Amer. Math. Soc. Trans.
(2) Vol. 105, 1976

SOME QUESTIONS FROM THE GENERAL THEORY OF DIFFERENCE SCHEMES

A. A. SAMARSKIĬ

One of the rapidly developing branches of modern mathematics is the theory of difference schemes for the solution of the differential equations of mathematical physics. Difference schemes are also widely used in the general theory of differential equations as an apparatus for proving existence theorems and investigating the differential properties of solutions. But here one is primarily interested only in the asymptotic (for $h \to 0$) properties of the difference approximations.

The theory of difference schemes has a number of special problems.

In the final analysis, of greatest importance from the point of view of numerical analysis is the determination of algorithms permitting one to obtain a solution of a differential equation on an electronic computer with a prescribed accuracy in a finite number of operations. One encounters in this connection the question of the quality of an algorithm, i.e. the manner in which the accuracy of the algorithm depends on 1) the amount of information on the original problem, and 2) the amount of computation (viz. the machine time spent in solving the problem with a prescribed accuracy). Experience with computers has stimulated the formulation of a number of special (for the theory of difference methods) problems: 1) the determination of the achievable order of accuracy of difference schemes for various classes of problems, 2) the construction of schemes for the solution of a wide class of problems with a certain guaranteed accuracy, 3) the construction of schemes giving increased accuracy in narrower classes of problems, 4) the development of methods for investigating the stability and convergence of difference schemes, 5) the formulation of general principles for constructing stable difference schemes and economizing the amount of computations (economical schemes), and others.

In the present article we dwell only on a circle of questions connected with such fundamental notions of the theory of difference schemes as stability and approximation.

The main purpose of the article is to show how the results of the general theory of difference schemes can be used to formulate principles for constructing

---

AMS (MOS) subject classifications (1970). Primary 65-02, 65M05, 65M10; Secondary 47A50.

difference schemes of a prescribed quality. This approach requires forsaking a detailed description of the structure of difference operators for concrete classes of differential equations and presenting the theory in the language of functional analysis. The difference schemes that are analogs of the nonstationary differential equations of mathematical physics are treated in this connection as difference (with respect to the variable t) equations with operator coefficients defined in an abstract space (of any number of dimensions). The difference schemes for elliptic equations are treated as operator equations of the first kind. It should be emphasized, however, that the indicated notions of schemes have a much more general meaning.

In §1 we give an account of the general theory of stability of two- and three-level operator difference schemes in Hilbert space. The study of stability is carried out independently of that of approximation for families of admissible difference schemes. Here we obtain necessary and sufficient stability conditions and corresponding a priori estimates. The sufficient conditions distinguishing classes of stable schemes are in the form of easily verifiable linear operator inequalities. Simple rules are formulated for verifying the stability of schemes of a particular form. To investigate the stability of two-level schemes we employ a method that is more sensitive than the energy method.

The stability theory is used to formulate a general principle for regularizing difference schemes in order to obtain stable schemes of a prescribed quality.

In §2 the theory of iterative methods for solving the equation $Au = f$, where $A \in (H \to H)$ is a linear operator in a Hilbert space H, is treated as a branch of the general theory of stability of operator difference schemes. Our main concern is with obtaining effective estimates for the rate of convergence of the iterations and with choosing optimal iterative parameters. We consider a class of implicit schemes with a factored operator B on the upper level of the form $B = (E + \omega R_1)(E + \omega R_2)$, where E is the identity operator, $\omega > 0$ is a parameter and R_1 and $R_2 = R_1^*$ are adjoint or "triangular" (with a triangular matrix) linear operators. A formula for the parameter ω is obtained from the condition that the number of iterations be minimized.

An estimate of the rate of convergence for the method of minimal corrections is obtained in the case when A is a nonselfadjoint operator, and others.

In §3 we consider the total approximation method as a constructive method for obtaining economical difference schemes for the multidimensional equations of mathematical physics. The notion of an additive scheme is introduced as a system of operator difference equations that approximates the original differential equation in the total sense. Two quite general heuristic methods (proposed earlier by the author) for obtaining additive economical schemes are discussed.

The additive schemes required a new technique for investigating convergence and a new type of a priori estimates that take into account the definition of the property of approximation.

The absence of a comparative analysis of the works of different authors (such an analysis would have required a substantial increase in the size of our article) is compensated to a certain extent by a rather extensive bibliography.

We have not had the chance to discuss the works on difference methods of an applied character, although such works best illustrate the possibilities of difference methods and are a constant source of stimulation for the formulation of new theoretical problems.

§1. General theory of stability of difference schemes

1. The basic a priori characteristics of a difference scheme are the error of approximation and the stability. By the error of approximation of a scheme one usually means the residual that arises upon substituting the solution of the differential equation into the difference equation. The stability of a difference scheme is defined as the property of continuous dependence of the solution of the difference problem on the input data (on the initial and boundary data and on the right side of the equation). In contrast to the case of a differential equation this continuity must be uniform relative to the admissible mesh widths. For a linear scheme the existence of stability implies the satisfaction of an a priori estimate of the solution of the difference problem in terms of the input data. In this case stability and approximation imply convergence of the difference scheme, the order of accuracy (rate of convergence) of the scheme being determined by the degree of approximation. The estimation of the degree of approximation is generally (see §3) a comparatively simple problem, while the investigation of stability involves significant difficulties and is the central problem of the general theory of difference schemes.

The first rigorous definition of the notion of stability of difference schemes was given by V. S. Rjaben'kiĭ and A. F. Filippov [1]–[3]. The general questions of stability theory have been subsequently considered in [4]–[22].

One of the basic methods of investigating the stability of difference schemes consists in the application of the Fourier transform to difference equations. The stability conditions in this connection are given in the form of various restrictions on the spectra of the scheme operators (or on the spectra of the Fourier transforms of these operators). Of relevance here, for example, are [4]–[15]. This approach to the study of stability has a number of distinguishing features: the Cauchy problem is normally considered, an assumption is made concerning the connection between the mesh widths of the schemes with respect to space and

time, the requirements on the smoothness of the coefficients of the corresponding differential equation are often excessive. Many stability criteria (such as, for example, in [6], [10], [12], [13]) cannot be readily used directly in the investigations of concrete difference schemes.

In the majority of papers in this direction the investigation of stability and convergence is carried out in the space of solutions of the differential equation, which does not correspond to the actual state of affairs since the solution of the difference problem is in fact a mesh function. The connection between stability, approximation and convergence in spaces other than the original space (in factor spaces) is discussed in [23], [24].

In [19]–[21] spectral methods are used to obtain necessary conditions for the stability of two-level difference schemes with boundary conditions of general form.

The stability of concrete schemes has been successfully investigated with the use of the energy method, which frees one from the need to carry out a detailed study of the spectral properties of the difference scheme operators (see, for example, [25]–[30]). This line of attack was initiated with the well-known paper of Courant, Friedrichs and Lewy [32]. The difference analogs of Sobolev's imbedding theorems [31] are also used.

2. The basic problem of stability theory is the derivation of sufficient stability conditions that are readily verifiable in the case of concrete schemes. Effective sufficient conditions for the stability of difference schemes with operators defined in an abstract Hilbert space have been obtained in [34]–[36].

Let us proceed to a presentation of some of the results of this theory. We first note that the stability of a difference scheme is an intrinsic property that does not depend on the approximation of some differential equation. It is therefore natural to study stability independently of approximation.

Difference schemes (which are analogs of the nonstationary problems of mathematical physics) are defined by us as difference (with respect to the variable t) equations with operator coefficients defined on abstract Hilbert spaces H_h (which are analogs of spaces of mesh functions depending on the mesh width h). No assumptions are made concerning the structure of the scheme operators. The original family of schemes is defined only by the requirements of positiveness and, possibly, selfadjointness of the scheme operators.

The following problem is posed: distinguish the class of stable schemes belonging to the original family. It turns out that sufficient conditions for the stability of (two- and three-level) schemes have the form of linear inequalities between the scheme operators and are readily verifiable.

Difference schemes are usually expected to 1) appoximate to within a

certain degree the original equation, 2) be stable, and 3) minimize (in some agreed-upon sense) the number of arithmetic operations required to determine the solution of the difference problem with a prescribed accuracy (in the case of one-dimensional equations of, for example, parabolic type a scheme is said to be economical if the number of operations required to determine the difference solution is proportional to the number of mesh points used in this connection). As was noted above, the convergence of a scheme is a consequence of stability and approximation. The indicated requirements compete with each other, and their simultaneous satisfaction is a difficult problem.

Once we have classes of stable schemes, it is natural to seek in these classes schemes of a desired quality. This can be done, since writing the schemes in canonical form permits one to distinguish the operators (regularizers) responsible for stability. By taking advantage of the arbitrariness in the choice of R and varying R so as to remain in the class of stable schemes, we can construct schemes of a desired quality. A general method for regularizing schemes is presented in [34].

Thus the proposed theory of stability of difference schemes bears a constructive character.

The question of what information is needed on the scheme operators in order to render a correct judgement concerning the existence of stability is investigated.

A method employed in the study of stability is that of reducing an implicit scheme to an explicit one and estimating the norm of the translation operator of the explicit scheme. This method is more sensitive than the energy method and permits one to obtain coincident necessary and sufficient stability conditions in the case when one of the scheme operators is nonselfadjoint.

We proceed to a presentation of stability theory for two-level schemes [33]–[36].

3. Let $\{H_h\}$ be a set of real Hilbert spaces depending on a parameter h, which is a vector with norm $|h| > 0$ of a certain normed space. We introduce on a segment $0 \leqslant t \leqslant t_0$ a uniform (for the sake of simplicity) net $\overline{\omega}_\tau = \{t_k = k\tau, k = 0, 1, \cdots, k_0, k_0\tau = t_0\}$ with mesh width $\tau = t_0/k_0$. Let $A_{h\tau}(t_k), B_{h\tau}(t_k), R_{h\tau}(t_k), C_{h\tau}(t_k)$, etc. be linear operators mapping H_h onto H_h for each value of the parameter $t_k \in \overline{\omega}_\tau$, let $\varphi_{h\tau}(t_k)$, $y_{h\tau}(t_k)$, $F_{h\tau}(t_k)$, etc. be abstract functions of $t_k \in \overline{\omega}_\tau$ with values in H_h and let $y_{0,h\tau}$ and $y_{1,h\tau}$ be arbitrary vectors of H_h. For the sake of simplicity the notation in the sequel will not, as a rule, indicate the dependence of the operators, functions and vectors on h and τ.

By an m-level difference scheme is meant an $(m-1)$th order difference (with respect to $t = t_k$) equation

$$B(t)\,y(t+\tau)=\sum_{k=0}^{m-2}C_k(t)\,y(t-k\tau)+F(t),\quad \text{where}\quad t=s\tau\geqslant(m-2)\,\tau,$$

with operator coefficients and $m-1$ initial conditions obtained by prescribing the vectors $y(0), y(\tau), \cdots, y((m-2)\tau)$.

We will consider here only two-level $(m=2)$ and three-level $(m=3)$ schemes. An important role will be played in the sequel by the canonical forms of these schemes.

A two-level scheme is

$$\begin{gathered} B(t)\frac{y(t+\tau)-y(t)}{\tau}+A(t)\,y(t)=\varphi(t),\qquad 0\leqslant t=k\tau<t_0,\\ y(0)=y_0\in H_h. \end{gathered}\tag{1}$$

A three-level scheme is

$$\left.\begin{gathered} B(t)\frac{y(t+\tau)-y(t-\tau)}{2\tau}+\tau^2R(t)\frac{y(t+\tau)-2y(t)+y(t-\tau)}{\tau^2}\\ +A(t)\,y(t)=\varphi(t),\\ 0<t=k\tau<t_0,\qquad y(0)=y_0,\qquad y(\tau)=y_1,\qquad y_0,\ y_1\in H_h. \end{gathered}\right\}\tag{2}$$

Schemes (1) and (2) are difference analogs of the following abstract Cauchy problems for first and second order equations:

$$\begin{gathered} \mathcal{B}\frac{du}{dt}+\mathcal{A}u=f(t),\qquad 0\leqslant t\leqslant t_0,\qquad u(0)=u_0,\\ \mathcal{B}\frac{du}{dt}+\mathcal{R}\frac{d^2u}{dt^2}+\mathcal{A}u=f(t),\qquad 0\leqslant t\leqslant t_0,\qquad u(0)=u_0,\\ \frac{du}{dt}(0)=u_1. \end{gathered}$$

In order to take into account the case of positive and nonselfadjoint operators B in (1) and (2) we consider here a real Hilbert space H. An analogous stability theory for schemes in a complex Hilbert space $\tilde{H}$ is developed in [37].

4. Varying h and τ, we obtain a set $\{y_{h\tau}(t)\}$ of the solutions of problems (1) and (2). Stability for schemes (1) and (2) is defined as the property of uniform in (h,τ) continuity of $\{y_{h\tau}(t)\}$ relative to the input data $\{\varphi_{h\tau}(0)\}$ and $\{y_{h\tau}(0)\}$ (and $\{y_{h\tau}(\tau)\}$ in the case of (2)). We will assume that schemes (1) and (2) are solvable for any input data, i.e. that there exist inverse operators B_k^{-1} for (1) and $(B_k+2\tau R_k)^{-1}$ for scheme (2). Let us give a definition of stability for the two-level scheme (1). The solution of problem (1) is the sum of the solutions of the problems

$$B_k \frac{y_{k+1} - y_k}{\tau} + A_k y_k = 0, \qquad k = 0, 1, \ldots, \qquad k_0 - 1, \quad y_0 \in H_h, \tag{1a}$$

$$B_k \frac{y_{k+1} - y_k}{\tau} + A_k y_k = \varphi_k, \qquad k = 0, 1, \ldots, \qquad k_0 - 1, \quad y_0 = 0. \tag{1b}$$

Suppose H_h is any normed linear space. We will say (see [36]) that *scheme* (1) *is stable with respect to the initial data* in the norm $\|\cdot\|_{(1_h,k)}$ if there exists a constant c_0 not depending on the choice of τ, h and y_0 such that the solution of problem (1a) satisfies the inequality

$$\|y_k\|_{(1_h,\,k)} \leqslant \rho^k \|y_0\|_{(1_h,\,0)}, \qquad k = 1, 2, \ldots, \tag{3a}$$

for any $y_0 \in H_h$, where $\rho = e^{c_0 \tau}$ and $\|\cdot\|_{(1_h,k)}$ is a norm in H_h depending possibly on k.

Scheme (1) *is stable with respect to the right side* if there exists a constant $M_2 > 0$ not depending on h, τ or φ_k such that the solution of problem (1b) satisfies the a priori estimate

$$\|y_k\|_{(1_h,\,k)} \leqslant M_2 \max_{0 \leqslant j < k} \|\varphi_j\|_{(2_h,\,j)}, \qquad k = 1, 2, \ldots, \tag{3b}$$

for all $\varphi_k \in H_h$, where $\|\cdot\|_{(2_h,j)}$ is a norm in H_h depending on j.

It is usually required that a scheme be stable for sufficiently small $\tau \leqslant \tau_0$ and $|h| \leqslant h_0$, where τ_0 and h_0 are constants not depending on either k or the input data. Scheme (1) is said to be conditionally stable if it is stable when some relation between τ and h holds. If on the other hand scheme (1) is stable for any $\tau > 0$ and $|h| > 0$ $(h_0 = \tau_0 = \infty)$ it is said to be absolutely stable.

The definitions of stability given above do not assume that H_h is a Hilbert space.

Scheme (1a) is often written in the form $y_{k+1} + S_k y_k$, where $S_k = E - \tau B_k^{-1} A_k$ is the translation operator from the kth level to the $(k+1)$th level. It follows that

$$y_k = T_k y_0, \qquad T_k = S_{k-1} S_{k-2} \ldots S_1 S_0,$$

where T_k is the solving operator, so that

$$\|y_k\|_{(1_h,\,k)} \leqslant \|T_k\| \, \|y_0\|_{(1_h,\,0)}.$$

Scheme (1a) is stable if

$$\|T_k\| \leqslant \rho^k = e^{c_0 \tau k} < e^{c_0 t_0} = M_1 \qquad \text{for} \quad c_0 \geqslant 0.$$

Thus the stability of scheme (1a) implies the boundedness of its solving operator.

The basic question is the following: *What properties must the operators A_k and B_k have in order to ensure the stability of scheme* (1)? An answer to this question can be obtained in the case when H_h is a Hilbert space.

5. In conjunction with a basic space H_h we will consider energy spaces H_D consisting of the same vectors as in H_h but having scalar products $(y, v)_D = (Dy, v)$ and norms $\|y\|_D = \sqrt{(Dy, y)}$, where $D = D^* > 0$ is a positive self-adjoint operator in H_h ($D > 0$ means that $(Dx, x) > 0$ for all $x \neq 0$ in H_h). The operator D can depend on t_k: $D = D_k = D(t_k)$.

