ON FINITE-DIFFERENCE SCHEMES FOR SOLVING THE DIRICHLET PROBLEM FOR AN ELLIPTIC EQUATION WITH VARIABLE COEFFICIENTS IN AN ARBITRARY REGION*

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HOMOGENEOUS difference schemes in non-uniform mashes are considered for an elliptic equation with variable coefficients in regions of a general type and with boundary conditions of the first kind.

Considerable attention has been paid, see e.g., [1–11], to finite-difference schemes for elliptic equations, and notably Poisson's equation.

The present paper considers homogeneous difference schemes in non-uniform meshes for an elliptic equation with variable coefficients and regions of a general type with boundary conditions of the first kind. The schemes are shown to be uniformly convergent at a rate $O(h^2 \ln(V_0/H_*))$ where V_0 is the volume of the considered region G, h is the maximum step in the spatial lattice $R_p{}^h$:

$$h = \max_{x_i \in \bar{G}} \max_{1 \leqslant \alpha \leqslant p} \hbar_\alpha(x_i),$$

$\hbar_\alpha(x_i)$ is the mean step at the base-point x_i of the lattice $R_p{}^h$ in the direction of the ox_α axis ($\alpha = 1, 2, \ldots, p$, p is the number of dimensions), and H_* is the minimum volume of a cell,

$$H_* = \min_{x_i \in \bar{G}} H(x_i), \qquad H(x_i) = \prod_{\alpha=1}^{p} \hbar_\alpha(x_i).$$

The accuracy to which the initial problem is solved is determined by the errors occurring both at the base-points of the boundary zone, and at strictly interior base-points. The error introduced by approximation of the equation at base-points of the boundary zone may be estimated either by means of the maximum principle (as e.g., in [5]), or by means of a majorant function [7]. For a uniform estimate of the error occurring at strictly interior base-points, both the majorant function method [6, 7]

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and the Green's function method [1] are used.

Schemes of a general type were considered on a uniform mesh for an elliptic equation with variable, reasonably smooth coefficients, in [1], and estimates of the convergence rate in the mesh norm of C were obtained for them by means of the Green's function. It was then assumed that the region boundary is a suface or curve which is as smooth as desired. The case of a piecewise-smooth boundary containing conical points required special investigation [7], and it proved necessary to utilize quasi-uniform meshes.

The aim of the present paper is to construct and examine homogeneous difference schemes, approximating the Dirichlet problem for the elliptic equation with variable coefficients

$$Lu = \sum_{\alpha=1}^{\overline{p}} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right) - q(x)u = -f(x),$$

$$k_\alpha(x) \geqslant c_1 = \text{const} > 0, \qquad q(x) \geqslant 0, \qquad x = (x_1, x_2, \ldots, x_p),$$

in non-uniform meshes.

It is natural to require that the self-conjugate and negative-definite operator L (in the case of a homogeneous boundary condition) be approximated by a finite-difference operator Λ, retaining the same properties in the space of mesh functions. Incidentally, the operator Λ, corresponding to the "cross" scheme, as used in [2, 5], is not in general self-conjugate in the case of an arbitrary region, nor is it negative-definite: a point that was overlooked in [11].

A "cross" scheme with a self-conjugate negative-definite operator Λ in an arbitrary region and non-uniform is devised in the present paper. The scheme is shown to be uniformly convergent for the case of continuous (and reasonably smooth) coefficients and a reasonably smooth solution of the initial problem in a sequence of non-uniform meshes, at a rate $O(h^2 \ln (V_0 / H_*))$. In the class of discontinuous coefficients, the scheme is shown to be uniformly convergent at the same rate in one particular case. Aspects of convergence in the mesh norms of L_2 and $W_2{}^1$ are also discussed.

The method of energy inequalities of the n-th rank developed in [10] proved suitable for uniform estimation of the accuracy (in the mesh norm of C) of the p-dimensional "cross" scheme. This method enabled an estimate to be obtained for the solution of the finite-difference problem in the mesh norm of L_{2^n}, where n is any integer. The convergence in the mesh norm of C follows from this estimate.

1. Formulation of the problem

1. *Statement of the initial problem*

Let $x = (x_1, x_2, \ldots, x_p)$ be a point of p-dimensional space R_p, G a bounded region with boundary Γ, and $\bar{G} = G \cup \Gamma$. The intersection of the region G with a straight line passing through a point $x \in G$ and parallel to an axis ox_α, $\alpha = 1, 2, \ldots, p$, is assumed to consist of a finite number of intervals.

Consider the following problem: to find a function $u = u(x)$, continuous in $\bar{G}$, and satisfying the conditions

$$(1) \qquad Lu = -f(x) \quad \text{for} \quad x \in G; \quad u = \nu(x) \quad \text{for} \quad x \in \Gamma,$$

where L is an elliptic differential operator containing no mixed derivatives:

$$(2) \qquad \begin{gathered} Lu = \sum_{\alpha=1}^{p} L_\alpha u, \qquad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x)\frac{\partial u}{\partial x_\alpha}\right), \\ k_\alpha(x) \geqslant c_1 = \text{const} > 0, \\ k_\alpha(x) \in C^3(\bar{G}), \qquad f(x) \in C^2(\bar{G}), \qquad \alpha = 1, 2, \ldots, p. \end{gathered}$$

A solution $u(x) \in C^4(\bar{G})$ of problem (1), (2) is assumed to exist. Problems with discontinuous coefficients $k_\alpha(x)$ and right-hand side $f(x)$ will be discussed in Paragraph 2 of Section 3.

2. *The mesh*

1. *The lattice.* Take p families of hyperplanes

$$x_\alpha = x_\alpha^{(i_\alpha)} \qquad i_\alpha = 0, \pm 1, \ldots, \qquad \alpha = 1, 2, \ldots, p, \qquad x_\alpha^{(i_\alpha)} > x_\alpha^{(i_\alpha - 1)}$$

Denote by $x_i = (x_1^{(i_1)}, x_2^{(i_2)}, \ldots x_p^{(i_p)})$ the points of intersection of these hyperplanes. The points x_i will be said to form a lattice R_p^h in the initial space R_p.

2. *The mesh. Interior and boundary base-points of the mesh.*
A mesh $\bar{\omega}_h$ of base-points will be constructed in the region $\bar{G}$. Points x_i of the lattice R_p^h, belonging to G will be called interior base-points of the mesh; the set of interior base-points will be denoted by

$$\omega_h \equiv \omega = \{x_i \in G \cap R_p^h\}.$$

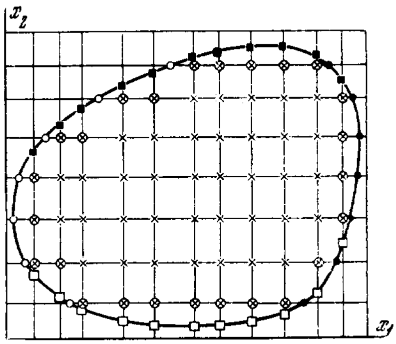

FIG. 1 ■ — for γ_2^+; □ — for γ_2^-; ● — for γ_1^+; ○ — for γ_1^-; ⊗ — for ω_γ; × — for ω^0

Draw straight lines parallel to the coordinate axes ox_α through the base-points $x_i \in \omega$. For simplicity, it will be assumed that the intersection of each of these straight lines with G consists of just one interval $\Delta_\alpha = \Delta_\alpha(x_i)$. The ends of this interval will be termed the boundary base-points in the direction x_α(along x_α). The set of all boundary base-points with respect to x_α will be written as γ_α; $\gamma_\alpha = \gamma_\alpha^+ \cup \gamma_\alpha^-$, where γ_α^+, γ_α^- are the sets of right- and left-hand boundary base-points with respect to x_α. Denote by $\gamma_h \equiv \gamma$ the set of all boundary base-points $\gamma = \bigcup_{\alpha=1}^{p} \gamma_\alpha$. The set of all interior and boundary base-points will be called the mesh $\bar\omega_h \equiv \bar\omega = \omega \cup \gamma$ (see Fig. 1).

3. *Chains of base-points.* Consider one of the intervals Δ_α. The set of base-points $x \in \omega$, lying in this interval, will be called a chain J_α. Denote by $\bar J_\alpha$ the set consisting of base-points $x \in J_\alpha$ and ends of intervals Δ_α. Following [5], denote by $x^{(+1_\alpha)}$ and $x^{(-1_\alpha)}$ the base-points nearest to the base-point $x \in \bar J_\alpha$ to the right and left and belonging to $\bar J_\alpha$. The base-points $x^{(\pm 1_\alpha)}$ will be termed the neighbours of x with respect to x_α, so that

$$x^{(\pm 1_\alpha)} = \left(x_1^{(i_1)}, \ldots, x_{\alpha-1}^{(i_{\alpha-1})}, x^{\alpha\,(i_\alpha \pm 1)}, x_{\alpha+1}^{(i_{\alpha+1})}, \ldots, x_p^{(i_p)}\right).$$

The interior base-points will be classified in detail.

4. *Near-boundary base-points.* We shall say that $x \in \omega$ is a near-boundary base-point with respect to x_α if at least one of its neighbours with respect to x_α (call it an x_α-neighbour) belongs to γ_α. Denote by ω_{γ_α} the set of near-boundary base-points with respect to x_α. Three types of such base-points are possible:

(1) if $x^{(-1_\alpha)} \in \gamma_\alpha^-$, $x^{(+1_\alpha)} \in \omega$, then $x \in \omega_{\gamma_\alpha}^-$;

(2) if $x^{(+1_\alpha)} \in \gamma_\alpha^+$, $x^{(-1_\alpha)} \in \omega$, then $x \in \omega_{\gamma_\alpha}^+$;

(3) if $x^{(+1_\alpha)} \in \gamma_\alpha^+$, $x^{(-1_\alpha)} \in \gamma_\alpha^-$, then $x \in \omega_{\gamma_\alpha}^0$.

The set of all near-boundary base-points is the sum of three sets:

$$\omega_{\gamma_\alpha} = \omega_{\gamma_\alpha}^{+} \cup \omega_{\gamma_\alpha}^{-} \cup \omega_{\gamma_\alpha}^{0}.$$

Denote by ω_α the complement of ω_{γ_α} up to ω, so that $\omega = \omega_\alpha \cup \omega_{\gamma_\alpha}$ by ω_γ the set of near-boundary base-points with respect to all directions:

$$\omega_\gamma = \bigcup_{\alpha=1}^{p} \omega_{\gamma_\alpha},$$

and ω^0 the complement of ω_γ up to ω, so that $\omega = \omega_\gamma \cup \omega^0$. Obviously, ω^0 consists of the base-points, all the neighbours of which are interior base-points (see Fig. 1).

5. *Regular and irregular base-points.* Let x be an interior base-point of the mesh $(x \in \omega)$, while $x^{(\pm 1_\alpha)} \in \bar{\omega}$ are its x_α-neighbours; either $x^{(\pm 1_\alpha)} \in \omega$, or $x^{(\pm 1_\alpha)} \in \gamma$. It will be said that:

(a) the base-point $x \in \omega$ is regular with respect to the direction x_α (or x_α-regular), if both the base-points $x^{(\pm 1_\alpha)}$ are points of the lattice R_p^h;

(b) the base-point $x \in \omega$ is x_α-irregular if one or both of $x^{(\pm 1_\alpha)}$ or $x^{(-1_\alpha)}$ does not belong to the lattice R_p^h.

