20. SHIZGAL, B. and KARLUS, M. Non-equilibrium contribution to the rate of reaction. I. Perturbation of the velocity distribution function. *J. Chem. Phys.*, **52**, 4, 4262–4278, 1970.

21. ASMOLOV, E. S. and MAKASHEV, N. K. On disturbance of the equilibrium kinetics of high-threshold chemical reactions. *Dokl. Akad. Nauk SSSR*, **261**, 3, 562–566, 1981.

22. GORDIETs, B. F., OSIPOV, A. N. and SHELEPIN, L. A. *Kinetic processes in gases, and molecular lasers.* Nauka, Moscow, 1980.

U.S.S.R. Comput. Maths. Math. Phys. Vol. 22, No. 4, pp. 160–178, 1982. 0041–5553/82$07.50+.00
Printed in Great Britain ©1983. Pergamon Press Ltd.

TWO-DIMENSIONAL DIFFERENCE SCHEMES OF MAGNETOHYDRODYNAMICS ON TRIANGLE LAGRANGE MESHES*

V. M. GOLOVIZNIN, V. K. KORSHUNOV and A. A. SAMARSKII

Moscow

(*Received* 27 *October* 1981)

BASED on the variational principle, two-dimensional difference schemes of magnetohydrodynamics on triangular Lagrange meshes are obtained, which are fully conservative, including the difference analogue of the law of conservation of moment of momentum. The meshes are described in terms of difference operators. From the conditions of spectral matching an artificial viscosity is introduced. Examples of numerical calculations are given.

It is essential to stipulate that the difference schemes used for solving non-linear problems of the mechanics of continuous media depict the precise features of the flows studied on real, i.e. comparatively rough, computational meshes [13].

For multi-dimensional problems this situation has a deeper content compared with the one-dimensional case.

Thus, in passing from one to several space dimensions one must take into account the additional law of conservation which in accordance with the principle of total conservation [1] should also satisfy difference schemes: the law of conservation of moment of momentum.

In addition, in multi-dimensional problems an essential part is played by the "geometrical instabilities", such as Rayleigh–Taylor and Helmholtz instabilities. The development of these instabilities in the early stages is described by the linear approximation of the corresponding equations; therefore, for a sufficient description of a continuous medium by a difference scheme it is necessary that the acoustic approximations of difference schemes should approximate the equations of acoustics quite well and depict their most characteristic properties.

A comparison of various methods for the numerical solution of problems in gas dynamics and magnetohydrodynamics, based on the Euler [2] and Lagrange [3] variables, shows that the latter method, in its range of application, has greater "resolving power".

*Zh. vychisl. Mat. mat. Fiz., **22**, 4, 926–942, 1982.

However, when solving two-dimensional problems using quadrangle Lagrange meshes [4 – 10] well-known difficulties arise [4, 8], due to "overlapping" of the Lagrange cells even when there are no strong shear deformations in the modelled flows. Difference schemes on triangular Lagrange meshes are free from this drawback [11–14].

In this paper we obtain and examine two-dimensional differences schemes of MHD on triangular Lagrange meshes, intended for computing both flat and axisymmetrical flows with two components of the frozen magnetic field lying in the plane of the flow. The construction of the schemes is based on Hamilton's variational principle [5].

It is shown that the system of differential-difference equations (differential with respect to time and difference with respect to the space variables), obtained from the variational principle has the property of complete conservatism [1] which includes a difference analogue of the law of conservation of moment of momentum, and has the first degree of approximation with respect to space. The linear approximation of the differential-difference equations approximates the equations of acoustics also to the first degree, and in addition the space part of the difference acoustic operator conserves the property of self-conjugate and of positive definiteness on the corresponding background flows.

The system of differential-difference equations and the corresponding linearized system are written in terms of difference analogues of differential operators [15]. Using the principle of spectral correlation a linear artificial viscosity is introduced.

The transition from differential-difference to difference equations is achieved by replacing the derivatives with respect to time by finite differences. This leads to a multiparameter family of difference schemes. It is shown that by writing the differential-difference equations in operator form we can preserve the property of complete conservatism in the case of time sampling (with appropriate choice of the parameters).

Examples of numerical computations are given.

1. The functional of action

Let G be a certain domain in Cartesian coordinates (x, y), occupied by a continuous medium (in the case of axial symmetry x corresponds to the r coordinate, which is a radius, and y to the z coordinate).

The region Ω corresponds to the region G in Lagrange variables (α, β).

The functional of action for a non-dissipative medium with an infinite electrical conductivity in the presence of the magnetic field $\mathbf{H}=(H_x, H_y)$ is defined by the expression [5]

$$S=\int_{t_0}^{t_1} L(t)\,dt=\int_{t_0}^{t_1}\left[\iint_{\Omega}\rho J\left(\frac{u^2+v^2}{2}-\varepsilon-\frac{H^2}{8\pi\rho}\right)d\alpha\,d\beta\right]dt, \tag{1}$$

where $L(t)$ is the Lagrangian, ρ, ϵ are the density and specific internal energy of the medium which occupies the domain G, $H^2=H_x^2+H_y^2$, $\mathbf{v}=(u, v)$ is the velocity, and

$$J=\frac{1}{l}\frac{\partial(x^l, y)}{\partial(\alpha, \beta)}>0 \tag{2}$$

is the Jacobian of transition from Euler to Lagrange coordinates ($l = 1$ for plane, and $l = 2$ for axial symmetry of the problem).

2. The variational principle in MHD

The dynamic equations of MHD follow from the condition that the first variation of functional (1) equals zero if we take into account the following additional differential relations [5]: the conditions for the magnetic field to be frozen:

$$H_x=\frac{1}{J}\frac{\partial(x, \psi)}{\partial(\alpha, \beta)}, \qquad H_y=\frac{1}{J}\frac{\partial(y, \psi)}{\partial(\alpha, \beta)}, \qquad \psi(\alpha, \beta)=\text{const}; \tag{3}$$

the continuity equations

$$\rho J=\rho_0(\alpha, \beta); \tag{4}$$

and the first law of thermodynamics

$$d\varepsilon=-\frac{P}{\rho_0(\alpha, \beta)}dJ. \tag{5}$$

Here $\psi(\alpha, \beta)$ is the magnitude of the magnetic flux, $\rho_0(\alpha, \beta)$ is the density of the medium in Lagrange variables, and P is the pressure of the medium.

3. Equations of adiabatic MHD

The dynamic equations of MHD in Lagrange variables have the form

$$\begin{aligned}
&\rho_0\frac{\partial u}{\partial t}+x^{l-1}\frac{\partial(P_H, y)}{\partial(\alpha, \beta)}-\frac{1}{4\pi}\frac{\partial(H_x, \psi)}{\partial(\alpha, \beta)}=0,\\
&\rho_0\frac{dv}{dt}+\frac{1}{l}\frac{\partial(x^l, P_H)}{\partial(\alpha, \beta)}-\frac{1}{4\pi}\frac{\partial(H_y, \psi)}{\partial(\alpha, \beta)}=0,
\end{aligned} \tag{6}$$

where $P_H=P+H^2/8\pi$.

The condition of adiabatic flow (5) gives an equation for the change in internal energy

$$\rho_0\frac{d\varepsilon}{dt}=-P\left(\frac{\partial(u, y)}{\partial(\alpha, \beta)}+\frac{\partial(x, v)}{\partial(\alpha, \beta)}\right), \tag{7}$$

which is equivalent to the law of conservation of entropy s.

The system of equations (6), (7), (3) and (4), together with the equation of state $P = P(\rho, \epsilon)$, the kinematic relations $dx/dt = u$, $dy/dt = v$ and the boundary conditions fully define the MHD model of a continuous medium.

Without loss of generality we can assume that on the boundary Γ of domain G we are given the conditions

$$\frac{dP}{d\mathbf{n}}\Big|_{\gamma_1}=0, \qquad P|_{\gamma_2}=P^*(x,y)\geqslant 0, \tag{8}$$

where $\gamma_1 \cup \gamma_2 = \Gamma$, and $\mathbf{n}$ is the internal normal to Γ.

4. The variational principle for the equations of acoustics

We will derive the equations describing the acoustic oscillations of the medium [9].

