CLASSES OF STABLE SCHEMES*

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IN [1] where the regularization method has been discussed, we made use of theorems on the sufficient stability conditions, and on the *a priori* estimates for two-layer and three-layer schemes. In the present paper we shall give the proofs of these theorems.

Let $\{H_N\}$ be a sequence of real unitary spaces, $\{y_{N,\tau}(t)\}$ a sequence of abstract functions $y_{N,\tau}(t)$ of a discrete argument $t \in \bar{\omega}_\tau$ with values in H_N, where $\bar{\omega}_\tau = \{t_j = j\tau,\ j = 0, 1, \ldots, j_0\}$ is a mesh over the interval $0 \leqslant t \leqslant t_0$ with a step τ. We shall consider linear two-point (three-point) operator equations, connecting the points $y_{N,\tau}(t+\tau)$ and $y_{N,\tau}(t)$ (the points $y_{N,\tau}(t+\tau)$, $y_{N,\tau}(t)$, $y_{N,\tau}(t-\tau)$) of the space H_N. It is natural to call these equations two-layer (three-layer) operator-difference schemes.

The starting point for these investigations is the canonical form of describing schemes. We shall consider a set of two-layer schemes

$$B(t)\frac{y(t+\tau)-y(t)}{\tau}+A(t)y(t)=\varphi(t), \qquad 0 \leqslant t = t_j < t_0, \tag{0.1}$$

$$y(0)=y_0, \qquad y_0 \in H_N,$$

and a set of three-layer schemes

$$B(t)\,\frac{y(t+\tau)-y(t-\tau)}{2\tau}+R(t)\,(y(t+\tau)-2y(t)+y(t-\tau))+ \tag{0.2}$$

$$+A(t)y(t)=\varphi(t), \qquad y(0)=y_0, \qquad y(\tau)=y_1, \qquad y_0, y_1 \in H_N.$$

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Here $y(t) = y_{N,\tau}(t)$ is the required function $t \in \bar{\omega}_\tau$; $\varphi_{N,\tau}(t)$ is a given abstract function $t \in \bar{\omega}_\tau$ with values in H_N; y_0 and y_1 are given vectors from H_N; $A(t) = A_{N,\tau}(t)$, $B(t) = B_{N,\tau}(t)$, $R(t) = R_{N,\tau}(t)$ are linear (additive and homogeneous) operators, mapping H_N into H_N.

By introducing some very general assumptions as to the operators of the scheme (for example, that B and A are positive, that A is self-adjoint for (0.1)), we shall separate out from the set all possible schemes of an initial family of two-layer schemes (IS-2) and three-layer schemes (IS-3).

Following [2, 3] the stability of the schemes will be defined as the property of equal-degree continuity with respect to N, τ of $\{y_{N,\tau}(t)\}$ with respect to the input data $\{\varphi_{N,\tau}(t)\}$, $\{y_{N,\tau}(0)\}$ and $\{y_{N,\tau}(\tau)\}$.

The problem is formulated as follows: sufficient information is to be revealed with respect to the operators of this scheme to make the scheme stable.

The necessary and sufficient stability conditions of two-layer schemes (Section 2) and the sufficient stability conditions of three-layer schemes (Section 3) are found. These conditions isolate classes of stable schemes. The sufficient stability conditions have a simple form, for example

$$(By, y)_N \geqslant 0.5\tau(Ay, y)_N \text{ for } (0.1), \tag{0.3}$$

$$(Ry, y)_N \geqslant 0.25(Ay, y)_N \text{ for } (0.2), \tag{0.4}$$

where $(,)_N$ is a scalar product in H_N.

In studying the stability of actual difference schemes, approximating equations of mathematical physics, it is necessary to reduce the scheme to the canonical form (0.1) or (0.2), to introduce the space of mesh functions H_N, to ascertain whether the scheme belongs to IS-2 or IS-3, and finally to verify that the sufficiency conditions (0.3) or (0.4) are satisfied.

The method used here to derive the *a priori* estimates is based on some elementary theorems of functional analysis and on energy inequalities. It is a natural development of the energy method of obtaining *a priori* estimates, used by many authors in studying the stability of specific difference schemes for differential equations of mathematical physics, and also for the difference analogues of these equations. Examples of this are, for example, the papers [4 - 23] and others (these papers contain references to other work).

Space does not permit a survey of this work. We shall make only one remark. The general tendency in the development of *a priori* estimates is to try to obtain a solution of the difference problem with the strongest possible norm through a right-hand part which has the weakest possible norm. This is important for investigating the rate of convergence of schemes for equations with discontinuous coefficients, over non-uniform meshes, etc.

As an example we mention the paper [8] where, for an implicit scheme approximating a Sobolev type equation, the author has obtained an estimate of the solution y in a mesh norm $W_2{}^1$ through $\|\eta\|_{L_2}$, at first representing the right-hand part in a divergent form $\varphi = \mathrm{div}_h\,\eta$, where div_h is the difference analogue of the operator div. This norm, as follows from Section 1, is identical with the norm $\|\varphi\|_{a^{-1}}=\sqrt{(A^{-1}\varphi,\ \varphi)}$ in the energy space $H_{A^{-1}}$ in the case of a conservative or divergent operator $A = T^*ST$ when $S = E$ (see [10]). The estimates $\|y\|_{W_2{}^1}$ through $\|\eta\|_{L_2}$ have been obtained for the difference elliptic problem in [9 - 10], the estimates $\|y\|_{W_2{}^1}$ and $\|y\|_C$ through $\|\eta\|_{L2}$ have been obtained in [18], and the estimates $\|y\|_C$ through $\|\eta\|_{L_1}$ in [24] (for the one dimensional problem). In [16] for weighted schemes, being difference analogues of parabolic and hyperbolic types of equations, estimates have been obtained containing a right-hand part in a norm of the kind $\|A^{-1}\varphi\| + \|A^{-1}\varphi\|$, etc. The simplest *a priori* estimates for the Rote scheme, approximating the abstract Cauchy problem, are given in [25].

Some *a priori* estimates for operator-difference schemes in a Hilbert space are given in [3, 26].

The basic results of this paper are given in Sections 2 and 3; Section 1 contains auxiliary material used in Sections 2 and 3, and also some *a priori* estimates for operator equations of the I kind $Ay = \varphi$.

1. Introduction

Before turning to the theory concerning the stability of evolutionary (two-layer and three-layer) schemes we shall outline the necessary mathematical apparatus and demonstrate the applicability of the energy method for obtaining the *a priori* estimates, using the example of equations of the I kind $Ay = \varphi$, where A is a linear operator defined in a real unitary space. The *a priori* estimates obtained in Sections 2 and 3 are of independent interest for the theory of difference approximations to the boundary problems for elliptical equations.

1. SOME INEQUALITIES AND IDENTITIES

Let H_N be a real unity space with a scalar product $(,)_N$ and norm $\| x \|_N = \sqrt{(x, x)_N}$, A_N a linear (homogeneous and additive) operator, defined over H_N, where N takes integer positive values. In the general case $N = (N_1, N_2, \ldots, N_m)$ is a composite index, i.e. the set of integers $N_1, N_2, \ldots, N_m$, $N_s > 0$ where m is finite. The condition $N \to \infty$ means that all $N_s \to \infty$, $s = 1, 2, \ldots, m$. We shall consider the sequence of spaces $\{H_N\}$ and of operators $\{A_N\}$. Whenever N is arbitrary, the subscript N will be omitted.

We shall need some elementary inequalities (x and y are arbitrary vectors from H) [27 - 29]:

1) the triangular inequality

$$\|x + y\| \leqslant \|x\| + \|y\|; \tag{1}$$

2) the Cauchy inequality

$$|(x, y)| \leqslant \|x\| \|y\|; \tag{2}$$

3) the generalized Cauchy inequality

$$(Ax, y)^2 \leqslant (Ax, x)(Ay, y), \tag{3}$$

or

$$|(Ax, y)| \leqslant \| x \|_a \| y \|_a, \quad \| x \|_a = \sqrt{(Ax, x)}, \tag{3'}$$

where A is an arbitrary non-negative ($(Ax, x) \geqslant 0$ or $A \geqslant 0$)) selfadjoint operator;

4) the inequality

$$\sum_{\alpha=1}^{p} \lambda_\alpha \mu_\alpha \leqslant \left(\sum_{\alpha=1}^{p} \lambda_\alpha^2 \right)^{1/2} \left(\sum_{\alpha=1}^{p} \mu_\alpha^2 \right)^{1/2}, \tag{4}$$

where λ_α, μ_α are arbitrary non-negative numbers;

5) the ε-inequality

$$|ab| \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \tag{5}$$

where a, b, $\varepsilon > 0$ are arbitrary numbers.

In addition, we shall use the following properties of linear operators (see [27 - 29]).

Let X and Y be linear normed spaces with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively, and let A be a linear operator from X into Y. In order that the inverse operator A^{-1} should exist and be linear (as the operator from Y into X), it is necessary and sufficient that a constant $\delta > 0$ should exist such that for all $x \in X$

$$\|Ax\|_2 \geqslant \delta \|x\|_1. \tag{6}$$

Here the following estimate is valid

$$\|A^{-1}\| \leqslant 1/\delta. \tag{7}$$

For the proof of this theorem see [27 - 29].

All subsequent lemmas and theorems refer to linear operators A in a real unitary space H.

We shall use the same notation as in [1] (x, y are arbitrary elements from H, δ and c_* are positive numbers):

A is non-negative, $A \geqslant 0$, if $(Ax, x) \geqslant 0$;
A is positive, $A > 0$, if $(Ax, x) > 0$ for all $x \neq 0$;
A is positive definite, $A \geqslant \delta E$, if $(Ax, x) \geqslant \delta \| x \|^2$;
A is semibound from below, $A \geqslant -c_* E$, if $(Ax, x) \geqslant -c_* \| x \|^2$;
A is selfadjoint, $A^* = A$, if $(Ax, y) = (x, Ay)$.
Here E is a unit operator.

Let us formulate a number of elementary lemmas. For the sake of completeness we shall also give their proofs.

Lemma 1.1

If A is a positive definite operator

$$A \geqslant \delta E, \tag{8}$$

the following inequalities are valid

$$\|Ax\|^2 \geqslant \delta(Ax, x), \tag{9}$$

$$\|Ax\| \geqslant \delta\|x\|, \tag{10}$$

$$\|A^{-1}\| \leqslant 1/\delta. \tag{11}$$

From (2) and (8) it follows that: $(Ax, x) \leqslant \|Ax\|\|x\| \leqslant \|Ax\|(1/\sqrt{\delta})\sqrt{(Ax, x)}$. After cancelling out by $\sqrt{(Ax, x)}$ we obtain (9). The inequality (10) follows from (9) and (8): $\|Ax\|^2 \geqslant \delta(Ax, x) \geqslant \delta^2\|x\|^2$; the inequality (11) follows from (10) and (7).

Thus, the positive definite nature of the operator A is sufficient for the boundedness of the inverse operator A^{-1}.

Lemma 1.2

If $A > 0$ and the condition

$$\| Ax \|^2 \leqslant \Delta(Ax, x), \quad \Delta > 0, \text{ for all } \quad x \in H, \tag{12}$$

is satisfied, the inverse operator A^{-1} is positive definite

$$A^{-1} \geqslant \frac{1}{\Delta} E. \tag{13}$$

Indeed, since $A > 0$ therefore $A^{-1} > 0$. Putting $x = Ay$, $y = A^{-1}x$ and using the condition (12), we obtain $\|x\|^2 = \|Ay\|^2 \leqslant \Delta(Ay, y) = \Delta(A^{-1}x, x)$, i.e. $A^{-1} \geqslant (1/\Delta)E$. Hence in view of Lemma 1.1 it follows that $\|A^{-1}x\| \geqslant (1/\Delta)\|x\|$. Comparing with (7), we find $\| A \| \leqslant \Delta$. Condition (12) indicates the boundedness of A.

Lemma 1.3

If $A^* = A$ is a linear bounded and non-negative operator,

$$\|Ax\|^2 \leqslant \|A\|(Ax, x). \tag{14}$$

The proof of this lemma is given in [33].

Lemma 1.4

Let $A^* = A$, $A > 0$. Then for any y and φ from H the following inequalities are satisfied

$$|(y, \varphi)| \leqslant \|y\|_a \|\varphi\|_{a^{-1}}, \tag{15}$$

$$|(y, \varphi)| \leqslant \varepsilon \|y\|_a^2 + \frac{1}{4\varepsilon} \|\varphi\|^2_{a^{-1}}, \tag{16}$$

where $\varepsilon > 0$ is any number, and

$$\|y\|_a = \sqrt{(Ay, y)}, \qquad \|\varphi\|_{a^{-1}} = \sqrt{(A^{-1}\varphi, \varphi)}.$$

Indeed, $(y, \varphi) = (A^{-1}\varphi, Ay) = (Ay, A^{-1}\varphi)$. We apply inequality (3′): $|(y, \varphi)| = (Ay, A^{-1}\varphi) \leqslant \|y\|_a \|A^{-1}\varphi\|_a = \|y\|_a \|\varphi\|_{a^{-1}}$, since $\|A^{-1}\varphi\|_a^2 = (A^{-1}\varphi, \varphi)$. Inequality (16) follows from (15) and (5).

Lemma 1.5

If A is an arbitrary linear operator, for any v, $u \in H$ the following identity is valid

$$(Av, v) + (Au, u) = {}^1/_2(A(v + u), v + u) + {}^1/_2(A(v - u), v - u). \tag{17}$$

Indeed, $(A(v + u), v + u) + (A(v - u), v - u) = [(Av, v) + (Av, u) + (Au, v) + (Au, u)] + [(Av, v) - (Av, u) - (Au, v) + (Au, u)] = 2[(Av, v) + (Au, u)]$.

Remark 1. If $A = A^* > 0$, (17) can be written in the form

$$\|v\|_a^2 + \|u\|_a^2 = \frac{1}{2}\|v + u\|_a^2 + \frac{1}{2}\|v - u\|_a^2. \tag{18}$$

Lemma 1.6

If $A^* = A$

$$(Av, u) = {}^1/_4(A(v + u), v + u) - {}^1/_4(A(v - u), v - u), \quad v, u \in H. \tag{19}$$

The lemma is proved by analogy with Lemma 1.5, if it is noted that $(Au, v) = (Av, u)$.

Lemma 1.7

If $A^* = A$

$$(A(v+u),\ v-u) = (A(v-u),\ v+u) = (Av,\ v) - (Au,\ u). \quad (20)$$

Lemma 1.8

If $A^* = A$,

$$(A(v-u),\ v) = {}^1/_2[(Av,\ v) - (Au,\ u)] + {}^1/_2(A(v-u),\ v-u). \quad (21)$$

To prove (20) and (21) it is sufficient to write down the expression for $(A(v+u),\ v-u)$ and, respectively, for $(A(v-u),\ v-u)$ and to take into account that $(Au,\ v) = (Av,\ u)$.

Lemma 1.9

For any linear non-negative operator $(A \geqslant 0)$ and any $u,\ v \in H$

$$(A(v-u),\ v-u) \leqslant 2[(Av,\ v) + (Au,\ u)]. \quad (22)$$

Indeed, in view of Lemma 1.5 we have $(A(v-u),\ v-u) = 2[(Av,\ v) + (Au,\ u)] - (A(v+u),\ v+u) \leqslant 2[(Av,\ v) + (Au,\ u)]$ since $(Ax,\ x) \geqslant 0$ for any $x = v - u$.

Remark 2. If $A^* = A > 0$, (22) can be written in the form

$$\| v-u \|_a^2 \leqslant 2[\| v \|_a^2 + \| u \|_a^2]. \quad (23)$$

2. *A priori* ESTIMATES FOR THE EQUATION $Ay = \varphi$

Let us consider a sequence of equations of the I kind

$$A_N y_N = \varphi_N, \quad \varphi_N \in H_N, \quad (24)$$

where A_N is a linear operator, mapping H_N into H_N.

The problem (24) is well-posed [27], if for any $N \geqslant N_0$: (1) its solution exists for all $\varphi_N \in H_N$, (2) there is such a positive number M independent of N that

$$\|y_N\|_{(1_N)} \leqslant M \|\varphi_N\|_{(2_N)}, \quad (25)$$

where $\|\cdot\|_{(1_N)}$ and $\|\cdot\|_{(2_N)}$ are some norms for the set H_N.

