ON THE CONVERGENCE OF A LOCALLY ONE-DIMENSIONAL SCHEME FOR SOLVING THE MULTIDIMENSIONAL EQUATION OF HEAT CONDUCTION ON NON-UNIFORM MESHES*

A. A. SAMARSKII and I. V. FRYAZINOV

Moscow

(*Received* 3 *September* 1970)

WE examine the convergence of a locally one-dimensional scheme (see [1-5]) for solving, in a sequence of non-uniform meshes, the first boundary value problem in an arbitrary region, and the second and third boundary value problems in stepped regions, for the equation of heat conduction containing no mixed derivatives.

It will be shown that the schemes considered are convergent in the mesh norm of C in a sequence of non-uniform meshes at the rate $O(h^2 \ln (V_0 / H_*) + \tau)$, where τ is the mesh time-step, V_0 the volume of the region G, and h the maximum step of the space mesh R_p^h:

$$h = \max_{x_i \in \overline{G}} \max_{1 \leqslant \alpha \leqslant p} \hbar_\alpha (x_i),$$

where $\hbar_\alpha(x_i)$ is the mean step of R_p^h at the base-point x_i in the direction of the coordinate axis ox_α, $\alpha = 1, 2, \ldots, p$, and p is the number of dimensions; H_* is the minimum cell volume,

$$H_* = \min_{x_i \in \overline{G}} H(x_i), \qquad H(x_i) = \prod_{\alpha=1}^{p} \hbar_\alpha (x_i).$$

The convergence rate in a non-uniform mesh is estimated in the present paper by using both the maximum principle and the method of energy inequalities of the n-th rank developed in [6] (see also [3]); this allows the solution of the finite-difference problem to be estimated in the mesh norm of L_{2^n}, where n is an arbitrary integer. From this estimate, the convergence in the mesh norm of C is obtained. If only the maximum principle and the associated theorems are used, too low a convergence rate estimate is obtained for the scheme. When

Zh. vȳchisl. Mat. mat. Fiz.*, **11, 3, 642-657, 1971.

obtaining *a priori* estimates of the n-th rank, the finite-difference operators Λ_α, corresponding to the differential operators $(\partial / \partial x_\alpha)(k_\alpha(x, t)\partial / \partial x_\alpha)$ (and to homogeneous boundary conditions) would have to be negative in any non-uniform mesh. The operators chosen in [3] do not retain their negative properties for all regions G and meshes R_p^h. In the present paper, negative operators (on any non-uniform mesh in any region), as introduced in [7], are used for Λ_α.

Section 1 and 2 will be concerned with constructing, and investigating the convergence of, a locally one-dimensional scheme in a non-uniform mesh in an arbitrary region, in the case of the first boundary value problem. In Section 3 a locally one-dimensional scheme is developed for solving the second and third boundary value problems in stepped regions. These schemes are also uniformly convergent at a rate $O(h^2 \ln(V_0 / H_*) + \tau)$.

1. Formulation of the problem

1. THE INITIAL PROBLEM

Let $x = (x_1, x_2, \ldots, x_p)$ be a point of p-dimensional space R_p, G a region bounded in R_p with boundary Γ, and $\bar{G} = G \cup \Gamma$ It is assumed that the intersection of G with a straight line through the point $x \in G$ and parallel to the axis ox_α consists of a finite number of intervals. For simplicity, it will be assumed that the intersection in question in fact consists of just one interval $\Delta_\alpha = \Delta_\alpha(x)$. We put $Q_T = G \times (0 < t \leqslant T)$, $\bar{Q}_T = \bar{G} \times (0 \leqslant t \leqslant T)$.

Consider the following problem: find the function $u(x, t)$, continuous in $\bar{Q}_T$, and satisfying the equation

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} (L_\alpha u + f_\alpha(x, t)), \qquad (x, t) \in Q_T, \tag{1}$$

and the boundary and initial conditions

$$\begin{aligned} u(x, t) &= v(x, t) \quad \text{if} \quad x \in \Gamma, \qquad 0 \leqslant t \leqslant T; \\ u(x, 0) &= u_0(x), \qquad x \in \bar{G}. \end{aligned} \tag{2}$$

Here, L_α, $\alpha = 1, 2, \ldots, p$, are one-dimensional elliptic operators

$$L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t)\frac{\partial u}{\partial x_\alpha}\right), \qquad k_\alpha(x, t) \geqslant c_1 = \text{const} > 0. \tag{3}$$

It will be assumed that the problem (1)-(3) has a unique solution, reasonably smooth in $\bar{Q}_T$. The same assumptions as in [3] will be made regarding the smoothness of the input data, i.e. the functions $k_\alpha(x, t)$, $f_\alpha(x, t)$, $\alpha = 1, 2, \ldots,$

p, $\nu(x, t)$, $u_0(x)$ and of the solution $u = u(x, t)$ of the problem (1)-(3).

2. THE MESH. CLASSIFICATION OF THE BASE-POINTS

The same notation as in [7] will be used. A mesh $\bar{\omega}$ is constructed in $\bar{G}$ and its base-points classified in the same way as in [7].

We draw p families of hyperplanes

$$x_\alpha = x_\alpha^{(i_\alpha)}, \quad i_\alpha = 0, \pm 1, \ldots, \quad \alpha = 1, 2, \ldots, p, \quad x_\alpha^{(i_\alpha)} > x_\alpha^{(i_\alpha - 1)}.$$

The points $x_i = (x_1^{(i_1)}, \ldots, x_p^{(i_p)})$ of intersection of the hyperplanes will be said to form a mesh $R_p{}^h$ in space R_p. Points x_i of the mesh belonging to G will be called interior base-points; the set of interior base-points is denoted by ω, $\omega = R_p{}^h \cap G$. The set of ends of the intervals $\Delta_\alpha(x)$ drawn through the base-points $x \in \omega$ will be called the set of boundary base-points with respect to the direction x_α (with respect to x_α) and will be denoted by γ_α; $\gamma_\alpha = \gamma_\alpha{}^+ \cup \gamma_\alpha{}^-$, where γ_α^+ and γ_α^- are the sets of right-hand and left-hand boundary base-points with respect to x_α, while $\gamma = \bigcup_{\alpha=1}^{p} \gamma_\alpha$ is the set of boundary base-points. The set of interior and boundary base-points will be termed the mesh ω in $\bar{G}$, $\bar{\omega} = \omega \cup \gamma$. The set of base-points lying in the interval Δ_α will be denoted by Z_α. The set consisting of the base-points $x \in Z_\alpha$ and of the ends of intervals Δ_α is denoted by $\bar{Z}_\alpha$. Denote by $x^{(+1_\alpha)}$ and $x^{(-1_\alpha)}$ the base-points nearest to $x \in \bar{Z}_\alpha$ on its right and left, and belonging to $\bar{Z}_\alpha$. They will be called the neighbours of x with respect to x_α:

$$x^{(\pm 1_\alpha)} = (x_1^{(i_1)}, \ldots, x_{\alpha-1}^{(i_{\alpha-1})}, x_\alpha^{(i_\alpha \pm 1)}, x_{\alpha+1}^{(i_{\alpha+1})}, \ldots, x_p^{(i_p)}).$$

We shall say that $x \in \omega$ is a near-boundary or frontier base-point if one or more of its neighbours, with respect to any direction x_α, $\alpha = 1, 2, \ldots, p$, belongs to γ. The set of frontier base-points is denoted by ω_γ, and the set of all other interior base-points by ω^0, so that $\omega = \omega_\gamma \cup \omega^0$.

