v_i. It is obvious that $p_hv_h\in H_0{}^1(0,l)$, $q_hv_h\in L^q(0,l)$, and

$$\|q_hv_h\|_{L^q(0,l)}=|v_h|_{h,q}, \qquad \|p_hv_h\|_{H_0{}^1(0,l)}=\|v_h\|_{h,2}.$$

In a similar way, for the net functions $v_{\tau,h}$, defined at the nodes of the net $\omega_\tau\times\omega_h$, we will introduce an operator of extension q_τ form the formulas $q_\tau p_hv_{\tau,h}=p_hv_h^{j+1}$, $q_\tau q_hv_{\tau,h}=q_hv_h^{j+1}$ for all $j\tau<t<(j+1)\tau$, $j=0,1,2,\dots,N$ (the net ω_τ in this case is assumed to be uniform).

We will denote (see /10/) by

$$\lambda_1{}^h=\frac{4}{h^2}\sin^2\left(\frac{\pi h}{2l}\right), \tag{1.14}$$

$$\psi_h(x)=\frac{\operatorname{tg}(\pi h/2l)}{h}\sin\left(\frac{\pi x}{l}\right),\qquad 0<x<l, \tag{1.15}$$

the first (least) eigenvalue and the corresponding eigenfunction of the difference problem

$$(\psi_h)_{\bar{x}x}+\lambda\psi_h=0,\qquad x\in\omega_h,\quad \psi_h\in V_h. \tag{1.16}$$

The function ψ_h in (1.15) is chosen so that $|\psi_h|_{h,1}=1$. Note that $\psi_h(x)>0$ in ω_h.

Throughout this paper the difference constants independent of h and τ are denoted by A_0, $A_1,\ldots$.

2. The sufficient conditions for the difference scheme to be solvable in a fixed time layer.

1. We will show initially that when $\beta<\sigma+1$, and also when $\beta=\sigma+1$, $\lambda_1^h>1$, the difference scheme (1.5) is solvable for the net function $\hat{u}$ (u in this case is assumed to be known) for any values of τ. To do this we will need the following assertion (see /10,16/).

Lemma 1. For any function $v_h\in H_h$ the following estimates hold:

$$|v_h|_{h,2(\sigma+1)}^{2(\sigma+1)}\leq\frac{1}{\lambda_1^h}\|v_h^{\sigma+1}\|_{h,2}^2, \tag{2.1}$$

$$|v_h|_{h,\beta+\sigma+1}^{\beta+\sigma+1}\leq A_0\|v_h^{\sigma+1}\|_{h,2}^{(\beta+\sigma+1)/(\sigma+1)},\qquad A_0=l^{1+(\beta+\sigma+1)/2(\sigma+1)}. \tag{2.2}$$

If the function $v(x)\geq0$ is such that $v^{\sigma+1}\in H_0^1(0,l)$, inequalities (2.1) and (2.2) hold when $h=0$.

Consider the continuous operator $P_h\colon \mathbf{R}^M\to\mathbf{R}^M$

$$P_h(\hat{u})=\left\{\frac{\hat{u}_k-u_k}{\tau}-(\hat{u}_k^{\sigma+1})_{\bar{x}x}-\hat{u}_k^\beta,\quad k=1,2,\ldots,M\right\}. \tag{2.3}$$

The existence of a root of the equation $P_h(\hat{u})=0$ will denote that scheme (1.5) is solvable.

Suppose initially that $1<\beta<\sigma+1$. Then

$$(P_h(\hat{u}),\hat{u}^{\sigma+1})_h=\frac{1}{\tau}(\hat{u}-u,\hat{u}^{\sigma+1})_h+\|\hat{u}^{\sigma+1}\|_{h,2}^2-|\hat{u}|_{h,\beta+\sigma+1}^{\beta+\sigma+1}. \tag{2.4}$$

Using inequality (2.2), and also the estimate /12/

$$(\xi-\eta)\xi^{\sigma+1}\geq\frac{1}{\sigma+2}(\xi^{\sigma+2}-\eta^{\sigma+2}),\qquad \xi,\quad \eta\in\mathbf{R}_+^1, \tag{2.5}$$

we obtain from (2.4)

$$(P_h(\hat{u}),\hat{u}^{\sigma+1})_h>-\frac{1}{\sigma+2}\frac{1}{\tau}|u|_{h,\sigma+2}^{\sigma+2}-|\hat{u}|_{h,\beta+\sigma+1}^{\beta+\sigma+1}+A_1|\hat{u}|_{h,\beta+\sigma+1}^{2(\sigma+1)},$$

where $A_1=l^{-(\beta+3\sigma+3)/(\beta+\sigma+1)}$. The second term on the right side can be estimated using Young's inequality /16/. As a result we obtain

$$|\hat{u}|_{h,\beta+\sigma+1}^{\beta+\sigma+1}\leq\frac{A_1}{2}|\hat{u}|_{h,\beta+\sigma+1}^{2(\sigma+1)}+A_2,$$

$$A_2=\frac{\sigma+1-\beta}{2(\sigma+1)}\left[\frac{\beta+\sigma+1}{A_1(\sigma+1)}\right]^{(\beta+\sigma+1)/(\sigma+1-\beta)}.$$

