A FINITE-DIFFERENCE METHOD FOR THE SOLUTION OF ONE-DIMENSIONAL NON-STATIONARY PROBLEMS IN MAGNETO-HYDRODYNAMICS*

A. A. SAMARSKII, P. P. VOLOSEVICH, M. I. VOLCHINSKAYA and S. P. KURDYUMOV

Moscow

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1. Introduction

IN theoretical investigations of a number of applied problems in magneto-hydrodynamics (various types of MHD-generators, problems of astrophysics etc.) there is particular interest in the study of interaction processes between a compressible electrically conducting gas and a magnetic field for arbitrary Reynolds numbers Re_m and the magnetic interaction parameters $R_H = H^2/8\pi p$, where H is the magnetic field strength and p the pressure. In this case and in physical experiments an important role is played by the investigation of mathematical models which take into account mainly the non-linear relations between the non-stationary processes of magneto-hydrodynamics. In the one-dimensional approximation numerical methods also enable us not only to study the quantitative sides of the processes, but also to establish a number of new qualitative regularities. Thus the use of numerical methods for equations of magneto-hydrodynamics, taking into account complex non-linear dissipative processes, has made possible the solution of a number of actual physical problems [1-6]. In [6] a new physical phenomenon is described, the so-called T-layer effect – a high-temperature, electrically-conducting, self-sustaining layer of gas, arising at a definite part of the mass due to Joule heating.

The present paper describes numerical methods of solving the equations of magneto-hydrodynamics which, are specifically used in the study of the T-layer phenomenon. It is assumed that the thermal and electrical conductivity may be arbitrary functions of the temperature and density. The method and its computer programs enable us to solve a large group of problems with various combinations of boundary conditions and equations of state of the material. It is also considered

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that the medium studies can consist of regions with various strongly changing physical parameters. Real physical viscosity is not taken into account.

The magneto-hydrodynamic set of equations is solved by the method of finite differences. The method for solving hydrodynamic and heat-conduction equations (without a magnetic field) developed by A. N. Tikhonov and A. A. Samarskii in 1952, is basically assumed.

Implicit conservative difference schemes, which are unconditionally stable, are considered. The conservativeness of the difference schemes is essential if we consider discontinuities in the solution (contact and shock waves),because it ensures the convergence of the difference schemes if discontinuities are present.

The method is applicable for the solution of multi-regional problems with strongly varying physical parameters for the medium. In this case the difference scheme requires high reliability in the sense of stability with respect to local disruptions of monotonicity.

The method of successive recursions for the solution of hydrodynamic and heat-conduction problems (without a magnetic field), in whose development N. N. Kalitkin took part, has been used since 1958. A similar method was put forward independently in [7].

The method of solving the magneto-hydrodynamic equations put forward in this paper was invented in 1962 and was first made public at the third Riga conference on magneto-hydrodynamics in 1964.

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2. The differential and difference equations of magneto-hydrodynamics

1. Let t be the time, $\mathbf{H}$ the magnetic field strength vector v the velocity, ρ the density of the substance, p the pressure and ϵ the internal energy. The set of equations of magneto-hydrodynamics taking account the non-linear electrical

and thermal conductivity in the absolute system of Gaussian units takes the form [8]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\nabla p/\rho - [\mathbf{H} \operatorname{rot} \mathbf{H}]/4\pi\rho, \quad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0,$$

$$\frac{\partial}{\partial t}(\rho v^2/2 + \rho\varepsilon + H^2/8\pi) = -\operatorname{div}\mathbf{q},$$

$$\mathbf{q} = \rho\mathbf{v}(v^2/2) + \varepsilon + p/\rho) + [\mathbf{H}[\mathbf{v}\mathbf{H}]]/4\pi - \nu_m[\mathbf{H} \operatorname{rot}\mathbf{H})/4\pi + W,$$

$$\partial\mathbf{H}/\partial t = \operatorname{rot}[\mathbf{v}\mathbf{H}] - \operatorname{rot}[\nu_m \operatorname{rot}\mathbf{H}), \quad W = -\varkappa\nabla T, \quad \operatorname{div}\mathbf{H} = 0, \tag{2.1}$$

where $\nu_m = c^2/4\pi\sigma$ is the magnetic viscosity and c the velocity of light.

The electrical and thermal conductivity, σ and $\varkappa$ respectively, are non-linear functions of the temperature T and density ρ and satisfy the conditions $\partial\sigma/\partial T \geqslant 0$, $\partial\varkappa/\partial T \geqslant 0$. The internal energy and pressure are functions of the density and temperature.

2. We shall denote the cylindrical or cartesian coordinates by r, ϕ, z. Let the one-dimensional motion of the medium be directed along the Eulerian r-axis. We shall assume that in the plane case a non-zero component H_r of the vector $\mathbf{H}$ may exist in the direction of motion and components H_φ and H_z perpendicular to the direction of motion. From the equation $\operatorname{div}\mathbf{H} = 0$ we have $H_r = H_{r_0} = \text{const}$ in the plane case and $H_{r_0} = 0$ in the case of axial symmetry. We shall denote the corresponding components of the velocity $\mathbf{v}$ by v_r, v_φ and v_z.

We now introduce into the direction of motion r a Lagrange mass coordinate x, connected with r by the formula $dx = \rho r^{\nu-1}dr$, where $\nu = 1$ corresponds to the plane case and $\nu = 2$ to the case of axial symmetry.