The distances between a base-point $x \in \bar{\omega}$ and its neighbours $x^{(\pm 1_\alpha)} \in \bar{\omega}$ will be termed the mesh steps and denoted by $h_\alpha{}^\pm$, $\alpha = 1, 2, \ldots, p$,

$$h_\alpha{}^+(x) = x^{(+1_\alpha)} - x, \quad h_\alpha{}^-(x) = x - x^{(-1_\alpha)}, \quad h_\alpha{}^+(x) = h_\alpha{}^-(x^{(+1_\alpha)}).$$

The steps

$$\hbar_\alpha(x) = \hbar_\alpha(x_\alpha^{(i_\alpha)}) = 0.5\,(x_\alpha^{(i_\alpha+1)} - x_\alpha^{(i_\alpha-1)}),$$

where $x_\alpha^{(i_\alpha \pm 1)}$ are the coordinates of points of the mesh $R_p{}^h$ (but not of the mesh

$\bar{\omega}$), will also be introduced. Obviously, $\hbar_\alpha$ depends only on the coordinate x_α (the index i_α). At base-points $x \in \omega^0$ we have $\hbar_\alpha = 0.5(h_\alpha^+ + h_\alpha^-)$. At frontier base-points

$$\hbar_\alpha \geqslant 0.5(h_\alpha^+ + h_\alpha^-) \equiv \hbar_\alpha^*.$$

Each $x \in \omega$ is associated with a volume $H(x)$, bounded by the pieces $s_\alpha^\pm$ of hyperplanes orthogonal to the axes ox_α and passing through the points $(x^{(\pm 1_\alpha)} + x)/2$, $x^{(\pm 1_\alpha)}$, $x \in R_p^h$:

$$H(x) = \prod_{\alpha=1}^{p} \hbar_\alpha(x), \qquad s_\alpha^\pm = \prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{p} \hbar_\beta(x).$$

Here and throughout, the same letters are used to denote a body and its volume, or a figure and its area. Write ω_α for the set of transitional base-points $x^{(\pm 0.5_\alpha)} \in G$ with respect to x_α, and $\tilde{Z}_\alpha$ for the chain of transition base-points $x^{(\pm 0.5_\alpha)} \in \tilde{\omega}_\alpha$ and lying in Δ_α:

$$x^{(\pm 0.5_\alpha)} = (x_1^{(i_1)}, \ldots, x_{\alpha-1}^{(i_\alpha - 1)}, x_\alpha^{(i_\alpha \pm 0.5)}, x_{\alpha+1}^{(i_\alpha+1)}, \ldots, x_p^{(i_p)}),$$

where

$$x_\alpha^{(i_\alpha \pm 0.5)} = x_\alpha^{(i_\alpha)} \pm 0.5 h_\alpha^\pm.$$

Each transitional base-point $x^{(\pm 0.5_\alpha)} \in \tilde{\omega}_\alpha$ is associated with a volume $H_\alpha^\pm = s_\alpha^\pm h_\alpha^\pm$. Denote by h the maximum mean step $\hbar_\alpha$ of the mesh ω:

$$h = \max_{x_i \in G} \max_{1 \leqslant \alpha \leqslant p} \hbar_\alpha(x_i).$$

Finally, introduce the time mesh ω_τ with step τ and fractional step τ/p:

$$\omega_\tau = \left\{ t_{j+\alpha/p} = t_j + \frac{\alpha}{p}\tau,\ j = 0, 1, \ldots, T/\tau - 1 = j_0,\ \alpha = 1, 2, \ldots, p \right\}.$$

All positive constants independent of h and τ will in future be denoted by the same letter M.

3. MESH FUNCTIONS. OPERATORS

Let $y = y(x)$ be a mesh function defined at the base-points of $\bar{\omega}$. We put

$$y^{(\pm 1_\alpha)} = y(x^{(\pm 1_\alpha)}), \quad y_{x_\alpha}^{+} = (y^{(+1_\alpha)} - y)/h_\alpha^+, \quad y_{\bar{x}_\alpha}^{-} = (y - y^{(-1_\alpha)})/h_\alpha^-.$$

With transitional base-points $x^{(\pm 0.5_\alpha)} \in \tilde{\omega}_\alpha$. will be associated the function

$y_{x_\alpha}^{\pm}$, representing the analogue of the derivative $\partial u/\partial x_\alpha$. With the coefficients $k_\alpha(x, t)$ will be associated mesh functions $a_{\alpha t} = a_\alpha(x^{(\pm 0.5_\alpha)}, t)$. It will be assumed that

$$a_\alpha^\pm = k_\alpha(x^{(\pm 0.5_\alpha)}, t) + O((h_\alpha^\pm)^2), \qquad a_\alpha^\pm \geqslant c_1 > 0. \tag{4}$$

With the functions $w_\alpha = k_\alpha(x, t)\partial u / \partial x_\alpha$, $\alpha = 1, 2, \ldots, p$, will be associated mesh functions W_α, defined at base-points of $\tilde{\omega}_\alpha$: $W_\alpha^\pm = W_\alpha(x^{(\pm 0.5_\alpha)}, t) \equiv a_\alpha^\pm y_{x_\alpha}^\pm$, $\alpha = 1, 2, \ldots, p$. We put $(\chi_\alpha)_{\hat{x}_\alpha} \equiv (\chi_\alpha^+ - \chi_\alpha^-) / \hbar_\alpha$, where $\chi_\alpha^\pm = \chi_\alpha(x^{(\pm 0.5_\alpha)}, t)$ is a function defined at transitional points of $\tilde{\omega}_\alpha$.

The differential operator L_α will be associated with the difference operator Λ_α:

$$\Lambda_\alpha y = (W_\alpha)_{\hat{x}_\alpha} = \frac{1}{\hbar_\alpha}\left[a_\alpha^+ \frac{y^{(+1_\alpha)} - y}{h_\alpha^+} - a_\alpha^- \frac{y - y^{(-1_\alpha)}}{h_\alpha^-}\right]. \tag{5}$$

With the functions f_α we associated the mesh functions ϕ_α, $\alpha = 1, 2, \ldots, p$, satisfying the approximation conditions

$$\varphi_\alpha = f_\alpha^{j+1} + (\mu_\alpha)_{\hat{x}_\alpha} + \bar{\mu}_\alpha, \qquad \mu_\alpha^\pm = O((h_\alpha^\pm)^2), \qquad \bar{\mu}_\alpha = O(\hbar_\alpha^2). \tag{6}$$

Let $\mathcal{H}$ be the space of mesh functions, defined on $\bar{\omega}$ and vanishing in γ, with the scalar product

$$(y, z) = \sum_\omega Hyz \quad \text{and norm} \quad \|y\| = \sqrt{(y, y)}.$$

It was shown in [7] that the operators Λ_α, $\alpha = 1, 2, \ldots, p$, are self-conjugate and negative-definite. Let the mesh functions $z(x)$ and $y(x)$ vanish at base-points $x \in \gamma$; then,

$$(\Lambda_\alpha z, y) = (z, \Lambda_\alpha y), \qquad -(\Lambda_\alpha z, z) \geqslant \delta_0 \|z\|^2, \tag{7}$$

where $\delta_0 = 4c_1 / 3D^2$, and D is the diameter of the region G.

4. THE FINITE-DIFFERENCE PROBLEM AND APPROXIMATION ERROR

The initial problem (1)-(3) will be associated with the finite-difference problem: find the mesh function $y(x, t)$, satisfying the conditions

$$y_0 \equiv y^j, \quad x \in \bar{\omega};$$

$$\frac{y_\alpha - y_{\alpha-1}}{\tau} = \Lambda_\alpha y_\alpha + \varphi_\alpha, \quad x \in \omega, \quad y_\alpha = \nu(x, t_{j+1}), \quad x \in \gamma_\alpha; \tag{8}$$

$$\alpha = 1, 2, \ldots, p, \quad y^{j+1} \equiv y_p, \quad x \in \bar{\omega}, \quad j = 0, 1, \ldots, j_0;$$

$$y^0 = u_0(x), \quad x \in \bar{\omega},$$

where the operators Λ_α are given by (5), and the mesh functions $a_\alpha^\pm$ and ϕ_α satisfy the approximation conditions (4) and (5), while $j_0 = T/\tau - 1$.

