THE NUMERICAL STABILITY OF TWO-LEVEL AND THREE-LEVEL ITERATIVE METHODS*

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THE problem of the numerical stability of two-level and three-level iterative processes in solving the linear operator equation of the first kind $Au = f$ in Hilbert space is considered.

One of the problems of the theory of iterative processes is that of obtaining quantitative characteristics enabling methods of different structure to be compared. In theoretical investigations the criterion of comparison of methods by the number of iterations, on the assumption that all the iterations are carried out exactly, is most often used.

However, in a practical computational method the process of rounding the results of arithmetical operations introduces some errors into the solution at each stage. This fact leads to the necessity to compare iterative methods by their numerical accuracy.

In the present paper this characteristic is considered for two-level (simple iteration) and three-level (semi-iterative Chebyshev and stationary) iterative processes. The numerical stability of Richardson's method was investigated in [1–3].

In the investigation it is assumed that the introduction of a rounding error is equivalent to a perturbation of the input data of the iterative scheme. This approach, which enables the problem of the numerical accuracy of the method to be reduced to a study of the stability with respect to the input data of some perturbed problem, was used in [4] when considering an abstract scheme of a two-level iterative process.

The estimates obtained (Theorems 1, 3, 5) prove the numerical stability of

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the iterative schemes considered. It is shown that the coefficients in the estimates depend only on the dimensionless parameter $\xi = \gamma_1/\gamma_2$, where γ_1 and γ_2 are the limits of the spectrum of the operator A or constants of equivalence of the operator A and a second operator B of the iterative scheme.

1. Two-level iterative schemes

1. In a real Hilbert space H let an operator equation of the first kind with a linear selfconjugate operator A be given ($A : H \to H$, $A = A^* \to 0$)

(1.1) $$Au = f,$$

where u is an unknown and f is a specified element of H.

Let B be an easily reversible operator satisfying the conditions

(1.2) $$B = B^* \geqslant \beta E, \ \gamma_1 B \leqslant A \leqslant \gamma_2 B, \ \gamma_1 > 0, \ \beta < 0,$$

where γ_1 and γ_2 are constants of energy equivalence of the operators A and B (see [2], ch. VIII).

For an approximate solution of problem (1.1) we consider an implicit two-level iterative scheme with the constant parameter $\tau > 0$ and arbitrary $y_0 \in H$:

(1.3) $$B(y_{k+1} - y_k)/\tau + Ay_k = f, \ k = 0, 1, \dots .$$

With the assumptions of (1.2) the optimal value of the parameter is [2]

(1.4) $$\tau = \tau_0 = 2/(\gamma_1 + \gamma_2).$$

The following estimates then hold:

(1.5) $$\|y_n - u\|_D \leqslant \rho_0^n \|y_0 - u\|_D, \ D = A \text{ or } B,$$

where $\|\cdot\|_D$ is the norm in the energy space H_D defined as follows: $\|x\|_D = (Dx, x)^{1/2}$ for $D = D^* > 0$, $\rho_0 = (1 - \xi)/(1 + \xi)$, $\xi = \gamma_1/\gamma_2$. In order to decrease the norm of the initial error $z_0 = y_0 - u$ in H_A (H_B) by the factor $1/\epsilon$ it is sufficient to perform $n \geqslant n(\epsilon, \xi)$ iterations, where

$$n(\epsilon, \xi) = \ln \epsilon / \ln \rho_0 \approx \ln(1/\epsilon)/2\xi.$$

The scheme (1.3), (1.4) is called an implicit method of simple iteration. In addition to the estimate (1.5), expressing the convergence of the iterative process, it is easy to obtain estimates of the stability of the right side of the scheme (1.3), (1.4):

$$\|y_n\|_B \leqslant \rho_0^n \|y_0\|_B + [(1-\rho_0^n)/\gamma_1] \|f\|_{B^{-1}},$$
$$\|y_n\|_A \leqslant \rho_0^n \|y_0\|_A + (1+\rho_0^n) \|f\|_{A^{-1}}.$$

2. The estimates obtained above imply the convergence and stability of the right side of an ideal numerical process. For a practical process it is necessary to investigate the stability with respect to the input data of some perturbed scheme, taking rounding errors into account.

It may be considered that the introduction of rounding errors is equivalent to a perturbation of the initial approximation, the right side and the operators A and B of the iterative process (1.3). Then the actual solution $\tilde{y}_k$ may be regarded as the exact solution of the following problem:

$$\|y_n\|_B \leqslant \rho_0^n \|y_0\|_B + [(1-\rho_0^n)/\gamma_1] \|f\|_{B^{-1}},$$
$$\|y_n\|_A \leqslant \rho_0^n \|y_0\|_A + (1+\rho_0^n) \|f\|_{A^{-1}}.$$

Assuming that the scheme (1.6) belongs to the original family of schemes, that is, conditions (1.2), or

$$\tilde{A} = \tilde{A}^* > 0, \ \tilde{B} = \tilde{B}^* > \tilde{\beta} E, \ \tilde{\beta} \geqslant 0, \tag{1.7}$$

are satisfied, we investigate its stability.

