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AN APPROACH TO THE COMPARISON OF SOLUTIONS OF PARABOLIC EQUATIONS*

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A METHOD for comparing the solutions of second-order parabolic equations is based on pointwise estimates of the highest derivative of the solution in terms of the lower derivatives.

1. Introduction

For the quasi-linear degenerate parabolic equation

$$u_t=\mathscr{L}(u)=[k(u)u_x]_x \tag{1.1}$$

we consider in $Q_T=\{(t, x):0<t\leqslant T,\ x\in\Omega\}$, $\Omega=\{x:0<x<\infty\}$, the first boundary value problem with the conditions

$$u(0,\ x)=u_0(x)\geqslant 0,\quad x\in\Omega,\qquad u(t,\ 0)=u_1(t)\geqslant 0,\quad 0\leqslant t\leqslant T, \tag{1.2}$$

where $u_0(x)$ and $u_1(t)$ are continuous functions of their arguments, $0\leqslant u_i\leqslant M<\infty$, $i=0,\ 1$, $u_0(0)=u_1(0)$. The function $k(u)$ is defined for $u\geqslant 0$, $k(u)>0$ for $u>0$, $k(0)=0$.

In particular, Eq. (1.1) describes the process of heat propagation in a medium with thermal conductivity $k(u)$, dependent on the temperature u of the medium.

In [1–5] the concept of metastable localization of heat was introduced, and for the case $k(u)=u^\sigma$, $\sigma>0$, conditions were defined for the existence or absence of localization in boundary value problems and in the Cauchy problem for Eq. (1.1). In the analysis, use was made of the similarity solutions obtained in [6, 7], and the theorems given in [8] on comparison with respect to the boundary conditions.

If $k(u)$ is not a power function, the class of group-invariant solutions of Eq. (1.1) (see [9]) contains none that has the heat localization property. Hence it becomes necessary to find a suitable means for comparing the solutions of such equations.

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We show in the present paper that the solutions of problems (1.1), (1.2) with different $k(u)$ can be compared in Q_T provided that certain comparison conditions are imposed on the coefficients. The boundary conditions (1.2) then have to satisfy certain requirements, ensuring that a pointwise estimate is satisfied in Q_T for the highest derivative of a solution in terms of the lower derivatives. Our solution comparison method extends the heat localization effect to a wider class of coefficients.

In Section 2 we give the existence conditions for special point-wise estimates of the leading derivative (conditions for criticality of boundary data), while a comparison theorem is proved in Section 3, some generalizations of the comparison method are discussed in Section 4, and finally, in Section 5 we use the comparison theorem to isolate the classes of coefficients $k(u)$ which admit the existence or absence of heat localization, depending on the form of the boundary conditions.

Our approach enables the solutions corresponding to different parabolic operators $\mathcal{L}(u)$ to be compared. As one such operator we can take e.g., an operator of simple type such that the corresponding solutions have familiar properties.

We shall assume that the functions k, u_0, u_1 satisfy the assumptions of the existence theorem for a generalized solution of problem (1.1), (1.2) in the sense of [8]. We introduce the notation: $Q_T^0=\{(t,x):(t,x)\in Q_T,\ u(t,x)>0\}$, where $u(t,x)$ is the generalized solution, and $S_T=\bar{Q}_T^0\setminus Q_T^0$, $P_T=Q_T\setminus S_T$. It was shown in [8] that $u(t,x)$ satisfies in P_T Eq. (1.1) in the ordinary sense, while in S_T, i.e., at points of degeneracy, the generalized solution may not have the smoothness predicated in (1.1). Put $Q_{t_1,t_2}=\{(t,x):\ t_1\leqslant t\leqslant t_2,\ x\in\Omega\}$, $S_{t_1,t_2}=\{(t,x):\ t_1\leqslant t\leqslant t_2,\ (t,x)\in S_T\}$, $P_{t_1,t_2}=\{(t,x):t_1\leqslant t\leqslant t_2,\ (t,x)\in P_T\}$, $0\leqslant t_1\leqslant t_2\leqslant T$.

2. Criticality conditions

Definition. We shall say that the boundary conditions (1.2) in problem (1.1), (1.2) are critical if, everywhere in P_T.

$$u_t(t,x)\geqslant 0. \tag{2.1}$$

The criticality conditions for the boundary data will be used below to derive *a priori* pointwise estimates for the highest derivative u_{xx} in terms of the lower derivatives u_x, u.

