SOME PROBLEMS OF DIFFERENCE SCHEME THEORY*

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1. DIFFERENCE schemes for solving various problems of mathematical physics have appeared in recent years. Economic methods for solving multidimensional problems have been particularly developed (alternating direction schemes, decomposition schemes, locally one-dimensional schemes etc.). A large number of different algorithms is now available for solving a given problem. Special interest is thus aroused by the theoretical and experimental comparison of difference schemes, the development of general principles for constructing families of difference schemes possessing given properties, and finding the schemes in the family that satisfy auxiliary optimality requirements (as regards accuracy, economy etc.).

There are various schemes which give the same results (are algebraically identical) provided the right-hand sides and boundary conditions are matched in a certain way. In this case our choice of one scheme rather than another must be based on practical convenience. This situation is typical for economic methods of solving multidimensional problems. Several papers [1, 2, 3, 4] have been concerned with comparing economic methods.

A comparison in [1, 2] with the operator decomposition method (for the equation of heat conduction in a rectangular region), to which the methods of [5 - 7] are reducible after eliminating the intermediate values, showed that the order of approximation of these methods depends, in the case of non-stationary boundary conditions (and for [7], in the case of stationary conditions also) on the method of specifying the boundary conditions at the intermediate step. It is shown that, by varying the right-hand side at the boundary base-points of the mesh, we can

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arrange for method [5] to be the same as the operator decomposition method $O(\tau^2 + h^2)$. In [7], boundary conditions are written for the method of [5] for the intermediate value $y^{n+\frac{1}{2}}$, leading to the same result. If the right-hand sides or boundary conditions are suitably matched, the methods of [2, 5, 7] and certain others can be treated, as shown in [3, 4], as computational algorithms realizing the same scheme with split operator. All these schemes are algebraically identical.

Additive (locally one-dimensional) schemes forming a sequence of intermediate (in the elementary case one-dimensional) schemes were considered in [8] for the parabolic equation in Banach space with the operator

$$A = \sum_{\alpha=1}^{p} A_\alpha \quad \text{or} \quad A = \sum_{\alpha=1}^{p} A_{\alpha3},$$

where A_α, $A_{\alpha\beta}$ are operators of simpler structure than A. For the additive scheme, the error of approximation is the sum of the approximation errors [9] for all the intermediate schemes (additivity of the approximation). From this point of view, the alternating-direction-implicit (ADI) scheme of [5] is a locally one-dimensional scheme of a special kind.

2. A basic problem in difference scheme theory is to find sufficient conditions for stability.

If we investigate difference schemes as operator equations in abstract space H with a scalar product, we can use the method of energy inequalities [10] to obtain sufficient conditions for two-layer and multi-layer schemes in a form that can easily be checked. We only make use here of general information on the operators such as positiveness (semiboundedness from below) and selfadjointness. For instance, the the two-layer scheme

$$By_{\bar{t}} + Ay = 0, \quad 0 < t = j\tau, \quad y(0) = y_0, \quad y_{\bar{t}} = (y - \check{y})/\tau,$$
$$y = y^{j+1}, \quad \check{y} = y^j,$$

where A and B are linear operators in H, and $\tau > 0$ is a constant, the sufficient conditions for stability, with which $\|y^j\|_a \leqslant \|y_0\|_a$ ($\|y\|_a = \overline{\sqrt{(Ay, y)}}$, A is independent of t), are

$$(By, y) \geqslant 0.5\tau(Ay, y), \quad (Ay, y) \geqslant 0, \quad (By, y) \geqslant 0,$$

if A is selfadjoint. Hence, in particular, we obtain for the scheme

$$y_{\bar{t}} + A(\sigma y + (1 - \sigma)\check{y}) = 0, \quad 0 < t = j\tau, \ldots, \quad y(0) = y_0,$$

the sufficient condition for stability $\sigma \geqslant 0.5 - 1/\tau\|A\|$, since $B = E + \sigma\tau A$.

More exact stability conditions involving the initial data are required to prove the convergence of iterational schemes.

3. In the present paper, we illustrate the above topics of the general theory (algebraic identity and stability) from the example of the iterational schemes for solving the stationary equation $(A_1 + A_2)v = f$, where A_1, A_2 are linear positive-definite operators in real Hilbert space H.

We show in Section 1 that the ADI scheme is algebraically identical with three extremely simple locally one-dimensional schemes.

The rate of convergence of the ADI scheme was investigated in [11] for the case when A_1 and A_2 are arbitrary positive definite operators.

In Section 2, which is a development of [11], we obtain new estimates of the rate of convergence of the ADI scheme. In addition to assuming that A_1 and A_2 are positive definite,

$$(Ay, y) \geqslant \delta\|y\|^2, \qquad \delta = \text{const} > 0,$$

we use the condition

$$\|Ay\|^2 \leqslant \Delta(Ay, y), \qquad \Delta = \text{const} > 0,$$

which involves no loss of generality, since, for any positive definite operator A,

$$\Delta \leqslant \frac{1}{\delta}\|A\|^2 \qquad (\Delta = \|A\| \text{ if } A^* = A).$$

In Section 3, we show by examples that Δ is easily evaluated for certain non-selfadjoint operators.

When assessing the rate of convergence of the LTS, the main role is played by a bound on the norm of the transition operator (Theorem 5):

$$\|(E + \sigma\tau A)^{-1}(E - (1 - \sigma)\tau A)\| \leqslant \sqrt{\rho},$$

$$\rho = \frac{1 - 2(1 - \sigma)\tau\delta + (1 - \sigma)^2 \tau^2\delta\Delta}{1 + 2\sigma\tau\delta + \sigma^2\tau^2\delta\Delta},$$

which implies, in particular, the stability condition $\sigma \geqslant 0.5 - 1/\tau\Delta$ for the above-mentioned schemes with weights, without the assumption

that A is selfadjoint.

We show that, when using the ADI scheme, the initial error can be re-duced in the ratio $1/\varepsilon$ ($\varepsilon > 0$ is the given accuracy) simply by perform-ing

$$\nu(\varepsilon) \geqslant \ln \frac{1}{\varepsilon} \left[\ln \frac{1 + \sqrt{\eta}}{1 - \sqrt{\eta}} \right]^{-1}$$

iterations, where $\eta = \delta / \Delta$ ($\delta = \min(\delta_1, \delta_2)$, $\Delta = \max(\Delta_1, \Delta_2)$ with $\tau = 1/\sqrt{\delta\Delta}$.

Notice that other authors (e.g. [14, 15, 17, 18]) studied the con-vergence of the ADI scheme on the assumption that A_1 and A_2 are finite-dimensional, positive definite, selfadjoint and commutative.

Some applications of the general theory of Section 2 are also con-sidered in Section 3.

We show that the ADI scheme is convergent for the Dirichlet differ-ence problem in an arbitrary region, whether the elliptic operator with variable coefficients on arbitrary meshes is selfadjoint or not. It follows from the bound for $\nu(\varepsilon)$ that $\nu(\varepsilon) \approx 1/h \ln 1/\varepsilon$ as $h \to 0$, where h is the mesh interval.

We also consider the application of the general theory to the trans-port equation. An iterational scheme was proposed in [12, 13] which may be seen from Section 1 to be algebraically identical with the ADI scheme. Hence its convergence for any τ, and the estimate of its rate of con-vergence, follow from [11] (we only need to use the fact that A_1 and A_2 are positive definite, and evaluate $\|A_1\|$, $\|A_2\|$). The convergence rate estimate of [11] can be improved by using Theorem 5 (of Section 1) while recalling that A_2 is selfadjoint.

Comparison with the method of simple iteration shows that extra study is needed before we can say that the ADI scheme is worth using as an economic algorithm for the radiation transport equation.

In paragraph 4 of Section 3 we consider the ADI scheme for solving the Dirichlet difference problem in an arbitrary p-dimensional region, obtained by splitting the Laplace difference operator into a sum of non-selfadjoint operators. V.K. Saul'ev used this scheme (with $p = 2$ and a rectangular region) for the equation of heat conduction (alternating method). We show that, in this case also, $\nu(\varepsilon) = O(1/h \ln 1/\varepsilon)$.

We have confined ourselves to a few examples in order to illustrate how the convergence rate estimates of Section 2 should be applied (though the results of Section 3 are of independent interest).

1. Comparison of various iterational schemes

Given the equation

$$(A_1 + A_2)v = f, \tag{1}$$

where A_1, A_2 are linear operators in real Hilbert space H. We can solve it by using the iterational ADI scheme

$$(E + \tau A_1)\bar{y} = (E - \tau A_2)y^k + \tau f,$$

$$(E + \tau A_2)y^{k+1} = (E - \tau A_1)\bar{y} + \tau f, \qquad k = 0, 1, 2, \ldots, \tag{2}$$

$$y^0 = y_0 \in H,$$

where E is the unit operator, k the number of iterations, $\bar{y} = y^{k+\frac{1}{2}}$ the intermediate value (subiteration), and τ the relaxation parameter ($\tau = \tau_k$ is in general dependent on k). In addition to (2), we consider the scheme

$$y_{(1)} = (E - \tau A_2)y^k + \tau f_1, \qquad (E + \tau A_1)y_{(2)} = y_{(1)} + \tau(f - f_1),$$

$$y_{(3)} = (E - \tau A_1)y_{(2)} + \tau f_2, \qquad (E + \tau A_2)y^{k+1} = y_{(3)} + \tau(f - f_2), \tag{3}$$

$$k = 0, 1, \ldots, \qquad y^0 = y_0,$$

where f_1 and f_2 are arbitrary elements of H.