As a measure of the perturbation of the operators A and B we will take the relative change of their energy $(0 < a_1, a_2 < 1)$

$$|((\tilde{A} - A)x, x)| \leqslant a_1 (Ax, x), \ |((\tilde{B} - B)x, x)| \leqslant a_2 (Bx, x). \tag{1.8}$$

We consider the scheme (1.6) with perturbed initial data, assuming that the iterative parameter τ is defined by formula (1.4) in terms of the unperturbed values of γ_1 and γ_2.

Theorem 1

If conditions (1.2), (1.7), (1.8) are satisfied and

$$a = (a_1 + a_2)/(1 - a_2) \leqslant 0.5\xi,$$

the scheme (1.6) with the parameters (1.4) satisfies the estimates

$$\|\tilde{y}_n - u\|_{\tilde{B}} \leqslant \tilde{\rho}^n \|\tilde{y}_0 - u\|_{\tilde{B}} + (1-\tilde{\rho}^n)\,[1/\tilde{\gamma}_1 \max_{1\leqslant j\leqslant n} \|\tilde{f}_j - f\|_{\tilde{B}^{-1}}$$
$$+ (1/\tilde{\xi}) \max_{1\leqslant j\leqslant n} \|\tilde{w}_j\|_{\tilde{B}^{-1}}] + [\alpha_1(1+\tilde{\rho}^n)/\tilde{\gamma}_1]\,\|f\|_{\tilde{B}^{-1}}, \tag{1.9}$$

where

$$\tilde{\gamma}_1 = \gamma_1 - \alpha\gamma_2 \geqslant 0.5\gamma_1, \qquad \tilde{\rho} = (1-\tilde{\xi})\,/\,(1+\tilde{\xi}) \leqslant 0.5(1+\rho_0),$$
$$\tilde{\xi} = (\xi - \alpha)\,/\,(1+\alpha) \geqslant \xi\,/\,3 \quad \text{for} \quad \alpha \leqslant 0.5\xi.$$

It must be emphasized that in the estimate (1.9) the coefficient of the perturbation $\tilde{w}$ on the right side does not exceed a quantity proportional to $1/\xi$ ("the conditionality number of the iterative process").

To prove the theorem we consider a problem for the error $z_k = \tilde{y}_k - u$ and pass to an equivalent explicit scheme, following [2]:

$$x_{k+1} - \varphi = \tilde{S}(x_k - \varphi) + \tau\varphi_{k+1} + \psi_{k+1}, \qquad k = 0, 1, \ldots,$$
$$x_0 \ \text{given}, \qquad \tilde{S} = E - \tau\tilde{C}.$$

Here we have adopted the notation

$$x_k = \tilde{B}^{1/2} z_k, \qquad \tilde{C} = \tilde{B}^{-1/2}\tilde{A}\tilde{B}^{-1/2}, \qquad \varphi_k = \tilde{B}^{-1/2}(\tilde{f}_k - f),$$
$$\psi_k = \tilde{B}^{-1/2}\tilde{w}_k, \qquad \varphi = \tilde{B}^{1/2}\tilde{A}^{-1}(A - \tilde{A})u. \tag{1.10}$$

Solving for x_k a difference equation of the first order, we find

$$x_k - \varphi = \tilde{S}^k(x_0 - \varphi) + \sum_{j=0}^{k-1} \tilde{S}^j(\tau\varphi_{k-j} + \psi_{k-j}),$$
$$\|x_k\| \leqslant \|\tilde{S}\|^k\|x_0\| + [\,(1-\|\tilde{S}\|^k)/(1-\|\tilde{S}\|)\,]\,(\tau \max_{1\leqslant j\leqslant k}\|\varphi_j\|$$
$$+ \max_{1\leqslant j\leqslant k}\|\psi_j\| + (1+\|\tilde{S}\|^k)\,\|\varphi\|.$$

We first estimate the norm of the operator S.

Lemma 1

If the conditions of Theorem 1 are satisfied the following estimate holds

$$\|\tilde{S}\| \leqslant \tilde{\rho} < 1.$$

Indeed, by (1.2), (1.7), (1.8) we have

$$\tilde{C} = \tilde{C}^*, \qquad \gamma_1^* E \leqslant \tilde{C} \leqslant \tilde{\gamma}_2 E, \qquad \gamma_1^* = \gamma_1(1-\alpha_1) / (1+\alpha_2),$$
$$\tilde{\gamma}_2 = \gamma_2(1+\alpha_1) / (1-\alpha_2) = \gamma_2(1+\alpha),$$

where $\alpha = (\alpha_1 + \alpha_2)/(1 - \alpha_2)$. We note that $\tau_0\tilde{\gamma}_2 - 1 = \tilde{\rho} = \rho_0 + \alpha(1 + \rho_0)$. We introduce the constant $\tilde{\gamma}_1 \leqslant \gamma_1^*$, putting

$$\tilde{\gamma}_1 = (1 - \tilde{\rho}) / \tau_0 = \gamma_1 - \alpha\gamma_2 \leqslant \gamma_1^*.$$