The accuracy of the scheme (8) is determined by the error $z_\alpha = y_\alpha - u(x, t_{j+1})$, $\alpha = 1, 2, \ldots, p$, $z^j = y^j - u(x, t_j)$, $j = 0, 1, \ldots, j_0$, where y is the solution of problem (8), and $u(x, t)$ the solution of problem (1)-(3). Substituting $y_\alpha = z_\alpha + u(x, t_{j+1})$, $y^j = z^j + u(x, t_j)$, for z in (8), we obtain the problem

$$z_0 \equiv z^j, \quad x \in \bar{\omega};$$

$$\frac{z_\alpha - z_{\alpha-1}}{\tau} = \Lambda_\alpha z_\alpha + \Psi_\alpha, \quad x \in \omega; \quad z_\alpha = 0, \quad x \in \gamma; \tag{9}$$

$$\alpha = 1, 2, \ldots, p, \quad z^{j+1} \equiv z_p, \quad x \in \bar{\omega}, \quad j = 0, 1, \ldots, j_0;$$

$$z^0 = 0, \quad x \in \bar{\omega},$$

where Ψ_α is the approximation error of the α-th equation of (8);

$$\Psi_\alpha = -\delta_{\alpha,1} \frac{u^{j+1} - u^j}{\tau} + \Lambda_\alpha u^{j+1} + \varphi_\alpha, \tag{10}$$

where $\delta_{\alpha,1}$ is the Kronecker delta, $\delta_{1,1} = 1$, $\delta_{\alpha,1} = 0$ for $\alpha > 1$..

Taylor's formula is used at the base-points $x \in \omega^0$. Subject to conditions (4) and (6),

$$\Lambda_\alpha u^{j+1} + \varphi_\alpha = (L_\alpha u + f_\alpha)^{j+1} + (\chi_\alpha)_{\hat{x}_\alpha} + O(\hbar_\alpha^2), \tag{11}$$

where

$$\chi_\alpha^\pm = \left\{ (a_\alpha^\pm - k_\alpha) \frac{\partial u}{\partial x_\alpha} + \frac{(h_\alpha^\pm)^2}{8} \left[\frac{a_\alpha^\pm}{3} \frac{\partial^3 u}{\partial x_\alpha^3} + \right.\right.$$
$$\left.\left. + \frac{\partial^2}{\partial x_\alpha^2} \left(k_\alpha \frac{\partial u}{\partial x_\alpha} \right) \right] \right\}^{j+1}_{x = x(\pm 0.5_\alpha)} + \mu_\alpha^\pm = O((h_\alpha^\pm)^2).$$

At frontier points of ω_γ, where $\hbar_\alpha \geqslant 0.5(h_\alpha^+ + h_\alpha^-) \equiv \hbar_\alpha^*$,

$$\Lambda_\alpha u^{j+1} + \varphi_\alpha = \frac{\hbar_\alpha^*}{\hbar_\alpha} \left(\frac{W_\alpha^+ - W_\alpha^-}{\hbar_\alpha^*} \right) + \varphi_\alpha = \frac{\hbar_\alpha^*}{\hbar_\alpha} (L_\alpha u)^{j+1} + \tag{12}$$
$$+ f_\alpha^{j+1} + O(\hbar_\alpha) = (L_\alpha u + f_\alpha)^{j+1} + \psi_\alpha^*,$$

where

$$\psi_\alpha^* = \frac{\hbar_\alpha^* - \hbar_\alpha}{\hbar_\alpha}(L_\alpha u)^{j+1} + O(\hbar_\alpha).$$

Hence

$$\psi_\alpha^* = O(1) \quad \text{for} \quad \hbar_\alpha > \hbar_\alpha^*, \qquad \psi_\alpha^* = O(\hbar_\alpha) \quad \text{for} \quad \hbar_\alpha = \hbar_\alpha^*.$$

Noting that

$$\frac{u^{j+1} - u^j}{\tau} = \left(\frac{\partial u}{\partial t}\right)^{j+1} + O(\tau),$$

and combining (10)-(12), the error Ψ_α can be written as

$$\Psi_\alpha = \overset{\circ}{\psi}_\alpha + \psi_\alpha + (\chi_\alpha)_{\hat{x}_\alpha} + \psi_\alpha^*, \tag{13}$$

where

$$\begin{aligned}
&\overset{\bullet}{\psi}_\alpha = \left[-\delta_{\alpha,1}\frac{\partial u}{\partial t} + L_\alpha u + f_\alpha\right]^{j+1}; \\
&\psi_\alpha = O(\hbar_\alpha^2 + \tau), \qquad \chi_\alpha^\pm = O((h_\alpha^\pm)^2), \qquad \psi_\alpha^* = 0 \quad \text{if} \quad x \in \omega_\alpha^0; \\
&\psi_\alpha^* = O(1) \quad \text{if} \quad x \in \omega_\gamma \text{and} \hbar_\alpha > \hbar_\alpha^*; \\
&\psi_\alpha^* = O(\hbar_\alpha) \quad \text{if} \quad x \in \omega_\gamma \text{and} \hbar_\alpha = \hbar_\alpha^*.
\end{aligned} \tag{14}$$

Since $\overset{\bullet}{\psi}_\alpha = O(1)$, $\alpha = 1, 2, \ldots, p$, it follows from (13) that $\Psi_\alpha = O(1)$, and the finite-difference equations (8) $\alpha = 1, 2, \ldots, p$) will not approximate the initial equation at all points $x \in \omega$. However, on the solution of the initial problem (1)-(3),

$$\sum_{\alpha=1}^{p} \overset{\circ}{\psi}_\alpha = 0, \qquad x \in \omega, \tag{15}$$

i.e.

$$\left\| \sum_{\alpha=1}^{p} \Psi_\alpha \right\| \to 0 \quad \text{as} \quad h \to 0, \qquad \tau \to 0, \tag{16}$$

i.e. the scheme approximates the initial problem in an over-all sense. In future, (8) will be regarded as a composite scheme, approximating the initial problem in an over-all sense, i.e. as an additive scheme (see [1, 2, 4] regarding over-all approximation and composite schemes).

2. Convergence and accuracy of the scheme

1. ISOLATION OF THE PRINCIPAL TERM OF THE ERROR

Following [1-5], the solution of problem (9), (13) and (14) will be written as

$$z_\alpha = v_\alpha + \eta_\alpha, \qquad \alpha = 1, 2, \ldots, p, \qquad z^j = v^j + \eta^j, \qquad (17)$$
$$j = 0, 1, \ldots, j_0,$$

where the function η is defined, at all base-points of $\bar{\omega}$, as the solution of the problem

$$\eta_0 \equiv \eta^j, \qquad \frac{\eta_\alpha - \eta_{\alpha-1}}{\tau} = \mathring{\psi}_\alpha, \qquad \alpha = 1, 2, \ldots, p, \qquad \eta^{j+1} \equiv \eta_p, \qquad (18)$$
$$j = 0, 1, \ldots, j_0, \qquad x \in \bar{\omega}, \qquad \eta^0 = 0 \text{ for } x \in \bar{\omega}.$$

The function v then satisfies the conditions

$$v_0 \equiv v^j, \qquad x \in \bar{\omega}, \qquad \frac{v_\alpha - v_{\alpha-1}}{\tau} = \Lambda_\alpha v_\alpha + \overline{\Psi}_\alpha, \qquad x \in \omega;$$
$$v_\alpha = -\eta_\alpha \quad \text{for } x \in \gamma, \qquad \alpha = 1, 2, \ldots, p, \qquad v^{j+1} \equiv v_p, x \in \bar{\omega}, \qquad (19)$$
$$j = 0, 1, \ldots, j_0, \qquad v^0 = 0 \quad \text{for } x \in \bar{\omega},$$

where

$$\overline{\Psi}_\alpha = \bar{\psi}_\alpha + (\chi_\alpha)_{\hat{x}_\alpha} + \psi_\alpha^*, \qquad \bar{\psi}_\alpha = \psi_\alpha + \Lambda_\alpha \eta_\alpha. \qquad (20)$$