We will say that scheme (1a) is 1) stable in H_D if (3a) is satisfied with $\|\cdot\|_{(1_h,k)} = \|\cdot\|_{(1_h,0)} = \|\cdot\|_D$, i.e. if

$$\|y_k\|_D \leqslant \rho^k \|y_0\|_D, \quad \rho = e^{c_0\tau} \qquad (D \text{ does not depend on } t_k);$$

and 2) stable in H_{D_k} if

$$\|y_{k+1}\|_{D_k} \leqslant \rho^{k+1} \|y_0\|_D \qquad (D \text{ depends on } t_k).$$

In the case of two-level schemes the norms $\|\cdot\|_A$ and $\|\cdot\|_B$ ($D = A$ or $D = B$) are natural. We will write $B \geqslant \gamma A$ if $(Bx, x) \geqslant \gamma(Ax, x)$ for all $x \in H_h$, where γ is a constant.

6. The original family of schemes (1) is defined by the conditions

$$B_k > 0, \qquad A_k = A_k^* \quad \text{for all} \quad k = 0, 1, \ldots, k_0 - 1,$$

i.e. B_k is a nonselfadjoint operator. We first consider the case of schemes with constant (i.e. not depending on k (on t_k)) operators A and B:

$$B\frac{y_{k+1} - y_k}{\tau} + Ay_k = 0, \qquad k = 0, 1, \ldots, k_0 - 1, \ y_0 \in H_h. \quad (1a^*)$$

It is assumed everywhere in the sequel that scheme ($1a^*$) belongs to the original family, i.e. $B > 0$, $A = A^*$.

THEOREM 1. *Suppose $A > 0$. Then the condition*

$$B \geqslant \frac{1}{2}\tau A \tag{4}$$

is necessary and sufficient for the stability with $\rho = 1$ ($c_0 = 0$) in H_A of scheme ($1a^*$), *i.e. for the satisfaction of the estimate*

$$\|y_k\|_A \leqslant \|y_0\|_A.$$

THEOREM 2. *Suppose* $B = B^* > 0$. *Then the conditions*

$$\frac{1-\rho}{\tau} B \leqslant A \leqslant \frac{1+\rho}{\tau} B \tag{5}$$

for any $\rho > 0$ *(for* $A > 0$*) are necessary and sufficient for the stability of scheme* (1a*) *in* H_B *(in* H_A*).*

REMARK 1. If $A > 0$ and $\rho \geqslant 1$, condition (5) is equivalent to the inequality

$$B \geqslant \frac{\tau}{1+\rho} A. \tag{5*}$$

In particular, when $\rho = 1$ we get $2B \geqslant \tau A$.

REMARK 2. It is nowhere assumed that the operators A and B are commutative.

If A and B are commutative, conditions (5) are necessary and sufficient for stability in H_h, H_{A^2}, H_{B^2}, etc.

7. Suppose $A_k = A(t_k)$ and $B_k = B(t_k)$ are variable operators. We will say that $A_k > 0$ is Lipschitz continuous in t_k if

$$|((A_k - A_{k-1})x, x)| \leqslant c_1 \tau (A_{k-1}x, x) \quad \text{for all} \quad x \in H_h \text{ and } k = 1, 2, \ldots,$$

where $c_1 = \text{const} > 0$ does not depend on τ or h.

THEOREM 3. *Suppose* $A_k > 0$ *and is Lipschitz continuous in* t_k. *Then the condition*

$$B_k \geqslant \frac{\tau}{1+\rho} A_k, \quad \rho = e^{c_0\tau} \qquad \textit{for all} \qquad k = 0, 1, \ldots, k_0 - 1,$$

with $c_0 \geqslant 0$ *is sufficient for the stability of scheme* (1a) *in* H_{A_k} *with* $\bar{c}_0 = c_0 + c_1/2$:

$$\|y_{k+1}\|_{A_k} \leqslant \bar{\rho}^k \|y_0\|_{A_0}, \quad \bar{\rho} = e^{\bar{c}_0\tau}.$$

THEOREM 4. *Suppose* $B_k = B_k^*$ *and is Lipschitz continuous in* t_k. *Then the condition*

$$\frac{1-\rho}{\tau} B_k \leqslant A_k \leqslant \frac{1+\rho}{\tau} B_k \qquad \textit{for all} \qquad k = 0, 1, \ldots, k_0 - 1$$

with $c_0 \geqslant 0$ *is sufficient for the stability of scheme* (1a) *in* H_{B_k} *with* $\bar{c}_0 = c_0 + c_1/2$, *i.e. for the satisfaction of the estimate*

$$\|y_{k+1}\|_{B_k} \leqslant \bar{\rho}^k \|y_0\|_{B_0}, \qquad \bar{\rho} = e^{\bar{c}_0\tau}.$$

Let us write scheme (1) in the form

$$(E+\tau R_k)\frac{y_{k+1}-y_k}{\tau}+A_k y=\varphi_k, \qquad B_k=E+\tau R_k, \tag{6}$$

where E is the identity operator. Then the condition $2B_k \geqslant \tau A_k$ will be satisfied if

$$R_k \geqslant \sigma_0^{(k)} A_k, \qquad \sigma_0^k=\frac{1}{2}-\frac{1}{\tau\|A_k\|}, \qquad k=0,1,2,\ldots. \tag{7}$$

The index k in the stability conditions will be dropped in the sequel. It follows from (7) that the condition $2R \geqslant A$ is also sufficient for the stability of scheme (1).

EXAMPLE 1. Consider the weighted scheme

$$\begin{gathered}\frac{y_{k+1}-y_k}{\tau}+A(\sigma y_{k+1}+(1-\sigma)y_k)=\varphi_k,\\ k=0,1,\ldots,k_0-1,\ y_0\in H_h,\end{gathered} \tag{8}$$

where σ is a parameter (the weight factor) and $A=A(t_k)>0$. Reducing scheme (8) to the canonical form (1) or (6), we find that $B=E+\sigma\tau A$ or $R=\sigma A$. If $A=A^*$ and estimate (5) holds, it follows by virtue of (7) and Theorem 4 that scheme (8), when $\varphi_k=0$, is stable for $\sigma \geqslant \sigma_0^{(k)}$. If $A(t_k)>0$ is a nonselfadjoint operator, we first apply the operator $A^{-1}>0$ to (8) and then reduce the result to the canonical form (1a) with $\widetilde{B}=A^{-1}+\sigma\tau E$, $\widetilde{A}=E$ and make use of Theorem 1, from which it follows that scheme (8), when $\varphi_k=0$ and $A(t)>0$ is even any nonselfadjoint operator, is stable in H_h with $\rho=1$ for $\sigma \geqslant ½$.

8. Conditions (4), (5) and (5*) are sufficient for the stability of scheme (1) with respect to the right side in the corresponding norms. Thus estimate (3b) holds with

$$\|y\|_{(1)}=\|y\|_B=\sqrt{(By,y)},\quad \|\varphi\|_{(2)}=\|\varphi\|_{B^{-1}}=\sqrt{(B^{-1}\varphi,\varphi)}$$

for (7) or with

$$\|y\|_{(1)}=\|y\|_A=\sqrt{(Ay,y)},\quad \|\varphi\|_{(2)}=\sqrt{(AB^{-1}\varphi,B^{-1}\varphi)}$$

for (5*). Of importance for the theory of difference schemes are estimates of the solution of the problem in H_A or H_B in terms of the right side taken in a possibly weaker norm. Such norms are $\|\varphi\|_{(2)}=\|\varphi\|_{A^{-1}}$ in $H_{A^{-1}}$ or the negative norm $\|\varphi\|_{(2)}=\sup[|(\varphi,x)|/\|x\|_A]$ (the functional norm), an analog of which is widely used in the general theory of differential equations. For example, we have

THEOREM 5. *If the conditions of Theorem* 4 *are satisfied, the solution of problem* (1b) *satisfies the estimate*

$$\|y_{k+1}\|_{A_k} \leqslant M_2 \max_{0<j\leqslant k} \left[\|\varphi_j\|_{A_j^{-1}} + \left\|\frac{\varphi_j - \varphi_{j-1}}{\tau}\right\|_{A_j^{-1}}\right]. \tag{9}$$

If on the other hand $2R \geqslant A$, *then scheme* (1b) *is stable in* H_{A_k} *and estimate* (3b) *holds with* $\|\varphi\|_{(2)} = \|\varphi\|$, $\|y\|_{(1)} = \|y\|_A$.

We note that estimate (3b) holds with

$$\|\cdot\|_{(1_h,\,k)} = \|\cdot\|_{(1_h,\,0)} = \|\cdot\|,\ \|\cdot\|_{(2_h,\,k)} = \|\cdot\|_{A_k^{-1}}$$

for the weighted scheme (8) when $y_0 = 0$, $\sigma \geqslant ½$ and $A(t) = A^*(t) > 0$.

9. Let us now consider the three-level scheme (2). We will assume that $B(t)$ is a nonselfadjoint operator, $R(t) = R^*(t) > 0$ and $A(t) = A^*(t) > 0$. These conditions define the original family of schemes.

The stability of scheme (2) with respect to the initial data is expressed by inequality (3a). But by the norm $\|y_{k+1}\|_{(1_h,k)}$ one should understand a functional depending on y_k and y_{k+1} and defined when $\rho = 1$ by the equality

$$\begin{aligned}\|Y_{k+1}\|^2_{(1_h,\,k)} = {} & \frac{1}{4}(A_k(y_{k+1} + y_k),\ y_{k+1} + y_k) \\ & + \left(\left(R_k - \frac{1}{4}A_k\right)(y_{k+1} - y_k),\ y_{k+1} - y_k\right).\end{aligned} \tag{10}$$

(An expression for the norm when $\rho \neq 1$ can be found in [38].)

We cite a theorem on the stability of scheme (2).

THEOREM 6. *Suppose* A *and* R *are constant operators,* $B \geqslant 0$ *and*

$$R > \frac{1}{4}A. \tag{11}$$

Then scheme (2) *is stable with respect to the initial data with* $\rho = 1$, *so that the solution of problem* (2) *when* $\varphi = 0$ *satisfies the a priori estimate*

$$\|Y_k\|_{(1_h)} \leqslant \|Y_1\|_{(1_h)}, \tag{12}$$

where $\|Y_k\|$ *is defined by formula* (10). *Under these same conditions the solution of problem* (2) *when* $\varphi \neq 0$ *satisfies the estimate*

$$\|Y_{k+1}\|_{(1_h)} \leqslant \|Y_1\|_{(1_h)} + M_2 \max_{0<j\leqslant k} \left[\|\varphi_j\|_{A^{-1}} + \left\|\frac{\varphi_j - \varphi_{j-1}}{\tau}\right\|_{A^{-1}}\right], \tag{13}$$

where M_2 *does not depend on* $|h|$ *or* τ.

REMARK 3. If $B \geqslant \delta E$, $\delta > 0$ and $4R > A$, the following estimate holds in place of (13):

$$\|Y_{k+1}\|_{(1_h)} \leqslant \|Y_1\|_{(1_h)} + M_2 \max_{0<j\leqslant k} \|\varphi_j\|. \tag{14}$$

Consider the three-level scheme

$$(E + \tau^2 R)\,\frac{y_{k+1} - 2y_k + y_{k-1}}{\tau^2} + B\,\frac{y_{k+1} - y_{k-1}}{2\tau} + Ay_k = \varphi_k \tag{2*}$$

which is in the second canonical form. It is obtained from (2) by formally replacing $\tau^2 R$ by $E + \tau^2 R$. The condition $4R > A$ is sufficient for the stability of scheme (2*) with $\rho = 1$ in the norm $\|Y\|_{(1^*)}$, where

$$\|Y\|^2_{(1^*)} = \|Y_{k+1}\|^2_{(1)} + \left\|\frac{y_{k+1} - y_k}{\tau}\right\|^2 \qquad \text{for} \quad \|\varphi\|_{(2)} = \|\varphi\|. \tag{15}$$

EXAMPLE 2. Consider the three-level weighted scheme

$$\frac{y_{k+1} - y_{k-1}}{2\tau} + A\,(\sigma_1 y_{k+1} + (1 - \sigma_1 - \sigma_2)\,y_k + \sigma_2 y_{k-1}) = 0, \tag{16}$$

where $A > 0$. We apply A^{-1} to scheme (16) and reduce the resultant equation to the canonical form (2) with operators

$$\tilde{B} = A^{-1} + (\sigma_1 - \sigma_2)\tau E,\ \tilde{R} = {}^1/_2\,(\sigma_1 + \sigma_2)E,\ \tilde{A} = E.$$

Since $\tilde{R}$ and $\tilde{A}$ are selfadjoint constant operators, it follows from Remark 3 that scheme (16) is stable for any nonselfadjoint $A = A(t) > 0$ if $\sigma_1 \geqslant \sigma_2$, $\sigma_1 + \sigma_2 > ½$.

EXAMPLE 3. The weighted scheme

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{\tau^2} + A\,(\sigma_1 y_{k+1} + (1 - \sigma_1 - \sigma_2)\,y_k + \sigma_2 y_{k-1}) = 0, \tag{17}$$

where $A = A^* > 0$, reduces to the canonical form (2*) with operators

$$R = \frac{\sigma_1 + \sigma_2}{2} A, \qquad B = (\sigma_1 - \sigma_2)\,\tau A.$$

The conditions $R + E/\tau^2 > ¼\,A$, $B \geqslant 0$ are satisfied for $\sigma_1 \geqslant \sigma_2$, $\sigma_1 + \sigma_2 \geqslant ½$.

10. From the preceding we obtain simple rules for verifying the stability of concrete schemes: 1) reduce the difference scheme to the canonical form (1) or (2) and thereby determine the operators B, A or B, R, A; 2) introduce the space H_h of mesh functions (depending on the structure of the scheme operators) and investigate the basic properties (positiveness, selfadjointness, etc.) of the

scheme operators as operators on H_h; and 3) verify the satisfaction of the sufficient stability conditions indicated above. If the sufficient stability conditions are satisfied, the given scheme belongs to the class of stable schemes and the a priori estimates obtained for two- and three-level schemes of general form can be used.

The stability of multilevel schemes has been considered by A. V. Gulin in [38]. Here it is shown in particular that the sufficient conditions obtained in [35] for the stability of three-level schemes are also necessary in the case of constant operators. The following necessary and sufficient stability conditions are found:

$$\| Y_{n+1} \|_{(1_\rho)} \leqslant \rho^n \| Y_1 \|_{(1_\rho)}, \qquad \rho = e^{c_0 \tau},$$

where $\| \cdot \|_{(1_\rho)}$ is an analog depending on ρ of the norm (10). These conditions have the form of linear operator inequalities connecting not only A and R but also B when $\rho \neq 1$.

11. The stability theory presented above bears a constructive character and can be used not only for investigating the stability of concrete schemes but also for constructing new schemes of a prescribed quality. This possibility is connected with the fact that 1) the writing of schemes in the canonical form (1) with $B = E + \tau R$ (see (6)) or (2), (2*) permits one to distinguish the operators R (regularizers) responsible for stability, and 2) the sufficient stability conditions, $R \geqslant \sigma_0 A$ or $2R \geqslant A$ for two-level schemes and $4R > A$ for three-level schemes, impose weak restrictions on the arbitrariness in the choice of the regularizers R.

If the given scheme (1) is unstable it can always be replaced, by varying only the operator R, by a stable scheme with the same operator A.

Once we have classes of stable schemes, it is natural to seek in these classes schemes of a desired quality satisfying the additional requirements that they 1) approximate the original equation to within a certain degree, and 2) are economical. The requirement that they be economical usually means in the case of the nonstationary problems of mathematical physics that the number of arithmetic operations used to solve the difference problem must be proportional to the number of mesh points used.

Basically, the method of regularization consists in passing from an original (for example, an explicit) scheme to another scheme of a desired quality by varying the operator R (and possibly also the operators A and B).

Since the stability conditions have the form of energy inequalities, viz. $(Rx, x) \geqslant \sigma_0 (Ax, x)$ for (6) and $(Rx, x) > (Ax, x)/4$ for (2), it is natural to choose as R operators of as simple a structure as possible that are energetically

equivalent (semisimilar (see [39]) and equivalent with respect to the spectrum (see [50])) to the operator A. Suppose A and A_0 are energetically equivalent, i.e.

$$\gamma_1 A_0 \leqslant A \leqslant \gamma_2 A_0, \quad \gamma_2 > \gamma_1 > 0. \tag{18}$$

Choosing $R = \sigma A_0$, we obtain a stable scheme (6) for $\sigma \geqslant \sigma_0 \gamma_2$ or a stable scheme (2) for $\sigma > \gamma_2/4$.

It should be noted that various forms of energetically equivalent operators are used in the theory of approximate methods of solution of differential equations and systems of algebraic equations (see, for example, [49]–[53]).

We indicate some examples of the choice of a regularizer R.

1) $R = \sigma E$, where E is the identity operator, $\sigma \geqslant \sigma_0 \|A\|$ for (6) and $\sigma > \|A\|/4$ for (2).