3. The solution of (2.1) is sought in the bounded region $0 \leqslant x \leqslant l$ where $x = 0$ is the left-hand boundary of the plane medium or the centre of axial symmetry and $x = l$ is the outer boundary of the medium.

For the gas-dynamic values on each of the boundaries we may be given either the speed or the pressure as arbitrary functions of time.

For the energy equation we may be given the temperature T or the thermal flux W on the boundary.

For the magnetic field equations on each of the boundaries $x = 0$ and $x = l$ we may be given either the components of the field vector H_φ and H_z or the functions

$$\Phi = \rho \nu_m \frac{\partial (r^{\nu-1} H_\varphi)}{\partial x}, \qquad \Psi = \rho \nu_m r^{\nu-1} \frac{\partial H_z}{\partial x}$$

in the form of arbitrary functions of time.

The components of the magnetic field vector on the boundaries $x = 0$ and $x = l$ can also be determined from the additional equations for electronic circuits.

If contact discontinuities are present in the medium (regions with different physical parameters) junction conditions are added to the system (2.1) and the boundary conditions: the continuity of the thermal flux to the left and right of the discontinuity and the continuity of the temperature give

$$W_{\text{л}} = W_{\text{п}}, \qquad T_{\text{л}} = T_{\text{п}}. \tag{2.2}$$

In addition, if $\sigma \neq 0$ we require the continuity of the functions Φ and Ψ to the left and to the right of the contact discontinuity and the condition for isomagnetism

$$\Phi_{\text{л}} = \Phi_{\text{п}}, \qquad \Psi_{\text{л}} = \Psi_{\text{п}}, \qquad H_{\varphi_{\text{л}}} = H_{\varphi_{\text{п}}}, \qquad H_{z_{\text{л}}} = H_{z_{\text{п}}}. \tag{2.3}$$

At the initial instant $t = 0$ the components of the vectors $\mathbf{v}$ and $\mathbf{H}$ are given, and also the density $\rho(0, x)$ (or the radius $r(0, x)$) and the temperature $T(0, x)$.

4. The system (2.1) is solved by the method of finite differences. A non-uniform network $\omega_{m,\tau} = \{(x_i, t^j)\}$ is constructed in the region $G = \{(x, t)\}$. We shall denote the steps of the network $\omega_{m,\tau}$ in space and time by $m_i = x_{i+1} - x_i$, $\tau^{j-1} = t^j - t^{j-1}$ We replace the functions under consideration by corresponding network functions. The values of the functions for the speed, the coordinate r and the thermal and magnetic fluxes $v_{r_i}{}^j$, $v_{\varphi_i}{}^j$, $v_{z_i}{}^j$, r_i^j, $W_i{}^j$, $\Phi_i{}^j$, $\Psi_i{}^j$ will be related to the "integral" (nodal) point of the network $(x_i t^j)$. The difference values of the function for the density, pressure, temperature, internal energy and the strength of the magnetic field $\rho_i{}^j$, $p_i{}^j$, $T_i{}^j$, $\varepsilon_i{}^j$, $H_{\varphi_i}{}^j$, $H_{z_i}{}^j$ will be related to the "semi-integral" point of the network $(x_{i+1/2}, t^j)$, где $x_{i+1/2} = 0.5(x_i + x_{i+1})$ the middle of the mass interval m_i. For simplicity we shall use only integral indices for the network functions. We shall use the notation $\bar{m}_i = x_{i+1/2} - x_{i-1/2} = 0.5(m_i + m_{i-1})$. The change from the set of differential equations to the set of difference equations at the internal points of the network $0 < x_i < x_N = l$ is made by replacing the derivatives with respect to x by two-sided (central) differences, and at the boundary points $x = 0$ and $x = l$ by one-sided (left-hand or right-hand) differences.

The equations of motion (for the plane case $\nu = 1$), and also the continuity and energy equations are considered in a divergent form, i.e. in the form of balance equations. Therefore in writing the corresponding difference equations it is natural to use the integro-interpolation (energy) method, by means of which the conservative difference schemes of [9-12] are constructed. The conservativeness, which ensures the convergence of the difference schemes with continuities, is very essential for obtaining discontinuous solutions (contact and shock waves).

The set of difference equations approximating to (2.1) takes the form

$$\frac{v_{r_i}^j - v_{r_i}^{j-1}}{\tau^{j-1}} = \frac{1}{m_i}\{\gamma_1 [r_i^{\nu-1}(P_{i-1} - P_i)]^j + (1-\gamma_1)[r_i^{\nu-1}(P_{i-1} - P_i)]^{j-1}\} - \frac{\nu-1}{8\pi}[\gamma_1 (H_{\varphi_{i-1}}^j + H_{\varphi_i}^j)^2 / (\rho_{i-1}^j + \rho_i^j) r_i^j + (1-\gamma_1)(H_{\varphi_{i-1}}^{j-1} + H_{\varphi_i}^{j-1})^2 / (\rho_{i-1}^{j-1} + \rho_i^{j-1}) r_i^{j-1}], \tag{2.4}$$

$$\frac{v_{z_i}^j - v_{z_i}^{j-1}}{\tau^{j-1}} = \frac{1}{m_i}\frac{H_{r_0}}{4\pi}[\gamma_1 (H_{z_i} - H_{z_{i-1}})^j + (1-\gamma_1)(H_{z_i} - H_{z_{i-1}})^{j-1}], \tag{2.5}$$