Adding Eqs. (18) for $\alpha = 1, 2, \ldots, p$, using condition (15) and the condition $\eta^0 = 0$ for $x \in \bar{\omega}$, we get

$$\eta^j = 0, \qquad j = 0, 1, \ldots, j_0, \qquad x \in \bar{\omega}, \qquad (21)$$

while on intermediate layers

$$\eta_\alpha = \tau \sum_{\beta=1}^{\alpha} \mathring{\psi}_\beta = O(\tau), \qquad \Lambda_\alpha \eta_\alpha = O(\tau). \qquad (22)$$

From (20), (22) and (14),

$$\bar{\psi}_\alpha = O(\tau + \hbar_\alpha^2). \qquad (23)$$

Problem (19) has thus been obtained for the function v, where the right-hand side of each equation $\Psi_\alpha \to 0$ as $h \to 0$ and $\tau \to 0$ at strictly interior base-points $x \in \omega^0$; $\|\overline{\Psi}_\alpha\| \to 0$ as $h \to 0, \tau \to 0$; $v_\alpha = -\eta_\alpha \to 0$ as $\tau \to 0$ at boundary base-points $x \in \gamma$; $\alpha = 1, 2, \ldots, p$.

Put

$$P_\alpha v \equiv (v_\alpha - v_{\alpha-1})/\tau - \Lambda_\alpha v_\alpha, \qquad v_0 \equiv v^j, \qquad v_p \equiv v^{j+1}.$$

The solution of problem (19), i.e. v, can be written as the sum

$$v = v^{(1)} + v^{(2)} + v^{(3)} + v^{(4)}, \tag{24}$$

where $v^{(n)}$, $n = 1, 2, 3, 4$, are the solutions of the problems

$$P_\alpha v^{(1)} = 0, \quad x \in \omega; \quad v_\alpha^{(1)} = -\eta_\alpha, \quad x \in \gamma, \quad \alpha = 1, 2, \ldots, p, \tag{25}$$
$$j = 0, 1, \ldots, j_0, \quad v^{(1)}|_{t=0} = 0, \quad x \in \bar{\omega};$$
$$P_\alpha v^{(2)} = \psi_\alpha^*, \quad x \in \omega; \quad v_\alpha^{(2)} = 0, \quad x \in \gamma, \quad \alpha = 1, 2, \ldots, p, \tag{26}$$
$$j = 0, 1, \ldots, j_0, \quad v^{(2)}|_{t=0} = 0, \quad x \in \bar{\omega};$$
$$P_\alpha v^{(3)} = \bar{\psi}_\alpha, \quad x \in \omega; \quad v_\alpha^{(3)} = 0, \quad x \in \gamma, \quad \alpha = 1, 2, \ldots, p, \tag{27}$$
$$j = 0, 1, \ldots, j_0, \quad v^{(3)}|_{t=0} = 0, \quad x \in \bar{\omega};$$
$$P_\alpha v^{(4)} = (\chi_\alpha)_{\hat{x}_\alpha}, \quad x \in \omega; \quad v_\alpha^{(4)} = 0, \quad x \in \gamma, \quad \alpha = 1, 2, \ldots, p, \tag{28}$$
$$j = 0, 1, \ldots, j_0, \quad v^{(4)}|_{t=0} = 0, \quad x \in \bar{\omega}.$$

2. THE MAXIMUM PRINCIPLE AND BOUNDS FOR $v^{(1)}$, $v^{(2)}$, $v^{(3)}$

In [1-3], the scheme 8 was considered with operators Λ_α, somewhat different from the operators considered in the present paper, in fact, the operators

$$\Lambda_\alpha' y = \frac{1}{\hbar_x^*}[a_x^+ y_{x_\alpha}^+ - a_\alpha^- y_{\bar{x}_\alpha}],$$

where $\hbar_\alpha^* = 0.5(h_\alpha^+ + h_\alpha^-)$ at all base-points of the mesh ω, were taken as Λ_α. The operators Λ'_α are the same as the Λ_α defined by (5) when the set of boundary base-points $\gamma \in R_p^h$ so that $\hbar_\alpha = 0.5(x_\alpha^{(i_\alpha+1)} - x_\alpha^{(i_\alpha-1)}) = \hbar_\alpha^*$ at all base-points of ω (at strictly interior base-points $x \in \omega^0$ the equation $\hbar_\alpha = \hbar_\alpha^*$ always holds). The maximum principle and associated theorems were used in [2] to obtain *a priori* estimates for the solution of problems (25)-(27) with operators $\Lambda_\alpha'(\alpha = 1, 2, \ldots, p)$ on a uniform mesh R_p^h with steps $h_\alpha =$ const, $\alpha = 1, 2, \ldots, p$ (the mesh ω is only non-uniform at frontier points where $\hbar_\alpha^* \neq h_\alpha$) . The method of obtaining these estimates was in no way connected with the uniformity or otherwise of the mesh. Each base-point $P = (x, t)$ of the space-time mesh $\Omega = \{\omega \times \omega_\tau\}$ was associated with a set of base-points $S(P)$ (the pattern of the scheme) and a set $S'(P) = S(P) \setminus P$ (the neighbourhood of base-point P). At each $P \in \Omega$, the finite-difference equation is written in the canonical form, while a boundary condition is imposed at the boundary base-points $P \in \Gamma = \{\gamma \times \omega_\tau\}$ We consider the problem

$$A(P)y(P) = \sum_{Q \in S'(P)} B(P,Q)\,y(Q) + F(P) \quad \text{for} \quad P \in \Omega, \tag{29}$$

$$y(P) = \nu(P) \quad \text{for} \quad P \in \Theta,$$

where $A(P)$ and $B(P, Q)$ are coefficients of the scheme. Introduce the norms

$$\|y\|_{c,\Omega} = \max_{P\in\Omega} |y(P)|, \qquad \|y\|_{c,\Theta} = \max_{P\in\Theta,} |y(P)|,$$
$$\|y\|_{c,\omega} = \max_{x\in\omega} |y(x,t)|$$

and denote by $D(P)$ the quantity

$$D(P) = A(P) - \sum_{Q\in S'(P)} B(P, Q).$$

The following theorems were proved in [2].

Theorem 1

Let $A(P) > 0$, $B(P, Q) > 0$, $D(P) \geqslant 0$. Then the following holds for problem (29) with $\nu(P) \equiv 0$.

$$\|y\|_{c,\Omega} \leqslant \|\nu\|_{c,\ni}. \tag{30}$$

Theorem 2

Let $A(P) > 0$, $B(P, Q) > 0$, $D(P) > 0$. Then

$$\|y\|_{c,\Omega} \leqslant \|F/D\|_{c,\Omega} \tag{31}$$

for problem (29) with $\nu(P) \equiv 0$.

Let $\gamma(P) = 0$. Then, with $P \in \{\omega_\gamma \times \omega_\tau\} \equiv \Omega_\gamma$ the summation in equation (29) is actually over interior base-points only of the mesh $Q \in S(P) \cap \Omega \equiv \bar{S}'(P)$. At strictly interior base-points $P \in \{\omega^0 \times \omega_\tau\} \equiv \Omega^0$ put $\bar{S}'(P) = S'(P)$. We put

$$\bar{D}(P) = A(P) - \sum_{Q\in S'(P)} B(P, Q).$$

Theorem 3

Let $\bar{D}(P) = 0$ and $F(P) = 0$ for $P \in \Omega^0$, $\bar{D}(P) > 0$ with $P \in \Omega_\gamma$; $A(P) > 0$, $B(P, Q) > 0$. Then

$$\|y\|_{c,\Omega} \leqslant \max_{P\in\Omega_\gamma} \left| \frac{F(P)}{\bar{D}(P)} \right|. \tag{32}$$

for the solution of problem (29) with $\nu(P)=0$.