Let us assume that a small perturbation $(\Delta x, \Delta y)$ of any background flow is given. Expanding (1), taking account of (2) – (5), in a Taylor series in terms of values of the perturbations up to the second order of smallness inclusive, we find

$$\Delta L=\Delta_1 L+\Delta_2 L+o((\Delta x)^3, (\Delta y)^3),$$

where, as a consequence of Eqs. (6) for the undisturbed flow, the first order term is $\Delta_1 L \equiv 0$, and

$$\begin{aligned}\Delta_2 L=\int\int_{\Omega}\Big\{&\rho_0\frac{(\Delta u)^2+(\Delta v)^2}{2}+P_H\Delta_2 J\\ &-\frac{1}{2}\frac{c_A^2\rho}{J}(\Delta_1 J)^2-\frac{1}{8\pi J}\Big[\Big(\frac{\partial(\Delta x,\psi)}{\partial(\alpha,\beta)}-H_x\Delta_1 J\Big)^2\\ &+\Big(\frac{\partial(\Delta y,\psi)}{\partial(\alpha,\beta)}-H_y\Delta_1 J\Big)^2\Big]\Big\}d\alpha d\beta.\end{aligned} \tag{9}$$

Here the terms of the expansion of the Jacobian J are

$$\Delta_1 J=\frac{\partial(x^{l-1}\Delta x, y)}{\partial(\alpha,\beta)}+\frac{1}{l}\frac{\partial(x^l,\Delta y)}{\partial(\alpha,\beta)},$$
$$\Delta_2 J=\frac{\partial(x^{l-1}\Delta x,\Delta y)}{\partial(\alpha,\beta)},$$

where $c_A^2=(\partial P/\partial\rho)_s$ is the adiabatic speed of sound. The condition for the first variation of the functional to be zero

$$\Delta_2 S=\int_{t_0}^{t_1}\Delta_2 L\,dt$$

gives the equations of MHD acoustics

$$\begin{aligned}&\rho_0\frac{d\Delta u}{dt}+(l-1)\Delta x\frac{\partial(P_H,y)}{\partial(\alpha,\beta)}+x^{l-1}\Big[\frac{\partial(\Delta P_H,y)}{\partial(\alpha,\beta)}\\ &\pm\frac{\partial(P_H,\Delta y)}{\partial(\alpha,\beta)}\Big]-\frac{1}{4\pi}\frac{\partial(\Delta H_x,\psi)}{\partial(\alpha,\beta)}=0,\end{aligned} \tag{10}$$

$$\rho_0\frac{d\Delta v}{dt}+\frac{1}{l}\frac{\partial(x^l,\Delta P_H)}{\partial(\alpha,\beta)}+\frac{\partial(x^{l-1}\Delta x,P_H)}{\partial(\alpha,\beta)}-\frac{1}{4\pi}\frac{\partial(\Delta H_y,\psi)}{\partial(\alpha,\beta)}=0,$$

where

$$\Delta P_H = -\frac{c_A^2 \rho}{J}\Delta_1 J + \frac{H_x \Delta H_x + H_y \Delta H_y}{4\pi},$$

$$\Delta H_x = \frac{1}{J}\left[\frac{\partial(\Delta x, \psi)}{\partial(\alpha, \beta)} - H_x \Delta_1 J\right],$$

$$\Delta H_y = \frac{1}{J}\left[\frac{\partial(\Delta y, \psi)}{\partial(\alpha, \beta)} - H_y \Delta_1 J\right].$$

It follows from Eq. (9) that in the operator equation

$$d^2\Delta\mathbf{r}/dt^2 = -A\Delta\mathbf{r}, \qquad \Delta\mathbf{r} = (\Delta x, \Delta y),$$

which corresponds to (10), the operator A is self-conjugate under the condition (8), and P_H = const. In stable background flows the operator A is non-negative definite.

5. Sampling of physical quantities

Since the choice of Lagrange variables is arbitrary, we shall assume that the domain Ω is a parallelogram with an angle $\pi/3$ (Fig. 1).

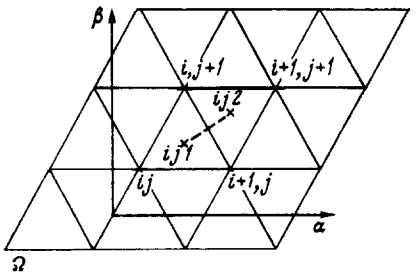

FIG. 1

We will divide Ω into right-angled triangles (let h_α be the side of a triangular cell, and h_β its height), and introduce in Ω two difference meshes: a set of vertices

$$\bar{\omega} = \{\alpha_{ij} = (i + j/2)h_\alpha,\ \beta_{ij} = jh_\beta,\ 0 \leqslant i \leqslant N,\ 0 \leqslant j \leqslant M\}$$

and a set of the centres of triangular cells

$$\omega = \{[\alpha_{ij1} = (i + (j+1)/2)h_\alpha,\ \beta_{ij1} = (j + {}^1/_3)h_\beta],$$
$$[\alpha_{ij2} = (i + 1 + j/2)h_\alpha,\ \beta_{ij2} = (j + {}^2/_3)h_\beta],\quad 0 \leqslant i \leqslant N-1,\quad 0 \leqslant j \leqslant M-1\}.$$

Thus, we introduce two subscripts corresponding to each node of the mesh, i and $j \in \bar{\omega}$, and three corresponding to each cell, i, j, and $k \in \omega$, $k = 1, 2$.

By the inverse transformation $\Omega \to G$ we obtain a difference mesh in the domain G.

We denote by $\mathscr{H}_{\bar{\omega}}$ and $\mathscr{H}_\omega$ the sets of mesh functions given on meshes $\bar{\omega}$ and ω respectively.

Let us ascribe the values of the coordinates, the velocities of the particles of the medium, and the value of the magnetic flux to the mesh $\bar{\omega}$:

$$x_{ij},\ y_{ij},\ u_{ij},\ v_{ij},\ \psi_{ij}\in\mathcal{H}_{\bar{\omega}},\qquad i,\ j\in\bar{\omega},$$

and the thermodynamic quantities and the components of the magnetic field to the mesh ω:

$$\rho_0(\alpha,\ \beta)_{ijk},\ P_{ijk},\ \rho_{ijk},\ \varepsilon_{ijk},\ (H_x)_{ijk},\ (H_y)_{ijk}\in\mathcal{H}_{\omega},\qquad i,\ j,\ k\in\omega.$$

Below, the subscripts ij and ijk will be omitted, using the notation $f_{ij}\equiv f_{\bar{\omega}}\equiv f$, $f_{ij}\in\mathcal{H}_{\bar{\omega}}$, $i,\ j\in\bar{\omega}$; $g_{ijk}\equiv g_{\omega}\equiv g$, $g_{ijk}\in\mathcal{H}_{\omega}$, $i,\ j,\ k\in\omega$.

For convenience we will introduce a local system of subscripts (see Fig. 2): III_0 (ij) = $\{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ for the set of cell centres to node ij (see Fig. 2a), III_1 (ij) = $\{1,\ 2,\ 3\}$ for the set of vertices of the cell ijk (see Fig. 2b for cell ij 1 and Fig. 2c for cell ij 2).

