MONOTONIC DIFFERENCE SCHEMES FOR ELLIPTIC AND PARABOLIC EQUATIONS IN THE CASE OF A NON-SELFADJOINT ELLIPTIC OPERATOR *

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(*Received 6 February 1965*)

THE present note proposes monotonic schemes, uniformly convergent at a rate $O(h^2)$, for a non-selfadjoint second order elliptic equation in an arbitrary region, and also monotonic locally one-dimensional schemes, uniformly convergent at a rate $O(h^2 + \tau)$ for a parabolic equation with non-selfadjoint elliptic operator.

1. We take the first boundary value problem in the domain $G + \Gamma$ of p-dimensional space $x = (x_1, \ldots, x_p)$ for an elliptic equation containing first derivatives

$$Lu = -f, \quad x \in G, \quad u|_\Gamma = f_1, \tag{1}$$

$$Lu = \sum_{\alpha=1}^{p} L_\alpha{}^0 u - qu, \quad L_\alpha{}^0 u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha}\right) + r_\alpha \frac{\partial u}{\partial x_\alpha}, \tag{2}$$

where $k_\alpha = k_\alpha(x) \geqslant c_1 > 0$, $q = q(x) \geqslant 0$, $r_\alpha = r_\alpha(x)$, $f = f(x)$, $f_1 = f_1(x)$, c_1 and c_1 is a constant.

We consider as usual a mesh $\omega_h(G)$ in the region G, formed by the intersection of the hyperplanes $x_\alpha = i_\alpha h_\alpha$, $i_\alpha = 0, \pm 1, \pm 2, \ldots, \alpha = 1, \ldots, p$, where h_α is the interval of the mesh; the base-points $x_i = (i_1 h_1, \ldots, i_p h_p) \in G$ are interior, the boundary γ_h of the mesh ω_h consists of the points of intersection of the hyperplanes $x_\alpha = i_\alpha h_\alpha$ with the boundary Γ of G. The set ω_h^* of interior base-points adjacent to the base-points $x_i \in \gamma_h$ will be termed the boundary zone, and the set $\tilde{\omega}_h$ of remaining base-points the fundamental region of the mesh.

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* *Zh. vȳchisl. Mat. mat. Fiz.* 5, 3, 548 - 551, 1965.

Let us now write the scheme for problem (1). As usual, the term containing the second derivative with respect to x_α in L_α^0 is replaced by a three-point uniform scheme. The natural replacement of the first derivative $\partial u/\partial x_\alpha$ by a two-sided difference ratio gives a scheme of the second order of approximation. This scheme is monotonic only for sufficiently small mesh intervals h_α ($\alpha = 1, \ldots, p$). It is clear from the one-dimensional example ($p = 1$) that the formulae are applicable for sufficiently small h_α, when $h_\alpha |r_\alpha| < k_\alpha$. If we use one-sided differences (the right-hand for $r_\alpha > 0$ and the left-hand for $r_\alpha < 0$), we obtain a monotonic scheme, for which the maximum principle always holds. It has the first order of accuracy, however.

Let us construct a monotonic scheme of the second order of accuracy, containing one-sided difference derivatives taking account of the sign of r_α.

Let $\Lambda_\alpha^* u$ be a three-point scheme of the second order for $\dfrac{\partial}{\partial x_\alpha}\left(k_\alpha \dfrac{\partial u}{\partial x_\alpha} \right)$:

i.e.
$$\Lambda_\alpha^* u = (a_\alpha u_{\bar{x}_\alpha})_{x_\alpha} = \frac{\partial}{\partial x_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha} \right) + O(h_\alpha^2),$$

where $\bar{u}_{x_\alpha}$ is the left-hand, and u_{x_α} the right-hand difference ratio with respect to the direction x_α. The coefficient a_α approximates k_α, in accordance with [1]

$$a_\alpha = k_\alpha - 0{,}5 h_\alpha \frac{\partial k_\alpha}{\partial x_\alpha} + O(h_\alpha^2).$$

The simplest expression for a_α is

$$a_\alpha = k_\alpha^{(-0.5\alpha)},$$

where $k_\alpha^{(-0.5\alpha)}$ is the value of k_α at the mid-point of the left-hand mesh interval, directed along x_α.