Denote by $\omega_{\alpha,\,\mathrm{reg}} \equiv \omega_{\alpha,\,\mathrm{r}}$ the set of x_α-regular base-points, by $\omega_{\alpha,\,\mathrm{irreg}} \equiv \omega_{\alpha,\,\mathrm{ir}}$ the set of x_α-irregular base-points, and by $\omega_{\mathrm{ir}} = \bigcup_{\alpha=1}^{p} \omega_{\alpha,\mathrm{ir}}$ the set of all irregular base-points (irregular with respect to at least one direction). Here, ω_{r} is the complement of ω_{ir} up to ω, so that $\omega = \omega_{\mathrm{ir}} \cup \omega_{\mathrm{r}}$. The base-point illustrated in Fig. 2a is x_1-irregular and x_2-regular, in Fig. 2b it is x_1- and x_2-regular, and in Fig. 2c, x_1- and x_2-irregular.

The base-points of Fig. 2 belong respectively to the sets $\omega_{\gamma_1}{}^-$, $\omega_{\gamma_2}{}^+$ (Fig. 2a), $\omega_{\gamma_1}{}^0$, $\omega_{\gamma_2}{}^+$ (Fig. 2b), and $\omega_{\gamma_1}{}^+$, $\omega_{\gamma_2}{}^+$ (Fig. 2c).

6. *The mesh steps.* Consider the base-points $x \in \bar{\omega}$ and $x^{(\pm 1_\alpha)} \in \bar{\omega}$. The distances between them will be termed the steps of mesh $\bar{\omega}$ and denoted by $h_\alpha^\pm$. If $x \in \omega_{\alpha,\,\mathrm{r}}$ is a regular base-point, then $h_\alpha^+ = x_\alpha^{(i_\alpha+1)} - x_\alpha^{(i_\alpha)}$, $h_\alpha^- = x_\alpha^{(i_\alpha)} - x_\alpha^{(i_\alpha-1)}$.

(Henceforth, $x_\alpha^{(i_\alpha)}$, $\alpha = 1, 2, \ldots, p$, is always used to denote the coordinates of a point of the lattice R_p^h.) In this case, the step $h_\alpha^\pm = h_\alpha^\pm(x_\alpha)$ depends on just one argument $x_\alpha = x_\alpha^{(i_\alpha)}$. Let $x \in \omega_{\alpha,\,\mathrm{ir}}$ be an irregular base-point. Its distance from the adjacent boundary base-point $x^{(+1_\alpha)}$ (or $x^{(-1_\alpha)}$), which is not a point of the lattice R_p^h, will be written as $h_{\alpha*}^+$ $(h_{\alpha*}^-)$

$$h_{\alpha*}^+ < x_\alpha^{(i_\alpha+1)} - x_\alpha^{(i_\alpha)}, \qquad h_{\alpha*}^- < x_\alpha^{(i_\alpha)} - x_\alpha^{(i_\alpha-1)}.$$

In general, the steps $h_{\alpha*}^\pm$ depend on all the arguments $x_1^{(i_1)}, x_2^{(i_2)}, \ldots, x_p^{(i_p)}$ (on all the indices $i_1, i_2, \ldots, i_p$).

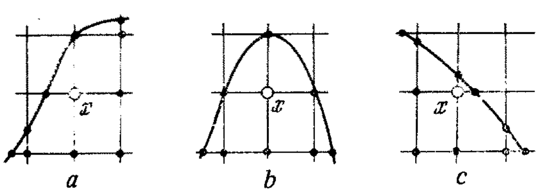

FIG. 2

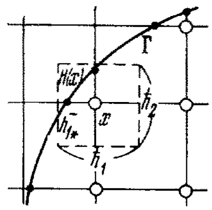

FIG. 3

The steps

$$\hbar_\alpha = 0.5(x_\alpha^{(i_\alpha+1)} - x_\alpha^{(i_\alpha-1)})$$

at interior base-points $x \in \omega$ will also be considered. Obviously, $\hbar_\alpha$ depends only on the coordinate x_α (the index i_α). At a regular base-point,

$$\hbar_\alpha = 0.5(h_\alpha^+ + h_\alpha^-).$$

If x is an irregular base-point, and, for example $x^{(-1_\alpha)}$ is not a lattice point, while $x^{(+1_\alpha)}$ is a regular base-point, then

$$\hbar_\alpha = 0.5\,(x_\alpha^{(i_\alpha+1)} - x_\alpha^{(i_\alpha-1)}) > \hbar_{\alpha*}, \text{ where } \hbar_{\alpha*} = 0.5\,(h_\alpha^+ + h_{\alpha,*}^-).$$

It may be mentioned that, in [11], $\hbar_\alpha$ at an irregular base-point was in fact the step $\hbar_{\alpha*}$, which in general depends on all the coordinates (see Fig. 3).

To standardize the notation, we put

$$h_\alpha^+ = \begin{cases} x_\alpha^{(i_\alpha+1)} - x_\alpha^{(i_\alpha)}, & x \in \omega_{\alpha,\,\mathrm{r}}\,, \\ h_{\alpha*}^+, & x \in \omega_{\alpha,\,\mathrm{ir}}\,; \end{cases}$$

$$h_\alpha^- = \begin{cases} x_\alpha^{(i_\alpha)} - x_\alpha^{(i_\alpha-1)}, & x \in \omega_{\alpha,\,\mathrm{r}}\,, \\ h_{\alpha*}^-, & x \in \omega_{\alpha,\,\mathrm{ir}}\,. \end{cases}$$

7. *Mesh cells.* Each base-point $x \in \omega$ will be associated with a closed region $H(x)$ (the cell volume), bounded by the pieces $s_\alpha^\pm$ of the hyperplanes passing through the points $\bar{x}^{(\pm 0.5_\alpha)} = (x_1^{(i_1)} \dots x_{\alpha-1}^{(i_\alpha-1)}, 0.5(x_\alpha^{(i_\alpha \pm 1)} + x_\alpha^{(i_\alpha)}), x_{\alpha+1}^{(i_\alpha+1)}, \dots, x_p^{(i_p)})$

and orthogonal to the axes ox_α $(\alpha = 1, 2, \dots, p)$.

Henceforth, the same letter will be used to denote a body and its volume, or a figure and its area, so that

$$(3) \qquad H(x) = \prod_{\alpha=1}^{p} \hbar_\alpha, \quad s_\alpha^\pm = s_\alpha = \prod_{\beta \neq \alpha}^{1 \div p} \hbar_\beta.$$

Consider the chain of base-points J_α. All the base-points of this chain have identical coordinates $x_1^{(i_1)}, x_2^{(i_2)}, \dots, x_{\alpha-1}^{(i_\alpha-1)}, x_{\alpha+1}^{(i_\alpha+1)}, \dots, x_p^{(i_p)}$ and distinct coordinates

$x_\alpha^{(i_\alpha)}$, $i_\alpha = i_\alpha^1, i_\alpha^2, \dots$ Hence the steps $\hbar_1, \hbar_2, \dots, \hbar_{\alpha-1}, \hbar_{\alpha+1}, \dots, \hbar_p$ are the same at all $x \in J_\alpha$. This enables every chain J_α to be associated with an area

$$(4) \qquad s_\alpha = \prod_{\beta \neq \alpha}^{1 \div p} \hbar_\beta \qquad (s_\alpha = \text{const}_\alpha \quad \text{for} \quad x \in J_\alpha).$$

8. *Intermediate base-points.* It will be convenient later if certain mesh functions are referred to base-points of the basic mesh $\bar{\omega}$, and others, such as the first difference "derivatives" y_{x_α} and $y_{\bar{x}_\alpha}$ (see Paragraph 3), to intermediate points $x^{(\pm 0.5_\alpha)} = 0.5(x + x^{(\pm 1_\alpha)})$; $x, x^{(\pm 1_\alpha)} \in \bar{\omega}$. The set of intermediate base-points with respect to the direction x_α, $x^{(\pm 0.5_\alpha)} \in G$ will be written as $\tilde{\omega}_\alpha$, and the set of intermediate base-points $x^{(\pm 0.5_\alpha)} \in \omega_\alpha$, lying on a straight line parallel to the ox_α axis and passing through the base-point $x^{(\pm 0.5_\alpha)} \in \tilde{\omega}_\alpha$, will be termed the chain $\mathcal{J}_\alpha$. The base-points $x^{(\pm 0.5_\alpha)} \in \tilde{\omega}_\alpha$, nearest to the boundary base-points $x^{(\pm 1_\alpha)} \in \gamma_\alpha$, will be termed near-boundary intermediate base-points. By anology with Paragraph 4, corresponding notation will be used for the sets of near-boundary intermediate base-points: $\tilde{\omega}_{\gamma_\alpha}^{+}$, $\tilde{\omega}_{\gamma_\alpha}^{-}$, $\tilde{\omega}_{\gamma_\alpha} = \tilde{\omega}_{\gamma_\alpha}^{+} \cup \tilde{\omega}_{\gamma_\alpha}^{-}$, $\tilde{\omega}_\alpha'$ is the complement of $\tilde{\omega}_{\gamma_\alpha}$ up to $\tilde{\omega}_\alpha$ $(\tilde{\omega}_\alpha = \tilde{\omega}_{\gamma_\alpha} \cup \tilde{\omega}_\alpha')$. Every base-point $x^{(\pm 0.5_\alpha)} \in \tilde{\omega}_\alpha$ will be associated with a volume

$$H_\alpha^\pm = s_\alpha h_\alpha^\pm.$$

Denote by h the maximum step with respect to the spatial variables, and let us agree to use the same letter M to denote all constants independent of h.

It will be assumed that every base-point $x \in \omega$ has at least one neighbour (with respect to some direction $\xi \in \omega$. This can always be arranged for if the steps $h_\alpha^\pm$ of the lattice R_p^h are small enough. Denote by $K'_\alpha(x)$ the set of x_α-neighbours of $x \in \omega$ belonging to ω:

$$\overline{K_\alpha}'(x) = \begin{cases} x^{(+1_\alpha)}, \; x^{(-1_\alpha)}, & x \in \omega_\alpha, \\ x^{(\pm 1_\alpha)}, & x \in \omega^{\mp}_{\gamma_\alpha}. \end{cases}$$

We put

$$\overline{K}'(x) = \bigcup_{\alpha=1}^{p} \overline{K_\alpha}'(x), \qquad x \notin \overline{K}'(x).$$

3. *Mesh functions and difference operators*

Henceforth, only base-points of the basic mesh $\bar{\omega}$ and of intermediate meshes $\tilde{\omega}_\alpha$, $\alpha = 1, 2, \ldots, p$. will be considered. Let x, $x^{(\pm 1_\alpha)}$ be base-points of $\bar{\omega}$, and $x^{(\pm 0.5_\alpha)}$ base-points of an intermediate mesh $\tilde{\omega}_\alpha$. Let $y = y(x)$, $x \in \bar{\omega}$ and $v_\alpha(x')$, $x' \in \tilde{\omega}_\alpha$, be mesh functions, specified respectively in the basic mesh $\bar{\omega}$ and the intermediate meshes $\tilde{\omega}_\alpha$. The introduction of intermediate mesh functions makes the derivation of the finite-difference schemes clearer and less abstract, and facilitates their investigation. We put

$$y^{(\pm 1_\alpha)} = y(x^{(\pm 1_\alpha)}) \qquad y_{x_\alpha}^{+} = (y^{(+1_\alpha)} - y)/h_\alpha^{+}, \qquad y_{\bar{x}_\alpha}^{-} = (y - y^{(-1_\alpha)})/h_\alpha^{-}.$$

The function $v_\alpha \equiv v_\alpha^{+} = y_{x_\alpha}^{+}$ ($v_\alpha^{-} = y_{\bar{x}_\alpha}^{-}$), representing the difference analogue of the derivative $\partial u / \partial x_\alpha$, will be referred to the intermediate base-points $x^{(\pm 0.5_\alpha)} \in \tilde{\omega}_\alpha$.