The final estimate takes the form

$$(P_h(\hat{u}),\hat{u}^{\sigma+1})_h>\frac{A_1}{2}|\hat{u}|_{h,\beta+\sigma+1}^{2(\sigma+1)}-\left(A_2+\frac{1}{\sigma+2}\frac{1}{\tau}|u|_{h,\sigma+2}^{\sigma+2}\right).$$

Hence, by virtue of Brauer's theorem /17/, we can conclude that the equation $P_h(\hat{u})=0$ in the sphere

$$|\hat{u}|_{h,\beta+\sigma+1}^{2(\sigma+1)}<\frac{2}{A_1}\left(A_2+\frac{1}{\sigma+2}\frac{1}{\tau}|u|_{h,\sigma+2}^{\sigma+2}\right) \tag{2.6}$$

has at least one solution (note that there are no solutions outside this sphere).

Now consider the case when $\beta=\sigma+1$. Then, we have from (2.4) and (2.1)

$$(P_h(\hat{u}),\hat{u}^{\sigma+1})_h>(\lambda_1^h-1)|\hat{u}|_{h,2(\sigma+1)}^{2(\sigma+1)}-\frac{1}{\sigma+2}\frac{1}{\tau}|u|_{h,\sigma+2}^{\sigma+2}.$$

Hence, when $\lambda_1^h>1$, the equation $P_h(\hat{u})=0$ has at least one solution such that

$$|\hat{u}|_{h,2(\sigma+1)}^{2(\sigma+1)}<\frac{1}{(\lambda_1^h-1)(\sigma+2)\tau}|u|_{h,\sigma+2}^{\sigma+2}. \tag{2.7}$$

We have thus proved the following theorem.

Theorem 1. Suppose $\beta<\sigma+1$ or $\beta=\sigma+1$, $\lambda_1^h>1$. Then, for any $\tau>0$ at least one solution $\hat{u}_h\in H_h$ of the scheme (1.5) exists, which belongs to the sets (2.6) or (2.7) respectively, and there are no solutions outside this set.

Note. 1. As estimates obtained in Sect.5 show, under the conditions of Theorem 1, difference scheme (1.5) has a unique solution for fairly small $\tau>0$.

2. The following result, similar to Theorem 1, holds for scheme (1.11) of general form. Suppose that for all $u>0$

$$Q(u)\leqslant\nu_1\varphi(u)+\nu_2,\qquad \nu_1=\mathrm{const}<\lambda_1^h,\quad \nu_2=\mathrm{const}. \tag{2.8}$$

Then in the set

$$|\varphi(\hat{u})|_{h,2}\leqslant\frac{1}{\lambda_1^h-\nu_1}\left[\nu_2 l^{1/2}+\frac{1}{\tau}|u|_{h,2}\right]$$

at least on solution $\hat{u}_h\in H_h$ of scheme (1.11) exists.

2. When $\beta>\sigma+1$ or $\beta=\sigma+1$, $\lambda_1^h\leqslant 1$ the operator of scheme (1.5) is not coercive, and Theorem 1 ceases to hold. In these cases we will seek a solution $\hat{u}$ of scheme (1.5) close to u for fairly small τ. We will put $\hat{u}-u=\hat{z}$ in ω_h and introduce the continuous operator F_h: $\mathbf{R}^M\to\mathbf{R}^M$:

$$F_h(\hat{z})=\{\tau[(\hat{z}_k+u_k)^{\sigma+1}]_{\bar{x}x}+\tau(\hat{z}_k+u_k)^\beta,\quad k=1,2,\ldots,M\}.$$

The presence in F_h of a fixed point denotes that scheme (1.5) is solvable. We have

$$|F_h(\hat{z})|_C\leqslant(|u|_C+|\hat{z}|_C)^\beta\tau+\frac{2}{h^2}(|u|_C+|\hat{z}|_C)^{\sigma+1}\tau.$$

Hence the operator F_h translates the set $X_{C_0}=\{\hat{z}\mid |\hat{z}|_C\leqslant C_0\}$ into itself (here C_0 is an arbitrary positive constant), if

$$\tau\leqslant\frac{C_0}{(|u|_C+C_0)^\beta+2h^{-2}(|u|_C+C_0)^{\sigma+1}}. \tag{2.9}$$

Then, by virtue of Schauder's theorem regarding the fixed point /18/, the following theorem holds.