Then $\gamma_1^* + \tilde{\gamma}_2 \geqslant \tilde{\gamma}_1 + \tilde{\gamma}_2 = 2/\tau_0$, and consequently,

$$\|\tilde{S}\| \leqslant \max_{\gamma_1^* \leqslant t \leqslant \tilde{\gamma}_2} |1 - \tau t| = \max(1 - \tau_0\gamma_1^*, \tau_0\tilde{\gamma}_2 - 1) = \tilde{\rho}.$$

The lemma is proved. We note that if $\alpha_1 = \alpha_2 = 0$, Lemma 1 implies that

$$\|S\| \leqslant \rho_0, \; S = E - \tau C, \; C = B^{-1/2}AB^{-1/2}. \tag{1.11}$$

We now estimate $\|\phi\|$ in terms of known quantities.

Lemma 2

Let $\phi = \tilde{B}^{1/2}\tilde{A}^{-1}(A - A)$, where u is the solution of problem (1.1). If conditions (1.2), (1.7), (1.8) are satisfied, we have $\|\phi\| < (\alpha_1/\tilde{\gamma}_1)\|f\|_{\tilde{B}^{-1}}$.

Indeed,

$$\|\varphi\| = \|\tilde{B}^{1/2}\tilde{A}^{-1}(A - \tilde{A})A^{-1}f\| \leqslant \|\tilde{B}^{1/2}(\tilde{A}^{-1} - A^{-1})\tilde{B}^{1/2}\| \, \|\tilde{B}^{-1/2}f\|,$$

and hence it is sufficient to estimate the norm of the selfconjugate operator

$$\tilde{B}^{1/2}(\tilde{A}^{-1} - A^{-1})\tilde{B}^{1/2}.$$

Because of the assumptions of the lemma the chain of inequalities

$$(1 - \alpha_1)A \leqslant \tilde{A} \leqslant (1 + \alpha_1)A,$$
$$(1 - \alpha_1)\tilde{A}^{-1} \leqslant A^{-1} \leqslant (1 + \alpha_1)\tilde{A}^{-1},$$
$$-\alpha_1\tilde{A}^{-1} \leqslant \tilde{A}^{-1} - A^{-1} \leqslant \alpha_1\tilde{A}^{-1},$$

is satisfied, and

$$|((\tilde{A}^{-1} - A^{-1})x, x)| \leqslant \alpha_1(\tilde{A}^{-1}x, x).$$

From this we obtain

$$|(B^{1/2}(\tilde{A}^{-1}-A^{-1})B^{1/2}x, x)| \leqslant \alpha_1(\tilde{C}^{-1}x, x) \leqslant \alpha_1 / \tilde{\gamma}_1(x, x).$$

The lemma is proved.

The statement of the theorem follows from Lemmas 1 and 2 when the inequality $1-\tilde{\rho} \geqslant \tilde{\xi}$ is taken into account.

Theorem 1 asserts that the iterative process (1.3), (1.4) is numerically stable and retains the theoretical rate of convergence if the perturbations of the right side and of the operators of the problem are quantities of order $o(\xi)$.

2. Three-level iterative schemes

1. For the approximate solution of problem (1.1) we consider an implicit three-level iterative scheme of standard type [2]:

$$By_{k+1} = \omega_{k+1}(B-\tau A)y_k + (1-\omega_{k+1})y_{k-1} + \tau\omega_{k+1}f, \quad k = 1, 2, \ldots, \tag{2.1}$$

$$By_1 = (B-\tau A)y_0 + \tau f \tag{2.2}$$

with an arbitrary $y_0 \in H$ and the parameters τ and $\{\omega_k\}$.

We first consider the numerical stability of the iterative process (2.1), (2.2) with the constant parameters τ and ω (the stationary method).

Considering that A and B satisfy conditions (1.2), we put (see [2])

$$\tau = \tau_0 = 2/(\gamma_1+\gamma_2), \ \omega_k = \omega_\infty = 1+\rho_1^2, \ k = 2, 3, \ldots,$$

$$\rho_1 = (1-\sqrt{\xi})/(1+\sqrt{\xi}), \ \xi = \gamma_1/\gamma_2. \tag{2.3}$$

Passing from (2.1), (2.2) to the equivalent explicit scheme, we obtain

$$x_{k+1} = \omega_{k+1}Sx_k + (1-\omega_{k+1})x_{k-1} + \tau\omega_{k+1}\phi, \ k = 1, 2, \ldots, \tag{2.4}$$

$$x_1 = Sx_0 + \tau\phi, \ S = E - \tau C, \ x_0 - \text{given}, \tag{2.5}$$

where $x_k = A^{1/2}y_k$, $C = C_1 = A^{1/2}B^{-1}A^{1/2}$, $\phi = CA^{-1/2}f$, or $x_k = B^{1/2}y_k$, $C = C_2 = B^{-1/2}AB^{-1/2}$, $\phi = B^{-1/2}f$.