We shall assume that the functions k, u_0, u_1 satisfy smoothness conditions, sufficient for term by term differentiation of Eq. (1.1) with respect to t or x once everywhere in P_T. We also assume that $u_0(x)\in C(\Omega)\cap C^2(P_{0,0})$, $u_1(t)\in C^1([0,T])$. Under these assumptions, we have:

Lemma 1

For criticality of conditions (1.2), it is necessary and sufficient that

$$\mathscr{L}(u_0(x))\geqslant 0,\quad x\in P_{0,0},\qquad u_1'(t)\geqslant 0,\ 0\leqslant t\leqslant T. \tag{2.2}$$

Proof. The necessity is obvious. Let us prove the sufficiency.

1. Some preliminary remarks need to be made about the properties of function $u(t, x)$. We shall show that the function $u_0(x)$, satisfying inequalities (2.2) and bounded in Ω, is non-increasing in Ω.

We first show that $u_0(x)$ cannot have a positive maximum in Ω. For, if, for some $x_m \in \Omega$, $u_0(x)$ has a positive maximum and is not identically equal to a constant, then $x_1 < x_m$, $x_2 > x_m$, exist such that $u_0(x) > 0$, $x_1 \leqslant x \leqslant x_2$, $u_0'(x_1) > 0$, $u_0(x_2) < 0$. Using the first inequality of (2.2), we obtain $k(u_0(x))u_0'(x) \,|_{x_1}^{x_2} \geqslant 0$, which leads to a contradiction.

Assume that $u_0'(x_3) > 0$, $u_0(x_3) > 0$ for some $x_3 \in \Omega$. Then, by what has been proved, $u_0'(x) \geqslant 0$ for $x > x_3$. Hence $\mathcal{L}(u_0)$ is defined for all $x > x_3$, and the first of inequalities (2.2) can be written as

$$\mathscr{L}(u_0(x)) = \alpha(x), \qquad x > x_3,$$

where $\alpha(x)$ is non-negative and continuous for $x > x_3$. Let $x_4 > x_3$ exist such that mes $\omega_\epsilon > 0$ for some $\epsilon > 0$, where $\omega_\varepsilon = \{x\colon\ x_3 < x < x_4,\ \alpha(x) \geqslant \varepsilon\}$. By integration of the above equation, we obtain

$$\int_{u_0(x_3)}^{u_0(x)} k(\eta)\,d\eta = -k(u_0(x_3))u_0'(x_3) + \int_{x_3}^{x} d\zeta \int_{x_3}^{\zeta} \alpha(\xi)\,d\xi, \qquad x > x_3.$$

On estimating the integral on the right-hand side for $x > x_4$, we get

$$\int_{x_3}^{x} d\zeta \int_{x_3}^{\zeta} \alpha(\xi)\,d\xi \geqslant \int_{x_4}^{x} d\zeta \int_{x_3}^{\zeta} \alpha(\xi)\,d\xi \geqslant \varepsilon \text{ mes } \omega_\varepsilon (x - x_4),$$

whence we see that $u_0(x) \to \infty$, $x \to \infty$, which contradicts the boundedness of the function in Ω. The case $\alpha(x) \equiv 0$, $x > x_3$, may be treated in the same way.

We have thus shown that

$$u_0'(x) \leqslant 0, \qquad x \in P_{0,0}.$$

Hence it follows that there is not more than one point $\xi(0) \in S_{0,0}$, and for any $0 < t \leqslant T$, the set $S_{t,t}$ consists of at most one element. Let us denote it by $\xi(t)$. It is easily seen that $S_T = \{(t, x)\colon\ 0 < t \leqslant T,\ x = \xi(t)\}$, $P_T = \{(t, x)\colon 0 < t \leqslant T, x \in \Omega, x \neq \xi(t)\}$.

It can be shown that the definition of generalized solution (see [8]) stipulates that, for any $0 < t \leqslant T$,

$$k(u(t, x))u_x(t, x) \to 0, \qquad x \to \xi(t). \tag{2.3}$$

Hence, given any $\delta > 0$, $\delta < \xi(t) \neq 0$, $0 < t \leqslant T$, there exists $x_0(t)$ such that

$$\xi(t)-\delta<x_0(t)<\xi(t), \qquad u_t(t, x_0(t))>0. \tag{2.4}$$

2. Denote $u_t(t, x)$ by $z(t, x)$. Everywhere in P_T the function z satisfies the equation

$$z_t=[k(u)z]_{xx}.$$

We put $z(t, x)=Y(t, x)e^{\alpha t}$, $\alpha>0$. The function Y satisfies in P_T the equation

$$[\alpha-k'(u)u_{xx}-k''(u)(u_x)^2]Y+Y_t=k(u)Y_{xx}+2k'(u)u_xY_x. \tag{2.5}$$

Consider the set N of points σ of the interval $(0, T)$ such that $Y(t, x)\geq 0$ for all $(t, x)\in Q_{0,\sigma}$. If $\sup\sigma = T$, the lemma is proved. Let $\sup\sigma = t_0 < T$.

