# NUMERICAL SIMULATION OF MHD-PROBLEMS ON THE BASIS OF VARIATIONAL APPROACH

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I. Many of modern scientific investigations deal with solution of multi-dimensional MHD-problems. For these problems the finite-difference methods are of the most universal and effective use.

The difference scheme may by considered as the discrete model of a medium. Obviously the scheme of better quality results in better simulation of the medium properties. For example the divergence schemes [I] provide an adequate accuracy of the discontinuous flow computations. The completely conservative schemes [2] may be used successfully even on crude grids.

The differential equations of nondissipative MHD and their most important properties with conservation laws included may be obtained brom variational principles similar to those of Hamilton-Ostrogradskii in classical mechanical systems [3, 4]. The completely conservative finite-difference schemes can be obtained through the analogous approach.

2. As an example we consider the liquid volume $G$ of an idealy conductive adiabatic plasma moving in space. Let $\Omega_3$ be the domain that corresponds to $G$ in Lagrangian coordinates $(\alpha,\beta,\gamma)$; $x=x(\alpha,\beta,\gamma,t)$, $y=y(\alpha,\beta,\gamma,t)$, $z=z(\alpha,\beta,\gamma,t)$ the Cartezian coordinates of particles of the medium. The Jacobian $D=D(x,y,z)/D(\alpha,\beta,\gamma)>0$; $\alpha,\beta,\gamma\in\Omega_3$. The Lagrangian of liquid volume $G$ is

$$\mathcal{L}(t)=\iiint_{\Omega_3}\rho D\left(\frac{\vec{V}^2}{2}-\varepsilon-\frac{\vec{H}^2}{8\pi\rho}\right)d\alpha\, d\beta\, d\gamma \qquad (1)$$

Here $\vec{V}=(u,v,w)$ is the velocity vector, $\vec{H}=(H_x,H_y,H_z)$ the vector of frozen-in magnetic field.

The law of mass conservation is

$$\rho D=\rho_0(\alpha,\beta,\gamma) \qquad (2)$$

The frozen-in condition for magnetic field yields the equations

$$H_x \frac{D(y,z)}{D(\alpha,\beta)} + H_y \frac{D(z,x)}{D(\alpha,\beta)} + \frac{D(x,y)}{D(\alpha,\beta)} H_z = \Phi_{\alpha\beta}$$

$$H_x \frac{D(y,z)}{D(\beta,\gamma)} + H_y \frac{D(z,x)}{D(\beta,\gamma)} + H_z \frac{D(x,y)}{D(\beta,\gamma)} = \Phi_{\beta\gamma} \qquad (3)$$

$$H_x \frac{D(y,z)}{D(\gamma,\alpha)} + H_y \frac{D(z,x)}{D(\gamma,\alpha)} + H_z \frac{D(x,y)}{D(\gamma,\alpha)} = \Phi_{\gamma\alpha}$$

In this paper we consider that all flow parameters depend on only tw( space coordinates $x$ and $y$ i.e. $\alpha$ and $\beta$ ,respectively,in Lagrangian coordinates. In this case we can accept

$$\Omega_3 = \Omega_2 \otimes \{0 \leq \gamma \leq 1\}$$

$$\frac{\partial}{\partial z} \equiv 0, \quad \frac{\partial x}{\partial \gamma} = \frac{\partial y}{\partial \gamma} \equiv 0, \quad \frac{\partial z}{\partial \gamma} \equiv 1$$

Equation (2) converts into

$$\rho D = \rho_0(\alpha,\beta) \qquad (2')$$

The relation

$$D\vec{H} = \frac{\partial \vec{z}}{\partial \alpha} \Phi_{\beta\gamma} + \frac{\partial \vec{z}}{\partial \beta} \Phi_{\gamma\alpha} + \frac{\partial \vec{z}}{\partial \gamma} \Phi_{\alpha\beta} \qquad (3')$$

follows from (3).

The time differentiation of (3') gives the equation of induction of magnetic field

$$\frac{d}{dt}(\vec{H}\cdot D) = (\vec{H}\cdot\vec{\nabla}_{\alpha\beta})\vec{\nabla}$$

where

$$\vec{\nabla}_{\alpha\beta} = \left\{ \frac{D(\cdot,y)}{D(\alpha,\beta)}, \frac{D(x,\cdot)}{D(\alpha,\beta)}, 0 \right\} = D\cdot\vec{\nabla}$$