Below we require an explicit representation of the solution of problem (2.4), (2.5). Let $\phi = 0$, x_j and x_{j+1} be given. Then the representation

$$(2.6)\qquad x_k = \rho_1^k \left[U_{k-j-1}\left(\frac{S}{\rho_0}\right)\left(\frac{x_{j+1}}{\rho_1^{j+1}} - \frac{S}{\rho_0}\frac{x_j}{\rho_1^j}\right) - T_{k-j}\left(\frac{S}{\rho_0}\right)\frac{x_j}{\rho_1^j}\right],$$

holds for the solution of (2.4), where $k \geqslant j$, and $T_k(t)$ and $U_k(t)$ are Chebyshev polynomials of degree k of the first and second kinds:

$$T_k(t) = \cos(k \arccos t), \qquad |t| \leqslant 1, \qquad \max_{|t|\leqslant 1} |T_k(t)| = 1,$$

$$U_k(t) = \frac{\sin((k+1)\arccos t)}{\sin(\arccos t)}, \qquad |t| \leqslant 1, \qquad \max_{|t|\leqslant 1} |U_k(t)| = k+1.$$

Putting $j = 0$ in (2.6) and taking account of (2.5), the estimate $\|S\| \leqslant \rho_0$ and the equation $\rho_0 = 2\rho_1/(1+\rho_1^2)$, we verify the validity of Theorem 2.

Theorem 2 (see [5])

If conditions (1.2) are satisfied, the iterative process converges and the following estimate holds:

$$\|y_n - u\|_D \leqslant \rho_1^n \left(1 + \frac{1-\rho_1^2}{1+\rho_1^2} n\right) \|y_0 - u\|_D, \qquad D = A \quad \text{or} \quad B.$$

2. The problem of the numerical stability of the iterative scheme (2.1), (2.2) is formulated as follows: Investigate the stability with respect to the input data of the perturbed scheme

$$\tilde{B}\tilde{y}_{k+1} = \omega_{k+1}(\tilde{B} - \tau\tilde{A})\tilde{y}_k + (1-\omega_{k+1})\tilde{y}_{k-1} + \tau\omega_{k+1}\tilde{f}_{k+1} + \tilde{w}_{k+1},$$

$$(2.7)\qquad \tilde{B}\tilde{y}_1 = (\tilde{B} - \tau\tilde{A})\tilde{y}_0 + \tau\tilde{f}_1 + \tilde{w}_1, \quad \tilde{y}_0 \text{ given}$$

Theorem 3

If the conditions (1.2), (1.7), (1.8) are satisfied and

$$a = (a_1 + a_2)/(1 - a_2) \leqslant 0.5\xi,$$

the estimates

$$\|\tilde{y}_n - u\|_{\tilde{B}} \leqslant \tilde{\rho}_1^{\,n}\left(1 + \frac{1-\rho_1^2}{1+\rho_1^2}\, n\right)\|\tilde{y}_0 - u\|_{\tilde{B}} + (1 - \sqrt{(\alpha/\xi)})^{-2}$$
$$\times\, [(1/\gamma_1) \max_{1\leqslant j\leqslant n} \|\tilde{f}_j - f\|_{\tilde{B}^{-1}} + (1/\xi) \max_{1\leqslant j\leqslant n} \|\tilde{w}_j\|_{\tilde{B}^{-1}}] +$$
$$+ \frac{2\alpha_1}{\gamma_1}\left[1 + \tilde{\rho}_1^{\,n}\left(1 + \frac{1-\rho_1^2}{1+\rho_1^2}\, n\right)\right]\|f\|_{\tilde{B}^{-1}},$$

hold for the scheme (2.7), (2.3), where

$$\tilde{\rho}_1 \leqslant \rho_1 + \sqrt{a}\,(1 + \rho_1) < 1.$$

To prove the theorem we consider the problem for the error $z_k = \tilde{y}_k - u$ and pass to the equivalent explicit scheme

$$x_{k+1} - \varphi = \omega_{k+1}\tilde{S}(x_k - \varphi) + (1 - \omega_{k+1})(x_{k-1} - \varphi)$$
$$+ \tau\omega_{k+1}\varphi_{k+1} + \psi_{k+1},$$

$$x_1 - \varphi = \tilde{S}(x_0 - \varphi) + \tau\varphi_1 + \psi_1, \qquad k = 1, 2, \ldots, \tag{2.8}$$

where x_k, $\tilde{S}$, ϕ_k, ψ_k, ϕ are defined in (1.10).