$$\frac{r_i^j - r_i^{j-1}}{\tau^{j-1}} = \gamma_2 v_{r_i}{}^j + (1 - \gamma_2) v_{r_i}^{j-1}, \qquad \rho_i{}^j = \nu m_i / (r_{i+1}^\nu - r_i{}^\nu), \tag{2.6}$$

$$\frac{E_i{}^j - E_i^{j-1}}{\tau^{j-1}} = \frac{1}{m_i}[\gamma_3 (q_i - q_{i+1})^j + (1-\gamma_3)(q_i - q_{i+1})^{j-1}], \tag{2.7}$$

$$\frac{H_{z_i}^j - H_{z_i}^{j-1}}{\tau^{j-1}} = \frac{H_{r_0}}{m_i}[\gamma_4 \rho_i^j (v_{z_{i+1}} - v_{z_i})^j + (1-\gamma_4)\rho_i^{j-1}(v_{z_{i+1}} - v_{z_i})^{j-1}] + \frac{1}{m_i}[\gamma_4 \rho_i^j H_{z_i}^j (r_i^{\nu-1} v_{r_i} - r_{i+1}^{\nu-1} v_{r_{i+1}})^j + (1-\gamma_4)\rho_i^{j-1} H_{z_i}^{j-1}(r_i^{\nu-1} v_{r_i} - r_{i+1}^{\nu-1} v_{r_{i+1}})^{j-1}] + \frac{1}{m_i}[\gamma_4 \rho_i{}^j (\Psi_{i+1} - \Psi_i)^j + (1-\gamma_4)\rho_i^{j-1}(\Psi_{i+1} - \Psi_i)^{j-1}], \tag{2.8}$$

where

$$P_i = p_i + (H_{\varphi_i}{}^2 + H_{z_i}{}^2)/8\pi, \qquad E_i = \varepsilon_i + [(v_{r_i} + v_{r_{i+1}})^2 + (v_{\varphi_i} + v_{\varphi_{i+1}})^2 + (v_{z_i} + v_{z_{i+1}})^2]/8 + (H_{\varphi_i}{}^2 + H_{z_i}{}^2)/8\pi\rho_i,$$

$$q_i = {}^1/_2 (P_{i-1} + P_i) r_i^{\nu-1} v_{r_i} - \frac{H_{r_0}}{8\pi}[(H_{\varphi_{i-1}} + H_{\varphi_i}) v_{\varphi_i} + (H_{z_{i-1}} + H_{z_i}) v_{z_i}] + N_i + W_i,$$

$$N_i = -\frac{r_i^{\nu-1}}{4\pi}(H_{\varphi_i}\Phi_i + H_{z_i}\Psi_i), \qquad 0 \leqslant i \leqslant N-1,$$

$\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are "weight" factors (constants).

The difference equations for the components of the velocity v_{φ_i} and magnetic field H_{φ_i} are written in a corresponding way.

The constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ in the system (2.4)–(2.8) have different values depending on the choice of difference scheme. If $\gamma_1 = 0$, $\gamma_2 = 1$ on the assumption that the difference value for the speed v_i^j is related to the intermediate time layer $(x_i, t_i^{j-1/2})$, we have an explicit "cross" scheme. If $\gamma = ½$ we have a symmetrical implicit scheme and if $\gamma_i = 1$ an implicit leading scheme.

5. The difference formula for the "magnetic flux" Ψ takes the form

$$\Psi_i = k_{z_i}(H_{z_i} - H_{z_{i-1}}), \tag{2.9}$$

where $k_{z_i} = 0.5k_{z_i}^{(-)}k_{z_i}^{(+)}/(k_{z_i}^{(+)}m_{i-1} + k_{z_i}^{(-)}m_i)$, $k_i^{(-)} = k_z(\rho_{i-1}, T_{i-1})$ is the coefficient of magnetic viscosity of the region to the left of the contact discontinuity, and $k_{z_i}^{(+)} = k_z(\rho_i, T_i)$ to the right. The expressions for the function Φ_i and the integral flux N_i are of the same form. In the derivation of the formula of form (2.9) possible discontinuities of the conductivity at the contact discontinuity and the junction conditions (2.3) are taken into account-

The difference formula for the thermal flux W is considered in the form $W_i = k_i(\Sigma_{i-1} - \Sigma_i)$, where $\Sigma = T^{\alpha}/a$ is a step function of the temperature [9, 13], and k_i is of a form similar to (2.9). The linearization of the thermal flux with respect to the function Σ enables us to calculate accuractely the temperature wave font on coarse space networks in the case where the thermal conductivity $\varkappa$ is a function of a high power of the temperature ($\varkappa \sim T^{\alpha-1}$). The possibility of linearization of the thermal flux with respect to the temperature T also occurs.

3. The method of direct calculation of the shock waves

1. In many magneto-hydrodynamic problems of practical interest discontinuities solutions, i.e. shock waves, may exist.

If $0 < \sigma < \infty$ the shock waves are isothermal and isomagnetic (on the assumption that their frontal structure is not taken into account), i.e. the temperature and strength of the magnetic field on the front of the shock wave are continuous, but the flux discontinuous [14].

