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VARIATIONAL SCHEMES OF MAGNETOHYDRODYNAMICS IN AN ARBITRARY COORDINATE SYSTEM*

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DIFFERENCE schemes are constructed for the three-dimensional MHD equations, on the basis of a variational principle similar to the principle of least action in classical mechanics. An arbitrary curvilinear coordinate system is used. The properties of the resulting difference equations are studied.

1. Differential equations

1. In the numerical simulation of the motion of a continuous medium it proves convenient to use a formal approach based on the introduction of the Lagrange function [1, 2]. In the absence of dissipative processes, the Lagrangian of the continuous medium, immersed in a magnetic field, can be written as

$$L=\int_\Omega \left(\frac{v_iv^i}{2}-\varepsilon-\frac{H_iH^i}{8\pi\rho}\right)\rho g^{1/2}\,d\Omega. \tag{1.1}$$

Here, as the Euler reference system we use any fixed curvilinear coordinate system x^i, $i=1, 2, 3$; ρ is the density, ϵ is the specific internal energy, v_i, v^i are the covariant and contravariant components of the velocity vector $\mathbf{v}$, H_i, H^i are the covariant and contravariant components of the magnetic field vector $\mathbf{H}$, Ω is the domain occupied by the medium, $d\Omega=dx^1dx^2dx^3$; g is the determinant of the metric tensor g_{ik}. We assume henceforth that the Latin indices take values from 1 to 3, while a repeated index denotes summation.

On transforming to Lagrange coordinates q^i, i.e. putting $x^i=x^i(q^1, q^2, q^3, t)$, $dx^i/dt=v^i$ (q^1, q^2, q^3, t), we can rewrite Lagrangian (1.1) as

$$L=\int_{\Omega_q} \left(\frac{v_iv^i}{2}-\varepsilon-\frac{H_iH^i}{8\pi\rho}\right)\rho J g^{1/2}\,d\Omega_q,$$

where $\Omega_q=\Omega(\mathbf{x}(\mathbf{q}))$, $d\Omega_q=dq^1dq^2dq^3$, $J=\partial(x^1, x^2, x^3)/\partial(q^1, q^2, q^3)>0$.

Specification of the Lagrangian along with the supplementary constraints reflecting the typical features of the flow, fully describes the MHD system. In our case, the role of the constraints will be played by the conditions for continuity, conservation of the magnetic flux, and adiabaticity.

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2. The continuity condition follows from the mass conservation law for an arbitrary fluid volume $\Omega' \subset \Omega$:

$$\frac{d}{dt}\int\limits_{\Omega} \rho g^{1/2}\, d\Omega = \frac{d}{dt}\int\limits_{\Omega_q} \rho J g^{1/2}\, d\Omega_q = 0,$$

or

$$\rho J g^{1/2} = \rho_0(\mathbf{q}). \tag{1.2}$$

If, as q^i, we take the coordinates of particles of the medium at the initial instant, then ρ_0 signifies the density at $t = 0$. On differentiating (1.2) with respect to time, we obtain the relation

$$\frac{d\rho}{dt} + \rho g^{-1/2} \frac{\partial g^{1/2} v^i}{\partial x^i} = 0,$$

which is the equation of continuity in the curvilinear coordinate system.

3. The condition for conservation of magnetic flux through any fluid surface Σ (frozen conditions for an infinitely electrically conducting medium) may be written as

$$\frac{d}{dt}\int\limits_{\Sigma} H^i g^{1/2}\, dS_i = 0;$$

here, dS_i are the components of the pseudo-vector of an elementary area [3]. The quantities $g^{1/2} dS_i$ are the components of the vector, directed along the normal to the surface element, and equal in absolute value to its area. Using the transformation rule for pseudo-vectors [4], we find that

$$\frac{d}{dt}\int\limits_{\Sigma_q} H^i J g^{1/2} \frac{\partial q^k}{\partial x^i}\, dS_{qk} = 0. \tag{1.3}$$

Here, dS_{qk} are the components of dS in the Lagrange coordinate system. From (1.3) we easily obtain the frozen conditions in the form

$$g^{1/2} J_i^k H^i = \Phi^k(\mathbf{q}), \tag{1.4}$$

where

$$J_i^k = J\frac{\partial q^k}{\partial x^i} = \frac{1}{2} e^{kmn} e_{irl} \frac{\partial x^r}{\partial q^m} \frac{\partial x^l}{\partial q^n},$$

e^{kmn}, e_{irl} are absolutely antisymmetric tensors.

Using the property of J_i^k:

$$J_i^k \frac{\partial x^m}{\partial q^k} = \delta_i^m J, \tag{1.5}$$

we can easily solve Eq. (1.4) for H^i:

$$J g^{1/2} H^i = \Phi^k \partial x^i / \partial q^k. \tag{1.6}$$

From (1.6) we obtain at once the equations of induction and magnetic field energy variation in the Lagrange variables:

$$Jg^{1/2}\frac{dH^i}{dt}=-\frac{dJ^{1/2}g}{dt}H^i+\Phi^k\frac{\partial v^i}{\partial q^k}, \tag{1.7}$$

$$\frac{d}{dt}Jg\;\frac{H^iH_i}{8\pi}=-\frac{H_iH^i}{8\pi}\frac{dJg^{1/2}}{dt}+\frac{H_i\Phi^k}{4\pi}\frac{\partial v^i}{\partial q^k}+\frac{Jg^{1/2}H^mH^n}{8\pi}\frac{\partial g_{mn}}{dt}. \tag{1.8}$$

4. The condition for the flow to be adiabatic is given by

$$d\varepsilon=-pd\rho^{-1}. \tag{1.9}$$

Jointly with the equation of continuity (1.2), condition (1.9) gives the law of variation of ϵ:

$$\rho\frac{d\varepsilon}{dt}=-p\frac{d}{dt}(Jg^{1/2}). \tag{1.10}$$

5. By the principle of least action, the motion of the medium occurs in such a way that the functional of action

$$F=\int_{t_1}^{t_2}L(t)\,dt$$

takes a stationary value [5, 6], i.e.

$$\delta F=\int_{t_1}^{t_2}\int_{\Omega_q}\Big\{\rho_0\Big[g_{il}v^l\delta v^i+\frac{v^iv^l}{2}\frac{\partial g_{il}}{\partial x^k}\delta x^k-\delta\varepsilon\Big]$$

$$-\delta(Jg^{1/2})\frac{H_iH^i}{8\pi}-\frac{Jg^{1/2}}{4\pi}g_{il}H^l\delta H^i-J\frac{g^{1/2}H^lH^i}{8\pi}\frac{\partial g_{li}}{\partial x^k}\delta x^k\Big\}\,d\Omega_q\,dt=0.$$

Using the supplementary conditions (1.2), (1.6), and (1.9), we can eliminate from this the variations $\delta\varepsilon$, δv^i, δH^i, $\delta(Jg^{1/2})$, by expressing them in terms of variation λx^i.