We represent x_k as the sum $x_k = v_k + \bar{x}_k$, where $\bar{x}_k$ is the solution of the following problem:

$$\bar{x}_{k+1} - \varphi = \omega_{k+1}\tilde{S}(\bar{x}_k - \varphi) + (1 - \omega_{k+1})(\bar{x}_{k-1} - \varphi),$$
$$k = 1, 2, \ldots,$$
$$\bar{x}_1 - \varphi = \tilde{S}(\bar{x}_0 - \varphi), \qquad \bar{x}_0 = x_0.$$

Using (2.6), we obtain

$$\bar{x}_k - \varphi = \rho_1^{\,k}\left[\frac{1-\rho_1^2}{1+\rho_1^2}\, U_{k-1}\left(\frac{\tilde{S}}{\rho_0}\right)\frac{\tilde{S}}{\rho_0} + T_k\left(\frac{\tilde{S}}{\rho_0}\right)\right](x_0 - \varphi). \tag{2.9}$$

For v_k we obtain the following problem

$$v_{k+1} = \omega_{k+1}\tilde{S}v_k + (1 - \omega_{k+1})v_{k-1} + \tau\omega_{k+1}\varphi_{k+1} + \psi_{k+1},$$
$$k = 1, 2, \ldots,$$
$$v_1 = \tau\varphi_1 + \psi_1, \qquad v_0 = 0,$$

the solution of which will be sought in the form $v_k = \sum_{j=0}^{k} Y_{k,j}$. Then for fixed $j = 0, 1, \ldots, Y_{k,j}$ is the solution of the homogeneous problem

$$Y_{k+1,\,j} = \omega_{k+1}\tilde{S}Y_{k,\,j} + (1-\omega_{k+1})Y_{k-1,\,j}, \qquad k \geqslant j+1,$$
$$Y_{j+1,\,j} = \tau\omega_{j+1}\varphi_{j+1} + \psi_{j+1}, \qquad Y_{j,\,j} = 0.$$

Here we have introduced formally $\omega_1 = 1$. Using (2.6), we find

$$Y_{k,j} = \rho_1^{k-j-1}\, U_{k-j-1}(\tilde{S}/\rho_0)\,(\tau\omega_{j+1}\,\varphi_{j+1} + \psi_{j+1}), \qquad Y_{k,k} = 0,$$

and consequently,

$$v_k = \sum_{j=0}^{k-1} \rho_1^{\,j} U_j(\tilde{S}/\rho_0)\,(\tau\omega_{k-j}\,\varphi_{k-j} + \psi_{k-j}). \tag{2.10}$$

To complete the proof of the theorem it remains to estimate the norms of the operator polynomials $U_j(\tilde{S}/\rho_0)$ and $T_j(\tilde{S}/\rho_0)$.

We introduce the following notation:

$$\tilde{\rho}_0 = \rho_0/\tilde{\rho} = (1-\tilde{\xi}_0)/(1+\tilde{\xi}_0),\ \tilde{\xi}_0 = a/(1+\alpha-\xi),$$

$$\tilde{\rho}_1 = (1-\sqrt{\tilde{\xi}_0})/(1+\sqrt{\tilde{\xi}_0}), \quad \tilde{q}_n = 2\bar{\rho}_1^{\,n}/(1+\bar{\rho}_1^{\,2n}); \tag{2.11}$$

where $\bar{\rho}_1 = \rho_1/\tilde{\rho}_1$.

Lemma 3

If the conditions of Theorem 3 are satisfied, the following estimates hold:

$$\|T_k(\tilde{S}/\rho_0)\| \leqslant \tilde{q}_k^{-1} \leqslant \tilde{\rho}_1^{-k}, \qquad \|U_k(\tilde{S}/\rho_0)\| \leqslant (k+1)\tilde{\rho}_1^{-k},$$
$$\|U_k(\tilde{S}/\rho_0)\tilde{S}/\rho_0\| \leqslant (k+1)\bar{\rho}_1^{-(k+1)}, \qquad \bar{\rho}_1 \leqslant \rho_1 + \sqrt{\alpha}(1+\rho_1) < 1.$$

Indeed, since $\|\tilde{S}/\rho_0\| \leqslant \tilde{\rho}/\rho_0 = \tilde{\rho}_0^{-1} \leqslant \tilde{\rho}_1^{-1}$, we have

$$\|T_k(\tilde{S}/\rho_0)\| \leqslant T_k(\tilde{\rho}/\rho_0) = T_k(\tilde{\rho}_0^{-1}) = \tilde{q}_k^{-1} \leqslant \rho_1^{-k},$$
$$\|U_k(\tilde{S}/\rho_0)\| \leqslant [T_{k+1}^2(\bar{\rho}_0^{-1}) - 1]^{1/2}[T_1^2(\bar{\rho}_0^{-1}) - 1]^{-1/2}$$
$$= (1-\bar{\rho}_1^{2(k+1)})(1-\bar{\rho}_1^2)^{-1}\bar{\rho}_1^{-k} \leqslant (k+1)\bar{\rho}_1^{-k}.$$