In the case where the medium is not a thermal conductor ($\varkappa = 0$) and has an infinite electrical conductivity ($\sigma = \infty$), some types of shock waves exist which differ from one another in their physical properties [8].

The method under consideration assumes that it is possible to calculate the shock waves directly without an explicit choice of the discontinuity front.

For this, by analogy with ordinary gas-dynamics [15, 16], we introduce an artificial viscosity mechanism (the so-called "pseudoviscosity"), which serves to "smear" the shock waves.

The forms of viscosity may be different.

On the right-hand side of the equation for the component of the velocity v_r and in the energy equation (see equations (2.4) and (2.7)) instead of the function P_i we consider the function $G_i = P_i + \omega_i$ where ω is a function of the form

$$\omega = -\nu_0 m_i^{1+\mu} (P/\rho)^{(1-\mu)/2} \left| \frac{\partial (r^{\nu-1} v_r)}{\partial x} \right|^{\mu} \times \tag{3.1}$$
$$\times \rho \left[\frac{\partial (r^{\nu-1} v_r)}{\partial x} - \nu_1 \left| \frac{\partial (r^{\nu-1} v_r)}{\partial x} \right| \right] \Big/ r^{(\nu-1)(\mu+1)}.$$

If $\mu = 1$ formula (3.1) corresponds to the so-called quadratic viscosity and if $\mu = 0$ to the linear viscosity, which is analogous to the second physical viscosity. From (3.1) it follows that if $\nu_1 = 1$ in the region where $\partial (r^{\nu-1} v_r) / \partial x \geqslant 0$, the viscosity $\omega = 0$, i.e. outside the shock wave zone there is no viscosity. The choice of the coefficient ν_0 depends essentially on the character if the motion of the medium being studied and is made by numerical experiments. (For more details of the choice of viscosity see [17]).

Besides viscosity of the form (3.1) we use a combined viscosity which has the form [18]

$$\omega = -\nu_0 m_i \rho \left(\frac{\partial v_r}{\partial x} - \nu_1 \left| \frac{\partial v_r}{\partial x} \right| \right) \left(\sqrt{(P/\rho)} + \nu_2 m_i \left| \frac{\partial v_r}{\partial x} \right| \right) \tag{3.2}$$

With large speed gradients it is the same as the quadratic viscosity and with small gradients it is the same as the linear viscosity.

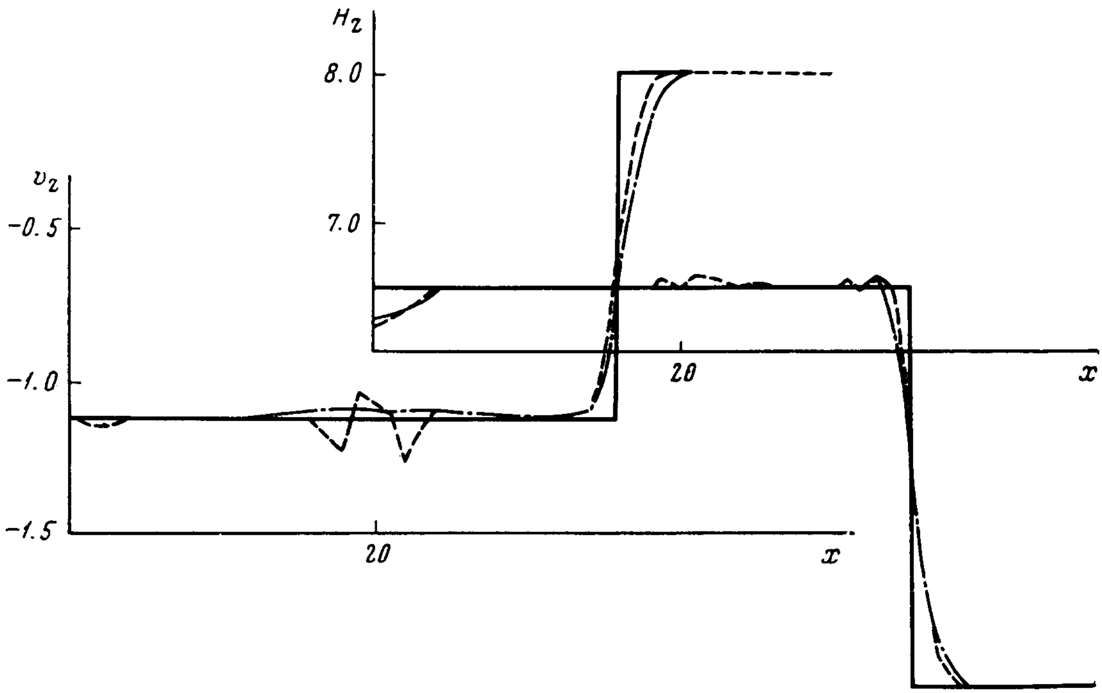

FIG. 1.

In the case when $\sigma = \infty$ (a frozen magnetic field) and $H_{r_0} \neq 0$ the set of equations of motion and equations of the magnetic field is hyperbolic. In this case for the direct calculation of the magneto-hydrodynamic discontinuities we also require to introduce artificial viscosities into the equations for the components of the velocity v_φ and v_z and into the equations for the magnetic field. It must be noted, however, that with the introduction of pseudoviscosities into the magneto-hydrodynamic equations the conditions of evolution of the magneto-hydrodynamic discontinuities must first be satisfied.

