A. N. TIKHONOV'S RESEARCHES ON MATHEMATICAL PHYSICS

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October 30, 1966, was the 60th birthday of Academician Andrei Nikolaevich Tikhonov, who was born in Gzhatsk, Smolensk oblast. In 1922 he entered the faculty of physics and mathematics at Moscow University, where over the years he progressed from student to head of the mathematics department; the latter post he held for more than 30 years. At present he is head of the department of computational mathematics in the faculty of mathematical mechanics at Moscow University, and also deputy director of the Institute of Applied Mathematics, Academy of Sciences of the USSR.

He began his research in mathematics at the age of 18 under the direction of P. S. Aleksandrov; his first area of interest was in topology, where he produced some fundamental results. Tikhonov topology has become one of the branches of modern mathematics. Classical theorems of his are the theorem of bicompactness for the products of any set of bicompacts and the theorem of the fixed point for convex bicompact spaces.

The second period in his research was concerned with the theory of the equations of mathematical physics; here he began work under the direction of Professor V. V. Stepanov when he was already famous, and soon became a leading worker in this area. In this period he combined natural science with research on fundamental mathematical problems; he always gave a clear mathematical formulation for a scientific problem, but the mathematical research was not restricted to solution of the particular physical problem, but was extended.

These researches in the general problems of mathematical physics arose in part from his interest in the fundamental problems of geophysics and electrodynamics. For example, the heat balance of the Earth led him to examine the uniqueness of Cauchy's problem for equations of parabolic type and to formulate methods for solving general functional equations of Volterre type. Inverse problems in geophysics and electrodynamics served as basis for a research into the recovery of a differential operator from the properties of its spectrum; this led him to develop methods of solving incorrectly-formulated problems in mathematical physics.

Space does not allow discussion of all his researches on mathematical physics, and even a simple list of his papers (over 100) would take too much space.

His researches on mathematical physics may be classified as follows: 1) theory of thermal conductivity and of equations of parabolic type, 2) electrodynamics, 3) theory of differential equations with small parameters, 4) research in computational mathematics, 5) research on inverse problems and incorrectly-formulated problems.

His first researches under 1) were concerned with determining the past climate of the Earth. A topic widely discussed in the 1930's was the origin of permafrost and its relation to previous cold periods, which raises the question of deducing the temperature history from the temperature as a function of depth. This inverse problem is not common for equations of parabolic type, so one of the first aspects to examine was the correctness of the scientific formulation. He very soon showed that there was a lack of clarity even regarding the uniqueness of the solution for the direct problem of thermal conduction, although the equation of thermal conductivity had been studied by many leading mathematicians for over a century (Fourier, Cauchy, Poisson, etc). He showed that the solution to Cauchy's problem for a parabolic equation exists and is unique in the class of functions that increase not more rapidly than $C \exp(x^2)$ towards infinity. He gave examples showing that the solution to Cauchy's problem is not unique in the class of functions that increase as $C \exp(x^{2+\varepsilon})$ at infinity. Subsequently he elucidated the conditions of uniqueness for solutions of inverse problems in thermal conduction, which provided a firm basis for geothermal research in the USSR. The thermal history of the Earth is naturally related to radiation in outer space, which leads to nonlinear boundary problems for equations of parabolic type. For the case of emission in accordance with Stefan's law, he posed the general problem of the cooling of a system of n moving bodies with a nonlinear radiation law. The solution was found via his theory of nonlinear functional equations, which he called equations of Volterra type. This theory influenced the development of the theory of equations with retarded argument.

His geophysical studies were far from being restricted to geothermal topics. He made a major contribution to the use of electromagnetic fields for studying the structure of the Earth's crust, especially the use of the natural electromagnetic field for obtaining a complete electrical section of the crust. His method consists of comparing the electric and magnetic components of the natural field at the surface, which gives the properties of the layers; if a broad frequency spectrum is used, it is possible to evaluate the electrical structure down to depths of hundreds of km. The basis is Tikhonov's demonstration of the uniqueness of the corresponding inverse problem.