We also have

$$\bar{\rho}_1 = \rho_1 + (1+\rho_1)\frac{\rho_1}{\tilde{\rho}_1}\frac{1-\tilde{\rho}_1}{1+\rho_1}, \quad \frac{\rho_1}{\tilde{\rho}_1}\frac{1-\tilde{\rho}_1}{1+\rho_1} = \frac{1-\sqrt{\xi}}{1+\sqrt{\xi_0}}\sqrt{\xi_0}$$
$$= \sqrt{\alpha}(1-\sqrt{\xi})/[\sqrt{(1+\alpha-\xi)}-\sqrt{\alpha}] \leqslant \sqrt{\alpha}.$$

The lemma is proved.

Using Lemma 3 and formula (2.9), we obtain the estimate

$$\|\bar{x}_k - \varphi\| \leqslant \bar{\rho}^k\left[1 + \frac{1-\rho_1^2}{1+\rho_1^2}k\right]\|x_0 - \varphi\|.$$

Also, by Lemmas 3 and 1,

$$\left\|\sum_{j=0}^{k-1} \rho_1^j U_j(\tilde{S}/\rho_0)\right\| \leqslant \sum_{j=0}^{\infty}(j+1)\bar{\rho}_1^j = (1-\bar{\rho}_1)^{-2}$$
$$\leqslant (1-\rho_1)^{-2}(1-\sqrt{(\alpha/\xi)})^{-2}.$$

Then taking into account the equation

$$\tau\omega_k = \tau_0\omega_\infty = (1-\rho_1)^2/\gamma_1, \tag{2.12}$$

we find from (2.10) that

$$\|v_k\| \leqslant (1-\sqrt{(\alpha/\xi)})^{-2}[(1/\gamma_1)\max_{1\leqslant j\leqslant k}\|\varphi_j\| + (1/\xi)\max_{1\leqslant j\leqslant k}\|\psi_j\|].$$

Theorem 3 is proved.

3. We now consider the numerical stability of the semi-iterative Chebyshev method (2.1), (2.2) with the set of parameters [6]

$$\tau = \tau_0 = 2/(\gamma_1+\gamma_2), \qquad \omega_k = 4(4-\rho_0^2\omega_{k-1})^{-1},$$
$$\omega_1 = 2, \quad k = 2, 3, \ldots . \tag{2.13}$$

If we use the notation

$$q_k = T_k^{-1}(1/\rho_0) = 2\rho_1^k/(1+\rho_1^{2k}),$$

we obtain for ω_k the representation

$$\omega_k = \frac{2}{\rho_0} T_{k-1}(1/\rho_0) T_k^{-1}(1/\rho_0) = (2/q_1)(q_k/q_{k-1}) =$$

(2.14)
$$(1+\rho_1^2)(1+\rho_1^{2(k-1)})/(1+\rho_1^{2k}).$$

This implies that

$$\lim_{k\to\infty} \omega_k = 1+\rho_1^2 = \omega_\infty \text{ and } 2 = \omega_1 > \omega_2 > \ldots > \omega_k > \ldots > \omega_\infty > 1,$$

and hence the method of (2.1)–(2.3) is a limiting case of the scheme (2.1), (2.2), (2.13).

A representation, similar to (2.6) of the solution by an equivalent explicit scheme (2.4), (2.5) with $\phi = 0$ holds for the method of (2.1), (2.13):

$$x_k = q_k\left[U_{k-j-1}\left(\frac{S}{\rho_0}\right)\left(\frac{x_{j+1}}{q_{j+1}} - \frac{S}{\rho_0}\frac{x_j}{q_j}\right) + T_{k-j}\left(\frac{S}{\rho_0}\right)\frac{x_j}{q_j}\right]. \tag{2.15}$$

This implies the following theorem.

Theorem 4

If the conditions (1.2) are satisfied, the iterative process (2.1), (2.2), (2.13) converges and the following estimate holds:

$$\|y_n - u\|_D \leqslant q_n \|y_0 - u\|_D,\ D = A \text{ or } B.$$

Therefore, the schemes (2.1)–(2.3) and (2.1), (2.2), (2.13) converge at the same rate.

4. The problem of the numerical stability of the Chebyshev iterative process reduces to the problem of obtaining estimates of the stability of the explicit scheme (2.8), (2.13). The following theorem holds.

