ECONOMIC DIFFERENCE SCHEMES FOR A HYPERBOLIC SYSTEM OF EQUATIONS WITH COMPOUND DERIVATIVES AND THEIR APPLICATION TO EQUATIONS IN THE THEORY OF ELASTICITY*

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1. IN this section we consider first of all additive difference schemes (see [1] - [5]) for a system of second order hyperbolic equations which contain compound derivatives. There are schemes with variable directions which are absolutely stable and convergent at least with a speed

$$O(|h|^2+ au)$$
, where $|h|^2 = \sum_{\alpha=1}^{r} h_{\alpha}^2$, h_{α} is the step in the variable a_{α}

and p is the number of dimensions. The numerical algorithm consists of the conversion of a three-point triangular operator, which reduces to the successive application of known formulae. The number of operations to determine a solution for a new time layer is proportional to the number of nodes of the space network and is a quantity of the same order as the number of operations for a purely explicit scheme. Thus the schemes put forward below are economic.

To construct economic schemes for an equation of the form

$$\frac{\partial^2 u}{\partial t^2} + \sum_{\alpha, \beta=1}^{\mathbf{p}} A_{\alpha\beta}(t) u = f$$

we use the common property of the operator

$$A = \sum_{\alpha, \beta=1}^{p} A_{\alpha\beta},$$

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i.e. its representation in the form of the sum of operators $A_{\alpha\beta}$ of simpler structure.

Additive economic schemes for a general second order hyperbolic system are then used to describe economic schemes for a system of equations in the theory of elasticity in the case of two or three space variables (p = 2, p = 3).

A resolving scheme is also constructed for equations of elasticity, which is absolutely stable and converges with a speed $O(|h|^2 + \tau^2)$. With regard to economy the resolving scheme is comparable with additive schemes, but for convergence it requires more smoothness for the solution of the differential equation.

In Section 7 an iteration scheme with alternating directions for the solution of a difference problem, corresponding to a stationary problem in the theory of elasticity, is considered.

The convergence of this scheme is proved for p = 2, 3, and it is shown that the number of iterations $v = O(h^{-2(p-1)/p} \ln (1/\epsilon))$, where ϵ is the required accuracy.

Economic schemes of another type are considered in the two-dimensional case (p = 2) for a dynamic problem of the theory of elasticity in [6] and for a static problem in the theory of elasticity in [7].

2. Let $\overline{G} = G + \Gamma = \{0 \leq x_{\alpha} \leq l_{\alpha}, \alpha = 1, ..., p\}$ be a *p*-dimensional parallelepiped with boundary Γ . In the cylinder $\overline{Q}_T = \overline{G} \times [0 \leq t \leq T]$ there is a solution of the problem

$$\frac{\partial^{2}\mathbf{u}}{\partial t^{2}} = \sum_{\alpha, \beta=1}^{p} L_{\alpha\beta}\mathbf{u} + \mathbf{f}(x,t), \qquad L_{\alpha\beta}\mathbf{u} = \frac{\partial}{\partial x_{\alpha}} \left(k_{\alpha\beta}(x,t) \frac{\partial \mathbf{u}}{\partial x_{\beta}} \right), \quad (1)$$

$$\mathbf{u}|_{\Gamma} = \mathbf{v}(x,t), \quad 0 \leq t \leq T, \quad \mathbf{u}(x,0) = \mathbf{v}_{0}(x), \\
\frac{\partial \mathbf{u}}{\partial t}(x,0) = \mathbf{v}_{1}(x), \quad x \in \overline{G}.$$
(2)

Here $x = (x_1, \ldots, x_p)$; $\mathbf{u} = \mathbf{u}(x, t) = (u^1, \ldots, u^s, \ldots, u^n)$, $\mathbf{f}, \mathbf{v}, \mathbf{v}_0, \mathbf{v}_1$ are vectors of dimensionaly *n*, and $k = (k_{\alpha\beta}) = (k_{\alpha\beta}s^m)$, *s*, $m = 1, \ldots, n$, is a partitioned $p \times p$ matrix with $n \times n$ submatrices, which satisfies the symmetry condition

$$k_{\alpha\beta}^{\ast m}(x,t) = k_{\beta\alpha}^{m\ast}(x,t)$$
(3)

and the condition of positive definiteness

$$\sum_{s,\ m=1}^{n} \sum_{\alpha,\ \beta=1}^{p} k_{\alpha\beta}^{sm}(x,t) \xi_{\beta}^{m} \xi_{\alpha}^{n} \ge c_{1} \sum_{s=1}^{n} \sum_{\alpha=1}^{p} (\xi_{\alpha}^{s})^{2}, \quad (x,t) \in \overline{Q}_{T}, \quad (4)$$

where $\xi_{\alpha} = (\xi_{\alpha}^{i_1}, \ldots, \xi_{\alpha}^{s_i}, \ldots, \xi_{\alpha}^{n_i}) \neq 0$ is an arbitrary real vector and c_1 is a positive constant.