Whatever f_1, f_2, scheme (3) is algebraically identical with scheme (2), i.e. the y^{k+1} found from (2) and (3) are the same. For, if we add the first to the second of equations (3), and the third to the fourth, we can eliminate $y_{(1)}$, $y_{(3)}$ and obtain equation (2) with $\bar{y} = y_{(2)}$.

We show that the schemes

$$\bar{y}_{(1)} = (E - \tau A_2)\bar{y}^k + \tau f_1, \qquad (E + \tau A_1)\bar{y}_{(2)} = (E - \tau A_1)\bar{y}_{(1)} + 2\tau(f - f_1),$$

$$\tag{4}$$

$$(E + \tau A_2)\bar{y}^{k+1} = \bar{y}_{(2)} + \tau f_1, \qquad k = 0, 1, 2, \ldots, \qquad \bar{y}^0 = y_0,$$

and

$$\tilde{y}_{(1)} = (E - \tau A_2)\tilde{y}^k + \tau f_1, \qquad \tilde{y}_{(2)} = (E - \tau A_1)\tilde{y}_{(1)} + \tau \tilde{f}_2,$$

$$\tag{5}$$

$$(E + \tau A_1)\tilde{y}_{(3)} = \tilde{y}_{(2)} + \tau[2(f - f_1) - \tilde{f}_2], \qquad (E + \tau A_2)\tilde{y}^{k+1} = \tilde{y}_{(3)} + \tau f_1,$$

$$\tilde{y}^0 = y_0.$$

are algebraically identical with scheme (2). For this, we only need to show that (4) and (5) are equivalent to scheme (3) with $f_2 = f_1$, provided $\bar{y}^k = y^k$, $\tilde{y}^k = y^k$. Comparing (4) and (3), we see that the first and last equations of (3) and (4) are the same, $\bar{y}_{(1)} = y_{(1)}$. We show that $\bar{y}_{(2)} = y_{(3)}$. Introducing $\bar{y} = 0.5(\bar{y}_{(1)} + \bar{y}_{(2)})$, we can replace the second equation of (4) by the two equations

$$\bar{y}_{(2)} = (E - \tau A_1)\bar{y} + \tau(f - f_1), \qquad (E + \tau A_1)\bar{y} = \bar{y}_{(1)} + \tau(f - f_1). \tag{6}$$

The resulting $\bar{y}_{(2)}$ is the same as obtained from the second of equations (4). For, subtracting the second of equations (6) from the first, we get $\bar{y} = 0.5(\bar{y}_{(1)} + \bar{y}_{(2)})$. After substituting this expression in the first of equations (6), we get the second of equations (4).

We have thus proved the algebraic identity of (4) with (3), and hence with the ADI scheme (2).*

Adding the second and third of equations (5), we get

$$(E + \tau A_1)\tilde{y}_{(3)} = (E - \tau A_1)\tilde{y}_{(1)} + 2\tau(f - f_1).$$

Since $\tilde{y}_{(1)} = \bar{y}_{(1)}$ we have $\tilde{y}_{(3)} \doteq \bar{y}_{(2)}$ and schemes (5), (4) are identical.

Schemes (3), (4) and (5) are thus algebraically identical with scheme (2), whatever the linear operators A_1, A_2 (this linearity was used in passing from (6) to the second of equations (4)). We are assuming here that $(E + \tau A_\alpha)^{-1}$, $\alpha = 1$, 2 exist. This is the only restriction on A_α.

On the other hand, the solution of problem (2) satisfies

$$(E + \tau A_1)(E + \tau A_2)y^{k+1} = (E - \tau A_1)(E - \tau A_2)y^k + 2\tau f. \tag{7}$$

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* The equivalence of schemes (3) and (4) can be proved by eliminating the intermediate points $y_{(1)}$, $y_{(2)}$, $y_{(3)}$ in (3) and $\bar{y}_{(1)}$, $\bar{y}_{(2)}$ in (4). Our present method of proof, proposed by I.V. Fryazinov, is simpler.

To prove this, we obtain from (2)

$$2\bar{y} = (E + \tau A_2)y^{k+1} + (E - \tau A_2)y^k \qquad (8)$$

and substitute this in the first or second of equations (2). If (7) has a unique solution, then (7), (2) are algebraically identical. For substituting $(E + \tau A_2)y^{k+1} = 2\bar{y} - (E - \tau A_2)y^k$ in (7), we obtain the first of equations (2); the second of equations (2) follows from this and from (8).

It was shown in [1, 4] that the decomposing operator scheme (7) is equivalent to several other schemes, in particular the ADI scheme. If A_1, A_2 are commutative, (7) is equivalent to the scheme [3], [1, 2]

$$(E + \tau A_1)\bar{y} = (E - \tau A_1)y^k, \qquad (9)$$

$$(E + \tau A_2)y^{k+1} = (E - \tau A_2)\bar{y} + 2\tau\varphi, \quad k = 0,1,\ldots, \quad y^0 = y_0, \quad (10)$$

where φ is the solution of the equation

$$(E + \tau A_1)\varphi = f \qquad (11)$$

[1, 4].

Thus there are many schemes equivalent to the ADI scheme (2).

When writing schemes in operator form, the question of non-homogeneous boundary conditions for the intermediate values $\bar{y}$, y_1 etc. does not arise, since the difference operators A_1, A_2 are now defined in the space of functions satisfying homogeneous boundary conditions, while the non-homogeneous conditions are taken into account by the right-hand side.

The schemes considered above can also be used for solving non-stationary equations, e.g. the equation of heat conduction (with interval 2τ).

Depending on the method of specifying the boundary conditions for the intermediate values ($\bar{y}$, $y_{(1)}$, $y_{(2)}$ etc.), we obtain different schemes, which are only equivalent when the boundary conditions are matched [1 - 4].

2. Convergence of the iterational scheme

1. Let v be a solution of the initial problem (1), y^{k+1} a solution of

problem (2). For the difference $z^{k+1} = y^{k+1} - v$ the Cauchy problem

$$(E + \tau A_1)\bar{z} = (E - \tau A_2)z^k, \qquad \bar{z} = \vec{y} - v, \qquad z^0 = y_0 - v,$$
$$(E + \tau A_2)z^{k+1} = (E - \tau A_1)\bar{z}, \qquad k = 0,1\ldots, \qquad (12)$$
$$z^0 = z_0 = y_0 - v.$$

is obtained. To assess the rate of convergence of method (12), we need to know how fast the initial discrepancy $z_0 = y_0 - v$ is damped as the number of iterations increases. For this, we have to investigate the stability of problem (12) with respect to its initial data.

Let $(.)$ be the scalar product, while $\|y\| = \sqrt{(y,y)}$ is the norm in real Hilbert space H.

We shall assume everywhere that the operators A_α are positive definite

$$(A_\alpha y, y) \geqslant \delta_\alpha \|y\|^2, \qquad \alpha = 1, 2, \qquad \delta_\alpha = \text{const} > 0, \qquad y \in \text{H}. \qquad (13)$$

The convergence of the ADI scheme was investigated in [14 - 15] in complex finite - dimensional space $\tilde{H}_N$. By a theorem of functional analysis [16, Chap. V], if a linear operator specified in complex Hilbert space is positive, it must be selfadjoint.

The ADI scheme was applied in [11] as an economic method of solving systems of differential and algebraic equations. It was assumed that the matrix A of the system can be written as $A = A_1 + A_2$, where A_1, A_2 are triangular matrices satisfying condition (13) only. The method of energy inequalities was used to prove the convergence, using only the fact that A_1 and A_2 are operators in a space with a scalar product satisfying (13).

Theorem 3 of [11] shows that the iterational scheme (2) is convergent at the rate of a geometric progression whatever $\tau > 0$, the ratio ρ of the progression being dependent on δ_1, δ_2 and $\|A_1\|$, $\|A_2\|$. It was shown how τ can be chosen so as to minimize ρ. The proof of Theorem 3 was performed for the case $D = E/\tau$, when equation (23) of [11] becomes (12).

To assist our future treatment, we state a corollary to Theorem 3 of [11] as a separate theorem, with a change of notation.

