THE NUMERICAL SOLUTION OF INTERIOR STATIONARY PROBLEMS OF ELECTRODYNAMICS*

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WE SHALL be concerned in this paper with aspects of the numerical solution of non-self-conjugate boundary value problems such as occur when determining electric current or temperature fields in a medium with anisotropic electrical and thermal conductivity [1]. By using a variational approach we are able to carry over the most important properties of the operator of the initial problem to the approximating difference operator. We devise a divergent second-order difference scheme for the divergent positive-definite operator of the initial boundary value problem. The scheme has been employed for the numerical solution of a number of concrete physical problems.

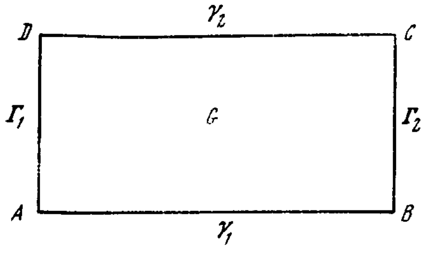

FIG. 1.

1. To find a function $u(x_1, x_2)$, continuous in a closed rectangular region $\bar{G}$ (see Fig. 1) with boundary $\Gamma = \Gamma_1 + \Gamma_2 + \gamma_1 + \gamma_2$ ($\gamma_1 = AB$, $\gamma_2 = CD$, $\Gamma_1 = AD$, $\Gamma_2 = BC$), and satisfying at interior points of G the equation

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$$\mathcal{A}u = f, \tag{1}$$

$$\mathcal{A}u = -\frac{\partial}{\partial x_1}\left(k\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(k\frac{\partial u}{\partial x_2}\right) - \frac{\partial}{\partial x_1}\left(r\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(r\frac{\partial u}{\partial x_1}\right) \tag{2}$$

with the boundary conditions

$$u = 0 \text{ on } \gamma_1 \text{ and } \gamma_2, \tag{3}$$

$$au = k\frac{\partial u}{\partial x_2} - r\frac{\partial u}{\partial x_1} = 0 \text{ on } \Gamma_1 \text{ and } \Gamma_2, \tag{4}$$

$$k(x_1, x_2) \geqslant c_1 > 0, \quad c_1 = \text{const.}$$

The quadratic form

$$[u, u] = \iint\limits_G k\left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right] dx_1 dx_2 \tag{5}$$

is positive definite:

$$[u, u] \geqslant \delta_0(u, u), \qquad \delta_0 = \text{const} > 0, \qquad (u, v) = \iint\limits_G uv\,dx_1 dx_2. \tag{6}$$

In the class of reasonably smooth coefficients k and r, and functions u, satisfying (3) and (4), the differential expression (2) specifies a linear positive definite operator $\mathcal{A}$, which is self-conjugate only when $r \equiv 0$. A discussion of the existence and uniqueness conditions for the solution of problems (1)–(4) may be found in [2].

2. There is a close connection between boundary value problems of the type (1)–(4) and variational principles for describing continuous media [3], so that it seems natural to use a variational approach [4], based on determination of the solution by means of integral identities, when devising an appropriate difference scheme .

Let $C_1^{(2)}$ be the family of functions twice continuously differentiable inside G and satisfying the boundary conditions (3). A function $u \in C_1^{(2)}$ will be a solution of problem (1)–(4) when and only when, whatever the $v \in C_1^{(2)}$, we have the identity

$$\Phi(u, v) = [u, v] - (f, v) \equiv 0, \tag{7}$$

$$[u, v] = \iint\limits_G I(u, v)\, dx_1 dx_2, \tag{8}$$

$$I(u, v) = k\left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right) + r\left(\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_2}\right). \tag{9}$$

The truth of this statement follows from the first Green's formula

$$\iint_G v\mathcal{A}u\,dx_1\,dx_2 = [u, v] - \int_C^D v\left(k\frac{\partial u}{\partial x_2} - r\frac{\partial u}{\partial x_1}\right)dx_1 - \int_A^B v\left(k\frac{\partial u}{\partial x_2} - r\frac{\partial u}{\partial x_1}\right)dx_1 \tag{10}$$

and the arbitrariness in selecting $v \in C_1^{(2)}$. In turn, we can use (7) as a definition of the generalized solution of problem (1)–(4).