2) $R = \sigma A_1$ or $R = \sigma A_2$, where A_1 and $A_2 = A_1^*$ are adjoint ("triangular") operators, $A_0 = A_1 + A_2$,

$$\sigma \geqslant \left(1 - \frac{2}{\tau \|A\|}\right) \gamma_2 \quad \text{for (6) and} \quad \sigma > \frac{1}{2} \gamma_2 \quad \text{for (2).}$$

3) R is chosen so that $\widetilde{B} = E + \tau \widetilde{R}$ for (1) and $\widetilde{B} + 2\tau \widetilde{R}$ for (2) are factorized operators that are representable in the form of a product of a finite number of operators of simpler structure:

$$\widetilde{B} = \prod_{\alpha=1}^{p} (E + \tau R_\alpha) \quad \text{for (1) and} \quad \widetilde{B} + 2\tau \widetilde{R} = \prod_{\alpha=1}^{p} (E + \tau R_\alpha) \quad \text{for (2).}$$

These schemes will be called factorized schemes. The following two special cases will be considered.

a) The R_α, $\alpha = 1, \cdots, p$, are positive, selfadjoint and pairwise commutative operators. Here a factorized scheme (1) is stable, for example, under the condition $R_0 = \Sigma_{\alpha=1}^{p} R_\alpha \geqslant \frac{1}{2} A$. For if scheme (1) with $B = E + \tau R_0$ is stable, the factorized scheme with $\widetilde{B} = \Pi_{\alpha=1}^{p} (E + \tau R)$ is also stable since $\widetilde{B} > B$.

b) $p = 2, R = R_1 + R_2, R_2 = R_1^*$. Here a factorized scheme is stable when $2R \geqslant A$.

The methods employed in practice to obtain stable schemes for concrete problems can be regarded as elementary examples of regularization. Thus the explicit scheme of Du Fort and Frankel [40] for the heat equation belongs to the class of stable schemes (2) with $B = E$, $R = \sigma E$ and $\sigma > \|A\|/4$, while the symmetric schemes of V. K. Saul'ev [41] belong to the class of stable schemes (6) with $R = \sigma A_1$ or $R = \sigma A_2$, $A_1^* = A_2$ and $A_1 + A_2 = A$.

These schemes are obtained from explicit schemes by means of transformations corresponding to the introduction of elementary regularizers (the identity operator or triangular operators). The economical methods of alternating directions (schemes with a splitting operator in the terminology of [42]; for the literature see [42] [48]) are based on the use of factorized schemes with the R_α being pairwise commutative or "almost commutative" (in the case of equations with variable coefficients) difference operators that correspond to elliptic operators containing derivatives only with respect to the variable x_α ("one-dimensional" operators).

The general principle of regularization permits one to obtain new absolutely stable economical factorized schemes for the basic equations of mathematical physics with variable coefficients [34]. Elliptic difference operators with constant coefficients are chosen as the regularizers R for this purpose. For example, an elliptic difference operator with a diagonal block matrix, the blocks of which are also diagonal matrices, is chosen as R in the case of a system of parabolic equations with mixed derivatives in a p-dimensional parallelepiped. A two- or three-level factorized scheme is then constructed. The process of solving the difference equations is reduced to the successive application of a standard three-point (in the case of second order equations) sweep algorithm.

Especially good opportunities for regularization are afforded by the use of three-level schemes, since in this case it is possible to preserve second order accuracy in τ (R is multiplied by τ^2). In the case of two dimensions absolutely stable factorized schemes of accuracy $O(\tau^2 + h^2)$ are obtained for parabolic equations with discontinuous coefficients when the lines of discontinuity are parallel to the coordinate axes. We note that the regularization of three-level schemes is generally carried out by varying not only the operator R but also the operator B (this is the case, for example, under certain methods of factorizing the operator $B + 2\tau R$).

To each operator A there can be put in correspondence a large number of operators of simpler structure that can be used as regularizers for two- and three-level schemes. The compliation of a catalog of regularizers and the choice of the best regularizers is an important problem of the theory of difference schemes. For various elliptic operators it is possible to use one and the same regularizer R. This makes it possible to create standard programs for solving classes of problems. In this connection the algorithm for determining the solution at a new level is not changed, but the concrete form of the operator A of the problem is taken into account in calculating the right side.

12. Until now we have considered the stability of schemes (1) and (2) relative to the right sides and the initial data. In the case of an actual computational algorithm for solving problem (1) (or (2)) the presence of rounding errors

means that one is actually finding the exact solution not of equation (1) but of an equation with perturbed operators $\widetilde{B}, \widetilde{A}$ and $\widetilde{\varphi}, \widetilde{y}_0$. Therefore the notion of stability must be widened. Clearly, by the stability of an actual scheme one should understand not only the continuous dependence of the solution on φ and y_0 but also its continuous dependence on $\|\widetilde{B} - B\|$ and $\|\widetilde{A} - A\|$. The method employed by us above for estimating the norm of the translation operator of an equivalent explicit scheme in conjunction with the energy method permits one to obtain a priori estimates expressing the (uniform in h and τ) stability (coefficient stability) of the solution of problem (1) relative to the operators of scheme (1). It has been determined that for this to be the case a scheme must have a certain "reserve of stability". In particular, scheme (1) has coefficient stability when $2R \geqslant A$.

§2. Iterative schemes

1. The theory of iterative methods for solving the equation

$$Au = f, \tag{1}$$

where A is a linear operator defined on a real Hilbert space H, is a branch of the general theory of stability of difference schemes.

Iterative schemes are written in the same canonical form as difference schemes for evolution equations.

A two-level (one-step) iterative scheme has the form

$$B_k(y_{k+1} - y_k)/\tau_{k+1} + Ay_k = f, \qquad k = 0, 1, 2, \cdots, \quad y_0 \in H_h, \tag{2}$$

where k is the iteration number, y_k is the iteration of number k, $\tau_{k+1} > 0$ is a parameter and B_k is an arbitrary operator having an inverse B_k^{-1}. If $B_k = E$, scheme (2) is explicit; if $B_k \neq E$, it is implicit. Since the solution u of equation (1) satisfies (2), the error $z_k = y_k - u$ satisfies the homogeneous equation

$$B_k \frac{z_{k+1} - z_k}{\tau_{k+1}} + Az_k = 0, \qquad k = 0, 1, 2, \ldots, \quad z_0 = y_0 - u. \tag{3}$$

The estimation of the rate of convergence of the iterations in scheme (2) reduces to the derivation of a priori estimates expressing the stability of scheme (3) with respect to the initial data.

The original family of schemes (2) is defined by the conditions

$$A = A^* > 0, \quad B_k = B_k^* > 0.$$

The implicit scheme (3) is equivalent to the explicit scheme

$$x_{k+1} = S_{k+1}x_k, \qquad k = 0, 1, \ldots, \qquad S_{k+1} = E - \tau_{k+1}C_k,$$
$$C_k = A^{\frac{1}{2}} B_k^{-1} A^{\frac{1}{2}}, \qquad x_k = A^{\frac{1}{2}} z_k, \tag{4}$$

where S_{k+1} is the translation operator. Hence

$$x_n = T_n x_0, \qquad T_n = S_n S_{n-1} \ldots S_1;$$
$$\|x_n\| \leqslant \|T_n\| \|x_0\|, \qquad \|x_k\| = \|z_k\|_A, \tag{5}$$

where T_n is the solving operator of scheme (4). Thus

$$\|x_n\| \leqslant q_n \|x_0\|, \qquad \|z_n\|_A \leqslant q_n \|z_0\|_A, \qquad \text{if} \quad \|T_n\| \leqslant q_n. \tag{6}$$

The iterations converge in H_A if $q_n \to 0$ as $n \to \infty$. The norm of the operator T_n depends on the B_k and τ_{k+1}, which should be chosen from the solution of the problem of determining $\inf\|T_n\|$ or $\inf q_n$. The basic problem of the theory of the iterative schemes (2) reduces to an estimation of the norm of the solving operator of an equivalent explicit scheme and a choice of the iterative parameters $\{\tau_{k+1}\}$ and operators $\{B_k\}$ from the minimum condition for this norm.

Only the bounds of operators or the equivalence constants of the scheme operators are used in the theory of iterative methods. All of the results obtained by spectral methods are naturally obtained by estimating the norm of the solving operator with the use of the definition of the norm of an operator function. The finite dimensionality of the space H_h is nowhere used in this connection.

2. In the case of "stationary" schemes with constants $B_{k+1} = B$ and $\tau_{k+1} = \tau$:

$$B\frac{z_{k+1} - z_k}{\tau} + Az_k = 0, \qquad k = 0, 1, 2, \ldots, \qquad z_0 = y_0 - u \in H_h, \tag{7}$$

the translation operator $S = E - \tau C$ is a constant, $I_n = S^n$ and the problem of determining $\inf\|T_n\|$ reduces to the problem of minimizing the norm of S. The solution of this problem is well known [60]; in fact, if γ_1 and γ_2 are bounds of the operator C such that

$$\gamma_1 E \leqslant C \leqslant \gamma_2 E, \qquad \gamma_2 \geqslant \gamma_1 > 0, \tag{8}$$

then $\inf\|S\|$ is achieved when $\tau = \tau_0 = 2/(\gamma_1 + \gamma_2)$ and is given by the relations

$$\inf_{\tau>0}\|S\| = \|E - \tau_0 C\| = \rho_0, \qquad \rho_0 = \frac{1-\xi}{1+\xi}, \ \xi = \frac{\gamma_1}{\gamma_2}, \qquad \tau_0 = \frac{2}{\gamma_1 + \gamma_2}. \tag{9}$$

In addition the estimate

$$\|z_n\|_D \leqslant \rho_0^n \|z_0\|_D, \qquad D = A \quad \text{or} \quad D = B, \tag{10}$$

holds for scheme (7) if

$$\gamma_1 B \leqslant A \leqslant \gamma_2 B, \tag{11}$$

since conditions (8) and (11) are equivalent when $C = A^{1/2}B^{-1}A^{1/2}$ or $C = B^{-1/2}AB^{-1/2}$ (see [36]).

The method of reducing the implicit scheme (7) to an explicit scheme permits one to prove the computational stability of scheme (2) with $\tau = \tau_0$. The choice of the operator B affects not only the number of operations needed to calculate a single iteration but also the number $n_0(\epsilon)$ of iterations, where $\epsilon > 0$ is the prescribed accuracy of the iterations. It is therefore natural to choose B from some admissible family of operators so that 1) the ratio $\xi = \gamma_1/\gamma_2$ is maximized (ρ_0 is minimized) and 2) B is an economical operator (the number of operations needed to solve the equation $By = \varphi$ for any $\varphi \in H$ is minimal in some sense, for example, with respect to an order relative to ξ for $\xi \to 0$).

In constructing B one usually starts from an operator $R = R^*$ (see [39], [49]–[53]) that is energy equivalent to A and B:

$$c_1 R \leqslant A \leqslant c_2 R, \qquad c_2 \geqslant c_1 > 0, \tag{12}$$

$$\mathring{\gamma}_1 B \leqslant R \leqslant \mathring{\gamma}_2 B, \qquad \mathring{\gamma}_2 > \mathring{\gamma}_1 > 0. \tag{13}$$

Then inequalities (11) with $\gamma_1 = c_1\mathring{\gamma}_1$ and $\gamma_2 = c_2\mathring{\gamma}_2$ are valid. We represent R in the form of a sum $R = R_1 + R_2$, where R_1 and $R_2 = R_1^*$ are disjoint ("triangular") operators and consider the factored operator

$$B = (E + \omega R_1)(E + \omega R_2), \quad B = B^* > 0, \tag{14}$$

where $\omega > 0$ is a parameter. The numbers $\mathring{\gamma}_1^0$ and $\mathring{\gamma}_2^0$ depend in this case on the parameter ω, which should be chosen so that the ratio $\xi = \gamma_1/\gamma_2 = c_1\mathring{\gamma}_1/c_2\mathring{\gamma}_2$ or, what is the same thing (since c_1 and c_2 do not depend on ω), the ratio $\mathring{\gamma}_1/\mathring{\gamma}_2 = f(\omega)$ is maximized [49]. We have

THEOREM 7. *Suppose* $R_2 = R_1^*, R = R_1 + R_2$ *and*

$$R \geqslant \delta E, \ \|R_2 x\|^2 \geqslant \frac{1}{4}\Delta(Rx, x), \qquad \Delta \geqslant \delta > 0. \tag{15}$$

Then the formulas

$$\mathring{\gamma}_1 = \frac{\delta}{2(1+\sqrt{\eta})}, \qquad \mathring{\gamma}_2 = \frac{\delta}{4\sqrt{\eta}}, \qquad \frac{\mathring{\gamma}_1}{\mathring{\gamma}_2} = \frac{2\sqrt{\eta}}{1+\sqrt{\eta}}, \qquad \eta = \frac{\delta}{\Delta}, \tag{16}$$

are valid when $\omega = 2/\sqrt{\delta\Delta}$.

Knowing $\mathring{\gamma}_1$ and $\mathring{\gamma}_2$, we find $\gamma_1 = c_1\mathring{\gamma}_1$, $\gamma_2 = c_2\mathring{\gamma}_2$ and $\tau_0 = 2/(\gamma_1 + \gamma_2)$. The following estimate of the number $n_0(\epsilon)$ of iterations as $n \to 0$ holds:

$$n_0(\varepsilon) = O\left(\frac{1}{\sqrt{\eta}}\ln\frac{1}{\varepsilon}\right).$$

In practice one often applies factorized operators of the form

$$B = (E + \omega_1 R_1)(E + \omega_2 R_2), \tag{17}$$

where R_1 and R_2 are commutative selfadjoint operators. An optimal choice of the parameters ω_1 and ω_2 from the condition that $\mathring{\gamma}_1/\mathring{\gamma}_2$ be maximized can be made without difficulty in this case.

3. We now consider the iterative schemes (2) for equation (1) in the case when $A > 0$ is a nonselfadjoint operator and

$$B = B^* > 0. \tag{18}$$

The equation for the error $z_k = y_k - u$ is equivalent to the explicit scheme

$$x_{k+1} = Sx_k, \qquad k = 0, 1, 2, \ldots, S = E - \tau C, \qquad x_0 \in H_h, \tag{19}$$

where $C = B^{-1/2}AB^{-1/2}$ and $x_k = B^{1/2}z_k$, so that $\|x_k\| = \|z_k\|_B$. Bearing in mind this connection between the explicit scheme (19) and the implicit scheme (7) for the z_k, we can confine ourselves to a study of the explicit scheme (19).

Suppose

$$C \geqslant \gamma_1 E,\ C^{-1} \geqslant \frac{1}{\gamma_2}E \quad \text{or} \quad \|Cx\|^2 \leqslant \gamma_2(Cx, x) \quad \text{for all} \quad x \in H_h. \quad (\gamma_2 > \gamma_1 > 0). \tag{20}$$

Then the estimate

$$\|S\| \leqslant \sqrt{1 - \xi}, \qquad \xi = \frac{\gamma_1}{\gamma_2} \tag{21}$$

of the norm of the translation operator holds when $\tau = 1/\gamma_2$. This estimate is rough, as can be seen from the example of scheme (7) with $B = E + \omega A$ and $\tau = 2\omega$. Suppose

$$A \geqslant \delta E, \qquad \|Ax\|^2 \leqslant \Delta(Ax, x), \qquad \Delta \geqslant \delta > 0. \tag{22}$$

The estimate

$$\|S\| \leqslant \sqrt{\frac{1 - \sqrt{\eta}}{1 + \sqrt{\eta}}},\ \eta = \frac{\delta}{\Delta},$$

was obtained in [49], whereas formula (21) implies

$$\|S\| \leqslant \sqrt{\frac{1-\eta}{1+\eta}}.$$

In order to improve the estimate it is necessary, following [51], to distinguish the symmetric and skewsymmetric parts of the operator C:

$$C = C_0 + C_1, \qquad C_0 = \frac{1}{2}(C + C^*), \qquad C_1 = \frac{1}{2}(C - C^*).$$

THEOREM 8. *Suppose*

$$\gamma_1 E \leqslant C_0 \leqslant \gamma_2 E, \ \|C_1\| \leqslant \gamma_3, \qquad \gamma_2 \geqslant \gamma_1 > 0, \qquad \gamma_3 \geqslant 0. \tag{23}$$

Then, when

$$\tau = \frac{\tau_0(1-\varkappa^2)}{1+\varkappa\rho_0}, \text{ where } \tau_0 = \frac{2}{\gamma_1+\gamma_2}, \ \varkappa = \frac{\gamma_3}{\sqrt{\gamma_1\gamma_2+\gamma_3^2}}, \ \rho_0 = \frac{1-\xi}{1+\xi}, \tag{24}$$

one has the estimate

$$\inf\|S\| \leqslant \|E - \tau C\| \leqslant \bar{\rho}, \text{ where } \bar{\rho} = \frac{\rho_0+\varkappa}{1+\varkappa\rho_0} < 1, \tag{25}$$

which when $\kappa = 0$ $(C = C^*)$ *goes over into the estimate*

$$\|S\| \leqslant \rho_0 \tag{26}$$

when $\tau = \tau_0$; *i.e. estimate* (25) *is unimprovable.*

In [51] approximate formulas have been obtained for $\bar{\tau}$ and $\bar{\rho}$ in the case when $\widetilde{H}_h$ is a finite-dimensional complex space with a scalar product and

$$C = C_0 + iC_1, \ C_0 = \operatorname{Re} C = {}^1/_2(C + C^*), \ C_1 = \operatorname{Im} C = \frac{1}{2i}(C - C^*).$$

Conditions (23) are equivalent to the conditions (cf. [50], [51])

$$\gamma_1 B \leqslant A_0 \leqslant \gamma_2 B, \qquad (B^{-1}A_1x, \ A_1x) \leqslant \gamma_3(Bx, \ x), \qquad x \in H_h, \tag{27}$$

where $A_0 = ½(A + A^*)$ and $A_1 = ½(A - A^*)$.