On equating to zero the factors with the independent variations, we arrive at the equations of motion of magnetohydrodynamics:

$$\rho_0\left(\frac{dv_i}{dt}-\frac{v^kv^l}{2}\frac{\partial g_{kl}}{\partial x^i}\right)$$
$$=-g^{1/2}\frac{\partial}{\partial a^l}(p^*J_i{}^l)+\frac{\partial}{\partial q^l}\frac{H_i\Phi}{4\pi}-\frac{H^kH^l}{8\pi}\frac{\partial q_{kl}}{\partial x^i}Ja^{1/2} \tag{1.11}$$

where $p^*=p+H_iH^i/8\pi$.

Equations (1.2) and (1.6), or (1.7), and (1.10) and (1.2), jointly with the kinematic relations $dx^i/dt=v^i$ and the equation of state $p=p(\rho,\epsilon)$, fully define the behaviour of a dissipationless MHD medium under the appropriate initial and boundary conditions.

For the distribution of the initial magnetic field, we need to require that the solenoidal condition be satisfied:

$$\operatorname{div} \mathbf{H} = g^{-1/2} \frac{\partial}{\partial x^i} (g^{1/2} H^i) = 0, \tag{1.12}$$

which can be expressed in terms of the fluxes Φ^i as follows: $\partial \Phi^i / \partial q^i = 0$.

2. The Discrete model

1. We shall assume that Ω_q is the unit cube in the space of Lagrange variables q^i. We introduce into Ω_q a rectangular difference mesh with steps $\Delta_q{}^i = h_i$. We shall use Greek indices to indicate mesh quantities. With each node we associate a triple of positive integers $(\alpha, \beta, \gamma) \in \omega_h = \{(\alpha, \beta, \gamma): \alpha = 0, 1, \ldots, N;\ \beta = 0, 1, \ldots, M;\ \gamma = 0, 1, \ldots, P\}$. The set of all nodes defining a mesh cell (elementary parallelepiped) is denoted by Z_1, assuming that the cell index is equal to the node index $(\alpha, \beta, \gamma) \in Z_1$, at which min $(\alpha + \beta + \gamma)$ is reached. The set of all cells containing the given node (α, β, γ) as a vertex will be written as $Z_2\ (\alpha, \beta, \gamma)$. We introduce the set of cells $\bar{\omega}_h$ and the set of all interior nodes $\mathring{\omega}_h$, and the spaces of mesh functions R_h and $\bar{R}_h$, defined on ω_h and $\bar{\omega}_h$ respectively.

The quantities x^i, v^i, v_i and g_{ik} are referred to the mesh nodes by denoting them respectively by $\{(x^i)_{\alpha\beta\gamma}\}$, etc. Then, the connection between the covariant and contravariant components of v is written as usual for each node:

$$v_i = g_{ik} v^k \quad \text{for} \quad (\alpha, \beta, \gamma) \in \omega_h \tag{2.1}$$

(to simplify the notation, we omit the index $\alpha\beta\gamma$).

The thermodynamic quantities, and also $H_i H^i$, J and $J_k{}^m$ will be referred to the centres of the Lagrange cells and indicated by the cell index. Since $\{(g_{ik})_{\alpha\beta\gamma}\} \in R_h$, the relationship between $(H_i)_{\alpha\beta\gamma}$ and $(H^i)_{\alpha\beta\gamma}$ is given by

$$(H_i)_{\alpha\beta\gamma} = \langle g_{ik} \rangle_{\alpha\beta\gamma} (H^k)_{\alpha\beta\gamma}, \tag{2.2}$$

where $\{\langle g_{ik} \rangle_{\alpha\beta\gamma}\} \in \bar{R}_h$ is an approximation of g_{ik} at the cell centre, e.g. of the type

$$\langle g_{ik} \rangle_{\alpha\beta\gamma} = \frac{1}{8} \sum_{\nu \in Z_1(\alpha\beta\gamma)} (g_{ik})_\nu. \tag{2.3}$$

2. Let us find the difference analogues for partial derivatives $\partial f / \partial q^i$. For the Lagrange cell (α, β, γ) we introduce the expressions

$$\begin{aligned}
\partial_{\bar{1}} f &= \frac{1}{4h_1} \sum_{\alpha_1, \beta_1, \gamma_1 = 0,\, 1} (-1)^{\alpha_1 + 1} f_{\alpha+\alpha_1, \beta+\beta_1, \gamma+\gamma_1}, \\
\partial_{\bar{2}} f &= \frac{1}{4h_2} \sum_{\alpha_1, \beta_1, \gamma_1 = 0,\, 1} (-1)^{\beta_1 + 1} f_{\alpha+\alpha_1, \beta+\beta_1, \gamma+\gamma_1}, \\
\partial_{\bar{3}} f &= \frac{1}{4h_3} \sum_{\alpha_1, \beta_1, \gamma_1 = 0,\, 1} (-1)^{\gamma_1 + 1} f_{\alpha+\alpha_1, \beta+\beta_1, \gamma+\gamma_1},
\end{aligned} \tag{2.4}$$

where $\{f_{\alpha\beta\gamma}\}\in R_h$. Expressions (2.4) approximate $\partial f/\partial q^i$ at the cell centre. For sufficiently smooth f,

$$\partial_{\bar{i}} f = \frac{\partial f}{\partial q^i}\bigg|_{\overline{\alpha\beta\gamma}} - O(h^2). \tag{2.4'}$$

The bar over an index means that the approximation is performed at the cell centre, $h^2=h_1^2+h_2^2+h_3^2$.

