DIFFERENCE SCHEMES ON "OBLIQUE" NETS*

A.A. SAMARSKII and A.V. GULIN

Moscow

(*Received 2 April 1965*)

1.* WHEN difference schemes are used for the equations

$$Lu = -f, \qquad \frac{\partial u}{\partial t} = Lu + f, \qquad \frac{\partial^2 u}{\partial t^2} = Lu + f, \tag{1}$$

where Lu is the elliptic operator of the general form

$$Lu = \sum_{\alpha,\,\beta=1}^{p} \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}\frac{\partial u}{\partial x_\beta}\right) + \sum_{\alpha=1}^{p} r_\alpha \frac{\partial u}{\partial x_\alpha} - qu, \tag{2}$$

$$\sum_{\alpha,\,\beta=1}^{p} k_{\alpha\beta}\xi_\alpha\xi_\beta > 0, \qquad \xi = (\xi_1, \xi_2, \ldots, \xi_p) \neq 0,$$

for an arbitrary space region G of change in $x = (x_1, x_2, \ldots, x_p)$ difficulties arise associated with the approximation of mixed derivatives on the net (see [1]).

If the coefficients $k_{\alpha\beta}$ = const., then we can always get rid of the mixed derivatives by introducing the new variables

$$s_\alpha = \sum_{\beta=1}^{p} b_{\alpha\beta}x_\beta, \qquad \alpha = 1, 2, \ldots, p, \qquad x_\alpha = \sum_{\beta=1}^{p} c_{\alpha\beta}s_\beta \tag{3}$$

and reducing the quadratic form $\sum\limits_{\alpha,\beta=1}^{p} k_{\alpha\beta}\xi_\alpha\xi_\beta$ to diagonal form over the whole region G. The operator Lu will take the form

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* *Zh. vȳchisl. Mat. mat. Fiz.* 5, 4, 773 - 776, 1965.

$$L'u = \sum_{\alpha=1}^{p}\left(b_\alpha \frac{\partial^2 u}{\partial s_\alpha^2} + \tilde{r}_\alpha \frac{\partial u}{\partial s_\alpha} \right) - qu. \tag{4}$$

where r_α and q may be variables. The operator Lu contains derivatives with respect to certain directions s_α. It is natural to introduce the net ω_h in G for the variables s_α, drawing the hyperplanes $s_\alpha^{(i_\alpha)} = i_\alpha h_\alpha$, $i_\alpha = 0, \pm 1, \pm 2, \ldots$, such that $\omega_h = \{s_i = (s_1^{(i_1)}, s_2^{(i_2)}, \ldots, s_p^{(i_p)})\}$.

We agree to call this an oblique net. If the transformation (3) is not orthogonal, then ω_h is oblique - angled (when $p = 2$ it is a parallelogram). On ω_h the operator (4) can be approximated to using ordinary difference schemes.

2. For two space variables ($p = 2$) the operator (2) can be reduced to the canonical form

$$L'u = \sum_{\alpha=1}^{2}\left[\frac{\partial}{\partial s_\alpha}\left(b_\alpha \frac{\partial u}{\partial s_\alpha} \right) + \tilde{r}_\alpha \frac{\partial u}{\partial s_\alpha} \right] - qu \tag{5}$$

in the whole region G and when the coefficients $k_{\alpha\beta}$ are variables

$$k_{\alpha\beta} = k_{\alpha\beta}(x_1, x_2), \qquad \alpha, \beta = 1, 2, \qquad k_{\alpha\beta} = k_{\beta\alpha},$$

provided the additional condition

$$k_{11} - k_{22} = \mu k_{12}, \tag{6}$$

where μ is an arbitrary constant, is satisfied.