By definition of t_0, there exist $t_0\leq t_1<T$ and $0<\delta_1\leq T-t_1$ such that

$$\min_{x\in\Omega} Y(t_1, x)=0 \tag{2.6}$$

and for all $t_1<t<t_1+\delta_1$ the function $Y(t, x)$ has negative values in Ω.

Consequently, by inequalities (2.2), Y has a negative minimum with respect to x for all $t_1<t<t_1+\delta_1$. Denote the minimum point by $(t, \bar{x}(t))$. Then, $0<\bar{x}(t)<\xi(t)$, $t_1<t<t_1+\delta_1$, since, for at least one t of the interval, the equation $\bar{x}(t) = \xi(t)$ contradicts (2.4). Notice that t_1 is chosen in such a way that

$$Y(t_1, \bar{x}(t_1))=0. \tag{2.7}$$

Since the function $z_t(t, \bar{x}(t))/z(t, \bar{x}(t))$ is not upper-bounded as $t \to t_1^+$,

$$Y_t(\bar{t}, \bar{x}(t))\leq 0, \qquad t_1<\bar{t}<t_1+\delta_2 \quad \forall\alpha>0, \quad 0<\delta_2\leq\delta_1. \tag{2.8}$$

On choosing a sufficiently large $\alpha > 0$ in (2.7) and using (2.8), we arrive at a contradiction, $\sup\sigma = T$; the lemma is proved.

If $\xi(t)=\infty$, $0<t\leq T$, in (2.3), the proof is similar.

Corollary. If functions $u_0(x)$ and $u_1(t)$ satisfy the criticality conditions (2.2), it follows from inequality (2.1) and the structure of operator $\mathcal{L}$ in (1.1) that, everywhere in P_T, we have the pointwise estimate for the leading derivative.

$$u_{xx} \geqslant -[k'(u)/k(u)](u_x)^2. \tag{2.9}$$

Note. 1. Under the assumptions of Lemma 1, it can be shown in the same way that

$$u_x(t, x) \leqslant 0, \qquad (t, x) \in P_T. \tag{2.10}$$

Note 2. For the case $u_0(x) \equiv 0,\ x \in \Omega,$ inequalities (2.1), (2.9) were obtained by a different method in [10].

3. Comparison theorem

Consider in Q_T, for the equations

$$u_t^{(\nu)} = \mathcal{L}^{(\nu)}(u^{(\nu)}) = [k^{(\nu)}(u^{(\nu)})u_x^{(\nu)}]_x, \qquad \nu = 1, 2, \tag{3.1}$$

the boundary value problems with the conditions

$$\begin{gathered} u^{(\nu)}(0, x) = u_0^{(\nu)}(x), \quad x \in \Omega, \qquad u^{(\nu)}(t, 0) = u_1^{(\nu)}(t), \\ 0 \leqslant t \leqslant T, \quad \nu = 1, 2. \end{gathered} \tag{3.2}$$

Let us find the conditions on operators $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ in (3.1), and on the boundary data (3.2), which ensure that the solutions of the problems, with $\nu = 1, 2$, can be compared in $\bar{Q}_T$, i.e., that the solution of the problem with $\nu = 1$ is majorized in $\bar{Q}_T$ by another solution, corresponding to $\nu = 2$. To obtain the conditions, we use pointwise estimates of the highest derivative of the solution, corresponding to $\nu = 2$, and (see Section 2), we assume that $u_0^{(2)}(x) \in C(\Omega) \cap C^2(P_{0,0}^{(2)})$, $u_1^{(2)}(t) \in C^1([0, T])$ and $u^{(2)}(t, x) \in C^{2,4}(P_T^{(2)})$. Moreover, let $u_0^{(1)}(x) \in C(\Omega)$, $u_1^{(1)}(t) \in C([0, T])$, $u^{(1)}(t, x) \in C^{1,2}(P_T^{(1)})$ and $0 \leqslant u_i^{(\nu)} \leqslant M$, $i = 0, 1$, $\nu = 1, 2$.

Theorem 1

Let the following assumptions hold:

1) $u_0^{(2)}(x) \geqslant u_0^{(1)}(x), \quad x \in \Omega, \quad u_1^{(2)}(t) \geqslant u_1^{(1)}(t), \quad 0 \leqslant t \leqslant T,$

2) $k^{(2)}(u) \geqslant k^{(1)}(u), \qquad [k^{(2)}(u)/k^{(1)}(u)]' \geqslant 0, \quad 0 < u \leqslant M,$

3) with $\nu = 2$, conditions (3.2) are critical.