The next task is to write the finite-difference approximation for the elliptic operator L in the mesh ω. As a preliminary, consider the one-dimensional analogue of problem (1):

$$Lu = (ku')' = -f(x), \quad 0 < x < l, \quad u(0) = \nu_1, \quad u(l) = \nu_2.$$

Let

$$\bar{\omega} = \omega \cup \gamma, \qquad \omega = \{x_i, \; i = 1, 2, \ldots, N-1\}, \qquad x_{i+1} > x_i.$$
$$\gamma = \{0, l\}.$$

The integro-interpolation method will be used to obtain the difference scheme. The equation $Lu = -f$ is integrated with respect to x from $x_i^{(-0.5)} \equiv 0.5\,(x_i + x_{i-1})$ to $x_i^{(+0.5)} \equiv 0.5\,(x_{i+1} + x_i)$:

$$\text{(5)} \qquad \frac{w^{+} - w^{-}}{\hbar} = -\frac{1}{\hbar} \int_{x^{(-0.5)}}^{x^{(+0.5)}} f\,dx, \text{ where } w^{\pm} = ku'\,|_{x=x^{(\pm 0.5)}}.$$

Replace $\omega\pm$ by the difference analogue

$$W^{\pm} = a^{\pm} y_x^{\pm},$$

where the mesh function $a\pm$ approximates the coefficient $k(x)$ at intermediate points $x^{\pm(0.5)}$ to the second order:

$$a^{\pm} = k(x^{(\pm 0.5)}) + O((h^{\pm})^2).$$

Obviously, $W\pm$ is an intermediate mesh function. Replace the right-hand side of (5) by a mesh function φ (e.g., put

$$\varphi = f(x), \quad \varphi = (1/\hbar) \int_{x^{(-0.5)}}^{(+0.5)} f\,dx$$

etc., see [4, 5]). As a result, we arrive at the difference equation

$$\Lambda y = -\varphi \quad \text{for} \quad x \in \omega,$$

where

$$\Lambda y = (a y_{\bar{x}})_{\hat{x}} = \frac{1}{\hbar}(a^+ y_x^+ - a^- y_{\bar{x}}^-).$$

The notation here is $\Phi_{\hat{x}} = (\Phi(x^{(+0.5)}) - \Phi(x^{(-0.5)}))/\hbar$, where Φ is any mesh function defined in $\bar{\omega}$ (with $x_i^{(\pm 0.5)} = 0.5(x_{i\pm 1} + x_i)$, $\hbar_i = 0.5(x_{i+1} - x_{i-1})$).

The procedure is similar in the p-dimensional case. First consider base-points $x \in \omega$, the volumes H corresponding to which belong to $\bar{G}$. Integrating (1) over the volume $H \in \bar{G}$, containing such a base-point x, we get the identity

$$\sum_{\alpha=1}^{p} \left(\frac{1}{s_\alpha} \int_{s_\alpha} w_\alpha ds_\alpha\right)_{\hat{x}_\alpha} + \frac{1}{H} \int_H f\,dH = 0, \tag{6}$$

where

$$ds_\alpha = \prod_{\beta \neq \alpha}^{1 \div p} dx_\beta, \qquad dH = \prod_{\alpha=1}^{p} dx_\alpha,$$

and $w_\alpha = k_\alpha \partial u / \partial x_\alpha$ is the x_α-flux. The coefficient k_α is associated with a mesh function a_α such that

$$a_\alpha^\pm = k_\alpha(x^{(\pm 0.5_\alpha)}) + O((h_\alpha^\pm)^2), \qquad a_\alpha^\pm \geqslant c_1 > 0. \tag{7}$$

The mean flux $(1/s_\alpha)\int_{s_\alpha} w_\alpha ds_\alpha$ through the area s_α is approximated by the finite-difference expression

$$W_\alpha^\pm \equiv W_\alpha(x^{(\pm 0.5_\alpha)}) = a_\alpha^\pm y_{x_\alpha}^\pm,$$

while the left-hand side of (6) is replaced by

$$\sum_{\alpha=1}^{p} (W_\alpha)_{\hat{x}_\alpha}, \text{where} \quad (W_\alpha)_{\hat{x}_\alpha} = (W_\alpha^+ - W_\alpha^-)/\hbar_\alpha.$$

The term $(1/H)\int_H f dH$ in (6) is replaced approximately by a mesh function φ (e.g., $\varphi = f(x)$). It is then required that the following approximation condition be satisfied:

$$\varphi - f(x) = \sum_{\alpha=1}^{p} ((\mu_\alpha)_{\hat{x}_\alpha} + \bar{\mu}_\alpha), \qquad \mu_\alpha^\pm = O((h_\alpha^\pm)^2), \qquad \bar{\mu}_\alpha = O(\hbar_\alpha^2). \tag{8}$$

As a result, the following operator Λ representing the difference analogue of L is obtained:

$$\Lambda y = \sum_{\alpha=1}^{p} \Lambda_\alpha y,$$

where

$$\Lambda_\alpha y = (a_\alpha y_{x_\alpha})_{\hat{x}_\alpha} = \frac{1}{\hbar_\alpha}\left(a_\alpha^+ \frac{y^{(+1_\alpha)} - y}{h_\alpha^+} - a_\alpha^- \frac{y - y^{(-1_\alpha)}}{h_\alpha^-}\right),$$

$$\hbar_\alpha = 0.5\,(x_\alpha^{(i_\alpha+1)} - x_\alpha^{(i_\alpha-1)}).$$

The operator Λ will be defined by the same expression at the remaining mesh base-points. We thus arrive at the difference equation

$$\Lambda y = -\varphi(x), \qquad x \in \omega.$$

This scheme is the same as the well-known scheme of [4] except at near-boundary irregular base-points, where the step $\hbar_\alpha$ has a different meaning, e.g., $\hbar_\alpha = \hbar_{\alpha*} = 0.5\,(h_\alpha^+ + h_{\alpha*}^-)$, if the base-point $x^{(-1_\alpha)}$ is not a point of the lattice R_p^h.

4. *Statement of the difference problem and error of the approximation*

The initial problem (1) is associated with the following difference problem: to find the mesh function $y(x)$ defined in $\bar{\omega}$, and satisfying the conditions

$$\Lambda y = -\varphi(x) \text{ for } x \in \omega, \quad y = \nu(x) \text{ for } x \in \gamma. \tag{9}$$

Here, Λ is the operator defined above, while the mesh functions $a_\alpha^\pm$, $\alpha = 1, 2, \ldots, p$, and φ satisfy the approximation conditions (7) and (8).

The accuracy of the difference scheme (9) is determined by the error $z = y - u$, where u is the solution of the initial problem and y is the solution of problem (9). Substituting $y = z + u$ in (9), the problem

$$\Lambda z = -\psi(x) \text{ for } x \in \omega, \quad z = 0 \text{ for } x \in \gamma, \tag{10}$$

is obtained for z, where $\psi = \Lambda u + \varphi$ is the approximation error of the scheme (9). Using Taylor's formula under the above smoothness conditions and conditions (8), the error ψ at regular base-points can be written as

$$\psi = \sum_{\alpha=1}^{p} \Psi_\alpha, \qquad \Psi_\alpha = (\chi_\alpha)_{\hat{x}_\alpha} + \psi_\alpha, \qquad x \in \omega_r,$$

where

$$\chi_\alpha^\pm = \left[(a_\alpha^\pm - k_\alpha)\frac{\partial u}{\partial x_\alpha} + \frac{(h_\alpha^\pm)^2}{8}\left(\frac{a_\alpha^\pm}{3}\frac{\partial^3 u}{\partial x_\alpha^3} + \frac{\partial^2}{\partial x_\alpha^2}\left(k_\alpha \frac{\partial u}{\partial x_\alpha}\right)\right)\right]_{x=x^{(\pm 0.5_\alpha)}} + $$
$$+ \mu_\alpha^\pm, \qquad \psi_\alpha = \bar{\mu}_\alpha + O(\hbar_\alpha^2).$$

From (7) and (8),

$$\chi_\alpha^\pm = O((h_\alpha^\pm)^2), \qquad \psi_\alpha = O(\hbar_\alpha^2) \quad \text{for} \quad x \in \omega_r.$$

At irregular base-points ($x \in \omega_{\mathrm{ir}}$) the approximation error is obviously $O(1)$.

For, since

$$(W_\alpha)_{\hat{x}_\alpha} = (W_\alpha^+ - W_\alpha^-)/\hbar_\alpha = \frac{\hbar_{\alpha*}}{\hbar_\alpha}\frac{W_\alpha^+ - W_\alpha^-}{\hbar_{\alpha*}} = \frac{\hbar_{\alpha*}}{\hbar_\alpha}L_\alpha u + {} + O(\hbar_\alpha) = O(1),$$

we have

$$\Lambda u = \sum_{\alpha=1}^{p}\frac{\hbar_{\alpha*}}{\hbar_\alpha}L_\alpha u + O(\hbar_\alpha) = Lu + \sum_{\alpha=1}^{p}\frac{\hbar_{\alpha*} - \hbar_\alpha}{\hbar_\alpha}L_\alpha u + O(h),$$

whence

$$\Lambda u - Lu = O(1), \quad \psi = O(1) \quad \text{for} \quad x \in \omega_{\mathrm{ir}}.$$

For purposes of standardization the error will be written as

(11)
$$\psi = \sum_{\alpha=1}^{p}\Psi_\alpha, \qquad \Psi_\alpha = (\chi_\alpha)_{\hat{x}_\alpha} + \psi_\alpha + \overset{*}{\psi}_\alpha,$$
$$\chi_\alpha^\pm = \chi_\alpha(x^{(\pm 0.5_\alpha)}) = O((h_\alpha^\pm)^2), \qquad \psi_\alpha = O(\hbar_\alpha^2),$$
$$\overset{*}{\psi}_\alpha = \begin{cases} O(1), & x \in \omega_{\alpha,\,\mathrm{ir}}, \\ 0, & x \in \omega_{\alpha,\,\mathrm{r}}. \end{cases}$$

In a uniform lattice R_p^h, when, whatever the direction α

$$\hbar_\alpha = 0.5\,(x_\alpha^{(i_\alpha+1)} - x_\alpha^{(i_\alpha-1)}) = \mathrm{const}_\alpha, \quad \chi_\alpha(x^{(\pm 0.5_\alpha)}) = 0.$$

The solution of problem (10), (11) will be sought in the form

(12) $$z = v_1 + v_2 + v_3,$$

where the functions v_k, $k = 1, 2, 3$, are the solutions of the problem

(13)
$$\Lambda v_k = -\overline{\Psi}_k \quad \text{for} \quad x \in \omega, \qquad v_k = 0 \quad \text{for} \quad x \in \gamma,$$
$$\overline{\Psi}_k = \sum_{\alpha=1}^{p}\psi_{\alpha,k}, \qquad \psi_{\alpha,1} = (\chi_\alpha)_{\hat{x}_\alpha}, \qquad \psi_{\alpha,2} = \psi_\alpha, \qquad \psi_{\alpha,3} = \overset{*}{\psi}_\alpha,$$
$$k = 1, 2, 3.$$

Asymptotic expressions for the functions $\chi_\alpha(x^{(\pm 0.5_\alpha)})$, ψ_α, ψ_α^* in the class of smooth functions $u(x)$, $k_\alpha(x)$, $f(x)$ are given by (11).