Notice that, for $\partial_{\bar{i}} f$, we have the relation

$$\partial_{\bar{i}} f = \sum_{\nu\in Z_1(\alpha\beta\gamma)} f_\nu \frac{\partial}{\partial (x^k)_\nu} (\partial_{\bar{i}} x^k).$$

The difference analogue of the derivative $\partial f/\partial q^i$, defined at a mesh node, is introduced as follows:

$$(\partial_i f)_{\alpha\beta\gamma} = -\sum_{\nu\in Z_2(\alpha\beta\gamma)} f_\nu \frac{\partial}{\partial (x^k)_{\alpha\beta\gamma}} (\partial_{\bar{i}} x^k)_\nu.$$

It is easily seen that $\partial_i f$, $f\in \bar{R}_h$, is the second-order approximation of $\partial f/\partial q^i$ at a node.

3. We discretize the quantities $J_j{}^i$ and J by using $\partial_{\bar{i}}$ and J by using $\partial_{\bar{i}}$. We associate with $J_j{}^i$ and J the difference expressions $S_j{}^i$ and S, obtained by formal replacement of the derivative $\partial/\partial q^i$ by $\partial_{\bar{i}}$:

$$S_j{}^i = \frac{1}{2} e^{imn} e_{jkl} \partial_{\bar{m}} x^k \partial_{\bar{n}} x^l, \tag{2.5}$$

$$S = \frac{1}{6} e^{imn} e_{jkl} \partial_{\bar{m}} x^k \partial_{\bar{n}} x^l \partial_{\bar{i}} x^j. \tag{2.6}$$

Here,

$$S_j^i = J_j^i|_{\overline{\alpha\beta\gamma}} + O(h^2), \qquad S = J|_{\overline{\alpha\beta\gamma}} + O(h^2)$$

and the difference analogue of identity (1.5) is satisfied:

$$S_j{}^i \partial_{\bar{i}} x^m = \delta_j{}^m S; \tag{2.7}$$

moreover, we have

$$\sum_{\nu\in Z_1(\alpha\beta\gamma)} \frac{\partial S_{\alpha\beta\gamma}}{\partial (x^m)_\nu} = 0.$$

Using (2.7), (2.4), and (2.4′), we can show that, if $f\in R_h$, and $\varphi\in\bar{R}_h$, then

$$\sum_{\nu\in Z_1(\alpha\beta\gamma)} f_\nu \frac{\partial S_{\alpha\beta\gamma}}{\partial (x^k)_\nu} = S_k{}^i \partial_{\bar{i}} f = J\frac{\partial f}{\partial x^k}\bigg|_{\overline{\alpha\beta\gamma}} + o(h^2), \tag{2.8}$$

$$\sum_{\nu\in Z_2(\alpha\beta\gamma)} \varphi_\nu \frac{\partial S_\nu}{\partial (x^k)_{\alpha\beta\gamma}} = [\partial_i(\varphi S_k^i)]_{\alpha\beta\gamma} = -J\frac{\partial \varphi}{\partial x^k}\bigg|_{\alpha\beta\gamma} + O(h^2).$$

Consequently, the expressions

$$\sum_{Z_1} f_\nu \frac{\partial S}{\partial (x^k)_\nu} \text{ and} - \sum_{Z_2} \varphi_\nu \frac{\partial S_\nu}{\partial x^k}$$

can be regarded as difference analogues of the operators $J\partial/\partial x^k$, referred respectively to the cell and a mesh node.

4. Let us dwell on the approximation of the expression $g^{1/2}$ in a cell. Since g_{ik} relate to the nodes, the value of $g^{1/2}$ at the cell centre can be found e.g. as follows:

$$\langle\langle g^{1/2} \rangle\rangle = \frac{1}{8} \sum_{\nu \in Z_1(\alpha\beta\gamma)} g_\nu^{1/2}, \tag{2.9}$$

where $g_\nu = \det (g_{ik})_\nu$, while $\langle\langle g^{1/2} \rangle\rangle = g^{1/2}|_{\overline{\alpha\beta\gamma}} + O(h^2)$.

On the other hand, if we know the difference expression for $V = Jg^{1/2}$ (see e.g. [7–9]), we can find the mean value of $g^{1/2}$, namely,

$$\langle g^{1/2} \rangle_{\alpha\beta\gamma} = V_{\alpha\beta\gamma}/S_{\alpha\beta\gamma}. \tag{2.10}$$

Obviously, $\langle g_{1/2} \rangle = g^{1/2}|_{\overline{\alpha\beta\gamma}} + O(h^2)$, provided that $V_{\alpha\beta\gamma}$ approximates $(g^{1/2}J)|_{\overline{\alpha\beta\gamma}}$ to second order.

3. Differential-difference equations of MHD

1. The set of Lagrange cells $\bar{\omega}_h$ can be regarded as a discrete model of the continuous medium. The state of each cell is defined by the quantities $p_{\alpha\beta\gamma}$, $\rho_{\alpha\beta\gamma}$, $V_{\alpha\beta\gamma}$, $H_{\alpha\beta\gamma}$, $\varepsilon_{\alpha\beta\gamma}$ and $(v^i)_\nu$, $\nu \in Z(\alpha\beta\gamma)$. With the variational approach, we can construct a class of differential-difference equations, describing the matched variation of all the MHD quantities.

For the discrete system of $\bar{\omega}_h$ the Lagrangian is defined as the difference between the kinetic and potential energies:

$$L^h = \sum_{(\alpha\beta\gamma) \in \bar{\omega}_h} \rho_{\alpha\beta\gamma} V_{\alpha\beta\gamma} (K_{\alpha\beta\gamma} - \Pi_{\alpha\beta\gamma});$$

here, $K_{\alpha\beta\gamma}$, $\Pi_{\alpha\beta\gamma}$ denote respectively the specific kinetic and potential energies of a cell:

$$K_{\alpha\beta\gamma} = \frac{1}{8} \sum_{\nu \in Z_1(\alpha\beta\gamma)} \left(\frac{v^k v_k}{2}\right)_\nu,$$

$$\Pi_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma} + \frac{(H^k H_k)_{\alpha\beta\gamma}}{8\pi\rho_{\alpha\beta\gamma}}.$$

Obviously, $L^h = L + O(h^2)$. Notice that there are other ways in which the kinetic cell can be approximated.

For the difference Lagrangian we introduce the functional of action

$$F_h = \int_{t_1}^{t_2} L^h(t)\,dt.$$

In order to obtain the supplementary conditions on the variations of the functions appearing in L^h, we consider the difference analogues of the conditions for continuity and freezing of the magnetic field.