Theorem 2. Suppose condition (2.9) is satisfied. Then the difference scheme (1.5) has a solution $\hat{u}\in H_h$, where $|\hat{u}-u|_C\leqslant C_0$.

Note 3. Assuming that $C_0=|u|_C$ in (2.9), we obtain the following expression for the maximum possible time interval τ_d, for which scheme (1.5) is solvable:

$$\tau_d=(2^\beta|u|_C^{\beta-1}+2^{\sigma+2}|u|_C^{\sigma}h^{-2})^{-1}. \tag{2.10}$$

In the last paragraph we will show that this estimate regarding the nature of the dependence of τ_d on $|u|_C$ is in a certain non-improvable (note that in the case of uniformly bounded $|u|_C$ the estimate $\tau_d=O(h^2)$ when $h\ll 1$).

3. Conditions for the non-uniqueness and non-existence of a solution of the difference scheme.

1. We will show that when $\beta\geqslant\sigma+1$ and for fairly small τ the implicit difference scheme (1.5) has, in addition to the solutions constructed in Theorem 2, one other solution, which lies "close" to the root $\hat{U}=\tau^{-1/(\beta-1)}$ of the difference equation

$$\hat{U}/\tau=\hat{U}^\beta. \tag{3.1}$$

Equation (3.1) is identical with (1.5) if in the latter we ignore the term $(\hat{u}^{\sigma+1})_{\bar{x}x}$ and put $u\equiv 0$. This second solution is such that $|\hat{U}|_C\to+\infty$ as $\tau\to 0$.

We will put $\hat{z}=\hat{u}-\tau^{-1/(\beta-1)}$ and determine the continuous operator G_h: $\mathbf{R}^M\to\mathbf{R}^M$ from the equation

$$G_h(\hat{z})=\{\tau(\hat{z}_k+\tau^{-1/(\beta-1)})^\beta+\tau[(\hat{z}_k+\tau^{-1/(\beta-1)})^{\sigma+1}]_{\bar{x}x}-\tau^{-1/(\beta-1)}+u_k-\hat{z}_k,\quad k=1,2,\ldots,M\}.$$

The existence of a root of the equation $G_h(\hat{z})=0$ will denote that scheme (1.5) is solvable. Consider the expression

$$(G_h(\hat{z}),\hat{z})_h=((\hat{z}+\tau^{-1/(\beta-1)})^\beta-\tau^{-\beta/(\beta-1)},\hat{z})_h\tau+$$

$$\tau([(\hat{z}+\tau^{-1/(\beta-1)})^{\sigma+1}]_{\bar{x}x},\hat{z})_h+(u,\hat{z})_h-|\hat{z}|_{h,2}^2=I_1+I_2+I_3-|\hat{z}|_{h,2}^2$$

On the sphere $|\hat{z}|_{h,2}=a_0>0$. It is obvious that $|z|_C\leqslant a_0h^{-1/2}$, and therefore by putting

$$\eta_0=a_0h^{-1/2}\tau^{1/(\beta-1)}. \tag{3.2}$$

where

$$I_2\geqslant-|\hat{z}|_{h,2}|(\hat{z}+\tau^{-1/(\beta-1)})_{\bar{x}x}^{\sigma+1}|_{h,2}\tau\geqslant-\tau^{[\beta-(\sigma+2)]/(\beta-1)}\frac{2a_0}{h^2}(1+\eta_0)^{\sigma+1}l^{1/2},$$

$$I_3\geqslant-|\hat{z}|_{h,2}|u|_{h,2}=-a_0|u|_{h,2}.$$

To estimate I_1 we will use the inequality

$$\eta[(1+\eta)^\beta-1]\geqslant\frac{\beta+1}{2}\eta^2, \tag{3.3}$$

which holds for any $\beta>1$, $|\eta|\leqslant C_*(\beta)<+\infty$. Then, by choosing τ to be so small that

$$\eta_0=a_0h^{-1/2}\tau^{1/(\beta-1)}\leqslant C_*(\beta), \tag{3.4}$$

using (3.3) we obtain

$$I_1\geqslant\frac{\beta+1}{2}|\hat{z}|_{h,2}^2=\frac{\beta+1}{2}a_0^2.$$

Hence, the following inequality is satisfied:

$$(G_h(\hat{z}),\hat{z})_h \geqslant \frac{\beta-1}{2}a_0\left\{a_0-\frac{2}{\beta-1}\left[|u|_{h,2}+\right.\right.$$
$$\left.\left.\frac{2}{h^2}\tau^{[\beta-(\sigma+2)]/(\beta-1)}[1+C_*(\beta)]^{\sigma+1}l^{1/2}\right]\right\}.$$