Theorem 5

If the conditions (1.2), (1.7), (1.8) and

$$\alpha = (\alpha_1 + \alpha_2)/(1 - \alpha_2) \leqslant 0.5\xi,$$

are satisfied, the estimate

$$\|\widetilde{y}_n - u\|_{\widetilde{B}} \leqslant \bar{q}_n \|\widetilde{y}_0 - u\|_{\widetilde{B}} + [1 - \sqrt{(\alpha/\xi)}]^{-2} (2/\gamma_1 \max_{1 \leqslant j \leqslant n} \|\widetilde{f_j} - f\|_{\widetilde{B}^{-1}}$$
$$+ (1/\xi) \max_{1 \leqslant j \leqslant n} \|\widetilde{w}_j\|_{\widetilde{B}^{-1}}) + (2\alpha_1/\gamma_1)(1 + \bar{q}_n) \|f\|_{\widetilde{B}^{-1}},$$

holds for the scheme (2.7), (2.8), where $\bar{q}_n = q_n/\widetilde{q}_n \leqslant 2\bar{\rho}_1^{\,n}/(1 + \bar{\rho}_1^{\,2n})$, $\widetilde{q}_n$ and the $\bar{\rho}_1$ are defined in (2.11).

Theorem 5 is proved in the same way as Theorem 3. Taking into account (2.14), (2.15), we obtain instead of (2.9), (2.10) the following representations:

$$\bar{x}_k - \varphi = q_k T_k (\widetilde{S}/\rho_0)(x_0 - \varphi),$$

$$v_k = \sum_{j=0}^{k-1} q_k/q_{k-j}\, U_j(\widetilde{S}/\rho_0)(\tau\omega_{k-j}\varphi_{k-j} + \psi_{k-j})$$

$$= \sum_{j=0}^{k-1} U_j(\widetilde{S}/\rho_0)\,[(2\tau/\rho_0)(q_k/q_{k-j-1})\varphi_{k-j} + q_k/q_{k-j}\psi_{k-j}].$$

The estimate of Theorem 5 follows from Lemma 3 and the inequalities

$$q_k/q_{k-j} \leqslant \rho_1^{\,j}(1 + \rho_1^{\,2}),\ q_k/q_{k-j-1} \leqslant 2\rho_1^{\,j+1},\ j < k - 1.$$

It remains to estimate $\bar{q}_n$.

Lemma 4

When the conditions of Theorem 5 are satisfied the following inequality holds:

$$\bar{q}_n \leqslant 2\bar{\rho}_1^{\,n}/(1 + \bar{\rho}_1^{\,2n}). \tag{2.16}$$

The inequality (2.16) is equivalent to the following:

$$(\widetilde{\rho}_1^{\,2n} - \rho_1^{\,2n})(1 - \widetilde{\rho}_1^{\,2n}) > 0.$$

Since $0 \leqslant \alpha \leqslant 0.5\xi$, we have $\widetilde{\rho}_1 \leqslant 1$, $\widetilde{\rho}_1 < \rho_1$. The lemma is proved. Theorems 3, 5 assert that the iteration processes (2.1), (2.3) considered with the choices of parameters (2.3) and (2.13) are numerically stable and retain the theoretical asymptotic rate of convergence, if the perturbation of the right side and of the operators of the problem is a quantity of order $o(\xi)$.

Theorems 1, 3, 5 imply that a two-level (simple iteration method) and the three-level (semi-iterative Chebyshev and stationary) iterative methods may be referred to a single class of numerical processes which are numerically stable in the energy spaces H_A and H_B (compare with Richardson's method [1–3]).

3. Stability with respect to the parameters γ_1 and γ_2

We consider the effect of the inaccurate specification of the input information, that is, of the constants γ_1 and γ_2 on the rate of convergence of the iterative processes, assuming that all the calculations are performed accurately.

Instead of the exact values of γ_1 and γ_2 in (1.2), let certain approximations $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be known. We introduce the following notation:

$$\tilde{\tau}_0 = 2/(\tilde{\gamma}_1 + \tilde{\gamma}_2), \qquad \tilde{\rho}_0 = (1-\tilde{\xi})/(1+\tilde{\xi}),$$
$$\tilde{\rho}_1 = (1-\sqrt{\tilde{\xi}})/(1+\sqrt{\tilde{\xi}}),$$
$$\tilde{q}_n = 2\tilde{\rho}_1^{\,n}/(1+\tilde{\rho}_1^{\,2n}), \qquad \tilde{\xi} = \tilde{\gamma}_1/\tilde{\gamma}_2.$$

Then the iterative parameters for the two-level and three-level schemes are selected as follows. For the method of (1.3) $\tau = \tilde{\tau}_0$, for the stationary method of (2.1), (2.2) $\tau = \tilde{\tau}_0$, $\omega_k = 1 + \tilde{\rho}_1^{\,2}$ $k = 2, 3, \ldots$; for the semi-iterative Chebyshev method $\tau = \tilde{\tau}_0$, $\omega_k = 4(4 - \tilde{\rho}_0^{\,2}\omega_{k-1})^{-1}$, $k = 2, 3, \ldots$, $\omega_k = 2$.