We shall assume that the problem (1) - (2) has a unique solution u = u(x, t), which is continuous in \overline{Q}_T and differentiable as many times as necessary for this work. To infer the *a priori* evaluations it is assumed that $k_{\alpha\beta}(x,t)$ satisfies a Lipschitz condition with respect to t and $x_{\alpha'}, \alpha' = 1, \ldots, p$.

The system of equations in the theory of elasticity

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu \Delta \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mathbf{f}(x, t), \qquad (5)$$

where $\Delta \mathbf{u} = \sum_{\alpha=1}^{p} \partial^2 \mathbf{u} / \partial x_{\alpha}^2$ is the Laplace operator, $\mathbf{u} = (u^i, \ldots, u^p)$,

 $\lambda = \text{const} > 0$ and $\mu = \text{const} > \text{are Lamé's coefficients, is obviously a particular case of the system of equations (1) with <math>n = p$ and

$$k_{\alpha\beta}^{sm} = \mu \delta_{\alpha\beta} \delta_{sm} + (\lambda + \mu) \delta_{\alpha s} \delta_{\beta m}, \qquad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$
(6)

where δ_{ij} is the Kronecker delta. Condition (3) is satisfied automatically. We shall show that condition (4) is also satisfied if $c_1 = \mu$. In fact

$$\sum_{s,m=1}^{p} \sum_{\alpha,\beta=1}^{p} k_{\alpha\beta}^{m} \xi_{\alpha}^{s} \xi_{\beta}^{m} = \mu \sum_{\alpha,s=1}^{p} (\xi_{\alpha}^{s})^{2} + (\lambda + \mu) \sum_{\alpha,s=1}^{p} \xi_{\alpha}^{\alpha} \xi_{s}^{s} =$$
$$= \mu \sum_{\alpha,s=1}^{p} (\xi_{\alpha}^{s})^{2} + (\lambda + \mu) \left(\sum_{\alpha=1}^{p} \xi_{\alpha}^{\alpha}\right)^{2} \ge \mu \sum_{\alpha,s=1}^{p} (\xi_{\alpha}^{s})^{2}.$$

3. Let us introduce the difference networks $\omega_{\tau} = \{t_j = j\tau \in [0, T], j = 0, 1, ...\}$ and $\overline{\omega}_h = \{x_i = (i_1h_1, \ldots, i_{\alpha}h_{\alpha}, \ldots, i_{p}h_p) \in \overline{G} = G + \Gamma;$ $i_{\alpha} = 0, 1, \ldots, N_{\alpha}, h_{\alpha} = l_{\alpha} / N_{\alpha}, \alpha = 1, 2, \ldots, p\}$ with steps τ for the variable t and h_{α} for the variable x_{α} , $\alpha = 1, \ldots, p$: let $\gamma_h = \{x_i \in \Gamma\}$ be the boundary of the network $\overline{\omega}_h$, $\overline{\omega}_h \setminus \gamma_h = \{x_i \in G\}$ be the set of inner

nodes, $|h|^2 = \sum_{\alpha=1}^{p} h_{\alpha}^2$. Following [1], we shall introduce the notation $\mathbf{y} = \mathbf{y} \left(x_i, t_{j+1} \right) = \mathbf{y}^{j+1}, \quad \check{\mathbf{y}} = \mathbf{y}^j,$

$$\begin{aligned} x_i^{(\pm 1_{\alpha})} &= (i_1 h_1, \dots, i_{\alpha-1} h_{\alpha-1}, (i_{\alpha} \pm 1) h_{\alpha}, i_{\alpha+1} h_{\alpha+1}, \dots, i_p h_p), \\ \mathbf{y}^{(\pm 1_{\alpha})} &= \mathbf{y} \left(x_i^{(\pm 1_{\alpha})}, t_{j+1} \right), \quad \mathbf{y}_{\bar{x}_{\alpha}} &= \left(\mathbf{y} - \mathbf{y}^{(-1_{\alpha})} \right) / h_{\alpha}, \quad \mathbf{y}_{x_{\alpha}} &= \left(\mathbf{y}^{(+1_{\alpha})} - \mathbf{y} \right) / h_{\alpha}. \end{aligned}$$