By analogy with ordinary gas dynamics we chose viscosity terms of the form

$$\omega_z = \left(\nu_0' m_i + \nu_0'' m_i^2 \left| \frac{\partial v_z}{\partial x} \right| \right) \left(\frac{\partial v_z}{\partial x} - \left| \frac{\partial v_z}{\partial x} \right| \right), \tag{3.3}$$

$$h_z = - \left(\mu_0' m_i + \mu_0'' m_i^2 \left| \frac{\partial H_z}{\partial x} \right| \right) \left(\frac{\partial H_z}{\partial x} - \left| \frac{\partial H_z}{\partial x} \right| \right) \tag{3.4}$$

In the corresponding difference formulae we add to the right-hand side of equation (2.5) a term of the form $(1/\bar{m}_i)(\omega_{z_i} - \omega_{z_{i-1}})$ and if $\Psi \equiv 0$ we add a term of the form

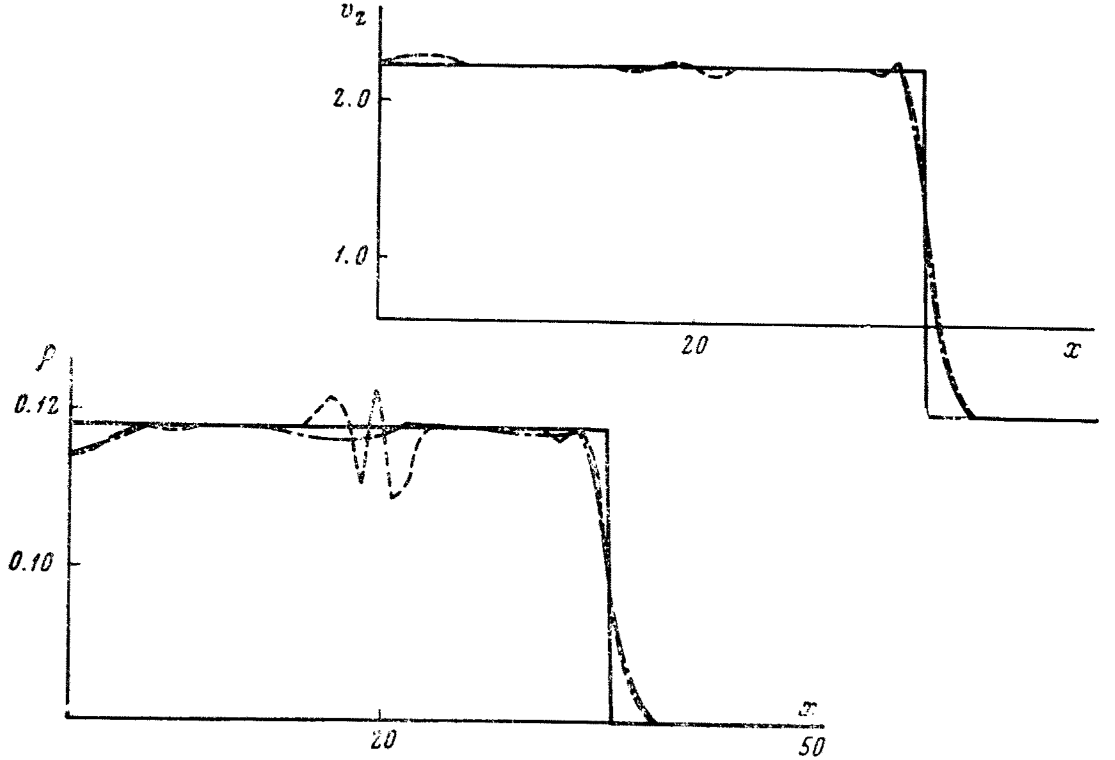

FIG. 2.

$$(1/m_i)\,(h_{z_i} - h_{z_{i+1}})$$

to the right-hand side of equation (2.8).

Reasonable values for the numerical coefficients of viscosity ν_0', ν_0'', μ_0', μ_0'' in formulae (3.3) and (3.4) are chosen from numerical experiments and depend on concrete solved problems.

2. We now give an example of a calculation on a computer of rapid magnetohydrodynamic shock waves for the case $\sigma = \infty$ (a frozen magnetic field), $H_r \neq 0$, $\varkappa \equiv 0$, $H_\varphi \equiv 0$, $v_\varphi \equiv 0$. The plane case is considered. Calculations are carried out by the implicit difference scheme for $\gamma_1 = \gamma_2 = 1$, $\gamma_3 = \gamma_4 = {}^1/_2$.

For simplicity an expression for the pressure of the form $p = \mathrm{const}\,\rho$ is used. The results of a comparison of the numerical solutions with the analytical solutions are given in Figs. 1 and 2. Here the continuous line denotes the analytic solution, the dot-dashed line the numerical solution with viscosities of the form (3.3), (3.4) and (3.1) for $\mu = 1$, and the dashed line the numerical solution without consideration of the viscous terms in the field equation.

The space network for the calculation was uniform and 50 mass intervals m_i were given.

The comparison given in Figs. 1 and 2 confirms the satisfactory accuracy of the calculations.

The series of calculations shows that the combined viscosity (3.2) possesses a definite advantage as compared with other forms of viscosities.

