U.S.S.R. Comput. Maths. Math. Phys. Vol. 20, No. 2, pp. 102–119, 1980. 0041–5553/80/020102–18$07.50/0
Printed in Great Britain.

USE OF EXACT DIFFERENCE SCHEMES FOR ESTIMATING THE RATE OF CONVERGENCE OF THE METHOD OF STRAIGHT LINES*

V. L. MAKAROV and A. A. SAMARSKII

(Kiev, Moscow)

(*Received* 18 *September* 1979)

THE METHOD of straight lines for partial differential equations, based on approximation of part of the differential operator by certain finite-difference relations, is considered. With the aid of the operator of exact difference schemes, the rate of convergence of the approximate solution is established, under natural conditions on the smoothness of the solution of the initial problem, whereby the existence of the solution is guaranteed.

Introduction

A lot of work has been published on the method of straight lines; the work up to 1965 is surveyed in [1]. All the schemes of the method can be roughly divided into two classes; the first contains schemes based on approximation of part of the differential operator by finite-difference relations, and the second, schemes in which part of the differential operator is approximated by means of a variational-projection method. As regards the second class of schemes, there are many publications in which convergence-rate estimates are obtained in the norm of L_2, under natural assumptions about the smoothness of the solution of the initial differential problem. (It may be mentioned that these results are only obtained for equations of.parabolic and elliptic type; see [2, 3] and the references cited there). No similar results have beeb published for schemes of the first class; the usual approach to obtaining convergence-rate estimates leads to excessive smoothness being demanded of the solution of the differential problem; these demands are not usually met in practice. The reason is that the approximation error appears in the *a priori* estimates in a form which contains high-order derivatives of the solution of the initial problem. In the present paper we offer a new approach to estimating the rate of convergence of schemes of the method of straight lines of the first class, based on the use of the exact difference schemes originally introduced in [4, 5]. With this approach convergence-rate estimates can be obtained, of the same order, and under the same assumptions about the smoothness of the solution of the initial differential problem, as for schemes of the method of straight lines of the second class [2, 3]. In addition, several new points arise which will deserve attention. Notably, the estimates are obtained in stronger norms than the norm of L_2. Also, on the basis of [6, 7], our approach allows similar results to be obtained for systems of partial differential equations and of high-order equations, and also, the results can be extended to quasi-linear equations.

---

*Zh. vȳchisl. Mat. mat. Fiz., 20, 2, 371–387, 1980.

1. Notation and auxiliary results

Let us first give some results from the theory of exact difference schemes.

Consider the boundary value problem.

$$\frac{d}{dx}\left[\frac{1}{p(x)}\frac{du}{dx}\right]-q(x)u=-f(x), \qquad x\in(0,1),$$
$$u(0)=a, \qquad u(1)=b, \qquad p(x)=1/k(x). \tag{1.1}$$

Let the following conditions hold:

Conditions A:

a) $0<\nu\leqslant k(x)\leqslant\mu,\ \nu,\ \mu=\text{const}$ and $k(x)$ is a summable function in the interval $[0, 1]$;

b) $\|q\|_{L_s(0,1)}\leqslant\mu<\infty,\ s\geqslant1,\ q(x)\geqslant0;$

c) $f(x)\in L_t(0,1),\ t\geqslant1,$ where $L_p(0,1)$ is the space of functions summable to p-th power.

When these conditions hold, the generalized solution of problem (1.1) exists, is unique, and belongs to the class $W_2^{\,1}(0,1)$. We shall prove this statement, since it holds under weaker constraints, and may be obtained (so it seems to us) by a simpler method than that used [8].

It is easily seen that, when conditions A hold, boundary value problem (1.1) is equivalent to the following Fredholm integral equation of the 2nd kind:

$$u(x)+\int_0^1 K(x,\eta)u(\eta)\,d\eta=F(x), \tag{1.2}$$

where

$$K(x,\eta)=\int_0^{|x,\eta|}\frac{d\xi}{k(\xi)}\int_{(x,\eta)}^1\frac{d\xi}{k(\xi)}\,q(\eta)\left[\int_0^1\frac{d\xi}{k(\xi)}\right]^{-1},$$
$$|x,\eta|=[x+\eta-|x-\eta|]/2, \qquad (x,\eta)=[x+\eta+|x-\eta|]/2, \tag{1.3}$$
$$F(x)=-\left[\int_0^1\frac{d\xi}{k(\xi)}\right]^{-1}\left[\int_0^x\frac{1}{k(\xi)}\int_0^1\int_\eta^\xi\frac{f(\zeta)}{k(\eta)}\,d\zeta\,d\eta\,d\xi-\int_0^x\frac{d\xi}{k(\xi)}\,(b-a)\right]+a.$$

and

$$l(x)=\|K(x,\eta)\|_{L_s(0,1)}\in C[0,1],$$
$$m(\eta)=\|K(x,\eta)\|_{C[0,1]}\in L_s(0,1) \tag{1.4}$$

$$F(x)\in C[0,1]. \tag{1.4'}$$

Using Theorem 1.16 of [9], p. 109, it follows from (1.4) and (1.4′) that the linear integral operator with kernel $K(x, \eta)$ acts from $L_{s/(s-1)}(0,1)$ into $C[0, 1]$, and is completely continuous. Since, moreover, the operator of problem (1.1) is self-adjoint and negative definite, we see also that the solution of integral equation (1.2) exists, is unique, and belongs to class $C[0, 1]$. If, additionally, we use the consequence of problem (1.1):

$$k(x)u'(x)=\left[\int_0^1\frac{d\xi}{k(\xi)}\right]^{-1}\left[\int_0^1\int_\eta^x\frac{f(\xi)}{k(\eta)}d\xi\,d\eta\right.$$

$$\left.+\int_0^1\frac{1}{k(\eta)}\int_\eta^x q(\xi)u(\xi)d\xi\,d\eta+b-a\right], \tag{1.5}$$

then we have $u(x)\in W_2^1(0,1)$, which is what we wished to prove. We introduce the uniform difference mesh $\omega_h=\{x_i=ih:\ i=1,2,\ldots,N-1,\ h=1/N\}$. If we take a non-uniform mesh, the working is simply more laborious. We shall require the following below:

Definition. The exact three-point difference scheme for problem (1.1) is the scheme

$$v_i=a_iv_{i+1}+b_iv_{i-1}+w_i,\quad i=1,2,\ldots,N-1,\qquad v_0=a,\qquad v_N=b,$$

of which the coefficients $a_i=a_i(k(\cdot),q(\cdot))$, $b_i=b_i(k(\cdot),q(\cdot))$, $w_i=w_i(k(\cdot),q(\cdot),f(\cdot))$ are functionals of $k(x)$, $q(x)$, and $f(x)$ in the interval $x_{i-1}\leqslant x\leqslant x_{i+1}$, dependent on the parameter h. The following conditions need to be satisfied:

$$v_i=u(x_i),\qquad i=1,2,\ldots,N-1.$$

We have:

Lemma 1

Let conditions A be satisfied; then there is a unique homogeneous three-point difference scheme for problem (1.1):

$$(u_{\bar{x}}/a)_x-du=-\varphi(x),\quad x\in\omega_h,\qquad u_0=a,\qquad u_N=b, \tag{1.6}$$

where $a(x)=h^{-1}v_1(x)$, $d(x)=T^x(q(\cdot))$, $\varphi(x)=T^x(f(\cdot))$. Here,

$$T^x(w(\cdot))=\frac{h^{-1}}{v_1(x)}\int_{x-h}^x v_1(\xi)w(\xi)d\xi+\frac{h^{-1}}{v_2(x)}\int_x^{x+h}v_2(\xi)w(\xi)d\xi,$$

and $v_1(x)$, $v_2(x)$ are pattern functions.

Proof. Let us show that, when conditions A hold, the pattern functions have the same properties as in [4, 5, 10], where it was assumed that

$$0<M_1 \leqslant \frac{1}{k(x)} \leqslant M_2, \quad 0 \leqslant q(x) \leqslant M_2, \quad k(x), q(x), f(x) \in Q^0[0, 1].$$

We shall show that the pattern functions $v_j(x)$, $j=1, 2,$ being solutions of the Cauchy problems

$$L^{(p,q)} v_j^i(x) \equiv \frac{d}{dx}\left[k(x) \frac{dv_j^i(x)}{dx}\right] - q(x) v_j^i(x) = 0,$$

$$x \in (x_{i-1}, x_{i+1}), \quad v_1^i(x_{i-1}) = 0, \quad k(x_{i-1}) \frac{dv_1^i(x_{i-1})}{dx} = 1,$$

$$v_2^i(x_{i+1}) = 0, \quad k(x_{i+1}) \frac{dv_2^i(x_{i+1})}{dx} = -1,$$

exist, are unique, and belong to class $W_2^1(x_{i-1}, x_{i+1})$.

The proof is similar to the above proof of the existence and uniqueness of the generalized solution of problem (1.1), belonging to class $W_2^1(0, 1)$. All we have to do, e.g. in the case of function $v_1^i(x)$, is replace Eqs. (1.2), (1.5) by the equations

$$\begin{aligned} v_1^i(x) &= F(x) + \int_{x_{i-1}}^{x} K(x, \xi) v_1^i(\xi) d\xi, \\ k(x) \frac{dv_1^i(x)}{dx} &= 1 + \int_{x_{i-1}}^{x} q(\xi) v_1^i(\xi) d\xi, \end{aligned} \tag{1.7}$$

where

$$K(x, \xi) = \int_{\xi}^{x} \frac{d\eta}{k(\eta)} q(\xi), \qquad F(x) = \int_{x_{i-1}}^{x} \frac{d\xi}{k(\xi)}. \tag{1.7'}$$

On then repeating the arguments of [5], we can see that Lemmas 1, 2, 3 of [5] hold. Using the imbedding theorem of [11], p. 64, any function of $W_2^1(0, 1)$ belongs to the class $C[0, 1]$, and hence the solution $u(x)$ of problem (1.1) is defined at the base-points of the mesh ω_h.

We use *reductio ad absurdum* to prove the uniqueness of the exact three-point difference scheme. Assume that there are two exact three-point difference schemes

$$\begin{aligned} &u_i = a_i^{(j)} u_{i+1} + b_i^{(j)} u_{i-1} + w_i^{(j)}, \quad i=1, 2, \ldots, N-1,, \quad j=1, 2, \\ &u_0 = a, \qquad u_N = b. \end{aligned} \tag{1.6'}$$

We fix arbitrary i, $1 \leqslant i \leqslant N-1$, and find for problem (1.1) the constants a and b such that $u(x_{i-1}) = u(x_{i+1}) = 0$. For this, putting $x = x_{i-1}$ and $x = x_{i+1}$ in (1.2) and (1.3), we obtain for a and b the system of linear algebraic equations with determinant

$$\Delta = \int_{x_{i-1}}^{x_{i+1}} \frac{d\xi}{k(\xi)} \left[\int_0^1 \frac{d\xi}{k(\xi)} \right]^{-1} \neq 0.$$

Then, from (1.6′), for this solution we have $w_i^{(1)} = w_i^{(2)}$. To obtain the equations $a_i^{(1)} = a_i^{(2)}$ and $b_i^{(1)} = b_i^{(2)}$ we have to repeat the same set of arguments, but choosing a and b from the conditions $u(x_{i+1}) = 1$, $u(x_{i-1}) = 0$ or $u(x_{i+1}) = 0$ and $u(x_{i-1}) = 1$. The system determinant Δ in the meantime remains unchanged. Since i was chosen arbitrarily, the schemes (1.6′) must be completely identical. The lemma is proved.

Note. From (1.7) and (1.7′), and the corresponding equations for $v_2{}^i(x)$, we obtain for sufficiently small h the inequalities

$$\frac{1}{\mu} \leqslant \frac{1}{h} \int_{x_{i-1}}^{x_i} \frac{d\xi}{k(\xi)} < \frac{1}{h} v_1{}^i(x)$$

$$\leqslant \frac{1}{h} \int_{x_{i-1}}^{x_i} \frac{d\xi}{k(\xi)} \left[1 - \int_{x_{i-1}}^{x_i} q(\xi)\, d\xi \, \frac{1}{h} \int_{x_{i-1}}^{x_i} \frac{d\xi}{k(\xi)} \right]^{-1}$$

$$\leqslant \nu^{-1} \left[1 - \frac{1}{\nu} \int_{x_{i-1}}^{x_i} q(\xi)\, d\xi \right]^{-1} \leqslant \nu_1^{-1}, \qquad \frac{1}{\mu} \leqslant \frac{1}{h} v_2{}^i(x) \leqslant \frac{1}{\nu_1},$$

$$\left\| \frac{1}{h} \frac{dv_j{}^i(x)}{dx} \right\|_{L_2(x_{i-1}, x_{i+1})} \leqslant C, \qquad j = 1, 2, \quad q(x) \in L_s(0, 1), \quad s > 1,$$

where the constants ν_1 and C are independent of h and i.