We replace r_α by the sum

$$r_\alpha = r_\alpha^+ + r_\alpha^-, \qquad r_\alpha^+ = \frac{r_\alpha + |r_\alpha|}{2} \geqslant 0, \qquad r_\alpha^- = \frac{r_\alpha - |r_\alpha|}{2} \leqslant 0$$

and approximate $r_\alpha(\partial u/\partial x_\alpha)$ by the expression

$$r_\alpha \frac{\partial u}{\partial x_\alpha} = \frac{r_\alpha}{k_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha} \right) \approx b_\alpha^+ a_\alpha^{(+1\alpha)} u_{x_\alpha} + b_\alpha^- a_\alpha u_{\bar{x}_\alpha}, \tag{3}$$

where $b_\alpha^\pm = r_\alpha^\pm / k_\alpha$, and $a_\alpha^{(+1\alpha)}$ is the value of a_α at the right-hand adjacent base-point in the direction of x_α.

To obtain a monotonic scheme of the second order of accuracy for equation (1), we have to write a monotonic scheme with one-sided first difference derivatives for the equation with disturbed coefficients

$$\tilde{L}u = \sum_{\alpha=1}^{p} \tilde{L}_\alpha u - qu = -f, \tag{4}$$

where

$$\tilde{L}_\alpha u = \varkappa_\alpha \frac{\partial}{\partial x_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha} \right) + r_\alpha \frac{\partial u}{\partial x_\alpha},$$

$$\varkappa_\alpha = \left(1 + \sum_{\beta \neq \alpha}^{1-p} R_\beta \right) \Big/ \left(1 + \sum_{\beta=1}^{p} R_\beta \right), \tag{5}$$

and $R_\beta = 0.5 h_\beta |r_\beta| / k_\beta$ is the "Reynolds' difference number". We obtain as a result the scheme

$$\Lambda y = \sum_{\alpha=1}^{p} \Lambda_\alpha y - qy = -f, \tag{6}$$

$$\Lambda_\alpha y = \varkappa_\alpha (a_\alpha y_{\bar{x}_\alpha})_{x_\alpha} + b_\alpha^{+} a_\alpha^{(+1_\alpha)} y_{x_\alpha} + b_\alpha^{-} a_\alpha y_{\bar{x}_\alpha}. \tag{6'}$$

Since $b_\alpha^{+} \geqslant 0,\ b_\alpha^{-} \leqslant 0,$ the maximum principle must hold for this scheme. It is easily verified that it has the second order of approximation on the solution $u = u(x)$ of equation (1)

$$\Psi = \Lambda u - Lu = O(|h|^2) \quad \text{for} \quad x \in \tilde{\omega}_h,$$

where $|h|^2 = \sum_{\alpha=1}^{p} h_\alpha^2$. In the case of constant coefficients monotonic schemes of a higher order of accuracy can be constructed.

The boundary conditions are specified by one of the methods guaranteeing the second order of accuracy for a $(2p + 1)$-point scheme in the case of Laplace's equation. A good method is $y = f_1$ on γ_h, while the difference scheme $\Delta y = -f$, allowing for the non-uniformity of the mesh, is written in the boundary zone ω_h^{*}.

We have thus obtained a monotonic scheme, uniformly convergent at a rate $O(|h|^2)$, provided conditions ensuring uniform approximation are satisfied.

2. We now consider the mixed problem for the parabolic equation in

the cylinder $\bar{Q}_T = (G+\Gamma)\times[0\leqslant t\leqslant T]$:

$$\frac{\partial u}{\partial t} = Lu + f, \quad u|_\Gamma = \nu(x,t), \qquad u(x,0) = u_0(x), \tag{7}$$

where Lu has the form (2), and $k_\alpha = k_\alpha(x, t)$, $q = q(x, t)$, $f = f(x, t)$. To obtain a scheme of the second order in space, we have to write a scheme of the second order of approximation at $r_\alpha = 0$ for the equation

$$\frac{\partial u}{\partial t} = \tilde{L}u + f \tag{8}$$

and replace the terms in the first derivative in accordance with (3).

We write $\tilde{L}$ as

$$\tilde{L}u = \sum_{\alpha=1}^{p}\tilde{L}_\alpha u,\ \tilde{L}_\alpha u = \varkappa_\alpha\frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x,t)\frac{\partial u}{\partial x_\alpha} - q_\alpha(x,t)u, \tag{9}$$

where $q_\alpha \geqslant 0, \sum_{\alpha=1}^{p} q_\alpha = q,$ e.g. $q_\alpha = q/p$, and $\varkappa_\alpha$ are given by (5).