Then, everywhere in $\bar{Q}_T$, we have

$$u^{(2)}(t, x) \geqslant u^{(1)}(t, x). \tag{3.3}$$

Proof. 1. Consider the set N of points σ of the interval $(0, T)$ such that $u^{(2)}(t, x) \geqslant u^{(1)}(t, x)$ for all $(t, x) \in Q_{0,\sigma}$. Assume that $\sup \sigma = t_0 < T$.

Let $\tilde{u}(t, x)$, $(t, x) \in Q_{t_0, T}$, be the solution of Eq. (3.2) with $\nu = 1$ with the conditions

$$\tilde{u}(t_0, x) = u^{(2)}(t_0, x), \quad x \in \Omega, \quad \tilde{u}(t, 0) = u_1^{(2)}(t), \quad t_0 \leqslant t \leqslant T.$$

By the boundary data comparison theorem (see [8]), we have

$$u^{(1)}(t, x) \leqslant \tilde{u}(t, x), \qquad (t, x) \in Q_{t_0, T}. \tag{3.4}$$

2. Put $z(t, x) = u^{(2)}(t, x) - \tilde{u}(t, x)$ and $z = Ye^{\alpha t}$, $\alpha > 0$. The function $Y(t, x)$ satisfies everywhere in $P^{(2)}_{t_0,T} = Q_{t_0,T} \backslash S^{(2)}_{t_0,T} \backslash \bar{S}_{t_0,T}$ the equation

$$\begin{aligned} \alpha Y + Y_t = {} & k^{(1)}(\tilde{u}) Y_{xx} + \{u_{xx}^{(2)} [k^{(2)}(u^{(2)}) \\ & - k^{(1)}(\tilde{u})] + (u_x^{(2)})^2 [k^{(2)\prime}(u^{(2)}) - k^{(1)\prime}(\tilde{u})]\} e^{-\alpha t} \\ & - k^{(1)\prime}(\tilde{u})(Y_x)^2 e^{\alpha t} + 2k^{(1)\prime}(\tilde{u}) Y_x u_x^{(2)}. \end{aligned} \tag{3.5}$$

From (2.6) and the definition of t_1 we obtain (2.7).

3. There are three possibilities.

a. Let $\delta_1 > 0$, $\delta_1 \leqslant \delta$ exist such that $(t, \bar{x}(t)) \in P^{(2)}_{t_1, t_1+\delta_1}$. Then, at points $(t, \bar{x}(t))$, $t_1 < t < t_1 + \delta_1$, Eq. (3.5) can be written as

$$\begin{aligned} & [\alpha + k^{(1)\prime}(\theta_1) u_{xx}^{(2)} + k^{(1)\prime\prime}(\theta_2)(u_x^{(2)})^2] Y + Y \\ & = k^{(1)}(\tilde{u}) Y_{xx} + \{u_{xx}^{(2)} [k^{(2)}(u^{(2)}) - k^{(1)}(u^{(2)})] \\ & + (u_x^{(2)})^2 [k^{(2)\prime}(u^{(2)}) - k^{(1)\prime}(u^{(2)})]\} e^{-\alpha t} \end{aligned} \tag{3.6}$$

where σ_1, σ_2 are functions of the variable t, and $\theta_1, \theta_2 \in [u^{(2)}(t, \bar{x}(t)), \tilde{u}(t, \bar{x}(t))]$. In the same way as when proving Lemma 1, we can show that (2.8) holds. Then, choosing sufficiently large α in (3.6), we arrive at a contradiction. For, at a suitably chosen point $(\bar{t}, \bar{x}(\bar{t}))$, the left-hand side of (3.6) is negative. But the right-hand side, by conditions 1) and 2) of the theorem, is non-negative, in view of the pointwise estimate (2.9) for the second derivative of function $u^{(2)}(t, x)$ at the point $(\bar{t}, \bar{x}(\bar{t})) \in P^{(2)}_{t_1, t_1+\delta_1}$.

b. Now let $(t, \bar{x}(t)) \in S^{(2)}_{t_1, t_1+\delta_2}$ for some $\delta_2 > 0$, $\delta_2 \leqslant \delta$. Then, function Y cannot have all the derivatives appearing in (3.6) at minimum points.