Lemma 2

Let conditions A hold with $s > 1$. Then,

1) we have the estimate

$$\| T^x(u(\cdot)) - u(x) \|_0 \leqslant Ch \left\| \frac{du}{dx} \right\|_{L_2(0,1)} \qquad \forall u(x) \in W_2{}^1(0, 1);$$

2) if $k(x) \in W_2{}^1(0, 1)$, then we have the *a priori* estimate

$$\| T^x(u(\cdot)) - u(x) \|_0 \leqslant Ch^2 \left\| \frac{d^2u}{dx^2} \right\|_{L_2(0,1)} \qquad \forall u(x) \in W_2{}^1(0, 1);$$

3) if $k(x) \in C_{0,\,1/s}[0, 1]$, then we have the *a priori* estimate

$$\| T^x(u(\cdot)) - u(x) \|_0 \leqslant Ch^{1+1/s} \qquad \forall u(x) \in C_{1,\,1/s}[0, 1],$$

where

$$\|w\|_0^2=\sum_{x_i\in\omega_h} h\omega^2(x_i).$$

Proof. It is easily shown that

$$T^\lambda(u(\cdot))-u(x)=\frac{h^{-1}}{v_1}\int_{x-h}^{x}\left(x-\frac{h}{2}-\xi\right)[v_1(\xi)u(\xi)]'\,d\xi$$

$$+\frac{h^{-1}}{v_2(x)}\int_{x}^{x+h}\left(x+\frac{h}{2}-\xi\right)[v_2(\xi)u(\xi)]'\,d\xi$$

$$=\frac{h^{-1}}{v_1(x)}\int_{x-h}^{x}\left[\frac{1}{2}\left(x-\frac{h}{2}-\xi\right)^2-\frac{h^2}{8}\right][v_1(\xi)u(\xi)]''\,d\xi$$

$$+\frac{h^{-1}}{v_1(x)}\int_{x}^{x+h}\left[\frac{1}{2}\left(x+\frac{h}{2}-\xi\right)^2-\frac{h^2}{8}\right][v_2(\xi)u(\xi)]''\,d\xi.$$

From this, using our Note, the lemma follows.

2. Convergence of the method of straight lines for equations of parabolic type

Let us consider the Cauchy problem for an abstract first-order differential equation in Hilbert space H:

$$\frac{du}{dt}+Au=f(t),\qquad t>0,\qquad u(0)=u_0, \tag{2.1}$$

where the linear operator $A: H\to H$ has domain of definition D, dense in H, and

$$A=A^*\geqslant\gamma E>0. \tag{2.2}$$

Assume that a linear operator $T: H\to H$ exists, with the following two properties:

1) if u and v are solutions of the equations

$$\begin{aligned}&Au=g,\\ &\tilde{A}v=[P(PTA)^{-1}]^{-1}v=PTg=\tilde{g},\qquad v\in X,\end{aligned} \tag{2.2'}$$

where P is an operator from H into Hilbert space X, then

$$Pu=v; \tag{2.2''}$$

2) $0<\nu_1 E\leqslant\tilde{A}=\tilde{A}^{*}$.

Instead of problem (2.1), (2.2), we consider an "approximation" of it, namely, the Cauchy problem

$$\frac{dv}{dt}+\tilde{A}(t)v=\tilde{f}(t),\qquad t>0,\qquad v(0)=PT(0)u_0.$$

The error $z=v-Pu$ is then found by solving the Cauchy problem

$$\frac{dz}{dt}+\tilde{A}(t)z=\frac{d\psi(t)}{dt},\qquad t>0,\qquad z(0)=P(T(0)u_0-u_0), \tag{2.3}$$

where

$$\psi(t)=\int_0^t P\left[T(\xi)\frac{du}{d\xi}-\frac{du}{d\xi}\right]d\xi.$$

Theorem 1

Assume that the linear operator $A(t)$ in problem (2.1), with domain of definition D, independent of t and dense in H, is differentiable and satisfies the conditions

$$0<\nu E\leqslant A(t)=A^{*}(t),\qquad t\geqslant 0,$$

where ν is independent of t. Then, if there exists a linear operator $T(t): H\to H$, having the properties (2.2′), (2.2″), such that

$$0\leqslant\tilde{A}'(t)+\mu_1\tilde{A}(t),\qquad t\geqslant 0.\qquad |\mu_1|<\infty, \tag{2.4}$$

then we have the estimate for $z=v-Pu$:

$$\begin{gathered}\int_0^t\|z(\xi)\|^2d\xi+\left\|\int_0^t\tilde{A}(\eta)z(\eta)\,d\eta\right\|^2_{\tilde{A}^{-1}(t)}\\ \leqslant\exp\left[t\max(\mu_1,\ 0)\right]\int_0^t\|\psi(\xi)+z(0)\|^2d\xi,\end{gathered} \tag{2.5}$$

where $\|\cdot\|$ is the norm in space X.

If the operator A is constant, we have the estimate

$$\left\{\int_0^t \| v - Pu \|^2 d\xi + \frac{1}{2}\left\| \int_0^t [v - Pu]\, d\xi \right\|_{\tilde{A}}^2 \right\}^{1/2} \leqslant \left\{\int_0^t \| P(Tu - u) \|^2 d\xi\right\}^{1/2}. \tag{2.6}$$

Proof. We integrate both sides of (2.3) from 0 to t, then multiply scalarly by $z(t)$; after transformations, we obtain

$$\begin{aligned} &\| z(t) \|^2 + \frac{1}{2}\frac{d}{dt}\left\| \int_0^t \tilde{A}(\xi) z(\xi)\, d\xi \right\|_{\tilde{A}^{-1}(t)}^2 \\ &= \frac{1}{2}\left\| \int_0^t \tilde{A}(\xi) z(\xi)\, d\xi \right\|_{[\tilde{A}^{-1}(t)]'}^2 + (\psi(t) - z(0),\, z(t)). \end{aligned} \tag{2.7}$$