In what follows we shall consider unitary real spaces H_N (generally speaking of infinite dimensions) with the scalar product $(,)_N$ and norm $\| x \|_N = \sqrt{(x,\ x)_N}$. If $A_N^* = A_N > 0$ it is possible to introduce over the set H_N a scalar product $(A_N x, y)_N$ and a norm $\| x \|_{a_N} = \sqrt{(A_N x, x)_N}$, i.e. to consider the energy space H_{A_N}. In order not to complicate the terminology, we shall say that over H_N we introduce the norm $\|\cdot\|_{(1_N)}$, $\|\cdot\|_{(2_N)}$ etc. for example $\|x\|_{(1_N)} = \|x\|_N$, $\|x\|_{(2_N)} = \|x\|_{a_N^{-1}}$ etc.

Below we shall prove a number of simple theorems about the stability of equation (24). It is assumed that all constants occurring in the *a priori* estimates are independent of N. The index N will now always be omitted and instead of (24) and (25) we shall write

$$Ay = \varphi, \tag{26}$$

$$\|y\|_{(1)} \leqslant M\|\varphi\|_{(2)}. \tag{27}$$

Theorem 1.1

If

$$A = \sum_{\alpha=1}^{m} A_\alpha, \qquad A_\alpha^* = A_\alpha > 0, \qquad \alpha = 1, \ldots, m,$$

then to solve equation (26) the following estimate is valid

$$\sum_{\alpha=1}^{m} \|y\|^2_{a_\alpha} \leqslant \sum_{\alpha=1}^{m} \|\varphi\|^2_{a_\alpha^{-1}}, \tag{28}$$

where φ_α are arbitrary vectors from H, satisfying the normalization condition $\varphi_1 + \ldots + \varphi_\alpha + \ldots + \varphi_m = \varphi$,

$$\|y\|^2_{a_\alpha} = (A_\alpha y, y), \qquad \|\varphi\|^2_{a_\alpha^{-1}} = (A_\alpha^{-1} \varphi_\alpha, \varphi_\alpha), \qquad \alpha = 1, 2, \ldots, m. \tag{29}$$

Multiplying (26) in a scalar fashion by y, we obtain the basic identity

$$(Ay, y) = (\varphi, y). \tag{30}$$

Substituting here $\varphi = \varphi_1 + \ldots + \varphi_m$ and using the estimate $|(\varphi_\alpha, y)| \leqslant \| y \|_{a_\alpha} \| \varphi_\alpha \|_{a_\alpha^{-1}}$ (Lemma 1.4) and then the inequality (4), we obtain (28). We note that from (30) and Lemma 1.4 the estimate $\| y \|_a \leqslant \| \varphi \|_{a^{-1}}$ immediately follows.

Theorem 1.2

Suppose we are given the operators A and $A_0^* = A_0 > 0$. If the inequality

$$(Ay,\ y) \geqslant c_1(A_0y,\ y) = c_1\|y\|_{a_0}^2, \text{where } c_1 > 0, \tag{31}$$

is satisfied, to solve equation (26) the following estimate is valid

$$\|y\|_{a_0} \leqslant \frac{1}{c_1}\|\varphi\|_{a_0^{-1}}. \tag{32}$$

Indeed from the identities (30), (31) and (15) it follows that

$$c_1\|y\|_{a_0}^2 \leqslant \|\varphi\|_{a_0^{-1}}\|y\|_{a_0}, \text{ i.e. } c_1\|y\|_{a_0} \leqslant \|\varphi\|_{a_0^{-1}}.$$

Theorem 1.3

Let $A = A_0 + A_1$, where $A_0^* = A_0 > 0$ and

$$|(A_1x, x)| \leqslant \gamma(A_0x, x), \quad 0 < \gamma < 1, \quad x \in H. \tag{33}$$

Then for the solution of equation (26) the following estimate is valid

$$\|y\|_{a_0} \leqslant \frac{1}{1-\gamma}\|\varphi\|_{a_0^{-1}}. \tag{34}$$

Proof. From the identity $(A_0y,\ y) = (\varphi,\ y) - (A_1y,\ y)$, Lemma 1.5 and (33) follows $\|y\|_{a_0}^2 \leqslant \|\varphi\|_{a_0^{-1}}\|y\|_{a_0} + \gamma\|y\|_{a_0}^2$ or $(1-\gamma)\|y\|_{a_0} \leqslant \|\varphi\|_{a_0^{-1}}$.

Theorem 1.4

Let $A = A_0 + A_1$, where $A_0^* = A_0 > 0$ and

$$\|A_1y\| \leqslant \gamma\|A_0y\|, \quad 0 < \gamma < 1, \quad y \in H. \tag{35}$$

Then for (26) we have the estimate

$$\|A_0y\| \leqslant \frac{1}{1-\gamma}\|\varphi\|. \tag{36}$$

We multiply (26) in a scalar fashion by A_0y: $(A_0y + A_1y,\ A_0y) = (\varphi, A_0y)$ or $\|A_0y\|^2 = (\varphi,\ A_0y) - (A_1y,\ A_0y)$. Taking into account (2) and (35), we find $\|A_0y\|^2 \leqslant \|\varphi\|\,\|A_0y\| + \gamma\|A_0y\|^2$. Cancelling out by $\|A_0y\|$, we obtain (36).

Remark. From Theorem 1.3 and 1.4 follow *a priori* estimates in the mesh norms W_2^1 and W_2^2 for the solution of the first difference boundary problem for schemes of an increased order of accuracy, approximating the Poisson equation in a p-dimensional parallelepiped ($p < 4$). These estimates have been obtained in [30, 23]. In the given case $\gamma = (p-1)/3$, where p is the number of measurements.

3. CONSERVATIVE OPERATORS

Let H and H_1 be two real unitary spaces with the scalar product $(,)$ and $(,]$ respectively. Consider the linear operators T, S, T^*, where T maps H into H_1, S maps H_1 into H_1, T^* maps H_1 into H. The operators T and T^* are mutual conjugates, so that

$$(Ty, v] = (y, T^*v) \quad \text{for any} \quad y \in H, \quad v \in H_1. \tag{37}$$

Operators of the kind

$$A = T^*ST, \tag{38}$$

given over H will be called conservative (divergent) [3]. The operator A is selfadjoint, $(Ay, z) = (y, Az)$ if the operator S is selfadjoint

$$(Sv, w] = (v, Sw], \quad v, w \in H_1.$$

Indeed, $(Ay, z) = (T^*STy, z) = (y, T^*S^*Tz)$, $y, z \in H$. From the identity

$$(Ay, y) = (STy, Ty] \tag{39}$$

it can be seen that A is positive definite if

$$(Sv, v] \geqslant c_1 \| v]|^2, \ \text{where} \| v]|^2 = (v, v], \quad v \in H_1, \quad c_1 > 0; \tag{40}$$

$$\| Ty]| \geqslant c_2 \| y \|, \quad c_2 > 0. \tag{41}$$

Theorem 1.5

If $S \geqslant c_1 E$ and there is an inverse operator $(T^*)^{-1}$ then for the solution of equation (26) with the operator (38) the following estimate is valid:

$$\| Ty]| \leqslant \frac{1}{c_1} \| (T^*)^{-1} \varphi]| \tag{42}$$

This theorem follows from Theorem 1.2, if we remember that

$$A \geqslant c_1 A_0, \ \text{where} \ A_0 = T^*T, \quad A_0^* = A_0, \tag{43}$$

and consequently

$$\| y \|_{A_0}^2 = (T^*Ty, y) = \| Ty]|^2, \ \text{i.e} \ \| y \|_{A_0} = \| Ty]|;$$

$$\| \varphi \|_{A_0^{-1}}^2 = (A_0^{-1}\varphi, \varphi) = (T^{-1}(T^*)^{-1}\varphi, \varphi) = \| (T^*)^{-1}\varphi]|^2, \ \text{i.e} \ \| \varphi \|_{A_0^{-1}} = \| (T^*)^{-1}\varphi]|.$$

Let us consider the case of a "multidimensional" conservative operator

$$A = \sum_{\alpha,\beta=1}^{p} T_\alpha{}^* S_{\alpha\beta} T_\beta, \tag{44}$$

which is the analogue of the elliptical difference operator in the space p of measurements. If $S_{\alpha\beta} = \delta_{\alpha\beta} S_{\alpha\alpha}$, the operator

$$A = \sum_{\alpha=1}^{p} T_\alpha{}^* S_\alpha T_\alpha, \text{ where } S_\alpha = S_{\alpha\alpha}, \tag{45}$$

can be interpreted as the analogue of the operator $\mathrm{div}_h(k\ \mathrm{grad}_h)$ not containing mixed products. Let H_α ($\alpha = 1, 2, \ldots, p$) be a space with the scalar product $(,]_\alpha$ and norm $\| v]|_\alpha = \sqrt{(v, v]_\alpha}$, and T_α, $S_{\alpha\beta}$, $T_\alpha{}^*$ be linear operators while T_α operates from H into H_α, $S_{\alpha\beta}$ operates from H_β into H_α and $T_\alpha{}^*$ operates from H_α into H and the following conditions are satisfied:

$$(T_\alpha y, v]_\alpha = (y, T_\alpha{}^* v), \qquad y \in H, \qquad v \in H_\alpha.$$

Condition (40) is replaced by the condition of positive definiteness of the matrix operator $S = (S_{\alpha\beta})$:

$$\sum_{\alpha,\beta=1}^{p} (S_{\alpha\beta} v_\beta, v_\alpha]_\alpha \geqslant c_1 \sum_{\alpha=1}^{p} \| v_\alpha]|_\alpha^2 \text{ for any } v_\alpha \in H_\alpha. \tag{46}$$

Theorem 1.6

If condition (46) is satisfied and there exist inverse operators $(T_\alpha{}^*)^{-1}$, $\alpha = 1, 2, \ldots, p$, then for the solution of equation

$$Ay = \sum_{\alpha,\beta=1}^{p} T_\alpha{}^* S_{\alpha\beta} T_\beta y = \varphi, \qquad y, \varphi \in H \tag{47}$$

the following *a priori* estimate is valid:

$$\sum_{\alpha=1}^{p} \| T_\alpha y]|_\alpha^2 \leqslant \frac{1}{c_1} \sum_{\alpha=1}^{p} \| (T_\alpha{}^*)^{-1} \varphi_\alpha]|_\alpha^2, \tag{48}$$

where φ_α are arbitrary vectors from H, satisfying the normalization condition

$$\varphi_1 + \varphi_2 + \ldots + \varphi_\alpha + \ldots + \varphi_p = \varphi.$$

The theorem follows from Theorem 1.2, if the proof of Theorem 1.1 is taken into account and we take

$$A_0 = \sum_{\alpha=1}^{p} A_\alpha, \quad A_\alpha = T_\alpha^* T_\alpha, \quad A_\alpha^* = A_\alpha. \tag{49}$$

Indeed, condition (46) when $v_\alpha = T_\alpha y$ gives

$$(Ay, y) = \sum_{\alpha,\beta=1}^{p} (T_\alpha^* S_{\alpha\beta} T_\beta y, y) = \sum_{\alpha,\beta=1}^{p} (S_{\alpha\beta} T_\beta y, T_\alpha y]_\alpha \geqslant c_1 \sum_{\alpha=1}^{p} \| T_\alpha y]|_\alpha^2 = c_1 (A_0 y, y.)$$

On the other hand, we have

$$(\varphi, y) = (Ay, y),$$

$$(\varphi_\alpha, y) \leqslant \| \varphi_\alpha \|_{a_\alpha^{-1}} \| y \|_{a_\alpha},$$

$$(\varphi, y) = \sum_{\alpha=1}^{p} (\varphi_\alpha, y) \leqslant \sum_{\alpha=1}^{p} \| \varphi_\alpha \|_{a_\alpha^{-1}} \| y \|_{a_\alpha},$$

where

$$\| \varphi_\alpha \|_{a_\alpha^{-1}} = \| (T_\alpha^*)^{-1} \varphi_\alpha]|_\alpha, \quad \| y \|_{a_\alpha} = \| T_\alpha y]|_\alpha .$$

Then using inequality (4), we obtain

$$(\varphi, y) \leqslant \left[\sum_{\alpha=1}^{p} \| \varphi_\alpha \|^2_{a_\alpha^{-1}} \right]^{1/2} \left[\sum_{\alpha=1}^{p} \| y \|^2_{a_\alpha} \right]^{1/2}.$$

Hence and from the inequality

$$c_1 (A_0 y, y) = c_1 \sum_{\alpha=1}^{p} \| T_\alpha y]|_\alpha^2 \leqslant (Ay, y) = (\varphi, y)$$

the estimate (48) follows.

Remark 1. Theorems 1.5 - 1.6 are proved without assuming the self-adjoint nature of operator A.

2. From Theorems 1.5 - 1.6 follow the *a priori* estimates in the norm W_2^1 for the difference analogues of the elliptic equations and sets of equations.

3. The analogue of Theorem 1.5 and 1.6 for selfadjoint elliptic equations have been obtained in [9, 10, 18].

4. Theorems 1.5 and 1.6 make it possible to estimate the rate of convergence in the norm W_2^1 of difference schemes over non-uniform meshes

for elliptic equations and sets of equations.

In order to use Theorems 1.1 - 1.6 in the theory of difference schemes it is at first necessary to introduce the corresponding spaces of mesh functions (depending on the structure of A) and to study the properties of the operator A in these spaces.

2. Classes of stable two-layer schemes

1. TWO-LAYER SCHEMES

Let $\{\mathcal{B}_2{}^N\}$ and $\{\mathcal{B}_1{}^N\}$ be sequences of normed linear spaces, $\bar{\omega}_\tau = \{t_j = j\tau,\ j = 0, 1, \ldots, j_0\}$ a mesh over the interval $0 \leqslant t \leqslant t_0$ with the step $\tau = t_0/j_0$; $A_N = A_{N,\tau}(t)$, $B_N = B_{N,\tau}(t)$ etc. linear (additive and homogeneous) operators mapping $\mathcal{B}_1{}^N$ into $\mathcal{B}_2{}^N$; $y_{N,\tau}(t)$ an abstract function of the discrete argument $t \in \bar{\omega}_\tau$ with values in $\mathcal{B}_1{}^N$ and $\varphi_{N,\tau}(t)$ a function of $t \in \bar{\omega}_\tau$ with values in $\mathcal{B}_2{}^N$.

A two-layer scheme is a linear operator equation relating two points $y_{N,\tau}(t + \tau)$ and $y_{N,\tau}(t)$ of the space $\mathcal{B}_1{}^N$:

$$B(t)y(t+\tau) = C(t)y(t) + \tau\varphi(t), \qquad t \in \omega_\tau, \qquad y(0) = y_0 \in \mathcal{B}_1{}^N, \tag{1}$$

where $\varphi(t)$ is a given function, $\omega_\tau = \{t_j,\ 0 \leqslant j < j_0\}$.

The dependence of the functions y and φ and also of the operators on τ and N will not be indicated, separating out only the dependence on t. We shall agree to relate the operator to the "lower layer" $t = t_j$, which is convenient to describe the scheme in an index free form.

Any two-layer scheme can be described in the canonical form

$$B(t)\,\frac{y(t+\tau) - y(t)}{\tau} + A(t)y(t) = \varphi(t). \tag{2}$$

Below we shall use the notation

$$t = t_j, \qquad t + \tau = t_{j+1}, \qquad t - \tau = t_{j-1}, \qquad y = y(t_j) = y^j,$$

$$\hat{y} = y(t_{j+1}) = y^{j+1}, \quad \check{y} = y^{j-1}, \qquad y_t = \frac{\hat{y} - y}{\tau} = \frac{y(t+\tau) - y(t)}{\tau},$$

$$y_{\bar{t}} = \frac{y - \check{y}}{\tau} = \frac{y(t) - y(t-\tau)}{\tau}, \qquad y_{\mathring{t}} = \frac{\hat{y} - \check{y}}{2\tau} = \frac{1}{2}(y_t + y_{\bar{t}}),$$

$$y_{\bar{t}t} = \frac{1}{\tau}(y_t - y_{\bar{t}}) = \frac{\hat{y} - 2y + \check{y}}{\tau^2}, \qquad A = A(t),$$
$$\check{A} = A(t - \tau), \qquad A_{\bar{t}} = \frac{A - \check{A}}{\tau}.$$

It is easily noted that $y_t = \hat{y}_{\bar{t}}$, $y_{\bar{t}} = \check{y}_t$.