For function $z(t, x)$ we obtain in $Q_{t_1,\, t_1+\delta_3}$ the problem

$$z_t=[k^{(2)}(u^{(2)})u_x^{(2)}]_x-[k^{(1)}(\tilde{u})\tilde{u}_x]_x,$$
$$z(t_1,\ x)=0,\quad x\in\Omega,\qquad z(t,\ 0)=0,\quad t_1\leqslant t\leqslant t_1+\delta_3. \tag{3.7}$$

Since $(t, \bar{x}(t))\in S^{(2)}_{t_1,t_1+\delta_3}$, there exists, for any $t_1<t<t_1+\delta_3$ a point $0 < \xi(t) < \bar{x}(t)$ such that $z(t, \xi(t)) = 0$. Then,

$$\int\limits_{\xi(t)}^{\infty} z(t,\eta)\,d\eta>0,\qquad t_1<t<t_1+\delta_3. \tag{3.8}$$

Noting that $z_x(t, \xi(t))\leqslant 0$, and the first of conditions 2), and integrating (3.7) with respect to the set $(t_1<t<t_1+\delta_2)\times(\xi(t)<x<\infty)$, we arrive at a contradiction with (3.8).

c. If, for any $\delta_4>0,\ \delta_4\leqslant\delta$ there are minimum points $(t, \bar{x}(t))$, belonging both to $P^{(2)}_{t_1,t_1+\delta_4}$, and to $S^{(2)}_{t_1,t_1+\delta_4}$, then the proof folllows similar lines to those in case a or case b.

Hence, throughout $Q_{t_0,T}$ we have $u^{(2)}(t,\ x)\geqslant\tilde{u}(t,x)$. From this and (3.4) we obtain (3.3), $\sup\sigma = T$. This proves the theorem.

Note 3. Conditions 2) of Theorem 1 are equivalent to the following:

$$k^{(1)}(u)\geqslant k^{(2)}(u)[1+\lambda(u)]^{-1},\qquad 0\leqslant u\leqslant M,$$

where $\lambda(u)\geqslant 0,\ \lambda'(u)\geqslant 0,\ 0<u\leqslant M$.

4. Some generalizations

1. When proving the propositions of Sections 2 and 3, we actually only used the assumptions that operators $\mathcal{L}$ and $\mathcal{L}^{(\nu)}$, $\nu = 1, 2$, are parabolic and sufficiently smooth. Hence our approach to the comparison of solutions is also valid for parabolic equations of general type

$$u_t=\mathscr{L}(u)=L(u,\ u_x,\ u_{xx}). \tag{4.1}$$

Consider in Q_T the first boundary value problem for Eq. (4.1) with conditions (1.2). We shall assume that a solution exists, and

$$\sup_{(t,x)\in Q_T} u(t,x)\leqslant M_1<\infty \text{ and } u(t,x)\in C^{2,1}(Q_T)$$

[11–13]. We shall also assume that function $L(p, q, r)$ is differentiable for $0<p\leqslant M_1,\ -\infty<q<\infty,\ -\infty<r<\infty$, so that the function $L^{-1}_{(3)}(p,\ q)$, which, since the operator $\mathcal{L}$ in (4.1) is parabolic, is uniquely defined by the equation

$$L(p, q, L_{(3)}^{-1}(p, q)) = 0, \quad 0 < p \leqslant M_1, \quad -\infty < q < \infty,$$

is also differentiable.

We define criticality of the boundary conditions of problem (4.1), (1.2) in the same way as in Section 2. Let $u_0(x) \in C^2(\Omega)$, $u_1(t) \in C^1([0, T])$. Under these assumptions, we have the following proposition, which can be proved in the same way as Lemma 1.

Lemma 2

For criticality of the boundary conditions (1.2) of problem (4.1), (1.2), it is necessary and sufficient that

$$\mathscr{L}(u_0(x)) \geqslant 0, \quad x \in \Omega, \qquad u_1'(t) \geqslant 0, \quad 0 \leqslant t \leqslant T.$$

Corollary. Under these assumptions of the lemma, we have throughout Q_T the pointwise estimate for the highest derivative:

$$u_{xx} \geqslant L_{(3)}^{-1}(u, u_x). \tag{4.2}$$

Lemma 2 is in fact equivalent to the following: if $u_t(t, x) \geqslant 0$ for $(t, x) \in \Gamma_T = \{(t, x) : t = 0, x \in \Omega\} \cup \{(t, x) : 0 < t \leqslant T, \ x = 0\}$, where Γ_T is the boundary of Q_T, then $u_t(t, x) \geqslant 0$ for all $(t, x) \in \bar{Q}_T$. This assertion holds for all operators in (4.1) that do not contain the variable t (otherwise, the operator has to satisfy a supplementary condition).