The idea for obtaining a locally one-dimensional scheme from [2] amounts to the following. The mesh $\omega_\tau = \{t^j = j\tau \in [0, T]\}$ is introduced and each interval (t^j, t^{j+1}) is divided into p parts by the points

$$t_\alpha^j = t_{\alpha-1}^j + \sigma_\alpha\tau, \quad \sigma_\alpha > 0, \quad \sum_{\alpha=1}^{p}\sigma_\alpha = 1,$$

e.g. $\sigma_\alpha = 1/p$ (see [2]). The one-dimensional problem

$$\sigma_\alpha\frac{\partial v_{(\alpha)}}{\partial t} = \tilde{L}_\alpha v + f_\alpha, \quad \sum_{\alpha=1}^{p} f_\alpha = f, \quad \alpha = 1, \ldots, p. \tag{10}$$

is solved in the interval $(t_{\alpha-1}^j, t_\alpha^j)$.

The initial conditions for $v_{(\alpha)}$ are specified for $t = t_{\alpha-1}^j$, and the boundary conditions on the part Γ_α of Γ which consists of points of intersection of Γ by arbitrary straight lines parallel to Ox_α and passing through points $x \in G$. It is easily verified that, for any σ_α, we have (cf. [2])

$$\max_{\bar{Q}_T} |v - u| = O(\tau).$$

It is even possible to quote examples when the strict equality $v^j = u^j$ holds for $t = t^j$. For instance, this is the case for the Cauchy problem or the first mixed problem with zero boundary conditions for the equations for the equation of heat conduction $\partial u/\partial t = \Delta u$, $G = G_0$, where Δ is the Laplace operator, and G_0 is a parallelepiped. Various two-layer or even three-layer schemes can be used for the numerical solution of (10). We prefer (especially in the present case) purely implicit schemes, for which the maximum principle holds for any τ, h_α. For $v^j = u^j$ schemes $O(h^4 + \tau^2)$ may be used.

Let us write a locally-one-dimensional scheme for problem (8) Let $\Lambda_\alpha = \Lambda_\alpha(t^*)$, $t^* \in (t_j, t_{j+1}$ = scheme (6'). Then a locally-one-dimensional scheme for (8) is

$$\frac{y_{(\alpha)}{}^j - \overset{j}{y}_{(\alpha-1)}}{\tau} = \Lambda_\alpha(t_\alpha^*)y_{(\alpha)}{}^j + f_\alpha(x, t_\alpha^*), \quad \alpha = 1, 2, \ldots, p, \quad j = 0, 1 \ldots, \tag{11}$$

$$y|_{t=0} = u_0(x),$$

where $y_{(p)}{}^j = y^{j+1}$, $y_{(0)}{}^j = y^j$. The boundary conditions for $y_{(\alpha)}{}^j = v(x, t_\alpha^{**})$ are taken on the part γ_h which belongs to Γ_α. In the boundary zone ω_h^* the value of $y_{(\alpha)}j$ is defined either in accordance with [2] (deflection by means of linear interpolation in the direction of x_α), or in accordance with [3] (an equation allowing for the non-uniformity of the three-dimensional mesh is written at the base-points ω_h^*, see above). We recall that $t_\alpha^* \in [t_j, t_{j+1}]$, $t_\alpha^{**} \in [t_j, t_{j+1}]$. We can recommend, e.g. $t_\alpha^* = t_\alpha^{**} = t_{j+1}$. The method of choosing t_α^* and t_α^{**} within the limits indicated does not affect the order of accuracy in τ.

With both methods of specifying the boundary conditions the maximum principle is satisfied and

$$\max_{\omega_h} y^j - u^j | = O(|h|^2 + \tau),$$

provided the approximation conditions indicated in [2] are satisfied. Thus the locally-one-dimensional scheme (11) is uniformly convergent at the rate $O(|h|^2 + \tau)$.

The above method can be used in the construction of monotonic schemes of the 2nd order of accuracy in h for quasilinear parabolic equations, and also for certain systems of differential equations (e.g. for an analogue of the Navier - Stokes equations).

The case of non-uniform meshes and discontinuous coefficients requires further investigation.

Translated by D.E. Brown

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