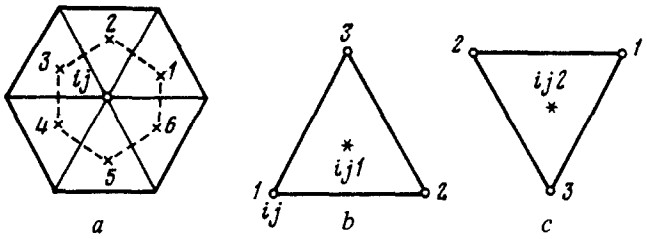

FIG. 2

6. Variational principle for discrete models of a continuous medium

We will approximate functional (1) and conditions (3) – (5) by the difference expressions with respect to space:

$$S_h=\int_{t_0}^{t_1}L_h(t)\,dt=\int_{t_0}^{t_1}\Big[\sum_{\omega}\langle\rho\rangle\langle J\rangle\Big(\frac{\langle u^2\rangle+\langle v^2\rangle}{2}-\langle\varepsilon\rangle-\frac{\langle H_x\rangle^2+\langle H_y\rangle^2}{8\pi\langle\rho\rangle}\Big)\frac{h_\alpha h_\beta}{2}\Big]dt, \tag{11}$$

$$\langle H_x\rangle=\frac{1}{\langle J\rangle}\Big\langle\frac{\partial(x,\psi)}{\partial(\alpha,\beta)}\Big\rangle,\qquad \langle H_y\rangle=\frac{1}{\langle J\rangle}\Big\langle\frac{\partial(y,\psi)}{\partial(\alpha,\beta)}\Big\rangle, \tag{12}$$

$$\langle\rho\rangle\langle J\rangle=\langle\rho_0(\alpha,\ \beta)\rangle, \tag{13}$$

$$d\langle\varepsilon\rangle=-\frac{\langle P\rangle}{\langle\rho\rangle\langle J\rangle}d\langle J\rangle. \tag{14}$$

Here $\langle\cdot\rangle\equiv\langle\cdot\rangle_{ijk}$, $i, j, k\in\omega$, are certain linear functionals for the corresponding mesh functions

$$\langle f\rangle=\sum_{l\in III(ij)}a_l f_l,$$

where $III\,(ij)$ is the approximation pattern, and a_l are certain coefficients ($III\,(ij)\subset\bar{\omega}$ if $f\in\mathscr{H}_{\bar{\omega}}$, and $III\,(ij)\subset\omega$ if $f\in\mathscr{H}_{\omega}$).

We shall assume that the following essential approximation matching conditions are satisfied [15]:

$$\frac{1}{l}\left\langle\frac{\partial(x^l, y)}{\partial(\alpha, \beta)}\right\rangle_{\omega}=\frac{1}{l}\left(\left\langle\frac{\partial x^l}{\partial\alpha}\right\rangle_{\omega}\left\langle\frac{\partial y}{\partial\beta}\right\rangle_{\omega}-\left\langle\frac{\partial x^l}{\partial\beta}\right\rangle_{\omega}\left\langle\frac{\partial y}{\partial\alpha}\right\rangle_{\omega}\right),$$

$$\left\langle\frac{\partial(P, y)}{\partial(\alpha, \beta)}\right\rangle_{\bar{\omega}}=\left\langle\frac{\partial}{\partial\alpha}\langle P\rangle_{\omega}\left\langle\frac{\partial y}{\partial\beta}\right\rangle_{\omega}\right\rangle_{\bar{\omega}}-\left\langle\frac{\partial}{\partial\beta}\langle P\rangle_{\omega}\left\langle\frac{\partial y}{\partial\alpha}\right\rangle_{\omega}\right\rangle_{\bar{\omega}}$$

etc.

Expressions (11) – (14) together with the equation of state $\langle P\rangle=P(\langle\rho\rangle, \langle\varepsilon\rangle)$ and the kinematic relationships $dx/dt=u$, $dy/dt=v$ completely define the properties of a discrete MHD medium.

From the condition that the first variation of the functional B_h equals zero, we obtain the dynamic differential-difference equations of MHD in the form

$$\frac{d}{dt}\frac{\partial L_h}{\partial u}-\frac{\partial L_h}{\partial x}=0,\quad \frac{d}{dt}\frac{\partial L_h}{\partial v}-\frac{\partial L_h}{\partial y}=0. \tag{15}$$

7. The differential-difference equations of MHD

Choosing as an approximation configuration of $III_1\,(ij)$ on $\bar{\omega}$ the set of vertices of the cell ijk (see Fig. 2), and on ω one point, namely the centre of the cell, we define the form of the functionals on the set $\mathscr{H}_{\bar{\omega}}$ by the expressions

$$\begin{aligned}
&\langle J\rangle=\frac{V}{h_\alpha h_\beta/2}+o(h_\alpha, h_\beta),\\
&\langle H_y\rangle=\frac{S(y, \psi)}{V}+o(h_\alpha, h_\beta),\quad \langle H_x\rangle=\frac{S(x, \psi)}{V}+o(h_\alpha, h_\beta),\\
&\langle u^2\rangle=\frac{(u_1^2+u_2^2+u_3^2)}{3}+o(h_\alpha^2, h_\beta^2),\\
&\langle v^2\rangle=\frac{(v_1^2+v_2^2+v_3^2)}{3}+o(h_\alpha^2, h_\beta^2)
\end{aligned} \tag{16}$$

and on the set $\mathscr{H}_{\omega}$ by

$$\langle\rho_0(\alpha, \beta)\rangle=\rho_0(\alpha, \beta),\quad \langle\rho\rangle=\rho,\quad \langle\varepsilon\rangle=\varepsilon,\quad \langle P\rangle=P. \tag{17}$$

Here, the volume and the cross sectional area of the cell in Euler coordinates are

$$V(x,\ y)\equiv V=[x_1'(y_2-y_3)+x_2'(y_3-y_1)+x_3'(y_1-y_2)]/2l,$$
$$S(x,\ y)=[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]/2. \tag{18}$$

Taking into account (16) and (17), expressions (11) – (14) take the form

$$S_h=\int_{t_0}^{t_1} L_h(t)\,dt=\int_{t_0}^{t_1}\left\{\sum_{\bar{\omega}} M\frac{u^2+v^2}{2}-\sum_{\omega}\left(m\varepsilon+V\frac{H^2}{8\pi}\right)\right\}dt, \tag{19}$$

$$H_x=S(x,\ \psi)/V, \qquad H_y=S(y,\ \psi)/V, \tag{20}$$

$$\rho V=m\equiv\rho_0(\alpha,\beta)\frac{h_\alpha h_\beta}{2}, \tag{21}$$

$$d\varepsilon=-\frac{P}{m}dV, \tag{22}$$

where

$$M=\frac{1}{3}\sum_{k\in\mathit{III}_0(ij)}' m_k, \qquad ij\in\bar{\omega}$$

(see Fig. 2a) is a consequence of (16).

On substituting $L_h(t)$ from Eq. (19) into (15) and taking into account expressions (20) – (22) we obtain the dynamic differential-difference equations of MHD on the triangular meshes. The complete system of equations is

$$M\frac{du}{dt}-\sum_{k\in\mathit{III}_0(ij)}'(P_H)_k\frac{\partial V_k}{\partial x}+\frac{1}{4\pi}\sum_{k\in\mathit{III}_0(ij)}'(H_x)_k\frac{\partial S_k(x,\psi)}{\partial x}=0,$$
$$M\frac{dv}{dt}-\sum_{k\in\mathit{III}_0(ij)}'(P_H)_k\frac{\partial V_k}{\partial y}+\frac{1}{4\pi}\sum_{k\in\mathit{III}_0(ij)}'(H_y)_k\frac{\partial S_k(y,\psi)}{\partial y}=0,$$
$$m\frac{d\varepsilon}{dt}=-P\sum_{k\in\mathit{III}_1(ij)}'\left(\frac{\partial V}{\partial x_k}u_k+\frac{\partial V}{\partial y_k}v_k\right),$$
$$H_x=\frac{S(x,\psi)}{V},\quad H_y=\frac{S(y,\psi)}{V}, \tag{23}$$
$$\rho V=m, \qquad P=P(\rho,\varepsilon), \qquad \frac{dx}{dt}=u,$$
$$\frac{dy}{dt}=v, \qquad P_H=P+\frac{H^2}{8\pi}.$$

It should be noted that the energy equation in system (23), which approximates the differential equation (7) can be treated as the law of conservation of entropy in a discrete medium.

8. Difference analogues of the differential operators

The dynamic equations (6) and the energy equations (7) in Euler coordinates have the form

$$\rho \frac{d\mathbf{v}}{dt} = -\operatorname{grad} P + \frac{1}{4\pi}[\mathbf{H} \times \operatorname{rot} \mathbf{H}], \tag{24}$$

$$\rho \frac{d\varepsilon}{dt} = -P \operatorname{div} \mathbf{v}.$$

We transform the dynamic equation into divergent form. Using the equation $\operatorname{div} \mathbf{H}=0$, we obtain

$$\rho \frac{d\mathbf{v}}{dt} = -\operatorname{grad} P_H + \frac{1}{4\pi}\mathbf{d}, \tag{25}$$

where $\mathbf{d}=(\operatorname{div} H_x\mathbf{H},\ \operatorname{div} H_y\mathbf{H})$.

The differential-difference equation of system (23) can be written in a form identical to Eqs. (24) and (25), apart from replacing the differential operators by difference operators.