3. It is a typical feature of the present problem that the operator $\mathfrak{A}$ is positive definite and divergent. It is natural to demand that the difference analogue A of $\mathfrak{A}$ have the same properties. We shall use (7)–(9) as our starting-point when devising the operator for the difference boundary value problem approximating the problem (1)–(4).

We construct in the rectangle $\bar{G} = (0 \leqslant x_\alpha \leqslant l_\alpha, \alpha = 1, 2)$ the mesh

$$\bar{\omega}_h = \{x_{1i_1} = i_1h_1,\ x_{2i_2} = i_2h;\ \ i_\alpha = 0, 1, \ldots, N_\alpha,\ h_\alpha = l_\alpha / N_2,\ \alpha = 1,2\}$$

with steps h_1 and h_2 in the x_1 and x_2 directions.

The integral bilinear form $[u, v]$ will be approximated by the difference expression

$$[u, v]_h = \sum_{i_1=0,\ i_2=0}^{N_1-1,\ N_2-1} (I_h)_{i_1i_2} h_1 h_2. \tag{11}$$

There are various ways in which $[u, v]$ can be approximated. But we shall only consider those which satisfy the following natural requirements:

(1) the local approximation of the bilinear form $I(u, v)$ by the difference analogue $I_h(u, v)$ is of the second order on smooth functions:

$$I(u, v) - I_h(u, v) = O(h^2), \qquad h = \max_{\alpha=1,2}(h_1, h_2); \tag{12}$$

(2) the quadratic form $[u, u]_h$ is positive:

$$[u, u]_h > 0, \qquad u \not\equiv 0. \tag{13}$$

The bilinear difference functional (11) defines the operator A by means of the identity

$$[u, v]_h = (v, Au)_h = \sum_{i_1=0,\, i_2=0}^{N_1-1,\, N_2} \varkappa_{i_2} v A u h_1 h_2, \tag{14}$$

$$\varkappa_{i_2} = \begin{cases} 1, & i_2 \neq 0, N_2, \\ 0.5, & i_2 = 0, N_2, \end{cases}$$

which is the difference analogue of Green's formula (10). The conjugate operator A^* is defined similarly:

$$[u, v]_h = (u, A^* v)_h = \sum_{i_1=0,\, i_2=0}^{N_1-1,\, N_2} \varkappa_{i_2} u A^* v h_1 h_2.$$

The difference operators A and A^* thus obtained are defined on any mesh functions y which satisfy the boundary condition $y = 0$ at the mesh base-points belonging to γ_1 and γ_2, and they are divergent. The possible construction of difference schemes on the basis of the integral identity (7) was examined in [5,6].

The following approximation of $I(u, v)$, satisfying (12) and (13), will be used:

$$I_h(u, v) = \frac{1}{2} k_{i_1 i_2} \sum_{\alpha=0}^{1} [(u_{x_1} v_{x_1})_{i_1,\, i_2+\alpha} + (u_{x_2} v_{x_2})_{i_1+\alpha,\, i_2}] +$$
$$+ r_{i_1 i_2} \sum_{\alpha,\, \beta=0}^{1} [(u_{x_2})_{i_1+\alpha,\, i_2} (v_{x_1})_{i_1,\, i_2+\beta} - (u_{x_1})_{i_1,\, i_2+\beta} (v_{x_2})_{i_1+\alpha,\, i_2}].$$

In accordance with (14), we obtain the following expressions for A:

$$Ay = -\sum_{\alpha=1}^{2} (k_- y_{\bar{x}_\alpha})_{x_\alpha} + \frac{1}{2} (p_- y_{\bar{x}_1} + p_+ y_{x_1}) - \frac{1}{2} (q^- y_{\bar{x}_2} + q^+ y_{x_2}) \tag{15}$$

at interior points,

$$Ay = -(k_- y_{\bar{x}_1})_{x_1} + \frac{2}{h_2} k^- y_{\bar{x}_2} - q^- y_{\bar{x}_2} - \frac{1}{h_2} (r_- y_{\bar{x}_1} + r_+ y_{x_1}) \tag{16}$$

on the part of the boundary Γ_1, and

$$Ay = -(k_- y_{\bar{x}_1})_{x_1} - \frac{2}{h_2} k^+ y_{x_2} - q^+ y_{x_2} + \frac{1}{h_2} (r_- y_{\bar{x}_1} + r_+ y_{x_1}) \tag{17}$$

on the part of the boundary Γ_2; here, $p = \partial r/\partial y$, $q = \partial r/\partial x$. The meaning of the notation ξ_-, ξ_+, ξ^-, ξ^+ for the functions $\xi(x)$ ($\xi(x)$ is one of the coefficients k, p, q or r) will be clear from Fig. 2.