Equation (2) can also be written in the form

$$B^j \frac{y^{j+1} - y^j}{\tau} + A^j y^j = \varphi^j. \qquad (2')$$

Using the notation introduced above instead of (2) and (28) we write

$$By_t + Ay = \varphi, \qquad t \in \omega_\tau, \qquad y(0) = y_0. \qquad (3)$$

A definition of the well-posed nature of the scheme (3) is given in [1]. We shall discuss in more detail the concept of stability. The definition of stability does not relate to the actual scheme (3) with fixed N, τ, but to a family of schemes corresponding to all possible values of N and τ. Thus, we consider a sequence of solutions $\{y_{N,\tau}(t)\}$ of problem (3), corresponding to the input data $\{\varphi_{N,\tau}(t)\}$and $\{y_{N,\tau}(0)\}$. The stability of the scheme (3) signifies that it is equally continuous (with respect to N and τ) as far as the solution of equation (3) is concerned relative to the right-hand part and the initial data.

Let $\|\cdot\|_{(1_N)}$ and $\|\cdot\|_{(2_N)}$ be the norms in $\mathscr{B}_1{}^N$ and $\mathscr{B}_2{}^N$. Usually $\mathscr{B}_1{}^N$ and $\mathscr{B}_2{}^N$ consist of the same vectors (mesh functions) and they differ only in their norms. The norms may be functions of t, so that $\|\cdot\|_{(1)} = \|\cdot\|_{(1,\,t)}$, $\|\cdot\|_{(2)} = \|\cdot\|_{(2,\,t)}$, and so on (the subscript N is omitted).

The scheme (3) is called stable [2, 1], if there are numbers $N_0 > 0$, $\tau_0 > 0$, and numbers $M_1 > 0$, $M_2 > 0$, $M_2{}' \geqslant 0$, independent of τ and N, so that when $\tau \leqslant \tau_0$ and $N \geqslant N_0$ the following inequality is satisfied for the solution of equation (3):

$$\|y(t+\tau)\|_{(1,\,t)} \leqslant M_1 \|y(0)\|_{(1,\,0)} + M_2 \max_{0 \leqslant t' \leqslant t} \|\varphi(t')\|_{(2,\,t')} + \qquad (4)$$
$$+ M_2{}' \max_{0 < t' \leqslant t} \|\varphi_{\bar{t}}(t')\|_{(2,\,t')}.$$

Here we do not attempt to give a completely general definition of stability, but only a definition which is to be used in this paper.

Examples of norms, which are functions of t, are supplied by the energy norms related to the operators of the scheme (3) (see point 4):

$$\|y(t+\tau)\|_{(1,t)} = \|y(t+\tau)\|_{a(t)} = \sqrt{(A(t)y(t+\tau),\ y(t+\tau))},$$
$$\|y(t+\tau)\|_{(1,t)} = \|y(t+\tau)\|_{b(t)},$$

or

$$\|y^{j+1}\|_{a_j} = \|\hat{y}\|_a = \sqrt{(A\hat{y},\hat{y})}, \qquad \|y^{j+1}\|_{b_j} = \|\hat{y}\|_b = \sqrt{(B\hat{y},\hat{y})}.$$

As to the dependence of $\|\cdot\|_{(1)}$ on t, sometimes we shall not indicate this, but shall write instead of $\|y^{j+1}\|_{a_j}$ simply $\|y^{j+1}\|_a = \|\hat{y}\|_a$, unless this can be misunderstood.

It is convenient to study the problem of stability with respect to the initial data and with respect to the right-hand part separately, i.e. to consider the problem

$$By_t + Ay = 0, \qquad t \in \omega_\tau, \qquad y(0) = y_0, \tag{3a}$$
$$By_t + Ay = \varphi, \qquad t \in \omega_\tau, \qquad y(0) = 0. \tag{3b}$$

The solution of problem (3) will be represented in the form of a sum $y = \tilde{y} + \bar{y}$ where $\tilde{y}$ is the solution of problem (3a), and $\bar{y}$ is the solution of problem (3b). In view of the triangular inequality

$$\|y\|_{(1)} \leqslant \|\tilde{y}\|_{(1)} + \|\bar{y}\|_{(1)}. \tag{5}$$

Therefore from the stability of the problems (3a) and (3b) follows the stability of (3). The scheme (3) is stable with respect to the initial data if the following estimate is valid for the solution of problem (3a)

$$\|y(t+\tau)\|_{(1,t)} \leqslant M_1\|y(0)\|_{(1,0)}; \tag{6}$$

and it is stable with respect to the right-hand part if the following estimate is valid for (3b)

$$\|y(t+\tau)\|_{(1,t)} \leqslant M_2 \max_{0\leqslant t'\leqslant t} \|\varphi(t')\|_{(2,t')} + M_2' \max_{0< t'\leqslant t} \|\varphi_{\bar{t}}(t')\|_{(2,t')}. \tag{7}$$

If scheme (3) is stable with respect to the initial data over any interval $[t' = j'\tau,\ t+\tau = (j+1)\tau]$, i.e.

$$\|y(t+\tau)\|_{(1,t)} \leqslant M_1\|y(t')\|_{(1,t')}, \quad t \geqslant t' \geqslant 0, \tag{8}$$

it is stable with respect to the right-hand part [2], so that for the solution of problem (3b) the following estimate holds:

$$\|y(t+\tau)\|_{(1,t)} \leqslant M_1 \sum_{t'=0}^{t} \tau \|\varphi(t')\|_{(2,t')} \tag{9}$$

with the condition that the norms $\|\cdot\|_{(2)}$ and $\|\cdot\|_{(1)}$ are coordinated as follows:

$$\|\varphi(t)\|_{(2,t)} = \|B^{-1}(t)\varphi(t)\|_{(1,t)}. \tag{10}$$

The proof of this theorem will be given by the method of superposition (by the Duhamel method). Let $Y(t, t') = Y^{j,j'}$ be the solution of the problem

$$\frac{1}{\tau} B(t)[Y(t+\tau,t') - Y(t,t')] + A(t)Y(t,t') = 0 \text{ when } t' < t; \tag{11}$$

$$Y(t',t') = 0, \qquad B(t')Y(t'+\tau,t') = \varphi(t'), \tag{12}$$

i.e. $Y(t'+\tau, t') = B^{-1}(t')\varphi(t')$. Then the solution of problem (3b) can be represented in the form

$$y(t) = \sum_{t'=0}^{t-\tau} \tau Y(t,t'). \tag{13}$$

This can be seen if we substitute (13) into (3) and take into account (11) and (12). In view of condition (8) we have

$$\|Y(t+\tau,t')\|_{(1,t)} \leqslant M_1 \|Y(t'+\tau, t')\|_{(1,t')} = M_1 \|B^{-1}(t')\varphi(t')\|_{(1,t')} = \\ = M_1 \|\varphi(t')\|_{(2,t')}.$$

Using the triangular inequality, we obtain

$$\|y(t+\tau)\|_{(1,t)} \leqslant \sum_{t'=0}^{t} \|Y(t+\tau,t')\|_{(1,t)} \leqslant M_1 \sum_{t'=0}^{t} \tau \|\varphi(t')\|_{(2,t')}.$$

The coordination condition of the norms (10) limits the field of applicability of this theorem, since for the theory of difference schemes it is important to estimate y with the strongest possible norm through a right-hand part in the weakest possible norm.

The fundamental problem of the theory is to formulate the sufficiency conditions for the stability of difference schemes.

If B^{-1} exists, from (1) it follows that

$$\hat{y} = B^{-1}Cy + \tau B^{-1}\varphi.$$

Hence it can be seen that scheme (3) is stable if there exists B^{-1} and

$$\|B^{-1}C\| \leqslant 1 + c_1\tau, \tag{14}$$

where c_1 = const. > 0 is independent of τ and N. Here the estimates (8) and (9) are valid.

Condition (14) signifies that $B^{-1}C$ is uniformly bounded with respect to the norm by the number $1 + c_1\tau$, with respect to N, τ. In general B and C (B and A) are unbounded, or non-uniformly bounded operators with respect to N, τ. What properties are to be possessed by the operators B and C for this condition to be satisfied? An answer to this question can be obtained in a testable form by considering the scheme (3) in a real unitary space H.

2. THE INITIAL FAMILY OF SCHEMES

Let H_N be a linear real system, $A_N = A_{N,\tau}(t)$, $B_N = B_{N,\tau}(t)$, $C_N = C_{N,\tau}(t)$ etc., be linear operators, mapping H_N into H_N for each $t \in \omega_\tau$. Over the linear system H_N we shall introduce a scalar product and a norm (the subscript N will be omitted):

$$(y, v) \text{ and } \|y\| = \sqrt{(y, y)},$$
$$(y, v)_{a(t)} = (A(t)y, v) \text{ and } \|y(t+\tau)\|_{a(t)} = \sqrt{(y(t+\tau), y(t+\tau))_{a(t)}},$$
$$(y, v)_{b(t)} = (B(t)y, v) \text{ and } \|y(t+\tau)\|_{b(t)} = \sqrt{(y(t+\tau), y(t+\tau))_{b(t)}} \quad \text{etc.},$$

where $A(t)$, $B(t)$ are selfadjoint positive operators. To estimate the right-hand part $\varphi(t)$ we shall use the norms $\|\varphi(t)\|_{(2)} = \|\varphi(t)\|$, $\|\varphi(t)\|_{(2)} = \|B^{-1}(t)\varphi(t)\|$, $\|\varphi(t)\|_{(2)} = \|\varphi(t)\|_{a^{-1}(t)} = \sqrt{(A^{-1}(t)\varphi(t), \varphi(t))}$ etc.

We shall consider the same initial family of two-layer schemes (IS-2) as in [1], assuming that the following conditions are satisfied:

1) $A = A(t)$ is selfadjoint, positive and Lipschts-continuous with respect to t:

$$A^*(t) = A(t) > 0, \tag{15}$$

$$|(A_{\bar t}y, y)| \leqslant c_2(\check{A}y, y) \quad \text{or} \quad (Ay, y) \leqslant (1 + c_2\tau)(\check{A}y, y), \ \check{A} = A(t-\tau), \tag{16}$$

c_2 = const. > 0 is independent of τ and N;

2) $B = B(t)$ is positive $B(t) > 0$.

We note that the operator $B(t)$ does not have to be selfadjoint. The condition $B(t) > 0$ is ensured by the solvability of the scheme (3), since when $B > 0$ the inverse operator B^{-1} exists.

3. ENERGY IDENTITIES AND INEQUALITIES

Using the obvious identity

$$y = \frac{1}{2}(\hat{y} + y) - \frac{1}{2}(\hat{y} - y) = \frac{1}{2}(\hat{y} + y) - \frac{\tau}{2} \cdot y_t, \tag{17}$$

we rewrite equation (3) in the form

$$(B - 0.5\tau A) y_t + 0.5 A (\hat{y} + y) = \varphi. \tag{18}$$

Scalar multiplying (18) by $2\tau y_t = 2(\hat{y} - y)$:

$$2\tau((B - 0.5\tau A) y_t, y_t) + (A(\hat{y} + y), \hat{y} - y) = 2\tau(\varphi, y_t). \tag{19}$$

Since A is a selfadjoint operator, therefore in view of Lemma 1.7,

$$(A(\hat{y} + y), \hat{y} - y) = (A\hat{y}, \hat{y}) - (Ay, y).$$

After substituting this expression into (9) we obtain the basic energy identity for the two-layer scheme:

$$2\tau((B - 0.5\tau A) y_t, y_t) + (A\hat{y}, \hat{y}) = (Ay, y) + 2\tau(\varphi, y_t). \tag{20}$$

Taking into account that $A = \check{A} + (A - \check{A}) = \check{A} + \tau A_{\bar{t}}$, we rewrite (20) in the form

$$2\tau((B - 0{,}5\tau A) y_t, y_t) + \hat{\mathscr{E}} = \mathscr{E} + \tau(A_{\bar{t}} y, y) + 2\tau(\varphi, y_t), \tag{21}$$

where

$$\hat{\mathscr{E}} = \mathscr{E}(t + \tau) = (A\hat{y}, \hat{y}), \qquad \mathscr{E} = \mathscr{E}(t) = (\check{A} y, y), \tag{22}$$

$\hat{\mathscr{E}}$ is the "energy" of the operator A. Using condition (16), we obtain the energy inequality

$$2\tau((B - 0.5\tau A) y_t, y_t) + \hat{\mathscr{E}} \leqslant (1 + c_2\tau)\mathscr{E} + 2\tau(\varphi, y_t). \tag{23}$$

In studying the stability of the scheme (3) we shall start out from the energy identity (21). Depending on the conditions imposed over B and A the term $2\tau(\varphi, y_t)$ can be evaluated by various methods. Let us indicate some of these (see Lemma 1.4):

$$2\tau|(\varphi, y_t)| \leqslant 2\tau\varepsilon\|y_t\|^2 + \frac{\tau}{2\varepsilon}\|\varphi\|^2, \tag{24}$$

$$2\tau|(\varphi, y_t)| \leqslant 2\tau\varepsilon(By_t, y_t) + \frac{\tau}{2\varepsilon}(B^{-1}\varphi, \varphi) \text{ when } B^* = B, \tag{25}$$

where ε = const. > 0 is any number.

We shall use the following

Lemma 2.1

If $A^*(t) = A(t) > 0$, for any $v(t)$, $\varphi(t) \in H$ we have the following inequality

$$2\tau(\varphi, v_t) = 2\tau(\varphi, \hat{v}_{\bar{t}}) \leqslant 2\tau(\varphi, \hat{v})_{\bar{t}} + \tau\varepsilon(Av, v) + \frac{\tau}{\varepsilon}(A^{-1}\varphi_{\bar{t}}, \varphi_{\bar{t}}). \tag{26}$$

We use the identity

$$2\tau(\varphi, \hat{v}_{\bar{t}}) = 2\tau(\varphi, \hat{v})_{\bar{t}} - 2\tau(\varphi_{\bar{t}}, v), \tag{27}$$

whose validity can easily be seen:

$$2\tau(\varphi, \hat{v})_{\bar{t}} - 2\tau(\varphi_{\bar{t}}, v) = 2(\varphi, \hat{v}) - 2(\check{\varphi}, v) - 2(\varphi - \check{\varphi}, v) =$$
$$= 2(\varphi, \hat{v} - v) = 2\tau(\varphi, v_t).$$

Using for the estimate $2\tau|(\varphi_{\bar{t}}, v)|$ Lemma 1.4, we obtain (26). In particular, when $v = y$ we have

$$2\tau(\varphi, y_t) \leqslant 2(\varphi, \hat{y}) - 2(\check{\varphi}, y) + 2\tau\varepsilon(Ay, y) + \frac{\tau}{2\varepsilon}(A^{-1}\varphi_{\bar{t}}, \varphi_{\bar{t}}). \tag{28}$$

Substituting into (21) the estimates (24) - (26), (28), we obtain the various energy inequalities.

We shall use the difference analogue of Grunwal's lemma to solve the inequalities.

Lemma 2.2

Let $\mathscr{E}(t)$ and $f(t)$ be two non-negative functions defined over the mesh $\{t_j = j\tau,\ j = s, s+1, \ldots, j_0,\ s = 0, 1\}$, and let the inequality

$$\mathscr{E}(t+\tau) \leqslant c_0 \sum_{t'=(s+1)\tau}^{t} \tau \mathscr{E}(t') + f(t), \qquad c_0 > 0, \qquad t = s\tau, (s+1)\tau, \ldots,$$

be satisfied. If $f(t)$ is a non-decreasing function ($f(t+\tau) \geqslant f(t)$), we have

$$\mathscr{E}(t+\tau) \leqslant e^{c_0 t} f(t).$$

If $f(t)$ is an arbitrary non-negative function, we have

$$\mathscr{E}(t+\tau) \leqslant f(t) + c_0 e^{c_0 t} \sum_{t'=st}^{t} \tau f(t').$$

Lemma 2.3

If $f(t) \geqslant 0$ and $\mathscr{E}(s\tau) = 0$, from the inequality

$$\mathscr{E}(t'+\tau) \leqslant (1 + c_0\tau)\mathscr{E}(t') + \tau f(t'), \qquad t' = s\tau, (s+1)\tau, \ldots,$$

we have the estimate

$$\mathscr{E}(t+\tau) \leqslant e^{c_0 t} \sum_{t'=s\tau}^{t} \tau f(t').$$

The proof of Lemmas 2.2 and 2.3 are given for example in [11, 16].