In fact, let us define the operators of the difference derivative:

$$\left\langle \frac{\partial f_\omega}{\partial x} \right\rangle_{\bar\omega} = -\frac{1}{V_{\bar\omega}} \sum_{k \in III_0(ij)} f_k \frac{\partial V_k}{\partial x_{\bar\omega}}, \qquad \mathcal{H}_\omega \to \mathcal{H}_{\bar\omega}, \tag{26}$$

$$\frac{1}{\langle x^{l-1} \rangle_\omega} \left\langle \frac{\partial \langle x^{l-1} \rangle_{\bar\omega} g_{\bar\omega}}{\partial x} \right\rangle_\omega = \frac{1}{V_\omega} \sum_{k \in III_1(ij)} g_k \frac{\partial V_\omega}{\partial x_k}, \qquad \mathcal{H}_{\bar\omega} \to \mathcal{H}_\omega, \tag{27}$$

and also

$$\left\langle\!\!\left\langle \frac{\partial f_\omega}{\partial x} \right\rangle\!\!\right\rangle_{\bar\omega} = -\frac{1}{S_{\bar\omega}} \sum_{k \in III_0(ij)} f_k \frac{\partial S_k}{\partial x_{\bar\omega}}, \qquad \mathcal{H}_\omega \to \mathcal{H}_{\bar\omega}, \tag{28}$$

$$\left\langle\!\!\left\langle \frac{\partial g_{\bar\omega}}{\partial x} \right\rangle\!\!\right\rangle_\omega = \frac{1}{S_\omega} \sum_{k \in III_1(ij)} g_k \frac{\partial S_\omega}{\partial x_k}, \qquad \mathcal{H}_{\bar\omega} \to \mathcal{H}_\omega, \tag{29}$$

where

$$V_{\bar\omega} = \frac{1}{3} \sum_{k \in III_0(ij)} V_k, \qquad S_{\bar\omega} = \frac{1}{3} \sum_{k \in III_0(ij)} S_k.$$

It can be shown by expanding in a Taylor series [15] that operators (26) and (28) approximate the corresponding derivatives to second-order accuracy with respect to h_α and h_β (on a six-point configuration), and operators (27) and (28) to first-order accuracy (on a triangular configuration).

With the help of the operators of the first-order difference derivatives (26) – (29) it is possible to form difference analogues of the operators grad, div and rot (see [7] and [15]), and also the operators of higher-order difference derivatives, (for example the Laplace operator [15]).

We shall define the following operators:

$$\mathrm{GRAD}_{\bar\omega} f_\omega = \left(\left\langle \frac{\partial f_\omega}{\partial x} \right\rangle_{\bar\omega}, \left\langle \frac{\partial f_\omega}{\partial y} \right\rangle_{\bar\omega} \right), \qquad \mathscr{H}_\omega \to \mathscr{H}_{\bar\omega},$$

$$\mathrm{DIV}_\omega \mathbf{g}_{\bar\omega} = \frac{1}{\langle x^{l-1} \rangle_\omega} \left\langle \frac{\partial \langle x^{l-1} \rangle_{\bar\omega} g_{x\bar\omega}}{\partial x} \right\rangle_{\bar\omega} \tag{30}$$

$$+ \frac{1}{\langle x^{l-1} \rangle_\omega} \left\langle \frac{\partial \langle x^{l-1} \rangle_{\bar\omega} g_{y\bar\omega}}{\partial y} \right\rangle_\omega, \qquad \mathscr{H}_{\bar\omega} \to \mathscr{H}_\omega;$$

$$\mathrm{GRAD}_\omega f_{\bar\omega} = \left(\left\langle\!\left\langle \frac{\partial f_{\bar\omega}}{\partial x} \right\rangle\!\right\rangle_\omega, \left\langle\!\left\langle \frac{\partial f_{\bar\omega}}{\partial y} \right\rangle\!\right\rangle_\omega \right), \qquad \mathscr{H}_{\bar\omega} \to \mathscr{H}_\omega,$$

$$\mathrm{DIV}_{\bar\omega} \mathbf{g}_\omega = \frac{1}{\langle x^{l-1} \rangle_{\bar\omega}} \left\langle\!\left\langle \frac{\partial \langle x^{l-1} \rangle_\omega g_{x\omega}}{\partial x} \right\rangle\!\right\rangle_{\bar\omega} + \tag{31}$$

$$+ \frac{1}{\langle x^{l-1} \rangle_{\bar\omega}} \left\langle\!\left\langle \frac{\partial \langle x^{l-1} \rangle_\omega g_{y\omega}}{\partial y} \right\rangle\!\right\rangle_{\bar\omega}, \qquad \mathscr{H}_\omega \to \mathscr{H}_{\bar\omega}.$$

Operators DIV_ω and $-\mathrm{GRAD}_{\bar\omega}$ in (30) are conjugate, i.e. $(f_\omega\ \mathrm{DIV}_\omega g_{\bar\omega}) = (-\mathrm{GRAD}_{\bar\omega} f_\omega g_{\bar\omega})$, where the scalar product in spaces $\mathscr{H}_\omega$ and $\mathscr{H}_{\bar\omega}$ are defined by the formulae

$$(f_\omega, g_\omega) = \sum_{k \in III_0(ij)} V_k f_k g_k, \qquad f, g \in \mathscr{H}_\omega,$$

$$(\mathbf{f}_{\bar\omega}, \mathbf{g}_{\bar\omega}) = \sum_{k \in III_1(ij)} \bar V_k (f_{xk} g_{xk} + f_{yk} g_{yk}), \qquad \mathbf{f}, \mathbf{g} \in \mathscr{H}_{\bar\omega}, \quad \bar V \equiv V_{\bar\omega}. \tag{32}$$

The operators $\mathrm{DIV}_{\bar\omega}$ and $-\mathrm{GRAD}_\omega$ in (31) are also conjugate if in (32) we assume that $V_\omega = \langle x^{l-1} \rangle_\omega S_\omega$, $V_{\bar\omega} = \langle x^{l-1} \rangle_{\bar\omega} S_{\bar\omega}$. We shall regard these formulae as a definition of $\langle x^{l-1} \rangle_\omega$ and $\langle x^{l-1} \rangle_{\bar\omega}$ respectively.

With the help of expressions (30) and (31), the corresponding equations of system (23) can be reduced to a form identical with (24) and (25):

$$\rho_{\bar\omega} \frac{d\mathbf{v}}{dt} = -\mathrm{GRAD}_{\bar\omega} P_H + \frac{1}{4\pi} \mathbf{D}_{\bar\omega}, \qquad \rho \frac{d\varepsilon}{dt} = -P\, \mathrm{DIV}_\omega \mathbf{v},$$

where

$$\rho_{\bar\omega} = \frac{1}{3} \frac{1}{V_{\bar\omega}} \sum_{k \in III_0(ij)} \rho_k V_k, \qquad \mathbf{D}_{\bar\omega} = (\mathrm{DIV}_{\bar\omega} H_x \mathbf{H}, \mathrm{DIV}_{\bar\omega} H_y \mathbf{H}).$$

Note that $\mathrm{DIV}_{\bar\omega} \mathbf{H} = 0$.

Using the difference operators introduced here, it is not difficult to obtain difference approximations on triangle meshes of any differential equations, including the equations of magnetic field diffusion and of thermal conductivity.

9. The differential-difference equations of acoustics

Let us obtain equations describing the acoustic oscillations of a continuous medium.