The difference analogue of the identity (7) is

$$\Phi_h(y, w) = [y, w]_h - (f, w)_h \equiv 0, \tag{18}$$

where
$$(f, w) = \frac{1}{4} \sum_{i_1=0,\ i_2=0}^{N_1-1,\ N_2-1} \sum_{\alpha=0}^{1} f_{i_1+\alpha,\ i_2+\alpha} W_{i_1+\alpha,\ i_2+\alpha} h_1 h_2$$

defines the difference solution y. Finding this solution is equivalent to solving the finite-difference boundary value problem

$$Ay = f, \tag{19}$$

where the operator A is given by (15)–(17). Let us consider the properties of A.

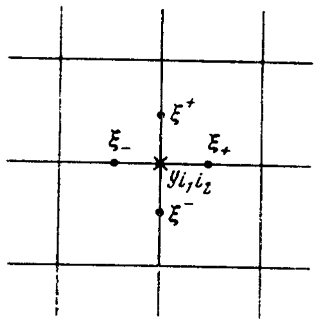

FIG. 2.

Notice that A, like the differential operator $\mathfrak{A}$, is positive definite:

$$(y, Ay)_h \geqslant \delta \|y\|^2, \qquad \delta > 0, \tag{20}$$

where δ = const is independent of h.

Let $A_0 = (A + A^*)/2$ be a self-conjugate, and $A_1 = (A - A^*)/2$ a skew-symmetric operator, such that $A = A_0 + A_1$.

The operators A_0 and A_1 have the same region of definition as A. Also, A_0 is positive definite,

$$(y, A_0 y)_h = (y, Ay)_h \geqslant \delta \|y\|^2,$$

so that we can introduce the norm

$$\|y\|_{A_0} = \sqrt{(y, A_0 y)}.$$

From (15)–(17),

$$\|y\|_{A_0}^{(2)} = h_1 h_2 \Bigg\{ \sum_{i_1=1,\ i_2=1}^{N_1,\ N_2-1} k_-(y_{\ _1})^2 + \sum_{i_1=1,\ i_2=1}^{N_1-1,\ N_2} k^-(y_{\bar{x}_2})^2 + \frac{1}{2} \sum_{i_1=1}^{N_1} [k_-(y_{\bar{x}_1})^2_{i_2=0} + \\ + k_-(y_{\bar{x}_1})^2_{i_2=N_2}] \Bigg\}. \tag{21}$$

The rate of convergence of the solution y of the difference problem to the solution of problem (1)–(4) may be estimated by means of the error

$$z = y - u. \tag{22}$$

Substituting $y = u + z$ in (19), we obtain the error equation

$$Az = \psi, \tag{23}$$

where

$$\psi = f - Au = (\mathcal{A}u)_h - Au$$

is the error in approximating the differential operator $\mathfrak{A}$ by the difference operator A on the solution u. If the initial problem has a reasonably smooth solution u, the local approximation error ψ (in a uniform metric) is $O(h^2)$; this follows because the bilinear form $[u, v]$ is approximated to $O(h^2)$ by the difference analogue $[u, v]_h$.

Hence

$$\|z\|_{A_0}^2 = (A_0 z, z)_h = (Az, z)_h = (\psi, z)_h. \tag{24}$$

Using the inequality of [7],

$$(\psi, z)_h \leqslant \|\psi\|_{A_0^{-1}} \|z\|_{A_0} \tag{25}$$

and, recalling (24),

$$\|z\|_{A_0} \leqslant \|\psi\|_{A_0^{-1}}. \tag{26}$$

Since $A_0 \geqslant \delta E$ (E is the unit operator) is positive definite,

$$A_0^{-1} \leqslant \delta^{-1} E,$$

whence

$$\|\psi\|^2_{A_0^{-1}} = (A_0^{-1}\psi, \psi) \leqslant \delta^{-1}(\psi, \psi) = \delta^{-1}\|\psi\|^2. \tag{27}$$

Using (27) and (26), we get the following *a priori* bound for problem (23):

$$\|z\|_{A_0} \leqslant \delta^{-1/2}\|\psi\|. \tag{28}$$

Since $\|\psi\| = O(h^2)$ and $\delta > 0$, the convergence rate of the difference scheme in the mesh norm W_2^1 must be $O(h^2)$.