Integrating both sides of (2.7) from 0 to t, and using inequality (2.4), we have

$$\begin{aligned} &\int_0^t \| z(\xi) \|^2 d\xi + \frac{1}{2}\left\| \int_0^t \tilde{A}(\xi) z(\xi)\, d\xi \right\|_{\tilde{A}^{-1}(t)}^2 \\ &\leqslant \frac{1}{2}\max(\mu_1, 0) \int_0^t \left\| \int_0^\xi \tilde{A}(\eta) z(\xi)\, d\eta \right\|_{\tilde{A}^{-1}(\xi)}^2 d\xi \\ &+ \frac{1}{2}\int_0^t \| z(\xi) \|^2 d\xi + \frac{1}{2}\int_0^t \| \psi(\xi) - z(0) \|^2 d\xi. \end{aligned}$$

On applying the generalized Gronwall lemma to the last inequality, we at once arrive at (2.5). If A is a constant operator, then (2.7) takes the form

$$\| z(t) \|^2 + \frac{1}{2}\frac{d}{dt}\left\| \int_0^t z(\xi)\, d\xi \right\|_{\tilde{A}}^2 = (P(Tu - u),\, z(t)).$$

On integrating both sides of this equation from 0 to t, and applying the Cauchy inequality to the right-hand side, we obtain (2.6). This proves the theorem.

As an example of the use of Theorem 1, consider the problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left[k(x)\frac{\partial u}{\partial x}\right] + q(x)u = f(x, t), \quad x \in (0, 1), \quad t > 0, \\ &u(0, t) = u(1, t) = 0, \qquad u(x, 0) = u_0(x). \end{aligned} \tag{2.8}$$

Assume that the coefficients of Eq. (2.8) satisfy conditions A; then, as the operator T, we can take the operator T^x of Section 1. Here, P is the operator of taking the trace of a function in mesh ω_h, while operator $\tilde{A}$ is given by

$$\tilde{A}v = -(v_{\bar{x}}/a)_x + dv,$$

and obviously, conditions (2.2′) and (2.2″) are satisfied.

Consequently, for the scheme of the method of straight lines

$$\frac{dv}{dt}-\left(\frac{1}{a}v_{\bar{x}}\right)_x+dv=PT^x(f(\cdot,T)),\quad x\in\omega_h,\quad t>0,$$
$$v(0,t)=v(1,t)=0,\quad v(x,0)=PT^x(u_0(\cdot)),\tag{2.9}$$

corresponding to problem (2.8), all the conditions of Theorem 1 hold, and from (2.6) we have the *a priori* estimate

$$\left\{\int_0^t\|v-Pu\|_0^2\,d\xi+\frac{1}{2}\left(\frac{1}{a},\left[\int_0^t(v-Pu)\,d\xi\right]_{\bar{x}}^2\right]\right\}^{1/2}$$
$$\leqslant\left\{\int_0^t\|PT^x(u(\cdot,\xi))-u(x,\xi)\|_0^2\,d\xi\right\}^{1/2}.$$

Using this estimate, along with Lemma 2, we have:

Theorem 2

Let conditions A hold with $s>1$. Then:

1) for $\|f\|_{q_1,r_1,Q_T}\leqslant\mu_2$, where $1/r_1+1/2q_1=5/4$, $q_1\in[1,2]$, $r_1\in[1,4/3]$, $u_0(x)\in L_2(0,1)$, we have the following estimate for the error of scheme (2.9) of the method of straight lines:

$$\left\{\int_0^t\|v-Pu\|_0^2\,d\xi+\frac{1}{2}\left(\frac{1}{a},\left[\int_0^t(v-Pu)\,d\xi\right]_{\bar{x}}^2\right]\right\}^{1/2}\leqslant Ch\left\|\frac{\partial u}{\partial x}\right\|_{2,Q_t};$$

2) for $\|f\|_{2,Q_T}\leqslant\mu_2$, $k(x)\in W_2^1(0,1)$, $u_0(x)\in W_2^1(0,1)$ we have the estimate for the error of scheme (2.9) of the method of straight lines:

$$\left\{\int_0^t\|v-Pu\|_0^2\,d\xi+\frac{1}{2}\left(\frac{1}{a},\left[\int_0^t(v-Pu)\,d\xi\right]_{\bar{x}}^2\right]\right\}^{1/2}\leqslant Ch^2\left\|\frac{\partial^2u}{\partial x^2}\right\|_{2,Q_t},$$

where

$$\|u\|_{q,r,Q_t}=\left\{\int_0^t\left[\int_0^1|u(x,t)|^q\,dx\right]^{r/q}dt\right\}^{1/r},\quad \|u\|_{2,2,Q_t}=\|u\|_{2,Q_t}.$$

In the proof of this theorem, we have to use results (see [12]) on the smoothness of the generalized solutions of equations of the parabolic type.

3. Convergence of the method of straight lines for equations of hyperbolic type

Consider the Cauchy problem for an abstract second-order differential equation in Hilbert space H:

$$\frac{d^2u}{dt^2}+A(t)u=f(t), \quad t>0, \qquad u(0)=u_0, \qquad u'(0)=u_0', \tag{3.1}$$

where $A(t)$: $H \to H$ is a linear operator with domain of definition D, independent of t and dense in H, while $A(t)=A^*(t)\geqslant \nu E>0$.

Assume that there is a linear operator $T(t)$: $H \to H$, with properties (2.2′), (2.2″). We introduce, instead of problem (3.1), the problem

$$\begin{gathered}\frac{d^2v}{dt^2}+\tilde{A}(t)v=\tilde{f}(t), \qquad t>0, \\ v(0)=PT(0)u_0, \qquad v'(0)=[PT(t)u(t)]'_{t=0},\end{gathered} \tag{3.2}$$

where $v \in X$, and P is an operator from H into X. For the error $z = v - Pu$ of (3.2), (3.1), we obtain the Cauchy problem

$$\begin{gathered}\frac{d^2z}{dt^2}+\tilde{A}(t)z=\frac{d^2\psi(t)}{dt^2}, \qquad t>0, \\ z(0)=P(T(0)u_0-u_0), \qquad z'(0)=[P(T(t)u-u)]'_{t=0},\end{gathered} \tag{3.3}$$