Theorem 1

If A_α are positive definite operators, then scheme (2) is convergent for any $\tau > 0$, so that the bound

$$\| z^{k+1} \|_{(2)} \leqslant \rho \| z^k \|_{(2)}, \qquad k = 0, 1, 2, \ldots, \qquad (14)$$

holds for the solution of problem (12), where

$$\|z\|_{(\alpha)}^2 = \|z\|^2 + \tau^2 \|A_\alpha z\|^2, \qquad \rho^2 = \rho_1^2 \rho_2^2, \qquad \rho_\alpha^2 = \frac{1 - \varkappa_\alpha}{1 + \varkappa_\alpha}, \qquad (15)$$

$$\varkappa_\alpha = \frac{2\tau\delta_\alpha}{1 + \tau^2 \|A_\alpha\|^2}, \qquad \alpha = 1, 2. \qquad (16)$$

The proof given in [11] is as follows. We evaluate the square of the norm of both sides of (12) and recall that

$$\|(E \pm \tau A_\alpha)z\|^2 = \|z\|_{(\alpha)}^2 \pm 2\tau(A_\alpha z, z), \qquad 2\tau(A_\alpha z, z) \geqslant \varkappa_\alpha \|z\|_{(\alpha)}^2$$

(Lemma 3 of [11]), since

$$\|z\|_{(\alpha)}^2 = \|z\|^2 + \tau^2 \|A_\alpha z\|^2 \leqslant (1 + \tau^2 \|A_\alpha\|^2) \|z\|^2 \leqslant \frac{1}{\delta_\alpha}(1 + \tau^2 \|A_\alpha\|^2)(A_\alpha z, z).$$

We now get

$$(1 + \varkappa_1)\|\bar{z}\|_{(1)}^2 \leqslant (1 - \varkappa_2)\|z^k\|_{(2)}^2, \qquad (1 + \varkappa_2)\|z^{k+1}\|_{(2)}^2 \leqslant (1 - \varkappa_1)\|\bar{z}\|_{(1)}^2,$$

$$\|\bar{z}\|_{(1)}^2 \leqslant \frac{1 - \varkappa_2}{1 + \varkappa_1}\|z^k\|_{(2)}^2, \qquad \|z^{k+1}\|_{(2)}^2 \leqslant \frac{1 - \varkappa_1}{1 + \varkappa_2}\|\bar{z}\|_{(1)}^2 \leqslant \rho_1^2 \rho_2^2 \|z^k\|_{(2)}^2.$$

From (14), $\|z^\nu\|_{(2)} \leqslant' \rho^\nu \|z_0\|_{(2)}$. Let $\varepsilon > 0$ be the required accuracy, so that $\|z^\nu\|_{(2)} \leqslant \varepsilon \|z_0\|_{(2)}$. It follows from this and the previous inequality that, to reduce the initial error in the ratio $1/\varepsilon$, $\nu(\varepsilon)$ iterations are needed, where

$$\nu(\varepsilon) \geqslant \ln \frac{1}{\varepsilon} \Big/ \ln \frac{1}{\rho}. \qquad (17)$$

Corollary. Let $\delta = \min(\delta_1, \delta_2)$, $\|A\| = \max(\|A_1\|, \|A_2\|)$. Then $\min \rho(\tau) = \rho(\tau^*) = \bar{\rho}$, $\tau^* = 1/\|A\|$, where $\rho = (1 - \eta)/(1 + \eta)$, $\eta = \delta/\|A\|$.

2. We now assume the following conditions in addition to (13):

$$\|A_\alpha y\|^2 \leqslant \Delta_\alpha(A_\alpha y, y), \qquad \Delta_\alpha = \text{const} > 0, \qquad \alpha = 1, 2, \qquad y \in H. \quad (18)$$

Lemma 1

For a linear positive $((Ay, y) \geqslant 0)$ and selfadjoint operator A,

$$\|Ay\|^2 \leqslant \|A\| (Ay, y), \quad y \in H. \tag{19}$$

In fact, putting $z = Ay$ in the generalized Cauch inequality: $(Ay, z)^2 \leqslant (Ay, y)(Az, z)$, we get

$$\|Ay\|^4 = (Ay, z)^2 \leqslant (Ay, y)(Az, z) \leqslant \|A\|\|z\|^2(Ay, y) =$$
$$= \|A\|\|Ay\|^2(Ay, y).$$

Cancelling by $\|Ay\|^2$, (19) follows.

Comparing (19) and (18), we see that, for a selfadjoint operator A_α,

$$\Delta_\alpha = \|A_\alpha\|.$$

If A_α is any positive definite operator, then $\Delta_\alpha \leqslant \|A_\alpha\|^2 / \delta_\alpha$, since

$$\|A_\alpha y\|^2 \leqslant \|A_\alpha\|^2 \|y\|^2 \leqslant \|A_\alpha\|^2 \frac{1}{\delta_\alpha}(A_\alpha y, y).$$

Condition (18) is thus satisfied for any positive definite operator.

Theorem 2

Let A_1, A_2 satisfy conditions (13), (18). Formulae (14), (15) now hold, provided

$$\varkappa_\alpha = \frac{2\tau\delta_\alpha}{1 + \tau^2\delta_\alpha\Delta_\alpha}, \quad \alpha = 1, 2. \tag{20}$$

It is sufficient to show that $2\tau(A_\alpha z, z) \geqslant \varkappa_\alpha \|z\|_{(\alpha)}{}^2$, where $\varkappa_\alpha$ is given by (20). For,

$$\|z\|_{(\alpha)}{}^2 = \|z\|^2 + \tau^2\|A_\alpha z\|^2 \leqslant \|z\|^2 + \tau^2\Delta_\alpha(A_\alpha z, z) \leqslant \left(\frac{1}{\delta_\alpha} + \tau^2\Delta_\alpha\right)(A_\alpha z, z)$$

and

$$(A_\alpha z, z) \geqslant \delta_\alpha(1 + \tau^2\delta_\alpha\Delta_\alpha)^{-1}\|z\|_{(\alpha)}^2.$$

Theorem 3

Let conditions (13) and (18) be satisfied. Then the iterational process (2) with parameter τ equal to

$$\tau = \tau^* = 1/\sqrt{\delta\Delta}, \quad \delta = \min(\delta_1, \delta_2), \quad \Delta = \max(\Delta_1, \Delta_2),$$

is convergent at the rate of the geometric progression with ratio

$$\bar{\rho} = \frac{1 - \sqrt{\eta}}{1 + \sqrt{\eta}}, \qquad \eta = \frac{\delta}{\Delta}, \tag{21}$$

so that

$$\|y^{k-1} - v\|_{(2)} \leqslant \bar{\rho}^{k+1} \|y_0 - v\|_{(2)}. \tag{22}$$

To prove this theorem, we only need to observe that

$$\rho \leqslant \bar{\rho} = (1 - \bar{\varkappa})/(1 + \bar{\varkappa}), \qquad \bar{\varkappa} = 2\tau\delta/(1 + \tau^2\delta\Delta),$$
$$\min \bar{\rho}(\tau) = \rho(\max \bar{\varkappa}), \qquad \max \bar{\varkappa} = \bar{\varkappa}(\tau^*) = \sqrt{\eta}.$$

Given a small characteristic parameter η, the number of iterations $\nu(\varepsilon)$ satisfies

$$\nu(\varepsilon) \approx \frac{1}{\sqrt{\eta}} \ln \frac{1}{\varepsilon}.$$

Notice that, given δ_α, Δ_α, $\alpha = 1, 2$, the value $\tau = \tau^*$ minimizing $\rho = \rho_1\rho_2$ can be found either graphically or by using tables of the functions $\rho^2(\tau)$.

It is clear from (14) that scheme (12) is convergent provided one of the operators, say A_1, is non-negative ($\delta_1 = 0$), while the second is positive definite ($\delta_2 > 0$); then

$$\rho = \sqrt{\bar{\rho}}, \; \bar{\rho} = (1 - \sqrt{\eta}) / (1 + \sqrt{\eta}), \; \eta = \delta_2/\Delta_2 \text{ with } \tau = 1/\sqrt{\delta_2\Delta_2}.$$

3. We obtain bounds for the solution of the Cauchy problem(12) by another method.

In some cases convergence rate estimates can be improved. We shall assume that the parameter $\tau = \tau_k > 0$ depends on k, so that

$$(E + \tau_{k+1}A_1)\bar{z} = (E - \tau_{k+1}A_2)z^k, \quad (E + \tau_{k+1}A_2)z^{k+1} = (E - \tau_{k+1}A_1)\bar{z},$$
$$k = 0, 1, 2, \ldots, \tag{23}$$
$$z^0 = z_0.$$

We obtain from this [15]
$$z^{k+1} = T_{k+1}z^k,$$
$$\tag{24}$$
$$T_{k+1} = (E + \tau_{k+1}A_2)^{-1}(E - \tau_{k+1}A_1)(E + \tau_{k+1}A_1)^{-1}(E - \tau_{k+1}A_2).$$

The operators $(E + \tau_1 A)^{-1}$ и $(E - \tau_2 A)$ are obviously commutable: $(E + \tau_1 A)^{-1}(E - \tau_2 A) = (E - \tau_2 A)(E + \tau_1 A)^{-1}$. In fact,

$$(E - \tau_2 A)(E + \tau_1 A)^{-1} = \left[\left(\frac{\tau_2}{\tau_1} + 1 \right) E - \frac{\tau_2}{\tau_1}(E + \tau_1 A) \right](E + \tau_1 A)^{-1} =$$

$$= \left(\frac{\tau_2}{\tau_1} + 1 \right)(E + \tau_1 A)^{-1} - \frac{\tau_2}{\tau_1}(E + \tau_1 A)^{-1}(E + \tau_1 A) = \qquad (25)$$

$$= (E + \tau_1 A)^{-1}\left[\left(\frac{\tau_2}{\tau_1} + 1 \right) E - \frac{\tau_2}{\tau_1}(E + \tau_1 A) \right] = (E + \tau_1 A)^{-1}(E - \tau_2 A)$$