4. An iterative method for successive recursions

1. The set of difference equations (2.4)–(2.8) is solved by the implicit difference schemes, taking into account the dissipation of energy due to electrical and thermal conduction, using the successive sweep method.

The idea of the method is to reduce the separate equations of the set to second-order difference equations and to apply the well-known sweep method [19] successively for their solution.

Implicit difference schemes for the equations of gas-dynamics with heat conduction (without magnetic field) were used by I. M. Gel'fand, O. V. Lokutsievskii and V. F. D'yachenko in 1957. The corresponding set of difference equations was solved by the matrix sweep method.

In the successive sweep-method only a one-dimensional sweep is used for three-point difference equations.

The order of the calculations in the separate equations of the set (2.4)–(2.8) may be different. The following sequence of calculations was chosen.

On each j-th layer the energy equation (2.7) is first solved on the assumption that the magneto-hydrodynamic quantities are known (sweep with respect to T). Then the set of gas-dynamic equations (2.4)–(2.6) is solved for a known temperature and fixed magnetic values (sweep with respect to v) and finally the set of equations for the diffusion of the magnetic field (see equation (2.8)) for known temperature and hydrodynamic values (sweep with respect to H).

Each separate sweep is calculated until the condition of convergence is satisfied. A single calculation of the first two sweeps (with respect to T and v) forms one cycle of a small loop. All three sweeps (with respect to T, v and H) form one cycle of a large loop. Each small loop is inside a large one and each large loop is calculated up to the given number of cycles.

Experiment showns that to satisfy the required accuracy two cycles of the small loop and two cycles of the large loop are sufficient. From a large number of numerical calculations it follows that the maximum number of iterations for the calculation of each separate sweep does not exceed three or four.

2. We shall dwell in more detail on the method of solving the energy equation.

Let us assume that on the j-th time layer the hydrodynamic values and also the strength of the magnetic field and magnetic flux N are known.

We linearize the function $\varepsilon^j = \varepsilon(\rho^j, T^j)$ by Newton's method, i.e. we put it in the form

$$\varepsilon^j = \varepsilon(\Sigma^{(s+1)}) = \varepsilon(\Sigma^{(s)}) + \left(\frac{\partial \varepsilon^{(s)}}{\partial \Sigma}\right) \delta\Sigma^{(s)}, \tag{4.1}$$

where $\Sigma = T^\alpha/\alpha$, $\delta\Sigma^{(s)} = \Sigma^{(s+1)} - \Sigma^{(s)}$, s is the number of the iteration. Substituting (4.1) in (2.7) we obtain the following second-order difference equation with respect to the function $\Sigma^{(s+1)}$:

$$a_i^{(s)}\Sigma_{i-1}^{(s+1)} - b_i^{(s)}\Sigma_i^{(s+1)} + c_i^{(s)}\Sigma_{i+1}^{(s+1)} + g_i^{(s)} = 0, \tag{4.2}$$

where the coefficients $a_i^{(s)}$, $b_i^{(s)}$, $c_i^{(s)}$ and $g_i^{(s)}$ depend on the function $T_i^{(s)}$, and also on v_i, ρ_i and H_i. The solution of equation (4.2) is found from the known sweep recurrence formulae.

On the assumption that the hydrodynamic and thermal values are fixed, the equation for the diffusion of the magnetic field is a linear second-order difference equation in H_φ and H_z and is solved by the sweep method with respect to the functions H_φ and H_z without iterations.

Calculations have shown that in the case $\sigma = 0$ for values of σ or near zero the calculation of the equations for the diffusion of the magnetic field by the sweep method with respect to the function H leads in a number of cases to unsatisfactory results. Similar difficulties encountered in the calculations , are noted in [5]. The fact is that if $\sigma = 0$ the derivatives $\partial(r^{\nu-1}H_\varphi)/\partial x$ and $\partial H_z/\partial x$ are zero and the fluxes Φ and Ψ becomes indeterminant. In a physical sense the functions Φ and Ψ have in this case a finite value. The indeterminancy of the type 0/0 arising in the difference equation for the diffusion of the magnetic field leads to a bump and in a number of cases to a significant distortion of the solution.

In [20] the method of flux sweep is put forward. In the case of the equation for the diffusion of the field by the sweep method the fluxes Φ and Ψ are first determined and then the field H. In the flux variant of the sweep method the functions Φ and Ψ are calculated more accurately than in the case of the ordinary sweep method with respect to H, which is very essential.

At the present time the flux sweep method is used both for the solution of the equation for the diffusion of the magnetic field and also for the solution of the energy equation for any range of variation of the values of the electrical conductivity $0 \leqslant \sigma \leqslant \infty$ and thermal conductivity $0 \leqslant \varkappa \leqslant \infty$. Now the authors together with N. N. Kalitkin, L. M. Degtyarev, A. P. Favorsk and Yu. P. Popov, propose to consider the equations of the magnetic field in the divergent form, i.e. in the form

$$\frac{\partial}{\partial t}(H_\varphi/r^{\nu-1}\rho) = H_{r_0}\frac{\partial v_\varphi}{\partial x} + \frac{\partial \Phi}{\partial x}, \qquad \frac{\partial}{\partial t}(H_z/\rho) = H_{r_0}\frac{\partial v_z}{\partial x} + \frac{\partial}{\partial x}(r^{\nu-1}\Psi), \tag{4.3}$$

and in the energy equation to explicitly choose the term denoting the Joule effect, i.e. to consider it in the form

$$\frac{\partial}{\partial t}\left(\varepsilon + \frac{v^2}{2}\right) = -\frac{\partial}{\partial x}(r^{\nu-1}v_r p) - \frac{\partial W}{\partial x} + Q + Fv, \tag{4.4}$$

where $Q = (\sigma / c^2\rho)(\Psi^2 + \Phi^2)$ is the Joule heating of the electric current, and $F = -(1/4\pi)[r^{\nu-1}H_z\partial H_z / \partial x + (H_\varphi \partial(r_\nu^{-1}H_\varphi) / \partial x)]$ of the Lorentz force.