where

$$\psi(t)=\int_0^t P[T(\xi)u''(\xi)-u''(\xi)](t-\xi)\,d\xi.$$

Theorem 3

Let the conditions of Theorem 2 hold, and let the operator $\tilde{A}(t)$ satisfy the operator inequality $[\tilde{A}^{-2}(t)]'\leqslant 2\mu_1\tilde{A}^{-2}(t)$ or $[\tilde{A}^2(t)]'+2\mu_1\tilde{A}^2(t)\geqslant 0$. Then, for the difference $z = v - Pu$ we have the estimate

$$\begin{gathered}\left[\left\|\tilde{A}^{-1}(t)\int_0^t \tilde{A}(\eta)z(\eta)\,d\eta\right\|^2+\left\|\int_0^t (t-\eta)\tilde{A}(\eta)z(\eta)\,d\eta\right\|^2_{\tilde{A}^{-1}(t)}\right] \\ \leqslant 2\varepsilon\exp\left\{\left[2\max(\mu_1,0)+\frac{1}{2\varepsilon}\right]t\right\}\int_0^t\|\tilde{\psi}(\xi)\|^2\,d\xi,\end{gathered} \tag{3.4}$$

where $\tilde{\psi}(\xi)=\psi(\xi)+\xi z'(0)+z(0)$, $\varepsilon>0$.

If A is a constant operator, we obtain, instead of (3.4),

$$\left\{\max_{0\leq t\leq T_1}\left[\left\|\int_0^t z(\xi)\,d\xi\right\|^2+\left\|\bar{A}^{1/2}\int_0^t (t-\xi)\,z(\xi)\,d\xi\right\|^2\right]\right\}^{1/2} \leq 2T_1^{1/2}\left[\int_0^{T_1}\|P(Tu-u)\|^2\,d\xi\right]^{1/2}. \tag{3.4'}$$

Proof. We integrate both sides of Eq. (3.3) twice with respect to t, then we multiply scalarly by

$$\bar{A}^{-1}(t)\int_0^t \bar{A}(\xi)\,z(\xi)\,d\xi$$

and we obtain

$$\begin{aligned}&\frac{d}{dt}\left\|\bar{A}^{-1}(t)\int_0^t \bar{A}(\xi)\,z(\xi)\,d\xi\right\|^2+\frac{d}{dt}\left\|\int_0^t (t-\xi)\,\bar{A}(\xi)\,z(\xi)\,d\xi\right\|^2_{\bar{A}^{-1}(t)}\\&=\left\|\int_0^t \bar{A}(\xi)\,z(\xi)\,d\xi\right\|_{[\bar{A}^{-2}(t)]'}+\left\|\int_0^t (t-\xi)\,\bar{A}(\xi)\,z(\xi)\,d\xi\right\|^2_{[\bar{A}^{-1}(t)]'}\\&+2\left(\tilde{\psi}(t),\ \bar{A}^{-1}(t)\int_0^t \bar{A}(\xi)\,z(\xi)\,d\xi\right).\end{aligned} \tag{3.5}$$

If we integrate both sides of (3.5) from 0 to t, use the conditions of the theorem, and apply the ϵ-inequality, we finally obtain

$$\begin{aligned}&\left\|\bar{A}^{-1}(t)\int_0^t \bar{A}(\xi)\,z(\xi)\,d\xi\right\|^2+\left\|\int_0^t (t-\xi)\,\bar{A}(\xi)\,z(\xi)\,d\xi\right\|^2_{\bar{A}^{-1}(t)}\\&\leqslant\left[2\max(\mu_1,0)+\frac{1}{2\varepsilon}\right]\int_0^t\left[\left\|\bar{A}^{-1}(\xi)\int_0^\xi \bar{A}(\eta)\,z(\eta)\,d\eta\right\|^2\right.\\&\left.+\left\|\int_0^\xi (\xi-\eta)\,\bar{A}(\eta)\,z(\eta)\,d\eta\right\|^2_{\bar{A}^{-1}(\xi)}\right]d\xi+2\varepsilon\int_0^t\|\tilde{\psi}(\xi)\|^2\,d\xi.\end{aligned} \tag{3.6}$$

Finally, integrating (3.6) with respect to t, and applying the generalized Gronwall lemma, we arrive at the *a priori* estimate (3.4).

If operator A is constant, then (3.5) becomes

$$\frac{d}{dt}\left\|\int_0^t z(\xi)\,d\xi\right\|^2+\frac{d}{dt}\left\|\int_0^t (t-\xi)\,z(\xi)\,d\xi\right\|^2_{\bar{A}}=2\left(\tilde{\psi}(t),\ \int_0^t z(\xi)\,d\xi\right),$$

whence we have

$$\left\| \int_0^t z(\xi)\, d\xi \right\|^2 + \left\| \int_0^t (t-\xi) z(\xi)\, d\xi \right\|_{\bar{A}}^2 \leqslant 2 \left[\int_0^t \| \tilde{\Psi}(\xi) \|^2 d\xi \right]^{1/2}$$

$$\times \left[\int_0^t \left\| \int_0^\xi z(\eta)\, d\eta \right\|^2 d\xi \right]^{1/2} \leqslant 2T_1^{1/2} \left[\int_0^{T_1} \| \tilde{\Psi}(\xi) \|^2 d\xi \right]^{1/2}$$

$$\times \max_{0 \leqslant t \leqslant T_1} \left[\left\| \int_0^t z(\xi)\, d\xi \right\|^2 + \left\| \int_0^t (t-\xi) z(\xi)\, d\xi \right\|_{\bar{A}}^2 \right]^{1/2},$$

which leads to (3.4′). The lemma is proved.