Let

$$\tilde{\rho} = \max_{\gamma_1 \leqslant t \leqslant \gamma_2} |1 - \tilde{\tau}_0 t|.$$

It is obvious that the estimate (1.5) and the estimates of Theorems 2, 4, in which ρ, ρ_1 and q_n are replaced by $\tilde{\rho}_0$, $\tilde{\rho}_1$ and $\tilde{q}_n$, are valid for the case $\tilde{\gamma}_1 \leqslant \gamma_1$, $\tilde{\gamma}_2 \geqslant \gamma_2$. We introduce the following quantities:

$$\rho_0^* = \tilde{\rho}_0/\tilde{\rho}, \qquad \rho_1^* = \rho_0^*/[1 + \sqrt{(1-\rho_0^{*2})}\,], \ q_n^* = 2\rho_1^{*n}/(1+\rho_1^{*2n}),$$
$$\bar{\rho}_1 = \tilde{\rho}_1/\rho_1^*, \qquad \bar{q}_n = \tilde{q}_n/q_n^*.$$

Since $\|S\| \leqslant \max_{\gamma_1 \leqslant t \leqslant \gamma_2} |1 - \tilde{\tau}_0 t| = \tilde{\rho}$ and $\tilde{\rho}_0 \leqslant \tilde{\rho}$, we obtain as in Lemma 3,

$$\|T_n(S/\tilde{\rho}_0)\| \leqslant (q_n^*)^{-1} \leqslant (\rho_1^*)^{-n}, \ \|U_{n-1}(S/\tilde{\rho}_0)S/\tilde{\rho}_0\| \leqslant n(\rho_1^*)^{-n}.$$

Therefore, instead of the estimate (1.5) and the estimates of Theorems 2, 4, the following inequalities hold, respectively:

$$\|y_n - u\|_D \leqslant \tilde{\rho}^n \|y_0 - u\|_D,$$
$$\|y_n - u\|_D \leqslant \tilde{\rho}_1^n \left(1 + \frac{1-\tilde{\rho}_1^2}{1+\tilde{\rho}_1^2} n\right) \|y_0 - u\|_D,$$
$$\|y_n - u\|_D \leqslant \bar{q}_n \|y_0 - u\|_D, \qquad D = A \text{ or } B. \tag{3.1}$$

It is obvious that if $\tilde{\rho} > 1$ the iterative methods may diverge. We assume that the condition $\tilde{\rho} < 1$ is satisfied. Then $\bar{\rho}_1 < 1$, since $\rho_0^* > \tilde{\rho}_0$, and consequently, $\rho_1^* > \tilde{\rho}_1$. By analogy with Lemma 4 we obtain

$$\bar{q}_n < 2\bar{\rho}_1^n (1 + \bar{\rho}_1^{2n}).$$

Since $\tilde{\rho} < 1$, the iterative methods converge. It follows from the estimates (3.1) that in the case considered there may be a sharp decrease in the rate of convergence in comparison with the case where γ_1 and γ_2 are known accurately.

As an example we consider the case where the condition $\tilde{\rho} < 1$ is satisfied. Let

$$\tilde{\gamma}_1 = (1 + a)\gamma_1, \; \tilde{\gamma}_2 = \gamma_2, \; a > 0.$$

Then direct calculations give

$$\tilde{\rho} = \frac{1-(1-\alpha)\xi}{1+(1+\alpha)\xi} = \rho_0 \{1 + 2\alpha\xi^2/[1 - \alpha\xi - (1+\alpha)\xi^2]\}$$
$$\approx \rho_0(1 + 2\alpha\xi^2),$$
$$\tilde{\rho}_1 = \frac{1 - \sqrt{[(1+\alpha)\xi]}}{1 + \sqrt{[(1+\alpha)\xi]}} = \rho_1[1 - 2\alpha\sqrt{\xi}(1 - \sqrt{\xi})^{-1}$$
$$\times [1 + \sqrt{(1+\alpha)}]^{-1}\{1 + \sqrt{[(1+\alpha)\xi]}\}^{-1}] \approx \rho_1(1 - \alpha\sqrt{\xi}),$$

$$\bar{\rho}_1 = [\sqrt{(1-\xi)} + \sqrt{(\alpha\xi)}]^2 \{1 + \sqrt{[(1+\alpha)\xi]}\}^{-2}$$
$$\approx \rho_1[1 + 2\sqrt{(\alpha\xi)}].$$

Here $\rho_0 = (1 - \xi)/(1 + \xi)$, $\rho_1 = (1 - \sqrt{\xi})/(1 + \sqrt{\xi})$, $\xi = \gamma_1/\gamma_2$. This implies that for the two-level method the asymptotic rate of convergence is preserved even in the case where $a = O(1)$. For the preservation of the asymptotic rate of convergence of three-level iterative schemes it is necessary that a be a quantity of order $o(\xi)$.

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