(τ_1, $\tau_2 > 0$ arbitrary). Hence [15]

$$T_{k+1} = (E + \tau_{k+1} A_2)^{-1}(E + \tau_{k+1} A_1)^{-1}(E - \tau_{k+1} A_1)(E - \tau_{k+1} A_2),$$

$$z^{k+1} = (E + \tau_{k+1} A_2)^{-1} B_1(\tau_{k+1}, \tau_{k+1}) B_2(\tau_k, \tau_{k+1}) B_1(\tau_k, \tau_k) \ldots$$
$$\ldots B_1(\tau_1, \tau_1)(E - \tau_1 A_2) z^0, \qquad (26)$$

where

$$B_\alpha(\tau_m, \tau_n) = (E + \tau_m A_\alpha)^{-1}(E - \tau_n A_\alpha), \qquad \alpha = 1, 2. \qquad (27)$$

From (26)

$$\|(E + \tau_{k+1} A_2) z^{k+1}\| \leqslant \prod_{j=1}^{k+1} \|B_1(\tau_j, \tau_j)\| \prod_{j=1}^{k} \|B_2(\tau_j, \tau_{j+1})\| \|(E - \tau_1 A_2) z^0\|. \qquad (28)$$

We thus require a bound for the norm of the operator

$$B(\tau_1, \tau_2) = (E + \tau_1 A)^{-1}(E - \tau_2 A) \qquad (29)$$

under the conditions

$$(Ay, y) \geqslant \delta \|y\|_2, \qquad (30)$$
$$\|Ay\|_2 \leqslant \Delta(A, y). \qquad (30')$$

This operator is the transition operator for the scheme

$$(E + \tau_1 A) y^{j+1} = (E - \tau_2 A) y^j. \qquad (31)$$

Putting $\tau = \tau_1 + \tau_2$, $\tau_1 = \sigma\tau$, $\tau_2 = (1 - \sigma)\tau$, where $\sigma = \tau_1/\tau$, we can rewrite the equation as

$$y_{\bar{t}} + A(\sigma y + (1 - \sigma) \check{y}) = 0, \qquad y_{\bar{t}} = \frac{y - \check{y}}{\tau}, \qquad (32)$$
$$y = y^{j+1}, \qquad \check{y} = y^j, \qquad j = 0, 1, 2, \ldots$$

We utilize the following equations, which can be verified directly:

$$\sigma y + (1-\sigma)\check{y} = 0.5(y + \check{y}) + (\sigma - 0.5)\tau y_{\bar{t}}, \tag{33}$$

$$(y_{\bar{t}},\ \sigma y + (1-\sigma)\check{y}) = 0.5(\|y\|^2)_{\bar{t}} + (\sigma - 0.5)\tau\|y_{\bar{t}}\|^2, \tag{34}$$

$$\|y\|^2 + \|\check{y}\|^2 = \frac{1}{2}\|y + \check{y}\|^2 + \frac{1}{2}\tau^2\|y_{\bar{t}}\|^2. \tag{35}$$

Lemma 2

We have

$$\|\sigma y + (1-\sigma)\check{y}\|^2 = \sigma\|y\|^2 + (1-\sigma)\|\check{y}\|^2 + (\sigma^2 - \sigma)\tau^2\|y_{\bar{t}}\|^2. \tag{36}$$

Using (33) and (35), we get

$$\|\sigma y + (1-\sigma)\check{y}\|^2 = (\sigma y + (1-\sigma)\check{y},\ \ \sigma y + (1-\sigma)\check{y}) = \frac{1}{4}\|y + \check{y}\|^2 +$$

$$+ (\sigma - 0.5)\tau\,(y + \check{y},\, y_{\bar{t}}) + (\sigma - 0.5)\tau^2\|y_{\bar{t}}\|^2 = (\sigma - 0.5)(\|y\|^2 - \|\check{y}\|^2) +$$

$$+ (\sigma^2 - \sigma)\tau^2\|y_{\bar{t}}\|^2 + \frac{1}{2}\left(\frac{1}{2}\|y + \check{y}\|^2 + \frac{1}{2}\tau^2\|y_{\bar{t}}\|^2\right) =$$

$$= (\sigma - 0.5)(\|y\|^2 - \|\check{y}\|^2) + (\sigma^2 - \sigma)\tau^2\|y_{\bar{t}}\|^2 +$$

$$+ 0.5(\|y\|^2 + \|\check{y}\|^2) = \sigma\|y\|^2 + (1-\sigma)\|\check{y}\|^2 + (\sigma^2 - \sigma)\tau^2\|y_{\bar{t}}\|^2.$$

We form the scalar product of (32) with $2y^{(\sigma)} = 2(\sigma y + (1-\sigma)\check{y})$, and recall (30), (34) and (36)

$$\|y\|^2 + 2\tau^2(\sigma - 0.5)\|y_{\bar{t}}\|^2 + 2\tau\delta[\sigma\|y\|^2 + (1-\sigma)\|\check{y}\|^2 +$$

$$+ (\sigma^2 - \sigma)\tau^2\|y_{\bar{t}}\|^2] \leqslant 0.$$

Hence follows the first energy inequality

$$(1 + 2\sigma\tau\delta)\|y\|^2 + 2\tau^2\omega\|y_{\bar{t}}\|^2 \leqslant (1 - 2(1-\sigma)\tau\delta)\|\check{y}\|^2, \tag{37}$$

$$\|y\|^2 \leqslant \frac{1 - 2(1-\sigma)\tau\delta}{1 + 2\sigma\tau\delta}\|\check{y}\|^2 \quad \text{for } \omega \geqslant 0, \tag{38}$$

where

$$\omega = \sigma - 0.5 - \tau\delta\sigma(1-\sigma). \tag{39}$$

Notice that $\omega \geqslant 0$ implies

$$\sigma \geqslant \frac{1}{2\tau\delta}[\tau\delta - 1 + \sqrt{1 + \tau^2\delta^2}] = f(\tau\delta), \text{ where } 0.5 \leqslant f(\tau\delta) \leqslant 1.$$

Now let $\omega < 0$.

In this case we need a second energy inequality. We apply the operator A^{-1} to (32)

$$A^{-1}y_{\bar{t}} + \sigma y + (1 - \sigma)\check{y} = 0. \tag{40}$$

We form the scalar product of (40) with $2\tau y_{\bar{t}}$ and recall (34)

$$2\tau(A^{-1}y_{\bar{t}},\ y_{\bar{t}}) + 2(\sigma - 0.5)\tau^2\|y_{\bar{t}}\|^2 + \|y\|^2 = \|\check{y}\|^2. \tag{41}$$

Lemma 3

If A satisfies (30) and (30'), then

$$(A^{-1}y,\ y) \geqslant \frac{1}{\Delta}\|y\|^2, \qquad y \in H. \tag{42}$$

For, putting $y = Av$, $v = A^{-1}y$ and using condition (30'), i.e.$\|Av\|^2 \leqslant \Delta(Av,\ v)$, we get

$$\|y\|^2 = \|Av\|^2 \leqslant \Delta(Av, v) = \Delta(A^{-1}y,\ y).$$

Using (42), we obtain from (41) the second energy inequality

$$2\tau\left(\frac{1}{\Delta} + (\sigma - 0.5)\tau\right)\|y_{\bar{t}}\|^2 + \|y\|^2 \leqslant \|\check{y}\|^2. \tag{43}$$

Hence follows at once

Theorem 4

The difference scheme (32) is stable with respect to its initial data, so that $\|y^j\| \leqslant \|y^0\|$ with

$$\sigma \geqslant 0.5 - \frac{1}{\tau\Delta}.$$

We consider (37) with $\omega < 0$ and (43) with

$$\sigma > 0.5 - \frac{1}{\tau\Delta}. \tag{44}$$

It is easily observed that the condition $\omega < 0$ and (44) are satisfied if

$$0.5 - \frac{1}{\tau\Delta} < \sigma < \frac{\tau\delta - 1 + \sqrt{1 + \tau^2\delta^2}}{2\tau\delta}. \qquad (45)$$

We multiply (43) by $C\tau$ and add the resulting inequality to (37)

$$(1 + 2\sigma\tau\delta + C\tau)\|y\|^2 \leqslant (1 - 2(1 - \sigma)\tau\delta + C\tau)\|\check{y}\|^2, \qquad (46)$$

where

$$C\tau = -\frac{\omega\tau\Delta}{1 + (\sigma - 0.5)\tau\Delta} = \frac{(\sigma - 0.5)\tau\Delta + \sigma(1 - \sigma)\tau^2\delta\Delta}{1 + (\sigma - 0.5)\tau\Delta}. \qquad (47)$$

After substituting (47) in (46), we get

$$\|y\|^2 \leqslant \frac{1 - 2(1 - \sigma)\tau\delta + (1 - \sigma)^2\tau^2\delta\Delta}{1 + 2\sigma\tau\delta + \sigma^2\tau^2\delta\Delta}\|\check{y}\|^2. \qquad (48)$$

From this, and the equation $y = B(\tau_1, \tau_2)\check{y} = (E + \tau_1 A)^{-1}(E - \tau_2 A)\check{y}$ we get

$$\|B(\tau_1, \tau_2)\| \leqslant \rho(\tau_1, \tau_2) = \sqrt{\frac{1 - 2\tau_2\delta + \tau_2{}^2\delta\Delta}{1 + 2\tau_1\delta + \tau_1{}^2\delta\Delta}}. \qquad (49)$$

In particular, with $\tau_1 = \tau_2 = \tau$ we have

$$\|(E + \tau A)^{-1}(E - \tau A)\| \leqslant \sqrt{\frac{1 - \varkappa}{1 + \varkappa}} = \rho(\tau), \qquad \varkappa = \frac{2\tau\delta}{1 + \tau^2\delta\Delta}. \qquad (50)$$