4. SUFFICIENT CONDITIONS OF STABILITY AND *a priori* ESTIMATES

Theorem 2.1

If the condition

$$B \geqslant 0.5\tau(1 - c_1\tau)A, \tag{29}$$

is satisfied, where c_1 = const. > 0 is independent of τ and of N, the scheme (3) is stable when

$$\tau \leqslant \tau_0, \quad \tau_0 < {}^1/_4 c_1$$

and the following estimate is valid for the solution of problem (3):

$$\|y(t+\tau)\|_{a(t)} \leqslant M_1\|y(0)\|_{a(0)} + M_2 \max_{0\leqslant t'\leqslant t} \|\varphi(t')\|_{a^{-1}(t')} + \\ + M_2' \max_{0<t'\leqslant t} \|\varphi_{\bar t}(t')\|_{a^{-1}(t')} \tag{30}$$

where M_1, M_2, M_2' are positive constants, depending only on c_1, c_2, and t_0.

Proof. Let us return to the identity (21). Condition (29) and Lemma 1.7 give

$$2\tau((B-0.5\tau A)y_t, y_t) \geqslant -c_1\tau^3(Ay_t, y_t) \geqslant -2c_1\tau[(A\hat{y}, \hat{y}) + (Ay, y)].$$

We substitute this estimate into (21):

$$(1-2c_1\tau)(A\hat{y}, \hat{y}) \leqslant (1+2c_1\tau)(Ay, y) + 2\tau(\varphi, y_t). \tag{31}$$

1. Stability with respect to the initial data. Putting $\varphi = 0$ and taking into account that $4c_1\tau < 1$ we obtain for the problem (3a) the inequality

$$(1-2c_1\tau)\mathscr{E}(t+\tau) \leqslant (1+2c_1\tau)(1+c_2\tau)\mathscr{E}(t) < (1+2(c_1+c_2)\tau)\mathscr{E}(t),$$
$$\mathscr{E}(t+\tau) < (1+4(c_1+c_2)\tau)\mathscr{E}(t) \quad \text{when } t>0, \tag{32}$$
$$\mathscr{E}(\tau) < (1+4(c_1+c_2)\tau)\|y(0)\|^2_{a(0)} \quad \text{when } t=0.$$

Hence we find $\mathscr{E}(t+\tau) < \exp[4(c_1+c_2)t]\,\|y(0)\|^2_{a(0)}$, i.e.

$$\|y(t+\tau)\|_{a(t)} \leqslant M_1\|y(0)\|_{a(0)}, \text{where } M_1 = \exp[2(c_1+c_2)t_0]. \tag{33}$$

2. Stability with respect to the right-hand part. To solve the problem (3a) instead of (32) we obtain the inequality

$$(1-2c_1\tau)(A(0)y(\tau), y(\tau)) \leqslant 2(\varphi(0), y(\tau)). \tag{34}$$

We substitute into (31) the estimate (26):

$$(1-2c_1\tau)(A\hat{y}, \hat{y}) \leqslant (1+2(c_1+\varepsilon)\tau)(Ay, y) + 2(\varphi, \hat{y}) - \\ -2(\check{\varphi}, y) + \frac{\tau}{2\varepsilon}(A^{-1}\varphi_{\bar t}, \varphi_{\bar t}).$$

Then taking into account the Lipschits-continuity of $A(t)$ and choosing ε so that $2(c_1+\varepsilon)\tau < 1$, we obtain

$$(1-2c_1\tau)\mathscr{E}(t'+\tau) \leqslant (1+2(c_1+c_2+\varepsilon)\tau)\mathscr{E}(t') + \\ + 2(\varphi(t'), y(t'+\tau)) - 2(\varphi(t'-\tau), y(t')) + \frac{\tau}{2\varepsilon}(A^{-1}\varphi_{\bar t}(t'), \varphi_{\bar t}(t')), \tag{35}$$

since $(1 + 2(c_1 + \varepsilon)\tau)(1 + c_2\tau) = 1 + [2(c_1 + \varepsilon) + c_2]\tau + 2(c_1 + \varepsilon)\tau c_2\tau < 1 + 2(c_1 + c_2 + \varepsilon)\tau$ when $2(c_1 + \varepsilon)\tau < 1$.

We sum (35) with respect to $t' = \tau, 2\tau, \ldots, t$, take into account (34) and use Lemma 1.4:

$$2(\varphi, \underset{\bullet}{\hat{y}}) \leqslant \varepsilon_1\hat{\mathscr{E}} + \frac{1}{\varepsilon_1}(A^{-1}\varphi, \varphi).$$

Choosing $\varepsilon_1 = ¼$, we arrive at the inequality

$$\mathscr{E}(t+\tau) \leqslant c_0\sum_{t'=\tau}^{t} \tau\mathscr{E}(t') + \Phi(t), \qquad t \geqslant 0,$$

where

$$\Phi(t) = 4(A^{-1}\varphi, \varphi) + \frac{2\tau}{\varepsilon}\sum_{t'=\tau}^{t}(A^{-1}(t')\varphi(t'), \varphi(t')),$$
$$c_0 = 4(2c_1 + c_2 + \varepsilon).$$

Now using Lemma 2.2 and choosing ε such that $2(c_1 + \varepsilon)\tau < 1$ we find

$$\mathscr{E}(t+\tau) \leqslant \Phi(t) + c_0e^{c_0t}\sum_{t'=0}^{t} \tau\Phi(t').$$

Hence we obtain directly

$$\|y(t+\tau)\|_{a(t)} \leqslant M_2 \max_{0<t'\leqslant t} \|\varphi(t')\|_{a^{-1}(t')} + M_2' \max_{0<t'\leqslant t} \|\varphi_{\bar{t}}(t')\|_{a^{-1}(t')}, \qquad (36)$$

where $y = \bar{y}$ is the solution of problem (3b). From (33) and (36) in view of (5) we obtain the estimate (30). Theorem 2.1 is now proved.

Theorem 2.2

If the condition

$$B \geqslant \varepsilon E + 0.5\tau A, \qquad \varepsilon > 0, \qquad (37)$$

is satisfied, to solve problem (3) the following *a priori* estimate holds:

$$\|y(t+\tau)\|_{a(t)} \leqslant M_1\left[\|y(0)\|_{a(0)} + \frac{1}{\sqrt{(2\varepsilon)}}\left(\sum_{t'=0}^{t} \tau\|\varphi(t')\|^2\right)^{1/2}\right]. \qquad (38)$$

Proof 1. When $\varphi = 0$ inequality (23) gives

$$\mathscr{E}(t+\tau) \leqslant (1+c_2\tau)\mathscr{E}(t) \quad \text{when } t>0,$$

$$\mathscr{E}(\tau) \leqslant (1+c_2\tau)\|y(0)\|^2_{a(0)} \quad \text{when } t=0.$$

Hence follows $\mathscr{E}(t+\tau) \leqslant e^{c_2 t}\mathscr{E}(\tau) \leqslant e^{c_2 t_0}\|y(0)\|^2_{a(0)}$, i.e.

$$\|y(t+\tau)\|_{a(t)} \leqslant M_1\|y(0)\|_{a(0)}, \qquad \text{where } M_1 = \exp(0.5c_2t_0). \tag{39}$$

2. When $y(0) = 0$ we have

$$2\tau\varepsilon\|y_t\|^2 + \mathscr{E}(t+\tau) \leqslant (1+c_2\tau)\mathscr{E}(t) + 2\tau(\varphi, y_t),$$

$$\frac{2\varepsilon}{\tau}\|y(\tau)\|^2 + \mathscr{E}(\tau) \leqslant 2(\varphi(0), y(\tau)).$$

We substitute into this inequality estimate (24) and obtain

$$\mathscr{E}(t'+\tau) \leqslant (1+c_2\tau)\mathscr{E}(t') + \frac{\tau}{2\varepsilon}\|\varphi(t')\|^2 \quad \text{when } t'>0,$$

$$\mathscr{E}(\tau) \leqslant \frac{\tau}{2\varepsilon}\|\varphi(0)\|^2 \quad \text{when } t'=0.$$

Summing with respect to $t' = \tau, 2\tau, \ldots, t$ and using Lemma 2.2, we find

$$\mathscr{E}(t+\tau) \leqslant \frac{1}{2\varepsilon}M_1^2\sum_{t'=0}^{t}\|\varphi(t')\|^2. \tag{40}$$

From (39) and (40) we obtain (38).

Note 1. It may appear that scheme (3) belongs to IS-2 for sufficiently small $\tau \leqslant \tau_0'$ and sufficiently large $N \geqslant N_0'$. Then Theorem 2.1 is valid when $\tau \leqslant \tau_0^*$ and $N \geqslant N_0'$, where $\tau_0^* = \min(\tau_0, \tau_0')$, and Theorem 2.2 is valid when $\tau \leqslant \tau_0'$ and $N \geqslant N_0'$. In order not to complicate the discussion, we shall assume that scheme (3) belongs to the initial family IS-2 for any τ and N. In view of the foregoing, this does not lead to any loss of generality.

2. Theorem 2.2 remains true if $A = A_0 + A_1$, where $A_0(t)$ satisfies (15) and (16), and $A_1(t)$ is a non-selfadjoint operator, subordinated to $A_0(t)$:

$$\|A_1(t)y\| \leqslant c_3\|y\|_{a_0(t)},$$

where $c_3 = \text{const.} > 0$ is independent of τ and N. Then in (38) instead of

$\| y \|_a$ we have $\| y \|_{a_0}$.

Condition (29) separates out from the class IS-2 the class K_0 of stable schemes. Schemes satisfying condition (37), obviously belong to K_0.

5. ON THE NECESSARY CONDITIONS OF STABILITY

We have now found the sufficient conditions of stability (29) of scheme (3) from IS-2. Let us now consider the problem of the necessary conditions of stability with respect to the initial data in the norm $\| y \|_{(1)} = \| y \|_a$. We separate out from IS-2 the class of schemes K_1, satisfying some additional requirements: B is a selfadjoint operator, $B^* = B$, A and B are independent of t.

Theorem 2.3

For the stability of any scheme (3) from the class K_1 with respect to the initial data it is necessary that condition (29) should be satisfied.

Proof. Let the scheme (3) from K_1 be stable with respect to the initial data, i.e. let such a number $M_1 > 0$ exist, independent of τ and of N, that for the solution of problem (3a) the inequality $\| y(t) \|_a \leqslant M_1 \| y_0 \|_a$ is satisfied for all $t \in \omega_\tau$, and in particular

$$\|y(t_0)\|_a \leqslant M_1 \|y_0\|_a \qquad (t_0 = j_0 \tau) \tag{41}$$

Since the operator A is positive and selfadjoint, an operator $A^{1/2} = (A^{1/2})^* > 0$ exists. Now we rewrite the scheme (3a) in the form $A^{1/2}\hat{y} = (E - \tau A^{1/2} B^{-1} A^{1/2}) A^{1/2} y$ and put

$$x = A^{1/2} y, \qquad \hat{x} = A^{1/2}\hat{y}, \qquad C = A^{1/2} B^{-1} A^{1/2}. \tag{42}$$

Then scheme (3a) is transformed into the explicit scheme

$$x_t + Cx = 0, \text{ or } \hat{x} = (E - \tau C)x, \qquad x(0) = x_0, \qquad x_0 = A^{1/2} y_0. \tag{43}$$

Inequality (41) takes the form

$$\|x(t_0)\| \leqslant M_1 \|x_0\|, \quad \text{since} \quad \|x\| = \|y\|_a. \tag{44}$$

Then using the equation $\hat{x} = (E - \tau C)x$, we find $x(t_0) = (E - \tau C)^{j_0} x_0$,

$\|x(t_0)\| = \|(E-\tau C)^{j_0}x_0\|$. Hence and from (44) it follows that

$$\|(E-\tau C)^{j_0}\| = \|E-\tau C\|^{j_0} \leqslant M_1, \text{ i.e. } \|E-\tau C\| \leqslant M_1^{1/j_0} = M_1^{\tau/t_0}.$$

For any M_1 one can find a number $c_1 > 0$, independent of τ and of N, such that the inequality $M_1^{\tau/t_0} \leqslant 1 + 2c_1\tau$ is satisfied. For this it is sufficient to put, for example, $c_1 = (1/2t_0)M_1 \ln M_1$. Thus, from (44) we have

$$\|E-\tau C\| \leqslant 1 + 2c_1\tau. \tag{45}$$

In view of a well-known theorem of functional analysis [28], $\|E-\tau C\| = \max\limits_{\|x\|} |((E-\tau C)x, x)|$ when $\|x\| = 1$, since $E-\tau C = (E-\tau C)^*$. Therefore $|((E-\tau C)x,x)| \leqslant (1+2c_1\tau\|x\|^2$, i.e.

$$-(1+2c_1\tau)E \leqslant E-\tau C \leqslant (1+2c_1\tau)E. \tag{46}$$

Hence we find $E \geqslant 0.5\tau C/(1+c_1\tau)$. Since $C^* = C$, therefore $C^{-1} \geqslant 0.5\,\tau E/(1+c_1\tau)$ or

$$(C^{-1}x, x) \geqslant \frac{0.5\tau}{1+c_1\tau}\|x\|^2.$$

Substituting (42) here, we obtain

$$(By, y) \geqslant \frac{0.5\tau}{1+c_1\tau}(Ay, y) \geqslant 0.5\tau(1-c_1\tau)(Ay, y),$$

i.e. condition (29) is satisfied. The theorem is thus proved.

Let us now examine the class of schemes K_2 (see Section 6), which is separated from IS-2 by the additional requirement that $B(t) = B^*(t)$. For schemes from K_2 the following statement can be made (it is proved by the same method as Theorem 2.3): let the scheme (3) from K_2 be stable with respect to the initial data. Then one can find such vectors $y \in H$ for which the following condition is satisfied

$$(By, y) \geqslant 0.5\tau(1-c_1\tau^\gamma)(Ay, y), \qquad 0 \leqslant \gamma \leqslant 1, \qquad c_1 > 0, \tag{46'}$$

where c_1 and γ are independent of τ and of N.

From this assertion it follows that the condition

$$(Bx, x) \leqslant 0.5\tau(1-c_1\tau^\gamma)(Ax, x) \text{ for all } x \in H \tag{47}$$

is sufficient for the instability of the scheme (3) from K_2.

We remark that condition (46′), satisfied for all $y \in H$ when $\gamma = 1$, is the same as the sufficient condition of stability (29)

6. THE CASE OF A SELFADJOINT OPERATOR $B(t)$

In defining the initial family IS-2 we have not assumed the self-adjoint nature and continuity with respect to t of the operator $B(t)$.

Let us now consider another initial family IS-2* of schemes (3), defined by the requirements:

1) $B^*(t) = B(t) > 0$;

2) $B(t)$ is Lipschits-continuous with respect to t with the constant c_2,

$$|(B_{\bar{t}}\, y, y)| \leqslant c_2 (\check{B}y, y); \tag{48}$$

3) $A^*(t) = A(t) \geqslant 0$.

Here we do not require the continuity of $A(t)$ with respect to t.