Expanding Eqs. (19) and (22) in a Taylor series with respect to any background flow to terms of the second order of smallness, we find

$$\Delta_2 L_h = \sum_{\bar{\omega}} M \frac{1}{2}[(\Delta u)^2 + (\Delta v)^2]$$
$$-\frac{1}{2}\sum_{\omega}\frac{c_A{}^2\rho}{V}\left[\sum_{l\in \mathit{III}_1(ij)}\frac{\partial V}{\partial x_l}\Delta x_l + \frac{\partial V}{\partial y_l}\Delta y_l\right]^2$$
$$+\frac{1}{2}\sum_{\omega} P_H \sum_{k,\, l\in \mathit{III}_1(ij)}\left(\frac{\partial^2 V}{\partial x_k \partial x_l}\Delta x_k \Delta x_l + \frac{\partial^2 V}{\partial y_l \partial x_k}\Delta y_l \Delta x_k\right) \tag{33}$$
$$-\sum_{\omega}\frac{1}{8\pi V}\left\{\left[\sum_{l\in \mathit{III}_1(ij)}\frac{\partial S(x,\psi)}{\partial x_l}\Delta x_l - H_x \sum_{l\in \mathit{III}_1(ij)}\left(\frac{\partial V}{\partial x_l}\Delta x_l + \frac{\partial V}{\partial y_l}\Delta y_l\right)\right]^2\right.$$
$$\left.+\left[\sum_{l\in \mathit{III}_1(ij)}\frac{\partial S(y,\psi)}{\partial y_l}\Delta y_l - H_y \sum_{l\in \mathit{III}_1(ij)}\left(\frac{\partial V}{\partial x_l}\Delta x_l + \frac{\partial V}{\partial y_l}\Delta y_l\right)\right]^2\right\}.$$

If $\Delta u \equiv 0$ and $\Delta v = 0$, then the quadratic form (33) describes the stability of the background flow: the flow is stable if the form is negative definite.

From the condition that the first variation of the functional

$$\Delta_2 S_h = \int_{t_0}^{t_1} \Delta_2 L_h \, dt$$

equals zero, there follow the dynamic equations of MHD acoustics.

In terms of difference operators, these equations have the form

$$\rho_{\bar{\omega}}\frac{d\Delta u}{dt} = -(l-1)\frac{\Delta x}{\langle x^{l-1}\rangle_{\bar{\omega}}}\mathrm{GRAD}_{x\bar{\omega}}P_H - \mathrm{GRAD}_{x\bar{\omega}}\Delta P_H$$
$$+\frac{1}{4\pi}\mathrm{DIV}_{\bar{\omega}}(\Delta H_x \mathbf{H})$$
$$-\left(\left\langle\!\left\langle\frac{\partial}{\partial x}P_H\left\langle\!\left\langle\frac{\partial \Delta y}{\partial y}\right\rangle\!\right\rangle_{\omega}\right\rangle\!\right\rangle_{\bar{\omega}} - \left\langle\!\left\langle\frac{\partial}{\partial y}P_H\left\langle\!\left\langle\frac{\partial \Delta y}{\partial x}\right\rangle\!\right\rangle_{\omega}\right\rangle\!\right\rangle_{\bar{\omega}}\right),$$
$$\rho_{\bar{\omega}}\frac{d\Delta v}{dt} = -\mathrm{GRAD}_{y\bar{\omega}}\Delta P_H + \frac{1}{4\pi}\mathrm{DIV}_{\bar{\omega}}(\Delta H_y \mathbf{H}) \tag{34}$$
$$-\left(\left\langle\!\left\langle\frac{\partial}{\partial x}P_H\left\langle\!\left\langle\frac{\partial\langle x^{l-1}\rangle_{\bar{\omega}}\Delta x}{\partial y}\right\rangle\!\right\rangle_{\omega}\right\rangle\!\right\rangle_{\bar{\omega}}\right.$$
$$\left.-\left\langle\!\left\langle\frac{\partial}{\partial y}P_H\left\langle\!\left\langle\frac{\partial\langle x^{l-1}\rangle_{\bar{\omega}}\Delta x}{\partial x}\right\rangle\!\right\rangle_{\omega}\right\rangle\!\right\rangle_{\bar{\omega}}\right),$$

where $\Delta P_H = -c_A{}^2\rho\,\mathrm{DIV}_{\omega}\Delta\mathbf{r} + (H_x\Delta H_x + H_y\Delta H_y)/4\pi$,

$$\Delta H_x = S(\Delta x, \psi)/V - H_x\mathrm{DIV}_{\omega}\Delta\mathbf{r},$$
$$\Delta H_y = S(\Delta y, \psi)/V - H_y\mathrm{DIV}_{\omega}\Delta\mathbf{r}, \quad \Delta\mathbf{r} = (\Delta x, \Delta y).$$

Comparing Eqs. (34) and (10) we see that the differential-difference equations of acoustics (34) approximate the same type of equations (10) with the first order of accuracy with respect to the steps of the space variables.

It follows from Eqs. (33) and (34) that the difference operator A_h on the right-side of (34) is self-conjugate on flows which satisfy the boundary conditions (8), and for P_H = const. On stable background flows A_h is positive definite.

In the basis of eigenfunctions of the operator A_h the equations of acoustics take the form

$$d^2a_i/dt^2+\lambda_i a_i=0, \quad i=1, 2, \ldots, N', \tag{35}$$

where λ_i are eigenvalues of the operator, N' is the dimensionality of the mesh space on which the operator is defined, and $a_i(t)$ are the coefficients of the expansion of $\Delta\mathbf{r}$ with respect to the eigenfunctions of the operator A_h.

10. Artificial viscosity

The system of differential-difference equations of MHD (23) is unsuitable for computing flows with shock waves since it conserves the entropy of a discrete medium. In order to avoid this limitation it becomes necessary to introduce some dissipative processes [1].

Following [8], we introduce into Eq. (23) an artificial viscosity in such a way that on an equilibrium background the viscosity introduced does not result in a redistribution of the energy of standing acoustic waves, i.e. we introduce viscosity into the discrete medium so that each mode $a_i(t)$ in Eq. (35) is damped independently:

$$\frac{d^2a_i}{dt^2}+\varkappa_i\lambda_i\frac{da_i}{dt}+\lambda_i a_i=0, \qquad i=1, 2, \ldots, N', \tag{36}$$

where $\varkappa_i$ are certain coefficients.

The inverse transition from (36) to (34) enables us to obtain the form of the artificial viscosity.

Below we give the complete system of differential-difference equations taking into account the artificial viscosity:

$$\begin{aligned}
&\rho_{\bar{\omega}}\frac{d\mathbf{v}}{dt}=-\mathrm{GRAD}_{\bar{\omega}}P_H^{\cdot}+\frac{1}{4\pi}\mathrm{D}_{\bar{\omega}}^{\cdot},\\
&\rho\frac{d\varepsilon}{dt}=-\left(P_H^{\cdot}-\frac{1}{8\pi}H^2\right)\mathrm{DIV}_{\omega}\mathbf{v}+q,\\
&\frac{d\rho}{dt}=-\rho\,\mathrm{DIV}_{\omega}\mathbf{v}, \qquad \mathbf{H}=S(\mathbf{r},\psi)/V, \qquad \frac{d\mathbf{r}}{dt}=\mathbf{v}, \qquad P=P(\rho,\varepsilon),
\end{aligned} \tag{37}$$

where

$$P_H^{\cdot}=P_H-\varkappa\frac{c_A^2\rho}{V}\mathrm{DIV}_\omega\mathbf{v}+\frac{\varkappa}{4\pi}\left(H_x\frac{dH_x}{dt}+H_y\frac{dH_y}{dt}\right),$$

$$\mathbf{D}_{\bar\omega}^{\cdot}=(\mathrm{DIV}_{\bar\omega}^{\cdot}H_x^{\cdot}\mathbf{H},\ \mathrm{DIV}_{\bar\omega}^{\cdot}H_y^{\cdot}\mathbf{H}),$$

$$q=\frac{\varkappa}{4\pi V}\left[H_x\frac{dH_x}{dt}S(u,y)+H_y\frac{dH_y}{dt}S(x,v)+(H_x^2+H_y^2)S(u,v)\right],$$

$$\mathrm{DIV}_{\bar\omega}^{\cdot}H_x^{\cdot}\mathbf{H}=-\frac{1}{V_{\bar\omega}}\sum_{k\in\mathit{III}_0(ij)}\langle x^{l-1}\rangle_k\left\{H_{xk}^{\cdot}\left[H_{xk}\frac{\partial S_k}{\partial x}+H_{yk}\frac{\partial S_k}{\partial y}\right]\right.$$

$$\left.+\varkappa_kH_{xk}\left[H_{xk}\frac{\partial S_k(x,v)}{\partial x}+H_{yk}\frac{\partial S_k(u,y)}{\partial y}\right]\right\},$$

$$\mathrm{DIV}_{\bar\omega}^{\cdot}H_y^{\cdot}\mathbf{H}=-\frac{1}{V_{\bar\omega}}\sum_{k\in\mathit{III}_0(ij)}\langle x^{l-1}\rangle_k\left\{H_{yk}^{\cdot}\left[H_{xk}\frac{\partial S_k}{\partial x}+H_{yk}\frac{\partial S_k}{\partial y}\right]\right.$$

$$\left.+\varkappa_kH_{yk}\left[H_{xk}\frac{\partial S_k(x,v)}{\partial x}+H_{yk}\frac{\partial S_k(u,y)}{\partial y}\right]\right\},$$

$$\mathbf{H}^{\cdot}=\mathbf{H}+\varkappa\frac{d\mathbf{H}}{dt}.$$

We shall choose the coefficients $\varkappa_{ij}$ using the condition of the most rapid attenuation of the highest harmonic of the acoustic operator corresponding to a certain partial mechanical system which includes the cell *ij* (see [8]). If, as in [13], we choose a triangle as the partial system, we find $\varkappa_\omega$ from the condition $\varkappa\leqslant 2\lambda_{\max}^{-1/2}$, where $\lambda_{\max}$ is the square of the maximum natural frequency of the system obtained by varying the functional $\Delta_2 B_h$ written for one triangular cell. The quantity $\lambda_{\max}$ can be estimated fairly accurately using Gershgorin's theorem [16].