The bound (49) is obtained under conditions (45) or with $\sigma > 0.5 - 1/\tau\Delta$ and $\omega = \sigma - 0.5 - \tau\delta\sigma(1 - \sigma) < 0$. It follows from this that

$$\frac{\tau_1}{1 + 2\tau_1\delta} < \tau_2 < \tau_1 + \frac{2}{\Delta}. \qquad (51)$$

We have now proved

Theorem 5

Let conditions (30), (30') and (51) be satisfied. Then

$$\|(E + \tau_1 A)^{-1}(E - \tau_2 A)\| \leqslant \rho(\tau_1, \tau_2), \qquad (52)$$

where

$$\rho(\tau_1, \tau_2) = \sqrt{\frac{1 - 2\tau_2\delta + \tau_2{}^2\delta\Delta}{1 + 2\tau_1\delta + \tau_1{}^2\delta\Delta}} < 1. \qquad (53)$$

We now return to inequality (28). Recalling that

$$\| (E + \tau_{k+1} A_2) z^{k+1} \|^2 \geqslant (1 + \varkappa(\tau_{k+1})) \| z^{k+1} \|^2,$$

$$\| (E - \tau_1 A_2) z^0 \|^2 \leqslant (1 - \varkappa_2(\tau_1)) \| z^0 \|_{(2)}^2, \quad \| z^0 \|_{(2)}^2 = \| z^0 \|^2 + \tau_1^2 \| A_2 z^0 \|^2,$$

we get

$$\| z^{(k+1)} \|_{(2)} \leqslant \prod_{j=1}^{k} \rho_1(\tau_j, \tau_j) \rho_2(\tau_j, \tau_{j+1}) \rho_1(\tau_{k+1}, \tau_{k+1}) \sqrt{ \frac{1 - \varkappa_2(\tau_1)}{1 + \varkappa_2(\tau_{k+1})} } \| z^0 \|_{(2)}, \quad (54)$$

where $\| z^k \|_{(2)} = (\| z^k \|^2 + \tau_k^2 \| A_2 z^k \|^2)^{1/2}$, $k = 0, 1, 2, \ldots$.

With $\tau_k = \tau = \text{const.}$, this gives us

$$\| z^{k+1} \|_{(2)} \leqslant \rho^{k+1} \| z^0 \|_{(2)}, \qquad \rho = \sqrt{\rho_1 \rho_2},$$

$$\rho_\alpha^2 = \frac{1 - \varkappa_\alpha}{1 + \varkappa_\alpha}, \qquad \varkappa_\alpha = 2\tau\delta_\alpha / (1 + \tau^2 \delta_\alpha \Delta_\alpha), \qquad \alpha = 1, 2, \tag{55}$$

which we obtained earlier (Theorem 2).

Noticing that $(1 + \tau^2 \delta_2^2) \| z \|^2 \leqslant \| z \|_{(2)}^2 \leqslant (1 + \tau^2 \| A_2 \|^2) \| z \|^2$; we obtain from (55)

$$\| z^{k+1} \| \leqslant \sqrt{ \frac{1 + \tau^2 \| A_2 \|^2}{1 + \tau^2 \delta_2^2} } \rho^{k+1} \| z^0 \|. \tag{56}$$

The bound (54) shows that iterations in accordance with scheme (2) are convergent for $\tau_k \geqslant \tau_{k-1}$, or $\tau_k \leqslant \tau_{k-1}$, provided that $\tau_{k-1}(1 + 2\tau_{k-1}\delta)^{-1} < \tau_k < \tau_{k-1} + 2/\Delta$. But the fastest convergence can be obtained for $\tau_k = \tau_{k-1} = \tau$ by a suitable choice of τ.

Note. An alternative, equivalent statement of Theorem 5 is: if conditions (30), (30′) and (45) are satisfied, then

$$\| (E + \sigma\tau A)^{-1} (E - (1 - \sigma)\tau A) \| \leqslant \rho, \qquad \rho = \left(\frac{1 - 2(1 - \sigma)\tau\delta + (1 - \sigma)^2 \tau^2 \delta\Delta}{1 + 2\sigma\tau\delta + \sigma^2 \tau^2 \delta\Delta} \right)^{1/2}$$

4. We consider some special cases.

Let $H = H_N$ be an N-dimensional space, and A a linear positive definite, selfadjoint operator in H_N. Then we have the familiar limit $\hfill (57)$

$$\| (E + \tau A)^{-1} (E - \tau A) \| \leqslant \max \left(\frac{1 - \tau\lambda_1}{1 + \tau\lambda_1}, \frac{\tau\lambda_N - 1}{\tau\lambda_N + 1} \right) \quad \text{for} \quad \frac{1}{\lambda_N} \leqslant \tau \leqslant \frac{1}{\lambda_1},$$

where $\lambda_1 = \delta$ and $\lambda_N = \|A\|$ are the minimum and maximum eigenvalues respectively of A. For, let $A\xi_s = \lambda_s \xi_s$, $s = 1, 2, \ldots, N$, where $\{\lambda_s, \xi_s\}$ is the system of eigenvalues and eigenfunctions of A, and $A, \breve{y} = \sum_{s=1}^{N} c_s \xi_s$. Then

$$y = (E + \tau A)^{-1}(E - \tau A)\breve{y} = \sum_{s=1}^{N} \frac{1 - \tau \lambda_s}{1 + \tau \lambda_s} c_s \xi_s,$$

$$\|y\| \leqslant \max_{1 \leqslant s \leqslant N} \left| \frac{1 - \tau \lambda_s}{1 + \tau \lambda_s} \right| \leqslant \max\left(\frac{1 - \tau \lambda_1}{1 + \tau \lambda_1}, \frac{\tau \lambda_N - 1}{\tau \lambda_N + 1} \right)$$

for $\dfrac{1}{\lambda_N} \leqslant \tau \leqslant \dfrac{1}{\lambda_1}$.

Taking $\tau^* = 1/\sqrt{\lambda_1 \lambda_N}$, we get

$$\|(E + \tau^* A)^{-1}(E - \tau^* A)\| \leqslant \rho^*, \quad \rho^* = \frac{1 - \sqrt{\eta}}{1 + \sqrt{\eta}}, \qquad \eta = \frac{\lambda_1}{\lambda_N}. \tag{58}$$

In this case, Theorem 5 gives us

$$\|(E + \tau^* A)^{-1}(E - \tau^* A)\| \leqslant \sqrt{\rho^*} \tag{59}$$

with the same value $\tau = \tau^*$, $\lambda_4 = \delta$, $\Delta = \|A\| = \lambda_N$. Let A_1, A_2 be commutable. Now, (26) gives us

$$z^{k+1} = \prod_{j=1}^{k+1} B_1(\tau_j, \tau_j) B_2(\tau_j, \tau_j) z^0.$$

If $H = H_N$ and at least the operator A_2 is selfadjoint, we have to use a cyclical system of parameters $\tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_{k_0}$, $\tau_{k+nk_0} = \tau_k$ with $1 \leqslant k \leqslant k_0$, $n = 1, 2, \ldots, n_0$. Let A_1 be positive. We can now employ a one-dimensional version of the $\{\tau_k\}$. We obtain for $\nu(\varepsilon)$

$$\nu(\varepsilon) \approx \ln \frac{1}{\varepsilon} \left/ \ln \frac{1}{\eta_2} \right., \qquad \eta_2 = \frac{\lambda_1^{(2)}}{\lambda_N^{(2)}},$$

where $\lambda_1^{(2)}$ and $\lambda_N^{(2)}$ are the minimum and maximum eigenvalues of A_2.

The case when $H = H_N$, $A_1^* = A_1$, $A_2^* = A_2$ and $A_1 A_2 = A_2 A_1$ has been fully investigated [17, 18], and we shall not dwell on it.

A theorem follows from the above that we shall utilize in Section 3, paragraph 3.

Theorem 6

Let $H = H_N$, A_1 and A_2 be positive - definite operators, and A_2 self-adjoint. The following limit now holds for the solution of problem (12):

$$\|z^{k+1}\|_{(2)} \leqslant \rho \|z^k\|_{(2)} \quad \text{for} \quad \tau = \tau^* = \frac{1}{\sqrt{\lambda_1^{(2)} \lambda_N^{(2)}}},$$

(60)

where $\|z\|_{(2)}^2 = \|z\|^2 + (\tau^*)^2 \|A_2 z\|^2$, $\quad \rho = \rho_1 \rho_2^*$, $\quad \rho_2^* = (1 - \sqrt{\bar{\eta}_2})/(1 + \sqrt{\bar{\eta}_2})$,

$\rho_1 = \sqrt{(1 - \varkappa_1^*)/(1 + \varkappa_1^*)}$, $\quad \varkappa_1^* = 2\tau^* \delta_1 / (1 + (\tau^*)^2 \delta_1 \Delta_1)$.

3. Examples

We consider some examples of the application of the limits obtained in Section 2. We use a standard method of investigation: we construct the mesh, then introduce the scalar product on the set of mesh functions and determine the operators A_1, A_2; using the difference analogues of Green's formulae and other simple devices, we then find the characteristic parameters δ_α, Δ_α. After this, we apply Theorem 2 or Theorem 6 (if e.g. A_2 is selfadjoint).