For schemes belonging to IS-2*, we can obtain *a priori* estimates in the norm

$$\| y(t+\tau)\|_{(1,\,t)} = \| y(t+\tau)\|_{b(t)} = \sqrt{(B(t)y(t+\tau), y(t+\tau))}$$

First we shall give the energy identity. For this we multiply equation (3) in a scalar fashion by $2\tau\hat{y}$:

$$2\tau(By_t, \hat{y}) + 2\tau(Ay, \hat{y}) = 2\tau(\varphi, \hat{y}). \tag{49}$$

Using Lemma 1.7 when $v = {}^1\!/_2(\hat{y} + y)$, $u = {}^1\!/_2(\hat{y} - y)$ and Lemma 1.8 when $v = \hat{y}$, $a = y$, we find

$$(Ay, \hat{y}) = {}^1\!/_4(A(\hat{y} + y), \hat{y} + y) - {}^1\!/_4(A(\hat{y} - y), \hat{y} - y) = \tag{50}$$
$$= \frac{1}{4}\|\hat{y} + y\|_a^2 - \frac{\tau^2}{4}\|y_t\|_a^2,$$

$$2\tau(By_t, \hat{y}) = \tau(B\hat{y}, \hat{y})_{\bar{t}} + \tau^2(By_t, y_t) - \tau(B_{\bar{t}}\, y, y). \tag{51}$$

Substituting (50) and (51) into (49), we obtain the energy identity

$$(B\hat{y},\hat{y})+\tau^2\left[(By_t,y_t)-\frac{\tau}{2}(Ay_t,y_t)\right]+\frac{\tau}{2}(A(\hat{y}+y),\hat{y}+y)= \tag{52}$$
$$=(\check{B}y,y)+\tau(B_{\bar{t}}y,y)+2\tau(\varphi,\hat{y}).$$

Hence it can be seen that the expression in square brackets is non-negative when

$$B\geqslant\frac{\tau}{2}A. \tag{53}$$

Under this condition, from identity (52) we have the inequality

$$\|\hat{y}\|_b^2+\frac{\tau}{2}\|\hat{y}+y\|_a^2\leqslant(1+c_2\tau)\|y\|_{\check{b}}^2+2\tau(\varphi,\hat{y}), \tag{54}$$

if we take into account that $(By,y)=(\check{B}y,y)+\tau(B_{\bar{t}}y,y)\leqslant$ $(1+\tau c_2)(\check{B}y,y)=(1+\tau c_2)\|y\|_{\check{b}}^2$ in view of condition (48). We rewrite (54) in the form

$$\mathscr{E}(t+\tau)+\frac{1}{2}\tau\|y(t+\tau)+y(t)\|_{a(t)}^2\leqslant(1+c_2\tau)\mathscr{E}(t)+2\tau(\varphi,\hat{y}).\ t>0, \tag{55}$$

where

$$\mathscr{E}(t+\tau)=\|y(t+\tau)\|_{b(t)}^2=\|\hat{y}\|_b^2. \tag{56}$$

Theorem 2.4

If the scheme (3) from IS-2* satisfies condition (53), to solve problem (3) we have the following *a priori* estimate:

$$\|y(t+\tau)\|_{b(t)}\leqslant M_1\|y(0)\|_{b(0)}+M_2\left[\sum_{t'=0}^{t}\tau\|\varphi(t')\|_{b^{-1}(t')}^2\right]^{1/2}, \tag{57}$$

where $M_1>0$, $M_2>0$ depend only on c_2 and t_0, $\|\varphi\|_{b^{-1}}^2=(B^{-1}\varphi,\varphi)$.

Proof. Since $A\geqslant 0$, from (55) it follows that

$$\mathscr{E}(t+\tau)\leqslant(1+c_2\tau)\mathscr{E}(t)+2\tau(\varphi(t),y(t+\tau))\quad\text{when } t>0. \tag{58}$$

1. Stability with respect to the initial data. If $\varphi=0$, we have $\mathscr{E}(t+\tau)\leqslant(1+c_2\tau)\mathscr{E}(t)\leqslant e^{c_2t}\mathscr{E}(\tau)\leqslant e^{c_2t_0}\|y(0)\|_{b(0)}^2$, since $\mathscr{E}(\tau)\leqslant\|y(0)\|_{b(0)}^2$.

Hence for (3a) we have the estimate

$$\| y(t+\tau)\|_{b(t)} \leqslant M_1 \| y(0)\|_{b(0)}, \text{ where } M_1 = \exp\,(0.5c_2t_0). \tag{59}$$

2. *Stability with respect to the right-hand part.* Let $y(0) = 0$. Then equation (3) when $t = \tau$ takes the form

$$By(\tau) = \tau\varphi(0).$$

Multiplying this by $y(\tau)$ we ootain

$$\mathscr{E}(\tau) = \tau(\varphi(0), y(\tau)).$$

Substituting into (58) the estimate $2\tau(\varphi, \hat{y}) \leqslant \tau\varepsilon(B\hat{y}, \hat{y}) + \frac{\tau}{\varepsilon}(B^{-1}\varphi, \varphi)$, we obtain

$$(1-\tau\varepsilon)\mathscr{E}(t'+\tau) \leqslant (1+\tau c_2)\mathscr{E}(t') + \frac{1}{\varepsilon}\tau\|\varphi(t')\|^2_{b^{-1}(t')} \text{ when } t' > 0,$$

$$(1-\tau\varepsilon)\mathscr{E}(\tau) \leqslant \frac{\tau}{4\varepsilon}\|\varphi(0)\|^2_{b^{-1}(0)} \qquad \text{when } t' = 0,$$

where $0 < \varepsilon < 1/\tau$.

Hence by summing with respect to $t' = 0, \tau, \ldots, t$ we find

$$\mathscr{E}(t+\tau) \leqslant \frac{c_2+\varepsilon}{1-\varepsilon\tau}\sum_{t'=\tau}^{t}\tau\mathscr{E}(t') + \frac{1}{\varepsilon(1-\varepsilon\tau)}\sum_{t'=0}^{t}\tau\|\varphi(t')\|^2_{b^{-1}(t')}.$$

Using Lemma 2.2 and choosing ε from the condition for minimizing M_2', we obtain

$$\mathscr{E}(t+\tau) \leqslant M_2' \sum_{t'=0}^{t}\|\varphi(t')\|^2_{b^{-1}(t')}, \tag{60}$$

where M_2' depends only on c_2 and t_0. Unifying (59) and (60), we obtain (57).

Theorem 2.5

If the scheme (3) from IS-2* satisfies the conditions

$$B \geqslant 0.5\tau(1+\varepsilon)A, \qquad A(t) > 0, \tag{61}$$

where $\varepsilon > 0$ is any number, the following *a priori* estimate is valid:

$$\|y(t+\tau)\|_{b(t)} \leqslant M_1\|y(0)\|_{b(0)} + M_2\left[\sum_{t'=0}^{t}\tau\|\varphi(t')\|^2_{a^{-1}(t')}\right]^{1/2}, \tag{62}$$

Proof. It is sufficient to find the estimate for the problem (3b). Taking into account that $\hat{y} = 0.5(\hat{y} + y) + 0.5(\hat{y} - y)$, and using Lemma 1.4, we have

$$2\tau(\varphi, \hat{y}) = \tau(\varphi, \hat{y} + y) + \tau(\varphi, \hat{y} - y),$$
$$2\tau(\varphi, \hat{y}) \leqslant \frac{1}{2}\tau(\|\hat{y} + y\|_a^2 + \|\varphi\|_{a^{-1}}^2) + \frac{\tau^3}{2}\varepsilon\|y_t\|_a^2 + \frac{\tau}{2\varepsilon}\|\varphi\|_{a^{-1}}^{2} \text{ when } t > 0.$$

Since $A \leqslant (2/\tau)B$, we have $\tau(\varphi(0), \ y(\tau)) \leqslant {}^1/_4\,\tau(A(0)y(\tau), \ \ y(\tau)) + \tau(A^{-1}(0)\varphi(0), \varphi(0)) \leqslant {}^1/_2(B(0)y(\tau), y(\tau)) + \tau\|\varphi(0)\|_{a^{-1}(0)}^2 = \frac{1}{2}\mathscr{E}(\tau) + \tau\|\varphi(0)\|_{a^{-1}(0)}^2$.

We substitute this estimate into (55)

$$\|\hat{y}\|_b^2 + \tau^2\left[\|y_t\|_b^2 - \frac{1}{2}\tau(1+\varepsilon)\|y_t\|_a^2\right] \leqslant (1 + \tau c_2)\|y\|_{\check{b}}^2 + \frac{1}{2}\tau\left(1 + \frac{1}{\varepsilon}\right)\|\varphi\|_{a^{-1}}^2,$$
$$\mathscr{E}(t' + \tau) \leqslant (1 + \tau c_2)\mathscr{E}(t') + \frac{1}{2}\tau\left(1 + \frac{1}{\varepsilon}\right)\|\varphi(t')\|_{a^{-1}(t')}^2 \quad \text{when } t' > 0,$$
$$\mathscr{E}(\tau) \leqslant 2\tau\|\varphi(0)\|_{a^{-1}(0)}^2 \quad \text{when } t' = 0. \tag{63}$$

Using Lemma 2.3, we obtain (62).

Note 1. If the operator B is constant, i.e. is independent of t, in (57) and (62) we have to put $M_1 = 1$ so that for the scheme (3a) we obtain the estimate

$$\|y(t+\tau)\|_b \leqslant \|y(t)\|_b \qquad \text{or} \quad \|y^{j+1}\|_b \leqslant \|y^j\|_b. \tag{64}$$

Note 2. An estimate of the solution of problem (3a) with respect to the norm $\|y\|_b$ in a finite dimensional case has been used [31, 32] to study the convergence of two-layer iterative schemes for the solution of sets of algebraic equations $Av = \varphi$. The spectral method was used to obtain the estimate

$$\|y^{j+1}\|_b \leqslant \rho\|y^j\|_b \tag{65}$$

under double-sided restrictions on the operator B of the form

$$\gamma_1 A \leqslant B \leqslant \gamma_2 A, \qquad \gamma_1 > 0, \qquad \gamma_2 > 0, \tag{66}$$

and the value

$$\tau = \frac{2\gamma_1\gamma_2}{\gamma_1 + \gamma_2}, \tag{67}$$

has been found, at which the minimum of ρ is reached, equal to

$$\rho_{\min} = \frac{\gamma_2 - \gamma_1}{\gamma_1 + \gamma_2}. \tag{68}$$

The same result can be obtained on the basis of Theorem 2.3 without assuming that H is finite dimensional.

Indeed, assuming $y(t_j) = \rho^j v(t_j)$, $t_j = j\tau$, we obtain for v the problem

$$\rho B v_t + \tilde{A} v = 0, \qquad v(0) = y(0), \qquad \tilde{A} = A - \frac{1-\rho}{\tau} B.$$

The conditions $\rho B \geqslant 0.5\,\tau \tilde{A}$, $\tilde{A} \geqslant 0$ give

$$\frac{\tau}{1+\rho} A \leqslant B \leqslant \frac{\tau}{1-\rho} A. \tag{69}$$

Equating (69) to (66), we ascertain that the minimum of ρ is reached when

$$\frac{\tau}{1+\rho} = \gamma_1, \qquad \frac{\tau}{1-\rho} = \gamma_2.$$

Hence (67) and (68) follow.

7. *A priori* ESTIMATES FOR A WEIGHTED SCHEME

Consider the single-parameter family of schemes (weighted schemes)

$$y_t + A(\sigma \hat{y} + (1-\sigma) y) = \varphi, \qquad t \in \omega_\tau, \qquad y(0) = y_0, \tag{70}$$

where $A(t)$ is a positive linear operator. The stability of the scheme (70) depends on the choice of the real parameter σ.

In order to use Theorem 2.1 and 2.2, we write the scheme in the canonical form (3)

$$(E + \sigma\tau A) y_t + Ay = \varphi, \qquad t \in \omega_\tau, \qquad y(0) = y_0, \qquad B = E + \sigma\tau A, \tag{71}$$

using the identity $\sigma\hat{y} + (1-\sigma) y = y + \sigma\tau y_t$. Since $A(t) > 0$, the inverse operator A^{-1} exists. Applying the operator A^{-1} to equation (71), we obtain

$$\tilde{B} y_t + \tilde{A} y = \tilde{\varphi}, \qquad t \in \omega_t, \qquad y(0) = y_0; \tag{72}$$

$$\tilde{B} = A^{-1} + \sigma\tau E, \qquad \tilde{A} = E, \qquad \varphi = A^{-1}\varphi. \tag{73}$$

In deriving the *a priori* estimates we shall use (70) in one of the forms (71) or (72).

Theorem 2.6

If $A^*(t) = A(t) > 0$ and

$$\sigma \geqslant \sigma_0, \qquad \sigma_0 = \frac{1}{2} - \frac{1}{\tau\|A\|}, \tag{74}$$

the scheme (70) is stable and the following estimate is valid for the solution of the problem

$$\|y(t+\tau)\| \leqslant \|y(0)\| + M_2 \max_{0\leqslant t'\leqslant t} \|A^{-1}(t')\varphi(t')\| + M_2' \max_{0<t'\leqslant t} \|(A^{-1}(t')\varphi(t'))_{\bar t}\|, \tag{75}$$

where M_2 and M_2' depend only on t_0.

Note 1. The requirement that $A(t)$ should be selfadjoint can be replaced by the requirement (see [34])

$$\|Ax\|^2 \leqslant \Delta(Ax, x), \qquad x \in H. \tag{76}$$

The theorem remains true when

$$\sigma \geqslant \sigma_0, \qquad \tau = \frac{1}{2} - \frac{1}{\tau\Delta}, \tag{77}$$

since from (76) it follows that $A^{-1} \geqslant (1/\Delta)E$ (Lemma 1.2).

Note 2. When $\sigma \geqslant 0.5$ the estimate remains true for any selfadjoint operator $A(t) > 0$.

Note 3. If $A(t)$ is a non-negative operator, in (74) we must formally put $1/\|A\| = 0$ so that $\sigma \geqslant 0.5$.

Note 4. If $A(t) = A^*(t) > 0$ and the condition that $A(t)$ is Lipschits-continuous is satisfied with respect to t, the estimate (38) is valid when

$$\sigma \geqslant \sigma_\varepsilon = \frac{1}{2} - \frac{1-\varepsilon}{\tau\|A\|}, \qquad 0 < \varepsilon \leqslant 1. \tag{78}$$

Theorem 2.7

If $A(t) = A^*(t) > 0$ and condition (78) is satisfied, for the solution of problem (70) we have the *a priori* estimate

$$\|y(t+\tau)\| \leqslant \|y(0)\| + \frac{1}{\sqrt{(2\varepsilon)}} \left[\sum_{t'=0}^{t} \tau \|\varphi(t')\|^2_{a^{-1}(t')} \right]^{1/2}. \tag{79}$$

Using the identity (20) for the scheme (72) with $\tilde{A} = E$:

$$2\tau((\tilde{B} - 0.5\tau\tilde{A})y_t, y_t) + \|\hat{y}\|^2 = \|y\|^2 + 2\tau(\tilde{\varphi}, y_t), \quad \tilde{\varphi} = A^{-1}\varphi.$$

Taking into account that from $\sigma \geqslant \sigma_\varepsilon$ we have $\tilde{B} - 0.5\tau\tilde{A} \geqslant \varepsilon A^{-1}$, since

$$\tilde{B} - 0.5\tau\tilde{A} = \varepsilon A^{-1} + (1-\varepsilon)A^{-1} + (\sigma - 0.5)\tau E \geqslant \varepsilon A^{-1} + \\ + \left[(1-\varepsilon)\frac{1}{\Delta\tau} + (\sigma - 0.5)\right] E \geqslant \varepsilon A^{-1},$$

we obtain

$$2\tau\varepsilon(A^{-1}y_t, y_t) + \|\hat{y}\|^2 \leqslant \|y\|^2 + 2\tau(A^{-1}\varphi, y_t).$$

Substituting here the estimate following from (1.3) and (1.5),

$$2\tau(A^{-1}\varphi, y_t) \leqslant 2\tau\varepsilon(A^{-1}y_t, y_t) + \frac{\tau}{2\varepsilon}(A^{-1}\varphi, \varphi), \tag{80}$$

we have

$$\|\hat{y}\|^2 \leqslant \|y\|^2 + \frac{\tau}{2\varepsilon}\|\varphi\|^2_{a^{-1}}.$$

Hence (79) follows.