2. A priori estimates

1. *The maximum principle and majorant functions. Auxiliary propositions*

Estimates for the solution of a finite-difference elliptic equation are obtained by using a method, based on the maximum principle, for constructing majorant functions (see e.g., [5, 7].

Estimates based on the maximum principle prove effective for investigating the accuracy of a scheme when no approximation is available at irregular near-boundary base-points.

Let us recall some well-known results. The difference equation is $L_h v = -\varphi$; the boundary condition $v|_\gamma = \nu$ (L_h is a linear operator) will be written as

$$A(x)v(x) = \sum_{\xi\in\overline{K}'} B(x,\xi)v(\xi) + F(x) \quad \text{for} \quad x \in \omega, \tag{14}$$

where $\overline{K}'(x)$ is the set of base-points defined in Para. 2 of Section 1.

Lemma 1

Let

$$\begin{gathered} A(x) > 0, \qquad B(x,\xi) > 0, \qquad A(x) - \sum_{\xi\in\overline{K}'} B(x,\xi) \equiv D(x) \geqslant 0, \\ D(x) > 0, \quad F(x) = F^*(x) \quad \text{for } x \in \omega^*, \\ D(x) \geqslant 0, \quad F(x) = 0 \text{ for } \quad x \in \omega\setminus\omega^*, \end{gathered} \tag{15}$$

where ω^* is a set of base-points $x \in \omega$. Then, the solution of problem (14) satisfies [4, 5]

$$\max_{\omega} |v| \leqslant \max_{\omega^*} |F^*(x)/D(x)|. \tag{16}$$

Lemma 1 will be used to obtain a bound for the function v_3 of (13). The expressions for A, B, D, and F in the present case are

$$A(x) = \sum_{\alpha=1}^{p} A_\alpha(x), \qquad A_\alpha(x) = \frac{1}{\hbar_\alpha}\left(\frac{a_\alpha^+}{h_\alpha^+} + \frac{a_\alpha^-}{h_\alpha^-}\right)$$

$$D(x) = \sum_{\alpha=1}^{p} D_\alpha(x),$$

(17)
$$D_\alpha(x) = \begin{cases} 0, & x \in \omega_\alpha, \\ a_\alpha^{\pm}/(h_\alpha^{\pm}\hbar_\alpha), & x \in \omega_{\gamma_\alpha}^{\pm}, \\ A_\alpha(x), & x \in \omega_{\gamma_\alpha}^{0}, \end{cases} \qquad B_\alpha(x, x^{(\pm 1_\alpha)}) = a_\alpha^{\pm}/(h_\alpha^{\pm}\hbar_\alpha)$$

$$\sum_{\xi \in \overline{K}'(x)} B(x, \xi)\, v(\xi) = \sum_{\alpha=1}^{p} \sum_{\xi \in \overline{K}'_\alpha(x)} B_\alpha(x, \xi)\, v(\xi),$$

$$\overline{K}'(x) = \bigcup_{\alpha=1}^{p} \overline{K}'_\alpha(x).$$

Note the following point. Fixing the direction x_α, denote by $Q_\alpha(\widetilde{Q}_\alpha)$ the sets of all chains J_α $(\widetilde{J}_\alpha)$ for the direction ox_α, $\alpha = 1, 2, \ldots, p$. Since any interior base-point $x_i(x^{(\pm 0.5_\alpha)})$ belongs to some chain $J_\alpha(x^{(\pm 0.5_\alpha)} \in \widetilde{J}_\alpha)$ for the direction ox_α, we have

$$\omega = \bigcup_{Q_1} J_1 = \bigcup_{Q_2} J_2 = \ldots = \bigcup_{Q_p} J_p, \quad \widetilde{\omega}_\alpha = \bigcup_{\widetilde{Q}_\alpha} \widetilde{J}_\alpha, \quad \alpha = 1, 2, \ldots, p.$$

Hence

$$\sum_{\omega} Hy = \sum_{Q_\alpha} \sum_{J_\alpha} Hy, \qquad \sum_{\widetilde{\omega}_\alpha} H_\alpha W_\alpha = \sum_{\widetilde{Q}_\alpha} \sum_{\widetilde{J}_\alpha} H_\alpha W_\alpha, \quad \alpha = 1, 2, \ldots, p.$$

Here, $H = s_1\hbar_1 = s_2\hbar_2 = \ldots = s_p\hbar_p$, $H_\alpha = H_\alpha^{\pm} = s_\alpha h_\alpha^{\pm}$, $\alpha = 1, 2, \ldots, p$. Since, whatever the base-point of the given chain J_α $(\widetilde{J}_\alpha)$, the area $s_\alpha = \prod_{\beta \neq \alpha} \hbar_\beta$ is independent of the coordinate x_α, and depends solely on the coordinates $x_1, x_2, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_p$, the last equations yield

Lemma 2

$$\sum_{\omega} Hy = \sum_{Q_\alpha} s_\alpha \Big(\sum_{J_\alpha} y\hbar_\alpha \Big), \qquad \sum_{\widetilde{\omega}_\alpha} H_\alpha W_\alpha = \sum_{\widetilde{Q}_\alpha} s_\alpha \Big(\sum_{J_\alpha} W_\alpha h_\alpha \Big).$$

2. *Spaces of mesh functions*

Denote by $\mathscr{H}$ the space of mesh functions defined in the mesh ω, with scalar product

$$(y, z) = \sum_{\omega} yzH \quad \text{and norm} \quad \|y\| = \sqrt{(y, y)}, \qquad y, z \in \mathscr{H}.$$

We shall also consider the spaces $\mathscr{H}_\alpha$, $\alpha = 1, 2, \ldots, p$, of mesh functions defined in meshes $\tilde{\omega}_\alpha$ with the scalar products

$$(v, w)_\alpha = \sum_{\tilde{\omega}_\alpha} vwH_\alpha \text{ и and norms } \|v\|_\alpha = \sqrt{(v, v)_\alpha}, \qquad v, w \in \mathcal{H}_\alpha.$$

The following notation will be used for sums along the chains J_α and $\tilde{J}_\alpha$ with fixed x_β $(\beta \neq \alpha)$:

$$(18) \qquad \begin{aligned} &(y, z)_{J_\alpha} = \sum_{J\alpha} \hbar_\alpha yz, \qquad \|y\|_{J_\alpha} = \sqrt{(y, y)_{J_\alpha}}, \\ &(v, w)_{\tilde{J}_\alpha} = \sum_{\tilde{J}_\alpha} h_\alpha vw, \qquad \|v\|_{\tilde{J}_\alpha} = \sqrt{(v, v)_{\tilde{J}_\alpha}}. \end{aligned}$$

Finally, we put

$$\|y\|_c = \max_\omega |y(x)|.$$

3. *The difference operators*

Denote by A the operator mapping space $\mathcal{H}$ into $\mathcal{H}$, and representing the sum of p operators A_α:

$$(19) \qquad A = \sum_{\alpha=1}^{p} A_\alpha.$$

Every operator A_α likewise maps $\mathcal{H}$ into $\mathcal{H}$ and is defined thus:

$$(20) \qquad A_\alpha y = -\frac{1}{\hbar_\alpha} \times \begin{cases} a_\alpha^{+} y_{x_\alpha}^{+} - a_\alpha y_{x_\alpha}, & x \in \omega_\alpha, \\ \mp a_\alpha^{\mp} y_{x_\alpha}^{\mp} - a_\alpha^{\pm} y/h_\alpha^{\pm}, & x \in \omega_{\gamma_\alpha}^{\pm}, \\ -(a_\alpha^{+}/h_\alpha^{+} + a_\alpha^{-}/h_\alpha^{-})\, y, & x \in \omega_{\gamma_\alpha}^{0}. \end{cases}$$

Obviously, on functions vanishing on γ.

$$A_\alpha y = -\Lambda_\alpha y, \ \ y|_\gamma = 0.$$

Further, let T_α be difference operators, mapping $\mathcal{H}$ into $\mathcal{H}_\alpha$; T_α^* operators mapping $\mathcal{H}_\alpha$ into $\mathcal{H}$; and S_α operators mapping $\mathcal{H}_\alpha$ into $\mathcal{H}_\alpha$, $\alpha = 1, 2, \ldots, p$.

$$(T_\alpha y)^{\pm} \equiv (T_\alpha y)^{(\pm 0.5_\alpha)} = \begin{cases} y_{\bar{x}_\alpha}^{\pm}, & x^{(\pm 0.5_\alpha)} \in \tilde{\omega}_\alpha', \\ \mp y/h_\alpha^{\pm}, & x^{(\pm 0.5_\alpha)} \in \tilde{\omega}_{\gamma_\alpha}^{\pm}, \end{cases}$$

$$y \in \mathcal{H}, \qquad T_\alpha y \in \mathcal{H}_\alpha;$$

$$T_\alpha^* w = -w_{\hat{x}_\alpha} \quad \text{for } x \in \omega, \qquad w \in \mathcal{H}_\alpha, \qquad T_\alpha^* w \in \mathcal{H};$$

$$(S_\alpha w)^{\pm} \equiv (S_\alpha w)^{(\pm 0.5_\alpha)} = a_\alpha^{\pm} w^{(\pm 0.5_\alpha)} \quad \text{for} \quad x^{(\pm 0.5_\alpha)} \in \tilde{\omega}_\alpha,$$

$$w \in \mathcal{H}_\alpha, \qquad S_\alpha w \in \mathcal{H}_\alpha.$$

By definition of the operators T_α, S_α, $T_\alpha{}^*$, A_α it follows that

$$A_\alpha = T_\alpha{}^* S_\alpha T_\alpha. \tag{21}$$

4. *Properties of the difference operators*

Lemma 3

The difference operators T_α and $T^*\alpha$ are conjugate to one another:

$$(T_\alpha{}^* w, y) = (w, T_\alpha y)_\alpha, \quad y \in \mathscr{H}, \quad w \in \mathscr{H}_\alpha. \tag{22}$$

Using Green's finite-difference formulae [5] , it is easily shown that

$$(y, T_\alpha^* w)_{J_\alpha} = \sum_{J_\alpha} y\,(T_\alpha^* w)\,\hbar_\alpha = \sum_{\mathcal{J}_\alpha} w\,(T_\alpha y)\,h_\alpha = (w, T_\alpha y)_{\tilde{J}_\alpha}. \tag{23}$$

Expression (22) is obtained by multiplying the last equation of (23) by s_α, summing over all chains $Q_\alpha(\tilde{Q}_\alpha)$ and recalling (18).