Suppose $R = R^* > 0$ and

$$c_1 R \leqslant A_0 \leqslant c_2 R, \qquad (R^{-1}A_1y, \ A_1y) \leqslant c_3^2(Ry, \ y), \qquad c_2 \geqslant c_1 > 0. \tag{28}$$

If $\mathring{\gamma}_1 B \leqslant R \leqslant \mathring{\gamma}_2 B$, then inequalities (27) hold with $y_i = c_i\mathring{\gamma}_i$, $i = 1, 2, 3$, $c_3 > 0$. The factorized operator (14) can be taken as B.

4. The application of the iterative methods described above assumes a knowledge of the constants $\gamma_1, \gamma_2, \gamma_3$ and c_1, c_2. In those cases when these constants are known inexactly or in general are unknown a priori it is expedient to make use of a variational iterative method such as the method of steepest

descent or the method of minimal residuals.

We first note that any iterative method

$$B\frac{y_{k+1}-y}{\tau_{k+1}}+Ay_k=f, \qquad k=0,1,\ldots, \quad y_0\in H_h, \tag{29}$$

can be interpreted as a method of corrections:

$$y_{k+1}=y_k-\tau_{k+1}w_k, \qquad w_k=B^{-1}r_k, \qquad r_k=Ay_k-f, \tag{30}$$

where r_k is a residual and w_k is a correction.

If $A=A^*>0$ and $B=B^*>0$, we can calculate τ_{k+1} by making use of the following formula from the method of steepest descent:

$$\tau_{k+1}=\frac{(w_k,\ r_k)}{(Aw_k,w_k)}\,. \tag{31}$$

This method converges in H_A at the same rate as scheme (29) when $\tau=\tau_0$ is constant (see [59]).

If $A>0$ is a nonselfadjoint operator, we apply the method of minimal corrections. The calculations are carried out according to formulas (30) with

$$\tau_{k+1}=\frac{(Aw_k,\ w_k)}{(B^{-1}Aw_k,\ Aw_k)}\,. \tag{32}$$

The equation $Au=f$ is equivalent to the equation

$$Cv=\varphi,\ v=B^{\frac{1}{2}}u,\ C=B^{-\frac{1}{2}}AB^{-\frac{1}{2}},\ \varphi=B^{-\frac{1}{2}}f, \tag{33}$$

which can be solved by the method of minimal residuals proposed by M. A. Krasnosel'skiĭ and S. G. Kreĭn [60]:

$$x_{k+1}=x_k-\tau_{k+1}\tilde{r}_k, \qquad \tilde{r}_k=Cx_k-\varphi, \qquad \tau_{k+1}=\frac{(\tilde{r}_k,\ C\tilde{r}_k)}{\|C\tilde{r}_k\|^2}\,. \tag{34}$$

By taking (33) into account, we can readily obtain (30) and (32) from (34). It therefore suffices to carry out all of the arguments for (34). If $C=C^*>0$ and $\gamma_1E\leqslant C\leqslant\gamma_2E$, the following estimate holds for (34):

$$\|Cx_n-\varphi\|\leqslant\rho_0^n\|Cx_0-\varphi\|, \qquad \rho_0=\frac{1-\xi}{1+\xi}, \qquad \xi=\frac{\gamma_1}{\gamma_2}\,. \tag{35}$$

In the case of a nonselfadjoint operator C we have

THEOREM 9. *Suppose C is a nonselfadjoint operator, $C=C_0+C_1$ and conditions* (23) *are satisfied. Then the following estimate holds for* (34):

$$\|Cx_n-\varphi\|\leqslant\bar{\rho}^n\|Cx_0-\varphi\|, \tag{36}$$

where $\bar{\rho}=(\rho_0+\kappa)/(1+\kappa\rho_0)$.(1)

(1) The case of a selfadjoint $C=C^*$ has been studied in [60].

The proof of this theorem makes use of the following

LEMMA. *Suppose C is a nonselfadjoint operator with bounds γ_1 and $\gamma_2 > \gamma_1 > 0$, so that $\gamma_1 E \leqslant C \leqslant \gamma_2 E$, and for some $\tau_* > 0$*

$$\|E - \tau_* C\| \leqslant \rho_*, \quad \textit{where} \quad \rho_* < 1. \tag{37}$$

Then, if

$$\gamma_1 \leqslant \frac{1-\rho_*^2}{\tau_*} \leqslant \gamma_2, \tag{38}$$

the inequality

$$(Cx, x)^2 \geqslant (1-\rho_*^2)\|Cx\|^2\|x\|^2 \tag{39}$$

holds for all $x \in H$.

PROOF. Conditions (38) imply

$$\|x\|^2 - 2\tau_*(Cx, x) + \tau_*^2\|Cx\|^2 \leqslant \rho^2\|x\|^2,$$
$$\tau_*^2\|Cx\|^2 \leqslant 2\tau_*(Cx, x) - (1-\rho_*^2)\|x\|^2$$

and

$$\|Cx\|^2 \leqslant \frac{2}{\tau_*}\frac{(Cx, x)}{\|x\|^2}\left[\frac{\|x\|^2}{(Cx, x)} - q\frac{\|x\|^4}{(Cx, x)^2}\right], \tag{40}$$

where $q = (1-\rho_*^2)/2\tau_*$.

Let us consider the function $\varphi(\alpha) = \alpha - q\alpha^2$, where $\alpha = \|x\|^2/(Cx, x) \in [1/\gamma_2, 1/\gamma_1]$, and find its maximum. The point $\alpha_0 = q/2$ is contained in the closed interval $[1/\gamma_2, 1/\gamma_1]$. Therefore $\max \varphi(\alpha) = \varphi(\alpha_0) = q/4$, and consequently

$$\frac{2}{\tau_*}\varphi(\alpha) \leqslant \frac{1}{2\tau_* q} = \frac{1}{1-\rho_*^2}.$$

But this together with (40) implies (39).

We proceed to the proof of Theorem 9. From (34) we find that

$$\tilde{r}_{k+1} = \tilde{r}_k - \tau_{k+1}C\tilde{r}_k;$$

$$\|\tilde{r}_{k+1}\|^2 = \|\tilde{r}_k\|^2 - 2\tau_{k+1}(C\tilde{r}_k, \tilde{r}_k) + \tau_{k+1}^2\|C\tilde{r}_k\|^2 = \left[1 - \frac{(C\tilde{r}_k, \tilde{r}_k)^2}{\|\tilde{r}_k\|^2\|C\tilde{r}_k\|^2}\right]\|\tilde{r}_k\|^2.$$

By virtue of the lemma the expression in brackets does not exceed $1 - (1-\rho_*^2) = \rho_*^2$, i.e.

$$\|\tilde{r}_{k+1}\| \leqslant \rho_*\|\tilde{r}_k\|, \quad \|\tilde{r}_n\| \leqslant \rho_*^n\|\tilde{r}_0\|.$$

But $\rho_* = \bar{\rho}$ when $\tau_* = \bar{\tau}$, which implies (36). The theorem is proved.

REMARK. If the conditions $C \geqslant \gamma_1 E$ and $C^{-1} \geqslant \gamma_2^{-1} E$ are satisfied, the estimate

$$\|Cx_n - \varphi\| \leqslant \tilde{\rho}^n \|Cx_0 - \varphi\|,$$

where $\tilde{\rho} = \sqrt{1 - \xi}$ and $\xi = \gamma_1/\gamma_2$, holds in place of (36).

The variational iterative methods with a factorized operator B have been applied by a number of authors (see, for example, [54]–[56]).

In the case of a nonselfadjoint operator A the factorized operator (14) can be chosen as B. The computational use of formulas (31) and (32) does not require a knowledge of the constants c_1, c_2 and c_3.

5. The operator B is sometimes given in explicit form and is sometimes constructed as a result of applying some (intrinsic) direct or iterative method. An example is provided by the so-called two-stage method (see [50], [50a], [51], [58]), which we formulate as the method of correctness. The correction w_k is calculated by solving the equation

$$Rw = r_k, \quad r_k = Ay_k - f \tag{41}$$

either by a direct method (in which case $B = R$) or by an iterative method with solving operator T_m, $\|T_m\| \leqslant q < 1$, under the zero initial conditions: $w^{(0)} = 0$. It is determined as the mth iteration: $w^{(m)} = w_k$, so that $w - w_k = T_m w$, where w is the exact solution of equation (41). Substituting $w = (E - T_m)^{-1} w_k$ into (41), we get $Bw_k = r_k$, where $B = R(E - T_m)^{-1}$. If $T_m = T_m^*$ and T_m and R are commutative, we get $B = B^* > 0$ and $\mathring{\gamma}_1 = 1 - q$, $\mathring{\gamma}_2 = 1 + q$ (see [50], [51]). The iteration y_{k+1} can be calculated by using scheme (30) with a constant parameter $\tau = \tau_0$ when $A = A^* > 0$, and with $\tau = \bar{\tau}$ when $A \neq A^*$, $A = A_0 + A_1$ (see (24)) if the constants c_1, c_2 and c_3 are known, so that

$$\gamma_1 = (1 - q)c_1, \ \gamma_2 = (1 + q)c_2, \ \gamma_3 = (1 + q)c_3.$$

If the constanta c_1, c_2 and c_3 are not known or are defined inexactly, it is expedient to use the two-stage method of steepest descent (τ_{k+1} is calculated by formula (31)) when $A = A^*$. As formula (32) shows, in the case of a nonselfadjoint operator $A \neq A^*$ the two-stage method of minimal corrections requires the successive solution with the use of the inner iterations T_m of the two equations

$$\begin{aligned} Rw &= r_k, \quad w^{(0)} = 0, \quad w_k = w^{(m)}, \\ Rv &= Aw_k, \quad v^{(0)} = 0, \quad v_k = v^{(m)}. \end{aligned} \tag{42}$$

The correction $w_k = w^{(m)}$ is first found, after which the second equation is solved and $v_k = B^{-1}Aw_k$ is determined as the mth iteration: $v^{(m)} = v_k$. Here $B = R(E - T_m)^{-1}$. Knowing w_k and $B^{-1}Aw_k$, we can calculate the parameter τ_{k+1} by formula (32). Theorem 9 implies the validity of estimate (36), in which one should put

$$\gamma_1 = c_1(1 - q), \quad \gamma_2 = c_2(1 + q), \quad \gamma_3 = c_3(1 + q), \quad c_3 > 0.$$

The requirement that R and T_m be commutative is a very strong restriction and substantially contracts the domain of applicability of the two-stage methods.

6. We now consider three-level (two-step) iterative schemes for solving the equation $Au = f$, where $A = A^* > 0$. A stationary iterative scheme can be written in the canonical form

$$B\left[\frac{y_{k+1} - y_{k-1}}{2\tau} + \varkappa(y_{k+1} - 2y_k + y_{k-1})\right] + Ay_k = f, \qquad k = 1, 2, \ldots, \tag{43}$$

in which $y_0 \in H_h$ and $y_1 \in H_h$ are arbitrary given vectors. Here $\tau > 0$ and $\kappa > 0$ are iterative parameters.

Alternatively the first approximation y_1 can be calculated by using the two-level scheme

$$B\frac{y_1 - y_0}{\tau_0} + Ay_0 = f, \qquad \tau_0 = \frac{2}{\gamma_1 + \gamma_2}. \tag{44}$$

For $z_k = y_k - u$, where u is the solution of problem (1), we have from (43) and (44) the scheme

$$B\left[\frac{z_{k+1} - z_{k-1}}{2\tau} + \varkappa(z_{k+1} - 2z_k + z_{k-1})\right] + Az_k = 0, \quad k = 1, 2, \ldots,$$
$$z_1 = y_1 - u \in H_h, \qquad z_0 = y_0 - u \in H_h. \tag{43*}$$

If y_1 is determined from (44), then

$$B\frac{z_1 - z_0}{\tau_0} + Az_0 = 0, \quad \tau_0 = 2/(\gamma_1 + \gamma_2), \quad z_0 \in H_h. \tag{44*}$$

We will assume that

$$\left.\begin{aligned} &A = A^* > 0, && B = B^* > 0, \\ &\gamma_1 B \leqslant A \leqslant \gamma_2 B, && \gamma_2 \geqslant \gamma_1 > 0. \end{aligned}\right\} \tag{45}$$

The optimal values of the parameters τ and κ can be obtained from the general theory of stability of three-level schemes (2) in §1 with $R = \kappa B$.

THEOREM 10. *The solution of problem* (43*) *when* $\tau = \tau_1$, $\kappa = \kappa_1$, *where*

$$\tau_1 = \frac{1}{\sqrt{\gamma_1\gamma_2}}, \qquad \varkappa_1 = \frac{1}{4}(\gamma_1 + \gamma_2), \tag{46}$$

satisfies the estimate

$$\| z_{n+1} \|_{(1_{\rho_1})} \leqslant \rho_1^n \| z_1 \|_{(1_{\rho_1})}, \tag{47}$$

where

$$\rho_1 = \frac{1 - \sqrt{\xi}}{1 + \sqrt{\xi}}, \qquad \xi = \frac{\gamma_1}{\gamma_2},$$

$$\| z_{n+1} \|^2_{(1_{\rho_1})} = \frac{1}{4}((A - \gamma_1 B)(z_{n+1} + \rho_1 z_n),\ z_{n+1} + \rho_1 z_n)$$
$$+ \frac{1}{4}((\gamma_2 B - A)(z_{n+1} - \rho_1 z_n),\ z_{n+1} - \rho_1 z_n).$$

This same theorem also holds for problem (43*)–(44*). We note that $\|z_{n+1}\|_{(\rho_1)}$ is generally a seminorm by virtue of formula (45).

If $B = E$, scheme (43) is said to be explicit. The explicit three-level scheme (43)–(44) was considered in [52]–[57] and [3], [41]. It was written in the form (for the equation $Cv = \varphi$)

$$\left.\begin{array}{lll} x_{k+1} = (1+\alpha)(E - \tau_0 C)x_k - \alpha x_{k-1} + (1+\alpha)\tau_0\varphi, & & k = 1, 2, \ldots \\ x_1 = (E - \tau_0 C)x_0 + \tau_0\varphi, & \gamma_1 E \leqslant C \leqslant \gamma_2 E, & C = C^*, \\ \gamma_1 > \gamma_2 > 0, & \tau_0 = 2/(\gamma_1 + \gamma_2). & \end{array}\right\} \tag{48}$$

The following estimate was obtained in [52] for $\alpha = \rho_1^2$:

$$\| x_n - v \| \leqslant q_n \| x_0 - v \|, \quad n = 1, 2, \ldots, \quad q_n = \rho_1^n \left(1 + \frac{1 - \rho_1^2}{1 + \rho_1^2}\, n\right). \tag{49}$$

Here v is a solution of the equation $Cv = \varphi$.

Reducing (48) to the canonical form (43), we get

$$\tau = \frac{1+\alpha}{1-\alpha}\tau_0 = \frac{1}{\sqrt{\gamma_1\gamma_2}} = \tau_1, \quad \varkappa = \frac{1}{2\tau_0} = \frac{1}{4}(\gamma_1 + \gamma_2) = \varkappa_1 \quad (\alpha = \rho_1^2),$$

i.e. the same values of τ_1 and κ_1 as in Theorem 10. But the rate of convergence in the metric of H_h is worse.

The implicit scheme (43) can be reduced to the explicit scheme (48) with

$$x_n = A^{\frac{1}{2}} y_n, \qquad C = A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}}, \qquad \varphi = A^{\frac{1}{2}} B^{-1} f$$

or with

$$x_n = B^{\frac{1}{2}} y_n, \qquad C = B^{\frac{1}{2}} A B^{-\frac{1}{2}}, \qquad \varphi = B^{-\frac{1}{2}} f.$$

For C we have $\gamma_1 E \leqslant C \leqslant \gamma_2 E$. Using now estimate (49), for $\tau = \tau_1$ and $\kappa = \kappa_1$ we get that either

$$1) \qquad \|z_n\|_D \leqslant q_n \|z_0\|_D, \qquad D = A \quad \text{or} \quad D = B, \tag{50}$$

if y_1 is calculated according to the two-level scheme (44) with $\tau_0 = 2/(\gamma_1 + \gamma_2)$, or

$$2) \qquad \|z_n\|_D \leqslant (\rho_0 q_{n+1} + q_n)\|z_0\|_D + q_{n-1}\|z_1\|_D, \tag{51}$$

if y_1 is an arbitrary vector, $z_0 = y_0 - u$ and $z_1 = y_1 - u$.