11. Complete conservatism

The system of differential-difference equations of MHD on triangular meshes (37) has the property of complete conservatism, namely:

1. For any subdomain Ω_1 of domain Ω such that $\Gamma\cap\Gamma_1=\varnothing$, where Γ and Γ_1 are the boundaries of Ω and Ω_1 respectively, the following relations hold:

a. The law of conservation of total energy

$$\sum_{\bar\omega_1}M\frac{d}{dt}\frac{u^2+v^2}{2}+\sum_{\omega_1}m\frac{d\varepsilon}{dt}+\sum_{\omega_1}\frac{d}{dt}\left(V\frac{H^2}{8\pi}\right)$$

$$=\sum_{\Gamma_1}\sum_{k\in\mathit{III}_0'(ij)}(P_H^*)_k\left(\frac{\partial V_k}{\partial x}u+\frac{\partial V_k}{\partial y}v\right)$$

$$-\frac{1}{4\pi}\sum_{\Gamma_1}\sum_{k\in\mathit{III}_0'(ij)}\left[H_{xk}^*u\frac{\partial S_k(x,\psi)}{\partial x}+H_{yk}^*v\frac{\partial S_k(y,\psi)}{\partial y}\right]$$

$$-\frac{1}{4\pi}\sum_{\Gamma_1}\sum_{k\in\mathit{III}_0'(ij)}\varkappa_k(H_{xk}u+H_{yk}v)\left[H_{xk}\frac{\partial S_k(x,\psi)}{\partial x}+H_{yk}\frac{\partial S_k(y,\psi)}{\partial y}\right],$$

where $\bar{\omega}_1$ is the set of nodes of mesh $\bar{\omega}$ in the region Ω_1, ω_1 is the set of cells ω in Ω_1; $S_0'(i, j)$ contains the cells of the configuration $S_0(i, j)$, $i, j \in G_1$ which are external with respect to Ω_1.

b. The law of conservation of momentum

$$\sum_{\bar{\omega}_1} M \frac{du}{dt} = \sum_{\Gamma_1} \sum_{k \in \mathrm{III}_0'(ij)} \left\{ P_{Hk}^* \frac{\partial V_k}{\partial x} - \frac{1}{4\pi} \left[H_{xk}^* \frac{\partial S_k(x, \psi)}{\partial x} + \varkappa_k H_{xk} \left(H_{xk} \frac{\partial S_k(x, v)}{\partial x} + H_{yk} \frac{\partial S_k(u, y)}{\partial y} \right) \right] \right\} + (l-1) \sum_{\omega_1} P_H^* S,$$

$$\sum_{\bar{\omega}_1} M \frac{dv}{dt} = \sum_{\Gamma_1} \sum_{k \in \mathrm{III}_0'(ij)} \left\{ P_{Hk}^* \frac{\partial V_k}{\partial y} - \frac{1}{4\pi} \left[H_{yk}^* \frac{\partial S_k(x, \psi)}{\partial y} + \varkappa_k H_{yk} \left(H_{xk} \frac{\partial S_k(x, v)}{\partial x} + H_{yk} \frac{\partial S_k(u, y)}{\partial y} \right) \right] \right\}, \tag{38}$$

where the second term in the right-hand part of Eq. (38) takes into account the contribution of pressure forces applied to the end faces of a cell (in the axisymmetrical case, for $l = 2$, the cell is part of a torus of triangular cross section of angle one radian, see Fig. 3).

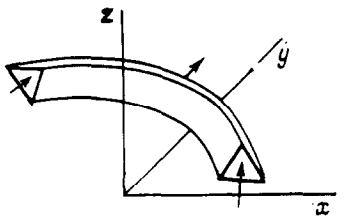

FIG. 3

c. The law of conservation of moment of momentum

$$\sum_{\bar{\omega}_1} \frac{d}{dt} M(uy - vx) = \sum_{\Gamma_1} \left(y \sum_{k \in \mathrm{III}_0'(ij)} P_{Hk}^* \frac{\partial V_k}{\partial x} - x \sum_{k \in \mathrm{III}_0'(ij)} P_{Hk}^* \frac{\partial V_k}{\partial y} \right) + \sum_{\omega_1} P_H^* \langle y \rangle - \frac{1}{4\pi} \sum_{\Gamma_1} \left(y \sum_{k \in \mathrm{III}_0'(ij)} H_{xk}^* \frac{\partial S_k(x, \psi)}{\partial x} - x \sum_{k \in \mathrm{III}_0'(ij)} H_{yk}^* \frac{\partial S_k(y, \psi)}{\partial y} \right) - \frac{1}{4\pi} \sum_{\Gamma_1} \sum_{k \in \mathrm{III}_0'(ij)} \varkappa_k (H_{xk} y - H_{yk} x) \times \left[H_{xk} \frac{\partial S_k(x, v)}{\partial x} + H_{yk} \frac{\partial S_k(u, y)}{\partial y} \right], \tag{39}$$

where

$$\langle y \rangle = \sum_{k \in \mathrm{III}_1(ij)} y_k \frac{\partial \langle x^{l-1} \rangle}{\partial x_k}$$

and the second term on the right-hand side of Eq. (39) takes into account the momentum of forces acting on the side faces of a cell for $l = 2$.

2. The energy equations written in entropy form [1].

3. For cells ω the law of conservation of mass $\rho V = m$, and the law of conservation of magnetic flux $\mathbf{H}=S(\mathbf{r}, \psi)/V$, $\psi(\alpha, \beta)=\text{const}$ are both applicable.

12. Time sampling

We shall replace the region of variation of the variable t, $0 \leqslant t \leqslant T$, by the discrete set of points $\omega_\tau=\{t_0, \ldots, t_n, \ldots, t_N;\ t_0=0,\ t_N=T;\ \tau_n=t_{n+1}-t_n\}$.

Approximating the derivatives with respect to time in Eq. (37), by finite differences, we obtain a closed multiparameter system of difference equations on triangular meshes:

$$
\begin{aligned}
&\rho_{\bar\omega}'\frac{\hat{\mathbf{v}}-\mathbf{v}}{\tau}=-\mathrm{GRAD}_{\bar\omega}^{\sigma_2}(P_H^*)^{\sigma_1}-\frac{1}{4\pi}\mathbf{D}_{\bar\omega}^*(\sigma_3),\\
&\rho\frac{\hat\varepsilon-\varepsilon}{\tau}=-[(P_H^*)^{\sigma_1}-\frac{1}{8\pi}(H_x\hat H_x+H_y\hat H_y)]\mathrm{DIV}_\omega^{\sigma_4}\mathbf{v}^{\sigma_5}+q(\sigma_3,\sigma_6),\\
&\frac{\hat\rho-\rho}{\tau}=-\rho\mathrm{DIV}_\omega^{\sigma_4}\mathbf{v}^{\sigma_5},\qquad \mathbf{H}^{\sigma_7}=S(\mathbf{r}^{\sigma_7},\psi)/V^{\sigma_7},\\
&\frac{\hat{\mathbf{r}}-\mathbf{r}}{\tau}=\mathbf{v}^{\sigma_8},\qquad P^{\sigma_9}=P(\rho^{\sigma_9},\varepsilon^{\sigma_9}),
\end{aligned}
\tag{40}
$$