Lemma 4

Operators S_α are self-conjugate and positive-definite:

$$(S_\alpha w, v)_\alpha = (w, S_\alpha v)_\alpha, \qquad (S_\alpha w, w)_\alpha \geqslant c_1 \|w\|_\alpha{}^2. \tag{24}$$

The lemma follows from the form of the operator S_α and the inequality (7). Some further notation is needed: we write the operator $B \geqslant B_1$, if $(By, y) \geqslant (B_1 y, y)$ for all $y \in \mathscr{H}$; for instance, $B \geqslant c_1 E > 0$, E where E is the unit operator, if $(By, y) \geqslant c_1 \|y\|^2$, where c_1 is a positive constant, and y is any mesh function defined in ω.

Lemma 5

The operators A_α, $\alpha = 1, 2, \ldots, p$, are self-conjugate and positive-definite:

$$A_\alpha = A_\alpha{}^* \geqslant \bar{\delta} E, \qquad A = A^* \geqslant \delta E,$$

where $\bar{\delta} \geqslant 4c_1 / 3D^2$, $\delta = \bar{\delta} p$, and D is the diameter of the region G.

The fact that A_α and A are self-conjugate follows from (19), (21), and Lemmas 3 and 4. To prove that A_α is positive-definite, write the identity

$$y(x) = \sum_{x' \in \tilde{J}_\alpha}^{x' = x^{(-0.5_\alpha)}} h_\alpha^- (T_\alpha y)^-,$$

where the summation is over $x' \in \widetilde{J}_\alpha$ to $x' = x^{(-0.5\alpha)}$. Squaring this equation and applying the Cauchy inequality to the right-hand side, then multiplying by $H(x)$ and summing over all base-points $x \in \omega$, we get

$$\|y\|^2 \leqslant \|T_\alpha y\|_\alpha^2\, 3D^2 / 4. \tag{25}$$

Using (21) and (24),

$$(A_\alpha y, y) = (S_\alpha T_\alpha y, T_\alpha y)_\alpha \geqslant c_1 \|T_\alpha y\|_\alpha^2.$$

From (25) and this last inequality,

$$(A_\alpha y, y) \geqslant \frac{4c_1}{3D^2} \|y\|^2.$$

From this and (19),

$$(Ay, y) \geqslant \delta \|y\|^2, \qquad \delta = \frac{4c_1}{3D^2} p. \tag{26}$$

We introduce the norm in $W_2^1(\omega)$:

$$\|y\|^2_{W_2^1} \equiv (\mathring{A} y, y) + \|y\|^2,$$

where

$$\mathring{A} = \sum_{\alpha=1}^{2} \mathring{A}_\alpha, \qquad \mathring{A}_\alpha = T_\alpha^* T_\alpha.$$

On functions y vanishing on γ, the norm becomes

$$\|y\|^2_{W_2^1} = \sum_{\alpha=1}^{p} \sum_{\widetilde{\omega}_\alpha} H_\alpha y_{x_\alpha}^2 + \sum_{\overline{\omega}} H y^2.$$

5. *A priori estimates of the n-the rank*

Consider the equation

$$Av = \theta, \text{where}\, \theta = \sum_{\alpha=1}^{p} \theta_\alpha, \quad A = \sum_{\alpha=1}^{p} A_\alpha. \tag{27}$$

This will be rewritten as

$$A_\alpha v = \theta_\alpha + F_\alpha, \text{where } F_\alpha = -\sum_{\beta \neq \alpha}^{1 \div p} (A_\beta v - \theta_\beta). \tag{28}$$

Obviously,

$$\sum_{\alpha=1}^{p} F_\alpha = 0. \tag{29}$$

The further procedure is the same as in [10]. Introduce functions $v^{(0)} = v$, $v^{(1)} = v^2$, ... , $v^{(n)} = v^{2^n}$ and write the equation for v^{2^n} (see [10])

$$(30)\qquad A_\alpha v^{(n)} + \frac{1}{\hbar_\alpha}\sum_{k=0}^{n-1} 2^{n-k-1} v^{\nu_n-\nu_{k+1}}[h_\alpha^+ a_\alpha^+ ((T_\alpha v^k)^{(+0.5_\alpha)})^2 +$$

$$+ h_\alpha^- a_\alpha^- ((T_\alpha v^{(k)})^{(-0.5_\alpha)})^2] = 2^n v^{\nu_n}(F_\alpha + \theta_\alpha), \qquad \nu_k = 2^k - 1.$$

Multiply this equation by $\hbar_\alpha$ and sum over $x \in J_\alpha$; we get

$$(31)\qquad I_\alpha^{(n-1)} = 2^n[(v^{\nu_n}, \theta_\alpha)_{J_\alpha} + (v^{\nu_n}, F_\alpha)_{J_\alpha}],$$

where

$$(32)\qquad I_\alpha^{(n-1)} = 2(A_\alpha v^{(n-1)}, v^{(n-1)})_{J_\alpha} +$$

$$+\sum_{k=0}^{n-2} 2^{n-k-1}\left(\frac{v^{\nu_n-\nu_{k+1}}}{\hbar_\alpha}, h_\alpha^+ a_\alpha^+ ((T_\alpha v^{(k)})^+)^2 + h_\alpha^- a_\alpha^- ((T_\alpha v^{(k)})^-)^2\right)_{J_\alpha}$$

Here, $(T_\alpha v^{(k)})^\pm \equiv (T_\alpha v^{(k)})^{(\pm 0.5\alpha)}$.

Lemma 6

$$(33)\qquad v^{(n)} \leqslant M_0 I_\alpha^{(n-1)},$$

where $M_0 = D / 2c_1$ (see Lemma 1* of [10]).

1. *First a priori estimate.* Consider the first term on the right-hand side of (31). Using Lemma 6.

$$|2^n (v^{\nu_n}, \theta_\alpha)_{J_\alpha}| \leqslant 2^n (1, |\theta_\alpha|)_{J_\alpha} \max_{J_\alpha} |v^{\nu_n}| \leqslant$$

$$\leqslant 2^n (1, |\theta_\alpha|)_{J_\alpha} (\max_{J_\alpha} |v^{(n)}|)^{\nu_n/2^n} \leqslant 2^n (1, |\theta_\alpha|)_{J_\alpha} (M_0 I_\alpha^{(n-1)})^{1-1/2^n},$$

$$\nu_n = 2^n - 1, \quad v^{(n)} = v^{2^n}.$$

For the right-hand side of the last inequality:

$$(34)\qquad |x_1 x_2| \leqslant \frac{|x_1|^{q_1}}{q_1} + \frac{|x_2|^{q_2}}{q_2}, \qquad \frac{1}{q_1} + \frac{1}{q_2} = 1,$$

where

$$q_1 = 2^n, \qquad q_2 = 1/(1 - 1/2^n), \qquad x_1 = 2^n (1, |\theta_\alpha|)_{J_\alpha} (2M_0/q_2)^{1/q_2},$$

$$x_2 = (I_\alpha^{(n-1)} q_2/2)^{1/q_2};$$

so that finally:

$$(35)\qquad |2^n (v^{\nu_n}, \theta_\alpha)_{J_\alpha}| \leqslant {}^1\!/_2 I_\alpha^{(n-1)} + (1, |\theta_\alpha|)_{J_\alpha}^{2^n} (2M_0(2^n - 1))^{2^n-1}.$$

From (31) and (35):

$$
{}^1/_2 I_\alpha^{(n-1)} \leqslant (1, |\theta_\alpha|)_{J_\alpha}^{2^n} (2M_0(2^n-1))^{2^n-1} + 2^n (v^{\nu_n}, F_\alpha)_{J_\alpha};
$$

this equation is multiplied by s_α and summed over all chains J_α.
On strengthening the inequality, we get

$$
\begin{aligned}
(36) \qquad & {}^1/_2 \sum_{Q_\alpha} s_\alpha I_\alpha^{(n-1)} \leqslant [2M_0(2^n-1)]^{2^n-1} M_s \max_{Q_\alpha} (1, |\theta_\alpha|)_{J_\alpha}^{2^n} + \\
& + \sum_{Q_\alpha} s_\alpha (2^n v^{\nu_n}, F_\alpha)_{J_\alpha}, \qquad M_s = \max_\alpha \sum_{Q_\alpha} s_\alpha .
\end{aligned}
$$

Applying Lemma 2, the second term on the right-hand side of (36) can be rewritten as

$$
\sum_{Q_\alpha} s_\alpha (2^n v^{\nu_n}, F_\alpha) J_\alpha = \sum_\omega 2^n v^{\nu_n} F_\alpha H .
$$

Sum (36) over α from 1 to p. Recalling (29), we get

$$
(37) \qquad \frac{1}{2} \sum_{\alpha=1}^{p} \sum_{Q_\alpha} s_\alpha I_\alpha^{(n-1)} \leqslant [2M_0(2^n-1)]^{2^n-1} M_s \max_\alpha \max_{Q_\alpha} (1, |\theta_\alpha|)_{J_\alpha}^{2^n} .
$$

From (32) and the inequalities of Lemmas 2 and 5,

$$
\begin{aligned}
(38) \qquad & \sum_{\alpha=1}^{p} \sum_{Q_\alpha} s_\alpha I_\alpha^{(n-1)} \geqslant 2 \sum_{\alpha=1}^{p} \sum_{Q_\alpha} s_\alpha (A_\alpha v^{(n-1)}, v^{(n-1)})_{J_\alpha} = \\
& = 2 \sum_{\alpha=1}^{p} (A_\alpha v^{(n-1)}, v^{(n-1)}) = 2 (A v^{(n-1)}, v^{(n-1)}) \geqslant 2\delta \| v^{(n-1)} \|^2 .
\end{aligned}
$$

From (37) and (38) with $n = 1$, we obtain

$$
(39) \qquad \|v\| \leqslant M_1 \max_\alpha \max_{Q_\alpha} (1, |\theta_\alpha|) J_\alpha, \qquad M_1 = \sqrt{\frac{2M_0 M_s}{\delta}} .
$$

Notice (as in [11] that

$$
(40) \qquad \|v^{(n-1)}\|^2 = \sum_\omega (v^{2^{n-1}})^2 H = \sum_\omega v^{2^n} H \geqslant \|v\|_c^{2^n} H_* ,
$$

where

$$
H_* = \min H(x).
$$

Combining inequalities (37), (38) and (40), and extracting the $2n$-th root from the result, we get

$$
\|v\|_c \leqslant M_2 2^n (M_3 H_*)^{-1/2^n} \max_\alpha \max_{Q_\alpha} (1, |\theta_\alpha|) J_\alpha ,
$$

where $M_2 = 2M_0$, $M_3 = \delta / M_s$. Hence

Lemma 7

The solution of the problem $Av = \sum_{\alpha=1}^{m} \theta_\alpha$ satisfies

$$\| v \|_C \leqslant M 2^n H_*^{-1/2^n} \max_\alpha \max_{Q_\alpha} (1, | \theta_\alpha |) J_\alpha, \tag{41}$$

where M is a constant independent of n and h.