7. The factorized operator (14) can be taken as the operator B. When $c_1 = c_2 = 1$ $(R = A)$ Theorem 8 implies

$$\xi = \frac{\gamma_1}{\gamma_2} = \frac{2\sqrt{\eta}}{1+\sqrt{\eta}}, \qquad \rho_1 = \frac{1-\sqrt{\xi}}{1+\sqrt{\xi}} = \frac{\sqrt{1+\sqrt{\eta}} - \sqrt{2\sqrt{\eta}}}{\sqrt{1+\sqrt{\eta}} + \sqrt{2\sqrt{\eta}}}$$

The number $n_0(\epsilon)$ of iterations satisfies the following asymptotic relation as $\eta \to 0$:

$$n_0(\varepsilon) \approx \frac{1}{2\sqrt{2}\sqrt[4]{\eta}} \ln \frac{1}{\varepsilon}.$$

In the case of a two-level scheme with the same operator B

$$n_0(\varepsilon) \approx \frac{1}{4\sqrt{\eta}} \ln \frac{1}{\varepsilon}.$$

If A is the Laplace difference operator, then $\eta = O(h^2)$ and

$$n_0(\varepsilon) = O\left(\frac{1}{\sqrt{h}} \ln \frac{1}{\varepsilon}\right)$$

for a three-level scheme with the factorized operator (14).

The indicated scheme is applicable for the Dirichlet difference problem in an arbitrary domain of any dimension.

Any three-level scheme can be formulated as a method of corrections:

$$y_{k+1} = (1+\alpha) y_k - \alpha y_k - \theta w_k, \quad \alpha = \rho_1^2, \quad \theta = \frac{1+\alpha}{2\varkappa_1} = \tau_0(1+\alpha),$$

$$\varkappa_1 = \frac{\gamma_1 + \gamma_2}{4}, \qquad k = 1, 2, \ldots,$$

where $w_k = B^{-1} r_k$ is a correction and $r_k = A y_k - f$ is a residual.

Let us formulate a two-stage three-level method. The correction w_k is found as a result of solving the equation

$$Rw = r_k$$

by the iterative method with solving operator T_m, $\|T_m\| \leqslant q < 1$, under the zero initial approximation: $w^{(0)} = 0$, so that $w_k = w^{(m)}$ and $B = R(E - T_m)^{-1}$. The number of iterations in this case satisfies the estimate

$$n_0(\varepsilon) = O\left(\frac{1}{\sqrt{\xi}} \ln \frac{1}{\varepsilon}\right), \tag{52}$$

where

$$\xi = \frac{c_1}{c_2} \frac{1-q}{1+q}.$$

In [51] a two-stage method was employed for an equation of elliptic type with variable coefficients. An alternating direction scheme was chosen as T_m. It was noted that the number of iterations was less than in the case of a two-stage two-level scheme. As can be seen from (52), an application of three-level schemes permits one to weaken the dependence of the rate of convergence on the ratio c_1/c_2, which is very important in the case of an elliptic equation with strongly varying coefficients.

A two-stage three-level variational iterative scheme was considered in [58].

§3. Total approximation method

1. In §§1 and 2 our main attention was directed to a study of the stability of difference schemes. A second important characteristic of a difference scheme, which establishes a connection between it and the original differential equation, is the error of the approximation. The sense in which the given scheme approximates the original problem governs 1) the choice of the method of investigating the accuracy of the scheme and 2) the type of a priori estimates expressing the stability with respect to the right side.

In the course of developing the theory of difference schemes a review was made of the approximation criterion. Thus, by studying the rate of convergence of homogeneous difference schemes in the class of discontinuous coefficients, it was determined [61], [62] that the local error of the approximation (or of the approximation in the mesh norm C or L_2) is not an adequate index of the accuracy of a scheme. A scheme was constructed (for the equation $(ku')' - qu = -f(x)$) which on an arbitrary nonuniform net does not approximate the differential equation at any point but has second order accuracy [63].

The error of the approximation should be understood in some integral or

total sense, i.e. one should estimate the error of the approximation in negative (weak) norms that take into account its indefinite (with respect to sign) or divergent character (see [61]–[66]). Let us explain this by an example. Suppose given a difference equation $Ay = \varphi$, where $A \in (H_h \to H_h)$, H_h is a Hilbert space and $A = A^* > 0$. The following exact estimate holds for it: $\|y\|_A = \|\varphi\|_{A^{-1}}$; i.e. the solution y in H_A can be expressed in terms of the right side φ in $H_{A^{-1}}$. If, say, $A \geqslant \gamma A_0$, $\gamma > 0$, and $A_0 = A_0^* > 0$, we obtain instead of an exact equality the estimate $\|y\|_{A_0} \leqslant \|\varphi\|_{A_0^{-1}}/\gamma$. Suppose H_h is the space of mesh functions defined on $\omega_h = \{x_i = ih,\ i = 0, 1, \cdots, N;\ h = 1/N\}$ and equal to zero for $i = 0, N$, and suppose $A_0 y = -y_{xx}$. Then

$$\|y\|_{A_0} = \left(\sum_{i=1}^{N} (y_{\bar{x},i})^2 h\right)^{1/2} \text{ is the analog of the norm in } \overset{\circ}{W}{}_2^1,$$

$$\|\varphi\|_{A_0^{-1}} = \left[\sum_{i=1}^{N-1} h \left(\sum_{k=i+1}^{N-1} h\varphi_k\right)^2\right]^{1/2} \text{ is the analog of the norm in } \overset{\circ}{W}{}_2^{-1}.$$

For nonstationary difference schemes, as was indicated in §1, the solution in H_A can be estimated in terms of $\|\varphi\|_A + \|\varphi_{\bar{t}}\|_{A^{-1}}$. A weakening of the approximation requirements (foresaking local approximation of a desired order) permitted us to substantially extend the domain of application of homogeneous difference schemes.

A priori estimates for concrete two- and three-level schemes having the property of total approximation of elliptic operators can be found in [26], [29], [64].

2. The notion of approximation played an important role in the development of the theory of economical methods of solving nonstationary problems of mathematical physics for the equations

$$\begin{aligned} &\frac{\partial u}{\partial t} = Lu + f(x, t), \qquad 0 < t < t_0, \\ &\frac{\partial^2 u}{\partial t^2} = Lu + f(x, t), \end{aligned} \tag{1}$$

where L is an elliptic operator and $x = (x_1, \cdots, x_p)$ is a point of a p-dimensional domain $\bar{G} = G + \Gamma$ with boundary Γ, as well as for the corresponding systems of equations (when u and f are vectors).

Let $\bar{\omega}_h = \{x_i\}$ be a net in the domain $\bar{G}$ and let $\bar{\omega}_\tau = \{t_j = j\tau\}$ be a net on the segment $0 \leqslant t \leqslant t_0$.

By an economical scheme one usually understands an unconditionally stable scheme such that the number of operations required to determine the solution is

proportional to the number $\omega_{h\tau} = \bar{\omega}_h \times c\bar{\omega}_\tau$ of mesh points used. This means that $O(N)$ operations, where N is the number ω_h of mesh points, are required to pass from the jth level to the $(j+1)$th level; in other words, there must be $O(1)$ operations at a single mesh point. The basic algorithmic idea of all of the economical methods consists in the writing of difference operations such that the process of solving them reduces to the successive application of standard algorithms (for example, the one-dimensional sweep algorithm) with the expenditure of $O(N)$ operations.

All of the two level (using for the determination of y^{j+1} only the value of y^j at the preceding level) economical methods can be written in the form

$$B_\alpha y^{j+\alpha/p} = \sum_{\beta=0}^{\alpha-1} C_{\alpha\beta} y^{j+\beta/p} + F^{j+\alpha/p}, \qquad \alpha = 1, 2, \ldots, p, \tag{2}$$

where the $y^{j+\alpha/p}$, $\alpha = 1, \cdots, p-1$, are intermediate values and the B_α and $C_{\alpha\beta}$ are linear difference operators acting on y as a function of $x \in \omega_h$, while each of the equations $B_\alpha y^{j+\alpha/p} = \varphi_\alpha$ with a known right side can be solved with the expenditure of $O(N)$ operations (the operators B_α are called economical).

The first economical methods for solving the heat equation in the case when G is a rectangle were proposed by the American mathematicians Peaceman, Rachford and Douglas [67]–[69].

Various economical methods have subsequently been considered for typical problems of mathematical physics by Baker and Oliphant [72], Douglas and Gunn [48], Saul'ev [41], Bagrinovskiĭ and Godunov [70], Janenko [71], [73], D'jakonov [42], Frjazinov [81]–[83], Andreev [84], Hubbard [94, [95] and others.

The economical methods are referred to by various names, for example, the alternating-direction implicit method [67]–[69], the decomposition method [71], the partial step method [73], the splitting operator method [42], the locally one-dimensional method [75], additive schemes [76] and the total approximation method. This terminology, while possibly transient and reflecting the individual approaches of the various authors to the construction and interpretation of the structure of the economical methods, affords a view of the various characteristic features of these methods (many of which coincide with respect to the algebraic structure).

From the point of the general theory it is expedient to differentiate the economical methods on the basis of the method of investigating them and, in particular, the notion of approximation. It should be emphasized in this connection that in most cases the method of investigation also determines principles

for the construction of economical schemes. The following two approaches are used for the investigation of economical algorithms.

1) The method of eliminating the intermediate values $y^{j+\alpha/p}$, $\alpha = 1, \cdots, p-1$, and reducing system (2) to an equivalent scheme "in whole steps"

$$B\frac{y^{j+1}-y^j}{\tau} + Ay^j = \varphi^j \tag{3}$$

with a factorized operator $B = B_1 \cdots B_p$. The properties of stability and approximation of an economical method (2) are verified for the factorized scheme (3). From this point of view the system (2) is interpreted as a method of realizing a factorized scheme.

2) The total approximation method, on which we dwell at length in this section.

The first approach, viz. the replacement of system (2) by an equivalent factorized scheme, has been applied in many papers ([67]–[69], [71], [73] and others). One can obviously start from a factorized scheme that is stable and approximates a multidimensional differential equation and solve the difference problem by using an algorithm of form (2) with economical operators B_α (see [42], [72], [74], [45] and others).

A general method for constructing stable factorized schemes was indicated in §1 (see [34]). Economical factorized schemes can be obtained by choosing various economical difference operators, depending on the actual problem, as the R_α.

The requirement that problems (2) and (3) be equivalent can be satisfied with the use of a special method of assigning 1) boundary conditions for the intermediate values $y^{j+\alpha/p}$, $\alpha = 1, \cdots, p-1$, as well as 2) the right sides $F^{j+\alpha/p}$. This was first pointed out by D'jakonov [88] (see also [76], [77]). Also, the elimination (without inverting the operators B_α) of the $y^{j+\alpha/p}$, $\alpha = 1, \cdots, p-1$, requires in a number of cases the pairwise commutativity of the operators B_α and $C_{\alpha\beta}$. We note that in a number of papers (see [67]–[69]) the economical algorithms are written so that the intermediate values can be eliminated without imposing additional restrictions on the scheme operators.

Finally, a factorized scheme (2) with operator $B = B_1 \cdots B_p$ is stable only under the condition of commutativity or "almost commutativity" of the selfadjoint operators $\{B_\alpha\}$.

A very interesting economical method (the splitting method [71]) has been proposed by Janenko for the multidimensional heat equation

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} L_\alpha u, \quad u = u(x, t), \quad x = (x_1, \ldots, x_p), \quad L_\alpha u = \frac{\partial^2 u}{\partial x_\alpha^2}. \tag{4}$$

The natural multidimensional weighted scheme

$$\frac{y^{j+1}-y^j}{\tau}=\Lambda(\sigma y^{j+1}+(1-\sigma)y^j),\qquad \Lambda=\sum_{\alpha=1}^{p}\Lambda_\alpha,\ \Lambda_\alpha\sim L_\alpha \tag{5}$$

is replaced by the system of homogeneous difference equations

$$\frac{y^{j+\alpha/p}-y^{j+\frac{\alpha-1}{p}}}{\tau}=\Lambda_\alpha\left(\sigma y^{j+\alpha/p}+(1-\sigma)\,y^{j+\frac{\alpha-1}{p}}\right),\qquad \alpha=1,2,\ldots,p, \tag{6}$$

or

$$B_\alpha y^{j+\alpha/p}=C_\alpha y^{j+(\alpha-1)/p},\quad B_\alpha=E-\sigma\tau\Lambda_\alpha,\quad C_\alpha=E+(1-\sigma)\tau\Lambda_\alpha. \tag{7}$$

The case when L_α is an arbitrary differential operator containing derivatives only with respect to x_α (the fractional step method) has been subsequently considered in [73].

System (6) can be reduced by the elimination method to the factorized scheme

$$\prod_{\alpha=1}^{p}B_\alpha y^{j+1}=\prod_{\alpha+1}^{p}C_\alpha y^j, \tag{8}$$

which does not coincide with the original scheme (5) and approximates equation (4).

The requirement that the factorized scheme (8) and system (6) be equivalent leads to the same difficulties as those discussed above. In particular, the passage from (6) to (8) is possible if the operators Λ_α are commutative (for (6) this means that G is a parallelepiped). Difficulties have also arisen with the assignment of right sides for equations (6) in the case of an inhomogeneous equation (4).

In all of the papers [67]–[73], [42]–[44] only domains of a special form (G is a p-dimensional parallelepiped) were considered. Among them it is obvious that the algorithms of [68], [69], [71] could also be used (upon formulating them more precisely) in the case of a domain G of more complicated form. It has turned out that the restrictions on the form of G connected with 1) the notion of approximation for the methods of [68], [71] and 2) the requirement of equivalence of (2) and (3) can be removed by introducing a new notion of scheme (the additive scheme) involving an approximation of the differential equation in a weaker sense (in the total sense) [75]. The renunciation of the classical notion of approximation and its replacement by the weaker condition of total approximation have substantially broadened the opportunities for constructing economical schemes and have permitted one to obtain economical additive schemes for a significantly wider class of linear and nonlinear problems of mathematical physics.

We proceed to a formulation of the notion of total approximation.

3. Let H_h be a normed linear space, let $\omega_i = \{t_j = j\tau, j = 0, 1, \cdots, j_0\}$ be a net with mesh width τ on the interval $0 \leqslant t \leqslant t_0$, and let $C_{\alpha\beta}$, $D_{\alpha\beta}, A_{\alpha\beta}, B$, etc. be linear operators from H_h into H_h that depend on h, τ and possibly $t \in \overline{\omega}_\tau$.

An n-level difference scheme was defined in §1 as a difference (with respect to the variable t) equation of $(n-1)$th order:

$$\sum_{\beta=0}^{n-1} C_\beta(t_j)\, y(t_j - \beta\tau) = f(t_j), \qquad (n-1)\tau \leqslant t_j \leqslant t_0, \tag{9}$$

with operator coefficients and the $n-1$ initial conditions

$$y(0) = y_0\,,\ y(\tau) = y_1, \ldots, \qquad y((n-2)\tau) = y_{n-2}.$$

We introduce a wider class of schemes.

An n-level composite scheme with period m (of order m) is a system of difference equations

$$\sum_{\beta=1}^{m} C_{\alpha\beta}(t_j)\, y(t_j + \beta\tau) = \sum_{\beta=0}^{n-2} D_{\alpha\beta}(t_j)\, y(t_j - \beta\tau) + f_\alpha(t_j), \tag{10}$$

where $\alpha = 1, \cdots, m$ and $(n-1)\tau \leqslant t_j \leqslant t_0$, with operator coefficients and the given initial values $y(k\tau)$, $k = 0, 1, \cdots, n-2$ (the number of levels is determined by the number of initial conditions). Here t_j takes the values

$$t_j = (n-1)\tau + km\tau, \quad k = 0, 1, \ldots, \quad j = n - 1 + km.$$

In order to find $y(t_j + m\tau) = y_{j+m}$, where $t_j = (m+n-1)\tau$, it is necessary to solve a system of m equations with the operator matrix $C = (C_{\alpha\beta})$ of order $m \times m$.

When $m = 1$ the composite scheme (10) goes over into the ordinary n-level scheme (9). When $n = 2$ we obtain a two-level composite scheme with period m (which we denote by $S(2, m)$):

$$\begin{gathered}\sum_{\beta=1}^{m} C_{\alpha\beta}(t_j)\, y(t_j + \beta\tau) = D_{\alpha 0} y(t_j) + f_\alpha(t_j), \\ \alpha = 1, 2, \ldots, m, \qquad y(0) = y_0.\end{gathered} \tag{11}$$

It is convenient for what follows to introduce the notstion $y^{j+\alpha/m} = y_{j+\alpha}$ and to replace τ by τ/m.