He also examined electrical surveys using the electromagnetic fields produced in the crust by dc and ac. His name is closely linked with progress in ac methods of electrical surveying. He solved for the transient electromagnetic field in a stratified half-space when a current arises in a wire at the surface of the medium. He devised a universal method of deducing the fields in stratified media that is applicable with fast computers. His work on electrical surveying stimulated major changes in equipment, and the leading place of the USSR in ac surveying is in no small part due to him.

Another research topic of his concerns waves in waveguides, a topic in which Kisun'ko in 1946 derived an expression for the solution to the inhomogeneous system of equations for a guide of arbitrary cross section. In 1947-9 Tikhonov and A. A. Samarskii set up a complete theory of the excitation of a guide by arbitrarily specified currents. It was shown that any field within a guide may be represented as a superposition of TE and TM waves, which gave a very simple method of constructing the source function for the Maxwell's equations within any waveguide. These studies provided the basis for many practical methods of calculating waveguide components.

The waveguide research raised the need to elucidate the conditions that uniquely determine the solution for the steady-state oscillations in an unbounded region. The analytic form of the conditions for Sommerfeld radiation is substantially dependent on the form of the boundary when this recedes to infinity. Tikhonov and Samarskii formulated the principle of the limiting amplitude, which indicates that the solution to the wave equation in an unbounded region is uniquely determined by the requirement that the solution is the limit of the solution to Cauchy's problem for the equation of the oscillations for $t \to \infty$.

He is also interested in chemical kinetics; is particular, he has published papers on the theory of adsorption from a current of air, which gives rise to nonlinear equations in partial derivatives. He has found exact analytic solutions for certain cases, while for others he has given conjugate numerical and asymptotic solutions.

His interest in inverse problems is very characteristic; amongst these he has studied ones in gravimetry, electrodynamics, and spectroscopy. This is because inverse problems arise in the study of objects via indirect effects. The solution of such problems is made difficult by instabilities; usually a solution is sought on the assumption that we possess full information, which is usually not the case.

Only a few years ago Courant claimed that correctness is an essential feature of any mathematical problem having physical meaning, and the approximate methods of solution should be devised only for correct problems. On this view, an incorrect problem is a purely mathematical exercise. One of the greatest services performed by Tikhonov is probably to overcome the veto on incorrect problems imposed first by Hadamard; he has demonstrated the utility of approximate solutions and has developed a method of solving a very general class of such problems, the method of regularization, which is not merely a set of theorems in functional analysis but also an efficient computing algorithm extremely well adapted to computers.

Academician Tikhonov and his students are applying computers to many problems, and this itself stimulates the posing of new problems, such as methods of processing observations to extract the maximal information without demanding increased experimental information (which complicates the apparatus).

In 1966 he was awarded a Lenin prize for his work on incorrect problems.

A characteristic feature has been to give an exhaustive solution, so he has naturally given much attention to approximate methods of solving problems in mathematical physics, which are frequently of economic importance. Such methods give solutions in usable forms; they have involved several new lines in mathematical research, especially asymptotic methods of studying differential equations and general computing algorithms. He is famous for his research on the asymptotic behavior of solutions to Cauchy's problem for a general system of differential equations with a small parameter (or system of parameters). Here he made several discoveries (stable and unstable roots, regions of influence of unstable roots), which have had considerable effects on asymptotic methods in the theory of ordinary differential equations and of equations in partial derivatives. In particular, he has given a complete asymptotic expansion for the frequently encountered integrals with kernals of δ-function type dependent on small parameters.

Tikhonov is a leading worker in computational mathematics. For over 30 years he has studied applied and computational mathematics. Under his direction, numerous universal programs have been drawn up for numerical solution of problems in geophysics, electrodynamics, plasma physics, gas dynamics, and so on; the basic approach here has been not so much the application of rapid arithmetic to particular detailed problems as the compilation of general programs for solving large classes of problems. There are often many different ways of solving a given mathematical problem. The primary objects of the theory of numerical methods are to describe families of computing algorithms, to compare algorithms, and to select the best ones.

In 1953 he was made a Hero of Socialist Labor for his outstanding work on mathematical physics.

His encouragement of young workers is famous. His abundant ideas, wide interests, and many-sided talents have led him to found several schools in different areas of his interests.

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