2. Using the expression for V introduced in Section 2, we can write

$$V_{\alpha\beta\gamma}\rho_{\alpha\beta\gamma} \equiv m_{\alpha\beta\gamma} = g^{1/2} J\rho\,|_{\overline{\alpha\beta\gamma}} + O(h^2), \tag{3.1}$$

where m signifies the time-invariant mass of the Lagrange cell (we assume that no mass exchange occurs between cells). From (3.1) we obtain the differential-difference equation of continuity

$$V\frac{d\rho}{dt} + \rho\frac{dV}{dt} = 0. \tag{3.2}$$

The time derivative of V can be approximated in two ways.

If the expression for the volume is regarded as a function of the coordinates, $V_{\alpha\beta\gamma} = V_{\alpha\beta}\,\{x^i_\nu,\ \nu \in Ш_1(\alpha, \beta, \gamma)\}$, then we have to put

$$\frac{dV}{dt} = \sum_{\nu \in Z_1(\alpha\beta\gamma)} \frac{\partial V}{\partial (x^k)_\nu}\frac{\partial x_\nu^k}{dt} = \sum_{\nu \in Z_1(\alpha\beta\gamma)} \frac{\partial V}{\partial (x^k)_\nu} v_\nu^k. \tag{3.3}$$

Since V is a known algebraic function, $\partial V/\partial (x^k)$ can be evaluated explicitly.

On the other hand, the equation $dg^{1/2}J/dt = J_k{}^i \partial g^{1/2} v^k/\partial x^i$ can be approximated with the aid of (2.8) by the expression

$$\frac{dV}{dt} = S_k{}^i \partial_{\bar{i}}(v^k g^{1/2}) = \sum_{\nu \in Z_1(\alpha\beta\gamma)} \frac{\partial S}{\partial (x^k)_\nu}(v^k g^{1/2})_\nu. \tag{3.4}$$

We will show that Eq. (3.2) approximates the differential equation of continuity (1.2) at the cell centre, if we use (3.4) or (3.3) for dV/dt. In the former case, this is obvious, since, from (2.5), we have

$$\frac{dV}{dt} = S_k{}^i \partial_{\bar{i}} (v^k g^{1/2}) = J^i_k \frac{\partial v^k g^{1/2}}{\partial x^i}\Big|_{\overline{\alpha\beta\gamma}} + O(h^2).$$

In the case of (3.3), using (2.8)–(2.10), we can write

$$\sum_{\nu\in Z_1(\alpha\beta\gamma)} \frac{\partial V}{\partial (x^k)_\nu} v_\nu{}^k = \sum_{\nu\in Z_1(\alpha\beta\gamma)} \left[\langle g^{1/2}\rangle \frac{\partial S}{\partial (x^k)_\nu} v_\nu{}^k + \frac{S\partial \langle g^{1/2}\rangle}{\partial (x^k)_\nu} v_\nu{}^k \right] = Jg^{1/2} \frac{\partial v^k}{\partial x^k}\Big|_{\overline{\alpha\beta\gamma}} + \frac{S}{8} \sum_{\nu\in Z_1(\alpha\beta\gamma)} v_\nu{}^k \frac{\partial (g^{1/2})_\nu}{\partial (x^k)_\nu}$$
$$+ O(h)^2 = J\left(g^{1/2}\frac{\partial v^k}{\partial x^k} + v^k \frac{\partial g^{1/2}}{\partial x^k}\right)\Big|_{\overline{\alpha\beta\gamma}} + O(h^2).$$

3. Let us discretize the equation for the magnetic field being frozen. The difference expression

$$\langle g^{1/2}\rangle H^k S_k{}^i = \Phi^i, \tag{3.5}$$

approximates, in view of (2.6) and (2.10), the frozen condition (1.4) at the cell centre to accuracy $O(h^2)$. It can be shown that Φ^k represents the "difference" fluxes through the planes passing through the cell centre at right angles to q^k. On performing the convolution of $\Phi^i \partial_i x^k$ in the light of (2.6), we can transform Eq. (3.5) to the difference analogue of Eq. (1.6):

$$VH^k = \Phi^i \partial_i x^k. \tag{3.5 }$$

Time-differentiation of (3.5′) leads to the difference equation of induction, corresponding to Eq. (1.8):

$$V\frac{dH^k}{dt} = -H^k \frac{dV}{dt} + \Phi^i \partial_i v^k, \tag{3.6}$$

where, by dV/dt we mean one of expressions (3.3) or (3.4).

Applying standard transformations to Eq. (3.6) and the equation of induction, written for the covariant components of H_k, we obtain the equation for the magnetic field energy of a cell in the form

$$\frac{d}{dt}\left(\frac{H^k H_k}{8\pi} V\right) = -\frac{H^k H_k}{8\pi} \frac{d}{dt} V_k + \frac{H_k \Phi^i \partial_i v^k}{4\pi} + \frac{VH^n H^m}{8\pi} \frac{d\langle g_{mn}\rangle}{dt}. \tag{3.7}$$

It can be easily be seen that Eq. (3.7) approximates Eq. (1.9) to accuracy $O(h^2)$. Various approximations are admissible for $d\langle g_{mn}\rangle/dt$, as they are for the derivative dV/dt. In the general case, obviously,

$$\frac{d\langle g_{ik}\rangle}{dt} = \sum_{\nu\in Z_1(\alpha\beta\gamma)} \frac{\partial \langle g_{ik}\rangle}{\partial x_\nu{}^l} v_\nu{}^l. \tag{3.8}$$

If $\langle g_{ik}\rangle$ is evaluated from expression (2.3), then (3.8) transforms to

$$\frac{d\langle g_{ik}\rangle}{dt} = \frac{1}{8} \sum_{\nu\in Z_1(\alpha\beta\gamma)} \left(\frac{\partial g_{ik}}{\partial x^l}\right)_\nu v_\nu{}^l. \tag{3.9}$$

Finally, let us consider the question of satisfying the condition for the magnetic field to be solenoidal.

Using the relations (2.8) for difference differentiation and relation (3.5), we can approximate expression (1.12) by the equation

$$\partial_i(\langle g^{1/2}\rangle H^k S_k^i) = \partial_i \Phi^i = 0. \tag{3.10}$$

But, by (2.8), the left-hand side of (3.10) is none other than the approximation of div **H**. Consequently, the condition

$$\partial_i \Phi^i = 0 \tag{3.11}$$

can be regarded as the condition for the magnetic field to be solenoidal. Hence, as in the differential case, if relation (3.11) is satisfied at the instant $t = t_0$, then the magnetic field will remain solenoidal at subsequent instants.