Scheme (11) can always be written in the canonical form

$$B\frac{y^{j+\alpha/m}-y^{j+(\alpha-1)/m}}{\tau}+\sum_{\beta=0}^{m}A_{\alpha\beta}y^{j+\beta/m}=\varphi^{j+\alpha/m},\qquad \alpha=1,\ldots,m,$$
$$j=0,1,\ldots,\quad y(0)=y_0. \tag{12}$$

It will be assumed below that scheme (12) is solvable and that the inverse operator B^{-1} exists. The stability of a composite scheme is defined by analogy with §1. We require, in particular, the following definition of stability. We will say that scheme (12) is stable if the a priori estimate

$$\|y^{j+1}\|_{(1)}\leqslant M_1\|y_0\|_{(1_0)}+M_2\max_{0\leqslant j\leqslant j}\sum_{\alpha=1}^{m}\|\varphi^{j+\alpha/m}\|_{(2)} \tag{13}$$

is satisfied for any y_0 and $\varphi^{j+\alpha/m}$, where the positive constants M_1 and M_2 do not depend on h, τ or the choice of y_0 and $\varphi^{j+\alpha/m}$, and $\|\cdot\|_{(1)}=\|\cdot\|_{(1_h)}$ and $\|\cdot\|_{(2)}=\|\cdot\|_{(2_h)}$ are certain norms on H_h.

4. In order to introduce the notions of accuracy and approximation for $S(2, m)$ it is necessary to consider a Banach space H_0 of the solutions $u=u(t)$ of the original continuous problem (cf. [49]). Suppose there exists a linear operator P_h from H_0 into H_h such that $u_h=P_hu\in H_h$ if $u\in H_0$ and the norm compatibility condition

$$\lim_{|h|\to 0}\|P_hu\|_{(1_h)}=\|u\|_0$$

is satisfied, where $\|\cdot\|_0$ is the norm in H_0. Let $\{y_h^j\}$ be the solution of problem (12) and let $u(t)$, $t\in[0, t_0]$, be the (continuous) solution of the original problem. The nearness of $y_h^j=y_h(t_j)$ to $u(t_j)$ is characterized by the quantity $\|z_h^j\|$, where $z_h^j=y_h^j-u_h^j$. Substituting $y_h^{j+\alpha/m}=z_h^{j+\alpha/m}+u_h^{j+\alpha/m}=u_h(t_j+\alpha/m\tau)$, into (12), we get for $z_h^{j+\alpha/m}$ the same system of equations (12) with right sides $\psi^{j+\alpha/m}=\psi_\alpha^j$, where ψ_α^j is the error of approximation at the equation of index α of (12) at the solution u.

A composite scheme (12) whose error of approximation is defined as the sum

$$\psi=\psi_1+\psi_2+\ldots+\psi_m \tag{14}$$

of the errors of approximation for the individual equations will be called an *additive scheme* and denoted by $AS(2, m)$. An additive scheme $AS(2, m)$ approximates the original continuous problem if

$$\max_{t_j \in \omega_\tau} \| \psi^j \|_{(2_h)} \to 0 \quad \text{for} \quad |h| \to 0, \quad \tau \to 0. \tag{15}$$

We represent ψ_α in the form of a sum

$$\psi_\alpha = \mathring{\psi}_\alpha + \psi_\alpha^* \quad \text{so that} \quad \sum_{\alpha=1}^{m} \mathring{\psi}_\alpha = 0. \tag{16}$$

The total approximation condition (15) will be satisfied if

$$\max_{t_j \in \omega_\tau} \| \psi_\alpha^*(t_j) \|_{(2_h)} \to 0 \quad \text{for} \quad |h| \to 0, \ \tau \to 0 \ \text{and} \ \alpha = 1, 2, \ldots, m. \tag{17}$$

We establish a connection between the properties of stability, approximation and convergence for $AS(2, m)$.

THEOREM 11. *Suppose an additive scheme has the properties of stability and approximation, the following "smoothness" condition for the solution* $u = u(t)$ *is satisfied*:

$$\sum_{\alpha=1}^{m=1} \Big\| \sum_{\beta=1}^{m} A_{\alpha\beta} \Big(\sum_{k=\beta+1}^{m} B^{-1} \mathring{\psi}_k \Big) \Big\|_{(2)} \leqslant M_0, \tag{18}$$

where $M_0 = \text{const} > 0$ *does not depend on* τ *or* h, *and* $y_0 = u_h(0)$. *Then the scheme converges and the following estimate holds for it*:

$$\| y_h^{j+1} - u_h^{j+1} \|_{(1)}$$

$$\leqslant M_2 \max_{0 \leqslant j' \leqslant j} \Big[\sum_{\alpha=1}^{m} \| \psi_\alpha^*(t_{j'}) \|_{(2)} + \tau \sum_{\alpha=1}^{m-1} \Big\| \sum_{\beta=1}^{m} A_{\alpha\beta} \Big(\sum_{k=\beta+1}^{m} B^{-1} \mathring{\psi}_k(t_{j'}) \Big) \Big\|_{(2)} \Big].$$

Effective a priori estimates can be obtained under weak restrictions on the scheme operators in the case when H_h is a real Hilbert space.

THEOREM 12. *Suppose* $A_{\alpha 0} = 0$, B *is a constant operator and the following conditions are satisfied*:

$$B = B^* > 0, \tag{20}$$

$$\sum_{\alpha, \beta=1}^{m} (A_{\alpha\beta} \xi_\beta, \xi_\alpha) \geqslant 0 \quad \textit{for any} \quad \xi_\alpha \in H_h, \ \alpha = 1, 2, \ldots, m. \tag{21}$$

Then the following estimate holds for the solution of problem (12):

$$\| y^{j+1} \|_B \leqslant \| y_0 \|_B + M_2 \max_{0 \leqslant j' \leqslant j} \Big[\Big\| \sum_{\alpha=1}^{m} \varphi^{j'+\alpha/m} \Big\|_{B^{-1}} + \sqrt{\tau} \sum_{\alpha=1}^{m} \| \varphi^{j'+\alpha/m} \|_{B^{-1}} \Big], \tag{22}$$

where

$$\|y\|_B = \sqrt{(By, y)}, \qquad \|\varphi\|_{B^{-1}} = \sqrt{(B^{-1}\varphi, \varphi)}.$$

COROLLARY. *An additive scheme* (12) *converges in* H_B *if* $y_0 = u_h(0)$ *and the conditions of Theorem* 12 *and the following total approximation conditions are satisfied*

$$\max_{0\leqslant j<j_0} \Big\| \sum_{\alpha=1}^{m} \psi^{j+\alpha/m} \Big\|_{B^{-1}} \to 0 \quad \textit{for} \quad |h| \to 0, \tau \to 0 \quad (\psi^{j+\alpha/m} = \psi_\alpha^j),$$

$$\max_{0\leqslant j<j_0} \sum_{\alpha=1}^{m} \|\psi^{j+\alpha/m}\|_{B^{-1}} \leqslant M_0,$$

where the positive constant M_0 *does not depend on* τ *or* h.

REMARKS. 1) If

$$y_0 = u_h(0), \quad \Big\| \sum_{\alpha=1}^{m} \psi_\alpha \Big\|_{B^{-1}} = O(|h|^l + \tau^k),$$

where $k > 0, l > 0$, and the conditions of Theorem 12 are satisfied, then $AS(2, m)$ converges in H_B at the rate of $O(|h|^l + \tau^{k_1})$, where $k_1 = \min(k, 1/2)$.

2) If, in addition, the smoothness condition (18) is satisfied with $\|\cdot\|_{(2)} = \|\cdot\|_{B^{-1}}$, then $AS(2, m)$ converges at the rate of $O(|h|^l + \tau^{k_2})$, where $k_2 = \min(k, 1)$.

5. The process of solving problem (12) is simplified if $A = (A_{\alpha\beta})$ is a lower triangular matrix ($A_{\alpha\beta} = 0$ for $\beta > \alpha$), so that

$$B\frac{y^{j+\alpha/m} - y^{j+(\alpha-1)/m}}{\tau} + \sum_{\beta=0}^{\alpha} A_{\alpha\beta} y^{j+\frac{\beta}{m}} = \varphi^{j+\frac{\alpha}{m}}. \tag{23}$$

The passage from the jth level to the $(j+1)$th level is effected by successively (from α to $\alpha+1$) solving the equations

$$B_\alpha y^{j+\alpha/m} = \Phi_\alpha, \qquad B_\alpha = B + \tau A_{\alpha\alpha},$$

$$\Phi_\alpha = By^{j+\frac{\alpha-1}{m}} - \tau \sum_{\beta=0}^{\alpha-1} A_{\alpha\beta} y^{j+\beta/m} + \tau\varphi^{j+\frac{\alpha}{m}}$$

for $\alpha = 1, \cdots, m$, which can be written in the form

$$y^{j+\alpha/m} = \sum_{\beta=0}^{\alpha-1} S_{\alpha\beta} y^{j+\beta/m} + \tau F_\alpha, \qquad \alpha = 1, 2, \ldots, m. \tag{24}$$

In this case $AS(2, m)$ has a lower triangular translation operator matrix $S = (S_{\alpha\beta})$. These $AS(2, m)$ with a triangular matrix $(S_{\alpha\beta})$ or $(A_{\alpha\beta})$ will be called triangular additive schemes.

It is not difficult to check that all of the economical algorithms of form (23) that correspond to two-level factorized schemes and can be interpreted as composite schemes belong to the family of triangular additive schemes and have the total approximation property. We indicate some special cases of triangular additive schemes (cf. [68], [71]):

$$A_{\alpha\beta} = A_\alpha \delta_{\alpha\beta}, \qquad \delta_{\alpha\beta} = \begin{cases} 1, \alpha = \beta, \\ 0, \alpha \neq \beta. \end{cases} \tag{25}$$

$$\begin{aligned} A_{\alpha\beta} &= \sigma A_{\alpha\beta}\delta_{\alpha\beta} + (1 - \sigma) A_\alpha \delta_{\alpha-1,\beta}, \\ A_{\alpha\beta} &= \frac{1}{2} A_\alpha \delta_{\alpha\beta} + \frac{1}{2} A_{\alpha-1}\delta_{\alpha-1,\beta}. \end{aligned} \tag{26}$$

For scheme (25), for example, condition (21) of Theorem 12 is satisfied if $(A_\alpha y, y) \geqslant 0$ for all $y \in H_h$, $\alpha = 1, \cdots, m$.

Additive schemes with a diagonal matrix $(A_{\alpha\beta})$ will be called *locally one dimensional.*

6. A basic question is the following: How can one construct additive difference schemes of a given order of accuracy?

The following method for constructing additive schemes, that guarantees total approximation, was proposed in [75].

Consider the equation

$$\begin{gathered} \frac{\partial u}{\partial t} = Lu + f(x, t), \\ x = (x_1, \ldots, x_p) \in G, \quad t \in [0, t_0], \quad u(x, 0) = u_0(x). \end{gathered} \tag{27}$$

Suppose the operator L acting on u as a function of x can be represented in the form of a sum of operators L_α of simpler structure (for example, "one-dimensional" operators, i.e. operators containing derivatives only with respect to x_α):

$$L = \sum_{\alpha=1}^{p} L_\alpha. \tag{28}$$

We represent f in the form of a sum

$$f = \sum_{\alpha=1}^{p} f_\alpha(x, t) \tag{29}$$

and rewrite equation (27) in the form

$$\sum_{\alpha=1}^{p} P_\alpha u = 0, \tag{30}$$

where $P_\alpha u = p^{-1}\partial u/\partial t - L_\alpha u - f_\alpha$.

We introduce on the segment $0 \leqslant t \leqslant t_0$ the net $\overline{\omega}_\tau = \{t_j = j\tau, j = 0, 1, \cdots, j_0\}$ with mesh width τ. Each mesh is divided into p equal parts by introducing the points $t_{j+\alpha/p} = (j + \alpha/p)\tau$, $\alpha = 1, \cdots, p-1$.

Instead of equation (30) we will solve on $[t_j, t_{j+1}]$ the system of equations

$$P_\alpha v_{(\alpha)} = 0 \quad \text{for} \quad t \in \Delta_\alpha, \text{ where } \Delta_\alpha = \left(t_{j+\frac{\alpha-1}{p}} < t \leqslant t_{j+\frac{\alpha}{p}}\right), \tag{31}$$

$$\alpha = 1, 2, \ldots, p,$$

which are connected by the conditions

$$v_{(\alpha)}\left(t_{j+\frac{\alpha-1}{p}}\right) = v_{(\alpha-1)}\left(t_{j+\frac{\alpha-1}{p}}\right), \qquad \alpha = 2, 3, \ldots, p, \qquad v_{(1)}(t_j) = v(t_j),$$

where $v(t_j) = v_{(p)}(t_j)$ and $v(0) = u(0) = u_0$.

Each of the equations $P_\alpha v_{(\alpha)} = 0$ is solved on its own interval Δ_α.

If each of the equations $P_\alpha v_{(\alpha)} = 0$ is approximated in the ordinary sense by a (two-level) difference scheme with mesh width τ/p, we obtain the following system of difference equations for determining y^{j+1}, given y^j:

$$\Pi_\alpha y_{(\alpha)} = 0, \qquad \alpha = 1, 2, \ldots, p, \qquad y_{(p)} = y^{j+1}. \tag{32}$$

We show that the composite scheme $\Pi_1 \to \Pi_2 \to \cdots \to \Pi_p$ approximates equation (30) in the total sense if each of the schemes (32) approximates the corresponding equation of (31) in the ordinary sense, i.e. if at any sufficiently smooth function the quantity

$$\Psi_\alpha = \Pi_\alpha u_h^{j+\alpha/p} - (P_\alpha u)_h^{j+\alpha/p} \tag{33}$$

tends (in some norm) to zero as $\tau \to 0$, $|h| \to 0$.

The error of approximation at the solution $u = u(x, t)$ of equation (30) for the scheme Π_α is obviously equal to

$$\psi_\alpha = \Pi_\alpha u_h^{j+\alpha/p} - (P_\alpha u)_h^{j+\alpha/p} + \Psi_\alpha.$$

Taking into account that $(P_\alpha, u)^{j+\alpha/p} = (P_\alpha, u)^{j+1/2} + O(\tau)$, we get $\psi_\alpha =$

$\mathring{\psi}_\alpha + \psi_\alpha^*$, where $\psi_\alpha^* = \Psi_\alpha + O(\tau)$ and $\mathring{\psi}_\alpha = (P_\alpha u)^{j+1/2}$.

Hence by virtue of equation (30) $\Sigma_1^p \mathring{\psi}_\alpha = 0$, and the total error of approximation for the additive scheme (32)

$$\psi = \sum_{\alpha=1}^{p} \psi_\alpha = \sum_{\alpha=1}^{p} \psi_\alpha^* \tag{34}$$

tends to zero as $|h| \to 0$ and $\tau \to 0$, since the ψ_α^* have this property.

Thus the additive scheme (32) approximates equation (27) (in the total sense) if each of the schemes (32) approximates the corresponding equation of (31) in the ordinary sense (on a net with mesh width τ/p).

This follows from the fact that the system of differential equations (31) approximates the multidimensional equation (27) in the total (integral) sense.

In fact, the error of approximation for the equation $P_\alpha v_{(\alpha)} = 0$ at the solution $u = u(x, t)$ of equation (27) is the residual $\Psi_\alpha(t) = P_\alpha u(t)$, where $t \in \Delta_\alpha$. Since $P_\alpha u = (P_\alpha u)^{j+1/2} + O(\tau)$ for $t \in [t_j, t_{j+1}]$, we get $\Psi_\alpha = \mathring{\Psi}_\alpha + \Psi_\alpha^*$, where $\mathring{\Psi}_\alpha = (P_\alpha u)^{j+1/2}$ and $\Psi_\alpha^* = O(\tau)$. Hence

$$\sum_{\alpha=1}^{p} \mathring{\Psi}_\alpha = 0, \quad \Psi = \sum_{\alpha=1}^{p} \Psi_\alpha = \sum_{\alpha=1}^{p} \Psi_\alpha^* = O(\tau),$$

i.e. the additive system of differential equations (31) approximates equation (27) with first order in τ.

Clearly the total error of approximation for system (31) can be determined by analogy with (34) as

$$\Psi = \sum_{\alpha=1}^{p} \frac{p}{\tau} \int_{t_{j+\frac{\alpha-1}{p}}}^{t_{j+\alpha/p}} \Psi_\alpha \, dt.$$

It is not difficult to note that the analogs of the above arguments remain valid, so that

$$\Psi = \sum_{\alpha=1}^{p} \frac{p}{\tau} \int_{t_{j+\frac{\alpha-1}{p}}}^{t_{j+\alpha/p}} \Psi_\alpha^* \, dt = O(\tau).$$

From the stability of system (31) and the total approximation we obtain the convergence of the solution of problem (31) to $u(x, t)$.

It should be emphasized that the total approximation for (32) and (31) at sufficiently smooth solutions of problem (27) is guaranteed by the satisfaction of conditions (28) and (29): the operator L is the sum $L_1 + \cdots + L_p = L$ and the right side f is the sum $f_1 + \cdots + f_p = f$.

The question of the nearness of the solutions of problems (27) and (31) has been studied by Janenko [89]. He considered the Cauchy problem in the half-space $|x| < \infty, t > 0$ for the system of equations

$$\frac{\partial u}{\partial t}(x, t) = L(x, t, D)u + f(x, t), \qquad u(x, 0) = u_0(x), \tag{35}$$

where $u(x, t)$ and $f(x, t)$ are vector functions of a vector argument, and $L(x, t, D)$ is a linear differential operator whose coefficients depend on x and t. It was assumed that L is representable in the form (28) and the Cauchy problem (35) was replaced by the composite Cauchy problem (31) in the particular case $f_\alpha = f/p$. By making use of the property of total approximation implied by condition (28) and interpreting it as the property of weak approximation of the coefficients of the differential equation the author proved that

$$\|v(x, t) - u(x, t)\| = O(\tau)$$

(under the condition of sufficient smoothness of $u(x, t)$).

