U.S.S.R. Comput. Maths. Math. Phys. Vol. 20, No. 6, pp. 52–99
Printed in Great Britain

0041–5553/80/060052–48$07.50/0

NUMERICAL METHODS FOR SOLVING MULTI-DIMENSIONAL PROBLEMS OF MECHANICS AND PHYSICS*

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(*Received* 22 *May* 1980)

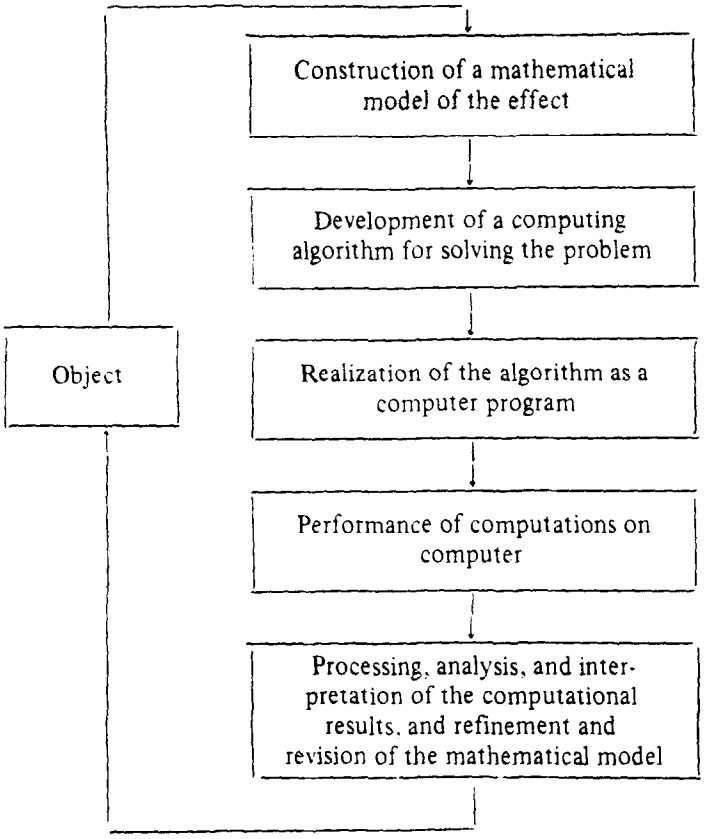
NUMERICAL methods for solving some multi-dimensional problems of gas dynamics, radiation gas dynamics, and plasma physics, are surveyed. The present state of computational experiments in these fields is discussed, and examples are given of the computation of some complex problems of mechanics and physics.

Introduction. Mathematical modelling and numerical methods

Theoretical studies in mechanics and physics have always been based on a mathematical foundations, and on obtaining basic quantitative characteristics of the object. As science has developed, there has been an increase both in the complexity of the phenomena, processes, and structures studied, and in the required accuracy of the results. Eventually the mathematical description of problems becomes so complicated that it is no longer possible to solve them by traditional means.

The invention of the high-speed electronic computer, the rapid development of computing methods, and direct numerical calculation of complex mathematical problems have marked a new stage in the application of mathematical methods to the solution of problems of science and engineering. The last two decades have seen the creation of the computational experiment (c.e.). which is a powerful new method of theoretical study, based on the use of a computer, and which has played an important role in the acceleration of scientific progress [1–4]. In essence, the experiment amounts to using a basic mathematical model to study, by computer, processes and systems of different kinds, to examine their behaviour under different conditions, and to find the optimal parameters and modes of actual or planned systems. With the aid of the c.e. we can predict mathematically the behaviour of complex effects and technical systems which it is difficult or impossible to study by other methods. The c.e. can be used effectively to study large-scale topical problems such as the theory and design of nuclear reactors, controlled thermonuclear synthesis, MHD energy conversion. topics in plasma physics, laser physics, or aerodynamics etc.

*Zh. vȳchisl. Mat. mat. Fiz., **20**, 6, 1416 – 1464, 1980.



The underlying ideas of the computational experiment can be seen from the scheme of Fig. 1, showing the stages in which the experiment is performed.

All the stages are closely linked and have the same basic aim – to obtain a result with the required accuracy in the shortest possible time. Even the present simplified scheme reflects a remarkable property of the c.e.: it operates effectively at the junction of different specialized fields and hence is irreplaceable for hybrid studies. It is often the only means, and not just an important means, for synthesizing knowledge and experience in different fields.

In the context of problems of mechanics and physics, we are concerned with uniting the efforts of specialists in mathematics, numerical methods, theoretical and experimental physics and, computer programming and design. These specialist fields are mutually enriched when the c.e. is performed. For instance, c.e. has stimulated the development of branches of mathematical physics concerned with the study of neutron transport and radiation equations, diffusion equations, systems of equations of hyperbolic type, equations with discontinuous coefficients, non-linear equations, kinetic equations describing a plasma, etc.

The present survey covers results obtained by the author and his associates during the last few years at the Institute of the Problems of Mechanics of the Academy of Sciences of the USSR, in connection with the development of numerical methods for solving complicated problems in mechanics and physics. The compilation of such algorithms is one stage in performing a c.e. With the experience so far accumulated, we can state certain conditions to be met by computing algorithms.

On the discretization of the equations of a continuous medium, i.e., changing from differential to difference equations, it is natural to demand that the resulting discrete model reflect correctly the main properties of the continuous medium. The satisfaction of this requirement has involved the statement and development of important and constructive concepts, such as conservative and completely conservative difference schemes [5]. Earlier, these schemes were obtained by means of certain semi-heuristic devices. The approach of the author and A.P. Favorskii to the construction of difference schemes, based on the use of general variational principles of mechanics, widens the scope for obtaining conservative and completely conservative schemes (see Section 1). Using this approach, for instance, we can automatically write schemes with given qualities in different coordinate systems in the multi-dimensional case. In Section 5 we shall consider the so-called completely neutral schemes of I.V. Fryazinov and B. D. Moiseenko for the Navier-Stokes equations, which preserve some important properties of the initial model and have many similarities to completely conservative schemes.

The choice of the numerical method for solving a problem is closely linked with the choice of mathematical model of the phenomenon in question. Experience shows that the model to be preferred is that best suited for numerical solution by computer, for which reliable economic algorithms are available. In Sections 3 and 4 we give examples of a successful choice of model in the framework of a given physical approximation, whereby substantial advances have been made in the numerical solution of certain problems of plasma physics. These results were obtained by L.M. Degtyarev and his colleagues and are concerned with problems of modelling plasma turbulence and studying the equilibrium configurations of a plasma.

In Section 2 we discuss topics connected with methods for multi-dimensional problems of radiation gas dynamics, developed under the guidance of B. N. Chetverushkin. The mathematical models used in studying these problems are extremely complex. For instance, they include as component parts the equations of gas dynamics and the kinetic equations of radiation transport. Further difficulties are created by the high dimensionality of the problem and the different scales of the physical processes that have to be taken into account. When devising the numerical method, various complex theoretical and practical problems had to be solved in connection with the choice of difference schemes, the construction of a method of solving the problems, and computer programming of the methods.

Let us briefly refer to some topics having direct reference to the c.e. but outside the framework of our present survey.

The mathematical models of many problems in mechanics and physics are often extremely complex and do not lend themselves to detailed theoretical study. But some of their important properties may be understood if the initial problem is divided into simpler units (modules). Modular analysis of a problem and preliminary study of the properties of individual modules demand the development of qualitative and analytic methods for studying problems of mathematical physics, such as e.g., the widely used method of constructing similarity solutions. The numerical algorithms and technology of performing the c.e. must reflect the modular structure of the initial problem. An important trend in c.e. development is the creation of packets of applied programs of mathematical physics [6]. The constructional principles of such program packets take account of the modular nature of the c.e., and of the fact that many different models may be used for studying the phenomenon to different degrees of approximation. Applied program packets are also convenient for the standardization, accumulation, and storage of numerical algorithms, etc.

Before an algorithm is widely used in practice, we need to study it theoretically for economy, accuracy, universality etc. In the theory of numerical methods two main topics can be distinguished: a) the construction and study of difference schemes, and *a priori* and *a posteriori* evaluation of their properties (convergence, accuracy, stability); b) the solution of difference equations. It can be claimed that there is now a very complete theory of difference schemes, both abstract and constructive, for linear problems [2, 7]. The difference scheme is a system of in general non-linear algebraic equations. When solving them, various iterative processes are used, which involves the repeated solution of a special type of system of linear algebraic equations of high order. The development of economic methods for solving such systems is one of the principal problems of the theory of numerical methods [8].

When performing a c.e. modelling the behaviour of some medium, we have to know its physical characteristics with reasonable accuracy; otherwise, the best computing methods will not yield a true picture of the actual phenomenon. The development of "physical software" for the c.e. is a vast problem in itself. For example, since the properties of substances cannot always be found directly by physical experiments, while the simplified models of a substance used in theoretical physics are crude, it becomes necessary to solve complex quantum-mechanical problems by numerical methods on a computer. In turn, modern physical experiments are so complex that their results cannot be correctly interpreted without using special methods of computer processing [9]. In both these cases we are in fact talking about performing independent computing experiments side by side with the main c.e.

1. Application of the variational approach to the construction of difference schemes

In numerical modelling, the difference scheme is interpreted as the discrete analogue of the physico-mathematical model of the phenomenon [2]. This means that the quality of the scheme has to be determined, not only by the canonical categories of the theory of numerical methods, but also by the extent to which the discrete model reflects the physical laws of the process and hence, the closely related properties of the equations. From this point of view it is natural, when constructing the difference algorithms, to be guided directly by the methods used in physics and mathematics for describing the processes.

Variational principles [10–12] are a universal and fundamental means for describing and studying problems of theoretical and mathematical physics. Due to their constructive features and relative simplicity they have often provided the basis for a theoretical consideration of many classical fields of physics. At the same time, the variational approach is widely used to construct generalized solutions of the equations of mathematical physics. In view of this, we can hope for the successful application of variational principles in cases where no proof is available of the existence and uniqueness of a solution. This is the situation that we often have to deal with when solving applied and notably non-linear problems.

Many vital properties of physical processes are bound up with the properties of the functionals being varied. For instance, the statement of the laws of conservation of momentum and energy is based on the absence of an explicit dependence of the Lagrangian on the space coordinates and time. A further merit of variational principles, that facilitates their use, is their invariance with respect to the measurement systems.

In [13–16] it was suggested that the variational approach should be used to construct finite-difference models of hydrodynamics, magnetohydrodynamics, and diffusion processes. This approach was further developed in [17–22].

1. *Hamilton's variational principle and the equations of hydrodynamics.* The Lagrange method of description is based on the strong analogy between a continuous medium and a continuous system of particles [23, 24]. This connection between continuum mechanics and the mechanics of a finite number of particles is best utilized when constructing the discrete model.

First consider the adiabatic motion of a finite volume V, bounded by the surface F, and consisting of the same continuous particles. Following the usual analogy, we can write the Lagrangian L of the system as

$$L(t)=\int\limits_{M(V)}\left(\frac{|W|^2}{2}-\varepsilon\right)dm.$$

Here, t denotes the time, $r(x, y, z)$ is the radius vector of a point, $\mathbf{W}(u, v, w)$ is the particle velocity vector, V is the volume, dV is an elementary volume, M is the mass, dm is an element of mass, ρ is the density, ϵ is the specific internal energy per unit mass, P is the pressure and d/dt is the substantial time derivative.

In accordance with Hamilton's principle of least action, the motion of the medium, as a mechanical system, produces the stationary value $\delta S = 0$ of the functional of action:

$$S=\int\limits_{t_0}^{t_1}L(t)\,dt. \tag{1.1}$$

The variation of the functional (1.1) has to be performed in the context of the relation: law of conservation of mass

$$\delta(dm)=\delta(\rho dV)=0, \tag{1.2}$$

the first law of thermodynamics

$$\delta\varepsilon=(P/\rho^2)\,\delta\rho, \tag{1.3}$$

and the kinematic connection

$$d\mathbf{r}/dt=\mathbf{W}. \tag{1.4}$$

Variation of (1 1) in the light of (1.2)–(1.4) shows that the stationarity condition (1.1) is satisfied if we have the equation, expressing the law of conservation of momentum:

$$\rho\frac{d\mathbf{W}}{dt}+\operatorname{grad}P=0. \tag{1.5}$$

Relation (1.2) transforms into the equation of continuity

$$\frac{d\rho}{dt}+\rho \operatorname{div} \mathbf{W}=0. \tag{1.6}$$

and (1.3) into the equation of energy variation

$$\rho \frac{d\varepsilon}{dt}+P \operatorname{div} \mathbf{W}=0. \tag{1.7}$$

In the process of variation we also use the law of conservation of volume

$$\frac{d}{dt}(dV)=dV \operatorname{div} \mathbf{W}. \tag{1.8}$$

The system (1.4)–(1.7) of equations of hydrodynamics is completed by the equation of state

$$R(P, \rho, \varepsilon)=0. \tag{1.9}$$

The existence of the laws of conservation for the equations of hydrodynamics is linked with the possibility of writing them in variational form. The absence of an explicit dependence of the Lagrangian on the coordinates and time leads to the law of conservation of momentum

$$\frac{d}{dt}\int_{M(V)} \mathbf{W}\, dm+\oint_{F(V)} P\cdot \mathbf{n}_F\, dF=0 \tag{1.10}$$

and of energy

$$\frac{d}{dt}\int_{M(V)}\left(\varepsilon+\frac{|\mathbf{W}^2|}{2}\right) dm+\oint_{F(V)} P(\mathbf{W}, \mathbf{n}_F)\, dF=0. \tag{1.11}$$

We recall that the law of conservation of mass and the condition that the process be adiabatic are introduced as connections.

Hamilton's variational principle extends without modification to the case when account is taken of the volume dissipative processes with "viscous pressure" q:

$$\delta\varepsilon=\frac{P+q}{\rho^2}\delta\rho.$$

For this non-adiabatic motion of the medium, Eqs. (1.5) and (1.6) become

$$\rho\frac{d\mathbf{W}}{dt}+\operatorname{grad}(P+q)=0, \qquad \rho\frac{d\varepsilon}{dt}+(P+q)\operatorname{div}\mathbf{W}=0.$$

2. *Construction of the equations of a discrete medium on the basis of a variational approach.* Assuming that the motion of the medium is plane and dependent only on the two space Cartesian coordinates x and y, we fix as the independent space variables the Lagrangian coordinates α and β. Using the well-known arbitrariness in the choice of α, β, we can assume that, corresponding to the intial volume V, bounded by the surface F, we have in the α, β plane the unit square G, bounded by the contour Γ (Fig. 2).

We introduce into G a rectangular difference mesh $\bar{\omega}_h$, uniform in each direction and consisting of nodes ω_{ij}:

$$\bar{\omega}_h=\{\omega_{ij}=(\alpha_i, \beta_j);\quad \alpha_i=i\Delta\alpha,\quad i=0, 1, \ldots, N;\quad \beta_j=j\Delta\beta,$$
$$j=0, 1, \ldots, M\}.$$

We denote by γ_h the set of nodes lying on Γ; the rest are called interior nodes, and the set of them is denoted by ω_h:

$$\omega_h=\{\omega_{ij}\in G\setminus\Gamma\},\qquad \bar{\omega}_h=\omega_h+\gamma_h,\qquad \gamma_h=\{\omega_{ij}\in\Gamma\}.$$

We also introduce the set of rectangular cells $\Omega_h=\{\Omega_{ij},\ i=0, 1, \ldots, N-1,\ j=0, 1, \ldots, M-1\}$, each of which has four nodes ω_{kl} as its vertices, forming the pattern $\mathrm{Pa}_\omega(\Omega)$ of the cell Ω_{ij}:

$$III_\omega(\Omega_{ij})=\{\omega_{ij},\ \omega_{i+1, j},\ \omega_{i+1, j+1},\ \omega_{i, j+1}\}.$$

The nodes in pattern $\mathrm{Pa}_\omega(\Omega_{ij})$ are always ordered in such a way that, as they are sampled in turn, the cell Ω_{ij} is circuited counter-clockwise. Any cell is defined one-to-one by its centre, which we therefore denote by the same symbol: $\Omega_{ij}=\{\alpha_{i+0.5}=(i+0.5)\Delta\alpha,\ \beta_{j+0.5}=(j+0.5)\Delta\beta\}$. This system of notation helps to prevent future misunderstandings.