Hence we conclude that $(G_h(\hat{z}),\hat{z})_h \geqslant 0$ for all

$$|\hat{z}|_{h,2}=a_0=\frac{2}{\beta-1}\left\{|u|_{h,2}+\tau^{[\beta-(\sigma+2)]/(\beta-1)}\frac{2}{h^2}[1+C_*(\beta)]^{\sigma+1}l^{1/2}\right\}. \tag{3.5}$$

It remains to show that conditions (3.4) and (3.5) are compatible for small τ. Substituting a_0 from (3.5) into (3.4) we have

$$\eta_0=\frac{2}{\beta-1}h^{-1/2}|u|_{h,2}\tau^{1/(\beta-1)}+\frac{4}{\beta-1}h^{-5/2}[1+C_*(\beta)]^{\sigma+1}l^{1/2}\tau^{[\beta-(\sigma+1)]/(\beta-1)}, \tag{3.6}$$

whence $\eta_0\to 0$ as $\tau\to 0$ if $\beta>\sigma+1$, i.e. condition (3.4) does not contradict (3.5) for small τ. We have thus established the following result.

Theorem 3. Suppose $\beta>\sigma-1$. Then, for fairly small τ the difference scheme (1.5), in addition to the solution constructed in Theorem 2, has one more solution. If $\beta=\sigma+1$, the previous conclusion holds when (see (3.6))

$$\frac{4}{\sigma}h^{-5/2}[1+C_*(\sigma+1)]^{\sigma+1}l^{1/2}<C_*(\sigma+1).$$

Note 4. Using the example of problem (3.1) it is easy to show that in any neighbourhood of the solution $\hat{U}=\tau^{-1/(\beta-1)}$ as small as desired, the operator

$$F_h(\hat{z})=\{\tau(\hat{z}_k+\tau^{-1/(\beta-1)})^\beta-\tau^{-1/(\beta-1)},\ k=1,2,\ldots,M\}$$

is not a compressing operator. Hence, we can obviously assert that the solution of scheme (1.5) constructed in Theorem 3 is unstable with respect to any iterational process (this conclusion is confirmed by a number of numerical calculations (1–6).

2. We will show below that when $\beta\geqslant\sigma+1$, and for certain restrictions on τ and h, difference scheme (1.5) can generally have no solution. To do this we will use the estimate

$$(\hat{u}^{\sigma+1})_{\bar{x}x}\geqslant-\frac{2}{h^2}\hat{u}^{\sigma+1},\qquad x\in\omega_h,$$

and taking this into account we can derive from (1.5) the system of inequalities

$$\hat{u}\geqslant u+\tau\hat{u}^{\sigma+1}\left(\hat{u}^{\beta-(\sigma+1)}-\frac{2}{h^2}\right),\qquad x\in\omega_h. \tag{3.7}$$

It is clear that it is sufficient to check that (3.7) holds at the point where $\max u$, is reached, i.e. to determine the conditions for which the inequality

$$\xi\geqslant|u|_C+\tau\xi^{\sigma+1}\left(\xi^{\beta-(\sigma+1)}-\frac{2}{h^2}\right) \tag{3.8}$$

has no solution in $\mathbf{R}_+^1$.

We will first consider the case when $\beta=\sigma+1$. Then, inequality (3.8) takes the form

$$\xi\geqslant|u|_C+\tau\xi^{\sigma+1}\left(1-\frac{2}{h^2}\right),\qquad \xi\in\mathbf{R}_+^1,$$

and, as is easily shown, has no solution if

$$h^2>2,\qquad \tau>\tau^*=\frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}}|u|_C^{-\sigma}\left(1-\frac{2}{h^2}\right)^{-1}. \tag{3.9}$$

Suppose now that $\beta>\sigma+1$. Using the Young inequality

$$\xi^{\sigma+1}\leqslant\frac{h^2}{4}\xi^\beta+\varepsilon,\qquad \xi\in\mathbf{R}_+^1,\quad \varepsilon=\frac{\beta-(\sigma+1)}{\beta}\left[\frac{4(\sigma+1)}{\beta h^2}\right]^{(\sigma+1)/[\beta-(\sigma+1)]},$$

we obtain that inequality (3.8) has no solutions if everywhere in $\mathbf{R}_+^1$

$$\xi<|u|_C-\frac{2\tau}{h^2}\varepsilon+\frac{\tau}{2}\xi^\beta.$$

Hence we obtain the condition for scheme (1.5) to be unsolvable when $\beta>\sigma+1$:

$$|u|_C\geqslant\frac{2\varepsilon}{h^2}\tau+\frac{\beta-1}{\beta}\left(\frac{2}{\beta\tau}\right)^{1/(\beta-1)}. \tag{3.10}$$

We have thus proved the following theorem.