Bounds can be found for the solutions of problems (25) and (27) by means of Theorems 1-3. In the case of functions $v^{(1)}$ and $v^{(2)}$, we select the four-point pattern

$$S(P)=\{P=(x,\ t_{j+\alpha/p}),\ \ (x^{(\pm 1_\alpha)},\ t_{j+\alpha/p}),\ \ (x,\ t_{j+(\alpha-1)/p})\}.$$

Equations (25) and (26) may be written in the form (29). It is easily shown that the conditions of Theorem 1 ($A(P)>0$, $B(P,Q)>0$, $D(P)=0$, and $F(P)=0$, are satisfied for problem (25). From (30), (25) and (22) (see also the bound (57) on p. 379 of [2]),

$$\|v^{(1)}\|_{c,\Omega}\leqslant M\tau. \tag{33}$$

The conditions of Theorem 3 ($\bar{D}(P)=F(P)=0$ for $P\in\Omega^0$; $F(P)=\psi_\alpha{}^*$, $\bar{D}(P)>0$ when $P\in\Omega_\gamma$; $A(P)>0$, $B(P,Q)>0$ and $\nu(P)=0$) are satisfied for problem (26). It turns out here that

$$\bar{D}(P)\geqslant c_1/h^2>0,\qquad P\in\Omega_\gamma. \tag{34}$$

From (32), (34) and (14) (see also the bound (58), on p. 380 of [2])

$$\|v^{(2)}\|_{c,\Omega}\leqslant Mh^2. \tag{35}$$

Finally, consider $v^{(3)}$. Choose the three-point pattern

$$S(P)=\{P=(x,t_{j+\alpha/p}),\quad (x^{(\pm 1_\alpha)},\ t_{j+\alpha/p})\}$$

and write (27) in the form (29). Then,

$$F(P)=\frac{(v^{(3)})^{j+\alpha/p}}{\tau}+\bar{\psi}^{j+\alpha/p},\qquad D(P)=1/\tau,\qquad A(P)>0, \tag{36}$$

$$B(P,\ Q)>0\quad\text{for}\quad P\in\Omega,\qquad \nu(P)=0\quad\text{for}\quad P\in\Theta.$$

The notation, here and below, is

$$(v^{(3)})^{j+(\alpha-1)/p}\equiv v^{(3)}_{\alpha-1},\qquad (v^{(3)})^{j+\alpha/p}\equiv v_\alpha{}^{(3)},\qquad \bar{\psi}^{j+\alpha/p}\equiv\bar{\psi}_\alpha\ \text{etc.}$$

From (36), the conditions of Theorem 2 are satisfied. From (36) and (31),

$$\|(v^{(3)})^{j+\alpha/p}\|_{c,\omega}\leqslant\|(v^{(3)})^{j+(\alpha-1)/p}\|_{c,\omega}+\tau\|\bar{\psi}^{j+\alpha/p}\|_{c,\omega}. \tag{37}$$

Adding inequalities (37) over all indices α and j, using (23) and the condition

$v^{(3)}|_{t=0} = 0$, we obtain (see also bound (59) on p. 381 of [2])

$$\|v^{(3)}\|_{c,\Omega} \leqslant M(h^2 + \tau). \tag{38}$$

If the solution of problem (28) is considered similarly, and we note that $(\chi_\alpha)_{\hat{x}_\alpha} = \mathcal{O}(\hbar_\alpha)$, on a non-uniform mesh, we obtain from (31):

$$\|v^{(4)}\|_{c,\Omega} \leqslant Mh. \tag{39}$$

The bound (39) is too crude, so that the method of energy inequalities of the n-th rank must be used when finding a bound for $v^{(4)}$.

3. *A PRIORI* ESTIMATES OF THE n-th RANK. A BOUND FOR $v^{(4)}$

Consider $v^{(4)}$ in the mesh norm of C. The *a priori* estimates of the n-th rank obtained in [6] will be used. Introduce the functions $\overset{0}{v} = v$, $\overset{1}{v} = v^2, \ldots,$ $\overset{n}{v} = v^{2^n}$ and write equation (28) for $(v^{(4)})^{2n}$. To simplify the notation, we shall simply write v instead of $v^{(4)}$. Then (see [6])

$$\frac{\overset{n}{v}_\alpha - \overset{n}{v}_{\alpha-1}}{\tau} - \Lambda_\alpha \overset{n}{v}_\alpha + \frac{1}{\hbar_\alpha} \sum_{k=0}^{n-1} 2^{n-k-1} v_\alpha^{\nu_n - \nu_{k+1}} \left\{ h_\alpha^+ a_\alpha^+ (\overset{k}{v}_{\alpha x_\alpha}^{+})^2 + \right. \tag{40}$$

$$\left. + h_\alpha^- a_\alpha^- (\overset{k}{v}_{\alpha x_\alpha}^{-})^2 + \left(\frac{\overset{k}{v}_\alpha - \overset{k}{v}_{\alpha-1}}{\tau} \right)^2 \right\} = 2^n v_\alpha^{\nu_n} (\chi_\alpha)_{\hat{x}_\alpha}, \quad \nu_k = 2^k - 1.$$

We put

$$(y, z)_{Z_\alpha} = \sum_{x \in Z_\alpha} yz\hbar_\alpha, \qquad (v, w)_{Z_\alpha} = \sum_{x^{(+0.5\alpha)} \in Z_\alpha} (vw)^{(+0.5\alpha)} h_\alpha^+,$$

$$\|y\|_{Z_\alpha} = \sqrt{(y, y)_{Z_\alpha}}, \qquad \|w\|_{Z_\alpha} = \sqrt{(w, w)_{Z_\alpha}},$$

$$\overset{n-1}{I}_\alpha = \sum_{k=0}^{n-1} 2^{n-k-1} \left(\frac{v_\alpha^{\nu_n - \nu_{k+1}}}{\hbar_\alpha}, \ h_\alpha^+ a_\alpha^+ (\overset{k}{v}_{\alpha x_\alpha}^{+})^2 + h_\alpha^- a_\alpha^- (\overset{k}{v}_{\alpha x_\alpha}^{-})^2 + \right.$$

$$\left. + \tau \left(\frac{\overset{k}{v}_\alpha - \overset{k}{v}_{\alpha-1}}{\tau} \right)^2 \right)_{Z_\alpha}$$

Equation (40) is multiplied by $\hbar_\alpha$ and the result summed over the chain Z_α:

$$\frac{(\overset{n}{v}_\alpha, 1)_{Z_\alpha} - (\overset{n}{v}_{\alpha-1}, 1)_{Z_\alpha}}{\tau} + \overset{n-1}{I}_\alpha = 2^n (v_\alpha^{\nu_n}, (\chi_\alpha)_{\hat{x}_\alpha})_{Z_\alpha}. \tag{41}$$

The following bound was obtained in [7] (see also [6]) for the right-hand side of (41):

$$2^n \left| (\overset{\nu}{v}{}_{\alpha}^{n}, (\chi_\alpha)_{\hat{x}_\alpha})_{\tilde{Z}_\alpha} \right| \leqslant {}_{1/2} \overset{n-1}{I}_{\alpha} + M_1 (2^n M_1)^{2^n} \| \chi_\alpha \|^{2^n}_{\tilde{Z}_\alpha}, \tag{42}$$

where $M_1 = \sqrt{(2D)}/c_1$ and D is the diameter of the region G. Using (42) to obtain a bound for the right-hand side of (41), the result is multiplied by $s_\alpha^{\pm}$ and summed over all the chains Z_α. We get

$$\frac{(\overset{n}{v}{}^{j+\alpha/p}, 1) - (\overset{n}{v}{}^{j+(\alpha-1)/p}, 1)}{\tau} \leqslant M_s M_1 (M_1 2^n)^{2^n} \max_{Q_\alpha} \| \chi^{j+\alpha/p} \|^{2^n}_{\tilde{Z}_\alpha}. \tag{43}$$