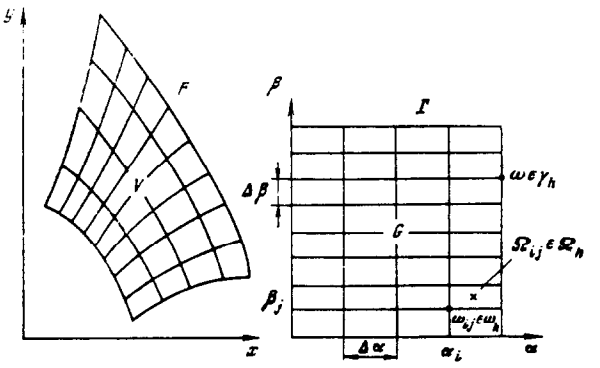


FIG. 2

We shall also use the pattern $\mathrm{Pa}_\Omega(\omega)$, composed of cells Ω_{kl}, having the node ω_{ij} as vertex:

$$\mathrm{Pa}_\Omega(\omega_{ij})=\{\Omega_{i, j},\ \Omega_{i-1, j},\ \Omega_{i-1, j-1},\ \Omega_{i, j-1}\};$$

here likewise the circuit rule is observed.

The images of cells $\Omega_{ij} \in \Omega_h$ in the x, y plane are plane rectangular volumes V_{ij} with curved boundaries. The vertices of any V_{ij} are the images of nodes $\omega_{kl} \in \mathit{III}_{\omega}(\Omega_{ij})$ in the x, y plane, for which it is pointless to introduce new notation, as it was for the centres of cells Ω_{ij}. The set $V_h = \{v_{ij}: i=0, 1, \ldots, N-1; j=0, \ldots, M-1\}$ forms a division of the initial volume V, which is thus associated with a finite system of particles of finite mass, formally identified with the "Lagrangian" set of cells Ω_h.

Henceforth, for typographical simplicity, we use the indexless notation $\Omega=\Omega_{ij}$, $\omega=\omega_{ij}$, $f=f_{ij}$, $\Delta_\alpha f=f_{i+1,j}-f_{ij}$, $\Delta_\beta f=f_{i,j+1}-f_{ij}$, $f_\alpha=\Delta_\alpha f/\Delta\alpha$, $f_\beta=\Delta_\beta f/\Delta\beta$. It will also be convenient to use the following symbols of averaging over patterns:

$$\langle\!\langle f\rangle\!\rangle_\Omega = 0.25 \sum_{\omega_{kl}\in \mathrm{Pa}\,\Omega} f_{kl}, \qquad f_{kl} \leftrightarrow \omega_{kl},$$

$$\langle\!\langle f\rangle\!\rangle_\omega = 0.25 \sum_{\Omega_{kl}\in \mathrm{Pa}\,\omega} f_{kl}, \qquad f_{kl} \leftrightarrow \Omega_{kl}.$$

We shall also use the abbreviated notation

$$\langle\Omega,\ f_\alpha\rangle = 0.5[(f_\alpha)_{ij}+(f_\alpha)_{i,j+1}], \quad \langle\Omega,\ f_\beta\rangle = 0.5[(f_\beta)_{ij}+(f_\beta)_{i+1,j}].$$

The Lagrangian L_h of the discrete system of particles modelling the continuous medium is naturally written as

$$L_h(t) = \sum_{\Omega_h} m\left(\frac{\langle\!\langle u^2+v^2\rangle\!\rangle_\Omega}{2} - \varepsilon\right).$$

The functional of action is

$$S_h = \int_{t_0}^{t_1} L_h(t)\,dt.$$

The law of conservation of mass is

$$\rho V = m. \tag{1.12}$$

The first law of thermodynamics, bearing in mind the viscous forces, is, for a volume element

$$m\,d\varepsilon = -(P+q)\,dV. \tag{1.13}$$

The kinematic relations, referred to the nodes, are

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v. \tag{1.14}$$

The law of conservation of volume V is conveniently written in "Jacobian" form

$$V = (\langle\Omega, x_\alpha\rangle\langle\Omega, y_\beta\rangle - \langle\Omega, x_\beta\rangle\langle\Omega, y_\alpha\rangle)\,\Delta\alpha\Delta\beta.$$

In this case, V is precisely equal to the area of the figure in the x, y plane, obtained by rectifying the curved sides of the image of Ω.

The laborious process of variation in the discrete case can be replaced by using directly the Euler–Lagrange extremality conditions:

$$\frac{d}{dt}\left(\frac{\partial L_h}{\partial u}\right)=\frac{\partial L_h}{\partial x}, \qquad \frac{d}{dt}\left(\frac{\partial L_h}{\partial v}\right)=\frac{\partial L_h}{\partial y},$$

which, in the light of the connection equations (1.12)–(1.14), become

$$\begin{aligned} \langle\!\langle m\rangle\!\rangle_\omega \frac{du}{dt} &= \sum_{\Omega_{kl}\in \mathrm{Pa}_{\Omega}(\omega_{ij})} (P_{kl}+q_{kl})\frac{\partial V_{kl}}{\partial x}, \\ \langle\!\langle m\rangle\!\rangle_\omega \frac{dv}{dt} &= \sum_{\Omega_{kl}\in \mathrm{Pa}_{\Omega}(\omega_{ij})} (P_{kl}+q_{kl})\frac{\partial V_{kl}}{\partial y}. \end{aligned} \tag{1.15}$$

Differentiating the volume V with respect to time, we obtain the differential-difference equation of the law of cell volume conservation

$$\frac{dV}{dt}=\sum_{\omega_{ik}\in \mathrm{Pa}_{\omega}(\Omega)}\left(u_{kl}\frac{\partial V}{\partial x_{kl}}+v_{kl}\frac{\partial V}{\partial y_{kl}}\right). \tag{1.16}$$

Associating this relation with (1.8), we obtain the difference analogue of the expression for the divergence:

$$D=\frac{1}{V}\sum_{\omega_{kl}\in \mathrm{Pa}_{\omega}(\Omega)}\left(u_{kl}\frac{\partial V}{\partial x_{kl}}+v_{kl}\frac{\partial V}{\partial y_{kl}}\right). \tag{1.17}$$

The energy equation in the "entropy" form follows from (1.13):

$$m\frac{d\varepsilon}{dt}=-(P+q)\frac{dV}{dt}.$$

After substituting (1.16) in this, we obtain the usual form:

$$m\frac{d\varepsilon}{dt}=-(P+q)\sum_{\omega_{kl}\in \mathrm{Pa}_{\omega}(\Omega)}\left(u_{kl}\frac{\partial V}{\partial x_{kl}}+v_{kl}\frac{\partial V}{\partial y_{kl}}\right).$$

The system of differential-difference equations is completed by the equation of state (1.9).

Without dwelling on the details, we observe that a direct check can be made of the difference analogues of the laws of conservation of momentum and energy (1.10), (1.11) for the differential-difference equations; the fact that these laws hold is a direct consequence of the variational method of obtaining them.

Expression (1.17) in fact defines an operator DIV on the set of mesh functions $\mathbf{W}_h=\{u_h,\ v_h\}$, specified in $\bar{\omega}_h$:

$$\underset{\bar{\omega}_h\to\Omega_h+\gamma_h}{\mathrm{DIV}}\ \mathbf{W}_h = \begin{cases} \dfrac{1}{V}\displaystyle\sum_{\omega_{kl}\in \mathrm{Pa}_{\omega_{ij}}\Omega_{ij}} \left(u_{kl}\dfrac{\partial V}{\partial x_{kl}}+v_{kl}\dfrac{\partial V}{\partial y_{kl}}\right) \text{ in } \Omega_h, \\ \dfrac{1}{\sigma}\displaystyle\sum_{\omega_{kl}} \left(u_{kl}\dfrac{\partial V^*}{\partial x_{kl}}+v_{kl}\dfrac{\partial V^*}{\partial y_{kl}}\right) \text{ in } \gamma_h. \end{cases} \tag{1.18}$$

Here, Pa_γ is the pattern composed of pattern composed of pairs of boundary nodes $\omega_{kl}\in\gamma_h$, ordered according to a counter-clockwise circuit of Γ: V^* is the zero volume of the fictitious cell, adjacent to Pa_γ, $\Omega^*\in\Omega_h^*$; σ is an element of the "difference" length of arc corresponding to Pa_γ.

The operator adjoint to DIV is then $(\mathrm{DIV})^* = -\mathrm{GRAD}$, where

$$\mathrm{GRAD}\, g_h = \left(-\frac{1}{\langle\!\langle V\rangle\!\rangle_\omega}\sum_{\Omega_{kl}\in \mathrm{Pa}_\omega(\omega)} g_{kl}\frac{\partial V_{kl}}{\partial x},\right.$$

$$\left.-\frac{1}{\langle\!\langle V\rangle\!\rangle_\omega}\sum_{\Omega_{kl}\in \mathrm{Pa}_\omega(\omega)} g_{kl}\frac{\partial V_{kl}}{\partial x}\right) \text{ in } \bar{\omega}_h.$$

The operator GRAD obviously acts in the set of mesh functions g_h, specified on $\Omega_h+\gamma_h$.

It can be shown that

$$(g_h,\ \mathrm{DIV}\,\mathbf{W}_h)=-((\mathrm{GRAD}\, g_h,\ \mathbf{W}_h)),$$

where the scalar products $(\ ,\ )$ and $((\ ,\ ))$ are defined as follows:

$$((\mathbf{W}_h^{(1)},\mathbf{W}_h^{(2)}))=\sum_{\omega_h}(u^{(1)}u^{(2)}+v^{(1)}v^{(2)})\langle\!\langle V\rangle\!\rangle_\omega,$$

$$(g_h^{(1)},g_h^{(2)})=\sum_{\Omega_h} g^{(1)}g^{(2)}V+\sum_{\gamma_h} g^{(1)}g^{(2)}\sigma.$$

Using GRAD and DIV, we can write the system of differential-difference equations in the form

$$\rho_h V_h=m_h,\quad V_h=V_h(\mathbf{r}_h),\quad \mathbf{r}_h=\{x_h,y_h\},\quad \frac{d\mathbf{r}_h}{dt}=\mathbf{W}_h,\quad \mathbf{W}_h=\{u_h,v_h\},$$

$$(\rho_\omega)_h\frac{d\mathbf{W}_h}{dt}=-\mathrm{GRAD}(P_h+g_h),\qquad (\rho_\omega)_h=\frac{\langle\!\langle m_h\rangle\!\rangle_\omega}{\langle\!\langle V_h\rangle\!\rangle_\omega},$$

$$\rho_h\frac{d\varepsilon_h}{dt}=-(P_h+q_h)\mathrm{DIV}\mathbf{W}_h,\qquad R(p_h,\rho_h,\varepsilon_h)=0,$$

$$\rho_h,\ \varepsilon_h\in\Omega_h,\quad m_h,\ V_h,\ P_h,\ q_h\in\Omega_h+\gamma_h,$$

$$\langle\!\langle V_h\rangle\!\rangle_\omega,\ (\rho_\omega)_h,\ \langle\!\langle m_h\rangle\!\rangle_\omega,\ \mathbf{r},\ \mathbf{W}_h\in\bar{\omega}_h.$$

The equations of volume conservation and of continuity are corollaries here:

$$\frac{dV_h}{dt} = V_h \operatorname{DIV} \mathbf{W}_h, \qquad \frac{d\rho_h}{dt} + \rho_h \operatorname{DIV} \mathbf{W}_h = 0.$$

On replacing the time derivatives by finite differences $df/dt \approx f_t = (\hat{f} - f)/\Delta t$, $f = f^n$, $\hat{f} = f^{n+1}$, where n is the number of the time layer, $\Delta t = t^{n+1} - t^n$, we obtain the family of difference schemes of hydrodynamics, which, given a suitable choice of weights, are completely conservative:

$$\begin{gathered}
\rho_h \mathbf{W}_{h_t} = m_h, \qquad V_h = V(\mathbf{r}_h), \qquad (\mathbf{r}_h)_t = \mathbf{W}_h^{(0.5)}, \\
(\tilde{\rho}_\omega)_h (\mathbf{W}_h)_t + \operatorname{GRAD}^{(0.5)} (P_h^{(\sigma_P)} + q_h^{(\sigma_q)}) = 0, \quad (\tilde{\rho}_\omega)_h = \frac{\langle\langle m_h \rangle\rangle_\omega}{\langle\langle V_h \rangle\rangle_\omega^{(0.5)}} \\
(\tilde{\rho}_\omega)_h (\mathbf{W}_h)_t + \operatorname{GRAD}^{(0.5)} (P_h^{(\sigma_P)} + q_h^{(\sigma_q)}) = 0, \qquad (\tilde{\rho}_\omega)_h = \frac{\langle\langle m_h \rangle\rangle_\omega}{\langle\langle V_h \rangle\rangle_\omega^{(0.5)}}, \\
(V_h)_t = V_h^{(0.5)} \operatorname{DIV}^{(0.5)} \mathbf{W}_h^{(0.5)}.
\end{gathered} \tag{1.19}$$

The operators $\operatorname{DIV}^{(0.5)}$, $\operatorname{GRAD}^{(0.5)}$ are obtained from (1.18) by replacing each factor of the operator coefficients by the corresponding half-sum with respect to the time layers. The parameters σ_p, σ_q remain arbitrary.

It can be shown that, for any $0 \leqslant \sigma_P, \sigma_q \leqslant 1$, the constructed completely conservative scheme has first order of approximation with respect to time and second with respect to space. The expressions DIV W_h and GRAD P_h then approximate div W and grad P respectively. With $\sigma_p = \sigma_q = 0.5$, the scheme has second order of approximation with respect to time also.

The stability of the family (1.19) is unconditional for $\sigma_P, \sigma_q \geqslant 0.5$; otherwise, the time step is restricted.

With the variational approach, we can construct without serious modifications completely conservative schemes in cylindrical, spherical, and any other coordinate systems [17, 18, 20].

The variational approach has also been used with success for constructing completely conservative schemes of ideal MHD. In this case extra terms are introduced into the Lagrangian, which correspond to the magnetic field energy, and also an extra connection, corresponding to the frozen condition.

It is quite obvious that the variational approach will also retain its force in the case of three space variables.

3. *Variational-discrete models of diffusion processes.* In the numerical modelling of applied problems it is not often that we can confine ourselves to dissipativeless motion of the medium. The motion is often accompanied by various physical effects such as diffusion transfer of heat or magnetic field etc., which in general have to be considered on moving curvilinear irregular meshes.

A typical feature of many problems is the strong space and time inhomogeneity of the coefficients in the equations. This is specially true for problems of plasma physics. As a result, the usual algorithms for solving equations of the heat conduction type prove to be inapplicable due to the critical loss of computational accuracy. In such situations it is best to introduce, along with the basic functions, the fluxes of these functions. With the flux form of the equations we can construct stable homogeneous computing algorithms, with low sensitivity to the spread of the coefficients and allowing degeneracy in the equations. Finding the fluxes is of independent interest. It is thus worth mentioning the algorithms by means of which we can simultaneously obtain both the initial functions themselves (temperature, magnetic field, etc.), and the corresponding fluxes (heat, electric field, etc.).

Variational principles may be effectively used to construct difference schemes for equations of the heat conduction type [21, 22]. Consider the equation of heat conduction in the flux form

$$\frac{\partial u}{\partial t} + \operatorname{div} \mathbf{W} = 0, \tag{1.20}$$

$$\mathbf{W} + k \operatorname{grad} u = 0 \tag{1.21}$$

in the simply connected plane domain V, in the plane of Cartesian coordinates (x, y); here, u is the temperature, and $\mathbf{W}$ the heat flux vector. Assume that we are given on the boundary Γ the homogeneous boundary condition for the normal component of flux; n is the outward unit normal to the contour Γ. At each fixed instant $t \geqslant 0$ the field of heat fluxes that satisfy the boundary condition stated minimizes the functional

$$\Phi(\mathbf{W}) = \int_V \frac{|\mathbf{W}|^2}{k} dV + \frac{\partial}{\partial t} \int_V u^2 \, dV, \tag{1.22}$$

in which functions u and k are assumed given and are not subject to variation. The variation of $\partial u/\partial t$ is found from Eq. (1.20), which plays the role of connection. If we eliminate $\partial u/\partial t$ from (1.22) with the aid of (1.20), Eq. (1.22) takes the form

$$\Phi(\mathbf{W}) = \int_V \left(\frac{|\mathbf{W}|^2}{k} - 2u \operatorname{div} \mathbf{W} \right) dV. \tag{1.23}$$

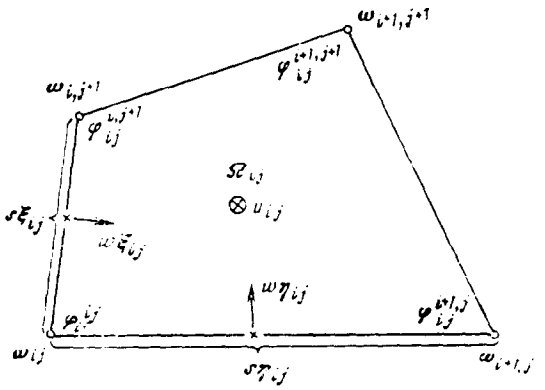


FIG. 3

Hence finding the field of fluxes W at each instant can be based on the minimization of the functional (1.23), while u is found from the balance equation (1.20). If boundary conditions of different types are specified, the corresponding integrals over the boundary Γ have to be introduced into the functional (1.23) as additive terms.