Theorem 4. Suppose $\beta=\sigma+1$. Then, when conditions (3.9) are satisfied the difference scheme (1.5) has no solution. When $\beta>\sigma+1$, the difference scheme (1.5) is unsolvable when condition (3.10) is satisfied.

Note 5. Inequalities (3.9) and (3.10) give specific estimates of the value of the time interval τ for which no iterational process for solving the implicit scheme (1.5) will converge. These estimates can be used in numerical calculations. In this connection we will consider inequality (3.10) in more detail.

Suppose

$$a_0=\frac{2[\beta-(\sigma+1)]}{\beta}\left[\frac{4(\sigma+1)}{\beta}\right]^{(\beta+1)/[\beta-(\sigma+1)]},\quad b_0=\frac{\beta-1}{\beta}\left(\frac{2}{\beta}\right)^{1/(\beta-1)}.$$

Then (3.10) takes the form

$$|u|_c\geqslant a_0\tau(h^2)^{-\beta/[\beta-(\sigma+1)]}+b_0\tau^{-1/(\beta-1)}$$

and is satisfied, for example, when

$$|u|_c=\beta a_0d_0(h^2)^{-1/[\beta-(\sigma+1)]},\quad d_0=\left[\frac{b_0}{a_0(\beta-1)}\right]^{\beta/(\beta-1)}, \tag{3.11}$$

$$\tau=\tau_{us}=d_0(h^2)^{(\beta-1)/[\beta-(\sigma+1)]}. \tag{3.12}$$

At the same time, condition (2.10) for the solvability of the scheme when the quantity $|u|_c$ is chosen from (3.11), takes the form

$$\tau_S=f_0(h^2)^{(\beta-1)/[\beta-(\sigma+1)]},\quad f_0=[2^\beta(\beta a_0d_0)^{\beta-1}+2^{\sigma+2}(\beta a_0d_0)^{\sigma+1}]^{-1}$$

and is identical with (3.12) in the form of the dependence on the interval h of the space net. Hence we can conclude that condition (2.10) for the solvability of scheme (1.5) when $\beta>\sigma+1$ cannot be improved for fairly large values of $|u|_c$ (when, for example, the difference solution is developed into a mode with accentuation (see Sect.4).

4. Bounded difference solutions. We will determine the sufficient conditions for the difference solution of problem(1.5), (1.6) to be bounded, in the sense of (1.8), when $\beta\geqslant\sigma+1$. (As will be shown in Sect.5, when $\beta<\sigma+1$ problem (1.5), (1.6) has no unbounded solutions.) To do this we will use the difference analog of the method, which was previously employed in the differential case to investigate the semilinear ($\sigma=0$) /19.20/, and the quasilinear /9,21/ equations of the form (1.1) (see also /22/).

1. We will put

$$\hat{E}=(\hat{u},\psi_h)_h,\quad t\in\omega_\tau, \tag{4.1}$$

where $\psi_h(x)$ is the eigenfunction (1.15) of problem (1.16) corresponding to the minimum eigenvalue (1.14). Miltiplying the system of equations (1.5) scalarly by ψ_h, we obtain the chain of equations

$$\frac{\hat{E}-E}{\tau_j}=-\lambda_1^h(\hat{u}^{\sigma+1},\psi_h)_h+(\hat{u}^\beta,\psi_h)_h,\quad t\in\omega_\tau, \tag{4.2a}$$

$$E^0=E_0=(u_{0h},\psi_h)_h, \tag{4.2b}$$

in deriving which we took into account the fact that /10/ (see (1.16))

$$((\hat{u}^{\sigma+1})_{\bar{x}x},\psi_h)_h=(\hat{u}^{\sigma+1},(\psi_h)_{\bar{x}x})_h=-\lambda_1^h(\hat{u}^{\sigma+1},\psi_h)_h.$$