1. The Dirichlet difference problem for the selfconjugate elliptic equation in an arbitrary region.

We consider the Dirichlet problem

$$(L_1 + L_2)u = -f(x), \qquad x = (x_1, x_2) \in G, \qquad u|_\Gamma = \mu(x);$$

$$h_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x) \frac{\partial u}{\partial x_\alpha}\right), \qquad 0 < c_1 \leqslant k_\alpha(x) \leqslant c_2, \qquad \alpha = 1, 2; \quad (61)$$

$$c_1, c_2 = \text{const} > 0.$$

in an arbitrary region $G + \Gamma$ of the (x_1, x_2) plane.

We construct in $\bar{G} = G + \Gamma$ the mesh $\bar{\omega}_{h,q} = \omega_{h,q} + \gamma_h$ by dividing the plane by the straight lines $x_\alpha = i_\alpha h_\alpha$, where the interval h_α in x_α is uniform.

The mesh boundary γ_h consists of the points of intersection of the straight lines $x_\alpha = i_\alpha h_\alpha$ with the boundary Γ of region G. We impose the condition

$$h_\alpha{}^* \geqslant q h_\alpha, \tag{62}$$

where q is a positive constant, independent of h_α. in the interval $h_\alpha{}^*$ close to the boundary Γ. Such a mesh can always be obtained e.g. by discarding the base-points at which condition (62) is violated (we assume as always that the set of base-points of this mesh $\omega_{h,q}$ is connected). It is in general non-uniform close to the boundary.

We write the second order difference scheme for problem (61)

$$(\Lambda_1 + \Lambda_2)v = -f(x), \qquad x \in \omega_{h,q}, \qquad v|_{\gamma_h} = \mu(x). \tag{63}$$

Here, $\Lambda_\alpha v = (a_\alpha(x)v_{\bar{x}_\alpha})_{\hat{x}_\alpha}$, $\alpha = 1, 2$, $0 < c_1 \leqslant a_\alpha(x) \leqslant c_2$ (see [19] for the notation).

This system of equations is solved by means of ADI iterations

$$\bar{y} - \tau\Lambda_1\bar{y} = y^k + \tau\Lambda_2 y^k + \tau f, \qquad k = 0, 1, 2 \ldots,$$

$$y^{k+1} - \tau\Lambda_2 y^{k+1} = \bar{y} + \tau\Lambda_1\bar{y} + \tau f, \qquad x \in \omega_{h,q}, \tag{64}$$

$$\bar{y} = \mu(x), \qquad y^{k+1} = \mu(x) \qquad \text{for } x \in \gamma_n.$$

We obtain problem (12) for the error $z^{k+1} = y^{k+1} - v$ $A_\alpha = -\Lambda_\alpha$, $\alpha = 1, 2$, are operators in the space Ω of mesh functions specified at $\omega_{h,q}$ and vanishing on the boundary γ_h. The scalar product in Ω is

$$(y, v) = \sum_{\omega_{h,q}} y(x) v(x) H, \qquad H = \hbar_1\hbar_2, \qquad \hbar_\alpha = 0.5\left(h_\alpha + h_\alpha{}^{(+1_\alpha)}\right).$$

To utilize Theorems 1 and 2. we have to prove that the A_α are positive - definite and selfadjoint. Green's difference formulae give

$$(A_\alpha y, y) \geqslant \delta_\alpha \|y\|^2, \qquad \delta_\alpha = \frac{4c_1}{D_\alpha{}^2}, \qquad (A_\alpha y, v) = (y, A_\alpha v). \tag{65}$$

Further, we obtain

$$\|A_\alpha\| \leqslant \frac{4c_2}{q h_\alpha{}^2} = \Delta_\alpha, \qquad \alpha = 1, 2, \text{ so that } \eta_\alpha = \frac{\delta_\alpha}{\Delta_\alpha} = \frac{c_1 q h_\alpha{}^2}{c_2 D_\alpha{}^2}. \tag{66}$$

Here D_α is the diameter of region G in the x_α direction. By Theorem 6, the convergence of the ADI scheme (64) follows at once from (65), (66). The parameter τ is obtained from the minimization condition for

$$\rho = \max \left(\left| \frac{1-\tau\delta_1}{1+\tau\delta_1} \right|, \ \left| \frac{1-\tau\Delta_1}{1+\tau\Delta_1} \right| \right) \max \left(\left| \frac{1-\tau\delta_2}{1+\tau\delta_2} \right|, \ \left| \frac{1-\tau\Delta_2}{1+\tau\Delta_2} \right| \right) . \quad (67)$$

A cruder method of fixing τ^* is as follows. Let $h = \min(h_1, h_2)$, $D = \max(D_1, D_2)$, $\delta = \min(\delta_1, \delta_2) = 4c_1/D^2$, $\Delta = 4c_2/qh^2$, $\Delta_1 \leqslant \Delta$, $\Delta_2 \leqslant \Delta$. We now get from (67)

$$\rho \leqslant \bar\rho = \left(\frac{1 - \sqrt{\tilde q}\, h/D}{1 + \sqrt{\tilde q}\, h/D} \right)^2 \ \text{with} \ \tau^* = \frac{1}{4\sqrt{c_1 c_2}} hD\sqrt{\tilde q},$$

$$\tilde q = q\,\frac{c_1}{c_2}, \quad (68)$$

while for $\nu(\epsilon)$ we have[*]

$$\nu(\epsilon) \approx \sqrt{\frac{c_2}{c_1}} \frac{D}{h\sqrt{q}} \ln \frac{1}{\epsilon} . \quad (69)$$

A formula $\nu(\epsilon) = O(1/h \ln 1/\epsilon)$ is also obtained when ω_h is any non-uniform mesh whose intervals $h_\alpha(x_i)$ satisfy at all base-points $x_i \in \bar\omega_h$ the condition $h_\alpha(x_i) \geqslant h$, where $h = \text{const.} > 0$. Here, $\Lambda_\alpha v = (a_\alpha v_{\bar x_\alpha})_{\hat x_\alpha}$ everywhere.

2. We now consider the Dirichlet problem for a non-selfadjoint elliptic operator with constant coefficients

$$(70)$$

$$(\tilde L_1 + \tilde L_2)u = -f(x), \qquad x \in G, \qquad u|_\Gamma = \mu(x), \qquad \tilde L_\alpha u = \frac{\partial^2 u}{\partial x_\alpha^2} + r_\alpha \frac{\partial u}{\partial x_\alpha} .$$

To solve this on the mesh $\omega_{h,q}$, we use a monotonic scheme of the second order of accuracy [20]

$$(\tilde\Lambda_1 + \tilde\Lambda_2)v = -f(x), \qquad x \in \omega_{h,q}, \qquad v = \mu(x) \ \text{for} \ x \in \gamma_h,$$

$$\tilde\Lambda_\alpha v = \nu_\alpha v_{\bar x_\alpha \hat x_\alpha} + b_\alpha^+ v_{\hat x_\alpha} + b_\alpha^- v_{\hat x_\alpha}, \quad (71)$$

$$b_\alpha^+ = \frac{r_\alpha + |r_\alpha|}{2} \geqslant 0, \qquad b_\alpha^- = \frac{r_\alpha - |r_\alpha|}{2} \leqslant 0,$$

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[*]   N.S. Bakhvalov kindly informed me that he previously obtained formula (69) for boundary conditions without drift (scheme $O(h)$) and $k_\alpha = $ const.

$$\nu_\alpha = 1 - \frac{R_\alpha}{1 + R}, \qquad R_\alpha = 0.5 h_\alpha |r_\alpha|, \qquad \alpha = 1, 2; \ R = R_1 + R_2.$$

The operators $\tilde{A}_\alpha = -\tilde{\Lambda}_\alpha$, defined on Ω are positive - definite

$$(\tilde{A}_\alpha z, z) \geqslant \frac{4}{D_\alpha^2} \|z\|^2, \qquad z \in \Omega, \qquad \delta_\alpha = \frac{4}{D_\alpha^2}, \tag{72}$$

and they satisfy

$$\| \tilde{A}_\alpha z \|^2 \leqslant \frac{4 (1 + c_3 h_\alpha)}{q h_\alpha^2} (\tilde{A}_\alpha z, z), \qquad c_3 = \max (|r_1|, |r_2|),$$

$$\Delta_\alpha = \frac{4 (1 + c_3 h_\alpha)}{q h_\alpha^2}, \qquad \alpha = 1, 2. \tag{73}$$

Property (72) can be proved by using Green's first difference formula [19]

$$(\tilde{A}_\alpha z, z) = (-\tilde{\Lambda}_\alpha z, z) = \nu_\alpha (1, z_{x_\alpha}^2]_\alpha + \frac{1}{2} (b_\alpha^+ - b_\alpha^-) h_\alpha (1, z_{x_\alpha}^{2}]_\alpha =$$

$$= (\nu_\alpha + R_\alpha) (1, z_{x_\alpha}^2]_\alpha \geqslant \left(1 - \frac{R_\alpha}{1 + R} + R_\alpha\right) \frac{4}{D_\alpha^2} \|z\|^2 \geqslant \frac{4}{D_\alpha^2} \|z\|^2, \qquad z \in \Omega,$$

since $\|z\|^2 \leqslant \frac{D_\alpha^2}{4} \|z_{\bar{x}_\alpha}\|_\alpha^2$ (for notation see [19]).