Equations (4.3) and (4.4) are equivalent to the equations for the diffusion of the magnetic field and energy in the system (2.1).

We cannot possibly dwell on a detailed justification for the proposed changes in the calculation of the equation for the diffusion of the magnetic field and energy equation.

5 . Analysis of the stability of the set of difference equations

The problem of investigating the stability of the complete set of difference equations (2.4)–(2.8), taking into account all the dissipative terms, is very complicated. Questions of the stability of parabolic equations are investigated quite fully in [9-12]. Experiment shows that the greatest restriction on the time step must be imposed in the limiting case $\nu_m \equiv 0$ and $\varkappa \equiv 0$, i.e. when the set of equations is hyperbolic.

Analysis of stability, carried out by the spectral method [15] for the case $\nu_m \equiv 0$, $\varkappa \equiv 0$, $\nu = 1$ on the assumption of the validity of the equations for the state of an ideal gas ($p = \rho\epsilon/(\gamma - 1)$, where γ is the ratio of the specific heats), leads to the following results (for more detail see [21]).

1. The implicit leading scheme ($\gamma_i = 1$), the symmetrical scheme ($\gamma_i = ½$) and also the implicit scheme ($\gamma_1 = \gamma_2 = 1$, $\gamma_3 = \gamma_4 = {}^1/_2$) are unconditionally stable.

2. The condition for stability of the explicit "cross" scheme ($\gamma_1 = 0$, $\gamma_2 = \gamma_3 = \gamma_4 = 1$), is of the form

$$\tau < \eta_0 m / c_{+0}, \tag{5.1}$$

where c_{+0} is the fast magneto-hydrodynamic speed of sound [8], $\eta = 1/\rho$.

Condition (5.1) is a generalization of the well-known ordinary gas-dynamic Courant condition for the set of difference equations of magneto-hydrodynamics.

Experiment shows that in a number of problems the quantity $c_+ = c(p, \rho, H^2)$ may be large and consequently the condition for stability (5.1) may substantially restrict the step in the time τ. Therefore it is inadvisable in practice to make use of explicit schemes for the equations of magneto-hydrodynamics.

3. We now consider implicit schemes similar to that considered in [7] for ordinary hydrodynamics. For the solution of the difference equations cited in [7] the successive sweep method is also used. In practice such schemes correspond to one cycle of successive sweeps in the leading scheme (see Section 4, para. 1).

The analysis carried out for the case $H_{r_0} = 0$ shows that, independently of the order of the application of the successive sweeps, any such scheme is inconditionally stable only if the condition

$$H^2/8\pi \leqslant (1 - \gamma/2)p, \tag{5.2}$$

is satisfied or

$$\gamma \leqslant 2(1 - R_H), \qquad R_H = H^2/8\pi p.$$

If

$$H^2/8\pi > (1-\gamma/2)p \qquad \text{or} \qquad \gamma > 2(1-R_H) \tag{5.3}$$

the condition for stability takes the form

$$\tau < \eta_0 m / \gamma (c_{+0}^2 - 2p_0\eta_0). \tag{5.4}$$

If $H \equiv 0$ the conditions for stability cited in [7] follow directly from the conditions (5.2) and (5.3).

If $H \neq 0$ and particularly in the case $R_H \geqslant 1$, i.e. for a wide class of practically interesting problems, the difference scheme considered in this section is scarcely more economic (in the sense of rapidity of speed of calculation) than the leading scheme, since a definite constraint on the step of the form (5.4) is required for stability. In addition it must be noted that in the approximation of a set of differential equations by such schemes divergence in time in the equation of motion and the energy equation i.e. the conservativeness of difference schemes is violated.

Numerical experiments show that when dissipative terms for thermal conductivity and finite conductivity are present in the medium an unconditionally stable implicit difference scheme, obtained with values of the weight factors $\gamma_1 = \gamma_2 = 1$, $\gamma_3 = \gamma_4 = 1/2$, is the most advantageous both in accuracy and economy, i.e. an implicit leading difference scheme for a set of hyperbolic equations of motion and continuity and a symmetrical implicit scheme for a set of parabolic equations for the energy and the diffusion of the magnetic field.

6. Comparison of numerical solutions with automodelling

The evaluation of the accuracy of the above numerical methods of solving the set of equations of magneto-hydrodynamics was carried out experimentally by the solution of a large number of model problems. To verify the method difficult problems were chosen with strongly varying physical parameters and essentially non-linear processes.