In view of the normalization of the function ψ_h and the fact that it is non-negative in ω_h, the Hölder inequality holds (we recall that $\beta\geqslant\sigma+1$ and the function $\psi_h\geqslant0$ is such that $|\psi_h|_{h,1}=1$)

$$(\hat{u}^\beta,\psi_h)_h=((\hat{u}^{\sigma+1})^{\beta/(\sigma+1)},\psi_h)_h\geqslant(\hat{u}^{\sigma+1},\psi_h)_h^{\beta/(\sigma+1)},$$

taking which into account we can derive from (4.2) the estimate

$$\frac{\hat{E}-E}{\tau_j}\geqslant(\hat{u}^{\sigma+1},\psi_h)_h^{\beta/(\sigma+1)}[1-\lambda_1^h(\hat{u}^{\sigma+1},\psi_h)_h^{[(\sigma+1)-\beta]/(\sigma+1)}],\quad t\in\omega_\tau.$$

Applying the Hölder inequality $(\hat{u}^{\sigma+1},\psi_h)_h\geqslant(\hat{u},\psi_h)_h^{\sigma+1}$ once again, we obtain

$$\frac{\hat{E}-E}{\tau_j}\geqslant(\hat{u}^{\sigma+1},\psi_h)_h^{\beta/(\sigma+1)}\left[1-\frac{\lambda_1^h}{\hat{E}^{\beta-(\sigma+1)}}\right],\quad t\in\omega_\tau. \tag{4.3}$$

Suppose the value of E_0 is such that

$$\mu_0=1-\lambda_1^hE_0^{(\sigma+1)-\beta}>0,\quad E_0=(u_{0h},\psi_h)_h \tag{4.4}$$

(note that when $\beta=\sigma+1$ this condition has the form $\lambda_1^h<1$). We can then conclude from (4.3) that $\hat{E}>E$ in ω_τ for fairly small τ_j, $j=0,1,\ldots$, and hence

$$\frac{\hat{E}-E}{\tau_j}\geqslant\hat{E}^\beta\left(1-\frac{\lambda_1^h}{E_0^{\beta-(\sigma+1)}}\right)=\mu_0\hat{E}^\beta,\quad t\in\omega_\tau.$$

Since

$$|\hat{u}|_c=\max_{1\leqslant k\leqslant M}\hat{u}_k\geqslant\hat{E},\quad t\in\omega_\tau, \tag{4.5}$$

to determine the conditions for the solution of problem (1.5), (1.6) to be bounded it is sufficient to find a system $\{\tau_j\}$ of time intervals such that

$$T_*=\sum_{j=0}^{\infty}\tau_j<+\infty \tag{4.6}$$

and from the inequalities

$$\hat{E}\geqslant E+\tau_j\mu_0\hat{E}^\beta,\quad t\in\omega_\tau,\quad E^0=E_0, \tag{4.7}$$

we obtain the condition

$$E^j\to+\infty,\quad j\to+\infty. \tag{4.8}$$

In view of (4.5), the correctness of (1.8) follows from (4.8).

Suppose

$$\tau_j = A\rho^{-\alpha j}, \quad j=0, 1, \ldots, \tag{4.9}$$

where $A>0$, $\alpha>0$, $\rho>1$ are constants which will be defined later. Then

$$T_* = \frac{A\rho^\alpha}{\rho^\alpha - 1} < +\infty, \tag{4.10}$$

i.e. condition (4.6) is satisfied.

We will show that when there are certain limitations on the quantities A, α, ρ, we obtain from (4.7) that the following estimate holds:

$$E^j \geqslant E_0\rho^j, \quad j=0, 1, \ldots. \tag{4.11}$$

To show this it is sufficient to establish that

$$E_0\rho^j + \tau_j\mu_0 E_0^\beta \rho^{\beta j} \geqslant E_0\rho^{j+1}, \quad j=0, 1, \ldots. \tag{4.12}$$

Substituting the values of τ_j from (4.9) into (4.12) and simplifying to obtain

$$1 + A\mu_0 E_0^{\beta-1}\rho^{j(\beta-\alpha-1)} \geqslant \rho, \quad j=0,1,\ldots.$$

Hence, when the following conditions are satisfied:

$$\alpha=\beta-1, \quad \rho=1+A\mu_0 E_0^{\beta-1} \tag{4.13}$$

estimate (4.11) holds. We have thus proved the following theorem.