2. *Second a priori estimate.* A bound will be obtained for the solution of the problem

$$Av = \sum_{\alpha=1}^{p} T_\alpha^* g_\alpha. \tag{42}$$

Using Lemma 3 (Eq. (23)), we have

$$2^n (v^{\nu_n}, T_\alpha^* g_\alpha)_{J_\alpha} = 2^n (T_\alpha v^{\nu_n}, g_\alpha)_{\tilde{J}_\alpha}. \tag{43}$$

Substituting for $T_\alpha v^{\nu_n}$ in terms of $T_\alpha v^{(k)}$ and ν (see [10], Section 2, (19)),

$$(T_\alpha v^{\nu_n})^{(\pm 0.5\alpha)} = \sum_{k=0}^{n-1} (v^{(\pm 1\alpha)})^{\nu_k} v^{\nu_n - \nu_{k+1}} (T_\alpha v^{(k)})^{(\pm 0.5\alpha)} \tag{44}$$

in the right-hand side of (43), we get

$$2^n | (T_\alpha v^{\nu_n}, g_\alpha)_{\tilde{J}_\alpha} | \leqslant 2^n \sum_{\tilde{J}_\alpha} h_\alpha^+ | g_\alpha^{(+0.5\alpha)} | \sum_{k=0}^{n-1} v^{(\nu_n - \nu_{k+1})/2} | (T_\alpha v^{(k)})^{+} | \times \tag{45}$$
$$\times | (v^{(+1\alpha)})^{\nu_k} v^{(\nu_n - \nu_{k+1})/2} | .$$

Using Lemma 2, the following inequalities are obtained for the last two factors in the right-hand side of (45):

$$\max_{J\alpha} v^{(+1\alpha)\nu_k} \max_{J\alpha} v^{(\nu_n - \nu_{k+1})/2} = (\max_{J\alpha} v^{(n)})^{(\nu_k + (\nu_n - \nu_{k+1})/2)/2^n} \leqslant \tag{46}$$
$$\leqslant (M_0 I_\alpha^{(n-1)})^{1/2 - 1/2^n}$$

Using the Cauchy inequality, together with (7) and expression (32) for I_α^{n-1} we get

$$\sum_{\tilde{J}_\alpha} h_\alpha^+ | g_\alpha^{(+0.5\alpha)} | \sum_{k=0}^{n-1} [v^{(\nu_n - \nu_{k+1})/2} \sqrt{a_\alpha^+} | (T_\alpha v^{(k)})^{(+0.5\alpha)} | 2^{(n-k-1)/2}] / \sqrt{a_\alpha^+} 2^{(n-k-1)/2} \leqslant \tag{47}$$

$$\leqslant \frac{1}{\sqrt{c_1}}\Big(\sum_{\widetilde{J}\alpha} h_\alpha g_\alpha^2\Big)^{1/2}\Big\{\sum_{k=0}^{n-1} 2^{n-k-1}\Big(\frac{v^{\nu_n-\nu_{k+1}}}{\hbar_\alpha},\ a_\alpha^+[(T_\alpha v^{(k)})^+]^2\Big)_{J_\alpha}\Big\}^{1/2}\times$$

$$\times\Big(\sum_{m=0}^{n-1} 1/2^{n-m-1}\Big)^{1/2}\leqslant\frac{1}{\sqrt{c_1}}\|g_\alpha\|_{\widetilde{J}_\alpha}\sqrt{I_\alpha^{(n-1)}}.$$

Combining (45)–(47), we obtain

$$(48)\qquad |2^n(v^{\nu_n}, T_\alpha^* g_\alpha)_{J_\alpha}|\leqslant\frac{2^n}{\sqrt{(M_0c_1)}}\|g_\alpha\|_{\widetilde{J}_\alpha}(M_0 I_\alpha^{(n-1)})^{1-1/2^n}.$$

Finally, to obtain a bound for the right-hand side of (48), we use (34) after putting

$$(49)\qquad q_1=2^n,\quad q_2=1/(1-1/2^n),\quad x_1=\frac{2^n\|g_\alpha\|_{J_\alpha}}{\sqrt{(c_1M_0)}}(2M_0/q_2)^{1/q_2},$$

$$x_2=q_2I_\alpha^{(n-1)}/2.$$

The final result is

$$(50)\qquad |2^n(v^{\nu_n}, T_\alpha^* g_\alpha)_{J_\alpha}|\leqslant{}^1\!/_2 I_\alpha^{(n-1)}+2\sqrt{\frac{M_0}{c_1}}\Big[2\sqrt{\frac{M_0}{c_1}}(2^n-1)\Big]^{2^n-1}\times$$

$$\times\|g_\alpha\|^{2^n}_{\widetilde{J}_\alpha}.$$

Repeating the arguments used when deriving the first a priori estimate (see (50) and (35)), it can be seen that

$$(51)\qquad \|v\|\leqslant M\max_\alpha\max_{\widetilde{Q}_\alpha}\|g_\alpha\|_{\widetilde{J}_\alpha}$$

and

Lemma 8

The solution of problem (42) satisfies

$$(52)\qquad \|v\|_C\leqslant M2^nH_*^{-1/2^n}\max_\alpha\max_{\widetilde{Q}_\alpha}\|g_\alpha\|_{\widetilde{J}_\alpha}.$$

So far, n has been arbitrary. Now take an n with the following dependence on H_*:

$$(52')\qquad 0.5\log_2 V_0/H_*<2^n<\log_2 V_0/H_*,$$

where V_0 is the volume of G. From this,

$$2^nH_*^{-1/2^n}\leqslant M\ln V_0/H_*.$$

Using this in conjunction with (41) and (52), we arrive at
Lemma 9

The solutions of problems (27) and (42) satisfy

$$\text{(53)} \qquad \| v \|_C \leqslant M \ln (V_0/H_*) \max_{\alpha} \max_{Q_\alpha} | (1, | \theta_\alpha |)_{J_\alpha} |,$$

$$\text{(54)} \qquad \| \bar{v} \|_C \leqslant M \ln (V_0/H_*) \max_{\alpha} \max_{\widetilde{Q}_\alpha} \| g_\alpha \|_{\widetilde{J}_\alpha}.$$

6. *A priori estimates in the norm of* $W_2{}^1(\omega)$

A priori estimates of the solutions of problems (27) and (42) will be obtained in the mesh norm of $W_2{}^1(\omega)$. From paragraph 4 of Section 2,

$$(\mathring{A}v, v) = \sum_{\alpha=1}^{p} \| T_\alpha v \|_\alpha{}^2 = \sum_{\alpha=1}^{p} \sum_{\widetilde{\omega}_\alpha} H_\alpha v_{x_\alpha}^2 \qquad \text{for } v|_\gamma = 0,$$

where $\mathring{A} = \sum_{\alpha=1}^{p} \mathring{A}_\alpha$, $\mathring{A}_\alpha = T_\alpha{}^* T_\alpha$. We have

$$\text{(55)} \qquad A \geqslant c_1 \mathring{A}, \qquad \mathring{A} \geqslant \delta E,$$

where $\delta = 4p / 3D^2$, D is the diameter of the region G, and $0 < c_1 \leqslant a_\alpha{}^\pm$, $\alpha = 1, 2, \ldots, p$. Put $\varepsilon = \delta / (1 + \delta)$, then (55) gives (see also Paragraph 4 of Section 2)

$$\text{(55}') \qquad (\mathring{A}v, v) \geqslant \varepsilon (\mathring{A}v, v) + (1 - \varepsilon) \delta \|v\|^2 = \frac{\delta}{1 - \delta} \|v\|^2_{W_2{}^1}.$$

Consider the solution of problem (27). Form the scalar product of (27) with v. Applying the generalized Cauchy inequality: $(Av, w)^2 \leqslant (Av, v) \times (Aw, w)$, if $A = A^* \geqslant 0$ (see [12]) and the inequalities (55), we get

$$c_1 (\mathring{A}v, v) \leqslant (Av, v) = (v, \theta) = (\mathring{A}v, \mathring{A}^{-1}\theta) \leqslant$$
$$\leqslant \sqrt{[(\mathring{A}v, v)(\mathring{A}^{-1}\theta, \theta)]}.$$

From this and (55), recalling that $\|\mathring{A}^{-1}\| \leqslant 1 / \delta$, we get

Lemma 10

The solution of problem (27) satisfies

$$\|v\|_{W_2{}^1} \leqslant \frac{\sqrt{(1 + \delta)}}{c_1 \delta} \|\theta\|.$$

Consider the solution of problem (42). Form the scalar product of (42) with v. Recalling (55) and (22), we have

$$c_1 (\mathring{A}v, v) \leqslant (Av, v) = \sum_{\alpha=1}^{p} (g_\alpha, T_\alpha v)_\alpha \leqslant \left(\sum_{\alpha=1}^{p} \|g_\alpha\|_\alpha{}^2 \right)^{1/2} \sqrt{(\mathring{A}v, v)}.$$

From this and (55) we have

Lemma 11

The solution of (42) satisfies

$$\|v\|_{W_2^1} \leqslant \frac{1}{c_1}\left(\frac{1+\delta}{\delta}\right)^{1/2}\left(\sum_{\alpha=1}^{p}\|g_\alpha\|_\alpha^2\right)^{1/2}.$$

Finally, consider the solution of the problem

$$Av = \psi^*$$

with the condition that the right-hand side ψ^* vanishes at all regular mesh base-points: $\psi^* = 0$ for $x \in \omega_r$. To obtain a bound for the solution, Lemma 1 is used, after putting $\omega^* = \omega_{1r}$, $F^* = \psi^*$,while D is given by (17). Since $D(x) \geqslant C_1 / h^2$(see (17)), we obtain from (11):

$$\|v\|_C \leqslant h^2\|\psi^*\|_C / c_1$$

and accordingly,

$$\|v\| \leqslant h^2\sqrt{V_0}\|\psi^*\|_C / c_1.$$

To obtain a bound for the solution of the problem $Av = \psi^*$ in the norm of $W_2^1(\omega)$, the problem is multiplied scalarly by v and the inequality obtained above for $\|v\|_C$ is used, in conjunction with (55). We get

$$c_1(\mathring{A}v, v) \leqslant (Av, v) = (v, \psi^*) \leqslant h^2\|\psi^*\|_C(1, |\psi^*|)/c_1 \leqslant$$
$$\leqslant h^2\|\psi^*\|_C^2\sum_{\omega_{ir}} H/c_1 \leqslant Mh^3\|\psi^*\|_C^2.$$

This, with (55′), yields

Lemma 12

The solutions of the problem $Av = \psi^*$, $\psi^* = 0$ for $x \in \omega_r$ satisfies

$$\|v\|_C \leqslant Mh^2\|\psi^*\|_C, \qquad \|v\| \leqslant Mh^2\|\psi^*\|_C,$$
$$\|v\|_{W_2^1} \leqslant Mh^{3/2}\|\psi^*\|_C.$$

The next topics to be discussed are the convergence and accuracy of the solution of the initial difference problem in the mesh norms of $C(\omega)$, $L_2(\omega)$, $W_2^1(\omega)$.

3. Order of accuracy of the difference schemes

1. *Order of accuracy in the class of smooth coefficients*

Problems (9), (10) and (13) (k = 1, 2, 3) will be written in new notation.