Here, $M_s = \max_{\alpha} \sum_{Q_\alpha} s_\alpha$, Q_α is the set of all chains $\tilde{Z}_\alpha$, and $\overset{n}{v}{}^{j+\alpha/p} \equiv \overset{n}{v}_{\alpha}$, $\overset{n}{v}{}^{j+(\alpha-1)/p} \equiv \overset{n}{v}_{\alpha-1}$, $\chi^{j+\alpha/p} \equiv \chi_\alpha$. We sum (43) over all α and j. Since $v \equiv v^{(4)} = 0$ at $t = 0$,

$$\sum_{\omega} (v^{j+\alpha/p})^{2^n}) H \leqslant T M_s M_1 (2^n M_1)^{2^n} \max_{j,\alpha} \max_{Q_\alpha} \| \chi^{j+\alpha/p} \|^{2^n}_{\tilde{Z}_\alpha}. \tag{44}$$

Next, a lower bound is obtained for the left-hand side of (44), in the same way as in [3, 7]. The 2^n-th root is extracted on both sides of the resulting inequality. As in [7], n is taken to be dependent on H_*. On strengthening the resulting inequality, the final result is

$$\| v^{(4)} \|_{c,\Omega} \leqslant M \ln \frac{V_0}{H_*} \max_{j,\alpha} \max_{Q_\alpha} \| \chi^{j+\alpha/p} \|_{\tilde{Z}_\alpha}, \tag{45}$$

where V_0 is the volume of G and $M = \text{const}$ is independent of τ and h. From (45) and (14), we obtain instead of (39):

$$\| v^{(4)} \|_{c,\Omega} \leqslant M h^2 \ln (V_0 / H_*). \tag{46}$$

Finally, from (44) with $n = 1$ and (14),

$$\| v^{(4)} \| \leqslant M n^2. \tag{47}$$

Since $\|v\| \leqslant \|v\|_{c,\Omega} \sqrt{V_0}$, the following estimates in the mesh norm of L_2 are obtained from (33), (35), (38) and (22):

$$\| v^{(1)} \| \leqslant M\tau, \qquad \| v^{(2)} \| \leqslant M h^2, \qquad \| v^{(3)} \| \leqslant M (h^2 + \tau), \tag{48}$$
$$\| \eta \| \leqslant M\tau.$$

4. ACCURACY OF THE LOCALLY UNIFORM SCHEME (8) ON NON-UNIFORM MESHES

The accuracy of the scheme (8) is determined by the error $z = y - u$, which satisfies conditions (9). In paragraph 1 of Section 2, the error z was written in the form

$$z_\alpha = v_\alpha^{(1)} + v_\alpha^{(2)} + v_\alpha^{(3)} + v_\alpha^{(4)} + \eta_\alpha. \tag{49}$$

The bounds (21), (22) and (48) have been obtained for η, the bounds (33) and (48) for $v^{(1)}$, the bounds (35) and (48) for $v^{(2)}$, the bounds (38) and (48) for $v^{(3)}$, and bounds (46), (47) and (39) for $v^{(4)}$. Combining these bounds and noting (49), the following results are obtained.

Theorem 4

The locally one-dimensional finite-difference scheme (8) is uniformly convergent on an arbitrary sequence of non-uniform meshes as the mesh steps tend independently to zero. Also,

$$\|y - u\|_{c,\Omega} \leqslant M[h^2 \ln (V_0 / H_*) + \tau]. \tag{50}$$

If the conditions

$$m_0 h^{\varkappa p} \leqslant H(x) \leqslant h^p, \tag{51}$$

where m_0 = const > 0 and $\varkappa$ = const > 0 are independent of h and τ, are satisfied, we then obtain from (50):

$$\|y - u\|_{c,\Omega} \leqslant M[h^2 \ln (D / h) + \tau], \tag{52}$$

where D is the diameter of the region G.

Theorem 5

The locally one-dimensional finite-difference scheme (8) is convergent in the mesh norm of $L_2(\omega)$ on any sequence of non-uniform meshes; also,

$$\|y - u\| \leqslant M(h^2 + \tau). \tag{53}$$

Note. Similar estimates apply for problem (1)-(2), where

$$L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha}\right) + b_\alpha \frac{\partial u}{\partial x_\alpha} - q_\alpha u, \qquad k_\alpha \geqslant c_1 > 0,$$

and for the corresponding monotonic [2] schemes. If $b_\alpha \neq 0$ or $q_\alpha < 0$, the inequalities (50) and (51) hold for $\tau \leqslant \tau_0$ = const.

3. A locally one-dimensional scheme for the second and third boundary value problems

1. FORMULATION OF THE PROBLEM

Let G be a region consisting of p-dimensional parallelepipeds with faces parallel to the coordinate planes (stepped region). The boundary Γ of the stepped region G consists of pieces Γ_α of hyperplanes orthogonal to the coordinate axes ox_α, $\Gamma_\alpha = \Gamma_\alpha^+ \cup \Gamma_\alpha^-$, where Γ_α^+ and Γ_α^- are the right- and left-hand pieces of the boundary Γ_α, $\alpha = 1, 2, \ldots, p$. As before,

$$Q_T = G \times (0 < t \leqslant T), \qquad \bar{Q}_T = \bar{G} \times (0 \leqslant t \leqslant T),$$

$$\bar{G} = G \cup \Gamma.$$

Consider the following problem: find the function $u = u(x, t)$, continuous in $\bar{Q}_T$ and satisfying the equation

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} \left\{ \frac{\partial}{\partial x_\alpha} \left(k_\alpha(x, t) \frac{\partial u}{\partial x_\alpha} \right) + f_\alpha(x, t) \right\}, \qquad (x, t) \in Q_T, \tag{54}$$

and the boundary and initial conditions

$$\mp k_\alpha(x, t) \frac{\partial u}{\partial x_\alpha} - \sigma_\alpha^{\pm}(x, t)\, u = \nu_\alpha^{\pm}(x, t) \quad \text{if} \quad x \in \Gamma_\alpha^{\pm}, \tag{55}$$

$$0 < t \leqslant T, \qquad \alpha = 1, 2, \ldots, p;$$

$$u(x, 0) = u_0(x) \quad \text{if} \quad x \in \bar{G},$$

where k_α, $\sigma_\alpha^{\pm}$, $\nu_\alpha^{\pm}$, f_α are given functions, and

$$k_\alpha(x, t) \geqslant c_1 = \text{const} > 0, \qquad \sigma_\alpha^{\pm} \geqslant 0. \tag{56}$$

It is assumed that the problem (54)-(56) has a unique and reasonably smooth solution.

2. THE MESH. CLASSIFICATION OF BASE-POINTS. NOTATION

The mesh R_p^h is constructed in such a way that the pieces $\Gamma_\alpha^{\pm}$ of the boundary Γ belong to the hyperplanes $x_\alpha = x_\alpha^{(i_\alpha^s)}$, $s = 1, 2, \ldots, M$, $\alpha = 1, 2, \ldots, p$. Consider the base-point mesh $\bar{\omega} = R_p^h \cap \bar{G}$. Denote by γ_α the set of $\bar{\omega}$ base-points belonging to Γ_α, and by ω_α^0 the set of the other base-points of $\bar{\omega}$. Notice that ω_α^0 includes all the base-points $x \in G$ $(x \in R_p^h \cap G)$, and also the base-points belonging to the pieces $\Gamma \setminus \Gamma_\alpha$ of the boundary $\Gamma(x \in R_p^h \cap (\Gamma \setminus \Gamma_\alpha))$. It will be said that $x \in \overset{*}{\gamma}_\alpha$, if $x \in \gamma_\alpha$ and the intervals $(x, x^{(\pm 1_\alpha)}) \in \bar{G}$. Write $\overset{\circ}{\gamma}_\alpha = \gamma_\alpha \setminus \overset{*}{\gamma}_\alpha$, where $\overset{*}{\gamma}_\alpha = \overset{*}{\gamma}_\alpha^+ \cup \overset{*}{\gamma}_\alpha^-$, $\overset{\circ}{\gamma}_\alpha = \overset{\circ}{\gamma}_\alpha^+ \cup \overset{\circ}{\gamma}_\alpha^-$, and $\overset{*}{\gamma}_\alpha^+, \overset{\circ}{\gamma}_\alpha^+$ and $\overset{*}{\gamma}_\alpha^-, \overset{\circ}{\gamma}_\alpha^-$ are the sets of right- and left-hand base-points of $\overset{*}{\gamma}_\alpha$ and $\overset{\circ}{\gamma}_\alpha$. Denote by ω_α the set of interior base-points with respect to the direction ox_α: $\omega_\alpha = \omega_\alpha{}^0 \bigcup \overset{*}{\gamma}_\alpha$.