We write the weighted scheme (70) in the form $(E + \sigma\tau A)\hat{y} = (E - (1-\sigma)\tau A)y + \tau\varphi$ and find $\hat{y} = (E+\sigma\tau A)^{-1}(E - (1-\sigma)\tau A)y + \tau(E + \sigma\tau A)^{-1}\varphi$. Hence we obtain

$$\|\hat{y}\| \leqslant \|(E+\sigma\tau A)^{-1}(E - (1-\sigma)\tau A)\| \, \|y\| + \tau\|(E+\sigma\tau A)^{-1}\| \, \|\varphi\|. \tag{81}$$

Let $\varphi = 0$. Then from Note 1 of Theorem 2.6 we have

$$\|(E+\sigma\tau A)^{-1}(E - (1-\sigma)\tau A)\| \leqslant 1 \quad \text{when } \sigma \geqslant 1/2 - 1/\tau\Delta, \tag{82}$$

if $\|Ax\|^2 \leqslant \Delta(Ax, x)$. From condition (76) it follows that

$$E \geqslant \frac{1}{\Delta}A, \qquad E + \sigma\tau A = (\sigma - \sigma_\varepsilon)\tau A + 0.5\tau A + \varepsilon E +$$

$$+(1-\varepsilon)\left(E-\frac{1}{\Delta}A\right)\geqslant 0.5\tau A+\varepsilon E>\varepsilon E$$

when

$$\sigma\geqslant\sigma_\varepsilon,\qquad \sigma_\varepsilon=\frac{1}{2}-\frac{1-\varepsilon}{\tau\Delta} \tag{78}$$

and consequently (Lemma 1.1),

$$\|(E+\sigma\tau A)^{-1}\|\leqslant\frac{1}{\varepsilon} \tag{83}$$

Substituting (83) and (82) into (81), we see that the following theorem holds.

Theorem 2.8

If conditions (76) and (78′) are satisfied, for the solution of problem (70) we have the estimate

$$\|y(t+\tau)\|\leqslant\|y(0)\|+\frac{1}{\varepsilon}\sum_{t'=0}^{t}\tau\|\varphi(t')\|. \tag{84}$$

If $A(t)>0$ is an arbitrary selfadjoint operator, when $\sigma\geqslant 0.5$ estimate (84) is valid with $\varepsilon=1$.

Note 5. Let $A(t)$ be semi-bounded from below, $A(t)\geqslant -c_*E$, $c_*=\text{const.}>0$. Then when $0.5\leqslant\sigma\leqslant 1$, $\tau\leqslant\tau_0$, $\tau_0<1/2\ c_*$ we have the estimate

$$\|y(t+\tau)\|\leqslant M_1\|y(0)\|+M_2\sum_{t'=0}^{t}\tau\|\varphi(t')\|, \tag{85}$$

where M_1 and M_2 depend only on c_* and t_0.

The proof of this theorem will be omitted.

Theorem 2.9

Let $A>0$ be a constant operator and $\sigma\geqslant 0.5$. Then for problem (70) we have the estimate

$$\|Ay(t+\tau)\|\leqslant\|Ay(0)\|+M_2\max_{0\leqslant t'\leqslant t}\|\varphi(t')\|+M_2'\max_{0<t'\leqslant t}\|\varphi_{\bar t}(t')\|, \tag{86}$$

where $M_2>0$, $M_2'>0$ depends only on t_0.

We rewrite (71) in the form

$$(E + (\sigma - 0.5)\tau A) y_t + 0.5A(\hat{y} + y) = \varphi \tag{87}$$

and scalar multiply (87) by $2\tau Ay_t = 2A(\hat{y} - y)$:

$$2\tau(Ay_t, y_t) + 2\tau^2(\sigma - 0.5) \| Ay_t \|^2 + \| \hat{A}y \|^2 = \| Ay \|^2 + 2\tau(\varphi, y_t).$$

If $\varphi = 0$, it follows immediately from this

$$\| Ay(t+\tau) \| \leqslant \| Ay(0) \| \qquad \text{when } \sigma \geqslant 0.5.$$

We transform

$$2\tau(\varphi, Ay_t) = 2\tau(\varphi, A\hat{y})_{\bar{t}} - 2\tau(\varphi_{\bar{t}}, Ay) \leqslant 2\tau(\varphi, A\hat{y})_{\bar{t}} +$$
$$+ \varepsilon\tau \| Ay \|^2 + \frac{\tau}{\varepsilon} \| \varphi_{\bar{t}} \|^2 \text{ when } t > 0,$$

so that

$$2\tau(Ay_t, y_t) + \| A\hat{y} \|^2 \leqslant (1 + \varepsilon\tau) \| Ay \|^2 + \frac{\tau}{\varepsilon} \| \varphi_{\bar{t}} \|^2 + 2\tau(\varphi, A\hat{y})_{\bar{t}}.$$

After this the argument is the same as in the proof of Theorem 2.1 when $y(0) = 0$.

Note 6. It is not required that operator A be selfadjoint. If $\|Ax\|^2 \leqslant \Delta(Ax, x)$, Theorem 2.9 is valid when $\sigma \geqslant \sigma_0$, $\sigma_0 = ½ - 1/\tau\Delta$. If $A = A^*$ the following estimate holds

$$\|Ay(t+\tau)\|^2 \leqslant \|Ay(0)\|^2 + \frac{1}{2} \sum_{t'=0}^{t} \tau \|\varphi(t')\|_a^2.$$

Note 7. Theorem 2.9 remains true for an operator $A(t)$ which satisfies the condition

$$\| A_{\bar{t}} y \| \leqslant c_2 \| \check{A} y \|.$$

A priori estimates of the kind (86) for multidimensional two-layer schemes, approximating the heat conduction equations, have been obtained in [23].

3. Classes of stable three-layer schemes

1. THE INITIAL FAMILY OF THREE-LAYER SCHEMES

Let us consider the set of three-layer schemes

$$By_{\overset{\circ}{t}} + \tau^2 R y_{\bar{t}t} + Ay = \varphi, \qquad 0 < t \in \omega_\tau, \qquad y(0) = y_0, \qquad y(\tau) = y_1, \quad (1)$$

where y_0 and y_1 are arbitrary vectors from H; B, R and A are linear operators from H into H.

The initial family of three-layer schemes IS-3 is specified by means of the following conditions (cf. [1]):

1) $A^*(t) = A(t)$, $A(t) > 0$;
2) $R^*(t) = R(t)$, $R(t) > 0$;
3) $A(t)$ and $R(t)$ are Lipschits-continuous with respect to t with constant c_2, independent of τ and N.

The requirements for the operator $B(t)$ are formulated in the course of the discussion. The basic estimates are obtained without assumptions about the selfadjoint nature of $B(t)$ and its continuity with respect to t.

2. ENERGY IDENTITY

An energy identity, corresponding to scheme (1), is used to verify the *a priori* estimates. Taking into account that

$$y = \frac{\hat{y} + \check{y}}{2} - \frac{\tau^2}{2} y_{\bar{t}t},$$

we rewrite scheme (1) in the form

$$By_{\overset{\circ}{t}} + \tau^2 (R - {}^1/_2 A) y_{\bar{t}t} + {}^1/_2 A(\hat{y} + \check{y}) = \varphi. \qquad (2)$$

We scalar multiply (2) by $2\tau y_{\overset{\circ}{t}} = \tau(y_t + y_{\bar{t}}) = \hat{y} - \check{y}$ and taking into account that $\tau y_{\bar{t}t} = y_t - y_{\bar{t}}$:

$$2\tau(By_{\overset{\circ}{t}}, y_{\overset{\circ}{t}}) + \tau^2((R - {}^1/_2 A)(y_t - y_{\bar{t}}),\ y_t + y_{\bar{t}}) + \qquad (3)$$
$$+ {}^1/_2 (A(\hat{y} + \check{y}),\ \hat{y} - \check{y}) = 2\tau(\varphi, y_{\overset{\circ}{t}}).$$

We shall use

Lemma 3.1

If A and R are selfadjoint operators, we have

$$(A(\hat{y} + \check{y}), \hat{y} - \check{y}) = (A\hat{y}, \hat{y}) - (A\check{y}, \check{y}), \qquad (4)$$

$$((R - {}^1/_2A)(y_t - y_{\bar t}),\ y_t + y_{\bar t}) = ((R - {}^1/_2A)\,y_t,\ y_t) - ((R - {}^1/_2A)\,y_{\bar t},\ y_{\bar t}). \quad (5)$$

This lemma is a consequence of Lemma 1.6, if we put $v = \hat y + \check y$, $u = \hat y - \check y$ and $v = y_t - y_{\bar t}$, $u = y_t + y_{\bar t}$ (replacing A by the operator $R - ½A$).

We add and subtract on the right in (3) (Ay, y), after which we use Lemma 1.5 when $v = \hat y$, $u = y$ and $v = y$, $u = \check y$

$$\begin{aligned}(A(\hat y + \check y),\ \hat y - \check y) &= [(A\hat y,\ \hat y) + (Ay,\ y)] - [(Ay,\ y) + (A\check y,\ \check y)] = \\ &= {}^1/_2\,[(A(\hat y + y),\ \hat y + y) + (A(\hat y - y),\ \hat y - y)] - \\ &\quad - {}^1/_2\,[(A(y + \check y),\ y + \check y) + (A(y - \check y),\ y - \check y)],\end{aligned}$$

so that

$$\begin{aligned}(A(\hat y + \check y),\ \hat y - \check y) = {}^1/_2\,[(A(\hat y + y),\ \hat y + y) + \tau^2(Ay_t,\ y_t)] - \\ - {}^1/_2\,[(A(y + \check y),\ y + \check y) + \tau^2(Ay_{\bar t},\ y_{\bar t})].\end{aligned} \quad (6)$$

Substituting expressions (4) and (6) into (3), we obtain

$$\begin{aligned}2\tau(By_{\overset{\circ}{t}},\ y_{\overset{\circ}{t}}) + \{{}^1/_4(A(\hat y + y),\ \hat y + y) + \tau^2((R - {}^1/_4A)\,y_t,\ y_t)\} = \\ = \{{}^1/_4(A(y + \check y),\ y + \check y) + \tau^2((R - {}^1/_4A)\,y_{\bar t},\ y_{\bar t})\} + 2\tau(\varphi,\ y_{\overset{\circ}{t}}).\end{aligned}$$

The operators A and R depend on t, so that

$$A = A(t), \qquad R = R(t), \qquad t = t_j.$$

Now we make use of the fact that

$$(A(t)\,v,\ v) = (\check A v,\ v) + ((A - \check A)\,v,\ v) = (\check A v,\ v) + \tau(A_{\bar t}v,\ v).$$

As a result we obtain the basic energy identity for a three-layer scheme

$$2\tau(By_{\overset{\circ}{t}},\ y_{\overset{\circ}{t}}) + \hat{\mathscr E} = \mathscr E + \tau F + 2\tau(\varphi,\ y_{\overset{\circ}{t}}), \quad (7)$$

where

$$\begin{aligned}\hat{\mathscr E} &= {}^1/_4(A(\hat y + y),\ \hat y + y) + \tau^2((R - {}^1/_4A)\,y_t,\ y_t), \qquad y_t = \frac{\hat y - y}{\tau}, \\ \mathscr E &= {}^1/_4(\check A(y + \check y),\ y + \check y) + \tau^2((\check R - {}^1/_4\check A)\,y_{\bar t},\ y_{\bar t}), \qquad y_{\bar t} = \frac{y - \check y}{\tau},\end{aligned} \quad (8)$$

$$F = {}^1/_4(A_{\bar t}(y + \check y),\ y + \check y) + \tau^2((R - {}^1/_4A)_{\bar t}\,y_{\bar t},\ y_{\bar t}). \quad (9)$$

Identity (7) holds for any three-layer scheme if it is assumed that $A(t)$ and $R(t)$ are selfadjoint operators.

The operators $A(t)$ and $R(t)$ are, according to the conditions, Lipschits-continuous with respect to t. Therefore the following estimate holds:

$$|F| \leqslant c_2\mathscr{E} + {}^1/_2 c_2\tau^2 (\check{A} y_{\bar{t}}, y_{\bar{t}}). \tag{9'}$$

Indeed,

$$|F| \leqslant \frac{c_2}{4} (\check{A}(y+\check{y}), y+\check{y}) + \tau^2 c_2 (\check{R} y_{\bar{t}}, y_{\bar{t}}) +$$
$$+ {}^1/_4 \tau^2 c_2 (\check{A} y_{\bar{t}}, y_{\bar{t}}) = c_2\mathscr{E} + {}^1/_2 c_2\tau^2 (\check{A} y_{\bar{t}}, y_{\bar{t}}).$$

If $R - \frac{1}{4} A = \tilde{R}$ satisfies the condition

$$|((R - {}^1/_4 A)_{\bar{t}} y, y)| \leqslant c_2 ((\check{R} - {}^1/_4 \check{A}) y, y),$$

instead of (9′) we obtain the estimate

$$|F| \leqslant c_2\mathscr{E}. \tag{9''}$$

From now on we shall assume $A(0) = A(\tau)$, $R(0) = R(\tau)$.

3. ON THE WELL-POSED NATURE OF THE SCHEME

The well-posed nature of the scheme (1), by analogy with Section 2, signifies that it is solvable and stable. We write (1) in the form

$$(B + 2\tau R)\hat{y} = \Phi(y, \check{y}, \varphi),$$

where Φ depends on φ and on the already-known vectors y, $\hat{y}$. Hence it can be seen that the scheme (1) is solvable, if an inverse operator $(B + 2\tau R)^{-1}$ exists. For this it is sufficient that

$$B + 2\tau R > 0. \tag{10}$$

Condition (10) is satisfied in particular when

$$B \geqslant 0.$$

We shall say that scheme (1) is stable with respect to the initial data and with respect to the right-hand part, or simply that it is stable, if numbers $\tau_0 > 0$ and $N_0 > 0$ exist such that when $\tau \leqslant \tau_0$ and $N \geqslant N_0$ for the solution of problem (1) we have one of the estimates:

$$\|y(t+\tau)\|_{(1,t)} \leqslant M_1 \|y(\tau)\|_{(1,0)} + M_2 \max_{0<t'\leqslant t} \|\varphi(t')\|_{(2,t')}, \tag{11}$$

$$\|y(t+\tau)\|_{(1,t)} \leqslant M_1 \|y(\tau)\|_{(1,0)} + M_2 \max_{0<t'\leqslant t} \|\varphi(t')\|_{(2,t')} + M_2' \max_{0<t'\leqslant t} \|\varphi_{\bar{t}}(t')\|_{(2,t')}, \tag{12}$$

where $\|\cdot\|_{(1)}$ is a norm of the kind

$$\|y(t+\tau\|^2_{(1)} = \|y(t+\tau)+y(t)\|^2_{(1_1)} + \|y(t+\tau)-y(t)\|^2_{(1_2)}, \tag{13}$$

$\|\cdot\|_{(1_1)}$, $\|\cdot\|_{(1_2)}$, $\|\cdot\|_{(2)}$ are some norms over the linear system H_N (depending, generally speaking, on t), and M_1, M_2, M_3' are positive constants, independent of τ and N.

If (11) or (12) are satisfied for any τ and N, scheme (1) is called absolutely stable (and it is assumed of course that scheme (1) belongs to IS-3 for any τ and N).

Let us now turn to the sufficient conditions for the stability of scheme (1) from IS-3. For convenience the solution of problem (1) will be represented in the form of the sum $y = \tilde{y} + \bar{y}$, where $\tilde{y}$ is the solution of the problem

$$By_{\overset{\circ}{t}} + \tau^2 R y_{\bar{t}t} + Ay = 0, \qquad 0 < t \in \omega_\tau, \qquad y(0) = y_0, \qquad y(\tau) = y_1, \tag{1a}$$

and $\bar{y}$ is the solution of an inhomogeneous equation with the homogeneous initial conditions

$$By_{\overset{\circ}{t}} + \tau^2 R y_{\bar{t}t} + Ay = \varphi, \quad 0 < t \in \omega_\tau, \qquad y(0) = 0,, \quad y(\tau) = 0. \tag{1b}$$

5. SUFFICIENT CONDITIONS FOR STABILITY. *A priori* ESTIMATES

In deriving the *a priori* estimates we shall start from the identity (7). We note first of all that

$$\hat{\mathscr{E}} \geqslant {}^1/_4 \|\hat{y} + y\|^2_a \quad \text{when } R \geqslant {}^1/_4 A, \tag{14}$$

$$\hat{\mathscr{E}} \geqslant \frac{1-\varepsilon_0}{4} \|\hat{y}+y\|^2_a + \frac{\varepsilon_0}{4}(\|\hat{y}+y\|^2_a + \tau^2\|y_t\|^2_a) \geqslant \tag{14'}$$

$$\geqslant \frac{1-\varepsilon_0}{4}\|\hat{y}+y\|^2_a + \frac{\varepsilon_0}{2}(\|\hat{y}\|^2_a + \|y\|^2_a) \quad \text{when } R \geqslant \frac{1+\varepsilon_0}{4} A,$$

where $\varepsilon_0 > 0$.