7. The technique indicated above of constructing additive schemes with a guaranteed approximation has subsequently been used to obtain economical schemes for many of the linear and nonlinear problems of mathematical physics (see [75], [76], [78], [45], [90], [91], [92]).

The total approximation method has permitted the extension of modified algorithms [68], [71] to the case of an arbitrary domain as well as the determination of a number of new homogeneous economical additive schemes for the linear and quasilinear equations and systems of equations of mathematical physics.

In this regard it became apparent in connection with a study of equations (27) with operators L_α depending on t that one must alter the composite system of differential equations approximating a multidimensional equation.

In [79] the following abstract Cauchy problem was considered in a Banach space B:

$$\frac{du}{dt} + \mathcal{A}(t)u = f(t), \qquad 0 \leqslant t \leqslant t_0, \qquad u(0) = u_0,\ u_0 \in B, \tag{36}$$

where $\mathcal{A}(t)$ is a linear operator with an everywhere dense in B domain of definition that is representable in the form of a sum

$$\mathcal{A}(t) = \sum_{\alpha=1}^{p} \mathcal{A}_\alpha(t), \tag{37}$$

and $u(t)$ and $f(t)$ are abstract functions of $t \in [0, t_0]$ with values in B.

The uniform net $\overline{\omega}_\tau = \{t_j = j\tau, j = 0, 1, \cdots, j_0\}$ with mesh width τ is introduced on the segment $0 \leqslant t \leqslant t_0$. Problem (36) is replaced by the system of Cauchy problems

$$\begin{aligned}
&\frac{dv_1}{dt}+\mathcal{A}_1 v_{(1)}=f_1(t), \qquad t\in[t_j,t_{j+1}], \qquad v_{(1)}(t_j)=v(t_j),\\
&\frac{dv_{(2)}}{dt}+\mathcal{A}_2 v_{(2)}=f_2(t), \qquad t\in[t_j,t_{j+1}], \qquad v_{(2)}(t_j)=v_{(1)}(t_{j+1}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{dv_{(\alpha)}}{dt}+\mathcal{A}_\alpha v_{(\alpha)}=f_\alpha(t), \qquad t\in[t_j,t_{j+1}], \qquad v_{(\alpha)}(t_j)=v_{(\alpha-1)}(t_{j+1}),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{dv_{(p)}}{dt}+\mathcal{A}_p v_{(p)}=f_p(t), \qquad t\in[t_j,t_{j+1}], \qquad v_{(p)}(t_j)=v_{(p-1)}(t_{j+1})
\end{aligned} \tag{38}$$

and it is required that

$$v(t_{j+1})=v_{(p)}(t_{j+1}), \qquad j=0,1,\ldots,j_0-1; \qquad v(0)=u_0.$$

The question of the proximity of the solutions of problems (36) and (38) was investigated. We indicate the main results.

1) If $A_\alpha(t')$ and $A_\beta(t'')$, $\alpha\neq\beta$, $\alpha,\beta=1,\cdots,p$, are commutative: $A_\alpha(t')A_\beta(t'')=A_\beta(t'')A_\alpha(t')$ for any $t',t''\in[0,t_0]$, then when $f=0$ the equality

$$v(t_j)=u(t_j) \tag{39}$$

holds for all $j=0,1,\cdots,j_0$.

If $f\neq 0$ it is possible to select f_p when $f_1=f_2=\cdots=f_{p-1}=0$ in such a way that (39) will be satisfied.

2). If A_α and A_β are noncommutative and the solution $u(t)$ of problem (36) satisfies a "smoothness" condition of the form

$$\|\mathcal{A}_\alpha\mathcal{A}_\beta u\|\leqslant M,$$

where $\|\cdot\|$ is the norm in B and $M=\text{const}>0$, then under condition (29)

$$\|v(t_j)-u(t_j)\|=O(\tau) \tag{40}$$

for all $j=1,\cdots,j_0$.

The following question arises: Is it possible to construct a system of partial Cauchy problems such that

$$\|v^j-u^j\|=O(\tau^2). \tag{41}$$

We write by convention the composite Cauchy problem (38) in the form of a chain: $A_1\to A_2\to\cdots\to A_p$. Consider now the symmetric chain

$$\frac{1}{2}\mathcal{A}_1\to\ldots\to\frac{1}{2}\mathcal{A}_{p-1}\to\frac{1}{2}\mathcal{A}_p\to\frac{1}{2}\mathcal{A}_p\to\frac{1}{2}\mathcal{A}_{p-1}\to\ldots\to\frac{1}{2}\mathcal{A}_1$$

or

$$\frac{1}{2}\mathcal{A}_1 \to \dots \to \frac{1}{2}\mathcal{A}_{p-1} \to \mathcal{A}_p \to \frac{1}{2}\mathcal{A}_{p-1} \to \dots \to \frac{1}{2}\mathcal{A}_1.$$

But this composite Cauchy problem (under an appropriate choice of the f_α and some additional "smoothness" conditions on u_0 and f) is such that estimate (41) is valid (see [72]).

The idea of symmetrization was developed by I. V. Frjazinov, who constructed and investigated a number of symmetrized additive schemes for equations of parabolic type in graduated domains composed of p-dimensional parallelepipeds (see [83]). Another symmetrization method has been applied to obtain economical schemes in the case of the equations of acoustics [93].

Let us show that the chain of Cauchy problems (38) approximates problem (36) in the total sense.

We put $z_{(\alpha)}(t) = v_{(\alpha)}(t) - u^{j+1}$ for $\alpha = 2, 3, \cdots, p$, $z_{(\alpha)}(t) = v_{(\alpha)}(t) - u(t)$, and write the equations for the $z_{(\alpha)}$:

$$\frac{dz_{(\alpha)}}{dt} + \mathcal{A}_\alpha z_{(\alpha)} = \psi_\alpha, \qquad t_j \leqslant t \leqslant t_{j+1}, \qquad \alpha = 1, 2, \dots, p,$$

$$z^j_{(\alpha)} = z^{j+1}_{(\alpha-1)} \quad \text{for } \alpha = 2, 3, \dots, p, \quad z^j_{(1)} = z^j, \ z^{j+1}_{(p)} = z^{j+1}, \ z(0) = 0.$$

The right side ψ_α is the error of approximation of equation (36) by the equation of index α of (38) in the class of solutions $u = u(t)$. Clearly, $\psi_\alpha(t) = f_\alpha(t) - A_\alpha(t)u^{j+1}$ for $\alpha > 1$ and $\psi_1(t) = f_1(t) - A_1(t)u(t) - du/dt$. We represent ψ_α in the form $\psi_\alpha(t) = \psi_\alpha^{j+1} + O(\tau)$ or $\psi_\alpha = \mathring{\psi}_\alpha + \psi_\alpha^*$, where $\mathring{\psi}_\alpha = (f_\alpha - A_\alpha u - \delta_{\alpha,1} du/dt)^{j+1}$, where $\delta_{\alpha,1}$ is the Kronecker delta. If $f_1 + \cdots + f_p = f$, then

$$\sum_{\alpha=1}^{p} \mathring{\psi}_\alpha = 0 \quad \text{and} \quad \left\| \sum_{\alpha=1}^{p} \psi_\alpha \right\| = \left\| \sum_{\alpha=1}^{p} \psi_\alpha^* \right\| = O(\tau),$$

i.e. problem (38) approximates problem (36) in the total sense.

The main difficulty in constructing $O(\tau^2)$ additive schemes is the assignment of boundary conditions for the intermediate values $y^{j+\alpha/p}$ (this problem does not arise under an abstract formulation); in order to satisfy the requirement of $O(\tau^2)$ total approximation it is necessary to introduce corrections to the natural boundary values.

The method described above for constructing economical additive schemes with a preliminary construction of the composite differential Cauchy problem (38)

and a suitable choice of the operators A_α followed by a difference approximation of each of the equations (38) is also very practical and is commonly used as a heuristic technique for obtaining economical additive schemes (see, for example, [72], [83], [55]). Let us cite an example.

Consider the case when $A = \Sigma_{\alpha,\beta=1}^{p} A_{\alpha\beta}$ and the matrix $(A_{\alpha\beta})$ is symmetric. We represent $(A_{\alpha\beta})$ in the form of a sum of two triangular matrices:

$$(\mathcal{A}_{\alpha\beta}) = (\mathcal{A}^-_{\alpha\beta}) + (\mathcal{A}^+_{\alpha\beta}), \qquad \mathcal{A}^-_{\alpha\beta} = 0 \quad \text{for } \beta > \alpha, \quad \mathcal{A}^+_{\alpha\beta} = 0 \quad \text{for } \beta < \alpha,$$

$$\mathcal{A}^-_{\alpha\alpha} = \mathcal{A}^+_{\alpha\alpha} = \frac{1}{2}\mathcal{A}_{\alpha\alpha}, \qquad \mathcal{A}_{\alpha\beta} = \mathcal{A}^-_{\alpha\beta} + \mathcal{A}^+_{\alpha\beta}$$

and introduce the operators

$$\mathcal{A}_\alpha = \sum_{\beta=1}^{\alpha} \mathcal{A}^-_{\alpha\beta} \qquad \text{for } \alpha = 1, 2, \ldots, p,$$

$$\mathcal{A}_\alpha = \sum_{\beta=\alpha}^{2p} \mathcal{A}^+_{2p+1-\alpha,\, 2p+1-\beta} \quad \text{for } \alpha = p+1, \ldots, 2p, \qquad \mathcal{A} = \sum_{\alpha=1}^{2p} \mathcal{A}_\alpha.$$

Following this we construct the chain $A_1 \to A_2 \to \cdots \to A_{2p}$ of problems (38) and approximate each operator $A_{\alpha\beta}$ by a difference operator $A_{\alpha\beta}$ depending on $h(u\tau)$ and defined in a normed linear space B_h. As a result, we obtain the following triangular additive scheme:

$$\frac{y^{j+\alpha/2p} - y^{j+(\alpha-1)/2p}}{\tau} + \tilde{A}_{\alpha\alpha} y^{j+\alpha/2p} + \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{2p} \tilde{A}_{\alpha\beta} y^{j+(\alpha-1)/2p} = \varphi^{j+\alpha/2p}, \qquad \alpha = 1, 2, \ldots, 2p,$$

where $\tilde{A}_{\alpha\beta} = A^-_{\alpha\beta}$ for $\alpha = 1, \cdots, p$ and $\tilde{A}_{\alpha\beta} = A^+_{2p+1-\alpha,2p+1-\beta}$ for $\alpha = p+1, \cdots, 2p$. If B_h is a Hilbert space and the matrix $A = (A_{\alpha\beta})$ is nonnegative, then the conditions of Theorem 12 are satisfied and the a priori estimate (22) holds for the triangular additive scheme.

BIBLIOGRAPHY

1. V. S. Rjaben'kiĭ, *On the application of the method of finite differences to the solution of Cauchy's problem*, Dokl. Akad. Nauk SSSR 86 (1952), 1071–1074. (Russian) MR 14, 477.

2. A. F. Filippov, *On stability of difference equations*, Dokl. Akad. Nauk SSSR 100 (1955), 1045–1048. (Russian) MR 16, 829.

3. V. S. Rjaben'kiĭ and A. F. Filippov, *On the stability of difference equations*, GITTL, Moscow, 1946; German transl., Mathematik für Naturwiss. und Technik, Band 3, VEB Deutscher Verlag, Berlin, 1960. MR 19, 865; 23 #A437.

4. P. D. Lax and R. D. Richtmyer, *Survey of the stability of linear finite difference equations*, Comm. Pure Appl. Math. 9 (1956), 267–293. MR 18, 48.

5. R. D. Richtmyer, *Difference methods for initial-value problems*, Interscience Tracts in Pure and Appl. Math., no. 4, Interscience, New York, 1957; Russian transl., IL, Moscow, 1960. MR 20 #438; 22 #3864.

6. P. D. Lax and B. Wendroff, *On the stability of difference schemes*, Comm. Pure Appl. Math. 15 (1962), 363–371. MR 27 #4375.

7. P. D. Lax and L. Nirenberg, *On stability for difference schemes: A sharp form of Gårding's inequality*, Comm. Pure Appl. Math. 19 (1965), 473–492; Russian transl., Matematika 11 (1967), no. 6, 3–20. MR 34 #6352.

8. H. O. Kreiss, *Über die Lösung des Cauchyproblem für lineare partielle Differentialgleichungen mit Hilfe von Differenzengleichungen. I. Stable Systeme von Differnzengleichungen* Acta Math. 101 (1959), 179–199. MR 24 #A335.

9. ———, *Über die approximative Lösung von linearen partiellen Differentialgleichungen mit Hilfe von Differenzengleichungen*, Kungl. Tekn. Högsk. Handl. Stockholm 1958, no. 128; Russian transl., Matematika 7 (1963), no. 2, 57–66.

10. ———, *Über die Stabilitätsdefinition für Differenzengleichungen die partielle Differentialgleichungen approximieren*, Nordisk. Tidskr. Informationsbehandling (BIT) 2 (1962 153 181. MR 29 #2992.

11. ———, *On difference approximations of the dissipative type for hyperbolic differential equations*, Comm. Pure Appl. Math. 17 (1964), 335–353. MR 29 #4210.

12. M. L. Buchanan, *A necessary and sufficient condition for stability of difference schemes for initial value problems*, J. Soc. Indust. Appl. Math. 11 (1963), 919–935. MR 28 #3537.

13. K. W. Morton and S. Schecter, *On the stability of finite difference matrices*, J. Soc. Indust. Appl. Math. Numer. Anal. Ser. B 2 (1965), 119–128. MR 31 #6393.

14. V. Thomée, *On maximum-norm stable difference operators*, Numerical Solution of Partial Differential Equations (Proc. Sympos. Univ. Maryland, 1965), Academic Press, New York, 1966, pp. 125–151. MR 35 #1225.

15. ———, *Parabolic difference operators*, Math. Scand. 19 (1966), 77–107. MR 35 #590.

16. ———, *Generally unconditionally stable difference operators*, SIAM J. Numer. Anal. 4 (1967), 55–69. MR 35 #3916.

17. S. I. Serdjukova, *On stability in the uniform metric of systems of difference equations*, Ž. Vyčisl. Mat. i Mat. Fiz. 7 (1967), 497–509 = USSR Comput. Math. and Math. Phys. 7 (1967), no. 3, 30–47. MR 35 #7023.

18. M. V. Fedorjuk, *On stability in C of the Cauchy problem for difference equations and partial differential equations*, Ž. Vyčisl. Mat. i Mat. Fiz. 7 (1967), 510–540 = USSR Comput. Math. and Math. Phys. 7 (1967), no. 3, 48–89. MR 35 #5147.

19. S. K. Godunov and V. S. Rjaben'kiĭ, *Introduction to the theory of difference schemes*, Fizmatgiz, Moscow, 1962; English transl., North-Holland, Amsterdam; Interscience, New York, 1964. MR 29 #724; 31 #5346.

20. ———, *Canonical forms of systems of linear ordinary difference equations with constant coefficients*, Ž. Vyčisl. Mat. i Mat. Fiz. 3 (1963), 211–222 = USSR Comput. Math. and Math. Phys. 3 (1963), 281–295. MR 27 #5054.

21. ———, *Spectral criteria for the stability of boundary-value problems for nonselfadjoint difference equations*, Uspehi Mat. Nauk 18 (1963), no. 3 (111), 3–14 = Math. Surveys 18 (1963), no. 3, 1–12. MR 28 #4261.

22. A. M. Il'in, *Stability of difference schemes for the Cauchy problem for systems of partial differential equations*, Dokl. Akad. Nauk SSSR 164 (1965), 491–494 = Soviet Math. Dokl. 6 (1965), 1252–1255. MR 32 #4864.

23. L. I. Jakut, *The problem of establishing the convergence of difference schemes*, Dokl. Akad. Nauk SSSR 151 (1963), 76–79 = Soviet Math. Dokl. 4 (1963), 954–958. MR 27 #5375.

24. S. G. Krein, *Linear differential equations in Banach space*, "Nauka", Moscow, 1967; English transl., Transl. Math. Monographs, vol. 29, Amer. Math. Soc., Providence, R. I., 1972. MR 40 #508.

25. O. A. Ladyženskaja, *The mixed problem for a hyperbolic equation*, GITTL, Moscow, 1953. (Russian) MR 17, 160.

26. V. I. Lebedev, *On the mesh method for a certain system of partial differential equations*, Izv. Akad. Nauk SSSR Ser. Mat. 22 (1958), 717–734. (Russian) MR 20 #5938.

27. M. Lees, *A priori estimates for the solutions of difference approximations to parabolic partial differential equations*, Duke Math. J. 27 (1960), 297–311. MR 22 #12725.

28. ———, *Energy inequalities for the solution of differential equations*, Trans. Amer. Math. Soc. 94 (1960), 58–73. MR 22 #4875.