As above, after introducing the steps h_α^+ and h_α^- we put

$$\hbar_\alpha = \begin{cases} 0.5\,(h_\alpha{}^+ + h_\alpha{}^-), & x \in \omega_\alpha, \\ 0.5 h_\alpha{}^{\mp}, & x \in \overset{\circ}{\gamma}_\alpha{}^{\pm} \end{cases} \tag{57}$$

Each base-point $x \in \bar{G}$ is associated with a volume $H(x) \in G$, bounded by pieces $s_\alpha^{\pm}$ of hyperplanes through the transitional points $x^{(\pm 0.5_\alpha)} \in G$ and orthogonal to the axes ox_α, and also, in the case $x \in \Gamma$, by the piece $s(x)$ of boundary Γ. If a base-point $x \in \bar{\omega}$ is a boundary base-point with respect to the directions $ox_{\alpha_1}, ox_{\alpha_2}, \ldots, ox_{\alpha_m}$, $m \leqslant p$, then

$$s\,(x) = \sum_{k=1}^{m} \bar{s}_{\alpha_k},$$

where

$$\bar{s}_\alpha = \begin{cases} s_\alpha{}^{\pm}, & x \in \overset{\circ}{\gamma}_\alpha{}^{\mp}, \\ |s_\alpha{}^+ - s_\alpha{}^-|, & x \in \overset{*}{\gamma}_\alpha. \end{cases}$$

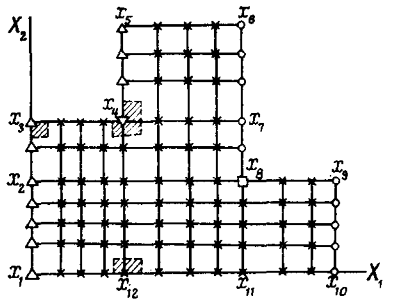

FIG. 1.

The volume $H(x)$ is obtained from

$$H\,(x) = \begin{cases} 0.5 h_\alpha{}^{\pm} s_\alpha{}^{\pm} \equiv \hbar_\alpha s_\alpha, & x \in \overset{\circ}{\gamma}_\alpha, \\ 0.5\,(h_\alpha{}^+ s_\alpha{}^+ + h_\alpha{}^- s_\alpha{}^-), & x \in \overset{*}{\gamma}_\alpha. \end{cases} \tag{58}$$

In Fig. 1, Δ denotes a base-point belonging to the set $\overset{\circ}{\gamma}{}_1^-$, $\circ$ a base-point belonging to $\overset{\circ}{\gamma}{}_1^+$, ∇ one belonging to $\overset{*}{\gamma}{}_1^-$, $\square$ one belonging to $\overset{*}{\gamma}{}_1^+$, and $\times$ one belonging to ω_1^0. The areas $H(x)$, corresponding to base-points x_3, x_4 and x_{12} are also indicated:

$$H\,(x_3) = {}^1/_4 h_1{}^+ (x_3)\, h_2{}^- (x_3),$$
$$H\,(x_4) = {}^1/_4\,[h_1{}^+ (x_4)\,(h_2{}^+ (x_4) + h_2{}^- (x_4)) + h_1{}^- (x_4)\, h_2{}^- (x_4)],$$
$$H\,(x_{12}) = {}^1/_4\,(h_1{}^+ (x_{12}) + h_1{}^- (x_{12}))\, h_2{}^+ (x_{12}).$$

A time mesh ω with step τ and fractional step τ/p is introduced in the same way as before. The space-time mesh is written as $\Omega = \{\bar{\omega} \times \omega_\tau\}$.

3. THE FINITE-DIFFERENCE OPERATORS AND THEIR PROPERTIES

Denote by Λ_α the operators corresponding to the differential operators L_α ($\alpha = 1, 2, \ldots, p$). As above, at the base-points $x \in \omega_\alpha^0$, we put

$$\Lambda_\alpha y = (a_\alpha y_{\bar{x}_\alpha})_{\hat{x}_\alpha} = \frac{1}{\hbar_\alpha}\left[a_\alpha^+ \frac{y^{(+1_\alpha)} - y}{h_\alpha^+} - a_\alpha^- \frac{y - y^{(-1_\alpha)}}{h_\alpha^-}\right], \quad x \in \omega_x^0. \quad (59)$$

Consider the base-points $x \in \mathring{\gamma}_\alpha$. At base-points of $\mathring{\gamma}_\alpha^+$, up to $O(h_\alpha^-)$,

$$L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha}\right) \approx \left[k_\alpha(x, t)\frac{\partial u}{\partial x_\alpha}(x, t) - k_\alpha(x^{(-0.5_\alpha)}, t) \times \right. \quad (60)$$
$$\left. \times \frac{\partial u}{\partial x_\alpha}(x^{(-0.5_\alpha)}, t)\right](0.5h_\alpha^-)^{-1}.$$

From (4) and Taylor's formula,

$$k_\alpha(x^{(-0.5_\alpha)}, t)\frac{\partial u}{\partial x_\alpha}(x^{(-0.5_\alpha)}, t) = a_\alpha^- \frac{u - u^{(-1_\alpha)}}{h_\alpha^-} + O((h_\alpha^-)^2). \quad (61)$$

From (55), (60) and (61),

$$L_\alpha u = \frac{-(\sigma_\alpha^+ u + \nu_\alpha^+) - a_\alpha^- u_{\bar{x}_\alpha}}{0.5h_\alpha^-} + O(h_\alpha^-) \quad \text{if} \quad x \in \mathring{\gamma}_\alpha^+. \quad (62)$$

Similarly, if $x \in \mathring{\gamma}_\alpha^+$

$$L_\alpha u = \frac{-(\sigma_\alpha^- u + \nu_\alpha^-) + a_\alpha^+ u_{\overset{+}{x}_\alpha}}{0.5h_\alpha^+} + O(h_\alpha^+) \quad \text{if} \quad x \in \mathring{\gamma}_\alpha^-. \quad (63)$$

Recalling (62) and (63),

$$\Lambda_\alpha y = \frac{\mp a_\alpha^\mp y_{\overset{\mp}{x}_\alpha} - (\sigma_\alpha^\pm u + \nu_\alpha^\pm)}{0.5h_\alpha^\mp} \quad \text{if} \quad x \in \mathring{\gamma}_\alpha^\pm. \quad (64)$$

Finally, consider base-points $x \in \overset{*}{\gamma}_\alpha$. We introduce weighting factors $\delta_\alpha = \delta_\alpha^\pm$ when $x \in \overset{*}{\gamma}_\alpha^\pm$:

$$\begin{aligned} \delta_\alpha^+ &= \frac{s_\alpha^+ \hbar_\alpha}{H}, \qquad 1 - \delta_\alpha^+ = \frac{\bar{s}_\alpha h_\alpha^-}{2H} \quad \text{if} \quad x \in \overset{*}{\gamma}_\alpha^+, \qquad s_\alpha^- > s_\alpha^+, \\ \delta_\alpha^- &= \frac{s_\alpha^- \hbar_\alpha}{H}, \qquad 1 - \delta_\alpha^- = \frac{\bar{s}_\alpha h_\alpha^+}{2H} \quad \text{if} \quad x \in \overset{*}{\gamma}_\alpha^-, \qquad s_\alpha^- < s_\alpha^+ \end{aligned} \quad (65)$$