By varying the functional of action $S = \int_{t_0}^{t_1} \mathcal{L}(t)\,dt$ and taking into account additional constraints (2'), (3'), and adiabaticity condition

$$\delta \mathcal{E} = \frac{P}{\rho^2}\delta\rho \qquad (4)$$

and by setting the first variation $\delta S$ equal to zero, we obtain the Euler equations

$$\rho D \frac{d\vec{V}}{dt} = -\nabla_{\alpha\beta}\left(P + \frac{H^2}{8\pi}\right) + \frac{\partial}{\partial\alpha}\frac{\vec{H}\,\Phi_{\beta\gamma}}{4\pi} + \frac{\partial}{\partial\beta}\frac{\vec{H}\,\Phi_{\gamma\alpha}}{4\pi} \qquad (5)$$

From (4) we obtain the equation for internal specific energy

$$\rho D \frac{d\varepsilon}{dt} = -P\left[\frac{D(u,y)}{D(\alpha,\beta)} + \frac{D(x,v)}{D(\alpha,\beta)}\right]$$

3. We assume that $\Omega_2(\alpha,\beta)$ is a square. Let us introduce the restangular difference grid with the intervals $h_\alpha$ and $h_\beta$; $\omega$ being the set of its cells and $\overline{\omega}$ the set of its nodes.

Let us approximate the Lagrangian $\mathcal{L}(t)$ by the discrete expression

$$\mathcal{L}_h = \sum_{(i,j)\in\omega} S_{ij}\,\rho_{ij}\left(\sum_{k\in Ш_1(i,j)} \frac{\vec{V}_k^2}{8} - \varepsilon_{ij} - \frac{\vec{H}_{ij}^2}{8\pi\rho_{ij}}\right) \qquad (6)$$

where $Ш_1(i,j) = \{(i,j),(i+1,j),(i+1,j+1),(i,j+1)\}$ and $S_{ij}$ is the volume of the Lagrangian cell.

The discrete analogs to the law of mass conservation and frozen-in condition are

$$\rho S = m \qquad (2'')$$

$$\begin{cases} H_x S = \Phi_k x_k \\ H_y S = \Phi_k y_k \\ H_z S - \Phi_k z_k = \Phi = const \;; \quad k \in Ш_1(i,j) \end{cases} \qquad (3'')$$

The repeated index denotes the summation. The value $\Phi_k = (H_x)_{ij}\frac{\partial S_{ij}}{\partial x_k} + (H_y)_{ij}\,\partial S_{ij}/\partial y_k$ is the magnetic flux across diagonals of the cell $(i,j)$ in the direction to the k-th node. According to (4) the energy change law is

$$m\frac{d\varepsilon}{dt} = -P\frac{dS}{dt} = -P\left(\frac{\partial S}{\partial x_k}u_k + \frac{\partial S}{\partial y_k}v_k\right) \qquad (4')$$

After the time differentiation frozen-in condition (3'') yields the magnetic induction equation

$$S\frac{d\vec{H}}{dt} = -\vec{H}\frac{dS}{dt} + \Phi_k \cdot \vec{V}_k$$

Varying the action functional with constraints (2")-(4") results in the MHD-equations

$$M\frac{du}{dt}-\left(P+\frac{\vec{H}^2}{8\pi}\right)_k\frac{\partial S_k}{\partial x}+\frac{(H_x)_k\Phi_k}{4\pi}=0$$

$$M\frac{dv}{dt}-\left(P+\frac{\vec{H}^2}{8\pi}\right)_k\frac{\partial S_k}{\partial y}+\frac{(H_y)_k\Phi_k}{4\pi}=0 \qquad (5')$$

$$M\frac{dW}{dt}+\frac{(H_z)_k\Phi_k}{4\pi}=0 \quad ; \quad k\in Ш_2(i,j)$$

$$M=\sum_{k\in Ш_2(i,j)} m_k/4$$

Here $Ш_2(i,j)=\{(i,j),(i-1,j),(i,j-1),(i-1,j-1)\}$

Schemes (5") are valid in the inner nodes of the grid and accurate to the second-order approximation in $h_\alpha$ and $h_\beta$. Simple generalization of the action functional extends (5*) into the boundary nodes. Differential-difference schemes (5*) are completely conservative [4].

4. The difference scheme is deduced from equations (5*) through replacing the time derivative by the finite-difference relations. By time centering the right hand side terms the difference schemes are made completely conservative.

5. The linearization of equations (2")-(5*) produces the differential-difference schemes for MHD-acoustics. The variational approach provides for a space operator of this schemes to be self-adjoint and positive [4]. The scheme stability depends on the implicitness of the scheme. The difference equations are solved by the Newton method.

To calculate the discontinuous flows artificial viscosity is used [4].

6. The variational approach has been used as a basis for numerical algorithms and techniques employed for solving a number of model and applied problems [5, 6, 7].

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