For, consider the orthogonal transformation (3) for $p = 2$ which reduces the quadratic form with matrix $(k_{\alpha\beta})$ to the principal axes. Let $x_\alpha = \sum_{\beta=1}^{2} c_{\alpha\beta} s_\beta$ — be the inverse transformation, with $c_{11} = c_{22} = c$, $c_{12} = - c_{21} = \bar{c}$. We calculate the derivatives

$$\frac{\partial u}{\partial s_1} = c\frac{\partial u}{\partial x_1} - \bar{c}\frac{\partial u}{\partial x_2}, \qquad \frac{\partial u}{\partial s_2} = \bar{c}\frac{\partial u}{\partial x_1} + c\frac{\partial u}{\partial x_2},$$

$$L_{11}u = \frac{\partial}{\partial s_1}\left(b_1 \frac{\partial u}{\partial s_1} \right) = c^2 \frac{\partial}{\partial x_1}\left(b_1 \frac{\partial u}{\partial x_1} \right) + \bar{c}^2 \frac{\partial}{\partial x_2}\left(b_1 \frac{\partial u}{\partial x_2} \right) -$$

$$- c\bar{c}\left[\frac{\partial}{\partial x_1}\left(b_1 \frac{\partial u}{\partial x_2} \right) + \frac{\partial}{\partial x_2}\left(b_1 \frac{\partial u}{\partial x_1} \right) \right],$$

$$L_{22}u = \frac{\partial}{\partial su}\left(b_2\frac{\partial u}{\partial s_2}\right) = \bar{c}^2\frac{\partial}{\partial x_1}\left(b_2\frac{\partial u}{\partial x_1}\right) + c^2\frac{\partial}{\partial x_2}\left(b_2\frac{\partial u}{\partial x_2}\right) + $$
$$+ c\bar{c}\left[\frac{\partial}{\partial x_1}\left(b_2\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(b_2\frac{\partial u}{\partial x_1}\right)\right],$$

where b_1 and b_2 are as yet unknown.

Equating the expressions

$$L_{11}u + L_{22}u = \sum_{\alpha,\beta=1}^{2}\frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}(x)\frac{\partial u}{\partial x_\beta}\right),$$

we obtain conditions to find c, $\bar{c}$, b_1 and b_2

$$k_{11} = c^2 b_1 + \bar{c}^2 b_2, \qquad k_{22} = \bar{c}^2 b_1 + c^2 b_2,$$
$$k_{12} = c\bar{c}(b_2 - b_1), \qquad \bar{c}^2 + c^2 = 1. \tag{7}$$

It is clear from this that condition (6) is necessary and sufficient for c and $\bar{c}$ to be independent of x. Solving equations (7) and taking the roots

$$\bar{c} = \sqrt{\frac{1+\sqrt{1-4q}}{2}}, \quad c = \sqrt{\frac{1-\sqrt{1-4q}}{2}}, \quad \text{where} \quad q = \frac{1}{4+\mu^2},$$

we find

$$b_1 = \frac{k_{11}+k_{22}}{2} - \frac{k_{12}}{2\sqrt{q}} \geqslant 0, \qquad b_2 = \frac{k_{11}+k_{22}}{2} + \frac{k_{12}}{2\sqrt{q}} \geqslant 0. \tag{8}$$

Note 1. Let $k_{11} = k_{22}$. Then $\mu = 0$, $q = 1/4$ and $b_1 = k_{11} - k_{12}$, $b_2 = k_{11} + k_{12}$.

2. Let $k_{11}k_{22} - k_{12}^2 = 0$. Then

$$b_1 = \begin{cases} 0, & k_{12} > 0, \\ k_{11}+k_{22}, & k_{12} < 0; \end{cases} \qquad b_2 = \begin{cases} k_{11}+k_{22}, & k_{12} > 0, \\ 0, & k_{12} < 0. \end{cases}$$

Using the orthogonal transformation mentioned above to reduce (2) to the form (5) we can reduce the solution of the first boundary problem

$$Lu = -f, \qquad x \in G, \qquad u\,|_\Gamma = \nu(x) \tag{9}$$

to the solution of the same problem in the variables (s_1, s_2) with the operator (5) containing no mixed derivative. The difference scheme can be constructed on the oblique net $\omega_h = \{s_i \in G\}$ as in [2]. It is monotonic and converges uniformly at the rate $O(|h|^2)$.

If the coefficients $k_{\alpha\beta}$ are variable, then condition (6) must be satisfied. If $k_{\alpha\beta}$ = const., then (6) is satisfied automatically.