Put

(56) $$\Phi = \varphi + \sum_{\alpha=1}^{p} \bar{\nu}_\alpha,$$

where

$$\bar{\nu}_\alpha = \frac{1}{\hbar_\alpha}\begin{cases} 0, & x \in \omega_\alpha, \\ a_\alpha^{\pm}\nu(x^{(\pm 1_\alpha)})/h_\alpha^{\pm}, & x \in \omega_{\gamma_\alpha}^{\pm}, \\ a_\alpha^{+}\nu(x^{(+1_\alpha)})/h_\alpha^{+} + a_\alpha^{-}\nu(x^{(-1_\alpha)})/h_\alpha^{-}, & x \in \omega_{\gamma_\alpha}^{0}. \end{cases}$$

Problems (9), (10) and (13) then become respectively

(57) $$Ay = \Phi, \qquad x \in \omega,$$

(58) $$Az = -\psi, \qquad x \in \omega,$$

(59) $$Av_k = -\overline{\Psi}_k, \qquad x \in \omega, \qquad k = 1, 2, 3.$$

Here, $\overline{\Psi}_k$, $k = 1, 2, 3$, are given by (11) and (13), and the error

$$z = y - u = v_1 + v_2 + v_3.$$

To obtain bounds for the functions v_k, $k = 1, 2, 3$, Lemmas 9–11 are used, together with the bounds (16), (39) and (51).

We will estimate v_1. From (11), we have

(60) $$\max_\alpha \max_{\widetilde{Q}_\alpha} \|\chi_\alpha\| J_\alpha \leqslant Mh^2, \qquad h = h_{\max}.$$

Lemmas 9 and 11 are now used after putting $g_\alpha \equiv \chi_\alpha$. From (54), (51), (60) and the inequality of Lemma 11 we have

(61) $$\|v_1\|_C \leqslant Mh^2 \ln(V_0/H_*), \qquad \|v_1\|_{W_2^1} \leqslant Mh^2, \qquad \|v_1\| \leqslant Mh^2.$$

To obtain a bound for v_2, Lemmas 9 and 10 are used, after putting $\theta_\alpha = \psi_\alpha$. From (11),

(62) $$\max_\alpha \max_{Q_\alpha} (1, |\psi_\alpha|) J_x \leqslant Mh^2.$$

From (39), (53), (62) and the inequality of Lemma 10 we obtain

(63) $$\|v_2\|_C \leqslant Mh^2 \ln(V_0/H_*), \qquad \|v_2\|_{W_2^1} \leqslant Mh^2, \qquad \|v_2\| \leqslant Mh^2.$$

Finally, consider v_3. The inequalities of Lemma 12 are used. Since

$$\psi^* = \sum_{\alpha=1}^{p} \psi_\alpha^* = 0 \quad \text{for} \quad x \in \omega_r, \qquad \|\psi^*\|_C = O(1),$$

these inequalities give

(64) $\|v_3\|_C \leqslant Mh^2, \qquad \|v_3\|_{W_2^1} \leqslant Mh^{3/2}, \qquad \|v_3\| \leqslant Mh^2.$

From (61), (63), (64), and (12),

(65) $\|z\|_C \leqslant Mh^2 \ln(V_0 / H_*), \qquad \|z\|_{W_2^1} \leqslant Mh^{3/2}, \qquad \|z\| \leqslant Mh^2.$

Theorem 1

The error $z = y - u$ in an arbitrary non-uniform mesh satisfies

(66) $\|y - u\|_C \leqslant Mh^2 \ln(V_0 / H_*).$

Theorem 2

The solution of the difference scheme (9) is convergent in the mesh norms of $L_2(\omega)$ and $W_2^1(\omega)$ in an arbitrary non-uniform mesh, and

(67) $\|y - u\| \leqslant Mh^2, \qquad \|y - u\|_{W_2^1} \leqslant Mh^{3/2}.$

Theorem 3

Let the lattice R_p^h be uniform with respect to each direction x_α, $\alpha = 1, 2, \ldots, p$. Then the difference scheme (9) is convergent in the mesh norms of $C(\omega)$, $L_2(\omega)$ and $W_2^1(\omega)$ and

$\|y - u\|_C \leqslant M|h|^2 \ln (V_0 / H), \qquad \|y - u\| \leqslant M|h|^2, \qquad \|y - u\|_{W_2} \leqslant M|h|^{3/2},$

where

$$|h|^2 = \sum_{\alpha=1} \hbar_\alpha^2, \qquad H = \prod_{\alpha=1}^{p} \hbar_\alpha.$$

Consider a sequence of meshes, non-uniform (as $h \to 0$) and such that

(68) $H_* \geqslant m_0 h^{\varkappa p}, \qquad m_0 = \text{const} > 0, \qquad \varkappa = \text{const} > 0.$

Inequalities (65) and (67), and the inequality $V_0 < D^p$ lead to

Theorem 4

With condition (68), the solution of the difference problem (9) is convergent to the solution of the initial problem in the mesh norm of $C(\omega)$. We have

(69) $\|y - u\|_C \leqslant Mh^2 \ln (D / h).$

Notes. 1. All our results remain true for the problem with operators

$$L_\alpha{}^q = L_\alpha - q_\alpha(x)E, \qquad q_\alpha(x) \geqslant 0, \qquad \alpha = 1, 2, \dots, p.$$

In this case, the operator

$$A_\alpha{}^q = A_\alpha + q_\alpha E.$$

has to be taken as A_α.

2. Let the boundary Γ be such that a sequence of meshes $R_p{}^h$, (as $h \to 0$) can be constructed, matched to the boundary Γ in the sense that $\gamma \in R_p{}^h$. Then, $\psi^* = 0$, $\omega_{1\Gamma} = \varnothing$ and

(70) $$\| y - u \|_{W_2{}^1} \leqslant Mh^2.$$

2. *Convergence of the scheme in the class of discontinuous coefficients*

Let the coefficients $k_\alpha(x)$, $\alpha = 1, 2, \dots, p$, and the function $f(x)$, have discontinuities on a finite number of non-intersecting surfaces Σ_s, $s = 1, 2, \dots, K$. The problem may then be stated as follows: to find the function $u = u(x)$, continuous in the closed region $\bar{G}$, and satisfying the equation and boundary condition

(71) $$Lu = -f(x) \quad \text{for} \quad x \in G \setminus (\bigcup_{s=1}^{k} \Sigma_s) \equiv Q,$$
$$u = \nu(x) \quad \text{for} \quad x \in \Gamma,$$

while on the surfaces Σ_s, $s = 1, 2, \dots, K$, it satisfies the conjugation conditions

(72) $$[u]_{\Sigma_s} = 0, \qquad \left[\sum_{\alpha=1}^{p} k_\alpha \frac{\partial u}{\partial x_\alpha} \cos(\widehat{n_s, x_\alpha}) \right]_{\Sigma_s} = 0, \qquad s = 1, 2, \dots, K.$$

Here, $[v(x)]_{\Sigma_s} = v^+(x) - v^-(x)$ denotes the difference between the limits of $\nu(x)$ on opposite sides of Σ_s, while n_s is the normal to Σ_s.

It will be assumed that problem (71) – (72) has a solution, i.e., there exists $u = u(x) \in Q^{(4)}(\bar{G})$, $k_\alpha \in Q^{(3)}(\bar{G})$, $f(x) \in Q^{(2)}(\bar{G})$ ($Q^{(n)}(\bar{G})$ is the class of functions having piecewise smooth n-th derivatives in $\bar{G}$). This problem was investigated in detail in [13, 16].

Some further classification of base-points is required. It will be said that a base-point $x \in \omega_\Sigma$, if the corresponding volume $H(x)$ intersects at least one of the surfaces Σ_s. Denote by ω_Q the complement of ω_Σ up to ω, so that

$$\omega = \omega_\Sigma \cup \omega_Q.$$

It will be said that an intermediate base-point $x^{(\pm 0.5\alpha)} \in \tilde{\omega}_{\Sigma,\alpha}$, if the corresponding volume $H_\alpha^\pm = H_\alpha(x^{(\pm 0.5\alpha)})$ intersects at least one of the surfaces Σ_S.

Denote by $\tilde{\omega}_{Q,\alpha}$, the complement of $\tilde{\omega}_{\Sigma,\alpha}$ up to $\tilde{\omega}_\alpha$ so that

$$\tilde{\omega}_\alpha = \tilde{\omega}_{\Sigma,\alpha} \cup \tilde{\omega}_{Q,\alpha}.$$

The convergence of homogeneous schemes of the type (9) will be examined. Let the coefficients $a_\alpha^\pm$, $\alpha = 1, 2, \ldots, p$, satisfy the approximation condition (7) at the base-points $x^{(\pm 0.5\alpha)} \in \tilde{\omega}_{Q,\alpha}$. At base-points $x^{(\pm 0.5\alpha)} \in \tilde{\omega}_{\Sigma,\alpha}$ it will only be required that

$$0 < c_1 \leqslant a_\alpha^\pm.$$

For instance, we put

$$a_\alpha^\pm = 0.5(k_\alpha^+(x^{(\pm 0.5\alpha)}) + k_\alpha^-(x^{(\pm 0.5\alpha)})) \quad \text{for} \quad x^{(\pm 0.5\alpha)} \in \tilde{\omega}_\alpha \tag{73}$$

($a_\alpha^\pm = k_\alpha(x^{(\pm 0.5\alpha)})$, if $k_\alpha^+ = k_\alpha^-$, the surfaces Σ_S do not pass through base-points $x^{(\pm 0.5\alpha)}$) At base-points $x \in \omega_Q$ the function $\varphi = \varphi(x)$ is assumed to satisfy the approximation condition (8), while when $x \in \omega_\Sigma$ it is merely assumed to be bounded by a constant independent of the mesh ω. For instance, it can be assumed that everywhere

$$\varphi = 0.5(f^+ + f^-) \quad \text{for} \quad x \in \omega. \tag{74}$$

The initial problem (71), (72) will be associated with the difference problem

$$Ay = \Phi \quad \text{for } x \in \omega, \tag{75}$$

where A and Φ are given by (19), (20), and (56). The problem

$$Az = -\Psi \quad \text{for} \quad x \in \omega, \tag{76}$$

is obtained for the error $z = y - u$, where Ψ is the approximation error:

$$\Psi = -Au + \Phi = \Lambda u + \varphi. \tag{77}$$

1. *Convergence in the mesh norm of* $W_2^{\,1}(\omega)$. While an arbitrary disposition of the surfaces Σ_S, $s = 1, 2, \ldots, K$, relative to the mesh ω base-points will be assumed, the treatment will be kept reasonably simple by confining the proof of the convergence of scheme (75) to regions G composed of p-dimensional parallelepipeds with boundaries parallel to the coordinate planes (stepped regions).