29. A. A. Samarskiĭ, *A priori bounds for difference equations*, Ž. Vyčisl. Mat. i Mat. Fiz. 1 (1961), 972–1000 = USSR Comput. Math. and Math. Phys. 1 (1961), 1138–1167. MR 26 #5746.

30. E. G. D'jakonov, *Difference schemes with splitting operator for general second-order parabolic equations with variable coefficients*, Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 278–291 = USSR Comput. Math. and Math. Phys. 4 (1964), no. 2, 92–110. MR 31 #1776.

31. S. L. Sobolev, *Applications of functional analysis in mathematical physics*, Izdat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1962; English transl., Transl. Math. Monographs, Vol. 7, Amer. Math. Soc., Providence, R. I., 1963. MR 29 #2624.

32. R. Courant, K. Friedrichs and H. Lewy, *Über die partiellen Differentialgleichungen der mathematischen Physik*, Math. Ann. 100 (1928), 32–74; Russian transl., Uspehi Mat. Nauk 8 (1941), 125–160; English transl., IBM J. Res. Develop. 11 (1967), 215–234. MR 35 #4621.

33. A. A. Samarskiĭ, *A contribution to the theory of difference schemes*, Dokl. Akad. Nauk SSSR 165 (1965), 1007–1010 = Soviet Math. Dokl. 6 (1965), 1558–1561. MR 34 #3815.

34. ———, *Regularization of difference schemes*, Ž. Vyčisl. Mat. i Mat. Fiz. 7 (1967), 62–93 = USSR Comput. Math. and Math. Phys. 7 (1967), no. 1, 79–120. MR 35 #3930.

35. ———, *Classes of stable schemes*, Ž. Vyčisl. Mat. i Mat. Fiz. 7 (1967), 1096–1133 = USSR Comput. Math. and Math. Phys. 1 (1967), no. 5, 171–223. MR 36 #4844.

36. ———, *Necessary and sufficient conditions for stability of two-level difference schemes*, Dokl. Akad. Nauk SSSR 181 (1968), 808–811 = Soviet Math. Dokl. 9 (1968), 946–949. MR 38 #554.

37. A. V. Gulin and A. A. Samarskiĭ, *Stability of difference schemes in complex Hilbert space*, Dokl. Akad. Nauk SSSR 181 (1968), 1042–1045 = Soviet Math. Dokl. 9 (1968), 966–969. MR 38 #555.

38. A. V. Gulin, *Necessary and sufficient conditions for the stability of triple level difference schemes* Ž. Vyčisl. Mat. i Mat. Fiz. 8 (1968), 899–902 = USSR Comput. Math. and Math. Phys. 8 (1968), no. 4, 278–284. MR 38 #1840.

39. S. G. Mihlin, *The numerical performance of variational methods*, "Nauka", Moscow, 1966; English transl., Wolters-Noordhoff, Groningen, 1971. MR 31 #3747; 43 #4236.

40. E. C. Du Fort and S. P. Frankel, *Stability conditions in the numerical treatment of parabolic differential equations*, Math. Tables and Other Aids to Computation 4 (1953), 135–152. MR 15, 474.

41. V. K. Saul'ev, *Integration of equations of parabolic type by the method of nets*, Fizmatgiz, Moscow, 1960; English transl., Internat. Ser. Monographs Pure Appl. Math., vol. 54, Pergamon Press, New York, 1964. MR 23 #A428; 33 #6153.

42. E. G. D'jakonov, *Difference schemes with splitting operator for higher-dimensional non-stationary problems*, Ž. Vyčisl. Mat. i Mat. Fiz. 2 (1962), 549–568 = USSR Comput. Math. and Math. Phys. 2 (1962), 581–607. MR 34 #3802.

43. ———, *Difference schemes of second order accuracy with a splitting operator for solving many-dimensional parabolic equations with variable coefficients*, Comput. Methods and Programming (Comput. Center Moscow Univ. Collect. Works III), Izdat. Moskov. Univ., Moscow, 1965, pp. 163–190. (Russian) MR 33 #891.

44. ———, *Efficient difference methods, based on splitting the difference operator, for certain systems of partial differential equations*, Comput. Methods Programming, VI, Izdat. Moscov. Univ., Moscow, 1967, pp. 76–120. (Russian) MR 35 #7598.

45. N. N. Janenko, *The method of fractional steps for solving multidimensional problems of mathematical physics*, "Nauka", Sibirsk. Otdel., Novosibirsk, 1967; English transl., Springer-Verlag, Berlin and New York, 1971. MR 36 #4815; 46 #6613.

46. A. A. Samarskiĭ, *Local one-dimensional difference schemes on nonuniform nets*, Ž. Vyčisl. Mat. i Mat. Fiz. 3 (1963), 431–466 = USSR Comput. Math. and Math. Phys. 3 (1963), 572–619. MR 36 #6165.

47. ———, *An economic algorithm for the numerical solution of systems of differential equations*, Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 580–585 = USSR Comput. Math. and Math. Phys. 4 (1964), no. 3, 263–271. MR 32 #610.

48. J. Douglas and J. E. Gunn, *A general formulation of alternating direction methods. I. Parabolic and hyperbolic problems*, Numer. Math. 6 (1964), 428–453. MR 31 #894.

49. A. A. Samarskiĭ, *Certain questions of the theory of difference schemes*, Ž. Vyčisl. Mat. i Mat. Fiz. 6 (1966), 665–686 = USSR Comput. Math. and Math. Phys. 6 (1966), no. 4, 74–102. MR 34 #7050.

50. E. G. D'jakonov, *On constructing iterative methods on the basis of the use of operators equivalent with respect to the spectrum*, Ž. Vyčisl. Mat. i Mat. Fiz. 6 (1966), 12–34 = USSR Comput. Math. and Math. Phys. 6 (1966), no. 1, 14–46. MR 33 #6807.

50a. ———, *An iteration method for solving systems of finite difference equations*, Dokl. Akad. Nauk SSSR 138 (1961), 522–525 = Soviet Math. Dokl. 2 (1961), 647–650. MR 25 #767.

51. J. E. Gunn, *The solution of elliptic difference equations by semi-explicit iterative techniques*, J. Soc. Indust. Appl. Math. Ser. B Numer. Anal. 2 (1965), 24–45. MR 31 #4199.

52. D. K. Faddeev and V. N. Faddeeva, *Computational methods in linear algebra*, Fizmatgiz, Moscow, 1960; English transl., Freeman, San Francisco, Calif., 1963. MR 28 #1742.

53. L. V. Kantorovič, *Functional analysis and applied mathematics*, Uspehi Mat. Nauk 3 (1948), no. 6 (28), 89–185; English transl., Nat. Bur. Standards Rep. no. 1509, U.S. Dept. of Commerce, Nat. Bur. Standards, Washington, D.C., 1952. MR 10, 380; 14, 766.

54. S. K. Godunov and G. P. Prokopov, *Variational approach to the solution of large systems of linear equations arising from strongly elliptic problems*, Preprint, Inst. Appl. Math. Acad. Sci. USSR, Moscow, 1968. (Russian)

55. G. I. Marčuk, *Numerical methods in weather forecasting*, Gidrometeorologičeskoe Izdat., Leningrad, 1967. (Russian) RŽMat 1968 #2Б747.

56. G. I. Marčuk and Ju. A. Kuznecov, *On optimal iteration processes*, Dokl. Akad. Nauk SSSR 181 (1968), 1331–1334 = Soviet Math. Dokl. 9 (1968), 1041–1044. MR 37 #7067.

57. S. P. Frankel, *Convergence rates of iteretive treatments of partial differential equations*, Math. Tables and Other Aids to Computation 4 (1950), 65–75. MR 13 #692.

58. E. L. Wachspress, *Extended application of alternating direction implicit iteration model problem theory*, J. Soc. Indust. Appl. Math. 11 (1963), 994–1016. MR 29 #6623.

59. L. V. Kantorovič and G. P. Akilov, *Functional analysis in normed spaces*, Fizmatgiz, Moscow, 1959; English transl., Internat. Series of Monographs in Pure and Appl. Math., vol. 46, Macmillan, New York, 1964. MR 22 #9837; 35 #4699.

60. M. A. Krasnosel'skiĭ and S. G. Kreĭn, *An iteration process with minimal residuals*, Mat. Sb. 31 (73) (1952), 315–334. (Russian) MR 14, 692.

61. A. N. Tihonov and A. A. Samarskiĭ, *On finite difference schemes for equations with discontinuous coefficients*, Dokl. Akad. Nauk SSSR 108 (1956), 393–396. (Russian) MR 18, 938.

62. ———, *Homogeneous difference schemes*, Ž. Vyčisl. Mat. i Mat. Fiz. 1 (1961), 5–63 = USSR Comput. Math. and Math. Phys. 1 (1961), 5–67. MR 29 #5391.

63. ———, *Homogeneous difference schemes on irregular meshes*, Ž. Vyčisl. Mat. i Mat. Fiz. 2 (1962), 812–832 = USSR Comput. Math. and Math. Phys. 2 (1962), 927–953. MR 29 #5392.

64. A. A. Samarskiĭ, *On the convergence and accuracy of homogeneous difference schemes for one-dimensional and higher-dimensional parabolic equations*, Ž. Vyčisl. Mat. i Mat. Fiz. 2 (1962), 603–634 = USSR Comput. Math. and Math. Phys. 2 (1962), 654–696. MR 26 #5747.

65. V. I. Lebedev, *Dirichlet's and Neumann's problems on triangular and hexagonal lattices*, Dokl. Akad. Nauk SSSR 138 (1961), 33–36 = Soviet Math. Dokl. 2 (1961), 519–522. MR 26 #4491.

66. ———, *Difference analogues of orthogonal decompositions, fundamental differential operators, and certain boundary-value problems of mathematical physics*, Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 449–465 = USSR Comput. Math. and Math. Phys. 4 (1964), no. 3, 69–92. MR 32 #6694.

———, *Difference analogues of orthogonal decompositions, fundamental differential operators, and certain boundary-value problems of mathematical physics*. II, Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 649–659 = USSR Comput. Math. and Math. Phys. 4 (1964), no. 4, 36–50. MR 32 #6695.

67. J. Douglas, *On the numerical integration of* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = \partial u/\partial t$ *by implicit methods*, J. Soc. Indust. Appl. Math. 3 (1955), 42–65. MR 17, 196.

68. D. W. Peaceman and H. H. Rachford, *The numerical solution of parabolic and elliptic differential equations*, J. Soc. Indust. Appl. Math. 3 (1955), 28–41. MR 17, 196.

69. J. Douglas and H. H. Rachford, *On the numerical solution of heat conduction problems in two and three space variables*, Trans. Amer. Math. Soc. 82 (1956), 421–439. MR 18, 827.

70. K. A. Bagrinovskiĭ and S. K. Godunov, *Difference schemes for multidimensional problems*, Dokl. Akad. Nauk SSSR 115 (1957), 431–433. (Russian) MR 21 #3640.

71. N. N. Janenko, *A difference method of solution in the case of the multi-dimensional equation of heat conduction*, Dokl. Akad. Nauk SSSR 125 (1959), 1207–1210. (Russian) MR 21 #3641.

72. G. A. Baker, Jr. and T. A. Oliphant, *An implicit, numerical method for solving the two-dimensional heat equation*, Quart. Appl. Math. 17 (1959/60), 361–373. MR 22 #1089a.

73. N. N. Janenko, *On economical implicit schemes (the method of fractional steps)*, Dokl. Akad. Nauk SSSR 134 (1960), 1034–1036 = Soviet Math. Dokl. 1 (1960), 1184–1186. MR 24 #B2557.

74. ———, *Implicit difference methods for the higher-dimensional heat equation*, Izv. Vysš. Učebn. Zaved. Matematika 1961, no. 4 (23), 148–157. (Russian) MR 27 #2125.

75. A. A. Samarskiĭ, *An efficient difference method for solving a multi-dimensional parabolic equation in an arbitrary domain*, Ž. Vyčisl. Mat. i Mat. Fiz. 2 (1962), 787–811 = USSR Comput. Math. and Math. Phys. 2 (1962), 894–926. MR 32 #609.

76. ———, *Difference methods for multi-dimensional differential equations in mathematical physics*, Apl. Mat. 10 (1965), 146–164. (Russian). MR 33 #8111.

77. ———, *Methods for increased order of accuracy for the higher-dimensional heat equation*, Ž. Vyčisl. Mat. i Mat. Fiz. 3 (1963), 812–840 = USSR Comput. Math. and Math. Phys. 3 (1963), 1107–1146. MR 36 #7355.

78. ———, *Local one-dimensional difference schemes for multidimensional hyperbolic equations in an arbitrary region*, Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 638–648 = USSR Comput. Math. and Math. Phys. 4 (1964), no. 4, 21–35. MR 30 #4393.

79. ———, *Additivity principle for the construction of efficient difference schemes*, Dokl. Akad. Nauk SSSR 165 (1965), 1253–1256 = Soviet Math. Dokl. 6 (1965), 1601–1604. MR 34 #7030.

80. A. N. Konovalov, *The fractional step method for solving the Cauchy problem for an N-dimensional oscillation equation*, Dokl. Akad. Nauk SSSR 147 (1962), 25–27 = Soviet Math. Dokl. 3 (1962), 1536–1539. MR 27 #2121.

81. I. V. Frjazinov, *On a difference approximation of the boundary conditions for the third boundary-value problem*, Ž. Vyčisl. Mat. i Mat. Fiz. 4 (1964), 1106–1112 = USSR Comput. Math. and Math. Phys 4 (1964), no. 6, 180–188. MR 31 #895.

82. ———, *Solution of the third boundary value problem for a two-dimensional equation of heat conduction in an arbitrary region by the locally one-dimensional method*, Ž. Vyčisl. Mat. i Mat. Fiz. 6 (1966), 487–502 = USSR Comput. Math. and Math. Phys. 6 (1966), no. 3, 103–125. MR 35 #5155.

83. ———, *Efficient symmetrized solution schemes of boundary value problems for a multi-dimensional equation of parabolic type*, Ž. Vyčisl. Mat. i Mat. Fiz. 8 (1968), 436–443 = USSR Comput Math. and Math. Phys. 8 (1968), no. 2, 271–283. MR 37 #2478.

84. V. B. Andreev, *On uniform convergence of certain differance schemes*, Ž. Vyčisl. Mat. i Mat. Fiz. 6 (1966), 238–250 = USSR Comput. Math. and Math. Phys. 6 (1966), no. 2, 59–77. MR 34 #967.

85. V. B. Andreev, *A method of numerical solution of the third boundary value problem for equations of parabolic type in a p-dimensional parallelepiped*, Comput. Methods Programming, VI, Izdat. Moskov. Univ., Moscow, 1967, pp. 64–75. (Russian) MR 35 #7612.

86. ———, *On difference schemes with a splitting operator for general p-dimensional parabolic equations of second order with mixed derivatives*, Ž. Vyčisl. Mat. i Mat. Fiz. 7 (1967), 312–321 = USSR Comput. Math. and Math. Phys. 7 (1967), no. 2, 92–104. MR 35 #7613.

87. G. I. Marčuk and N. N. Janenko, *Application of the method of splitting (frectional steps) for the solution of problems of mathematical physics*, Some Questions of Computational and Appl. Math., "Nauka", Novosibirsk, 1966, pp. 5–22. (Russian) RŽMat. 1966 #11Б498.

88. E. G. D'jakonov, *Difference schemes with splitting operator for higher-dimensional non-stationary problems*, Ž. Vyčisl. Mat. i Mat. Fiz. 2 (1962), 549–568 = USSR Comput. Math. and Math. Phys. 2 (1961), 581–607. MR 34 #3802.

89. N. N. Janenko, *Weak approximation of systems of differential equations*, Sibirsk. Mat. Ž. 5 (1964), 1431–1434. (Russian) MR 30 #1413.

90. G. I. Marčuk, *Formulation of some converse problems*, Dokl. Akad. Nauk SSSR 156 (1964), 503–506 = Soviet Math. Dokl. 5 (1964), 675–678. MR 29 #3887.

91. N. N. Janenko, *The method of fractional steps; the solution of problems of mathematical physics in several variables*, Izdat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1966; English transl., Springer-Verlag, Berlin and New York, 1971. MR 36 #4815; 46 #6613.

92. Ju. E. Bojarincev, *The convergence of the splitting method and a local correctness creiterion for difference equations with variable coefficients*, Certain Problems Numer. Appl. Math., "Nauka", Sibirsk. Otdel., Novosibirsk, 1966, pp. 92–100. (Russian) MR 35 #5145.

93. S. K. Godunov and A. V. Zabrodin, *Difference schemes of second-order accuracy for higher-dimensional problems*, Ž. Vyčisl. Mat. i Mat. Fiz. 2 (1962), 706–708 = USSR Comput. Math. and Math. Phys. 2 (1962), 790–792. MR 27 #2118.

94. B. E. Hubbard, *Alternating direction schemes for the heat equation on a general domain*, SIAM J. Numer. Anal. 2 (1965), 448–463. MR 33 #5136.

95. ———, *Some locally one-dimensional difference schemes for parabolic equations in an arbitrary region*, Math. Comp. 20 (1966), 53–59. MR 32 #4867.