4. Let us find the conditions for the first variation of the functional of action F_h:

$$\begin{aligned}\delta F_h = \int_{t_0}^{t_1} \sum_{(\alpha\beta\gamma)\in\bar{\omega}_h} \Big\{ & m_{\alpha\beta\gamma}\Big[\sum_{\nu\in Z_t(\alpha\beta\gamma)} (v_k)_\nu \delta v_\nu^k + \Big(\frac{v^i v^k}{2}\delta g_{ik}\Big)_\nu - \delta\varepsilon_{\alpha\beta\gamma}\Big] \\ & - \delta V_{\alpha\beta\gamma}\frac{(H^k H_k)_{\alpha\beta\gamma}}{8\pi} - V_{\alpha\beta\gamma}\frac{(H_k \delta H^k)_{\alpha\beta\gamma}}{4\pi} \\ & - \frac{(VH^k H^l)_{\alpha\beta\gamma}}{8\pi}\delta\langle g_{kl}\rangle_{\alpha\beta\gamma}\Big\} h_1 h_2 h_3\, dt.\end{aligned} \tag{3.12}$$

to vanish. We take $\delta(x^i)_{\alpha\beta\gamma}$ as independent variations, and find their connection with the variations of the other quantities, appearing in (3.12). The connection is given by

$$\begin{aligned}&\delta v^i = \delta\frac{dx^i}{dt} = \frac{d}{dt}\delta x^i, \qquad \delta g_{ik} = \frac{\partial g_{ik}}{\partial x^n}\delta x^n, \\ &V\delta H^k = -H^k \delta V + \Phi^i \partial_{\bar{i}} \delta x^k, \qquad \delta\varepsilon = \Big(\frac{\partial\varepsilon}{\partial\rho^{-1}}\Big)\Big(\frac{\partial\rho^{-1}}{\partial V}\Big)\delta V = -\frac{p}{m}\delta V.\end{aligned} \tag{3.13}$$

For δV and $\delta\langle g_{ik}\rangle$ we use two types of relations, corresponding to (3.3), (3.8) or (3.4), (3.9):

$$\delta V = \sum_{\nu\in Z_t(\alpha\beta\gamma)} \frac{\partial \bar{V}}{\partial(x^k)_\nu}\delta x_\nu^k, \tag{3.14}$$

$$\delta\langle g_{ik}\rangle = \sum_{\nu\in Z_t(\alpha\beta\gamma)} \frac{\partial\langle g_{kl}\rangle}{\partial(x^n)_\nu}\delta x_\nu^n$$

and

$$\begin{aligned}&\delta V = S_k^i \partial_{\bar{i}}(\delta x^k g^{1/2}) = \sum_{\nu\in Z_t(\alpha\beta\gamma)} \frac{\partial S}{\partial(x^n)_\nu}(g^{1/2}\delta x^n)_\nu, \\ &\delta\langle g_{kl}\rangle = \frac{1}{8}\sum_{\nu\in Z_t(\alpha\beta\gamma)} \delta(g_{kl})_\nu = \frac{1}{8}\sum_{\nu\in Z_t(\alpha\beta\gamma)} \Big(\frac{\partial g_{kl}}{\partial x^n}\Big)_\nu \delta x_\nu^n.\end{aligned} \tag{3.15}$$

It is easily seen that the presence of the different approximations of δV and $\delta\langle g_{kl}\rangle$ leads to different dynamic equations, which are obtained by substituting (3.13) and (3.14), or (3.15), into (3.12), and performing the appropriate transformations. Let us write these equations in their final form:

$$M\left(\frac{\partial v_i}{\partial t}-\frac{v^k v^l}{2}\frac{\partial g_{kl}}{\partial x^i}\right)=\sum_{\nu\in Z_2(\alpha\beta\gamma)} P_\nu^{\cdot}\frac{\partial V_\nu}{\partial x^i} -\partial_n\frac{H_i\Phi^n}{4\pi}+\sum_{\nu\in Z_2(\alpha\beta\gamma)}\frac{\partial\langle g_{ik}\rangle_\nu}{\partial x^i}\frac{(VH^iH^k)_\nu}{8\pi} \tag{3.16}$$

(this corresponds to the case with (3.14)),

$$M\left(\frac{dv_i}{dt}-\frac{v^k v^l}{2}\frac{\partial g_{kl}}{\partial x^i}\right)=g^{1/2}\sum_{\nu\in Z_2(\alpha\beta\gamma)} p_\nu^{\cdot}\frac{\partial S_\nu}{\partial x^i} -\partial_n\frac{H_i\Phi^n}{4\pi}+\frac{1}{8}\frac{\partial g_{ik}}{\partial x^i}\sum_{\nu\in Z_1(\alpha\beta\gamma)}\frac{(VH^kH^l)_\nu}{8\pi} \tag{3.17}$$

(corresponds to case with (3.15)). Here,

$$M=\hat{m}_{\alpha\beta\gamma}=\frac{1}{8}\sum_{\nu\in Z_2(\alpha\beta\gamma)} m_\nu.$$

Since $M=\rho Jg^{1/2}|_{\alpha\beta\gamma}+O(h^2)$, the fact that the approximation for Eqs. (3.17) is to second order is easily proved on the basis of the relations of Section 2. In Eq. (3.16) we have to estimate the order of approximation of the expression

$$\sum_{\nu\in Z'_2(\alpha\beta\gamma)} p_\nu\frac{\partial V_\nu}{\partial x^i}=\sum_{\nu\in Z_2(\alpha\beta\gamma)}\left(p_\nu^{\cdot}\langle g^{1/2}\rangle_\nu\frac{\partial S_\nu}{\partial x^i}+p_\nu^{\cdot}S_\nu\frac{\partial\langle g^{1/2}\rangle_\nu}{\partial x^i}\right).$$

Using (2.8) and the fact that

$$\langle g^{1/2}\rangle_{\alpha\beta\gamma}=\frac{1}{8}\sum_{\nu\in Z_1(\alpha\beta\gamma)}(g^{1/2})_\nu+O(h^2),$$

we obtain

$$\sum_{\nu\in Z_2(\alpha\beta\gamma)} p_\nu^{\cdot}\frac{\partial V_\nu}{\partial x^i}=\frac{\partial g^{1/2}}{\partial x_i}\sum_{\nu\in Z_2(\alpha\beta\gamma)} p_\nu^{\cdot}S_\nu-\frac{J\,\partial p^{\cdot}g^{1/2}}{\partial x^i}\Big|_{\alpha\beta\gamma}+O(h^2)$$
$$=\left(p^{\cdot}\frac{J\,\partial g^{1/2}}{\partial x^i}-\frac{J\,\partial p^{\cdot}g^{1/2}}{\partial x^i}\right)\Big|_{\alpha\beta\gamma}+O(h^2)=-\frac{g^{1/2}\,\partial p^{\cdot}J_i^{\ n}}{\partial q^n}\Big|_{\alpha\beta\gamma}+O(h^2).$$

Hence the second order of approximation of the dynamic equations (3.16) and (3.17) has been proved.