We prove (73) by means of the triangle inequality

$$\|\tilde{A}_\alpha z\| \leqslant \nu_\alpha \|z_{\bar{x}_\alpha \hat{x}_\alpha}\| + b_\alpha^+ \|z_{\hat{x}_\alpha}\| + |b_\alpha^-| \|z_{\bar{x}_\alpha}\| \leqslant \nu_\alpha \frac{2}{\sqrt{q} h_\alpha} (1, z_{x_\alpha}^2]_\alpha^{1/2} +$$

$$+ \frac{|r_\alpha|}{\sqrt{q}} (1, z_{x_\alpha}^2]_\alpha^{1/2} = \frac{1}{\sqrt{q}} \left(\frac{2\nu_\alpha}{h_\alpha} + |r_\alpha|\right) (1, z_{x_\alpha}^2]_\alpha^{1/2} = \frac{2 (\nu_\alpha + R_\alpha)}{\sqrt{q} h_\alpha} (1, z_{x_\alpha}^2]_\alpha^{1/2}.$$

If we now recall the identity $(A_\alpha z, z) = (\nu_\alpha + R_\alpha) (1, z_{x_\alpha}^2]_\alpha$, we get

$$\| A_\alpha z \|^2 \leqslant \frac{4}{q h_\alpha^2} (\nu_\alpha + R_\alpha) (A_\alpha z, z).$$

From (72), (73), the ADI scheme for (71) is convergent for any $\tau > 0$. If $h_1 = h_2 = h$, $D = \max(D_1, D_2)$, then

$$\rho \leqslant \bar{\rho} = \frac{1 - \sqrt{\bar{\eta}}}{1 + \sqrt{\bar{\eta}}}, \qquad \eta = \frac{qh^2}{(1 + c_3h) D^2} \quad \text{with} \quad \tau = \tau^* = \frac{1}{\sqrt{\delta \Delta}}, \quad (74)$$

where $\delta = 4/D^2$, $\Delta = 4(1 + c_3h)/qh^2$. The number of iterations obeys the asymptotic formula

$$\nu(\varepsilon) = \ln \frac{1}{\varepsilon} \bigg/ \ln \frac{1}{\rho} \approx \frac{D}{\sqrt{q} \, h} \ln \frac{1}{\varepsilon}. \qquad (75)$$

A similar formula can be obtained for the Dirichlet difference problem in the case of the non-selfadjoint elliptic operator with variable coefficients

$$\sum_{\alpha=1}^{2} \left[\frac{\partial}{\partial x_\alpha} \left(k_\alpha(x) \frac{\partial u}{\partial x_\alpha} \right) + r_\alpha(x) \frac{\partial u}{\partial x_\alpha} \right] - q_0(x) u = -f(x),$$

$$q_0(x) \geqslant c_1' > 0.$$

However, extra restrictions have to be imposed here on the coefficients and the mesh interval.

3. As our third example illustrating the application of the general theory of Section 2, we consider iterational schemes for solving the integro-differential equation for radiation transport in a layer

$$\mu \frac{\partial \varphi}{\partial x} + \sigma \varphi = \frac{1}{2} \sigma_s \int_{-1}^{1} \varphi(x, \mu') d\mu' + f(x, \mu), \qquad 0 \leqslant x \leqslant H, \qquad -1 \leqslant \mu \leqslant 1,$$

$$(76)$$

$$\varphi(0, \mu) = \xi(\mu), \qquad \mu > 0; \qquad \varphi(H, \mu) = \eta(\mu), \qquad \mu < 0$$

$$(77)$$

$$(\sigma > 0, \ \sigma > \sigma_s > 0).$$

This problem was considered in [12, 13]. Following [13], we introduce the mesh $\omega_{hl} = \{(x_i = ih, \ \mu_j = jl), \ i = 0, 1, \ldots, n, \ -1 \leqslant j \leqslant m, \ j \neq 0,$ $h = H/n, \ l = 1/m\}$ and approximate the operators

$$L_1\varphi = \mu \, \partial\varphi / \partial x, \qquad L_2\varphi = \sigma\varphi - \frac{1}{2} \sigma_s \int_{-1}^{1} \varphi(x, \mu') d\mu'$$

by means of operators Λ_1, Λ_2:

$$\Lambda_1 v = \begin{cases} \mu v_{\bar{x}}, & \mu > 0, \\ \mu v_x, & \mu < 0; \end{cases} \qquad \Lambda_2 v = (\Lambda_2 v)_{ij} = \sigma v_{ij} - \frac{1}{2} \sigma_s \sum_{\substack{\nu=-m \\ \nu \neq 0}}^{m} c_\nu v_{i\nu},$$

where $(v_{\bar{x}})_{ij} = (v_{ij} - v_{i-1,j}) / h, \quad (v_x)_{ij} = (v_{i+1,j} - v_{ij}) / h$.

We associate problem (76) - (77) with the equation $\Lambda_1 v + \Lambda_2 v = f$ with boundary conditions (77). Let Ω be the space of functions specified on the mesh ω_{hl} and vanishing on its boundary, i.e. with $i = 0$, $1 \leqslant j \leqslant m$ and $i = n$, $-m \leqslant j \leqslant -1$. We introduce the operators $A_1 v = \Lambda_1 v$, $A_2 v = \Lambda_2 v$ with $v \in \Omega$. The difference scheme for problem (76) - (77) can be written as

$$(A_1 + A_2) v = \tilde{f}, \qquad v \in \Omega \tag{78}$$

(the right-hand side $\tilde{f}$ takes account of the boundary conditions).

The scalar product

$$(z, v) = (z, v)^- + (z, v)^+ \text{ and the norm } \|z\| = \sqrt{(z, z)}, \tag{79}$$

can be introduced naturally into space Ω, where

$$(z, v)^- = \sum_{i=0}^{n-1} \sum_{j=-m}^{-1} z_{ij} v_{ij} hl, \qquad (z, v)^+ = \sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij} v_{ij} hl.$$

The operators A_1, A_2 are positive definite in Ω. If the c_ν are fixed in accordance with [13] (e.g. $c_\nu = l$), then A_2 is selfadjoint.

It can easily be shown that

$$(A_1 z, z) \geqslant \delta_1 \|z\|^2, \qquad \|A_1 z\|^2 \leqslant \Delta_1 (A_1 z, z), \qquad \delta_1 = \frac{hl}{H^2}, \qquad \Delta = \frac{2}{h}, \tag{80}$$

$$\sigma_c \|z\|^2 \leqslant (A_2 z, z) \leqslant \sigma \|z\|^2, \qquad \sigma_c = \sigma - \sigma_s > 0. \tag{81}$$

The bounds (81) are obtained in [13]. We prove (80). Using the identities $z z_{\bar{x}} = 0.5 (z^2)_{\bar{x}} = 0.5 h z_{\bar{x}}^2$, $z z_x = 0.5 (z^2)_x - 0.5 h z_x^2$ and the inequalities

$$z_i^2 \leqslant ih \sum_{k=1}^{n} h z_{\bar{x}, k}^2, \qquad \sum_{i=1}^{n-1} z_i^2 h \leqslant \frac{H (H - h)}{2} \sum_{k=1}^{n} h z_{\bar{x}, k}^2,$$

we get

$$(\mu z_{\bar{x}}, z)^+ = 0.5 \sum_{j=1}^{m} \mu_j (z_n^j)^2 l + 0.5 h (\mu, z_{\bar{x}}^2)^+ \geqslant \frac{h}{H^2} (\mu, z^2)^+ \geqslant \frac{hl}{H^2} (1, z^2)^+ \tag{82}$$

and similarly

$$(\mu z_x, z)^- = -0.5 \sum_{j=-m}^{-1} \mu_j (z_0{}^j)^2 l - 0.5h (\mu, z_x{}^2)^- \geqslant \frac{hl}{H^2} (1, z^2)^-. \quad (83)$$

Combining (82) and (83), we get $(A_1z, z) \geqslant \delta_1 \|z\|^2$. If we recall that $(A_1z, z) \geqslant 0.5h [(\mu, z_x{}^2)^+ - (\mu, z_x{}^2)^-]$, we obtain at once

$$\| A_1z \|^2 = (\mu^2, z_{\bar{x}}{}^2)^+ + (\mu^2, z_x{}^2)^- \leqslant$$
$$\leqslant (\mu, z_{\bar{x}}{}^2)^+ + (|\mu|, z_x{}^2)^- \leqslant \frac{(A_1z, z)}{0.5h} = \Delta_1 (A_1z, z), \qquad \Delta_1 = \frac{2}{h}.$$

The scheme (5) with $f_1 = 0$, $f_2 = 0$ is used in [13] to solve problem (78). As shown in Section 1, this scheme is algebraically identical with the ADI scheme (2). Its convergence for any τ therefore follows from [11]. We estimate its rate of convergence. The estimate of [11] is too crude, since it takes no account of the fact that A_2 is selfadjoint. Theorem 5 gives

$$\|z^\nu\|_{(2)} \leqslant' \rho_\nu \|z_0\|_{(2)}, \qquad \|z\|_{(2)}{}^2 = \|z\|^2 + \tau^2 \|A_2z\|^2,$$

where

$$\rho = \rho_1 \rho_2, \qquad \rho_1 = \sqrt{\frac{1 - \varkappa_1}{1 + \varkappa_1}}, \qquad \varkappa_1 = 2\delta_1\tau, \qquad \delta_1 \Delta_1 = \frac{2l}{H^2},$$

$$\rho_2 = \frac{1 - \sqrt{\eta_2}}{1 + \sqrt{\eta_2}}, \qquad \eta_2 = \frac{\sigma_c}{\sigma} \text{ with } \tau = \frac{1}{\sqrt{\sigma\sigma_c}}.$$

Hence it follows that

$$\nu(\varepsilon) = \ln \frac{1}{\varepsilon} \Big/ \ln \left(\rho_1 (1 - \sqrt{\eta_2})(1 + \sqrt{\eta_2})^{-1} \right).$$

By Theorem 2, the ADI scheme is also convergent with $\sigma_c = 0$. In this case $\rho = \rho_1 = ((1 - \sqrt{\eta_1}) / (1 + \sqrt{\eta_1}))^{1/2}$, $\eta_1 = h^2 l / 2H^2$, so that, as $h \to 0$, $l \to 0$ we have $\nu(\varepsilon) \approx H / h\sqrt{l}$. For comparison, we mention that the special methods used in [13] only enabled a bound to be obtained for the spectral radius $\rho(T)$ of the non-selfadjoint transition operator

$$T = (E + \tau A_2)^{-1} (E + \tau A_1)^{-1} (E - \tau A_1)(E - \tau A_2),$$
$$\rho(T) \leqslant \frac{1 - \sqrt{\eta_2}}{1 + \sqrt{\eta_2}}.$$

The economy of a method is characterized by the amount of computational work needed to obtain a solution to a given accuracy.