As an example we shall consider the automodelling plane problem of the motion of a gas in front of a piston in a magnetic field with non-linear heat conduction and conductivity [22]. It is assumed that the speed of the piston and the temperature on it vary according to a power law with respect to the time ($v \sim t^{n-1}$, $T \sim t^{2(n-1)}$) and the axial magnetic field given on the piston is constant: $\dot{H}_z = \text{const} < 0$. In front of the piston the gas is considered with initial conditions

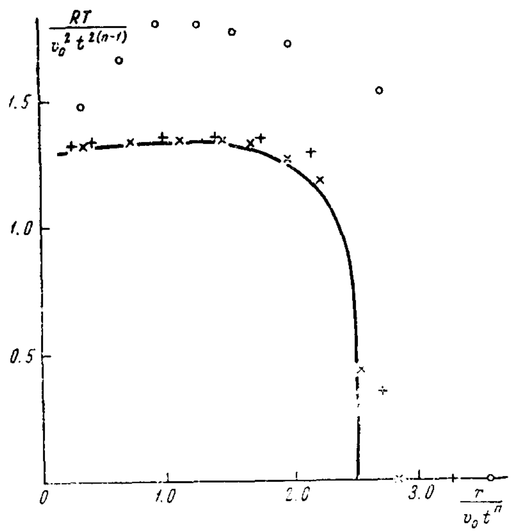

FIG. 3.

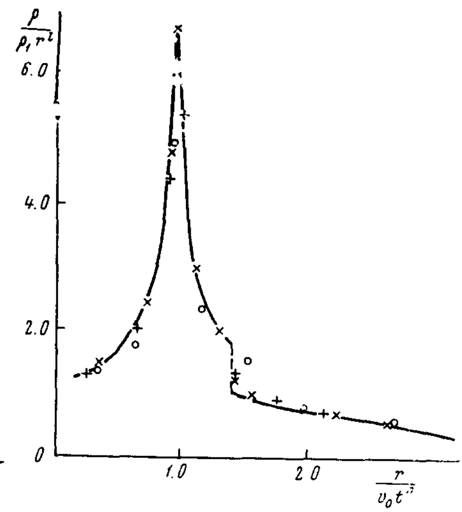

FIG. 4.

$$v(r, 0) = 0, \qquad T(r, 0) = 0, \qquad \rho(r, 0) = \rho_1 r^l, \qquad H(r, 0) = \text{const} > 0.$$

The thermal conductivity $\varkappa$ and electrical conductivity σ depend on the temperature and density in accordance with a power law:

$$\varkappa = \varkappa_0 T^{m_0} \rho^{-\sigma_0}, \qquad \sigma = \sigma_0' T^{m_1} \rho^{-\sigma_1}, \qquad m_0 > 0, \qquad m_1 > 0. \tag{6.1}$$

For definite relations between the constants n, l, m_0, m_1, σ_0 and σ_1 the problem is considered to be automodelling.

Figures 3 and 4 give comparative graphs of the relation between the dimensionless temperature $T / v_0^2 t^{2(n-1)}$ (see Fig. 3) and density $\rho/\rho_1 r^l$ (see Fig. 4) and the dimensionless coordinate $r/v_0 t^n$ (Here v_0 and ρ_1 are dimensional constants). The continuous lines in Fig. 3 and 4 depict the automodelling solution, and the circles and crosses the corresponding values of the numerical solution at various instants t. The calculation by means of the set of equations (2.4)–(2.8) began with $t = 0$ (with zero and constant initial data), and then an entry into the automodelling system was effected.

In spite of some "exotic" initial and boundary conditions substantial non-linearity of the electrical conductivity ($\sigma \sim T^{3/2}$) and thermal conductivity ($\varkappa \sim T^5$) is introduced into the automodelling problem.

The calculation is carried out by the successive sweep method using the implicit scheme with $\gamma_1 = \gamma_2 = 1$, $\gamma_3 = \gamma_4 = ½$. The circles in Figs. 3 and 4 correspond to the instant $t = t_1$, at which the perturbing wave (temperature wave) is packed on 9 mass intervals of the network, the horizontal crosses to the instant $t = t_2$ at which the temperature wave embraces 14 mass intervals, and the diagonal crosses correspond to the instant $t = t_3$ with 24 mass intervals of the network in the temperature wave zone.

The numerical solution shows that entry into the automodelling system is accomplished fairly quickly and accurately.

A large number of practically important magneto-hydrodynamic problems were computed by the above method. The discovery of the new physical phenomenon - the so-called *T*-layer effect (temperature layer) [6] - by means of calculations on a computer serves as one of the striking examples of the effective use of numerical methods in magneto-hydrodynamics. The essence of the *T*-layer phenomenon lies in the fact that in a compressible medium with defined conditions a local, comparatively narrow, zone of increased temperature and electrical conductivity may arise which represents a self-sustaining and stable macro-formation. The *T*-layer effect produces substantially new peculiarities in the behaviour of a plasma:

Firstly, the interaction of the plasma with the magnetic field is increased many times. Thus low-temperature plamsma can effectively interact with a magnetic field by means of the *T*-layer despite low conductivity;

Secondly, because of the *T*-layer the magnetic field can play the role of a catalyst, enabling comparatively cold plasma to transform its energy into radiation intensively.

It must be noted that the numerical solutions carried out in the study of the *T*-layer effect stimulate the formulation of physical experiments. The analysis of calculations enables us to indicate the range of variation of the physical parameters for which physical experiment can lead to positive results.

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