3. The equation for the specific internal energy can be obtained in the same way as in [1]:

$$m\frac{d\varepsilon}{dt}=-p\frac{dV}{dt}=\frac{m}{\rho^2}\frac{d\rho}{dt}. \tag{3.18}$$

Equation (3.18) is in the entropic form; for dV/dt we can use either of expressions (3.3), (3.4).

To calculate flows with shock waves, accompanied by an increase of entropy, we have to introduce artificial dissipative processes. This can be done in the ways described in [10, 11].

It should be mentioned that, in the case of two space variables, the schemes obtained are the same as those of [1, 2, 7, 8] for the cases of Cartesian, cylindrical, and spherical coordinates.

The technique of time-discretization is just the same as that developed in [1, 2, 12].

4. Some properties of differential-difference MHD equations

In this section we shall examine the properties of the difference system of MHD equations, for the case when dynamic equation (3.16) is used, and expression (3.3) is used for dV/dt. All our results are easily extended to the case when Eqs. (3.17) and (3.4) are used.

1. Let us write the complete system of MHD differential-difference equations:

$$M\left(\frac{dv_i}{dt}-\frac{v^k v^l}{2}\frac{\partial g_{kl}}{\partial x^i}\right)=\sum_{\nu\in Z_{\gamma}(\alpha\beta\gamma)}\left[p^{\cdot}\frac{\partial V_\nu}{\partial x^i}-\frac{\partial\langle g_{kl}\rangle_\nu}{\partial x^i}\frac{(VH^kH^l)_\nu}{8\pi}\right] - \partial_k\frac{H_i\Phi^k}{4\pi}, \tag{4.1}$$

$$\rho V\frac{d\varepsilon}{dt}=-p\frac{dV}{dt}=-p\sum_{\nu\in Z_{\gamma}(\alpha\beta\gamma)}\frac{\partial V}{\partial (x^k)_\nu}(v^k)_\nu, \tag{4.2}$$

$$VH^k=\Phi^{i}\partial_{\bar{i}}x^k, \tag{4.3}$$

$$V\frac{dH^k}{dt}=-H^k\frac{dV}{dt}+\Phi^n\partial_{\bar{n}}v^k, \tag{4.4}$$

$$\frac{d}{dt}\frac{H^kH_k}{8\pi}V=-\frac{H_kH^k}{8\pi}\frac{dV}{dt}+\frac{H_k\Phi^n\partial_{\bar{n}}v^k}{4\pi}-\frac{VH^kH^l}{8\pi}\frac{d\langle g_{kl}\rangle}{dt},\quad m=V\rho, \tag{4.5}$$

$$dx^i/dt=v^i,\qquad p=p(\varepsilon,\ \rho). \tag{4.6}$$

Notice that system (4.1)–(4.6) is simplified by utilizing the mixed components of the velocity and magnetic field vectors; though, merely by having recourse to the transformation expressions (2.1) and (2.2), the equations can easily be written in a form using only the covariant, or only the contravariant, components.

2. Let us turn to the conservation laws for system (4.1)–(4.6). It follows from the differential dynamic equation (1.11) that the law of change of momentum for an isolated fluid volume $\Omega\subset\Omega'$ has the form

$$\frac{d}{dt}\int_{\Omega_q}\rho_0 v_i\,d\Omega_q=\int_{\Omega_q}\rho_0\frac{\partial g_{kl}}{\partial x^i}\left(\frac{v^kv^l}{2}-\frac{H^kH^l}{8\pi}\right)d\Omega_q + \int p\,\frac{\partial g^{1/2}}{\partial x^i}J\,d\Omega_q-\oint\left[p^{\cdot}\delta_i^{\,k}+\frac{1}{4\pi}(H_iH^k-H_lH^l)\right]dS_k. \tag{4.7}$$

Assume that, in the difference case, the set $\omega_h' = \{\alpha_1 \leq \alpha \leq \alpha_2,\ \beta_1 \leq \beta \leq \beta_2,\ \gamma_1 \leq \gamma \leq \gamma_2\} \subset \bar{\omega}_h$, corresponds to Ω_q. The dynamic equations (4.1) give the momentum variation of this set. We sum (4.1) with respect to all $(\alpha\beta\gamma) \subset \omega_h'$ and use the relation

$$\sum_{\nu \in Z_1(\alpha\beta\gamma)} \frac{\partial V_{\alpha\beta\gamma}}{\partial (x^i)_\nu} = S_{\alpha\beta\gamma} \sum_{\nu \in Z_1(\alpha\beta\gamma)} \frac{\partial \langle g^{1/2} \rangle_{\alpha\beta\gamma}}{\partial (x^i)_\nu}.$$

As a result we obtain the difference analogue of (4.7):

$$\frac{d}{dt} \sum_{(\alpha\beta\gamma) \in \omega_h'} M_{\alpha\beta\gamma} (v_i)_{\alpha\beta\gamma} = \sum_{(\alpha\beta\gamma) \in \omega_h'} \left[M_{\alpha\beta\gamma} \left(\frac{v^k v^l}{2} \frac{\partial g_{kl}}{\partial x^i} \right)_{\alpha\beta\gamma} \right.$$
$$\left. - \sum_{\nu \in Z_2(\alpha\beta\gamma)} \frac{\partial \langle g_{kl} \rangle_\nu}{\partial (x^i)_{\alpha\beta\gamma}} \frac{(VH^l H^k)_\nu}{8\pi} \right] + \sum_{(\alpha\beta\gamma) \in \bar{\omega}_h'} (\mathrm{Sp}^*)_{\alpha\beta\gamma} \sum_{\nu \in Z_2(\alpha\beta\gamma)} \frac{\partial \langle g^{1/2} \rangle_{\alpha\beta\gamma}}{\partial (x^i)_\nu} + F_{ex},$$

where F_{ex} is the difference analogue of the surface integral in (4.7). Hence the difference equations (4.1) are conservative with respect to momentum.