We shall use Theorem 3 to find the rate of convergence of the method of straight lines for the first boundary value problem, in the case of a hyperbolic equation.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(k(x) \frac{\partial u}{\partial x} \right) + q(x) u = f(x,t), \qquad x \in (0,1), \qquad t > 0,$$

$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = u_0(x), \qquad \left. \frac{\partial u}{\partial t} \right|_{t=0} = u_1(x).$$

Using the operators T^x and P of Section 2, we construct the scheme

$$\begin{aligned} &\frac{d^2 v}{dt^2} - \left(\frac{1}{a} v_{\bar{x}} \right)_x + dv = PT^x(f(\cdot,t)), \qquad x \in \omega_h, \qquad t > 0, \\ &v(0,t) = v(1,t) = 0, \\ &v(x,0) = PT^x(u_0(\cdot)), \qquad \left. \frac{dv}{dt} \right|_{t=0} = PT^x(u_1(\cdot)). \end{aligned} \tag{3.7}$$

In accordance with inequality (3.4′), for the error $z = v - Pu$ we have the *a priori* estimate

$$\begin{aligned} &\left\{ \max_{0 \leqslant t \leqslant T_1} \left[\left\| \int_0^t z(\xi) d\xi \right\|_0^2 + \left(\frac{1}{a}, \left[\int_0^t (t-\xi) z(\xi) d\xi \right]_{\bar{x}}^2 \right) \right] \right\}^{1/2} \\ &\leqslant 2T_1^{1/2} \left[\int_0^{T_1} \| PT^x(u(\cdot,\xi)) - u(x,\xi) \|_0^2 d\xi \right]^{1/2}. \end{aligned} \tag{3.8}$$

This estimate, in conjunction with Lemma 2, gives us:

Theorem 4

Let conditions A hold. Then:

1) for $\max |q(x)| \leqslant \mu_1$, $f(x,t) \in L_{2,1}(Q_{T_1})$, $u_0(x) \in \mathring{W}_2^1(0,1)$, $u_1(x) \in L_2(0,1)$ we have the estimate for the error of the scheme (3.7)

$$\left\{ \max_{0 \leqslant t \leqslant T_1} \left[\left\| \int_0^t z(\xi) d\xi \right\|_0^2 + \left(\frac{1}{a}, \left[\int_0^t (t-\xi) z(\xi) d\xi \right]_{\bar{x}}^2 \right) \right] \right\}^{1/2} \leqslant Ch \left\| \frac{\partial u}{\partial x} \right\|_{2,Q_{T_1}}.$$

2) if the condition of Para. 1) and the conditions $|k'(x)|<\mu_2$, $f_t'(x,t)\in L_{2,1}(Q_{T_1})$, $u_0(x)\in W_2^2(0,1)\cap \mathring{W}_2^1(0,1)$, $u_1(x)\in \mathring{W}_2^1(0,1)$, hold, we have the estimate for the error of scheme (3.7):

$$\left\{\max_{0\leqslant t\leqslant T_1}\left[\left\|\int_0^t z(\xi)d\xi\right\|_0^2+\left(\frac{1}{a},\left[\int_0^t (t-\xi)z(\xi)d\xi\right]_{\bar{x}}^2\right]\right\}^{1/2}\leqslant Ch^2\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{2,Q_{T_1}}.$$

To prove the theorem, we make use of the results of [12], Chapter 4, Sections 3 and 4, on the smoothness of generalized solutions of a hyperbolic equation.

4. Convergence of the method of straight lines for equations of elliptic type

Consider the first boundary value problem for an abstract second-order differential equation in Hilbert space H:

$$\frac{d^2u}{dx^2}-A(x)u=f(x),\qquad x\in(0,1),\qquad u(0)=u_0,\qquad u(1)=u_1. \tag{4.1}$$

Here, $A(x)$: $H\to H$ is a linear operator, with domain of definition independent of x and dense in H, which satisfies the condition $A(x)=A^*(x)\geqslant \nu E>0$.

Assume, as above, that there is a linear operator $T(x)$: $H\to H$, with the properties (2.2'), (2.2''). Instead of problem (4.1) we consider an "approximation" of it in Hilbert space X ("simpler" than space H):

$$\frac{d^2v}{dx^2}-\tilde{A}(x)v=\tilde{f}(x),\qquad x\in(0,1),\qquad v(0)=PT(0)u_0,\qquad v(1)=PT(1)u_1,$$

where all the notation has the same meaning as before. Then, for the error $z=v-Pu$ we have the problem

$$\begin{gathered}\frac{d^2z}{dx^2}-\tilde{A}(x)z=\frac{d^2\psi(x)}{dx^2},\qquad x\in(0,1),\\ z(0)=P[T(0)u_0-u_0],\qquad z(1)=P[T(1)u_1-u_1],\end{gathered} \tag{4.2}$$

where

$$\psi(x)=\int_0^1 G(x,\xi)P[T(\xi)u''(\xi)-u''(\xi)]d\xi,$$

$$G(x,\xi)=\begin{cases}(1-x)\xi, & x\geqslant\xi,\\ (1-\xi)x, & x<\xi.\end{cases}$$

We replace the variable x by ξ in Eq. (4.2), multiply both sides by the function $G(x, \xi)$, and integrate with respect to ξ from 0 to 1:

$$z(x)-\bar{A}(x)\int_0^1 G(x,\xi)z(\xi)d\xi=-\int_0^1 G(x,\xi)[\bar{A}(x)-\bar{A}(\xi)]z(\xi)d\xi+\bar{\psi}(x).$$

Here, $\bar{\psi}(x)=\psi(x)+xz(1)+(1-x)z(0)$. From the last equation we have the estimate

$$\begin{aligned}\|z(x)\|_*^2&=\left\|\left[\frac{d^2}{dx^2}-\bar{A}(x)\right]\int_0^1 G(x,\xi)z(\xi)d\xi\right\|^2\\&\leqslant 2\|\bar{\psi}(x)\|^2+\mu_1\int_0^1\|z(\xi)\|^2d\xi,\end{aligned}\tag{4.3}$$

where it is assumed that

$$\mu_1=2\max_{0\leqslant x\leqslant 1}\int_0^1 G^2(x,\xi)\|\bar{A}(x)-\bar{A}(\xi)\|^2d\xi<1.\tag{4.4}$$

Inequality (4.3), along with (4.4) and the inequality $\|z(x)\|_*\geqslant\|z(x)\|$ leads to the estimate

$$\max_{0\leqslant x\leqslant 1}\|z(x)\|\leqslant\max_{0\leqslant x\leqslant 1}\|z(x)\|_*\leqslant\left(\frac{2}{1-\mu_1}\right)^{1/2}\max_{0\leqslant x\leqslant 1}\|\bar{\psi}(x)\|.\tag{4.5}$$

We have thus proved:

Theorem 5

Let the linear operator $A(x)$ in problem (4.1) have domain of definition D, independent of x and dense in H, and let it satisfy the condition

$$0<\nu E\leqslant A(x)=A^*(x),\qquad x\in[0,1],\tag{4.6}$$

where ν is independent of x. Then, if a linear operator $T(x)$: $H\to H$ exists, which satisfies conditions (2.2′) and (2.2″), and relation (4.4) holds, then we have the estimate (4.5) for $z(x)$.