We replace the domain G by a discrete set of points of a difference mesh. Assume that functions $\xi(x, y)$, $\eta(x, y)$ exist, realizing a smooth one-to-one mapping of domain V into the unit square $0 \leqslant \xi \leqslant 1$, $0 \leqslant \eta \leqslant 1$. On taking ξ, η as curvilinear coordinates, we cover the domain V by a quadrangular mesh, representing the image of the rectangular uniform mesh in the square in the (ξ, η) plane. The values of the mesh function u^h and of the coefficient k^h will be assumed constant inside each mesh cell. We shall describe the field of fluxes by means of the pair of mesh functions $W\xi^h$, $W\eta^h$, which are the projections of the heat flux vector on the normal to the mid-points of the mesh sides, in the directions of increasing ξ and η respectively. The indexing of the mesh quantities is illustrated in Fig. 3. This discretization enables the integro-interpolation heat balance relation for cell Ω_{ij} to be written as

$$Vu_t = -\nabla_\xi(s\xi\, W\xi) - \nabla_\eta(s\eta\, W\eta). \tag{1.24}$$

Here, symbols ∇_ξ, ∇_η denote the operators of taking the (forward) difference with respect to directions ξ and η. The quantity V^h is the volume of the cell Ω, while the coefficients $s\xi$, $s\eta$ approximate the lengths of the sides of the cell. We can write Eq. (1.24) in the operator form

$$D(u_h)_t = R\mathbf{W}_h, \tag{1.25}$$

where $\mathbf{W}_h = \{W\xi_h, W\eta_h\}$ is the mesh vector function, R is a block operator acting from the space of mesh functions W_h into the space of mesh functions U_h, and D is the diagonal operator. It can be shown [21] that (1.24) approximates differential equation (1.20) to second order with respect to space and first order with respect to time.

To find the field of fluxes, we approximate the functional in the difference mesh by the expression

$$\Phi_h(W_h) = \sum_{\Omega_h} \left\{ \frac{|W_h|^2}{k} V - 2u[\Delta_\xi(s\xi\, W\xi) + \Delta_\eta(s\eta\, W\eta)] \right\}. \tag{1.26}$$

The modulus of the flux vector at the centre of cell Ω is found from (see [25])

$$(W_h)_{ij}^2 = \Big[0.25 \sum_{s,l=0}^{1} (W\xi_{i+s,j})^2 + (W\eta_{i,j+l})^2 + \\ + (-1)^{s+l} \cos\varphi_{ij}^{i+s,j+l} W\xi_{i+s,j} W\eta_{i,j+l} \Big] (\sin^2 \varphi_{ij}^{i+s,j+l})^{-1},$$

where $\varphi_{ij}^{i+s,j+l}$ is the angle between the sides of the cell (see Fig. 3).

On writing the minimization conditions for functional (1.26), we obtain the difference analogue of Eqs. (1.21):

$$\begin{aligned}
&0.25\sum_{s_1,s_2=0}^{1}\frac{(V/k)_{i-s_1,j}}{\sin^2\varphi^{i,j-1}_{i-s_1,j}}\left[W^{\xi}_{\xi ij}+(-1)^{s_2+1}\cos\varphi^{i,j+1}_{i-s_1,j}W\eta_{i-s_1,j+1}\right]\\
&+s^{\xi}_{\xi ij}(u_{ij}-u_{i-1,j})=0,\\
&0.25\sum_{s_1,s_2=0}^{1}\frac{(V/k)_{i,j-1}}{\sin^2\varphi^{i+s_1,j}_{i,j-1}}\left[W\eta_{ij}+(-1)^{s_1+1}\cos\varphi^{i+s_1,j}_{i,j-1}W\xi_{i+s_1,j-1}\right]\\
&+s\eta_{ij}(u_{ij}-u_{i,j-1})=0,
\end{aligned}\tag{1.27}$$

or, in operator form,

$$L\mathbf{W}_h=Gu_h. \tag{1.28}$$

Equations (1.27) approximate (1.21) to second order with respect to space. Operators D, R, L, G have the properties

$$D=D^*\geqslant\delta_1 E,\qquad \delta_1>0,\qquad R^*=-G,\qquad L=L^*\geqslant\delta_2 E,\qquad \delta_2>0. \tag{1.29}$$

On eliminating the temperature from system (1.25), (1.28), we can obtain the equation for the fluxes in divergence form:

$$L(\mathbf{W}_h)_t+A\mathbf{W}_h=0,\qquad A=-GD^{-1}R. \tag{1.30}$$

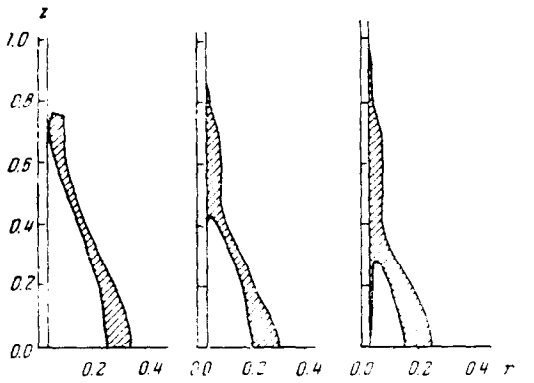


FIG. 4

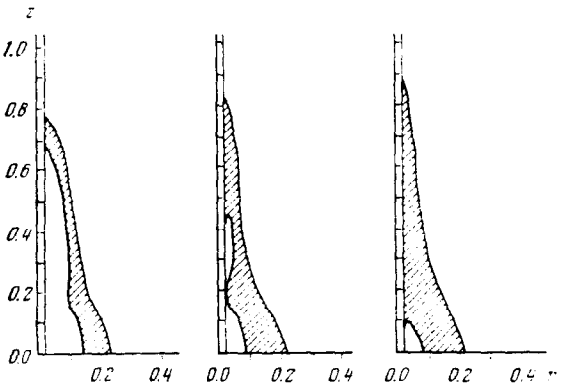


FIG. 5

It follows from (1.29) that $A = A^* > 0$. It was shown in [21] that Eq. (1.30) approximates, to second order with respect to space, and first order with respect to time, the differential equation

$$\frac{\partial (\mathbf{W}/k)}{\partial t} = \operatorname{grad}(\operatorname{div} \mathbf{W}).$$

To examine the stability of this algorithm, the results of the general theory of the stability of difference schemes [1] may be used. In view of the above-mentioned properties of the operators R, G, and L, scheme (1.30) is absolutely stable and has second order accuracy with respect to space and first order with respect to time.

Our discussion extends without serious modifications to the case of any coordinate system [21], and also to the three-dimensional case. Along with the completely implicit equation (1.30) we can consider the analogous equation with weight σ, thus enabling the approximation with respect to time to be improved to second order with $\sigma = 0.5$. The algorithm is also applicable in the case of moving space meshes; all the properties of the difference operators, and the accuracy of approximation, are then retained.

4. *Solution of multi-dimensional applied problems.* The completely conservative schemes obtained by the variational approach have been used with success for the numerical solution of some multi-dimensional applied problems. In particular, problems concerning the magneto-hydrodynamic delayed confinement of a plasma have been considered. One such system is based on the idea of quasi-spherical compression of the plasma by a heavy cylindrical liquid-metal liner converging to the axis. Numerical modelling has shown the possibility of the formation of cumulative jets, arising during the advance collapse of the liner ends (Fig. 4). The jet development substantially reduces the efficiency of compression and can lead to breakdown of the plasma, situated under the liner. During a c.e. [26] an optimal mode of liner collapse was found, in which the intensity of the cumulative jets is only slight and does not inhibit the required degree of compression of the plasma (Fig. 5).

On the basis of this method, studies were made of the stability of the magnetic cumulation process [27], the plasma transport in a magnetic conductor channel has been modelled numerically [28], and spontaneous development of magnetic fields [29] etc. have been modelled.

2. Methods for solving a two-dimensional problem of radiating gas dynamics

High-temperature gas-dynamic effects are now being encountered more and more frequently in science and engineering. Examples are 1) processes occurring in stellar atmospheres, 2) high-velocity entry of flight vehicles into the atmospheres of planets, 3) heavy-current radiating discharges, and 4) effects in the laser plasma.

Since radiation usually starts to have a substantial effect on the development of a process if the gas temperature reaches 10 000°K, it is often impossible to find the gas-dynamic fields without knowing the radiation fields.

Let us write the system of equations of radiating gas dynamics (see [30]):

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{u} = 0, \tag{2.1}$$

$$\rho \frac{d\mathbf{u}}{dt} = -\operatorname{grad}(P+\omega), \tag{2.2}$$

$$\rho \frac{d\varepsilon}{dt} = -(P+\omega)\operatorname{div}\mathbf{u} + \operatorname{div}\lambda \operatorname{grad} T - \operatorname{div}\mathbf{W} + Q, \tag{2.3}$$

$$\boldsymbol{\Omega} \operatorname{grad} I_\nu = \varkappa_\nu (I_{\nu p} - I_\nu), \tag{2.4}$$

$$W = \int_0^\infty d\nu \int \boldsymbol{\Omega} I_\nu \, d\Omega. \tag{2.5}$$

Here, $\varepsilon = \varepsilon(T, \rho)$, $P = P(T, \rho)$, $\lambda = \lambda(T, \rho)$, $\varkappa_\nu = \varkappa(\nu, T, \rho)$.

We have used the following notation: t is the time, $\mathbf{u}$ is the velocity vector, ρ is the density, P is the pressure, ϵ is the internal energy, ω is the artificial viscosity, λ is the thermal conductivity, $\mathbf{W}$ is the radiation energy flux vector, Q is the contribution to the energy equation of the different heat sources, I_ν is the radiation energy intensity, Ω is the unit vector of the photon transit direction, ν is the photon frequency, and $\varkappa_\nu$ is the coefficient of absorption of photons of frequency ν, while the spectral intensity of absolutely black body radiation is

$$I_{\nu p} = \frac{2h}{c^2} \frac{\nu^3}{\exp(h\nu/kT) - 1}.$$

To illustrate the difficulties that arise when solving problems of radiating gas dynamics, we choose the relatively simple transport equation for a plane layer

$$\mu \frac{dI_\nu}{dx} + \varkappa_\nu I_\nu = \varkappa_\nu I_{\nu p},$$

where μ is the cosine of the angle between the direction of motion of the photon of frequency ν and the x axis.

Let us choose 10 nodes for the frequency ν and 10 nodes for the angle μ. Hence, in order to find the radiation field I_ν, and then the radiation energy flux $\mathbf{W}$ in each time step, we have to solve 100 ordinary differential equations of the type

$$\mu_p \frac{dI_k}{dx} + \varkappa_k I_k = \varphi_k, \text{ where } \varphi_k = \varkappa_k \int_{\nu_k}^{\nu_{k+1}} \frac{4\pi h \nu^3}{c^2[\exp(h\nu/kT) - 1]} d\nu,$$

whereas there are just three equations of gas dynamics (2.1)–(2.3).

Hence the main part of the computing time needed to solve the entire problem of radiating gas dynamics is expanded in solving the transport equation.

This difficulty is linked with the extra dimensionality of the system (2.1)–(2.5) as compared with the equations of gas dynamics. There are further difficulties that are perhaps not so obvious, specific to problems of radiation gas dynamics (r.g.d.). Hence it is not a matter of coincidence that computations of r.g.d. non-stationary problems are 10–15 years behind those of problems of ordinary gas dynamics.

The close interconnection and mutual influence of gas-dynamic fields and the radiation field, and the need to solve the multi-dimensional kinetic equation at each time step, prevent us from constructing algorithms for solving g.r.d. problems by a mechanical combination of methods of neutron transport theory and methods of gas dynamics. This fact, and the fact that there are many important practical problems that need to be solved, suggest that numerical methods for solving such problems should be treated as an independent field of study.

1. *Methods for solving one-dimensional problems of g.r.d.* The first work on the numerical solution of non-stationary one-dimensional problems of r.g.d. appeared in the late sixties and early seventies.

When solving these problems special attention was paid to minimizing the computer time needed to deal with the transport equation. Important contributions to the development of methods for the effective handling of the radiation field dependence on the transit photon direction and energy were made in [31–34, 37–40]. Also of vital importance for performing practical computations, was the construction of stable schemes for the joint solution of the equations of gas dynamics and the radiation transport equation [35, 38, 41–44].

During these years, in the Institute of the Problems of Mechanics of the Academy of Sciences of the USSR, a method was devised for solving one-dimensional non-stationary problems of magnetic radiation gas dynamics (m.r.g.d.), on the basis of which heavy-current radiating discharges in lithium plasma were later computed [45–46]. As a result, several interesting qualitative and quantitative laws were discovered in plasma discharges.

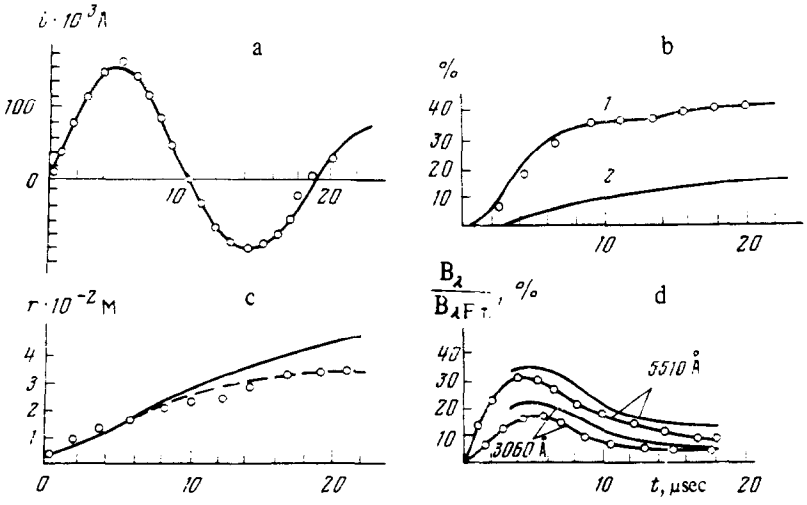


FIG. 6. O experimental data, — theoretical; a) for total current; b) position of shock wave, - - - boundary of luminous domain; c) curve 1 shows the proportion of energy imbedded in the plasma relative to total energy stored in the capacitor; curve 2 shows the fraction of energy liberated from a plasma in the form of radiation; d) brightness of the radiation at different frequencies.

Subsequently, the method for solving m.r.g.d. problems was considerably improved [40, 42, 47], the equations of state and absorption coefficients were more accurately computed, and general principles were developed for performing c.e.'s. As a result of all this, high accuracy could be achieved in solving complex one-dimensional m.r.g.d. problems.

As an illustration, consider some computed results for heavy-current radiating discharges in xenon plasma [48]. In Fig. 6 we plot data from [48] on the comparison of theory and experiment for convergent discharges. It can be seen that the divergence is at most a few percent. In the survey [49], experience in the field of solving one-dimensional r.g.d. problems is generalized.

2. *Methods for solving two-dimensional r.g.d. problems.* In view of the much greater complexity of two-dimensional problems, the difficulties of solving them are even greater. Most published work [50 - 51] is concerned with methods for solving stationary problems. The solution of non-stationary two-dimensional problems of radiating gas dynamics is at the limit of the capability, not only of the BESM-6, but of more powerful computers [52]. At present, the only way to solve these problems is to devise high-efficiency numerical methods.

Progress in the numerical modelling of one-dimensional problems has made it possible to come close to the solution of non-stationary two-dimensional r.g.d. problems. Numerical methods of solution began to be developed at the Institute of the Problems of Mechanics in the mid-seventies.