To transform the expression (77) for the approximation error, the identity (6) is subtracted from (77), yielding

$$\psi = \sum_{\alpha=1}^{p} (\chi_\alpha)_{\hat{x}_\alpha} + \theta, \tag{78}$$

where

$$\chi_\alpha^\pm = a_\alpha^\pm u_{x_\alpha}^\pm - \frac{1}{s_\alpha^\pm}\int_{s_\alpha^\pm} w_\alpha^- \, ds_\alpha^\pm, \qquad \theta = \varphi - \frac{1}{H}\int_H f\, dH.$$

Obviously, under our assumptions regarding the coefficients $a_\alpha^\pm$, and φ.

$$(79) \qquad \chi_\alpha^\pm = \begin{cases} O(h), & x^{(\pm 0.5_\alpha)} \in \tilde{\omega}_{Q,\alpha}, \\ O(1), & x^{(\pm 0.5_\alpha)} \in \tilde{\omega}_{\Sigma,\alpha}, \end{cases} \qquad \theta = \begin{cases} O(h), & x \in \omega_Q, \\ O(1), & x \in \omega_\Sigma. \end{cases}$$

To obtain a bound for the solution of problem (76) – (79), the function z is written as

$$(80) \qquad z = v_1 + v_2,$$

where v_1 and v_2 are the solutions of the problems

$$(81) \qquad Av_1 = -\theta \text{ for } x \in \omega,$$

$$(82) \qquad Av_2 = -\sum_{\alpha=1}^{p} T_\alpha{}^* \chi_\alpha \qquad \text{for} \qquad x \in \omega.$$

Lemmas 10 and 11 are used to find bounds for v_1 and v_2, Recalling (80), we obtain

$$(83) \qquad \|z\|_{W_2^1} \leqslant M' \left(\|\theta\| + \sum_{\alpha=1} \|\chi_\alpha\|_\alpha \right).$$

From (79),

$$(84) \qquad \|\theta\| \leqslant M\sqrt{h}, \qquad \|\chi_\alpha\|_\alpha \leqslant M\sqrt{h}.$$

From (83) and (84) we obtain

Theorem 5

The scheme (75) in the class of discontinuous coefficients is convergent to the solution of problem (71), (72). We have

$$(85) \qquad \|y - u\|_{W_2^1} \leqslant M\sqrt{h}.$$

Notice that, though the bound (85) is extremely crude, it proves the convergence of homogeneous schemes, the coefficients of which can be evaluated from very simple expressions, e.g.,

$$a_\alpha^\pm = 0.5\left(k_\alpha^+(x^{(\pm 0.5\alpha)}) + k_\alpha^-(x^{(\pm 0.5\alpha)})\right),$$

$$\varphi = 0.5(f^+(x) + f^-(x)).$$

In the case of an arbitrary region also, a bound with the accuracy of (85) holds for the scheme (75) in the class of discontinuous coefficients.

2. *On uniform convergence.* When discussing whether the scheme (75) is uniformly convergent in the class of discontinuous coefficients, the treatment will be confined to one particular case. It will not be assumed that G is a stepped region.

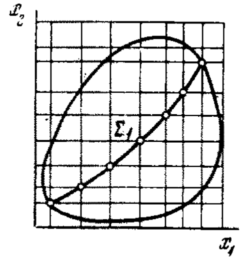

FIG. 4

Consider the problem (71), (72) with $p = 2$, with the following assumptions:

(a) the coefficients k_α, $\alpha = 1, 2$, and f have just one line of discontinuity Σ_1,

(b) the line Σ_1 is such that a sequence of meshes ω (with $h \to 0$) can be constructed, matched to the line Σ_1 in the following sense: any line C_α, parallel to a coordinate axis ox_α, $\alpha = 1, 2$, and passing through a base-point $x \in \omega$, cuts Σ_1 only at base-points of ω,

(c) every straight line C_α cuts the curve Σ_1 at just one point (see Fig. 4).

It should be said at once that assumptions (a) and (c) are made merely in order to simplify the treatment.

The same assumptions as in paragraph 1 are made regarding the existence and smoothness of the solution and coefficients of the problem. The base-points $x \in \bar{\omega} \cap \Sigma_1$ divide the curve Σ_1 into arcs of lengths $\sigma_1, \sigma_2, \ldots, \sigma_n$. The mesh is chosen in such a way that

$$(86) \qquad |\sigma_{i+1} - \sigma_i| \leqslant Mh \max_i \sigma_i.$$

The convergence of the scheme discussed below will be considered on the basis of a sequence of meshes $\bar{\omega}$ matched with Σ_1 and satisfying condition (86).

Take the scheme (75) in which the coefficients $a_\alpha^\pm$ and the function φ are given by the elementary expressions

$$(87) \qquad a_\alpha^\pm = k_\alpha(x^{(\pm 0.5_\alpha)}), \qquad \varphi = 0.5(f^+(x) + f^-(x)).$$

The error $z = y - u$ satisfies (76) with right-hand side ψ given by (77). Using a Taylor expansion, the approximation error ψ can be written as

$$(88) \qquad \psi = \sum_{\alpha=1}^{2} \Psi_\alpha, \qquad \Psi_\alpha = T_\alpha^* \chi_\alpha + \psi_\alpha + \psi_\alpha^*,$$

where, with condition (86),

$$(89) \qquad \psi_\alpha = \begin{cases} O(\hbar_\alpha^2), & x \in \omega \setminus \Sigma_1, \\ O(\hbar_\alpha), & x \in \Sigma_1, \end{cases}$$

$$\psi_\alpha^* = \begin{cases} 0, & x \in \omega_r, \\ O(1), & x \in \omega_{1r}, \end{cases} \qquad \chi_\alpha^\pm = O((h_\alpha^\pm)^2).$$

As before, the error z is written as a sum

$$(90) \qquad z = v_1 + v_2 + v_3,$$

where ν_1 is the solution of problem (76) with right-hand side $-\sum_{\alpha=1}^{p} (\chi_\alpha)_{\hat{x}_\alpha} \equiv \sum_{\alpha=1}^{p} T_\alpha^* \chi_\alpha$, ν_2 is the solution with right-hand side $\sum_{\alpha=1}^{p} \psi_\alpha$, and ν_3 is the solution with right-hand side $\psi^* = \sum_{\alpha=1}^{p} \psi_\alpha^*$. Bounds can be found for ν_1, ν_2 and ν_3 by means of Lemmas 9 – 12 and (39) and (51). From (89), (51), (39), (53),(54) and the inequalities of Lemmas 10 and 11,

$$\|v_k\|_C \leqslant Mh^2 \ln(V_0 / H_*), \qquad \|v_k\| \leqslant Mh^2,$$

$$\|v_k\|_{W_2^1} \leqslant Mh^2, \qquad k = 1, 2.$$

As regards ν_3, the following is obtained from (89) and the inequalities of Lemma 12, as in the case of continuous coefficients:

$$\|v_3\|_C \leqslant Mh^2, \qquad \|v_3\| \leqslant Mh^2, \qquad \|v_3\|_{W_2^1} \leqslant Mh^{3/2}.$$

Note. If condition (86) is not satisfied, then in (89) $\psi_\alpha = O(1)$ for $x \in \Sigma_1$ and the following bounds are obtained for ν_1:

$$\| v_1 \|_C \leqslant Mh \ln (V_0 / H_*), \qquad \| v_1 \| \leqslant Mh.$$

Combining the bounds for ν_1, ν_2 and ν_3, and recalling (90), we arrive at

Theorem 6

The solution of problem (75), (87) is convergent on a sequence of meshes ω, matched to the curve Σ_1 on which the problem coefficients are discontinuous, and satisfying conditions (a),(b) and (c), to the solution of problem (71), (72). When condition (86) is satisfied,

$$\|y-u\|_C \leqslant Mh^2 \ln (V_0 / H_*), \qquad \|y-u\| \leqslant Mh^2,$$
$$\|y-u\|_{W_2^1} \leqslant Mh^{3/2},$$

while when conditions (86) and (68) are satisfied,

$$\|y-u\|_C \leqslant Mh^2 \ln (D / h).$$

Notes. 1. Let the mesh $\bar{\omega}$ be matched with the boundaries $\Sigma_1 \ldots \Sigma_K$ of discontinuities in the case of $p \geqslant 2$ dimensions. Then,

$$\|y-u\|_C \leqslant Mh \ln (V_0 / H\ ), \qquad \|y-u\|_{W_2^1} \leqslant Mh,$$

while when condition (68) is satisfied,

$$\|y-u\|_C \leqslant Mh \ln (D / h).$$

2. If the mesh $\bar{\omega}$ is not matched with the line (or surface, if $p > 2$) of discontinuity, it is possible to show in a number of particular cases, with additional conditions on the mesh steps and their ratios, that

$$\|y-u\|_C \leqslant M\sqrt{h} \ln (V_0 / H\ ),$$

while if (68) is satisfied,

$$\|y-u\|_C \leqslant M\sqrt{h} \ln (D / h).$$

REFERENCES

1. BRAMBLE, J. H., HUBBARD, B. E. and VIDAR, T. Convergence estimates for essentially positive-type discrete Dirichlet problems, *Math. comput.*, **23**, 108, 695–709, 1969.

2. MIKELADZE, Sh. E. Numerical integration of elliptic and parabolic equations, *Izv. Akad. Nauk SSSR, Ser. matem,*, **5**, 1, 57–74, 1941.

3. WASOW, W. and FORSYTHE, G. E. Finite-difference methods for partial differential equations, Wiley, 1960.

4. TIKHONOV, A. N. and SAMARSKII, A. A. The equations of mathematical physics *(Uravneniya matematicheskoi fiziki)*, Nauka, Moscow, 1966.

5. SAMARSKII, A. A. Lectures on the theory of finite-difference schemes *(Lektsii po teorii raznostnykh skhem), Chap. IV, VTs Akad, Nauk SSSR,* Moscow, 1969.

6. GERSCHGORIN, S. A. Fehlerschätzung für das Differenzverfahren zür Lösung partiellen Differentialgleichungen, *Z. angew. Math. and Mech.*, **10**, 4, 373–382, 1930.

7. VOLKOV, E. A. A non-uniform mesh method for finite and infinite regions with conical points, *Differents. ur-niya,* **11**, 10, 1358–1373, 1966.

8. ANDREEV, V. B. On uniform convergence of some difference schemes, *Zh. vȳchisl. Mat. mat. Fiz.*, **6**, 2, 238–250, 1966.

9. GREENSPAN, D. Lectures on the numerical solution of linear singular and non-linear differential equations, Univ. of Michigan, New York, 1968.

10. SAMARSKII, A. A. A priori estimates for the solution of the difference analogue of a parabolic differential equation, *Zh. vȳchisl. Mat. mat. Fiz.*, **1**, 3, 441–460, 1961.

11. SAMARSKII, A. A. Logically homogenous difference schemes on non-uniform meshes, *Zh. vȳchisl. Mat. mat. Fiz.*, **3**, 3, 431–465, 1963.

12. KANTOROVICH, A. V. and AKILOV, G. P. Functional analysis in normed spaces *(Funktsional'nyi analiz v normirovannykh prostranstvakh)*, *Fizmatgiz*, Moscow, 1959.

13. IL'IN, V. A. On the solvability of the Dirichlet and Neumann problems for a linear elliptic operator, *Dokl. Akad, Nauk SSSR*, 1961, **137**, 1, 28–31, 1961.

14. IL'IN, V. A. and SHISHMARIV, V. A. A potential method for Dirichlet and Neumann problems when the equations have discontinuous coefficients, *Siberskii matem. Zh.* 2, 1, 46–58, 1961.

15. OLEINIK, O. A. Solution of the basic boundary value problems for second-order equations with discontinuous coefficients, *Dokl. Akad, Nauk, SSSR*, **124**, 6, 1219–1223, 1959.

16. LADYZHENSKAYA, O. A. and URAL'TSEVA, N. N. Linear and quasi-linear elliptic equations *(Lineinye i kvazilineinye uravneniya ellipticheskogo tipa)*, Nauka, Moscow, 1964.