Let us use Theorem 5 to estimate the rate of convergence of the method of straight lines for the Dirichlet problem in the case of the elliptic equation

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial}{\partial y}\left[k(y)\frac{\partial u}{\partial y}\right]-q(y)u=f(x,y),\qquad(x,y)\in G,$$

$$u(x,y)=\varphi(x,y),\qquad(x,y)\in\Gamma,$$

where Γ is the boundary of the square $G=\{(x,y):x\in(0,1),\ y\in(0,1)\}$.

Using the operators T^y and P of Section 2, we construct the scheme

$$\begin{gathered}\frac{d^2v}{dx^2}+\left(\frac{1}{a}v_{\bar y}\right)_y-dv=PT^y(f(x,\ )),\qquad x\in(0,1),\qquad y\in\omega_h,\\ v(x,0)=\varphi(x,0),\qquad v(x,1)=\varphi(x,1),\\ v(0,y)=PT^y(\varphi(0,\cdot)),\quad v(1,y)=PT^y(\varphi(1,\cdot)).\end{gathered}\tag{4.7}$$

In accordance with inequality (4.6), we have the estimate

$$\begin{gathered}\left\{\int_0^1\|z(\xi)\|_0^2\,d\xi+\int_0^1\left(\frac{1}{a},\left[\int_0^1G(x,\xi)z(\xi)d\xi\right]_{\bar y}^2\right]dx\right\}^{1/2}\\ \leqslant 2\cdot\left\{\int_0^1\|P[T^y(u(\xi,\cdot))-u(\xi,y)]\|_0^2\,d\xi\right\}^{1/2}.\end{gathered}\tag{4.8}$$

Inequality (4.8) in conjunction with Lemma 2 gives us:

Theorem 6

Let conditions A hold. Then:

1) for $\max\limits_{0\leqslant y\leqslant 1}|q(y)|\leqslant\mu$, $f(x,y)\in L_2(G)$, $\varphi(x,y)\in W_2^1(G)$ we have the estimate for the error of scheme (4.7) of the method of straight lines:

$$\left\{\int_0^1\|z(\xi)\|_0^2\,d\xi+\int_0^1\left(\frac{1}{a},\left[\int_0^1G(x,\xi)z(\xi)d\xi\right]_{\bar y}^2\right]dx\right\}^{1/2}\leqslant Ch\left\|\frac{\partial u}{\partial y^2}\right\|_{2,G};$$

2) if conditions of Para. 1) hold, and also the conditions $|k'(y)|<\mu_2$, $\varphi(x,y)\in W_2^2(G)$, then we have the error estimate for scheme (4.7):

$$\left\{\int_0^1\|z(\xi)\|_0^2\,d\xi+\int_0^1\left(\frac{1}{a},\left[\int_0^1G(x,\xi)z(\xi)d\xi\right]_{\bar y}^2\right]dx\right\}^{1/2}\leqslant Ch^2\left\|\frac{\partial^2u}{\partial y^2}\right\|_{2,G}.$$

To prove the theorem, we have to use the results obtained in [8], Chapter 3, Sections 5 and 10, on the smoothness of generalized solutions of an elliptic equation.

5. Conclusion

The above results can be extended to systems of second-order partial differential equations, to higher-order partial differential equations, to the multi-dimensional case, to the quasi-linear case, and to the mesh method. Use is then made of results on exact difference schemes for systems of ordinary second-order equations [6] and for higher-order ordinary differential equations [7]. Without dwelling on all the possible extensions mentioned above, to which later papers will be devoted, let us briefly give the idea of the proof of convergence of the method of straight lines under minimal conditions on the smoothness of the solution of the initial differential problem, for the quasi-linear case. For clarity, we shall consider the method of straight lines applied to quasi-linear equations of parabolic type.

The starting point is the abstract Cauchy problem

$$\frac{du}{dt}+Au=f(t,u), \qquad t>0, \qquad u(0)=u_0,$$

where $A: A\to H$ is a self-adjoint positive definite operator, with domain of definition $D(A)$ dense in H. For simplicity, we shall assume that operator A is independent of t. Let the function $f(t, u)$ satisfy the Lipschitz condition

$$\|f(t, u_1)-f(t, u_2)\|_H\leqslant L\|u_1-u_2\|_H$$

with constant L, independent of t.

Using the notation of Section 2, we consider the "simpler" Cauchy problem

$$\frac{dv}{dt}+\tilde{A}v=PTf(t,\Lambda v), \qquad t>0, \qquad v(0)=PTu_0, \tag{5.1}$$

where $\Lambda: X\to H$ is a linear operator with bounded norm: $\|\Lambda\|\leqslant c_1$. Using the notation $z(t)=v(t)-Pu(t)$ for the error of the solution of problem (5.1), we arrive at the Cauchy problem

$$\frac{dz}{dt}+\tilde{A}z=\frac{d}{dt}P(Tu-u)+PT[f(t,\Lambda v)-f(t,u)],$$

$$z(0)=P(Tu_0-u_0).$$

Performing working similar to that of Section 2, we arrive at the *a priori* estimate

$$\begin{aligned}\left\{\int_0^t\|z(\xi)\|^2d\xi\right\}^{1/2}\leqslant 2^{1/2}\exp[Tc_1^2L^2t]\left\{\left[\int_0^t\|P(Tu-u)\|^2d\xi\right]^{1/2}\right.\\ \left.+\left[\int_0^t\eta\int_0^\eta\|PT(f(\xi,\Lambda Pu)-f(\xi,u))\|^2d\xi\,d\eta\right]^{1/2}\right\}, \qquad 0\leqslant t\leqslant T_1.\end{aligned} \tag{5.2}$$