($\delta_\alpha^+ = \delta_\alpha^- = 1$ if $s_\alpha^+ = s_\alpha^-$). At base-points $x \in \overset{*}{\gamma}_\alpha^\pm$ we approximate $L_\alpha u$

by adding expressions (59) and (64) taken with the weights δ_α and $1 - \delta_\alpha$ respectively. Put

$$\Lambda_\alpha y = \delta_\alpha^{\pm} (a_\alpha y_{x_\alpha})_{\hat{x}_\alpha} + (1 - \delta_\alpha^{\pm}) \left[\frac{\mp a_\alpha^{\mp} y_{x_\alpha}^{\mp} - (\sigma_\alpha^{\pm} y + \nu_\alpha^{\pm})}{0.5 h_\alpha^{+}} \right]$$

if $x \in \overset{*}{\gamma}_\alpha^{\pm}$. Using (65),

$$\Lambda_\alpha y = \frac{1}{H} (s_\alpha^{+} a_\alpha^{+} y_{x_\alpha}^{+} - s_\alpha^{-} a_\alpha^{-} y_{x_\alpha}^{-} - \bar{s}_\alpha (\sigma_\alpha^{\pm} y + \nu_\alpha^{\pm})) \quad \text{if} \quad x \in \overset{*}{\gamma}_\alpha^{\pm}. \tag{66}$$

The operator L_α is associated with the finite-difference operator Λ_α, given by (59), (64) and (66). Operators $\tilde{\Lambda}_\alpha$ will now be defined, corresponding to the operator L_α and the homogeneous boundary conditions (55):

$$\tilde{\Lambda}_\alpha y = \begin{cases} (a_\alpha y_{x_\alpha})_{\hat{x}_\alpha}, & x \in \overset{\circ}{\omega}_\alpha; \\ \dfrac{\mp a_\alpha^{\mp} y_{x_\alpha}^{\mp} - \sigma_\alpha^{\pm} y}{0.5 h_\alpha^{\mp}}, & x \in \overset{\circ}{\gamma}_\alpha^{\pm}; \\ \dfrac{1}{H} (s_\alpha^{+} a_\alpha^{+} y_{x_\alpha}^{+} - s_\alpha^{-} a_\alpha^{-} y_{x_\alpha}^{-} - \bar{s}_\alpha \sigma_\alpha^{\pm} y), & x \in \overset{*}{\gamma}_\alpha^{\pm}. \end{cases} \tag{67}$$

We also introduce the function

$$\tilde{\varphi}_\alpha = \begin{cases} \varphi_\alpha, & x \in \overset{\circ}{\omega}_\alpha, \\ \varphi_\alpha - \dfrac{\nu_\alpha^{\pm}}{0.5 h_\alpha^{\mp}}, & x \in \overset{\circ}{\gamma}_\alpha^{\pm}, \\ \varphi_\alpha - \dfrac{\bar{s}_\alpha \nu_\alpha^{\pm}}{H}, & x \in \overset{*}{\gamma}_\alpha^{\pm}. \end{cases} \tag{68}$$

Comparison of (67), (68) and (59), (64), (66), shows that

$$\tilde{\Lambda}_\alpha y + \tilde{\varphi}_\alpha \equiv \Lambda_\alpha y + \varphi_\alpha.$$

Let $\mathcal{H}$ be the space of mesh functions defined on the mesh $\bar{\omega}$, with scalar product

$$(y, z) = \sum_{\bar{\omega}} yzH, \qquad \| y \| = \sqrt{(y, y)}.$$

(The summation is over the closed region $\bar{\omega} = \omega \cup \gamma$.) It can be shown by direct evaluation that the operators $\tilde{\Lambda}_\alpha$ in $\mathcal{H}$ are self-conjugate and non-positive:

$$(\tilde{\Lambda}_\alpha z, y) = (z, \tilde{\Lambda}_\alpha y), \qquad (\tilde{\Lambda}_\alpha y, y) \leqslant 0. \tag{69}$$

When $\sigma_\alpha^{\pm} \geqslant 0$, $\sigma_\alpha^{+} + \sigma_\alpha^{-} > 0$ the operators $\tilde{\Lambda}_\alpha$ are negative-definite. We also introduce the norm

$$\| y \|_{c, \bar{\Omega}} = \max_{(x, t) \in \bar{\Omega}} | y(x, t) |.$$

4. THE FINITE-DIFFERENCE PROBLEM. ACCURACY OF THE SCHEME

The initial problem (54)-(56) may be associated with the finite-difference problem

$$y_0 \equiv y^j \quad \text{if} \quad x \in \bar{\omega};$$
$$\frac{y_\alpha - y_{\alpha-1}}{\tau} = \tilde{\Lambda}_\alpha y_\alpha + \tilde{\varphi}_\alpha, \qquad x \in \bar{\omega}, \qquad \alpha = 1, 2, \ldots, p; \tag{70}$$
$$y^{j+1} \equiv y_p, \qquad x \in \bar{\omega}, \qquad j = 0, 1, \ldots, j_0, \qquad y^0 = u_0(x), \qquad x \in \bar{\omega},$$

where $j_0 = T/\tau - 1$; $\tilde{\Lambda}_\alpha$, $\tilde{\varphi}_\alpha$ are given by (67) and (68).

Notice that, in each chain $\bar{Z}_\alpha$, the α-th equation of (70) connects, at base-points $x \in \gamma_\alpha$ the values of the required function y_α at two neighbouring base-points x and $x^{(-1\alpha)}$ (x and $x^{(+1\alpha)}$), if $x \in \overset{\circ}{\gamma}_\alpha^{+}$ ($x \in \overset{\circ}{\gamma}_\alpha^{-}$), or at base-points of Z_α belonging to the set ω_α, at three neighbouring base-points $x^{(-1\alpha)}$, x, $x^{(+1\alpha)}$. The solution of (70) is found by the pivotal condensation method in each chain Z_α. The α-th equation of (70) has to be solved in all the chains $\bar{Z}_\alpha$, including those whose base-points belong to the boundary Γ. For instance, with $p = 2$ and the region shown in Fig. 1, Eq. (70) with $\alpha = 1$ must also be solved in the chains of base-points belonging to the intervals $[x_1, x_{10}]$, $[x_2, x_9]$, $[x_3, x_7]$, $[x_5, x_6]$, and with $\alpha = 2$, in the chains of base-points belonging to the intervals $[x_1, x_3]$, $[x_2, x_4]$, $[x_{11}, x_6]$, $[x_{10}, x_9]$.

By using Theorems 1-3 and the *a priori* estimates of the n-th rank, the locally one-dimensional scheme (70) can be shown to be uniformly convergent in a sequence of non-uniform meshes. We have

Theorem 6

The locally one-dimensional scheme (70) is convergent in a sequence of non-uniform meshes as $h \to 0$ and $\tau \to 0$ to the solution of problem (54)-(56); also,

$$\| y - u \|_{c,\bar{\Omega}} \leqslant M\,[h^2 \ln (V_0/H_*) + \tau], \qquad \| y - u \| \leqslant M\,(h^2 + \tau).$$

With condition (51),

$$\| y - u \|_{c,\bar{\Omega}} \leqslant M\,[h^2 \ln (D/h) + \tau].$$

If $\sigma_\alpha^{\pm} \leqslant c_2 = \text{const} < 0$, the bounds of Theorem 6 only hold when $\tau \leqslant \tau_0(c_1, c_2)$ is small. The note on Theorems 4 and 5 again applies.

Translated by D. E. Brown

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