Radiation transport has been described by means of the multi-group diffusion approximation. Gas-dynamic flow was considered in Lagrange coordinates. The system of equations of radiating gas dynamics, for the axisymmetric case, is then

$$
\begin{aligned}
&\frac{d\rho}{dt} = -\rho \frac{1}{r} \frac{\partial (ru)}{\partial r} - \rho \frac{\partial v}{\partial z}, \\
&\rho \frac{du}{dt} = -\frac{\partial (P+\omega)}{\partial r}, \\
&\rho \frac{dv}{dt} = -\frac{\partial (P+\omega)}{\partial z}, \\
&\rho \frac{d\varepsilon}{dt} = -(P+\omega)\left(\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{\partial v}{\partial z}\right) + \frac{1}{r}\frac{\partial (rW_r)}{\partial r} + \frac{\partial W_z}{\partial z}, \\
&\frac{1}{r}\frac{\partial (rW_r^k)}{\partial r} + \frac{\partial W_z^k}{\partial z} + \varkappa_k U_k = \varkappa_k U_{pk}, \\
&\frac{l_k}{3}\frac{\partial U_k}{\partial r} = -W_r^k, \qquad \frac{l_k}{3}\frac{\partial U_k}{\partial z} = -W_z^k; \\
&W = \sum_{k=1}^{N_k} W^k.
\end{aligned}
\tag{2.6}
$$

The development of a joint algorithm for solving these equations required the consideration of a range of problems:

1) solution of the equations of gas dynamics,

2) constuction of a difference approximation of equations of elliptic type in non-orthogonal meshes,

3) development of iterative methods for solving the elliptic difference equations,

4) development of methods for averaging the diffusion equations over the photon energies,

5) development of methods for the simultaneous solution of the equations of gas dynamics and equations of radiation transport.

Instead of dwelling on methods for solving the equations of gas dynamics, we shall focus on the problems specific to r.g.d. problems.

The last three equations of (2.6) can be reduced to a system describing the behaviour of the radiation energy group density:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{rl_k}{3}\frac{\partial U_k}{\partial r}\right)-\frac{\partial}{\partial z}\frac{l_k}{3}\frac{\partial U_k}{\partial z}+\varkappa_k U_k=\varkappa_k U_{pk}. \tag{2.7}$$

Application of the Lagrangian coordinate system leads to the appearance of non-orthogonal meshes, in which we have to approximate the elliptic equation (2.7).

In [53, 54], on the basis of the integro-interpolation method, it was suggested that the fluxes in a non-orthogonal mesh could be found by an analytic transformation from the orthogonal mesh coordinates to the non-orthogonal mesh coordinates. With this approach we can partially take into account the strong variations of the absorption coefficients on the thermal and shock wave fronts. The resulting difference scheme may be written in the nine-point pattern as

$$\begin{aligned}&A_{in}^k U_{i-1,n-1}^k+B_{in}^k U_{i,n-1}^k+L_{in}^k U_{i+1,n-1}^k+K_{in}^k U_{i-1,n}^k\\&-C_{in}^k U_{in}^k+E_{in}^k U_{i+1,n}^k+P_{in}^k U_{i-1,n+1}^k+V_{in}^k U_{i,n+1}^k\\&+D_{in}^k U_{i+1,n+1}^k+F_{in}^k=0,\qquad k=1,2,\ldots,N_n.\end{aligned} \tag{2.8}$$

In orthogonal meshes, scheme (2.8) becomes the usual five-point scheme.

To solve the difference scheme (2.8), we use the non-linear iterative method described in [35]. When constructing this iterative algorithm, it is assumed that the solution of the difference system of equations (2.8) satisfies the conditions

$$U_{in}=U_{i+1,n}\alpha_{i+1,n}+\beta_{i+1,n},\qquad U_{in}=U_{i-1,n}\gamma_{i-1,n}+d_{i-1,n},$$
$$U_{in}=U_{i,n+1}\bar{\alpha}_{i,n+1}+\bar{\beta}_{i,n+1},\qquad U_{in}=U_{i,n-1}\bar{\gamma}_{i,n-1}+\bar{d}_{i,n-1}.$$

The pivotal condensation coefficients α, β, γ, d, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{d}$ are found from the appropriate system of equations, which is solved by iterations (see [54–56]):

$$\alpha_{i+1,n}=(E_{in}+\bar{\alpha}_{i+1,n}L_{in}+\bar{\gamma}_{i+1,n}D_{in})/\psi_{in\alpha},$$

$$\psi_{in\alpha}=C_{in}-\bar{\alpha}_{in}B_{in}-\bar{\gamma}_{in}V_{in}-\alpha_{in}\varphi_{in\alpha}, \quad \varphi_{in\alpha}=K_{in}+\bar{\alpha}_{i-1,n}A_{in}+\bar{\gamma}_{i-1,n}P_{in},$$

$$\alpha_{2n}=\varphi_{2n}, \quad i=2,3,\ldots,N_i-1, \quad n=2,3,\ldots,N_n-1;$$

$$\gamma_{i-1,n}=(K_{in}+\bar{\alpha}_{i-1,n}A_{in}+\bar{\gamma}_{i-1}P_{in})/\psi_{in\gamma},$$

$$\psi_{in\gamma}=C_{in}-\bar{\alpha}_{in}B_{in}-\bar{\gamma}_{in}V_{in}-\gamma_{in}\varphi_{in\gamma}, \quad \varphi_{in\gamma}=E_{in}+\alpha_{i-1,n}L_{in}+\bar{\gamma}_{i-1,n}D_{in},$$

$$\gamma_{N_i-1,n}=\varphi_{N_i-1}, \quad i=N_i-1,\ldots,3,2, \quad n=2,3,\ldots,N_n-1;$$

$$\bar{\alpha}_{i,n-1}=(V_{in}+\alpha_{i,n+1}P_{in}+\gamma_{i,n-1}D_{in})/\psi_{in\bar{\alpha}},$$

$$\psi_{in\bar{\alpha}}=C_{in}-\alpha_{in}K_{in}-\gamma_{in}E_{in}-\bar{\alpha}_{in}\varphi_{in\bar{\alpha}}, \quad \varphi_{in\bar{\alpha}}=B_{in}+\alpha_{i,n-1}A_{in}+\gamma_{i,n-1}L_{in},$$

$$\bar{\alpha}_{i,2}=\varphi_{i,2}, \quad i=2,3,\ldots,N_i-1, \quad n=2,3,\ldots,N_n-1;$$

$$\bar{\gamma}_{i,n-1}=(B_{in}+\alpha_{i,n-1}A_{in}+\gamma_{i,n-1}L_{in})/\psi_{in\bar{\gamma}},$$

$$\psi_{in\bar{\gamma}}=C_{in}-\alpha_{in}K_{in}-\gamma_{in}E_{in}-\bar{\gamma}_{in}\varphi_{in\bar{\gamma}}, \quad \varphi_{in\bar{\gamma}}=V_{in}+\alpha_{i,n+1}P_{in}+\gamma_{i,n+1}D_{in},$$

$$\bar{\gamma}_{i,N_n-1}=\varphi_{i,N_n-1}, \quad i=2,3,\ldots,N_i-1, \quad n=N_n-1,\ldots,3,2.$$

We write in a similar way the system for finding the coefficients β, α, $\bar{\beta}$, $\bar{d}$.

As distinct from other iterative methods, instead of solving the initial system (2.8), we here solve a new system for the coefficients α, β, ..., $\bar{d}$. Comparison with other methods has shown that the non-linear iterative algorithm can be employed successfully to solve difference linear equations in the case when *a priori* information about the limits of the difference operator spectrum is either absent or is known with insufficient accuracy. The method can be generalized: a matrix form of the algorithm was described in [57] and used to compute problems of viscous fluid dynamics.

The solution of the multi-group system of equations (2.8) in each time step sharply increases the computer time needed for the complete problem of radiating gas dynamics. To achieve greater economy, the following algorithm has been described and realized.

At some j-th time step we solve the multi-group system of diffusion equations. Then, using the numerical solution obtained, we construct an averaged difference equation. The coefficients of this equation are constructed in the light of the solution of the multi-group system. During N_j steps the integral flux and radiation energy density are found from the averaged equation. At step $j+N_j$ the procedure is repeated.

As distinct from the earlier methods of this type [34, 37], the present method is not tied to a one-dimensional geometry, equally applicable to solving both one- and two-dimensional r.g.d. problems. It likewise does not require that the absorption coefficients be smooth.

The simplest scheme for the joint solution of the energy equations of (2.6) and the radiation transport equations is the explicit scheme, when the radiation energy flux is found from the data of the previous time step. However, when domains of high optical thickness are present, the explicit scheme for joint solution demands restrictions on the time step, similar to the restrictions in the explicit scheme for computing the equation of heat conduction. Notice that it quite often happens that there are instants when part of the plasma domain becomes opaque.

The implicit scheme of [42, 43] for the joint solution of the energy equation and equations of diffusion type is as follows:

$$(1+g_i)\,(A_{in}U^j_{i-1,n-1}+B_{in}U^j_{i,n-1}+L_{in}U^j_{i+1,n-1}+K_{in}U^j_{i-1,n}$$

$$-C_{in}U^j_{in}+E_{in}U^j_{i+1,n}+P_{in}U^j_{i-1,n+1}+V_{in}U^j_{i,n+1}+D_{in}U^j_{i+1,n+1})$$

$$-\varkappa_{in}U^j_{in}+F^{j-1}_{in}+\left(\frac{\partial F}{\partial T}\right)^{j-1}\frac{\tau^{j-1}}{(\partial\varepsilon/\partial T)^{j-1}}Q_i^j=0,$$

$$g=\left(\frac{\partial F}{\partial T}\right)^{j-1}\tau^{j-1}\left[\left(\frac{\partial\varepsilon}{\partial T}\right)^{j-1}\rho\right]^{-1}$$

$$T=T^{j-1}+\frac{\tau^{j-1}}{(\partial\varepsilon/\partial T)^{j-1}}[DW^j+Q_i^j],$$

where DW is the difference approximation for div W, obtained on the basis of the last three equations of (2.6).

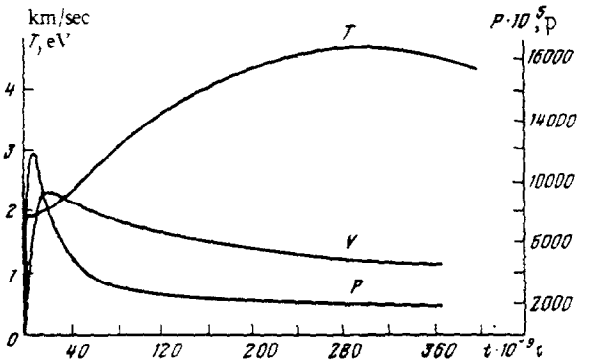


FIG. 7

This scheme has shown high efficiency when solving both one- and two-dimensional problems of r.g.d. For the model problem, linear with respect to T^4, it can be shown that the scheme is absolutely stable.

The diffusion approximation is fully applicable for a correct description of the radiation field in most r.g.d. problems. For a more complete description of the radiation field, it was proposed in [58] to use Vladimirov's self-conjugate equation [59]

$$-(\Omega\,\mathrm{grad})^2U_\nu+\varkappa_\nu U_\nu=\varkappa_\nu I_{\nu p}.$$

Computations have shown that, by using this equation, we can successfully counter the negative computational effect of the ray, which can occur when the radiation field is found directly from the transport equation (2.4).

3. *Examples of solutions.* The approach described for solving two-dimensional non-stationary problems of radiating gas dynamics has been realized in a program complex utilizing the extended working memory of the BESM-6 (see [60]). The approach has been used to solve a whole range of important practical problems. The program has provision for using the actual equations of state of a substance and the actual absorption coefficients.

As an example, consider the solution of the two-dimensional problem on the interaction of laser radiation with nitrogen plasma of high density close to a metal surface [63]. Effects of this type were studied experimentally in [61, 62]. It was pointed out there that, on the one hand, the zone of thermal action on the metal is much greater than the laser radiation focussing spot, while on the other hand, this thermal action does not lead to crater formation on the metal surface.

Consider some computational data for the version in which the initial nitrogen pressure was taken to be equal to 10^7 Pa, the laser radiation power was 5×10 W/cm^2, and the laser radiation focussing spot on the metal had a radius $r_0 = 250$ μm.

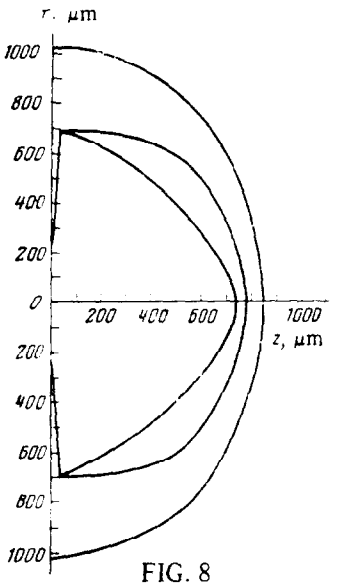


FIG. 8

In Fig. 7 we plot the characteristic velocities in the shock wave travelling in the cold nitrogen, and the characteristic pressures and temperatures in the hot plasma. Due to the high initial density of the nitrogen, the velocity in the shock wave is relatively low; hence it does not screen the laser radiation, which is entirely absorbed in the hot plasma.

In Fig. 8 we show space profiles of the temperature at the instant $t = 0.5$ μsec. It can be seen that the radius of the domain occupied by the hot plasma is much greater than the radius of the laser radiation focusing spot. The radiation generated in this domain has direct heating action on the metal plate. On the other hand, the hot plasma entirely screens the metal plate from the action of laser radiation. Hence the thermal action on the plate is quite gentle, which explains the absence of mechanical damage.

3. Modelling of turbulent processes in plasma

As a result of studies of plasma heating by heavy-current beams of relativistic electrons or high-power light beams, the topic of collisionless mechanisms of energy dissipation in a plasma has become vitally important. Initially, the energy of the external source is stored in the plasma in the form of long-wave electron oscillations with Langmuir frequency ω_p. The transformation of the energy of long-wave Langmuir oscillations into the short-wave part of the spectrum (the domain of their absorption by particles) is linked with what is known in plasma physics as modulation instability [64]. As a result, the energy of the Langmuir oscillations becomes localized in the domains of reduced plasma density (caverns). In the long run, a turbulent state (Langmuir turbulence) is established, when the energy of the original long-wave Langmuir oscillations becomes concentrated in a large number of randomly situated caverns of different characteristic size. As distinct from hydrodynamic turbulence of incompressible fluid, Langmuir turbulence admits of a one-dimensional model. This is bound up with the fact that the tendency for modulation instability to develop and for caverns to form is retained in the one-dimensional case.

Nevertheless, there is an important difference between one- and three-dimensional turbulence. In the actual three-dimensional geometry, the phenomenon of Langmuir collapse, predicted in [64], is present; it may be described as follows. During the energy localization in the caverns, energy compression occurs, and accordingly, short-wave pumping occurs of the energy of the Langmuir oscillations, forbidden in the caverns, to scales at which some dissipation mechanism becomes significant. In the one-dimensional geometry, the energy of the plasma oscillations is localized in caverns of finite size (solitons), with the result that the one-dimensional case is degenerate. This degeneracy is partially removed in the presence of a continuously operating source of Langmuir oscillations, when forced collapse is possible: as a result of absorption of energy from the source, the cavern can be compressed to a size at which dissipative processes come into action.

Strict analytic solutions are exceptionally difficult in the problem of strong Langmuir turbulence and so far none are available. However, on the basis of numerical modelling, it is possible to construct approximate models that are logically feasible.

Two approaches may be used for numerical modelling of Langmuir turbulence. In the first, the mathematical model is based on the Wlasow kinetic equations [65, 66], whereby both the wave and the kinetic processes in the plasma can be taken into account simultaneously. But computational realization of this model involves two serious difficulties. First, the characteristic space-time scales of the effect are determined by the ion dynamics and usually greatly exceed the space-time scales of the model (the Debye radius r_D i.e. the electron displacement relative to the ions, and the plasma period ω_p^{-1}). Moreover, for Wlasow's equations, the particle velocities as well as the space variables are independent variables, so that the dimensionality of the problem is increased. Hence practical realization of the kinetic approach is only possible if we take $m/M \sim 10^{-2}$ as the electron to ion mass ratio in the model; but this distorts the true picture of the turbulence.

In view of this, the second approach to constructing mathematical models of Langmuir turbulence is preferable; it is based on the approximate dynamic equations obtained by averaging the Wlasow equations over the particle velocities and a time interval greater than the plasma period. This model does not contain the plasma period as characteristic quantity and has natural space-time scales of the modulation instability.