Theorem 5. Suppose $\beta\geqslant\sigma+1$, and the initial function u_{0h} in (1.6) is such that condition (4.4) is satisfied. Suppose the difference problem (1.5), (1.6) is solvable in its sequence of time intervals (4.9), where the constants A, α, ρ satisfy relations (4.13). Then the difference solution of the problem exists in a time (4.10) where

$$\|u^j\|_C \geqslant E_0\rho^j \to +\infty, \quad j\to+\infty.$$

Corollary. Suppose $\sigma+1\leqslant\beta<\sigma+3$ and we are given the function $u_0(x)\geqslant 0$, $x\in R_+^1$. We fix an arbitrary $h>0$. Then, for fairly large M we can find a set of time intervals which satisfies condition (4.6), such that the solution of the difference problem (1.5), (1.6) for $l=(M+1)h$ and $u_{0h}\neq 0$ is unbounded in the sense of (1.8). If $|u_{0h}|_{h,1}>2$ for certain M, a similar conclusion holds in the case when $\beta=\sigma+3$.

This assertion holds for the differential problem (1.1), (1.3), (1.4), see /2,4,7/.

Proof. For large l, for the value of μ_0 in (4.4) the following estimate holds:

$$\mu_0 \simeq 1 - \left[\frac{|u_{0h}|_{h,1}}{2}\right]^{(\sigma+1)-\beta}\left(\frac{\pi}{l}\right)^{(\sigma+3)-\beta}$$

Hence, with the assumptions made $\mu_0>0$ for fairly large l, which, in view of Theorem 5, ensures that the above assertion holds.

Note 6. The assertion of Theorem 5 and the corollary to it also hold for an explicit difference scheme corresponding to Eq. (1.1)

$$\frac{\hat{u}-u}{\tau_j} = (u^{\sigma+1})_{\bar{x}x} + u^\beta, \quad (t,x)\in\omega_\tau\times\omega_h \tag{4.14}$$

with the conditions (1.6). Naturally, in this case, there is no need to stipulate specially that the difference problem is sôlvable. It is easy to show that when $\beta\geqslant\sigma+1$ and four fairly small τ/h^2 the solution of scheme (4.14) is subject to the maximum principle. Note that in the semilinear case ($\sigma=0$) the conditions for the solution of the difference scheme (4.14) to be bounded were obtained in /22,23/.

Note 7. It is well known /9/, that in the case of the differential equation of the problem considered there are unbounded solutions where $\beta=\sigma+1$ if $\lambda_1^0=(\pi/l)^2<1$. If $\lambda_1^0>1$, it is always solvable "as a whole". Hence, we can conclude from (4.4) and from the inequality $\lambda_1^h<\lambda_1^0$ (see (1.14)) that when $\beta=\sigma+1$ the difference problem may have unbounded solutions when its differential analog of such solutions is not permissible in principle.

Note 8. The following result, similar to that obtained in Theorem 5, holds for the general form of problem (1.11), (1.6). Suppose that for all $u>0$

$$\varphi(u)\leqslant\nu_1 Q(u)+\nu_2, \quad \nu_1=\text{const}<\lambda_1^h, \quad \nu_2=\text{const}>0$$

(this condition is the oppostive of that of (2.8)). Suppose, in addition, that the function $Q(u)$ is convex: $Q''(u)\geqslant 0$ when $u>0$. Then, if the initial function u_{0h} in (1.6) is such that

$$Q(E_0) > \frac{\lambda_1^h\nu_2}{\lambda_1^h-\nu_1},$$

the choice of the system of time intervals $\{\tau_j\}$ in the form

$$\tau_j = \frac{AE_0\rho^j}{(\lambda_1^h-\nu_1)Q(E_0\rho^j)-\lambda_1^h\nu_2}, \quad j=0,1,\ldots, \tag{4.15}$$

where $A>0$ is an arbitrary constant, and $\rho=1+A>1$, ensures that the solution of the difference problem (1.11), (1.6) is unbounded in the sense of (1.8), and estimate (4.11) holds. The series $\sum_{j=0}^{\infty}\tau_j$ converges if

$$\lim_{j\to\infty}\frac{\tau_j}{\tau_{j+1}}=\lim_{y\to\infty}\frac{1}{\rho}\frac{Q(\rho y)}{Q(y)}=q=\text{const}>1. \quad (4.16)$$

We will show that it follows from (4.16) that (1.10) holds. In fact, in view of (4.16), $Q(\rho y)>\rho q_*Q(y)$ for all $y>y_*$, where $q_*>1$ is a certain constant. Then

$$\int_{\rho y_*}^{\rho y}\frac{d\eta}{Q(\eta)}<\frac{1}{q_*}\int_{y_*}^{y}\frac{d\eta}{Q(\eta)},\quad y>y_*,$$

and hence, (we recall that $\rho>1$)

$$\left(1-\frac{1}{q_*}\right)\int_{y_*}^{y}\frac{d\eta}{Q(\eta)}+\int_{y}^{\rho y}\frac{d\eta}{Q(\eta)}<\int_{y_*}^{\rho y_*}\frac{d\eta}{Q(\eta)}\quad \forall y>y_*.$$

Hence while making the limit transition $y\to\infty$ we obtain (1.10).