where $\sigma_k\in[0, 1]$, $k=1, \ldots, 9$, $\sigma_3=(\sigma_3^1, \sigma_3^2, \sigma_3^3, \sigma_3^4)$, $f^\sigma=\sigma\hat f+(1-\sigma)f$, $f=f^n$, $\hat f=f^{n+1}$, $\tau=\tau_n$ and

$$
\begin{aligned}
&\mathrm{GRAD}_{\bar\omega}^{\sigma_1}f_\omega^{\sigma_2}=\left(-\frac{1}{V_{\bar\omega}}\sum_{k\in\mathfrak{M}_0(ij)}\frac{\partial V_k^{\sigma_1}}{\partial x}f_k^{\sigma_2},\ -\frac{1}{V_{\bar\omega}}\sum_{k\in\mathfrak{M}_0(ij)}\frac{\partial V_k^{\sigma_1}}{\partial y}f_k^{\sigma_2}\right),\\
&\mathrm{DIV}_\omega^{\sigma_1}\mathbf{v}^{\sigma_2}=\frac{1}{V_\omega}\sum_{k\in\mathfrak{M}_1(ij)}\left(\frac{\partial V^{\sigma_1}}{\partial x_k}u_k^{\sigma_2}+\frac{\partial V^{\sigma_1}}{\partial y_k}v_k^{\sigma_2}\right),\\
&\mathbf{D}_\omega^*(\sigma_3)=(\mathrm{DIV}_\omega^*H_x^*\mathbf{H}(\sigma_3),\ \mathrm{DIV}_\omega^*H_y^*\mathbf{H}(\sigma_3)),\\
&q(\sigma_3,\sigma_6)=\frac{\varkappa^{\sigma_3^2}}{4\pi V}\left[H_x^{\sigma_3^3}\frac{dH_x^{\sigma_3^2}}{dt}S(u^{\sigma_6},y^{\sigma_3^3})+H_y^{\sigma_3^3}\frac{dH_y^{\sigma_3^2}}{dt}S(x^{\sigma_3^4},v^{\sigma_6})\right.\\
&\left.+(H_x^{\sigma_3^2}H_x^{\sigma_3^4}+H_y^{\sigma_3^2}H_y^{\sigma_3^3})S(u^{\sigma_3^4},v^{\sigma_3^4})\right],\\
&\mathrm{DIV}_\omega^*H_x^*\mathbf{H}(\sigma_3)=-\frac{1}{V_{\bar\omega}}\sum_{k\in\mathfrak{M}_0(ij)}\langle x^{l-1}\rangle_k^{\sigma_3^1}\\
&\times\{(H_x^*)_k^{\sigma_3^2}\left[H_{xk}^{\sigma_3^3}\frac{\partial S_k^{\sigma_3^4}}{\partial x}+H_{yk}^{\sigma_3^3}\frac{\partial S_k^{\sigma_3^4}}{\partial y}\right]\\
&+\varkappa_k^{\sigma_3^2}H_{xk}^{\sigma_3^2}\left[H_{xk}^{\sigma_3^3}\frac{\partial S_k(x,v)^{\sigma_3^4}}{\partial x}+H_{yk}^{\sigma_3^3}\frac{\partial S_k(u,y)^{\sigma_3^4}}{\partial y}\right]\Big\}.
\end{aligned}
$$

$\mathrm{DIV}_\omega^*H_y^*\mathbf{H}(\sigma_3)$ is expressed by a similar formula.

Here we assume that for $(H_x^*)^\sigma$ and $(H_y^*)^\sigma$ the following expressions hold:

$$(H_x^*)^\sigma = H_x^\sigma + \varkappa^\sigma \frac{dH_x^\sigma}{dt} = H_x^\sigma + \varkappa^\sigma \left[\frac{S(u^\sigma, \psi)}{V^\sigma} - \frac{H_x^\sigma}{V^\sigma} \frac{\hat{V} - V}{\tau} \right],$$

and also

$$\frac{\partial V^\sigma}{\partial x} = \frac{\partial}{\partial x^\sigma} V^\sigma, \qquad \frac{\partial V^\sigma}{\partial y} = \frac{\partial}{\partial y^\sigma} V^\sigma,$$
$$\langle x'^{-1} \rangle^\sigma = V^\sigma / S^\sigma, \qquad V^\sigma = V(x^\sigma, y^\sigma), \qquad S^\sigma = S(x^\sigma, y^\sigma).$$

For $\sigma_k = 0.5$, $k = 2, \ldots, 9$ (σ_1 is arbitrary) the system of difference equations (40) has the property of complete conservation.

It should be noted that the presentation of the system of difference equations in conservative form (for $\sigma_k = 0.5$) with matched viscosity has become possible thanks to the representation of the system of differential-difference equations in operator form.

System (40) approximates the system of differential equations (6), (7), (3) and (4), generally speaking to the first order of accuracy with respect to time, and, for $\sigma_k = 0.5$, the second order of accuracy is achieved.

13. Examples of numerical computations

An explicit scheme for system (40) for $\sigma_k = 0$, $k = 1, 2, 3, 4, 6, 7, 9$, $\sigma_5 = 0.5$, $\sigma_8 = 1$ was tried out in numerous test computations.

When using the scheme it is convenient to use as a domain in Lagrange variables not only a parallelogram but also domains of other forms, e.g. a right-angled triangle or a rectangle. In the latter case, in the domain Ω a rectangular mesh of nodes $\bar{\omega}$ is introduced. To introduce a mesh of cells ω we use criterion $I(i, j) = \{0, 1\}$, in accordance with which a quadrangular cell ij is divided by one or other diagonal into two triangular cells. Here the shape of the configuration III_0 (i, j) is not fixed, but can vary from one point to another. Figure 4 shows some possible configurations of III_0 (i, j), from a four-point one to an eight-point. The second order of approximation with respect

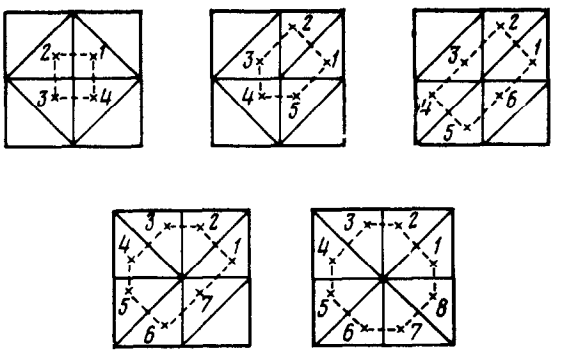

FIG. 4

to space variables is retained in this case only for configurations with two axes of symmetry; on the remaining configurations the first order is achieved.

Consider two examples.

1. The problem of compressing a spheroid $(\mathbf{H}\equiv 0)$: for $t = 0$ a gas $\rho=\rho_0=1$, $\varepsilon=\varepsilon_0=10^{-3}$, $P=\rho\varepsilon(\gamma-1)$, $\gamma=2$, occupies a domain in the shape of a flattened spheroid (because of the symmetry, only the part located in the first quadrant is considered). On the boundary of the spheroid the internal pressure

$$P=\begin{cases} t & \text{for} \quad t\leqslant t^*, \\ P^*=\text{const} & \text{for} \quad t>t^*, \end{cases}$$

is given, where t^* is the time taken by a shock wave to travel a distance equal to the semi-minor axis of the spheroid. The results of computations (see Fig. 5, $t = 1.1473$) were compared with