Using Eqs. (4.2) and (4.4), we can similarly obtain expressions for the variation of the specific internal and magnetic energy of the discrete set ω'_h:

$$\frac{d}{dt} \sum_{(\alpha\beta\gamma) \in \bar{\omega}_h'} m_{\alpha\beta\gamma} \varepsilon_{\alpha\beta\gamma} = - \sum_{(\alpha\beta\gamma) \in \bar{\omega}_h'} p_{\alpha\beta\gamma} \frac{dV_{\alpha\beta\gamma}}{dt}, \tag{4.8}$$

$$\frac{d}{dt} \sum_{(\alpha\beta\gamma) \in \bar{\omega}_h'} \left(\frac{H^k H_k}{8\pi} V \right)_{\alpha\beta\gamma} = - \sum_{(\alpha\beta\gamma) \in \omega_h} \left[\left(\frac{H^k H_k}{8\pi} \frac{dV}{dt} \right)_{\alpha\beta\gamma} \right.$$
$$\left. - \left(\frac{H_k \Phi^i \partial_i v^k}{4\pi} \right)_{\alpha\beta\gamma} - \frac{(VH^k H^l)_{\alpha\beta\gamma}}{8\pi} \frac{d\langle g_{kl} \rangle_{\alpha\beta\gamma}}{dt} \right]. \tag{4.9}$$

These equations express the balance of the different types of energy.

3. Let us show that the following analogue of the law of conservation of total energy holds for the differential-difference system (4.1)–(4.6):

$$\frac{d}{dt} \int_{\Omega_q} \rho_0 \left(\frac{v_k v^k}{2} + \varepsilon + \frac{H^k H_k}{8\pi\rho} \right) d\Omega_q$$
$$= \oint_S \left[p^* \delta_{ik} + \frac{1}{4\pi} (H_i H^k - H_l H^l) \right] v^i g^{1/2} dS_k.$$

As a preliminary, we shall obtain an expression for the change of kinetic energy of ω'_h. For this, we write the equation of motion, using the contravariant component:

$$M \left(\frac{dv^i}{dt} + \Gamma_{kl}{}^i v^k v^l \right)$$
$$= g^{ik} \sum_{\nu \in \varpi_2(\alpha\beta\gamma)} \left(p_\nu^* \frac{\partial V_\nu}{\partial x^k} + \frac{\partial \langle g_{kl} \rangle_\nu}{\partial x^k} \frac{(H^k H^l)_\nu}{8\pi} - \frac{g^{ik} \partial_l H_k \Phi^l}{4\pi} \right). \tag{4.10}$$

where $\Gamma_{kl}{}^i$ is the Christoffel symbol of the 2nd kind.

On performing the convolution of (4.1) with $v^i/2$, and of (4.10) with $v_i/2$, and summing with respect to all $(\alpha\beta\gamma)\in\omega_h'$, we obtain the following expression for the kinetic energy variation of the nodes:

$$\frac{d}{dt}\sum_{(\alpha\beta\gamma)\in\omega_h'}\left(\frac{Mv^iv_i}{2}\right)_{\alpha\beta\gamma}=\sum_{\alpha\beta\gamma\in\omega_h'}\left\{\sum_{\nu\in Ш_1(\alpha\beta\gamma)}\left[\dot{p}_{\alpha\beta\gamma}\frac{\partial V_{\alpha\beta\gamma}}{\partial(x^k)_\nu}(v^k)_\nu\right.\right.$$
$$\left.\left.+\frac{(VH^kH^l)_{\alpha\beta\gamma}}{8\pi}\frac{\partial\langle g_{kl}\rangle_\nu}{\partial(x^k)_\nu}(v^k)_\nu\right]-\left(\frac{H_k\Phi^l\,\partial_{\bar{l}}v^k}{4\pi}\right)_{\alpha\beta\gamma}\right\}+A_{\text{ex}}. \tag{4.11}$$

Here, A_{ex} is the work done by the pressure and magnetic field forces from the cells bounding ω'_h from outside.

On next adding Eqs. (4.8), (4.9) and (4.11), we obtain the required relation:

$$\frac{d}{dt}\left[\sum_{(\alpha\beta\gamma)\in\omega_h'}\left(\frac{Mv^iv_i}{2}\right)_{\alpha\beta\gamma}+\sum_{(\alpha\beta\gamma)\in\omega_h'}m_{\alpha\beta\gamma}\left(\varepsilon+\frac{H_kH^k}{8\pi\rho}\right)_{\alpha\beta\gamma}\right]=A_{\text{ex}}.$$

In view of the results of Sections 2 and 3, and the fact that the magnetic field equation and the entropy form of the equation for the specific internal energy (4.2) are mutually matched, the system of differential-difference equations (4.1)–(4.6) has the property of complete conservativeness [13].

5. Examples of numerical computations

We consider the motion of an ideal infinitely conducting gas, when all quantities depend on the variables t and $x^1=x$. At the initial instant, let $\rho=\rho_0=\text{const}$, $p=p_0=\text{const}$, $v_x\equiv v^1=0$. We choose the initial values of the velocity components $v_y\equiv v^2$, $v_z\equiv v^3$ as follows:

$$v_y=\begin{cases}0, & x<0,\\ W_0\sin\dfrac{\pi x}{l}, & 0\leqslant x\leqslant l,\\ 0, & x>l,\end{cases}$$

$$v_z=\begin{cases}W_0, & x<0,\\ \dfrac{W_0}{2}\left(1+\cos\dfrac{\pi x}{2}\right), & 0\leqslant x\leqslant l,\\ 0, & x>l.\end{cases}$$

We specify the initial magnetic field by the expressions

$$H_x\equiv H^1=H_{x0},$$

$$H_y\equiv H^2=\begin{cases}0, & x<0,\\ H_0\sin\dfrac{\pi x}{l}, & 0\leqslant x\leqslant l,\\ 0, & x>l,\end{cases}$$

$$H_z\equiv H^3=\begin{cases}-H_0, & x<0,\\ -H_0\cos\dfrac{\pi x}{l}, & 0\leqslant x\leqslant l,\\ H_0, & x>l.\end{cases}$$

Let $W_0=2H_0(4\pi\rho_0)^{-1/2}$. It is easily seen that, in a reference system moving with respect to the laboratory system at a velocity $v_{rel} = (a_A, 0, W_0/2)$, where $a_A=H_\infty(4\pi\rho_0)^{-1/2}$, we have the relations $H_\tau^2=H_y^2+H_z^2=\text{const.}$ $v_\tau^2=v_y^2+v_z^2=\text{const}$, $\mathbf{v}=-\mathbf{H}(4\pi\rho)^{-1/2}$.