From this point of view, it is interesting to compare the scheme of [13] with the simple iterational scheme

$$(\Lambda_1 + \sigma E) \, y^{k+1/2} = \widetilde{\Lambda}_2 y^k + f, \qquad (\widetilde{\Lambda}_2 y^k)_{ij} = \frac{\sigma_s}{2} \sum_{\substack{\nu=-m \\ \nu \neq 0}}^{m} c_\nu y_{i\nu}^{\ m},$$

$$(85)$$

$$(\Lambda_1 + \sigma E) \, y^{k+1} = \widetilde{\Lambda}_2 y^{k+1/2} + f.$$

Using the fact that the maximum principle applies for $\Lambda_1 + \sigma E$, we can easily show that scheme (85) is convergent in a uniform metric with ratio $\rho_0 = (\sigma_s/\sigma)^2$, so that

$$\| y^{k+1} - v \|_0 \leqslant \rho_0 \| y^k - v \|_0, \qquad \| y \|_0 = \max_{i,j} |y_{ij}|, \qquad \rho_0 = (\sigma_s/\sigma)^2.$$

Following [13], we take as the characteristic convergence rate of the scheme of [13] the ratio $\rho^* = (1 - \sqrt{\eta_2}) / 1 + \sqrt{\eta_2})$, $\eta_2 = \sigma_c/\sigma = 1 - \sigma_s/\sigma$. Comparing ρ^* and ρ_0, we see that there exists a region of variation of the parameter $\eta = (\sigma_c/\sigma)$ with $4\eta^3 + 4\eta - \eta^2 > 4$, in which $\rho_0 < \rho^*$.

Of course it would be premature to draw any final conclusion regarding the advantage of one method over another from majorant bounds. However, our comparison shows that further investigation is needed (experimental as well as theoretical) into whether the ADI scheme is worth while for the transport equation. We need a more precise concept of the economy of a scheme for this equation. In any case, the "economy of a scheme of the first order in τ" discussed in Section 5 of [13] is not economy in the sense of minimizing the computational work: in this case, the formula for the number of iterations is $\nu(\epsilon) = (1/\epsilon) \ln (1/\epsilon)$, since $\tau = O(\epsilon)$.

4. Theorems 2 and 6 offer an estimate of the convergence of the ADI scheme (12) of Section 1 and enable optimal (more precisely, ρ-minimizing) values to be obtained for the relaxation parameter τ for many other problems.

Decomposition of the operator A of the initial stationary problem into a sum of two positive definite operators A_1, A_2 can be performed in various ways. We illustrate this by the example of the Dirichlet difference problem for the Poisson equation

$$\Delta u = \sum_{\alpha=1}^{p} \frac{\partial^2 u}{\partial x_\alpha^2} = -f$$

in an arbitrary p-dimensional region $G = G + \Gamma$ on the mesh $\omega_{h,q}$ (see paragraph 1, Section 3). We write the operator

$$\Lambda y = -\sum_{\alpha=1}^{p} y_{\bar{x}_\alpha \hat{x}_\alpha}$$

as the sum $\Lambda = \Lambda^- + \Lambda^+$, where

$$\Lambda^- y = -\sum_{\alpha=1}^{p} \frac{1}{\hbar_\alpha} y_{\bar{x}_\alpha}, \qquad \Lambda^+ y = \sum_{\alpha=1}^{p} \frac{1}{\hbar_\alpha} y_{x_\alpha}. \tag{86}$$

In this case the iterational ADI scheme is

$$\bar{y} - \tau \Lambda^- \bar{y} = y^h + \tau \Lambda^+ y^h + \tau f, \qquad x \in \omega_{h,q}; \qquad \bar{y} = \mu(x) \text{ for } x \in \gamma_h, \tag{87}$$

$$y^{h+1} - \tau \Lambda^+ y^{h+1} = \bar{y} + \tau \Lambda^- \bar{y} + \tau f, \qquad x \in \omega_{h,q}, \qquad y^{h+1} = \mu(x) \text{ for } x \in \gamma_h.$$

This scheme was first proposed by V.K. Saul'ev for solving the equation of heat conduction (with $p = 1, 2$); he called it the alternating method [21]. It is economic, since $O(1/h^p)$ arithmetical operations are required for transition from y^k to y^{k+1}.

Introducing the operators $A_1 y = -\Lambda^- y$, $A_2 y = -\Lambda^+ y$ into the space of mesh functions specified on $\omega_{h,q}$ and vanishing on the mesh boundary γ_h, we can rewrite (87) as

$$(E + \tau A_1)\bar{y} = (E - \tau A_2)y^h + \tau f, \qquad (E + \tau A_2)y^{h+1} = (E - \tau A_1)\bar{y} + \tau \tilde{f}.$$

The right-hand side $\tilde{f}$ takes account of the non-homogeneous boundary conditions and only differs from f at boundary base-points of the mesh.

The operators A_1, A_2 are non-selfadjoint and positive - definite in Ω. We evaluate δ and Δ for them.

Using Green's difference formula as in paragraph 3, Section 3, we get $(A_\alpha y, y) = 0.5(Ay, y)$, $\alpha = 1, 2$. After simple working similar to that of paragraph 1, Section 3, we get

$$\| A_\alpha y \|^2 \leqslant \frac{2p}{q^2 h^2}(A_\alpha y, y), \text{ i.e. } \Delta = \frac{2p}{q^2 h^2}.$$

Since $(Ay, y) \geqslant (4p \mathbin{/} D^2) \|y\|^2$, we have

$$(A_\alpha y, y) > \frac{2p}{D^2} \|y\|^2, \qquad \delta = \frac{2p}{D^2},$$

where D is the diameter of the region G. Hence it follows that

$$\eta = \frac{\delta}{\Delta} \frac{q^2 h^2}{D^2}.$$

We now apply Theorem 2

$$\rho \leqslant \frac{1 - \sqrt{\eta}}{1 + \sqrt{\eta}} = \frac{1 - qh \mathbin{/} D}{1 + qh \mathbin{/} D}, \qquad \tau = \tau^* = \frac{Dqh}{2p}. \tag{88}$$

Thus the same asymptotic formula as in paragraphs 1, 2 (for scheme (64)) holds for the number of iterations $\nu(\varepsilon)$ in the case of scheme (86) – (87), namely,

$$\nu(\varepsilon) = O\left(\frac{1}{h} \ln \frac{1}{\varepsilon}\right) \tag{89}$$

which is independent of the number of dimensions (p does not appear in the expression for ρ, see (88)). The total amount of computation for scheme (87) is $O\left(\dfrac{1}{h^{p+1}} \ln \dfrac{1}{\varepsilon}\right)$.

Given any ρ, we can define A_1, A_2 by analogy with (86) for the non-selfadjoint problem

$$A_1 y = \sum_{\alpha=1}^{p} \left(\frac{\nu_\alpha}{\hbar_\alpha} y_{\bar{x}_\alpha} - b_\alpha^- y_{\overset{\vee}{x}_\alpha}\right), \qquad A_2 y = \sum_{\alpha=1}^{p} \left(-\frac{\nu_\alpha}{\hbar_\alpha} y_{x_\alpha} - b_\alpha^+ y_{\hat{x}_\alpha}\right). \tag{90}$$

The bound (88) still holds in this case. If the mesh ω_h is non-uniform at regular base-points and $h_\alpha(x_i) = h(h = \text{const.} > 0)$ everywhere in ω_h, the bound (88) holds if qh is replaced by h.

Note. When comparing different iterational methods, in addition to the number of iterations, we need to consider the error resulting from rounding-off errors when determining each iteration. For instance, the method of successive substitutions is used for realizing scheme (64). Hence, as follows from N.S. Bakhvalov's bounds, an error $O(\varepsilon_0/h^2)$ occurs when finding y^{k+1}, where ε_0 is the rounding-off error. In the case of scheme (87), this error obviously does not exceed $O(\varepsilon_0/h)$. As we have

seen, schemes (64) and (87) are similar as regards the number of iterations. The iterational ADI scheme can be used to solve equations with variable coefficients.

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