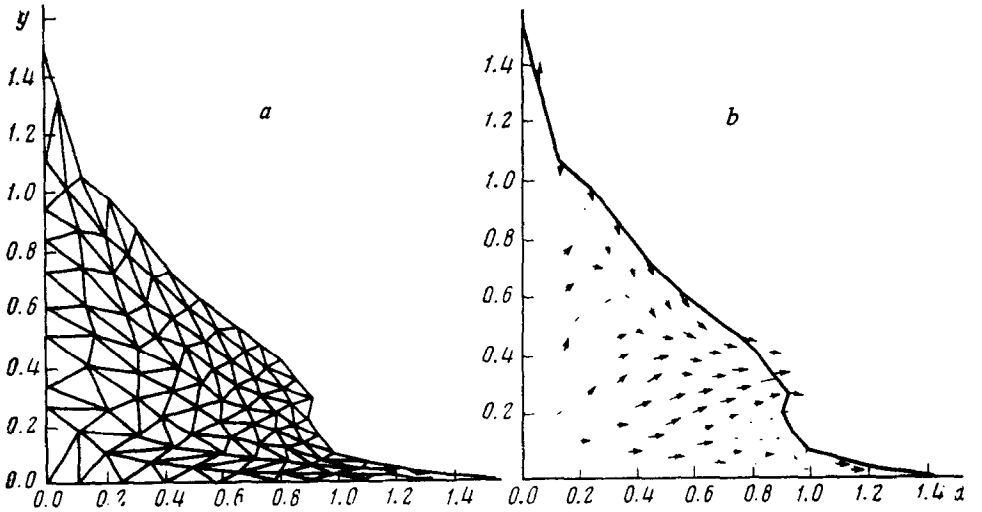

FIG. 5. a – computational mesh, b – field of velocities

those given in (10). In this paper the computations relating to the problem were carried out on quadrangular meshes with reinterpolation for long computation times. The comparison shows that in computation on triangular meshes the cumulative effect is more pronounced; this can probably be explained by the absence of diffusion of momentum, which appears during rearrangement of a mesh.

2. A flat domain occupied by gas with a frozen magnetic field has the shape of a spheroid stretched along the y axis. Figure 6 shows a quarter of this domain, divided by the difference mesh,

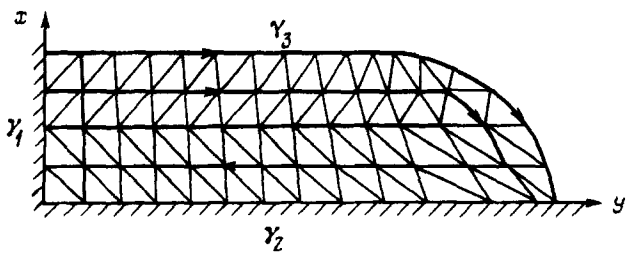

FIG. 6

occupying the top right hand quadrant in the (x, y) plane. On the boundaries γ_1 and γ_2 the conditions of symmetry are specified, and on γ_3 the external pressure $P^* = \text{const}$. The arrows represent the magnetic lines of force. Along these the density and the specific internal energy were assumed to be invariable, satisfying the condition $P + H^2/8\pi = P^*$ for $y = 0$. The equation of state was $P=\rho\varepsilon(\gamma-1)$, $\gamma=2$, and it was taken that at the initial instant the gas was at rest.

Since the initial state is unbalanced, the gas begins to be compressed along the y axis due to the action of the magnetic lines of force. Figure 7 shows the results of numerical modelling of this

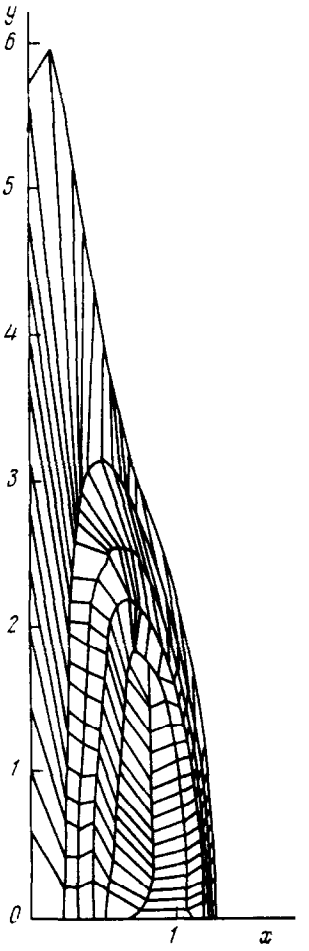

FIG. 7

process. The number of time steps was around 39,000. The lines dividing the mesh into triangles are not shown.

Translated by W. Chrzczonowicz

REFERENCES

1. SAMARSKII, A. A. and POPOV, Yu. P., *Difference methods of solving problems of gas dynamics* (Raznostnye metody reshenia zadach gazovoi dinamiki), Nauka, Moscow, 1980.

2. BELOTSERKOVSKII, O. M. (Ed.), *Numerical investigation of problems in gas dynamics* (Chislennoe issledovanie zadach gazovoi dinamiki), Nauka, Moscow, 1974.

3. BABENKO, K. I. (Ed.), *Theoretical principles and design of numerical algorithms in problems of mathematical physics* (Teoreticheskie osnovy i konstruirovanie zadach matematicheskoi fiziki), Nauka, Novosibirsk, 1979.

4. SHUL'TS, U. D. Two-dimensional finite-difference hydrodynamic equations in Lagrange variables, in: *Computational methods in hydrodynamics* (Vychisl. metody v gidrodinamike), Moscow, Mir, 1967.

5. GOLOVIZNIN, V. M., SAMARSKII, A. A. and FAVORSKII, A. P. The use of the principle of least action for building discrete mathematical models in MHD. *Dokl. Akad. Nauk SSSR*, **246**, 5, 1083–1088, 1979.

6. GOLOVIZNIN, V. M., *et al.* Variational schemes of MHD in an arbitrary system of coordinates. *Zh. vȳchisl. Mat. mat. Fiz.*, **21**, 1, 54–68, 1981.

7. FAVORSKII, A. P. Variational discrete models of MHD equations. *Diff. Ur-niya* **16**, 7, 1308–1321, 1980.

8. GOLOVIZNIN, V. M. *et al.* On certain properties of variational-difference equations of MHD and artificial dissipative processes of the 'pseudoviscosity' type. Preprint No. 25, IPM AN SSSR, Moscow, 1976.

9. GOLOVIZNIN, V. M. On a method of introducing artificial dissipative processes into MHD variational-difference schemes. Preprint No. 50, IPM AN SSSR, Moscow, 1980.

10. GOLOVIZNIN, V. M., TISHKIN, V. F. and FAVORSKII, A. P. Variational approach to the construction of difference schemes for the equations of MHD in spherical coordinates. Preprint No. 16, IPM AN SSSR, Moscow, 1977.

11. GOL'DIN, V. Ya., *et al.* Numerical computation of the equations of two-dimensional gas dynamics with detonation, in: *Numerical methods of the mechanics of a continuous medium* (Chisl. metody mekhan. sploshnoi sredy) Vol. 4, No. 3. Novosibirsk, Nauka, 1973.

12. VOLCHINSKAYA, M. N. and CHETVERUSHKIN, B. N. Methods for solving two-dimensional non-stationary problems of the dynamics of a radiating gas. Preprint No. 98, IPM AN SSSR, Moscow, 1977.

13. GOLOVIZNIN, V. M. and KORSHUNOV, V. K. On two-dimensional variational-difference schemes of gas dynamics on triangular meshes. Preprint No. 17, IPM AN SSSR, Moscow, 1980.

14. GOLOVIZNIN, V. M. and KORSHUNOV, V. K. On two-dimensional variational-difference schemes of MHD on triangular meshes. Preprint No. 99, IPM AN SSSR, Moscow, 1980.

15. GOLOVIZNIN, V. M., KORSHUNOV, V. K. and TARAN, M. D. The approximation of differential operators on irregular oblique-angled computational meshes. Preprint No. 157, IPM AN SSSR, Moscow, 1980.

16. GANTMAKHER, F. P. *Theory of matrices* (Teoriya matrits), Nauka, Moscow, 1968.

U.S.S.R. Comput. Maths. Math. Phys. Vol. 22, No. 4, pp. 178–185, 1982. 0041–5553/82$07.50+.00
Printed in Great Britain ©1983. Pergamon Press Ltd.

ON THE CORRECTNESS OF THE DIFFERENCE APPROXIMATION OF THE BOUNDARY CONDITIONS IN PENETRATION PROBLEMS*

V. V. BASHYROV and V. A. SVIDINSKII

Chelyabinsk

(*Received* 4 *July* 1980)

THE EQUATIONS of motion of the mechanics of a continuous medium in an arbitrary mixed Euler–Lagrange system of coordinates are examined. The boundary condition on the body-medium boundary is formulated, as it applies to the problem of an absolutely rigid body penentrating into a

*Zh. vȳchisl. Mat. mat. Fiz., 22, 4, 943–949, 1982.