2. Consider in Q_T two boundary value problems for the uniformly parabolic equations

$$u_t^{(\nu)} = \mathscr{L}^{(\nu)}(u^{(\nu)}) = L^{(\nu)}(u^{(\nu)}, u_x^{(\nu)}, u_{xx}^{(\nu)}), \quad \nu = 1, 2, \tag{4.3}$$

with boundary conditions (3.2). Let $u^{(1)}(t, x) \in C^{1,2}(Q_T)$, $u^{(2)}(t, x) \in C^{2,4}(Q_T)$,

$$\max\{\sup_{(t,x) \in Q_T} u^{(1)}(t, x), \quad \sup_{(t,x) \in Q_T} u^{(2)}(t, x)\} = M.$$

In addition, let the functions $L^{(\nu)}(p, q, r)$ be differentiable with respect to all their arguments for $0 < p \leqslant M$, $-\infty < q < \infty$, $-\infty < r < \infty$, $\nu = 1, 2$.

Theorem 2

Let assumptions 1) and 3) of Theorem 1 hold, and also, let

$$\begin{aligned} &L_3^{(2)}(p, q, r) - L_3^{(1)}(p, q, r) \geqslant 0, \\ &L^{(1)}(p, q, L_{(3)}^{(2)-1}(p, q)) \leqslant 0, \quad 0 < p \leqslant M, \ -\infty < q < \infty \end{aligned} \tag{4.4}$$

(here, $L_3^{(\nu)} = \partial L^{(\nu)}/\partial r$). Then inequality (3.3) holds everywhere in Q_T.

The proof is similar to the proof of Theorem 1; we use the estimate (4.2) for the highest derivative of the solution $u^{(2)}(t, x)$.

3. Let us indicate the form taken by conditions (4.4) for some concrete operators $\mathcal{L}^{(\nu)}$.

a. Let $\mathcal{L}^{(\nu)}(v^{(\nu)}) = \varphi^{(\nu)}(v^{(\nu)})v_{xx}^{(\nu)}$, where $\varphi^{(\nu)}(v^{(\nu)}) > 0$, $v^{(\nu)} > 0$, $\nu = 1, 2$. Inequalities (4.4) reduce to the condition

$$\varphi^{(2)}(p) \geqslant \varphi^{(1)}(p), \qquad 0 < p \leqslant M. \tag{4.5}$$

In this case, Eqs. (4.1) describe the heat propagation in a medium with fixed thermal conductivity and with heat capacity $c(v) = 1/\varphi(v)$, dependent on the temperature v; hence comparison condition (4.5) has a simple physical meaning.

Notice that Eqs. (3.1) can be reduced to the above by the substitution

$$u^{(\nu)} = V^{(\nu)-1}(v^{(\nu)}), \qquad V^{(\nu)}(u^{(\nu)}) = \int_0^{u^{(\nu)}} k^{(\nu)}(\eta)\, d\eta,$$

where $V^{(\nu)-1}$ are the inverse functions to $V^{(\nu)}$. Here,

$$\varphi^{(\nu)}(v^{(\nu)}) = k^{(\nu)}(V^{(\nu)-1}(v^{(\nu)})), \qquad \nu = 1, 2.$$

By comparison with conditions 2) of Theorem 1, conditions (4.3) are much simpler, and they contain no differential connections between the participating functions.

b. Let $\mathcal{L}^{(\nu)}(u^{(\nu)}) = [k^{(\nu)}(u^{(\nu)})u_x^{(\nu)}]_x + Q^{(\nu)}(u^{(\nu)})$, $k^{(\nu)}(u^{(\nu)}) > 0$, $u_x^{(\nu)} > 0$, $\nu = 1, 2$. This example is of special importance for studying the topics considered in [14–19]. Equations (4.1) then describe the heat and combustion propagation in a medium with non-linear heat conduction and volumetric separation of heat ($Q(u)$ is the power of the volumetric energy sources).

In view of the independence of the variation of p and q in the second of inequalities (4.4), the latter split up into three conditions:

$$k^{(2)}(p) \geqslant k^{(1)}(p), \qquad k^{(1)}(p)k^{(2)\prime}(p) \geqslant k^{(1)\prime}(p)k^{(2)}(p),$$
$$Q^{(2)}(p)k^{(1)}(p) \geqslant Q^{(1)}(p)k^{(2)}(p), \qquad 0 < p \leqslant M.$$

4. Our solution comparison method, the scope of which has been illustrated by the example of the first boundary value problem in an unbounded domain, is also applicable for problems in bounded domains, and for the Cauchy problem. Moreover, our results can be extended to problems of these types in multi-dimensional domains for parabolic equations with isolated Laplacian:

$$u_t = \mathcal{L}(u) = L(u, u_{x_1}, \ldots, u_{x_N}, \Delta u), \qquad \Delta u = \sum_{j=1}^{N} u_{x_j x_j}.$$

5. Metastable localization of heat

Our comparison theorems will be used in this section to study the effect of metastable heat localization in a medium with non-linear heat conduction.