3. Let us consider the mixed problem in the cylinder $\bar{Q}_T = (G+\Gamma) \times [0 \leqslant t \leqslant T]$ for the quasilinear parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{\alpha, \beta=1}^{2} \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}(x_1, x_2, t, u)\frac{\partial u}{\partial x_\beta}\right) + \sum_{\alpha=1}^{2} r_\alpha(x_1, x_2, t, u)\frac{\partial u}{\partial x_\alpha} + f(x, t, u), \quad (10)$$

$$u|_\Gamma = \nu(x_1, x_2, t), \qquad u(x, 0) = u_0(x). \quad (11)$$

$$k_{\alpha\beta} = k_{\beta\alpha}, \qquad \sum_{\alpha, \beta=1}^{2} k_{\alpha\beta}\xi_\alpha\xi_\beta \geqslant 0$$

for any $(x, t) \in \bar{Q}_T$, $|u| \leqslant M_0$, where M_0 is an arbitrary constant.

If condition (6) is satisfied, then using the rotation of axes mentioned above we can transform equation (10) to the form

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{2} L_\alpha u + \tilde{f}(s, t, u), \quad (12)$$

$$L_\alpha u = \frac{\partial}{\partial s_\alpha}\left(b_\alpha(s, t, u)\frac{\partial u}{\partial s_\alpha}\right) + \tilde{r}_\alpha(s, t, u)\frac{\partial u}{\partial s_\alpha}, \quad \tilde{f}(s, t, u) = f(x, t, u), \ b \geqslant 0;$$

and b_α can be expressed in terms of $k_{\alpha\beta}$ using formulae (8).

To solve equation (12) with the conditions (11) we use local one-dimensional schemes [3], [4] on the oblique space net $\omega_h = \{s_i \in G\}$ and the net $\omega_\tau = \{t^j \in (0, T), j = 0, 1, \ldots\}$. The corresponding schemes are given in [3], [4] and there is no need to write them out here.

When the operator L_α is not selfconjugate, i.e. when $r_\alpha \neq 0$, besides the schemes described in [3], [4], as in [2] we construct a monotonic local one-dimensional scheme which is uniformly stable with respect to the right-hand side, the boundary and the initial data. This scheme converges uniformly at a rate $O(|h|^2 + \tau)$, $|h|^2 = h_1^2 + h_2^2$, where h_2 is the step in the direction s_χ, and τ the step with respect to t.

The stability and convergence can be proved with the help of the maximum principle.

The boundary conditions on the net ω_h will be set by one of the methods described in [3], [4] which give second order accuracy with respect to h (with or without deflexion with respect to s_α).

The argument u in the coefficients $b_\alpha(x, t, n)$, $\tilde{r}_\alpha(s, t, u)$ and the right-hand side $\tilde{f}(x, t, u)$ is taken either with $u = y_\alpha$ (see [4]) or with the known value $u = y_{(\alpha-1)}$.

In the latter case the scheme is linear with respect to $y_{(\alpha)}$, and in the former an iterational method is used to solve the system of non-linear equations.

4. Let us consider the case of an arbitrary number of measurements. If the coefficients $k_{\alpha\beta}$ = const., then (2) can be reduced to the normal form (4) over the whole region by the transformation (3). This makes it possible to solve the first boundary problem on the oblique net $\omega_h = \{s_i \in G\}$ using monotonic schemes of order $O(|h|^2)$, the mixed problem for the parabolic equation $\partial u/\partial t = Lu + f$ and for the hyperbolic equation $\partial^2 u/\partial t^2 = Lu + f$ using the local one-dimensional schemes studied earlier in [3] - [5].

When $p > 2$ it is, generally speaking, very difficult to calculate the coefficients of the orthogonal transformation (3). Therefore, in practice, we can use any transformation (3) (which reduces (2) to the canonical form (4)) with easily calculated coefficients. However by doing this we obtain oblique - angled nets which "flatten out" as the discriminant of the quadratic form $\sum_{\alpha,\beta=1}^{p} k_{\alpha\beta}\xi_\alpha\xi_\beta$. becomes smaller. For an orthogonal transformation the net ω_h is rectangular and the resulting difference schemes are suitable even in the degenerate case $(k_{11}k_{22} - k_{12}^2 = 0)$.

Translated by R. Feinstein

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