2. We will give an example of an unbounded solution of the difference problem (1.5), (1.6) that can be represented in explicit form. This example, in particular, shows that the requirement that the problem is solvable in a sequence of time steps (4.9), indicated in Theorem 5, is not too burdensome.

Suppose $\beta=\sigma+1$. The difference solution of problem (1.5), (1.6) will be sought in the form

$$u_k^j=S^j\theta_k,\qquad (t,x)\in\omega_\tau\times\omega_h.$$

Substituting u_k^j into (1.5), for the net functions S^j and θ_k defined in ω_τ and ω_h respectively, we obtain the following problems:

$$\frac{\hat S-S}{\tau_j}=\frac{1}{\sigma}\hat S^{\sigma+1},\qquad t\in\omega_\tau, \quad (4.17)$$

$$(\theta^{\sigma+1})_{\bar x x}+\theta^{\sigma+1}=\frac{1}{\sigma}\theta,\qquad x\in\omega_h,\quad \theta\in H_h. \quad (4.18)$$

Suppose we are given the system of time intervals (4.9), where $\rho>1$, $\alpha=\sigma$. Then the solution of problem (4.17) will be the net function

$$S^j=\rho^j,\quad j=0,1,\dots,\qquad A=\sigma\rho^{-(\sigma+1)}(\rho-1). \quad (4.19)$$

We will formulate the solution of problem (4.18) in the special case when $\sigma=2$. We will fix an arbitrary interger $M>0$ and we will put $h=2\sin[3\pi/2(M+1)]$. In this case the length of the section l is

$$l=\frac{3\pi h}{2}\arcsin^{-1}\frac{h}{2},\qquad 0<h\leqslant 2. \quad (4.20)$$

The solution of problem (4.18) then has the form

$$\theta_k=\left\{2\left[3\left(1-\frac{4}{h^2}\sin^2\frac{a_h h}{2}\right)\right]^{-1}\right\}^{1/2}\sin(a_h kh),\quad k=0,1,\dots,M+1, \quad (4.21)$$

where $a_h=\pi/l$.

The functions (4.19) and (4.21) define an unbounded difference solution of problem (1.5), (1.6) when $\sigma=2$, $\beta=3$. When $\tau, h\to 0$ this solution converges to the solution of the differential equation (1.1), constructed in /1,4/.

Note that the function (4.21) is not the projection onto ω_h of the solution of the differential analogue of problem (4.18) when $\sigma=2$, although it has a similar structure. For example, in the case when $\sigma=1$, Eq. (4.18) also has the solution

$$\theta_k=A_h\sin^2(a_h kh)+B_h,\qquad k=0,1,\dots,M+1,$$

where

$$a_h=\frac{1}{2h}\arcsin\frac{h}{2},\qquad 0<h\leqslant 2,\qquad A_h=\frac{1}{\varkappa_h}[2(2\varkappa_h-1)]^{1/2},$$

$$B_h=\frac{1}{2\varkappa_h}\{1-[2(2\varkappa_h-1)]^{1/2}\},\qquad \varkappa_h=1-\frac{2}{h^2}\left[1-\left(1-\frac{h^2}{4}\right)^{1/2}\right],$$

which, however, does not satisfy the boundary conditions, and $\theta_k>0$ for any k when $h>0$. Only in the limit as $h\to 0$, when $\varkappa_h\to 3/4$, $B_h\to 0$ is the function θ_k a solution of the (differential) problem (4.18) when $l\geqslant 4\pi$.

In the differential case, the carrier l_0 of the analytical solution $\theta(x)$ of problem (4.18).

$$\theta(x)=\left\{\frac{2(\sigma+1)}{\sigma(\sigma+2)}\sin^2\left[\frac{\sigma}{2(\sigma+1)}x\right]\right\}^{1/\sigma},\qquad 0<x<l_0=2\pi\frac{\sigma+1}{\sigma},$$

gives the value of the so-called fundamental length, which defines the dimensions of that part of space in which the unbounded solution of the Cauchy problem for Eq. (1.1) increases to infinity /1–6/. Note that the fundamental difference length (4.20), corresponding to $\sigma=2$, differs slightly when $h\ll 1$ from $l_0=3\pi$. This difference may be considerable for fairly large h, for example, $l=9$ when $h=1$, and $l=6$ when $h=2$.

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