4. Iterative schemes based on the method of alternating directions were used for the numerical solution of (19). For this, A was written as the sum of two one-dimensional operators A_1 and A_2:

$$A = A_1 + A_2.$$

Two iterative schemes, namely, the longitudinal-transverse scheme (LTS) of [8] and the Douglas–Rachford (SDR) scheme [9], were compared from the point of view of the number of iterations needed to achieve a given accuracy. The computations were performed with a fixed iterative parameter τ. It is clear from Fig. 3, which refers to the solution of (19) with $k = r = 1$, that the LTS (curve 1) converges after fewer iterations than the SDR (curve 2) whatever the value of τ. Figure 4 shows the results of solving (19) with strongly space-variable coefficients $k = 1$, $r = a \sin \gamma x \sin \gamma y + b$, a, b and γ are constants. It is evident that each scheme has its own critical τ_*, beyond which the iterations diverge; the τ_* for the SDR (curve 2) is greater than for the LTS (curve 1). This restriction of the parameter τ with strongly variable coefficients comes from the fact that the operators A_1 and A_2 cease to have a definite sign, in spite of their sum being positive-definite. The curve of the number of iterations against τ has a plateau in the case of the SDR. This fact facilitates selection of the optimal iterative parameter and enables fewer iterations to be used with the SDR than with the LTS.

5. Results of the numerical solution of some typical problems are given below, for purposes of illustration and not with any attempt at completeness.

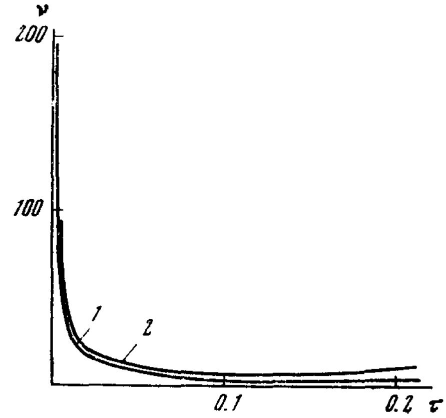

FIG. 3.

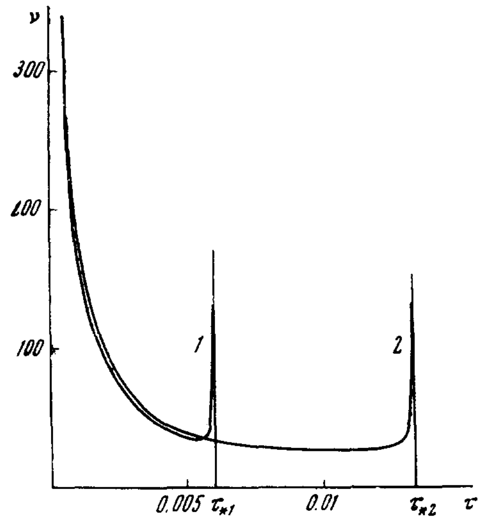

FIG. 4.

The different types of two-dimensional effect in the boundary zones of the magnetic field or close to the electrodes [1] are of great interest. Several exact solutions are available for elementary cases in this region. But the scope for an analytic approach is limited even when we confine ourselves to a very simplified discussion of problems in which the conducting liquid has given fixed parameters. In the general case of variable coefficients, numerical solution is the only practical possibility.

Consider the flow of an electrically conducting medium with constant velocity $v(v_1 = 1, v_2 = 0)$ and constant conductivity $\sigma = 1$.

The flow takes place in a flat channel $-\infty \leqslant x_1 \leqslant \infty$, $0 \leqslant x_2 \leqslant 1$ with parallel walls (see Fig. 5); the walls consist of a perfect dielectric, except for the pieces $-0.5 \leqslant x \leqslant 0.5$ occupied by ideal electrodes. A fixed potential difference $U = 0.7$ is maintained between the electrodes. The magnetic field is fixed and equal to $H = 1$ in the electrode zone, while outside this zone it falls exponentially to the power 2. The Hall parameter $\Omega = 3\,\mathrm{H}$.

By solving this problem numerically, the electric current distribution may be plotted, see Fig. 5, and integral energy characteristics such as the plasma power and electrical efficiency etc. may be found. A series of such computations is described in [10], aimed at discovering the influence of the magnetic field and electrode spatial distributions, and Hall parameter etc., on the current distribution and the integral energy characteristics of a magneto-hydrodynamic channel.