The initial conditions thus specified define the well-known [14] stationary gas motion, known as rotational or Alfven simple wave motion. In this wave the transverse components of the vectors v and H rotate without changing their absolute values. It can be shown that, in the present case, all the phases of the wave move with constant speed and the waves profile is not deformed. Notice that, with $l = 0$, the simple Alfven wave becomes a rotational discontinuity.

The results of a numerical computation of a simple Alfven wave are shown in Fig. 1. We plot quantities H_z, H_y, v_y, v_z against the Euler coordinate at different instants for the following parameter values: $H_0=1$, $\rho=1$, $p_0=0.01$, $H_\infty=2$, $l=1$. The adiabatic exponent $\gamma = 2$.

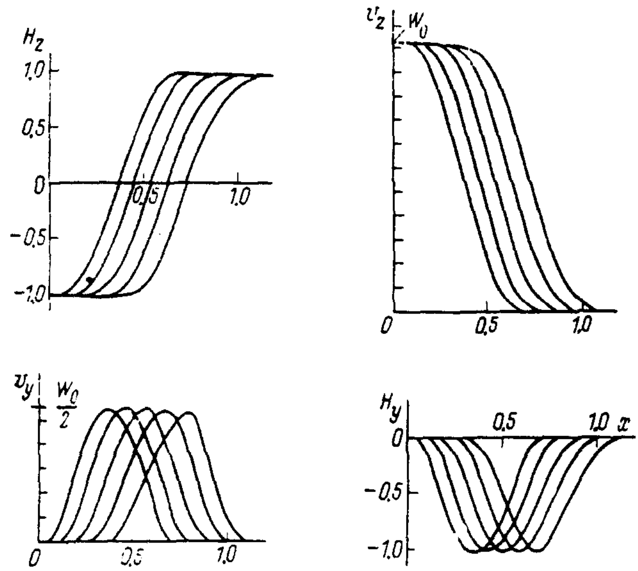

FIG. 1

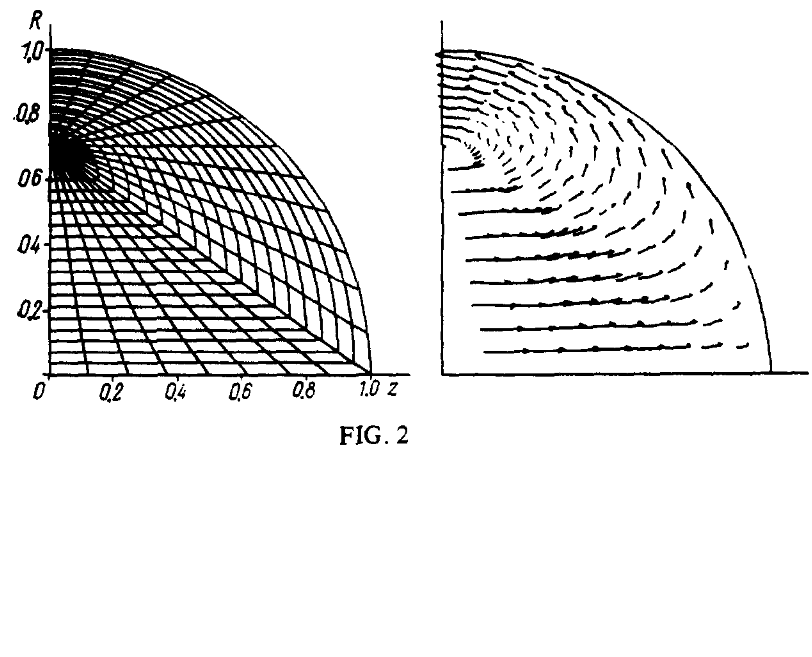

FIG. 2

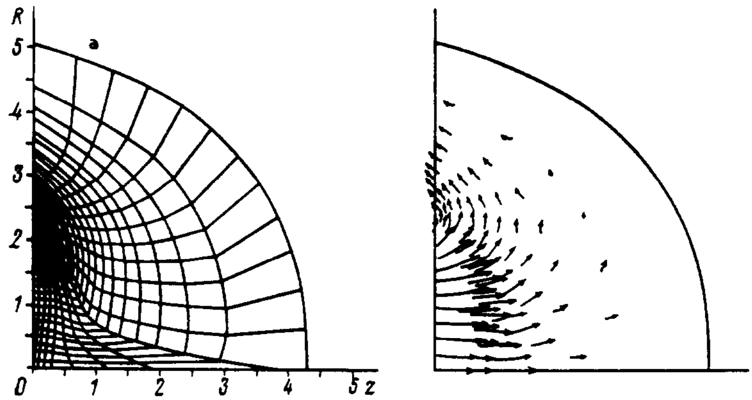

FIG. 3

As a second example we consider the axisymmetric problem of the divergence of a cluster of ideal infinitely conducting gas in vacuo. As coordinate system we use the cylindrical coordinates $x^1=r$, $x^2=\varphi$, $x^3=z$.

At the initial instant the gas is at rest and in the form of a sphere. In Fig. 2 we plot the configuration of the computational mesh and the magnetic field distribution at $t = 0$. The values of **H** were found from the expression

$$\mathbf{H}=\operatorname{rot}\mathbf{A}, \qquad A^1=A_r=0, \qquad A^2=A_\varphi=(H_0/2)-(1-r^2-z^2), \quad A^3=A_z=0,$$

where H_0 is a constant, corresponding to the maximum value of the magnetic field. The initial pressure was found from the condition $p+\mathbf{H}^2/8\pi=H_0^2/8\pi$. The initial density was taken equal to unity for the entire cluster; the adiabatic exponent $\gamma = 2$.

In Fig. 3 we show the computational mesh configuration and the magnetic field distribution at the instant $t = 2$.

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