1. *Initial equations.* On the basis of the dynamic system of equations of plasma turbulence [64], we consider the problem in a restricted interval L with periodic boundary conditions with respect to all the unknowns. For a single space dimension the equations take the form (in dimensionless variables)

$$\mathscr{E} = \frac{1}{2}[E(x,t)\exp(-i\omega_p t) + E^*(x,t)\exp(i\omega_p t)].$$

The complex amplitude of the high-frequency electric field satisfies the Schrödinger type of equation

$$2iE_t + \hat{\Gamma}E + E_{xx} = \delta n(E+E_0) - \langle \delta n E\rangle = 0. \tag{3.1}$$

In the linear approximation the quasi-neutral variation of density $\delta n = n - n_0$ and of the velocity w of slow movements satisfies the ordinary system of equations of gas dynamics, allowing for the high-frequency pressure of the Langmuir oscillations:

$$\delta n_t + w_x = 0, \tag{3.2}$$

$$w_t + \hat{\gamma} w + (\delta n + |E+E_0|^2)_x = 0, \qquad \langle \delta n E\rangle = \frac{1}{L}\int_0^L \delta n E\, dx. \tag{3.3}$$

The last term on the right-hand side of (3.1) corresponds to the source, which maintains a homogeneous electric field in the plasma at a constant level E_0, so that

$$\langle E\rangle = \frac{1}{L}\int_0^L E(x,t)\,dx = E_0.$$

In (3.3) we introduce terms describing the resonance absorption of plasma and sound waves by electrons. The terms $\hat{\Gamma}E$ and $\hat{\gamma}w$ are integral operators of convolution type:

$$\hat{\Gamma}E = \frac{1}{L}\int_0^L \Gamma(x-x')E(x',t)\,dx',$$

$$\hat{\gamma}w = \frac{1}{L}\int_0^L \gamma(x-x')w(x',t)\,dx';$$

here,

$$\Gamma(\xi) = \sum_{n=-\infty}^{\infty} \Gamma_n e^{ik\xi}, \qquad \gamma(\xi) = \sum_{n=-\infty}^{\infty} \gamma_n e^{ik\xi}, \qquad k = \frac{2\pi n}{L}$$

are the Fourier images of the damping decrements Γ_n, γ_n of the n-th terms of the corresponding Fourier series

$$E(x,t) = \sum_{n=-\infty}^{\infty} E_k e^{ikx}, \qquad w(x,t) = \sum_{n=-\infty}^{\infty} w_k e^{ikx},$$

where

$$E_k = \frac{1}{L}\int_0^L E(x,t)\,e^{ikx}\,dx, \qquad w_k = \frac{1}{L}\int_0^L w(x,t)\,e^{ikx}\,dx.$$

For the system of Eqs. (3.1)–(3.3) in a finite interval L with periodic boundary conditions, we have the integral corollary (energy balance)

$$\begin{gathered}\partial W/\partial t + R = Q,\\ W = \frac{1}{L}\int_0^L |E|^2\,dx, \qquad R = \sum_{n=-\infty}^{\infty} 2\Gamma_k |E_k|^2, \qquad Q = \operatorname{Im} E_0 \langle \delta n E^* \rangle.\end{gathered} \tag{3.4}$$

In the absence of a source and of damping, $E_0 = \hat{\Gamma}E = \hat{\gamma}w = 0$, in (3.1)–(3.3), we have the integrals of motion

$$\begin{gathered}w = \frac{1}{L}\int_0^L |E|^2\,dx,\\ I = \int_0^L \left[|E_x|^2 + \frac{1}{2}(n^2+w^2) + n|E|^2 \right] dx.\end{gathered} \tag{3.5}$$

In this case, system (3.1)–(3.3) has, on the unbounded straight line the family of exact solutions of solitary wave type (solitons):

$$\begin{gathered}E(x,t) = \frac{E_m \exp[iE_m^2 t/4 + q(x-x_0)]}{\operatorname{ch}[E_m(x-x_0-qt)/2(1-q^2)]},\\ \delta n = -E^2/(1-q^2), \qquad w = q\delta n.\end{gathered} \tag{3.6}$$

3. *Difference scheme.* Langmuir turbulence is a set of randomly situated solitons (3.6) with different amplitudes. The external field energy E_0 is absorbed by the Langmuir waves, whose length has a lower bound given by the condition

$$\lambda^2 > 2\pi^2 |E_0|^2.$$

They develop by non-linear evolution into the soliton solutions (3.6). These fairly narrow solitons include the absorption of waves by electrons (the terms $\hat{\Gamma}E$ and $\hat{\gamma}w$ in the initial system become substantial). Hence the problem contains all the main features of the turbulent process: energy absorption in the long waves, energy spectrum transformation in the short wave domain, and finally, damping of the short-wave part of the solution. The main interest here is not in the details of the space distributions $E(x, t)$ and $\delta n(x, t)$, but in the variation of the time-averaged energy of the plasma oscillations W, the frequency ν_{eff} of its particle absorption, and the energy of the sound waves W_s.

The computational algorithm used for numerical modelling of this process has to satisfy the following two requirements. First, for correct transmission of the statistical properties of the turbulence, the problem must be solved for sufficiently large values of L, in order for the solution to have sufficiently many solitons. Hence, in order to limit the meshes to a realistic number of nodes, it is essential to use algorithms with as high an order of space approximation as possible. This condition means that we have to employ spectral methods [67, 68], such as came to be developed after the appearance of the fast Fourier transformation algorithm [69]. A further advantage of the spectral method is that, after Fourier transformation of Eqs. (3.1) and (3.3), the non-local operators of convolution type, connected with the damping, transform into local operators of multiplication of the Fourier coefficients by the appropriate decrements. Second, in addition to high space approximation, the algorithm must yield a solution at asymptotic times, when the initial stage of turbulence is completely forgotten and a dynamically time-stationary solution is established. The need arises to construct conservative schemes [5, 70], possessing the difference analogue of the conservation laws (3.4).

Let us give the difference scheme for (3.1)–(3.3) that satisfies the above conditions. For simplicity, we confine ourselves to the case $E_0=\hat{\Gamma}E=\hat{\gamma}w=0$. We introduce the meshes

$$\omega_\tau=\{t_j=j\tau,\ j=0, 1, \ldots\}. \qquad \omega_h=\{x_i=ih,\ i=0, 1, \ldots, N-1\},$$
$$\omega_{\tau h}=\omega_\tau\times\omega_h.$$

On mesh ω_h, with the aid of Fourier finite sums, we introduce the mesh function

$$f_i(t)=\sum_{n=-N/2+1}^{N/2} f_n \exp(ikx_i), \qquad k=\frac{2\pi n}{L},$$

where

$$f_n=\frac{h}{L}\sum_{i=0}^{N-1} f_i \exp(-ikx_i).$$

We approximate the derivatives $\partial^p f/\partial x^p$ by the expression

$$\left(\frac{\partial^p f}{\partial x^p}\right)_i=\sum_{n=-N/2+1}^{N/2}(ik)^p f_n \exp(ikx_i).$$

On thus approximating E, δn, and U, and their derivatives, and substituting into (3.1)–(3.3), we obtain the following system of ordinary differential equations for the Fourier coefficients:

$$2\dot{u}_n-k^2v_n-\bar{c}=0, \qquad 2\dot{v}_n+k^2u_n+\bar{s}=0,$$
$$E_n=u_n+iv_n, \qquad \delta\dot{n}_n+ikw_n=0, \qquad \dot{w}_n+ik(w_n+\varepsilon_n)=0. \tag{3.7}$$

Here $\tilde{c}_n+i\tilde{\tilde{c}}_n$ and ε_n are the Fourier coefficients of the discrete Fourier series for $\delta n_i E_i$ and $\varepsilon_i=|E_i|^2$,

$$\tilde{c}_n+i\tilde{\tilde{c}}_n=\frac{h}{L}\sum_{i=0}^{N-1}n_iE_i\exp(-ikx_i),$$

$$\varepsilon_n=\frac{h}{L}\sum_{i=0}^{N-1}|E_i|^2\exp(-ikx_i).$$

For an analytic function f we have the equation

$$\frac{\partial^p f}{\partial x^p}(x_i)-\left(\frac{\partial^p f}{\partial x^p}\right)_i=O\left(h^{-p}\exp\left(-\frac{L}{h}\right)\right), \tag{3.8}$$

and for an $R+1$ times differentiable function, the equation

$$\frac{\partial^p f}{\partial x^p}(x_i)-\left(\frac{\partial^p f}{\partial x^p}\right)_i=O(h^{R-p}). \tag{3.9}$$

Consequently, system (3.7) approximates the initial system to order (3.8) or (3.9) in the case $p=2$. We next approximate (3.7) with respect to time to order $O(\tau^2)$:

$$\begin{gathered}2\frac{u_n-u_n}{\tau}-k^2\frac{v_n+\check{v}_n}{2}-c_n=0,\qquad 2\frac{v_n-v_n}{\tau}+k^2\frac{u_n+\check{u}_n}{2}+\tilde{c}_n=0,\\ \frac{\delta n_n-\delta\check{n}_n}{\tau}+\frac{ik}{2}(w_n+\check{w}_n)=0,\quad \frac{w_n-\check{w}_n}{\tau}+\frac{ik}{2}(\delta n_n+\varepsilon_n+\delta\check{n}_n+\varepsilon_n)=0,\end{gathered} \tag{3.10}$$

where

$$\begin{aligned}&f_{1n}=\tilde{c}_n+i\tilde{\tilde{c}}_n=\frac{h}{L}\sum_{i=0}^{N-1}f_{1i}\exp(-ikx_i),\\ &f_{1i}=\delta n_i^{1/2}E_i^{1/2},\\ &\delta n_i^{1/2}\equiv\frac{1}{2}(\delta n_i+\delta\check{n}_i),\qquad E_i^{1/2}\equiv\frac{1}{2}(E_i+\check{E}_i),\\ &\delta\check{n}_i\equiv\delta n_i(t_{j-1}),\qquad \check{E}_i\equiv E_i(t_{j-1}),\qquad \check{w}_i\equiv w_i(t_{j-1}).\end{aligned} \tag{3.11}$$

and hence approximate problem (3.1)–(3.3) to order $O(\tau^2+h^{-2}e^{-L/h})$, if the solution is an analytic function, or to order $O(\tau^2+h^{R-2})$ for an $R+1$ times differentiable solution. For mesh functions E_i, n_i, w_i the difference analogues of conservation laws (3.5) for the differential problem are satisfied:

$$W_h=\sum_{h=-N/2+1}^{N/2}|E_n|^2, \tag{3.12}$$

$$I_h = \sum_{n=-N/2+1}^{N/2} \left(k^2 |E_n|^2 + \frac{1}{2} |\delta n_n|^2 + |w_n|^2 + \delta n_n \varepsilon_n^* \right). \tag{3.13}$$

The difference scheme (3.10), (3.11) thus satisfies the above two conditions on approximation and conservativeness.

3. *Method of solving the difference problem.* To solve the non-linear algebraic problem (3.10), (3.11), we use the iterative process

$$\begin{aligned}
&2\frac{u_n^{s+1}-\check{u}_n}{\tau} - k^2 \frac{v_n^{s+1}-\check{v}_n}{2} - c_n^s = 0,\\
&2\frac{v_n^{s+1}-\check{v}_n}{\tau} + k^2 \frac{u_n^{s+1}+\check{u}_n}{2} + \hat{c}_n^s = 0,\\
&\frac{\delta n_n^{s+1}-\delta\check{n}_n}{\tau} + \frac{ik(w_n^{s+1}+\check{w}_n)}{2} = 0,\\
&\frac{w_n^{s+1}-\check{w}_n}{\tau} + \frac{ik}{2}(\delta n_n^{s+1} + \varepsilon_n^s + \delta\check{n} + \check{\varepsilon}_n) = 0.
\end{aligned} \tag{3.14}$$

It can be shown that, given a restriction on the time step, process (3.14) is convergent.

Experience shows that, if the difference analogues of the integrals of motion (3.12) are satisfied sufficiently accurately during the entire computing time, roughly 10 iterations are required. But this number can be reduced. We make the transition to the next time layer in two stages. At the first, predictor, stage, we confine ourselves to a finite number of iterations $s + 1 = p$ in (3.14). At the second (corrector) stage we perform a supplementary iteration in accordance with the relations

$$\begin{aligned}
&\frac{u_n^2-\check{u}_n^2}{\tau} - k^2 (v_n^{1/2})^p (u_n^{1/2})^p - (u_n^{1/2})^p (c_n)^p = 0,\\
&\frac{v_n^2-\check{v}_n^2}{\tau} + k^2 (u_n^{1/2})^p (v_n^{1/2})^p + (v_n^{1/2})^p (\hat{c}_n)^p = 0,\\
&\frac{\delta n_n-\delta\check{n}_n}{\tau} + ik w_n^{1/2} = 0,\\
&\frac{w_n-\check{w}_n}{\tau} + ik[\delta\check{n}_n + (\varepsilon_n^{1/2})^p] = 0.
\end{aligned} \tag{3.15}$$

On adding the first two equations (3.15) and summing the result on n, we see that, after the second stage, Eq. (3.12) is satisfied exactly. A similar corrector was used in [71] in the explicit scheme for modelling turbulence in an incompressible fluid. If the predictor iterations (3.14) are convergent the correcting iteration (3.15) does not disturb the solution, i.e. the corrector is matched with the predictor. Conversely, restriction of the predictor to an insufficient number of iterations (e.g. $p = 1$) with a sufficiently coarse time step leads to the appearance of negative energy for certain harmonics: $|E^p|^2=(u_n^2)^p+(v_n^2)^p<0$. This situation prevents us from restoring the electric field $E_n=u_n+iv_n$ and passing to the next time step. In order to exclude the case $|E_n|^2 < 0$ from (3.14), it is sufficient to perform a finite number of iterations (in practical computations $p \approx 3$, and from this condition the time step may be chosen). After the corrector, the signs of u_n, v_n are determined by the signs of u_n^p, v_n^p after the predictor. If, for some n, we have $u_n^2 < 0$ or $v_n^2 < 0$, but $u_n^2+v_n^2>0$, then

$$u_n=(|E_n|^2)^{1/2}\cos\varphi_n^p,$$

$$v_n=(|E_n|^2)^{1/2}\sin\varphi_n^p,$$

where φ_n^p is the phase of the electric field after the predictor. Then, in addition to (3.12) being satisfied exactly, (3.13) is satisfied with sufficient accuracy.

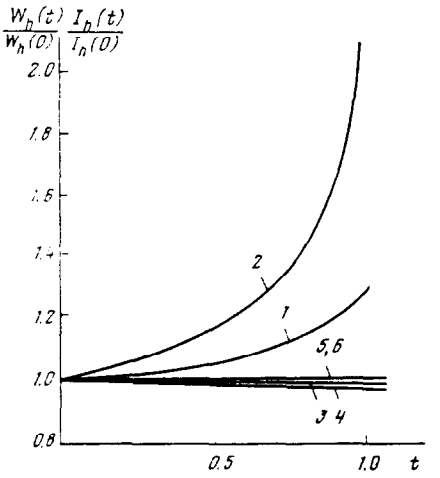


FIG. 9

Consider the application of this algorithm for computing the exact solution of (3.6). As the initial data in the length $L=4\pi$ $(N=512)$ we specify functions (3.6) with $E_m=35$, $q=0$, $x_0=2\pi$. We choose the time step $\tau = 10^{-3}$, which amounts to $\approx 0.02\ T$, where $T=8\pi / E_m^2$ is the period of the time oscillations of the soliton electric field. In Fig. 9 we show curves of $W_h(t) / W_h(0)$ and $I_h(t) / I_h(0)$ against time for different numbers of iterations: curves 1, 2 are for $p = 6$ without corrector; 3, 4 are for $p = 8$ without corrector; 5, 6 are for $p = 4$ with corrector. It can be seen that, by using the corrector (3.15), we can considerably reduce the number of iterations in (3.14).

The difference scheme (3.14), (3.15) extends in a natural way to the initial system of equations (3.1)–(3.4); we then have the difference analogue (3.4). It also extends in a natural way to the case of several space dimensions.

4. *Some computational results.* Using the results of a numerical integration of (3.1)–(3.3), we can trace the main stages in the development of Langmuir turbulence.

The initial stage of the modulation of the Langmuir pumping field is described by the dispersion equation, connecting the frequency ω and the wave number k of the modulating wave. In the dimensionless units quoted above, the equation is

$$\omega^2-k^2=2|E_0|^2k^2\frac{k^2}{(2\omega+k^2)(2\omega-k^2)}. \tag{3.16}$$

It follows from (3.16) that the domain of modulation instability is bounded by the condition $k^2<2|E_0|^2$. In the model examined below of plasma turbulence, $L=4\pi$, $k=n/2$, n is the number of a harmonic. In the domain of instability, e.g. for $E_0=2$, there appear then the harmonics $n=1,\ldots,5$ (the increment $\gamma_4=3.46$, corresponding to $n=4$, has maximum value).