We shall use everywhere the norm

$$\|y(t+\tau)\|_{(1,\,t)} = \|\hat{y}\|_{(1)} = \sqrt{(\mathscr{E}(t+\tau))}, \tag{15}$$

where $\mathscr{E}(t+\tau) = \hat{\mathscr{E}}$ is determined by formula (8).

Theorem 3.1

Let the following conditions be satisfied:

$$B \geqslant -c_1\tau^2 A, \tag{16}$$

$$R \geqslant \frac{1+\varepsilon_0}{4} A, \tag{17}$$

where ε_0 = const. > 0, c_1 = const. > 0 is independent of τ and N. Then the scheme (1) from IS-3 is stable for sufficiently small $\tau \leqslant \tau_0(c_1)$ and for the solution of problem (1) estimate (12) holds, in which $\|\cdot\|_{(1)}$ is defined by formula (15),

$$\|\varphi(t)\|_{(2,\,t)} = \sqrt{(A^{-1}(t)\varphi(t), \varphi(t))} = \|\varphi(t)\|_{a^{-1}(t)} \quad \text{or} \quad \|\varphi\|_{(2)} = \|\varphi\|_{a^{-1}}, \tag{18}$$

M_1, M_2 and M_2' depend only on c_1, c_2, t_0, ε_0 and $\tau < 1/4\, c_1$.

Proof. We use condition (16). For this we need Lemma 1.9, in view of which

$$(Bv_{\bar{t}}, v_{\bar{t}}) \geqslant -c_1\tau^2 (Av_{\bar{t}}, v_{\bar{t}}) \geqslant -2c_1 [(Av, v) + (A\check{v}, \check{v})].$$

Substituting here $v = \hat{y} + y$ we shall have

$$2\tau(By_{\mathring{t}}, y_{\mathring{t}}) = \frac{\tau}{2}(B(\hat{y}+y)_{\bar{t}}, \ (\hat{y}+y)_{\bar{t}}) \geqslant \tag{19}$$

$$\geqslant -c_1\tau\,[(A(\hat{y}+y), \hat{y}+y) + (A(y+\check{y}), y+\check{y})].$$

Noting that

$$\mathscr{E} \geqslant \frac{1}{4}\|y+\check{y}\|_a^2 + \frac{1}{4}\varepsilon_0\tau^2\|y_{\bar{t}}\|_a^2 \quad \text{when } R \geqslant \frac{1+\varepsilon_0}{4}A, \tag{20}$$

and using estimate (9′), we obtain

$$|F| \leqslant c_2\left(1 + \frac{2}{\varepsilon_0}\right)\mathscr{E}. \tag{21}$$

We substitute (19) and (21) into (7):

$$-c_1\tau(A(\hat{y}+y),\hat{y}+y)+\hat{\mathscr{E}}\leqslant\left(1+c_2\left(1+\frac{2}{\varepsilon_0}\right)\tau\right)\mathscr{E}+ \\ +c_1\tau(A(y+\check{y}),y+\check{y})+2\tau(\varphi,y_{\mathring{t}}). \tag{22}$$

1. Stability with respect to the initial data. Let $\varphi = 0$. Using (20) and formula (16) from Section 2, we shall have

$$(1-4c_1\tau)\hat{\mathscr{E}} \leqslant (1+c_2(1+2/\varepsilon_0)\tau)\mathscr{E} + 4c_1\tau(1+c_2\tau)\mathscr{E} \leqslant (1+c_2'\tau)\mathscr{E}, \tag{23}$$

where $c_2' = 2c_2(4 + 1/\varepsilon_0) + 4c_1$, while $4c_1\tau < 1$. From (23), by analogy with paragraph 5 Section 2, we obtain

$$\mathscr{E}(t+\tau) \leqslant M_1'\mathscr{E}(\tau), \qquad \tau \leqslant \tau_0, \qquad \tau_0 < 1/4c_1, \tag{24}$$

and M_1' depends on c_1, c_2, ε_0, t_0 and τ_0. From (24) we have

$$\|y(t+\tau)\|_{(1,t)} \leqslant M_1\|y(\tau)\|_{(1,0)}. \tag{25}$$

2. Stability with respect to the right-hand part. Let $y(0) = t(\tau) = 0$. Then $\varepsilon(\tau) = 0$. Using Lemma 21 when $v = \hat{y} + y$ we estimate

$$2\tau(\varphi,y_{\mathring{t}})=\tau(\varphi,(\hat{y}+y)_{\bar{t}})\leqslant\tau(\varphi,\hat{y}+y)_{\bar{t}}+ \\ +\varepsilon_1(A(y+\check{y}),y+\check{y})+\frac{1}{4\varepsilon_1}(A^{-1}\varphi_{\bar{t}},\varphi_{\bar{t}}), \tag{26}$$

where $\varepsilon_1 > 0$ is an arbitrary number. Since $(Av, v) \leqslant (1 + c_2\tau)(\check{A}v, v)$ we have

$$-c_1\tau(A(\hat{y}+y),\hat{y}+y)+\hat{\mathscr{E}}\leqslant\left(1+c_2\left(1+\frac{2}{\varepsilon_0}\right)\tau\right)\mathscr{E}+ \\ +\tau(\varepsilon_1+c_1)(1+c_2\tau)(\check{A}(y+\check{y}),y+\check{y})+\frac{\tau}{4\varepsilon_1}(A^{-1}\varphi_{\bar{t}},\varphi_{\bar{t}})+\tau(\varphi,\hat{y}+y)_{\bar{t}}$$

Using (20) and assuming that $4(\varepsilon_1 + c_1)\tau < 2$, we obtain

$$(1-4c_1\tau)\hat{\mathscr{E}}\leqslant(1+c_2'\tau)\mathscr{E}+(\varphi,\hat{y}+y)-(\check{\varphi},y+\check{y})+\frac{1}{4\varepsilon_1}\tau(A^{-1}\varphi_{\bar{t}},\varphi_{\bar{t}}), \tag{27}$$

where $c_2' = c_2(3 + 2/\varepsilon_0) + 4(\varepsilon_1 + c_1)$. This inequality is solved by analogy with inequality (23). Repeating the reasoning of Section 2.5 we obtain

$$\mathscr{E}(t+\tau)\leqslant\max_{0<t'\leqslant t}[\overline{M}_2(A^{-1}(t')\varphi(t'),\varphi(t'))+\overline{M}_2'(A^{-1}(t')\varphi_{\bar{t}}(t'),\varphi_{\bar{t}}(t'))]. \tag{28}$$

Here it is assumed that $\varphi(0) = \varphi(\tau)$, so that $\varphi_{\bar{t}}(\tau) = 0$. From (24) and (28) we have (12) when $\tau \leqslant \tau_0$, $\tau_0 < 1/4\, c_1'$.

Note 1. If instead of (16) B is non-negative:

$$B \geqslant 0,$$

Theorem 3.1 holds for any $\tau > 0$. This can be seen if in (23) and (27) we put $c_1 = 0$.

Note 2. From (15) it follows that scheme (1) is stable with respect to the right-hand part in the norm

$$\|y(t+\tau)\|^2_{(1,t)} = \|y(t+\tau)\|^2_{a(t)} + \|y(t)\|^2_{a(t)} \tag{29}$$

Estimate (28) remains true if instead of $\mathcal{E}\,(t+\tau)$ we substitute (29).

Note 3. The solvability condition $B + 2\tau R > 0$ is satisfied when $\tau \leqslant {}^{1/_4}\, c_1$, since $B + 2\tau R \geqslant (1 + \varepsilon_0 - 2c_1\tau)\tau A / 2$.

Theorem 3.2

For scheme (1) from IS-3 let the conditions (17) and

$$B \geqslant \delta E, \qquad \delta = \text{const} > 0, \tag{30}$$

be satisfied where E is a unit operator.

Then for the solution of problem (1) we have the estimate (11), in which $\|y(t+\tau)\|_{(1,\,t)}$ is defined by the formulae (15) and (8), while

$$\|\varphi\|_{(2)} = \|\varphi\|;$$

the constants M_1 and M_2 depend only on c_2, ε_0, δ and t_0.

1. Stability with respect to the initial data. In view of $B \geqslant \delta E$ we have

$$2\tau\delta\|y_{\mathring{t}}\|^2 + \hat{\mathscr{E}} \leqslant (1 + c_2(1 + 2/\varepsilon_0)\,\tau)\mathscr{E} + 2\tau(\varphi,\, y_{\mathring{t}}). \tag{31}$$

When $\varphi = 0$ it follows from here that

$$\begin{gathered}\mathscr{E}\,(t+\tau) \leqslant (1 + c_2(1 + 2/\varepsilon_0)\tau)\mathscr{E}\,(t) \leqslant M_1\mathscr{E}\,(\tau),\\ \|y(t+\tau)\|_{(1,\,t)} \leqslant M_1\|y(\tau)\|_{(1,\,0)}.\end{gathered} \tag{32}$$

2. *Stability with respect to the right-hand part*. We substitute the estimate $2\tau(\varphi, y_{\overset{\circ}{t}}) \leqslant 2\tau\delta\|y_{\overset{\circ}{t}}\|^2 + (\tau/2\delta)\|\varphi\|^2$ into (31):

$$\hat{\mathscr{E}} \leqslant \left(1 + c_2\left(1 + \frac{2}{\varepsilon_0}\right)\tau\right)\mathscr{E} + \frac{\tau}{2\delta}\|\varphi\|^2. \tag{33}$$

We apply Lemma 2.3 to (32):

$$\mathscr{E}(t+\tau) \leqslant M_2 \sum_{t'=\tau}^{t} \tau\|\varphi(t')\|^2. \tag{34}$$

Combining the estimates (32) and (34), we obtain (11) with $\|\varphi\|_{(2)} = \|\varphi\|$. The theorem is thus proved.

6. SCHEMES WITH CONSTANT OPERATORS A and R

From the proof of Theorems 3.1 and 3.2 it can be seen that these theorems remain true when

$$R \geqslant {}^1/_4 A,$$

if the operator $\tilde{R} = R - {}^1/_4 A$ is Lipschits-continuous with respect to t (see (9')).

Let us now consider the case when A and R are constant operators. Putting $c_2 = 0$ in (23), (27) and (33), we obtain

$$(1 - 4c_1\tau)\hat{\mathscr{E}} \leqslant (1 + 4c_1\tau)\mathscr{E}, \tag{23'}$$

$$(1 - 4c_1\tau)\hat{\mathscr{E}} \leqslant (1 + 4(\varepsilon_1 + c_1)\tau)\mathscr{E} + (\varphi, \hat{y} + y) - (\check{\varphi}, y + \check{y}) + \frac{1}{4\varepsilon_1}\tau(A^{-1}\varphi_{\bar{t}}, \varphi_{\bar{t}}) \tag{27'}$$

$$\hat{\mathscr{E}} \leqslant \mathscr{E} + \frac{\tau}{2\delta}\|\varphi\|^2, \tag{33'}$$

where

$$\hat{\mathscr{E}} = \|\hat{y} + y\|^2_{(1)} = {}^1/_4\|\hat{y} + y\|^2_a + \tau^2\|y_t\|^2_{\tilde{r}}, \qquad \|v\|^2_{\tilde{r}} = ((R - {}^1/_4 A)v, v),$$

while A and R are independent of t.

From these inequalities it can be seen, for example, that we have

Theorem 3.3

Let A and R be independent of t and

$$B \geqslant 0, \qquad R \geqslant {}^1/_4 A.$$

Then for the solution of problem (1a) the following estimate is valid

$$\|y(t+\tau)\|_{(1)} \leqslant \|y(\tau)\|_{(1)}. \tag{35}$$

If $B \geqslant \delta E$, where δ = const. > 0, for the solution of problem (1) the following inequality is satisfied

$$\|y(t+\tau)\|_{(1)} \leqslant \|y(\tau)\|_{(1)} + \frac{1}{\sqrt{(2\delta)}}\left[\sum_{t'=\tau}^{t} \tau \|\varphi(t')\|^2\right]^{1/2}. \tag{36}$$

7. *A priori* ESTIMATES FOR THE WEIGHTED SCHEME

The following weighted schemes occur very often in practice

$$y_{\overset{\circ}{t}} + Ay^{(\sigma_1,\sigma_2)} = \varphi(t), \qquad 0 < t \in \omega_\tau, \qquad y(0) = y_0, \qquad y(\tau) = y_1, \tag{37}$$

where

$$y^{(\sigma_1,\sigma_2)} = \sigma_1 \hat{y} + (1-\sigma_1-\sigma_2)y + \sigma_2 \check{y}, \qquad A = A(t) > 0.$$

Here σ_1 and σ_2 are real numbers on whose choice the stability and accuracy of the schemes depend.

Let us reduce scheme (37) to the canonical form (1). Using the formulae

$$\hat{y} = y + \tau y_t = y + \frac{\tau}{2} y_{\overset{\circ}{t}} + {}^1/_2\tau^2 y_{\bar{t}t}, \qquad \check{y} = y - {}^1/_2\tau y_{\overset{\circ}{t}} + {}^1/_2\tau^2 y_{\bar{t}t}$$

we find

$$y^{(\sigma_1,\sigma_2)} = y + (\sigma_1 - \sigma_2)\tau y_{\overset{\circ}{t}} + 0.5(\sigma_1+\sigma_2)\tau^2 y_{\bar{t}t}. \tag{38}$$

Substituting (38) into (37) we obtain

$$\begin{gathered}(E + (\sigma_1-\sigma_2)\tau A) y_{\overset{\circ}{t}} + 0.5\tau^2(\sigma_1+\sigma_2) A y_{\bar{t}t} + Ay = \varphi, \\ y(0) = y_0, \qquad y(\tau) = y_1.\end{gathered} \tag{39}$$

A comparison of (39) with (1) gives

$$B = E + (\sigma_1 - \sigma_2)\tau A, \qquad R = 0.5(\sigma_1+\sigma_2)A.$$

Since $A > 0$, therefore A^{-1} exists. Applying A^{-1} to (39):

$$By_{\overset{\circ}{t}} + \tau^2 \widetilde{R} y_{\bar{t}t} + \widetilde{A} y = \widetilde{\varphi}, \qquad \widetilde{\varphi} = A^{-1}\varphi, \qquad y(0) = y_0, \qquad y(\tau) = y_1. \tag{40}$$

where

$$\tilde{B} = A^{-1} + (\sigma_1 - \sigma_2)\tau E, \qquad \tilde{R} = 0.5(\sigma_1 + \sigma_2)E, \qquad \tilde{A} = E. \qquad (41)$$

The schemes (39) and (40) are obviously equivalent to scheme (37). Scheme (39) (or (37)) belongs to IS-3, if $A = A(t)$ is selfadjoint, positive, and Lipschits-continuous with respect to t and $\sigma_1 + \sigma_2 > 0$. Scheme (40) belongs to IS-3 if $A = A(t) > 0$, $\sigma_1 + \sigma_2 > 0$, since the operators $\tilde{A} = E$ and $\tilde{R} = 0.5(\sigma_1 + \sigma_2)E$ are constant and selfadjoint; in this case the operator $\dot{A}$, and consequently $\tilde{B} = A^{-1} + (\sigma_1 - \sigma_2)\tau E$ are not selfadjoint. Depending on the properties of the operator A we can use the general Theorems 3.1 and 3.2 for scheme (37) in the form (40), or in the form (39) (see Section 2.7).