1. Let problem (1.1), (1.2) be considered in $Q_T^1=\{(t, x):0<t<T, \quad x\in\Omega\}$, and let $u_1(t)$ be such that

$$u_1(t)\to+\infty, \qquad t\to T. \tag{5.1}$$

Definition. Following [1–5], we shall say that metastable heat localization occurs in problem (1.1), (1.2), (5.1) if $x_0<\infty$ exists such that $\operatorname{mes}\operatorname{supp} u(t, x)\leqslant x_0, \quad 0<t<T$. Otherwise, metastable heat localization is not present.

In short, if heat localization is present in problem (1.1), (1.2), (5.1), then, in spite of an unbounded temperature rise at the point $x = 0$, disturbances do not travel beyond a finite domain.

2. Consider, in $Q_T^2=\{(t, x):0<t<T, \ x\in\Omega_2\}, \ \Omega_2=\{x:-\infty<x<\infty\}$, the Cauchy problem for Eq. (1.1) with the initial condition

$$u(0, x)=u_0(x), \qquad x\in\Omega_2. \tag{5.2}$$

Definition. Metastable heat localization is present in problem (1.1), (5.2) if

$$\operatorname{supp} u(t, x)=\operatorname{supp} u_0(x), \qquad 0<t<t^*.$$

Heat localization in the Cauchy problem implies that the domain with non-zero temperature remains unchanged for a finite time.

3. *Theorem* 3

Assume that, in problem (1.1), (1.2), (5.1),

$$u_0(x)\leqslant T^n(1-x/x_0)^{2/\sigma}, \qquad x_0=[2(\sigma+2)/\sigma]^{1/2}T^{(1+n\sigma)/2}, \quad x\leqslant x_0,$$
$$u_0(x)=0, \qquad x>x_0$$

(where σ, n are constants, $\sigma>0, \ n<0, \ 1+n\sigma\geqslant0$),

$$\begin{aligned} &u_1(t)\leqslant(T-t)^n, \qquad 0\leqslant t<T,\\ &k(u)=u^\sigma[1+\lambda(u)]^{-1}, \qquad \lambda(u)\geqslant0, \ \ \lambda'(u)\geqslant0, \ \ 0<u<\infty. \end{aligned} \tag{5.3}$$

Then metastable heat localization occurs in the problem, while

$$\operatorname{mes}\operatorname{supp} u(t, x)\leqslant x_0, \qquad 0<t<T,$$
$$u(t, x)\leqslant T^{(1+n\sigma)/\sigma}(T-t)^{-1/\sigma}(1-x/x_0)^{2/\sigma}, \qquad 0<t<T, \quad x\leqslant x_0,$$
$$u(t, x)=0, \qquad 0<t<T, \quad x>x_0.$$

Theorem 4

Assume that, in problem (1.1), (1.2), (5.1),

$$u_1(t) \geqslant (T-t)^n, \qquad 0 \leqslant t < T, \quad n<0,$$
$$k(u) = u^\sigma[1+\lambda(u)], \qquad \lambda(u) \geqslant 0, \quad \lambda'(u) \geqslant 0, \quad 0<u<\infty$$

(where $\sigma>0$, $1+n\sigma<0$). Then heat localization is not present in the problem. Moreover, for any $x \in \Omega$,

$$u(t, x) \to \infty, \qquad t \to T.$$

Theorem 5

Assume that, in problem (1.1), (5.2),

$$0<u_0(x) \leqslant u_m(1-|x|/x_m)^{2/\sigma}, \quad |x|<x_m, \qquad u_0(x)=0, \quad |x| \geqslant x_m$$

(where u_m, x_m, σ are positive constants), and that (5.3) holds. Then metastable heat localization occurs, and

$$\operatorname{supp} u(t, x) = \operatorname{supp} u_0(x), \qquad 0<t<t^*,$$
$$0<u(t, x) \leqslant [x_m^2\sigma/2(\sigma+2)]^{1/\sigma}(t^*-t)^{-1/\sigma}(1-|x|/x_m)^{2/\sigma},$$
$$0<t<t^*, \quad |x|<x_m,$$
$$u(t, x)=0, \qquad 0<t<t^*, \quad |x| \geqslant x_m,$$

where $t^* = x_m^2\sigma/2u_m^\sigma(\sigma+2)$.