As initial data we shall use the following noise distributions of the electric field and density variation:

$$E(x,0)=\varepsilon\sum_{-N/2+1}^{N/2}(A_n^{(1)}+iB_n^{(1)})\exp\left[i\left(\frac{2\pi n}{L}x+\alpha_n^{(1)}\right)\right],$$
$$\delta n(x,0)=\nu\sum_{-N/2+1}^{N/2}(A_n^{(2)}+iB_n^{(2)})\exp\left[i\left(\frac{2\pi n}{L}x+\alpha_n^{(2)}\right)\right]. \tag{3.17}$$

Here, $A_n^{(2)}=A_{-n}^{(2)}$, $B_n^{(2)}=-B_{-n}^{(2)}$, $\alpha_n^{(2)}=-\alpha_n^{(2)}$; $A_n^{(1)}$, $A_n^{(2)}$, $B_n^{(1)}$, $B_n^{(2)}$ are the random amplitudes, uniformly distributed in $[0, 1]$, $\alpha_n^{(1)}$, $\alpha_n^{(2)}$ are the random phases, uniformly distributed in $[0, 2\pi]$, and ϵ, ν are fairly small coefficients. At the initial instant the ion velocity is zero, $w=0$.

Let $N=40$, $\varepsilon=\nu=0.1$, $E_0=2.0$ in (3.17), which corresponds to growth in the linear stage with maximum increment of the fourth harmonic of the solution. In Fig. 10 we show the space distributions of δ_n and $|E|^2$. Up to $t\approx3.5$, we have an exponential rise of the 4th harmonic amplitude. At the instant $t\approx10/\gamma\approx4.0$ a mesh of four solitons is formed from it. The energy absorbed from the pump is localized in the caverns. At other values of the pump field, the number of such solitons is likewise equal to the number of the most unstable harmonic. Then the mesh is disrupted. The presence of sound disturbances from the absorbed solitons and the pump action lead rapidly to a turbulent picture (see Fig. 10, $t=11.90$). Here the Landau damping decrement Γ_n is independent of time and corresponds to a Maxwellian velocity distribution of the electrons. Important characteristics of the plasma in the turbulent state are the mean oscillatory energy level

$$W=\frac{1}{L}\int_0^L\frac{|E|^2}{8\pi}dx=\sum_n|E_k|^2,\qquad W_k=|E_k|^2$$

and the effective collision frequency ν_{eff}, characterizing the rate of absorption by the plasma of external source energy $dW/dt=\nu_{\text{eff}}|E_0|^2$.

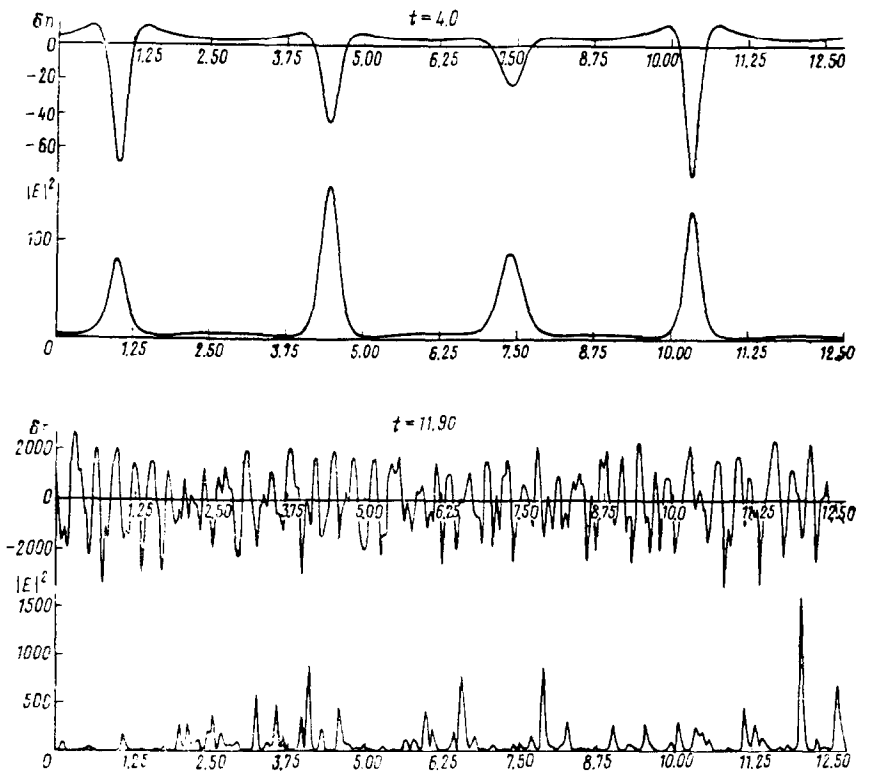


FIG. 10

In the course of time, solitons of fairly high amplitude are formed, and hence, by (3.6), they are fairly narrow, so that absorption by electrons starts. In the long run a quasi-stationary state of turbulence is established, with balance between the pump and the energy absorption:

$$\frac{dW}{dt} = \nu_{\text{eff}} \frac{|E_0|^2}{8\pi} - \sum_n \Gamma_k |E_k|^2 = 0,$$

whence

$$\nu_{\text{eff}} = |E_0|^{-2} \sum_n \Gamma_k |E_k|^2.$$

Details of the physical results may be found in [72, 73].

4. Solution of boundary value problems of MHD equilibrium of toroidal plasma

1. *Equation of equilibrium.* One of the main problems arising in the mathematical modelling of physical processes used for the long-term magnetic insulation of high-temperature plasma, is the study of possible equilibrium plasma configurations. For later study of such properties of the plasma configuration as the most dangerous magnetohydrodynamic instabilities of the plasma filament, and the transport process etc., an exact description of the equilibrium state is needed.

Equilibrium in plasma configurations is described in the approximation of single-fluid magnetohydrodynamics by the equations

$$\nabla P=(\mathbf{j}\times\mathbf{B}), \qquad \mathrm{rot}\,\mathbf{B}=\mathbf{j}, \qquad \mathrm{div}\,\mathbf{B}=0, \tag{4.1}$$

where P is the plasma pressure, j the electric current density, and B the magnetic field. In axisymmetric configurations (max current setting) we can introduce the so-called flux function $\psi = rA_\varphi$, where A is the magnetic vector potential, and (r, φ, z) is the cylindrical coordinate system. Apart from a constant, $\psi = \Phi_P/2\pi$, where Φ_P is the poloidal (transverse) magnetic flux. Put $f = rB_\varphi$; then the total magnetic field is given by

$$\mathbf{B}=(\nabla\psi\times\mathbf{e}_\varphi)/r+f\mathbf{e}_\varphi/r,$$

where $\mathbf{e}_\varphi$ is the unit vector along φ axis.

It follows from Eqs. (4.1) that $P=P(\psi)$, $f=f(\psi)$, while the function ψ satisfies the equation of equilibrium, which has the form [74]

$$r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}=-rj_\varphi(\psi,r), \tag{4.2}$$

where the longitudinal component of the electric current density is

$$j_\varphi(\psi,r)=r\frac{dP}{d\psi}+\frac{1}{r}f\frac{df}{d\psi}. \tag{4.3}$$

Equations (4.2), (4.3) hold in the domain Ω_p occupied by the plasma (Fig. 11). In the vacuum layer Ω_v, we have the homogeneous equation of equilibrium

$$r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}=0, \qquad (r,z)\in\Omega_v. \tag{4.4}$$

The system of equations (4.2)–(4.4) is completed by suitable boundary conditions. On the ideally conducting casing Γ_0, the flux function ψ is constant. To be specific, let

$$\psi(r,z)=0. \qquad (r,z)\in\Gamma_0. \tag{4.5}$$

The plasma–vacuum boundary Γ_p is defined by the level line $\psi = \psi_0 = \text{const}$, if there is no surface current on it, and ψ and its normal derivative are continuous. Hence we have the boundary value problem (4.2)–(4.5) with unknown boundary Γ_p and discontinuous right-hand side.

Two approaches may be used to solve such a problem numerically. The first is based on a solution of the initial equation of equilibrium on a fixed Euler mesh. Many publications have dealt with this type of approach (see e.g. the surveys [75, 76]). The second is based on a passage to new independent variables, one of which is the unknown solution. This approach takes the most complete account of the specific features of the problem and in particular, the unknown boundary of the plasma can be considered. The approach is developed in [77–80].

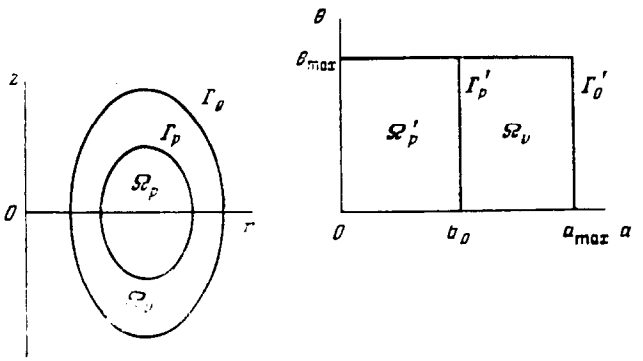


FIG. 11

Many publications [75, 76] have dealt with the solution of equilibrium problems with given dependences $P(\psi)$ and $f(\psi)$. However, problem of much greater interest are those when, instead of $f(\psi)$, we are given the so-called stability margin factor $q(\psi)$ with respect to spiral disturbances of the plasma filament:

$$q(\psi)=\frac{d\Phi_T}{d\Phi_\Pi}=-\frac{d\Phi_T}{2\pi\, d\psi};$$

here, Φ_T is the toroidal (longitudinal) magnetic flux. It is easily shown that the function $f(\psi)$ is expressible in terms of $q(\psi)$ as follows:

$$f(\psi)=4\pi^2 q(\psi)\left(\oint\limits_{\psi=\text{const}}\frac{1}{r^2}\frac{ds}{|\nabla\psi|}\right)^{-1}. \tag{4.6}$$

The integration in (4.6) is over the surface of a torus with cross-section ψ = const in the (r, z) plane.

The equilibrium problem is thus reduced to solving (4.2)–(4.6) with given functions $P(\psi)$, $q(\psi)$ and ψ on the plasma-vacuum boundary Γ_p and on the magnetic axis, i.e. with the conditions

$$\psi=\psi_0, \quad (r, z)\in\Gamma_p, \quad \psi=\psi_{max}, \quad (r, z)\in\{(r, z)\mid\nabla\psi=0\}. \tag{4.7}$$

Let us now show that, with this statement of the problem, the second of conditions (4.7) must in fact be stipulated. We average the equilibrium equation (4.2) over the volume between the two adjacent magnetic surfaces ψ and $\psi+d\psi$ = const. Denote by $V(\psi)$ the volume inside the magnetic surface defined by the condition ψ = const. In the averaging we use the equations

$$\int\limits_{\psi}^{\psi+d\psi} a\,d\tau=\oint\limits_{\psi} a\frac{ds}{|\nabla\psi|}d\psi, \qquad \int\limits_{\psi}^{\psi+d\psi}\operatorname{div}\mathbf{a}\,d\tau=\frac{d}{d\psi}\oint\limits_{\psi}(\mathbf{a}\nabla\psi)\frac{ds}{|\nabla\psi|}d\psi,$$

where $d\tau = r\,dr\,d\varphi\,dz$ is a volume element in cylindrical coordinates. Here, $V'(\psi)=-\oint ds/|\nabla\psi|$, while the average value $\langle a\rangle$ on the magnetic surface ψ = const is given by

$$\langle a\rangle=-[V'(\psi)]^{-1}\oint\limits_{\psi} a\frac{ds}{|\nabla\psi|}.$$

Noting that

$$\Delta^* = r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} = r^2 \operatorname{div}\frac{1}{r^2}\operatorname{grad},$$

we obtain after averaging Eq. (4.2) the ordinary second-order equation

$$\frac{d}{dV}\left(\left\langle\frac{|\nabla V|^2}{r^2}\right\rangle\frac{d\psi}{dV}\right) = -\frac{dP}{d\psi} - 4\pi^2 q(\psi)\frac{d}{dV}\left(4\pi^2 q(\psi)\left\langle\frac{1}{r^2}\right\rangle^{-1}\frac{d\psi}{dV}\right). \tag{4.8}$$

For the magnetic axis, where $V = 0$, we have $\langle|\nabla V|^2/r^2\rangle = 0$, on the left-hand side of (4.8), and hence $\psi = \psi_{max}$ is an interior point of the operator Δ^*. But, in view of the right-hand side of Eq. (4.8), we obtain an ordinary second-order differential equation, solvable for given ψ at both ends both when $V = V_0$ (the volume of the entire domain occupied by the plasma), and when $V = 0$, i.e. on the magnetic axis. Notice that it is the last term on the right-hand side of (4.8) that is the principle part of the corresponding differential operator. This fact has to be allowed for in the relevant numerical algorithms. Such a non-standard boundary value problem demands the development of suitable methods of solution, one of which is the method of inversion of the variables.

2. *Orthogonal flux coordinates.* Complete solution of problem (4.2)–(4.7) is only possible by using numerical methods. The most convenient for solving problems of evolution is the formulation of equilibrium in so-called natural (flux) coordinates, in which, as one of the independent coordinates, we choose the flux function ψ itself, or in the general case, a connected function $a = a(\psi)$, while as the other we take some auxiliary function θ. The unknowns in this case are $r(a, \theta)$, $z(a, \theta)$, and hence the coordinate surfaces a = const and ψ = const yield directly the geometry of of the magnetic surfaces.

The choice of the function θ is determined by the specific features of the problem. Below we consider orthogonal flux coordinates (ψ, θ) such that the coordinate surfaces a = const and θ = const form two families of orthogonal surfaces.

We write the orthogonality condition for ψ and θ as

$$\frac{\partial\psi}{\partial r}\frac{\partial\theta}{\partial r} + \frac{\partial\psi}{\partial z}\frac{\partial\theta}{\partial z} = 0. \tag{4.9}$$

Condition (4.9) is a consequence of the dependence $\mu = \mu(\psi, \theta)$, for which

$$\mu r\frac{\partial\theta}{\partial z} = \frac{\partial\psi}{\partial r}, \qquad \mu r\frac{\partial\theta}{\partial r} = -\frac{\partial\psi}{\partial z}, \tag{4.10}$$

where $\mu = \mu(\psi, \theta)$ is an arbitrary function. We choose the function μ in such a way that $\psi(r, z)$ satisfies the equilibrium equation (4.2) in the domain Ω_p, occupied by the plasma, and the homogeneous (4.4) in Ω_v. From (4.10) we obtain

$$r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = r\left(\frac{\partial\mu}{\partial r}\frac{\partial\theta}{\partial z} - \frac{\partial\mu}{\partial z}\frac{\partial\theta}{\partial r}\right). \tag{4.11}$$

Consider the right-hand side of this equation. Substituting

$$\frac{\partial \mu}{\partial r}=\frac{\partial \mu}{\partial \psi}\frac{\partial \psi}{\partial r}+\frac{\partial \mu}{\partial \theta}\frac{\partial \theta}{\partial r} \quad \text{and} \quad \frac{\partial \mu}{\partial z}=\frac{\partial \mu}{\partial \psi}\frac{\partial \psi}{\partial z}+\frac{\partial \mu}{\partial \theta}\frac{\partial \theta}{\partial z}$$

into the right-hand side of (4.11), we obtain

$$\frac{\partial \mu}{\partial r}\frac{\partial \theta}{\partial z}-\frac{\partial \mu}{\partial z}\frac{\partial \theta}{\partial r}=\frac{\partial \mu}{\partial \psi}\left(\frac{\partial \psi}{\partial r}\frac{\partial \theta}{\partial z}-\frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial r}\right). \tag{4.12}$$

In addition, Eqs. (4.10) give

$$\frac{\partial \psi}{\partial r}\frac{\partial \theta}{\partial z}-\frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial r}=\frac{|\nabla \psi|^2}{\mu^2}. \tag{4.13}$$

Using (4.12) and (4.13), Eq. (4.11) can be rewritten as

$$r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial \psi}{\partial r}+\frac{\partial^2 \psi}{\partial z^2}=\frac{\partial \mu}{\partial \psi}\frac{|\nabla \psi|^2}{\mu} \tag{4.14}$$

In the domain Ω_p, Eq. (4.14) must be identical with the equilibrium equation (4.2). Hence, for $\psi_0<\psi<\psi_{max}$, the function $\mu(\psi, \theta)$ is given by the condition

$$\frac{\partial \mu}{\partial \psi}=-\frac{\mu^2}{|\nabla \psi|^2}j_\varphi(\psi, r). \tag{4.15}$$

In the vacuum domain Ω_v, in accordance with (4.4), we obtain from (4.14):

$$\partial \mu/\partial \psi=0, \qquad 0<\psi<\psi_0. \tag{4.16}$$

The equilibrium is thus defined by the system of equations (4.10) with function $\mu(\psi, \theta)$, which satisfies (4.15), (4.16). We introduce the general flux coordinates $(a(\psi), \theta)$. On $a(\psi)$ we impose the requirement of monotonicity. We shall assume for clarity that $a(\psi_{max})=0$ on the magnetic axis, where $\psi=\psi_{max}$, $a(\psi_0)=a_0$ on the plasma boundary Γ_p, where $\psi=\psi_0$, and $a(0)=a_{max}$ on the casing, where $\psi=0$. Let $\nu(a)=-(da/d\psi)^{-1}$, $\mu_a(a, \theta)=-\mu\nu^{-1}(a)$, and let the surface functions (pressure P, function f, stability margin factor q) be functions of a, i.e. $P=P(a)$, $f=f(a)$, $q=q(a)$. Then, the system of equations (4.10) can be rewritten as

$$\mu_a r\frac{\partial \theta}{\partial z}=\frac{\partial a}{\partial r}, \qquad \mu_a r\frac{\partial \theta}{\partial r}=-\frac{\partial a}{\partial z}; \tag{4.17}$$

in the plasma, (4.15) transforms into the integro-differential equation

$$\frac{\partial}{\partial a}(\nu^2\mu_a^2)+\frac{d}{da}\left[\int_0^{\theta_{max}}\frac{d\theta}{Ir}\right]^{-2}=-2r^2\frac{dP}{da}, \qquad 0<a\leqslant a_0, \tag{4.18}$$

and in vacuum, to

$$\frac{\partial}{\partial a}(\nu^2\mu_a^2)=0, \qquad a_0<a<a_{\max}. \tag{4.19}$$

Next, we transform in Eqs. (4.17)–(4.19) from the variables (r, z) to (a, θ). We then take as unknown functions $r=r(a, \theta)$, $z=z(a, \theta)$. After the transformation, we arrive at the following system of equations for $r(a, \theta)$, $z(a, \theta)$:

$$\mu_a r\frac{\partial r}{\partial a}=\frac{\partial z}{\partial \theta}, \qquad \mu_a r\frac{\partial z}{\partial a}=-\frac{\partial r}{\partial \theta}. \tag{4.20}$$

The problem in (a, θ) variables is thus reduced to the simultaneous solution of Eqs. (4.18)–(4.20). The domain $\Omega=\Omega_p+\Omega_v$ in the (r, z) plane maps into the rectangular (see Fig. 11) $\Omega'=\{[0, a_{\max}]\times[0, \theta_{\max}]\}$ in the (a, θ) plane.