For the solvability of scheme (37) the inverse operator $(B + 2\tau R)^{-1} = (E + 2\sigma_1\tau A)^{-1}$ must exist. Since $E \geqslant A / \|A\|$, therefore $E + 2\sigma_1\tau A \geqslant (1 / \|A\| + 2\sigma_1\tau)A > 0$ when $\sigma_1 > -1 / 2\tau\|A\|$ and scheme (37) is solvable when $\sigma_1 \geqslant 0$. It is easy to note that

$$\|(E + 2\sigma_1\tau A)^{-1}\| \leqslant 1 \quad \text{when} \quad \sigma_1 \geqslant 0, \qquad (42)$$

since $E + 2\sigma_1\tau A \geqslant E$ when $\sigma_1 \geqslant 0$.

Theorem 3.4

Scheme (37) is stable and we have:

1) the *a priori* estimate (12) for any $\tau > 0$ and

$$\|\hat{y}\|_{(1)}^2 = \frac{1}{4}\|\hat{y} + y\|_a^2 + \left(\frac{\sigma_1 + \sigma_2}{2} - \frac{1}{4}\right)\tau^2\|y_t\|_a^2, \qquad \|\varphi\|_{(2)} = \|\varphi\|, \qquad (43)$$

if the following conditions are satisfied

$$A(t) = A^*(t) > 0, \qquad A(t) \text{ Lipschits-continuous with respect to } t, \qquad (44)$$

$$\sigma_1 + \sigma_2 \geqslant 0.5, \qquad \sigma_1 \geqslant \sigma_2; \qquad (45)$$

2) the estimate (12) with $\|y\|_{(1)}$, determinable according to formula (43), and $\|\varphi\|_{(2)} = \|\varphi\|_{a^{-1}}$, if the conditions (44) - (45) are satisfied, and $\sigma_1 + \sigma_2 \geqslant 0.5$, $\sigma_1 \geqslant \sigma_2 - 1 / \tau\|A\| - c_1\tau$, $c_1 = \text{const} \geqslant 0$, $\tau \leqslant \tau_0(c_1)$ is sufficiently small ($\tau_0 = \infty$ when $c_1 = 0$).

The first statement follows from Theorem 3.2, and the second from Theorem 3.1, if we apply them to weighted schemes in the canonical form (39).

If the scheme (37) is symmetrical, i.e. $\sigma_1 = \sigma_2 = \sigma$, for its stability it is sufficient that $\sigma \geqslant 1/4$.

We show that a weighted scheme is stable in another norm without assuming that the operator $A(t)$ is selfadjoint and Lipschits-continuous with respect to t.

Theorem 3.5

If $A(t) > 0$, when (45) holds scheme (37) is stable and the following *a priori* estimate is valid:

$$\| y(t+\tau)\|_{(\bar{1})} \leqslant \| y(\tau)\|_{(\bar{1})} + \max_{0<t'\leqslant t} (M_2 \| A^{-1}(t')\varphi(t')\| + M_2' \|(A^{-1}(t')\varphi(t'))_{\bar{t}}\|), \tag{46}$$

where $M_2 > 0$ and $M_2' > 0$ depend only on t_0,

$$\| y(t+\tau)\|^2_{(\bar{1})} = \|\hat{y}\|^2_{(\bar{1})} = 1/4 \|\hat{y} + y\|^2 + \left(\frac{\sigma_1+\sigma_2}{2} - 1/4\right)\tau^2 \| y_t \|^2. \tag{47}$$

In order to prove this we apply Theorem 3.1 to scheme (40) with the constant operators $\tilde{A} = E$ and $\tilde{R} = 0.5(\sigma_1 + \sigma_2)E$ when $c_1 = 0$.

Theorem 3.6

If the conditions of Theorem 3.5 are satisfied we have for scheme (37) the *a priori* estimate

$$\| y(t+\tau)\|_{(\bar{1})} \leqslant \| y(\tau)\|_{(\bar{1})} + \sqrt{(2(\sigma_1+\sigma_2))} \sum_{t'=\tau}^{t} \tau \| \varphi(t')\|.$$

Scheme (37) is stable with respect to the initial data. For scheme (40) the identity (7), when $\varphi = 0$, takes the form

$$2\tau(\tilde{B}y_{\overset{\circ}{t}}, y_{\overset{\circ}{t}}) + \|\hat{y}\|^2_{(\bar{1})} = \| y \|^2_{(\bar{1})}. \tag{49}$$

Since $\tilde{B} = A^{-1} + \tau(\sigma_1 - \sigma_2)E \geqslant A^{-1} > 0$, therefore

$$\|\hat{y}\|^2_{(\bar{1})} \leqslant \| y \|^2_{(\bar{1})}$$

and consequently

$$\| y(t)\|_{(\bar{1})} \leqslant \| y(t')\|_{(\bar{1})} \quad \text{when } \tau \leqslant t' < t. \tag{50}$$

Let us now consider problem (40) when $y(0) = 0$, $y(\tau) = 0$. Its solution will be sought in the form of a sum

$$y(t) = \sum_{t'=\tau}^{t} \tau Y(t, t'), \tag{51}$$

where $Y(t, t')$ as a function of t satisfies the homogeneous equation (37) and the initial conditions

$$Y(t'+\tau, t') + 2\sigma_1 \tau A(t') Y(t'+\tau, t') = 2\varphi(t'), \qquad Y(t', t') = 0. \tag{52}$$

Substituting (51) into (37) and using the equation for $Y(t, t')$ and condition (52), we ascertain that (51) is a solution of problem (37). When $t = 0$ and $t = \tau$, as can be seen from (51) and (52), we obtain $y(0) = y(\tau) = 0$. For the function $w(t) = Y(t, t')$ according to (50), we have the estimate

$$\|Y(t, t')\|_{(\bar{1})} \leqslant \|Y(t'+\tau, t')\|_{(1)} \qquad \text{for fixed} \qquad t' \leqslant t. \tag{53}$$

From (52) we find $Y(t'+\tau, t') = 2(E + 2\sigma_1\tau A)^{-1}\varphi(t')$. Since $\sigma_1 > 0$, therefore according to (42), we have

$$\|Y(t'+\tau, t')\| \leqslant 2\|\varphi(t')\|. \tag{54}$$

In view of the initial condition $Y(t', t') = 0$ and

$$\begin{aligned} \|Y(t'+\tau, t')\|^2_{(\bar{1})} &= {}^1/_4 \|Y(t'+\tau, t)\|^2 + \\ &+ \left(\frac{\sigma_1+\sigma_2}{2} - {}^1/_4\right) \|Y(t'+\tau, t')\|^2 = \frac{\sigma_1+\sigma_2}{2} \|Y(t'+\tau, t')\|^2, \end{aligned}$$

$$\|Y(t'+\tau, t')\|_{(\bar{1})} = \sqrt{(0.5(\sigma_1+\sigma_2))}\, \|Y(t'+\tau, t')\|. \tag{55}$$

Substituting (55) and (54) into the right-hand part of the inequality

$$\|y(t)\|_{(\bar{1})} \leqslant \sum_{t'=\tau}^{t-\tau} \tau \|Y(t, t')\|_{(\bar{1})} \leqslant \sum_{t'=\tau}^{t-\tau} \tau \|Y(t'+\tau, t')\|_{(\bar{1})},$$

we obtain an estimate for the solution of problem (37) when $y(0) = y(\tau) = 0$:

$$\|y(t+\tau)\|_{(\bar{1})} \leqslant \sqrt{(2(\sigma_1+\sigma_2))} \sum_{t'=\tau}^{t} \tau \|\varphi(t')\|. \tag{56}$$

Combining (50) and (56), we arrive at (48). The theorem is proved.

Theorem 3.7

If $A(t) = A^*(t) > 0$ and

$$\sigma_1 \geqslant \sigma_2 - \frac{1-\varepsilon}{\tau\|A\|}, \qquad \sigma_1 + \sigma_2 \geqslant 1/2, \qquad \varepsilon \in (0,1], \tag{57}$$

then for the scheme (37) the following *a priori* estimate is valid:

$$\|y(t+\tau)\|_{(\bar{1})} \leqslant \|y(\tau)\|_{(\bar{1})} + \frac{1}{\sqrt{(2\varepsilon)}} \Big[\sum_{t'=\tau}^{t} \tau \|\varphi(t')\|^2_{a^{-1}(t')}\Big]^{1/2}. \tag{58}$$

It is of course sufficient to prove this theorem when $y(0) = y(\tau) = 0$. We write for scheme (40) the identity

$$2\tau(\tilde{B}y_{\overset{\circ}{t}}, y_{\overset{\circ}{t}}) + \|\hat{y}\|^2_{(\bar{1})} = \|y\|^2_{(\bar{1})} + 2\tau(\tilde{\varphi}, y_{\overset{\circ}{t}}), \text{where } \tilde{\varphi} = A^{-1}\varphi. \tag{59}$$

Substituting here the estimate (80) from Section 2 for $2\tau(A^{-1}\varphi, y_{\overset{\circ}{t}})$ and assuming that $\tilde{B} = A^{-1} + (\sigma_1 - \sigma_2)\tau E \geqslant \varepsilon A^{-1}$ when $\sigma_1 \geqslant \sigma_2 - (1-\varepsilon)/\tau\|A\|$ we obtain

$$\|\hat{y}\|^2_{(\bar{1})} \leqslant \|y\|^2_{(\bar{1})} + \frac{\tau}{2\varepsilon}(A^{-1}\varphi, \varphi).$$

Hence (58) follows.

Thus if $A(t)$ is selfadjoint and Lipschits-continuous with respect to t, we have for scheme (37) the *a priori* estimates with respect to the norm (43). If, however, $A(t)$ is an arbitrary positive operator, the estimates are valid with respect to the norm (47). Then the requirement that $A(t)$ be selfadjoint (with Lipschits-continuity for t) enables us to obtain the estimate for $\|y\|_{(\bar{1})}$ through $\|\varphi\|_{a^{-1}}$, while $\|y\|_{(1)}$ is estimated, according to Theorem 3.1, through $\|\varphi\|_{a^{-1}}$ and $\|\varphi_{\bar{t}}\|_{a^{-1}}$.

Theorem 3.8

Let $A > 0$ be a constant operator and $\sigma_1 \geqslant \sigma_2$, $\sigma_1 + \sigma_2 \geqslant 1/2$. Then for the scheme (37) we have the estimate (12), where

$$\|y(t+\tau)\|^2_{(1)} = \|\hat{y}\|^2_{(1)} = 1/4\|A(\hat{y}+y)\|^2 + \left(\frac{\sigma_1+\sigma_2}{2} - 1/4\right)\tau^2\|Ay_t\|^2, \tag{60}$$

$\|\varphi\|_{(2)} = \|\varphi\|$, and $M_1 = 1, M_2 > 0$ depend only on t_0.

Scalar multiplying (39) by $2\tau A y_{\mathring{t}}$ and reasoning in the same way as in paragraph 4 of the present section, we obtain the identity

$$2\tau(Ay_{\mathring{t}},\, y_{\mathring{t}}) + 2(\sigma_1 - \sigma_2)\tau^2\|Ay_{\mathring{t}}\|^2 + \|\hat{y}\|^2_{(1)} = \|y\|^2_{(1)} + 2\tau(\varphi,\, Ay_{\mathring{t}}). \quad (61)$$

When $\varphi = 0$ we have immediately $\|\hat{y}\|_{(1)} \leqslant \|y\|_{(1)}$. Substituting the estimate

$$2\tau(\varphi,\, Ay_{\mathring{t}}) = 2\tau(\varphi,\, A(\hat{y}+y)_{\bar{t}}) \leqslant \tau(\varphi,\, A(\hat{y}+y))_{\bar{t}} + \tau\varepsilon\|y\|^2_{(1)} + \frac{\tau}{\varepsilon}\|\varphi_{\bar{t}}\|^2$$

into (61), we have

$$\|\hat{y}\|^2_{(1)} \leqslant (1+\tau\varepsilon)\|y\|^2_{(1)} + (\varphi,\, A(\hat{y}+y)) - (\check{\varphi},\, A(y+\check{y})) + \frac{1}{\varepsilon}\tau\|\varphi_{\bar{t}}\|^2. \quad (62)$$

Hence, by analogy with paragraph 5 of this section, we obtain the required estimate, if we take into account that

$$|(\varphi,\, A(\hat{y}+y))| \leqslant \|\varphi\|\|A(\hat{y}+y)\| \leqslant 2\|\varphi\|\|\hat{y}\|_{(1)} \leqslant {}^1/_2\|\hat{y}\|^2_{(1)} + 2\|\varphi\|^2.$$

The fact that the operator A is selfadjoint is not utilized. The theorem remains true for $A = A(t)$ if $\|A_{\bar{t}}y\| \leqslant c_2\|\check{A}y\|$. If $A^* = A > 0$, we have estimate (11), where $\|\varphi\|_{(2)} = \|\varphi\|_a$

8. *A priori* ESTIMATES FOR A THREE-LAYER SCHEME OF THE SECOND TYPE

Let us now consider a three-layer scheme in the second canonical form

$$y_{\bar{t}t} + \tau^2 R y_{\bar{t}t} + By_{\mathring{t}} + Ay = \varphi, \qquad 0 < t \in \omega_\tau, \qquad y(0) = y_0, \qquad y(\tau) = y_1. \quad (63)$$

The energy identity (7) in this case takes the form

$$2\tau(By_{\mathring{t}},\, y_{\mathring{t}}) + \|\hat{y}\|_*^2 = \|y\|_*^2 + \tau F + 2\tau(\varphi,\, y_{\mathring{t}}), \quad (64)$$

where F has the form (8), and

$$\|\hat{y}\|_*^2 = {}^1/_4\|\hat{y}+y\|_a^2 + \tau^2((R - {}^1/_4 A)y_t,\, y_t) + \|y_t\|^2. \quad (65)$$

Hence it can be seen that $\|\hat{y}\|_*^2 \geqslant {}^1/_4\|\hat{y}+y\|_a^2 + \|y_t\|^2$ when $R \geqslant {}^1/_4 A$.

Theorem 3.9

Let the conditions $B \geqslant 0$, $R^* = R > 0$, $A^* = A > 0$ be satisfied, and let A and R be constant operators and $R \geqslant \frac{1}{4} A$. Then for the scheme (63) we have the *a priori* estimate

$$\|y(t+\tau)\|_* \leqslant \|y(\tau)\|_* + e\sqrt{t_0}\left[\sum_{t'=\tau}^{t} \|\varphi(t')\|^2\right]^{1/2}. \tag{66}$$

Proof. Since A and R are independent of t, we have $F = 0$. When $\varphi = 0$ identity (64) takes the form $\|\hat{y}\|_*^2 \leqslant \|y\|_*^2$, i.e. for problem (63) when, $\varphi = 0$ we have the estimate

$$\|y(t+\tau)\|_* \leqslant \|y(t)\|_*. \tag{67}$$

Let $y(0) = y(\tau) = 0$ and consequently $\|y(\tau)\|_* = 0$. We transform

$$2\tau(\varphi, y_{\overset{\circ}{t}}) = \tau(\varphi, y_t + y_{\bar{t}}) \leqslant \tau\varepsilon(\|y_t\|^2 + \|y_{\bar{t}}\|^2) + \frac{1}{2\varepsilon}\tau\|\varphi\|^2 \leqslant$$

$$\leqslant \tau\varepsilon(\|\hat{y}\|_*^2 + \|y\|_*^2) + \frac{1}{2\varepsilon}\tau\|\varphi\|^2.$$

Substituting this estimate into $\|\hat{y}\|_*^2 \leqslant \|y\|_*^2 + 2\tau(\varphi, y_{\overset{\circ}{t}})$, we obtain

$$(1-\tau\varepsilon)\|y\|_*^2 \leqslant (1+\tau\varepsilon)\|y\|_*^2 + \frac{1}{2\varepsilon}\tau\|\varphi\|^2.$$

Choosing $\varepsilon = \frac{1}{2} t_0$ we arrive at the estimate

$$\|y(t+\tau)\|_*^2 \leqslant e^2 t_0 \sum_{t'=\tau}^{t} \tau\|\varphi(t')\|^2. \tag{68}$$

Combining (67) and (68), we arrive at the inequality (66).

Using the techniques developed earlier it is easy to obtain an analogous estimate for the case of variable operators $A(t)$ and $R(t)$. We remark that for the scheme (60) Theorems 3.1 and 3.2 are valid.

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