Ionization instability in a low-temperature magnetized plasma is of great interest to physicists as well as engineers. With certain assumptions, the mathematical description of this phenomenon amounts in essence to solving the set of equations

$$\begin{gathered}
\frac{\partial}{\partial x_1}\left(\frac{1}{\sigma}\frac{\partial\psi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{\sigma}\frac{\partial\psi}{\partial x_2}\right) + \frac{\partial}{\partial x_1}\left(\frac{H}{\sigma\mathbf{v}}\frac{\partial\psi}{\partial x_2}\right) - \\
- \frac{\partial}{\partial x_2}\left(\frac{H}{\sigma\mathbf{v}}\frac{\partial\psi}{\partial x_1}\right) = 0, \\
\frac{\partial n}{\partial t} = \frac{1}{\sigma}\left[\left(\frac{\partial\psi}{\partial x_1}\right)^2 + \left(\frac{\partial\psi}{\partial x_2}\right)^2\right] - nT\mathbf{v},
\end{gathered} \tag{29}$$

where n is the electron density per unit volume, and $\sigma = n/\nu$ is the electrical conductivity. A typical feature mathematically is the strong dependence of ν on n, which means that the coefficients in Eqs. (29) are strongly dependent on the space coordinates. Numerical solution of the problem provides both a qualitative picture and quantitative information on the process whereby the experimentally observed strata originate and develop. Figure 6 shows $n = 1$ level lines of the electron density at the instant $t = 1.8$ when the strata are developing strongly with a Hall parameter $\Omega = 10\,H$. The region (stratum) $n > 1$ is shaded. The vector field of the electric current density is represented by arrows. There is a marked tendency for the current to bunch round the strongly ionized stratum. The geometry of the problem is clear from Fig. 6. The boundary $ABCD$ of the region is perfect dielectric, except for the ideally sectionalized electrodes ab and cd, on which the normal component of the electric current has a given fixed density: $j_{x_2} = -\psi_{x_1} = -1$. At the initial instant $t = 0$ the electron density is fixed at $n = 1$. The connection between the temperature T and the electron density was provided by Saha's formula. Electron collisions with ions and neutral atoms were taken into account. As the process develops, the electron density and electric current distributions cease to be regular and recall the turbulent picture observed experimentally.

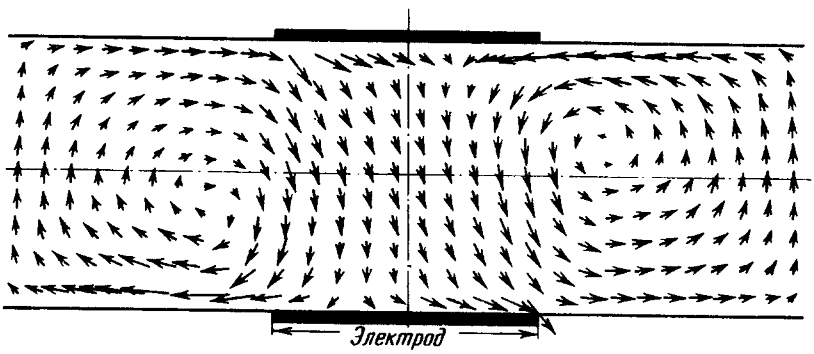

FIG. 5

A more detailed analysis of the results obtained by a numerical solution of such problems will be found in [11].

As a third example, consider the numerical solution of the two-dimensional problem of the entry of a supersonic conducting gas flow into a magnetic field. The phenomenon is described by the equations of magnetohydrodynamics. The required stationary solution is obtained by integration of the non-stationary set of equations. The following statement of the problem will be discussed. Given a channel of fixed cross-section $d = 1$. The magnetic field is spatially distributed as follows:

$$H = \begin{cases} 1, & x_1 > 0, \\ e^{15x_1}, & x_1 < 0. \end{cases}$$

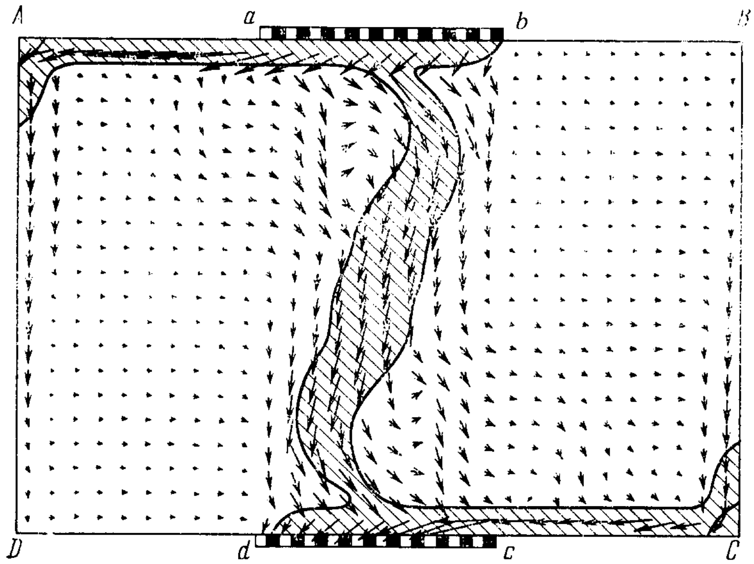

FIG. 6.

The region in which the problem is solved will be bounded by the sections $x_1 = \pm 1.5$ (see Fig. 7). At the channel entry (section $x_1 = -1.5$) we are given a supersonic gas flow with the following parameters: density $\rho = \rho_0$, temperature $T = T_0$, velocity $u = u_0$, pressure $p = p_0$, and adiabatic constant $\gamma = 1.12$. The conductivity is determined by the caesium additive electron density, and calculated from Saha's formula

$$\sigma = \sigma_0 \left(\frac{T}{T_0}\right)^{1/4} \left(\frac{\rho}{\rho_0}\right)^{-1/2} \exp\left[-\frac{I}{2}\left(\frac{1}{T} - \frac{1}{T_0}\right)\right].$$

The main dimensionless parameters characterizing the flow are the Mach number M and the parameter R_M of hydromagnetic interaction. These have the values $M = 2.92$ and $R_M = 0.5$ in the section $x_1 = -1.5$. The stationary values of the spatially distributed parameters are shown in Fig. 7. The ratio of the parameter value on the level line to its value at the entry cross-section is given. See [12] for more details. The current vortex that arises when the conducting gas enters the magnetic field is seen in Fig. 7. The existence of such a vortex is familiar [1] from the solution of the problem when the gas motion is specified in advance, i.e., in the approximation $R_M \ll 1$.

The trend of the level lines in Fig. 7 reveals that substantial braking and compression towards the axis occur when a supersonic conducting gas flow enters a magnetic field with $R_M = 0.5$. The variation of the flow parameters leads in turn to strong damping of the electric current in the region of the fixed magnetic field with $x_1 > 0$.

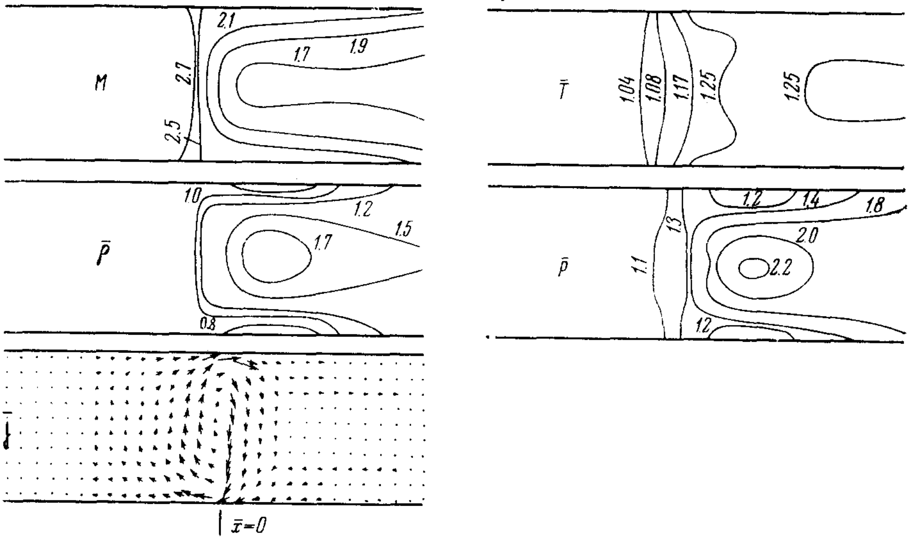

FIG. 7.

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