For numerical solution of the problem, it is preferable to transform from system (4.14) to the two elliptic equations

$$\begin{aligned}\frac{\partial}{\partial a}\left(\mu_a r\frac{\partial r}{\partial a}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{\mu_a r}\frac{\partial r}{\partial \theta}\right)=0,\\ \frac{\partial}{\partial a}\left(\mu_a r\frac{\partial z}{\partial a}\right)+\frac{\partial}{\partial \theta}\left(\frac{1}{\mu_a r}\frac{\partial z}{\partial \theta}\right)=0.\end{aligned} \tag{4.21}$$

Consider the boundary conditions for (4.21). With respect to the variable θ, we pose natural conditions of periodicity, period $\theta_{\max}$:

$$r(a, \theta+\theta_{\max})=r(a, \theta), \qquad z(a, \theta+\theta_{\max})=z(a, \theta). \tag{4.22}$$

In the general case, the position of the magnetic axis is unknown. However, we do not need to specify boundary conditions here, since, by (4.20),

$$\mu_a r\frac{\partial r}{\partial a}=0, \qquad \mu_a r\frac{\partial z}{\partial a}=0 \tag{4.23}$$

for $a = 0$, and hence, on the magnetic axis, (4.21) are degenerate elliptic equations.

Given the function μr, system (4.21) is equivalent to system (4.20), provided that we suitably pose the boundary conditions on Γ_0 (with $a = a_{\max}$). Since a and θ are orthogonal, the functions θ and μ cannot be arbitrary on any magnetic surface a = const, including on the outer surface $a = a_{\max}$. With a = const we can arbitrarily specify just one of functions θ or μ, the other then being uniquely determined by the condition that a and θ be orthogonal. For instance, it follows from (4.20) that, given θ, we can define the function $\mu_a(a, \theta)$ on a magnetic surface a = const from the condition

$$\mu^2 r^2\left[\left(\frac{\partial r}{\partial a}\right)^2+\left(\frac{\partial z}{\partial a}\right)^2\right]=\left(\frac{\partial r}{\partial \theta}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2. \tag{4.24}$$

This expression gives the initial condition for (4.18), (4.19). It can be substituted into the last magnetic surface $a = a_{\max}$ or in some interior magnetic surface $a = a^*$. In the latter case, (4.18), (4.19) have to be solved separately in $(0, a^*)$ and $(a^*, a_{\max})$.

After the function θ has been specified on the outer boundary (casing), on using the given coordinates of the casing Γ_0 for (4.21), we can pose boundary conditions of the 1st kind with $a = a_{\max}$:

$$r(a_{\max}, \theta) = r_0(\theta), \qquad z(a_{\max}, \theta) = z_0(\theta). \tag{4.25}$$

Hence the solution of the initial problem (4.2)–(4.7) of the equilibrium of an ideal plasma with given pressure and stability margin factor, reduces to the solution of the non-linear boundary value problem (4.21)–(4.23), (4.25) with coefficient μ, which satisfies (4.18), (4.19), (4.24). This statement of the equilibrium problem was taken as basis for the method of numerical solution. We shall not dwell on the details of the numerical modelling.

3. *Examples of computations.* The method developed above for solving problems of MHD equilibrium, based on the method of inversion of the variables, is specially convenient for solving problems of the evolution of equilibrium configurations and in the study of their stability. Consider as an example the solution of the problem of the evolution of an ideally conducting plasma as the pressure rises. For specific realistic set-ups, dimensionless parameters such as the stability margin factor and the quantity β are of definitive importance:

$$\beta = 2\langle P\rangle / \langle B_\varphi^2\rangle,$$

where

$$\langle P\rangle = \left[(S_{\Omega_p})^{-1} \int_{\Omega_p} P^2\, dr\, dz \right]^{1/2},$$

$$\langle B_\varphi^2\rangle = (S_{\Omega_p})^{-1} \int_{\Omega_p} B_\varphi^2\, dr\, dz, \qquad S_{\Omega_p} = \int_{\Omega_p} dr\, dz.$$

The condition $q > 1$ ensures stability of the equilibrium configuration with respect to particularly dangerous spiral disturbances. Set-ups with high values of β are energy-wise more favourable from the point of view of the thermonuclear reactor. The equilibrium configurations with sufficiently high values $\beta \approx 0.1$ can be obtained e.g., by increasing the pressure while keeping the magnetic fluxes constant.

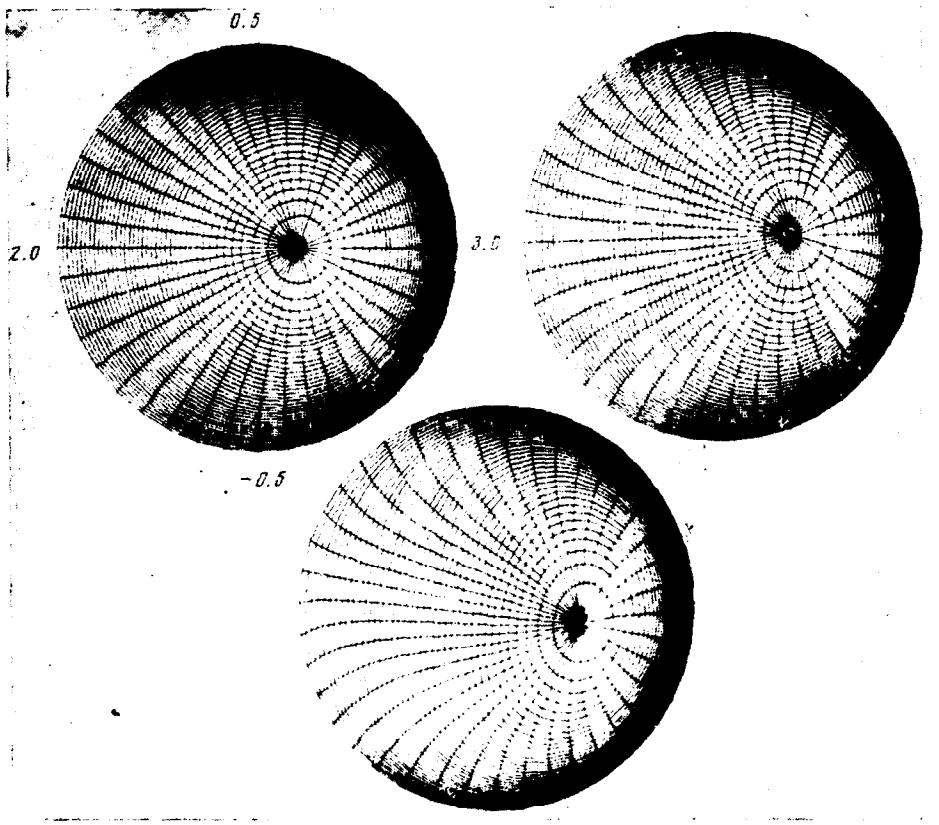


FIG. 12

In the examples below, as the flux variable we choose the function $a(\psi)=\Phi_1$, $a(\psi)\in[0, 1]$, $-da/d\psi=-2\pi q(\psi)=\nu^{-1}(a), \Phi_{1\,\max}=1$. The stability margin factor $q(a)$ is taken to be the linear function $q(a)=q_0+q_1a$, so that $q(0)=q_0$, $q(1)=q_0+q_1$, while the pressure $P(a)=\alpha(1-a), dP/da=-\alpha$.

In Fig. 12 we illustrate the results of computations with different α (pressure) for $q_0=1.5$, $q_1=2.5$, when the plasma touches the casing of circular section. It can be seen here that, due to the balloon effect (widening of the plasma filament as the pressure rises), the magnetic axis is displaced towards high values of r, while the magnetic increase the ellipticity of the plasma cross-section.

The method here described can be extended to non-orthogonal coordinates [79], and applied to problems with free boundary for elliptic equations and systems [78].

5. Completely neutral difference schemes for Navier–Stokes equations

In the numerical solution of problems of gas dynamics and hydrodynamics, it is vital to construct the difference schemes in such a way as to satisfy in them analogues of the conservation laws which are imposed in the initial differential equations. Schemes of this kind have been called conservative [2]. Further development of the principle of conservativeness has led to the concept of complete conservativeness of difference schemes, in which are reflected the auxiliary relations expressing the balance of different kinds of energy [2, 5].

In spite of the fact that numerical methods of hydrodynamics of incompressible viscous fluid have found wide application for different classes of applied problems, the theory of the difference schemes, and in particular, aspects of their stability, have not been adequately studied. For instance, when computing fluid flows in long time intervals, some of the usual difference schemes exhibit instability, and computation becomes impossible in practice. This situation, often referred to in the literature as "non-linear instability," was pointed out by Arakawa in [81]. Similar stability problems were encountered by Rozhdestvenskii *et al.* in [71] when numerically modelling turbulence. In this case, according to linear theory, an exponentially increasing mode is present in the solution, and in schemes which do not have the properties to be described below, computational instability develops extremely rapidly.

In [81], when considering the transport equations, some types of difference approximation (on a square mesh) of certain transport terms were proposed; these terms are the Jacobians $D=\partial(\omega, \psi)/\partial(x, y)$, which do not give a contribution to the energy balance $(E=\int\int [(U^2+V^2)/2]dxdy)$ or enstropy balance $(\int\int(\omega^2/2)dxdy)$, but reflect the properties of the initial equations at zero velocities on the domain boundary. We shall call such schemes energy-wise neutral and enstropy-wise neutral. Energy-wise neutral schemes for the equations of hydrodynamics, based on the use of Galerkin's method, were previously described in [71]. This latter fact justifies our introducing the special term "neutral" for characterizing the property in question, since Galerkin's method is implicitly conservative by definition. Moreover, the term "neutral" suggests the approximation of just some precisely defined terms of the equation.

We shall propose [83] some families of energy-wise and enstropy-wise neutral schemes for the systems of equations of viscous incompressible fluid hydrodynamics on meshes that are non-uniform with respect to x and y, and we shall construct a completely neutral scheme. Our discussion of aspects of the approximation of the Navier–Stokes equations will be carried out in a rectangular domain. On the rectangle boundary we specify velocities U and V in the directions of the coordinate axes O_x and O_y.

Study of approximation of the boundary conditions in the variables ψ, ω has shown that the well-known approximation of vorticity at boundary nodes in accordance with Thom's formula (see e.g., [82]) enables us to construct conservative implicit schemes. The mesh operators then retain the important properties of the differential operators of the initial problem, and the schemes are homogeneous. For these schemes we obtain *a priori* energy estimates, from which it follows that the schemes are stable (in the sense of stability of the zero solution). We also give a scheme which is linear in the required functions on the upper time layer, and which does not violate the energy estimates of stability.

1. *Statement of the initial problem.* In the rectangle $G=\{0<x<a,\ 0<y<b\}$, with boundary Γ, we consider the system of Navier–Stokes equations for an incompressible fluid with density $\rho = 1$ for $0 < t \leqslant T$, written in variables ψ, ω, Q, where $Q=P+(U^2+V^2)/2$ is the head, and P is the pressure.

We write the system of equations as

$$\omega_t=W_{2x}-W_{1y}, \quad W_{1x}+W_{2y}=0, \quad \Delta\psi=-\omega. \tag{5.1}$$

Here, the "fluxes" W_1 and W_2 have the form

$$W_1=-(\nu\omega_y-D_2+Q_x), \qquad D_2=V\omega, \qquad V=-\psi_x,$$
$$W_2=\nu\omega_x-D_1-Q_y, \qquad D_1=U\omega, \qquad U=\psi_y.$$

The boundary and initial conditions for system (5.1) are

$$\begin{aligned} &W_1=U_{0t}, \qquad W_2=V_{0t}, \qquad \psi=\psi_0, \qquad (x,\ y)\in\Gamma, \quad 0<t\leqslant T, \\ &U=U^0, \qquad V=V^0, \qquad (x,\ y)\in G, \quad t=0; \end{aligned} \tag{5.2}$$

U^0, V^0 are the given boundary velocities, which are obviously connected with ψ_0.

2. *Meshes and mesh functions.* In the intervals $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$ we introduce the points $x_0=0, x_1, \ldots, x_M=a$, $y_0=0, y_1, \ldots, y_N=b$. Denote the distances between adjacent points by $h^{(1)}, h^{(2)}$. We also introduce the step mid-points $\hbar^{(1)}, \hbar^{(2)}$ (see [2]). In the rectangle we introduce the mesh $\bar{\Omega}_h=\Omega_h\cup\Gamma_h$ of nodes $X=(x_i, y_k)$, $i=0, 1, \ldots, M$, $k=0, 1, \ldots, N$. The mesh functions and corresponding functions of the initial problem will be denoted by the same letters. At nodes of the mesh $\bar{\Omega}_h$ we define the quantities ψ_{ik} and ω_{ik}. The mesh functions $Q=Q_{i+1/2,\ k+1/2}$ will be referred to nodes $X_0=(x_{i+1/2}, y_{k+1/2})\in\Omega_{h_0}$, and also to boundary nodes $(x_{i+1/2}, y_k)$, $k=0, N$, $(x_i, y_{k+1/2})$, $i=0, M$, the set of which we denote by Γ_{h_0}.

We introduce the mesh analogues of the derivatives with respect to x and y. For the mesh functions $z_{is}=z(x_i, y_s)$, $s=k,\ k+1/2$, and $z_{i+1/2,\ s}=z(x_{i+1/2}, y_s)$, $s=k,\ k+1/2$, we put respectively

$$z_{h,\,i+1/2,\,s}=(z_{i+1,s}-z_{is})/h^{(1)}_{i+1/2}, \qquad s=k,\ k+1/2,$$

$$z_{\bar{x}is}=\begin{cases}(z_{1/2,\,s}-z_{0s})/\hbar^{(1)}_0, & i=0,\\ (z_{i+1/2,\,s}-z_{i-1/2,\,s})/\hbar^{(1)}_i, & 0<i<M,\\ (z_{Ms}-z_{M-1/2,\,s})/\hbar^{(1)}_M, & i=M, \quad s=k,\ k+1/2.\end{cases}$$

The difference ratios $z_{\bar{y}}$ and $z_{\hat{y}}$ are similarly defined.