The application of estimate (5.2) to prove the convergence of the method of straight lines for quasi-linear equations of parabolic type, is virtually the same as in the linear case, except for the operations with the second term on the right-hand side of (5.2). We shall therefore merely quote the result, similar to Lemma 2, for this term. Let the scalar function $f(x, y)$ satisfy with respect to its second argument the Lipschitz condition $|f(x,u_1)-f(x,u_2)|\leqslant L|u_1-u_2|$ with constant L, independent of x. Then, under the condition $\Lambda\equiv E$, and using the operator T^x of Section 1, we have

$$
\begin{aligned}
&|PT^x[f(\cdot,u(\;))-f(\;,u(x))]| \\
&\leqslant \frac{Lh^{-1}}{v_1(x)}\int_{x-h}^{x} v_1(\xi)\left|\int_{x}^{\xi}u'(\eta)\,d\eta\right|d\xi+\frac{Lh^{-1}}{v_2(x)}\int_{x}^{x+h} v_2(\xi)\left|\int_{x}^{\xi}u'(\eta)\,d\eta\right|d\xi \\
&\leqslant \frac{2}{3}Lh^{1/2}\left\{\int_{x-h}^{x+h}[u'(\eta)]^2\,d\eta\right\}^{1/2}, \qquad x\in\omega_h,
\end{aligned}
\tag{5.3}
$$

which leads to the estimate

$$
\|PT^x[f(\cdot,u(\cdot))-f(\cdot,u(x))]\|^2\leqslant\frac{8}{9}L^2h^2\int_0^1[u'(x)]^2\,dx. \tag{5.4}
$$

Estimate (5.4), in conjunction with Lemma 2, enables us to prove the convergence at a rate $O(h)$ of the method of straight lines for a quasi-linear parabolic equation, provided that its solution belongs, for each fixed t, to the class $W_2{}^1$ with respect to x.

Now let the solution of the quasi-linear parabolic equation belong, for each fixed t, to the class $W_2{}^2$ with respect to x. As the operator Λ we take the linear interpolation

$$
\Lambda u=\frac{x-x_{i-1}}{2h}u(x_{i+1})+\frac{x_{i+1}-x}{2h}u(x_{i-1}), \qquad x\in[x_{i-1},x_{i+1}].
$$

We then have

$$
\begin{aligned}
u(x)-\Lambda u&=\int_{x_{i-1}}^{x_{i+1}}K(x,t)u''(t)\,dt \\
&=\int_{x_{i-1}}^{x_{i+1}}\left[|x-t|_+-\frac{1}{2h}(x-x_{i-1})(x_{i+1}-t)\right]u''(t)\,dt,
\end{aligned}
$$

where $|\xi|_+=\xi$ for $\xi>0$, $|\xi|_+=0$ for $\xi<0$. Hence, recalling (5.3), we obtain

$$
\begin{aligned}
|PT^x[f(\cdot,u(\cdot))-f(\cdot,\Lambda Pu)]|&\leqslant\frac{Lh^{-1}}{v_1(x)}\int_{x_{i-1}}^{x_i}v_1(\xi)\left|\int_{x_{i-1}}^{x_{i+1}}K(\xi,t)u''(t)\,dt\right|d\xi \\
&+\frac{Lh^{-1}}{v_2(x)}\int_{x_i}^{x_i+h}v_2(\xi)\left|\int_{x_{i-1}}^{x_{i+1}}K(\xi,t)u''(t)\,dt\right|d\xi\leqslant Lh^{3/2}\left\{\int_{x_i-h}^{x_i+h}[u''(\eta)]^2\,d\eta\right\}^{1/2},
\end{aligned}
$$

which leads to the estimate

$$
\|PT^x[f(\cdot,u(\cdot))-f(\cdot,\Lambda u(x))]\|\leqslant 2^{1/2}Lh^2\left\{\int_0^1[u''(\eta)]^2\,d\eta\right\}^{1/2},
$$

which, in conjunction with Lemma 2, enables us to prove the convergence at a rate $O(h^2)$ of the method of straight lines for the quasi-linear parabolic equation.

Translated by D. E. Brown.

REFERENCES

1. LISKOVETS, O. A., The method of straight lines (survey), *Differents. ur-niya,* **1**, No. 12, 1662–1678, 1965.

2. AKOPYAN, Yu. R., and OGANESYAN, L. A., The rate of convergence of variational difference schemes for two-dimensional linear parabolic equations, in: *Variational difference methods for solving problems of mathematical physics* (Variatsionno-raznostnye metody resheniya zadach matem. fiz.,), VTs SO Akad. Nauk SSSR, 27–36, Novosibirsk, 1976.

3. ZLOTNIK, A. A., Convergence-rate estimates in L_2 for projection-difference schemes for parabolic equations, *Zh. vȳchisl. Mat. mat. Fiz.,* **18**, No. 6, 1454–1465, 1978.

4. TIKHONOV, A. N., and SAMARSKII, A. A., On homogeneous difference schemes of high order accuracy, *Dokl. Akad. Nauk SSSR,* **131**, No. 3, 514–517, 1960.

5. TIKHONOV, A. N., and SAMARSKII, A. A., Homogeneous difference schemes of high order of accuracy on non-uniform meshes, *Zh. vȳchisl. Mat. mat. Fiz.,* **1**, No. 3, 425–440, 1961.

6. MAKAROV, V. L., MAKAROV, I. L., and PRIKAZCHIKOV, V. G., Homogeneous difference schemes of high order of accuracy for systems of second-order differential equations, *Dokl. Akad. Nauk UkSSR, Ser. A,* No. 4, 302–305, 1978.

7. BURKHANOV, Sh. A., *et al.*, Exact difference schemes for ordinary fourth-order differential equations, *Dokl. Akad. Nauk UkSSR, Seri. A,* No. 9, 778–781, 1978.

8. LADYZHENSKAYA, O. A., and URAL'TSEVA, N. N., *Linear and quasi-linear elliptic equations* (Lineinye i kvazilineinye uravneniya ellipticheskogo tipa), Nauka, Moscow, 1964.

9. ZABREIKO, P. P., *et al., Integral equations* (Integral'nye uravneniya), SIB, Nauka, Moscow, 1968.

10. SAMARSKII, A. A., *Introduction to the theory of difference schemes* (Vvedenie v teoriya raznostnykh skhem), Nauka, Moscow, 1971.

11. SOBOLEV, S. L., *Applications of functional analysis to mathematical physics* (Nekotorye primeneniya funktsional'nogo analiza v matematicheskoi fizike), Izd-vo SO Akad. Nauk SSSR, Novosibirsk, 1962.

12. LADYZHENSKAYA, OA., SOLONNIKOV, V. A., and URAL'TSEVA, N. N., *Linear and quasi-linear parabolic equations* (Lineinye i kvazilineinye uravneniya parabolicheskogo tipa), Nauka, Moscow, 1967.