Theorems 3–5 are proved in [5] for the case $t^* = x_m^2\sigma/2u_m^\sigma(\sigma+2)$. If $\lambda(u) \not\equiv 0$, the theorems follow from our Theorem 1 and Note 1 on it.

Sufficient conditions for localization in a multi-dimensional domain can be stated in a similar way; here, the results of [1, 4] are used.

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REFERENCES

1. SAMARSKII, A. A., *et al.*, Effect of metastable heat localization in a medium with non-linear heat conduction, *Dokl. Akad. Nauk SSSR*, **223**, No. 6, 1344–1347, 1975.

2. KURDYUMOV, S. P., Non-linear processes in dense plasma, in: *Collected papers read at the 2nd international conference on plasma theory* (Sb. dokl. na II Mezhdunar. konf. po teorii plazmy), Kiev, 1974, Naukova Dumka, p. 278, Kiev, 1976.

3. ZMITRENKO, N. V., *et al.*, Metastable heat localization in a medium with non-linear heat conduction and conditions for it to occur experimentally, *IPM Akad. Nauk SSSR, preprint*, no. 103, 1977.

4. KURDYUMOV, S. P., MIKHAILOV, A. P., and PLOKHOTNIKOV, K. E., Heat localization in multi-dimensional problems of non-linear heat conduction, Thermal crystal, *IPM Akad. Nauk SSSR, preprint*, no. 22, 1977.

5. MIKHAILOV, A. P., Metastable localization of heat disturbances in a medium with non-linear heat conduction, *IPM Akad. Nauk SSSR, preprint*, No. 64, 1977.

6. SAMARSKII, A. A., and SOBOL', I. M., Examples of numerical computation of temperature waves, *Zh. vychisl. Mat. mat. Fiz.*, **3**, No. 4, 703–719, 1963.

7. GALAKTIONOV, V. A., and MIKHAILOV, A. P., On a similarly problem for the equation of non-linear heat conduction, *IPM Akad. Nauk SSSR, preprint*, No. 53, 1977.

8. OLEINIK, O. A., KALASHNIKOV, A. S., and CHU YU LIN, The Cauchy problem and boundary value problems for equations of non-stationary porous flow, *Izv. Akad. Nauk SSSR, Ser. matem.*, **22**, No. 5, 667–704, 1958.

9. OVSYANNIKOV, L. V., Group properties of the equation of non-linear heat conduction, *Dokl. Akad. Nauk SSSR*, **125**, No. 3, 492–495, 1959.

10. LIKHT, M. K., On propagation of disturbances in problems connected with degenerate quasi-linear equations of the parabolic type, *Differents. urniya*, **2**, No. 7, 953–957, 1969.

11. OLEINIK, O. A., and KRUZHKOV, S. N., Second-order quasi-linear parabolic equations with several independent variables, *Usp. Mat. Nauk*, **16**, No. 5, 115–155, 1961.

12. LADYZHENSKAYA, O. A., SOLONNIKOV, V. A., and URAL'TSEVA, N. N., *Linear and quasi-linear equations of parabolic type* (Lineinye i kvazilineinye uravneniya parabolicheskogo tipa), Nauka, Moscow, 1967.

13. FRIEDMAN, A., *Partial differential equations of parabolic type*, Prentice-Hall, 1964.

14. SAMARSKII, A. A., *et al.*, Thermal structures and fundamental length in a medium with non-linear heat conduction and volumetric heat sources, *Dokl. Akad. Nauk SSSR*, **227**, No. 2, 321–324, 1976.

15. SAMARSKII, A. A., *et al.*, Combustion of a non-linear medium in the form of compound structures, *Dokl. Akad. Nauk SSSR*, **237**, No. 6, 1330–1333, 1977.

16. ZMITRENKO, N. V., and KURDYUMOV, S. P., N- and S-modes of compression of a finite mass of plasma and singularities of modes with peaking, *Prikl. mekhan., tekhn. fiz.*, No. 1, 3–23, 1977.

17. ELENIN, G. G., and KURDYUMOV, S. P., Conditions for the complication of the organization of a non-linear dissipative medium, *IPM Akad. Nauk SSSR, preprint* No. 106, 1977.

18. SAMARSKII, A. A., Numerical methods in plasma physics, *IPM Akad. Nauk SSSR, preprint*, No. 107, 1977.

19. KURDYUMOV, S. P., *et al.*, Interaction of thermal structures, *IPM Akad. Nauk SSSR, preprint*, No. 77, 1978.