We introduce the mesh $\Omega_\tau=\{t_j=j\tau,\ j=0, 1, \ldots, j_0=T/\tau\}$ with step τ. We shall refer the function Q to the time layer $t_{j+1/2}=0.5(t_{j+1}+t_j)$. We put

$$\psi^{j+1/2}=(\psi^{j+1}+\psi^j)/2, \qquad \omega^{j+1/2}=(\omega^{j+1}+\omega^j)/2 \tag{5.3}$$

etc.

3. *Difference scheme.* Using the integro-interpolation method for approximating the initial problem (5.1), (5.2), we arrive at the following difference system of equations:

$$\omega_t^{j+1/2} = W_{2x}^{j+1/2} - W_{1y}^{j+1/2}, \qquad X \in \overline{\Omega}_h, \tag{5.4}$$

$$W_{1x}^{j+1/2} + W_{2y}^{j+1/2} = 0, \qquad X_0 \in \Omega_{h_0}, \tag{5.5}$$

$$\Delta_h \psi^{j+1} = -\omega^{j+1}, \qquad X \in \Omega_h, \tag{5.6}$$

with the boundary conditions at the nodes on Γ:

$$W_1^{j+1/2} = \overline{U}_{0i}^{j+1} \qquad W_2 = \overline{V}_{0i}^{j+1}, \qquad \psi^{j+1} = \overline{\psi}_0^{j+1}, \quad j = 0, 1, \ldots, j_0, \tag{5.7}$$

and the initial conditions

$$\psi^0 = \overline{\psi}^0, \quad \omega^0 = \overline{\omega}^0, \quad X \in \overline{\Omega}_h. \tag{5.8}$$

The quantities $\overline{U}_0$, $\overline{V}_0$ are the means of U_0, V_0 over the appropriate intervals $h^{(2)}$, $h^{(1)}$ or $\hbar^{(2)}$, $\hbar^{(1)}$; $\overline{\psi}_0$ is expressible in a natural way in terms of them.

The fluxes W_1, W_2 in (5.4), (5.5) will be given by the relations

$$\begin{aligned} W_{1,\, i,\, k+1/2} &= -(\nu\omega_y - D_2 + Q_{\hat{x}})_{i,\, k,\, +1/2}, \\ W_{2,\, i+1/2,\, k} &= (\nu\omega_x - D_1 - Q_{\hat{y}})_{i+1/2,\, k}. \end{aligned} \tag{5.9}$$

We specify W_1, W_2 at points of Γ as follows:

$$W_1 = \overline{U}_{0t}, \quad W_2 = \overline{V}_{0t}, \qquad t = t_{j+1/2}.$$

We introduce the notation $\alpha_i^{(1)} = h_{i+1/2}^{(1)}/(2\hbar_i^{(1)})$, $\alpha_k^{(2)} = h_{k+1/2}^{(2)}/(2\hbar_k^{(2)})$; then, $h_{i-1/2}^{(1)}/(2\hbar_i^{(1)}) = 1-\alpha_i^{(1)}$, $h_{k-1/2}^{(2)}/(2\hbar_k^{(2)}) = 1-\alpha_k^{(2)}$.

We shall give three ways of approximating the quantities $D_1 = U\omega$ and $D_2 = V\omega$, leading to schemes on a nine-point pattern for functions ω, ψ (we shall omit the index $j + 1/2$ of functions ω, ψ, U, V).

Method I:

$$D_{1,\, i+1/2,\, k}^{(1)} = (\overline{U}_{i+1,k}\omega_{i+1,k} + \overline{U}_{ik}\omega_{ik})/2, \qquad D_{2,\, i,\, k+1/2}^{(1)} = (\overline{V}_{i,k+1}\omega_{i,k+1} + \overline{V}_{ik}\omega_{ik})/2,$$

$$\overline{U}_{ik} = \alpha_k^{(2)} U_{i,k+1/2} + (1-\alpha_k^{(2)}) U_{i,k-1/2}, \qquad 0 < k < N,$$
$$\overline{U}_{i0} = U_{i,\,1/2}, \quad \overline{U}_{iN} = U_{i,\,N-1/2},$$

$$\overline{V}_{ik} = \alpha_i^{(1)} V_{i+1/2,k} + (1-\alpha_i^{(1)}) V_{i-1/2,k}, \qquad 0 < i < M, \quad \overline{V}_{0k} = V_{1/2,k},$$
$$\overline{V}_{Mk} = V_{M-1/2,k}.$$

Method II:

$$D^{(2)}_{1,i+1/2,0}=\bar{U}_{i+1/2,1/2}\bar{\omega}_{i+1/2,1/2}, \qquad D^{(2)}_{1,i+1/2,N}=\bar{U}_{i+1/2,N-1/2}\bar{\omega}_{i+1/2,N-1/2},$$

$$D^{(2)}_{1,i+1/2,k}=\alpha^{(2)}_k\bar{U}_{i+1/2,k+1/2}\omega_{i+1/2,k+1/2}+(1-\alpha^{(2)}_k)U_{i+1/2,k-1/2}\omega_{i+1/2,k-1/2},$$

$0<k<N$;

$$D^{(2)}_{2,0,k+1/2}=\bar{V}_{1/2,k+1/2}\bar{\omega}_{1/2,k+1/2}, \qquad D^{(2)}_{2,M,k+1/2}=\bar{V}_{M-1/2,k+1/2}\bar{\omega}_{M-1/2,k+1/2},$$

$$D^{(2)}_{2,i,k+1/2}=\alpha^{(1)}_i\bar{V}_{i+1/2,k+1/2}\bar{\omega}_{i+1/2,k+1/2}+(1-\alpha^{(1)}_i)\bar{V}_{i-1/2,k+1/2}\bar{\omega}_{i-1/2,k+1/2},$$

$0<i<M$;

$$\bar{U}_{i+1/2,k-1/2}=(U_{i+1,k-1/2}+U_{i,k-1/2})/2.$$

$$\bar{V}_{i-1/2,k+1/2}=(V_{i-1/2,k+1}+V_{i+1/2,k})/2.$$

$$\bar{\omega}_{i+1/2,k+1/2}=(\omega_{i+1,k+1}+\omega_{i+1,k}+\omega_{i,k+1}+\omega_{ik})/4.$$

Method III:

$$D^{(3)}_{1,i+1/2,k}=\bar{U}_{i+1/2,k}(\omega_{i+1,k}+\omega_{ik})/2,$$

$$D^{(3)}_{2,i,k+1/2}=\bar{V}_{i,k+1/2}(\omega_{i,k+1}+\omega_{ik})/2,$$

$$\bar{U}_{i+1/2,0}=U_{i+1/2,0}, \qquad U_{i+1/2,N}=\bar{U}_{i+1/2,N-1/2},$$

$$\bar{U}_{i+1/2,k}=\alpha^{(2)}_k\bar{U}_{i+1/2,k+1/2}+(1-\alpha^{(2)}_k)\bar{U}_{i+1/2,k-1/2}, \qquad 0<k<N;$$

$$\bar{V}_{0,k+1/2}=\bar{V}_{1,k+1/2}, \qquad \bar{V}_{M,k+1/2}=\bar{V}_{M-1,k+1/2},$$

$$\bar{V}_{i,k+1/2}=\alpha^{(1)}_i\bar{V}_{i+1/2,k+1/2}+(1-\alpha^{(1)}_i)\bar{V}_{i-1/2,k+1/2}, \quad 0<i<M.$$

The quantities $\bar{U}$, $\bar{V}$ are expressible in terms of U, V, while the latter are expressible with the aid of the equations $U_{i,k+1/2}=\psi_{y,i,k+1/2}$, $V_{i+1/2,k}=-\psi_{x,i+1/2,k}$ in terms of ψ_{ik}. Hence $D_s=D^{(n)}_s(\psi^{j+1/2}, \omega^{j+1/2})$, $s=1, 2$, $n=1, 2, 3$.

Equation (5.4), written at points Γ_h, contains the boundary values of the variable Q. At interior nodes, the variable Q in (5.4) falls out. It can be shown [83] that (5.4)–(5.8) is algebraically equivalent to the well-known problem in ψ, ω variables, and to the problem for Q with the appropriate conditions. For ψ, ω, the problem can be solved independently of the variable Q; then, instead of Eq. (5.4), we have to use at the boundary nodes the well-known Thom condition, while noting that it connects the boundary conditions $\bar{\omega}^0$ and $\bar{\psi}^0$ on Γ_h: $\bar{\omega}^0=-\Delta_h\bar{\psi}^0$, $x\in\Omega_h$.

We thus arrive at the following scheme in ψ, ω variables:

$$\begin{aligned}&\omega_i^{j+1}+D(\omega^{j+1/2}\psi^{j+1/2})=\nu\Delta_h\omega^{j+1/2}, \qquad X\in\Omega_h,\\ &\Delta_h\psi^{j+1}=-\omega^{j+1}, \qquad X\in\bar{\Omega}_h, \quad j=0, 1, \ldots, j_0.\end{aligned} \tag{5.10}$$

Here, Δ_h is the natural approximation of the Laplace operator at interior nodes, and the Thom approximation at boundary nodes:

$$D(\omega, \psi) = D_{1\hat{x}_1} + D_{2\hat{x}_2}. \tag{5.11}$$

4. *Family of neutral schemes. Completely neutral scheme.* Let us study the properties of schemes (5.10) (or (5.4)–(5.8)) for different approximations D_1 and D_2 in accordance with the expressions of methods I–III in the case of homogeneous boundary conditions

$$\overline{U}_0^j = \overline{V}_0^j = 0, \quad \overline{\psi}_0^j = \text{const}, \quad (x, y) \in \Gamma. \tag{5.12}$$

Since $W_{2ik}^{j+1/2} = 0$, $i = 0, M$, $W_{1ik}^{j+1/2} = 0$, $k = 0, N$, we have to put in (5.11): $D_{1ik}^{j+1/2} = 0$, $i = 0, M$, $D_{2ik}^{j+1/2} = 0$, $k = 0, N$. At nodes $(x_{i+1/2}, y_k)$, $k = 0, N$, $(x_i, y_{k+1/2})$, $i = 0, M$, we shall define $W_1^{j+1/2}$, $W_2^{j+1/2}$ from (5.9). In this connection, we shall also compute $D_{1,i+1/2,k}^{j+1/2}$, $k = 0, N$, $D_{2,i,k+1/2}^{j+1/2}$, $i = 0, M$, in accordance with the relevant expressions (under conditions (5.12)).

For the initial problem (5.1), (5.2) under homogeneous boundary conditions $u_0 = v_0 = 0$, we have the identity

$$\frac{\partial E}{\partial t} + \nu \int_G \omega^2 dx dy = 0, \qquad \int_G D\psi dx dy = 0, \tag{5.13}$$

where $D = \partial D_1/\partial x + \partial D_2/\partial y$, $D_1 = U\omega$, $D_2 = V\omega$.

In the case of homogeneous boundary conditions $u_0 = v_0 = 0$,

$$\int_G D\omega dx dy = 0. \tag{5.14}$$

Equations (5.13), (5.14) mean that the transport terms yield no contribution to the energy and enstropy balance.

We shall say that the difference scheme is energy-wise or enstropy-wise neutral if the mesh analogues of the transport terms $D^{j+1/2} = (D_1^{j+1/2})_{\hat{x}} + (D_2^{j+1/2})_{\hat{y}}$ do not yield a contribution to the energy balance $(D^{j+1/2}, \psi^{j+1/2}) = 0$ or the enstropy balance $(D^{j+1/2}, \omega^{j+1/2}) = 0$. If both these equations hold, we call the difference scheme completely neutral. Here we introduce the scalar product

$$(\omega, \Psi) = \sum_{\tilde{\Omega}_h} \omega^{(1)} \Psi^{(1)} h^{(2)}, \qquad \| \omega \| = (\omega, \omega)^{1/2}.$$

To discover the family of energy-wise neutral schemes, we have to compute under condition (5.12) the scalar product $(D^{(n),\,j+1/2},\,\psi^{j+1/2})$, $n=1,\,2,\,3$. We find as a result that $(D^{(n),\,j+1/2},\,\psi^{j+1/2})=0$, $n=1,\,2$, i.e., schemes with $D^{j+1/2}=D^{(1),\,j+1/2}$, $D^{j+1/2}=D^{(2),\,j+1/2}$ (expressions of methods I and II) are energy-wise neutral. For schemes with $D^{j+1/2}=D^{(3),\,j+1/2}$, we have the enstropy-wise neutral case $(D^{(3),\,j+1/2},\,\omega^{j+1/2})=0$. For energy-wise neutral schemes we have the analogue of (5.13):

$$E_1^{j+1}+\nu_1\|\omega^{j+1}\|^2=0,$$
$$E=\frac{1}{2}\left(\sum_{\Omega_{h_1}}\psi_x^2 h_1 h_2+\sum_{\Omega_{h_2}}\psi_y^2 h_2 h_1\right), \tag{5.15}$$
$$(x_{i-1},\,x_i)\in\Omega_{h_1},\qquad (x_i,\,y_{k-1/2})\in\Omega_{h_2}.$$

Consider the one-parameter family of schemes (5.10)–(5.12):

$$D_{1,i+1/2,k}^{j+1/2}=\beta D_{1,i+1/2,k}^{(1),j+1/2}+(1-\beta)D_{1,i+1/2,k}^{(2),j+1/2},$$
$$D_{2,i,k+1/2}^{j+1/2}=\beta D_{2,i,k+1/2}^{(1),j+1/2}+(1-\beta)D_{2,i,k+1/2}^{(2),j+1/2}. \tag{5.16}$$

Obviously, this family of schemes is energy-wise neutral. We compute the scalar products of $\omega_{ik}^{j+1/2}$ and $D_{ik}^{j+1/2}$ for $D_{ik}^{j+1/2}=D_{ik}^{(1),j+1/2}$ and $D_{ik}^{j+1/2}=D_{ik}^{(2),j+1/2}$:

$$(D^{(1),\,j+1/2},\,\omega^{j+1/2})=F^{j+1},\qquad (D^{(2),\,j+1/2},\,\omega^{j+1/2})=-0.5F^{j+1}.$$

On comparing the last two equations, we see that the energy-wise neutral scheme (5.10)–(5.12) of family (5.16) with $\beta=1/3$ is likewise enstropy-wise neutral. Thus, the scheme (5.10)–(5.12) with

$$D_{1,i+1/2,k}^{j+1/2}=\frac{1}{3}D_{1,i+1/2,k}^{(1),j+1/2}+\frac{2}{3}D_{1,i+1/2,k}^{(2),j+1/2}\equiv\bar{D}_{1,i+1/2,k}^{j+1/2},$$
$$D_{2,i,k+1/2}^{j+1/2}=\frac{1}{3}D_{2,i,k+1/2}^{(1)}+\frac{2}{3}D_{2,i,k+1/2}^{(2),j+1/2}\equiv\bar{D}_{2,i,k+1/2}^{j+1/2}$$

is completely neutral. The one-parameter family of enstropy-wise neutral schemes can be constructed on the basis of $D_1^{(3)}$, $D_2^{(3)}$ and $\bar{D}_1$, $\bar{D}_2$.

Similar approximations were previously obtained in [81] on a square mesh.

Notes. 1. Instead of (5.3) we can introduce

$$\omega_{ik}^{j+1/2}=\sigma\omega_{ik}^{j+1}+(1-\sigma)\omega_{ik}^{j},\qquad \psi_{ik}^{j+1/2}=\sigma\psi_{ik}^{j+1}+(1-\sigma)\psi_{ik}^{j}. \tag{5.17}$$

In this case, to the left-hand side of (5.15) we have to add the term $\tau(\sigma-0.5)\div(\|\psi_{x\bar{i}}^{j+1}\|_1^2+\|\psi_{y\bar{i}}^{j+1}\|_2^2)$, $E^{j+1}\to 0$ as $t_{j+1}\to\infty$, if $\sigma\geqslant 0.5$. The previous assertions remain true.

2. Schemes (5.4)–(5.8) with $D=D^{(1)}$ and $D=D^{(2)}$ remain energy-wise neutral if $D^{(1)}$ and $D^{(2)}$ are evaluated with respect to $\psi_{ik}{}^{j+1/2}$ of (5.17) (with $\sigma \geqslant 0.5$), and with respect to $\omega_{ik}{}^{j}$. The schemes with $D=\bar{D}=(\bar{D}_1)_{\bar{x}}+(\bar{D}_2)_{\bar{y}}$, $D=D^{(3)}$ are also enstropy-wise neutral, if $\bar{D}$ and $D^{(3)}$ are evaluated with respect to $\omega_{ik}{}^{j+1/2}$ from (5.17) (with $\sigma \geqslant 0.5$) and with respect to $\psi_{ik}{}^{j}$. In these cases, a linear system of equations is obtained for ω^{j+1} and ψ^{j+1}.

Translated by D. E. Brown.

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