We shall replace the operator

$$L_{\alpha\beta}\mathbf{u} = \frac{\partial}{\partial x_{\alpha}} \left(k_{\alpha\beta} \left(x, t \right) \frac{\partial \mathbf{u}}{\partial x_{\beta}} \right)$$
(7)

in the difference network ω_h by the same scheme of the second order of approximation as in [2], assuming that

$$\Lambda_{\alpha\beta}\mathbf{y} = \frac{1}{2} \left[\left(a_{\alpha\beta} \mathbf{y}_{\bar{\mathbf{x}}_{\beta}} \right)_{\mathbf{x}_{\alpha}} + \left(a_{\alpha\beta}^{(+1\beta)} \mathbf{y}_{\mathbf{x}_{\beta}} \right)_{\bar{\mathbf{x}}_{\alpha}} \right] \text{ for } \beta \neq \alpha,$$

$$\Lambda_{\alpha\alpha} \mathbf{y} = \left(a_{\alpha\alpha} \mathbf{y}_{\bar{\mathbf{x}}_{\alpha}} \right)_{\mathbf{x}_{\alpha}},$$
(8)

where $(a_{\alpha\beta})$ is a matrix-functional of the matrix $(k_{\alpha\beta})$ with pattern $\{-1 \leqslant s_{\beta} \leqslant 0, \beta = 1, ..., p\}$. The coefficients $a_{\alpha\beta} = (a_{\alpha\beta}^{sm})$ satisfy the conditions

$$a_{\alpha\beta}^{sm} = a_{\beta\alpha}^{ms}, \tag{9}$$

for sufficiently small $|h| \leqslant h_0$, the condition

$$\sum_{s,m=1}^{n}\sum_{\alpha,\beta=1}^{p}a_{\alpha\beta}^{sm}\xi_{\beta}^{m}\xi_{\alpha}^{s} \geqslant c_{1}'\sum_{\alpha=1}^{p}\sum_{s=1}^{n}(\xi_{\alpha}^{s})^{2}, \quad (x,t) \in \overline{\omega}_{h} \times \omega_{\tau}, \quad (10)$$

where $c_1 \leq c_1$ is a positive constant which does not depend on the network, and the condition obtained from (10) after replacing $a_{\alpha\beta}^{sm}$ by the coefficinet $(a_{\beta\beta}^{sm})^{(1_{\beta})}$ at the point $x_i^{(+1_{\beta})}$.

In the case of constant coefficients, $k_{\alpha\beta} = \text{const.}$, obviously $a_{\alpha\beta} = k_{\alpha\beta}$ and instead of (8) we obtain

$$\Lambda_{\alpha\beta} \mathbf{y} = \frac{1}{2} k_{\alpha\beta} (\mathbf{y}_{\bar{x}_{\beta}x_{\alpha}} + \mathbf{y}_{\bar{x}_{\alpha}x_{\beta}}), \qquad \Lambda_{\alpha\alpha} \mathbf{y} = k_{\alpha\alpha} \mathbf{y}_{x_{\alpha}x_{\alpha}}; \tag{11}$$

condition (10) is satisfied on any network.

Note. For $\Lambda_{\alpha\beta}y$ instead of (8) we can consider other representations also, e.g.

$$\Lambda_{\alpha\beta} y = \frac{1}{2} \left[\left(a_{\alpha\beta} \mathbf{y}_{\bar{\mathbf{x}}\beta} \right)_{\bar{\mathbf{x}}_{\alpha}} + \left(a_{\alpha\beta}^{(+1\beta)} \mathbf{y}_{\mathbf{x}\beta} \right)_{\mathbf{x}_{\alpha}} \right],$$

$$\Lambda_{\alpha\beta} y = \frac{1}{2} \left[\left(a_{\alpha\beta} \mathbf{y}_{\bar{\mathbf{x}}\beta} \right)_{\bar{\mathbf{x}}_{\alpha}}^{*} + \left(a_{\alpha\beta}^{(+1\beta)} \mathbf{y}_{\mathbf{x}\beta} \right)_{\bar{\mathbf{x}}_{\alpha}}^{*} \right],$$
(12)

where $\mathbf{v}_{\hat{x}_{\alpha}} = \frac{1}{2} (\mathbf{v}_{\bar{x}_{\alpha}} + \mathbf{v}_{x_{\alpha}})$ is the central difference derivative. In all cases the condition (10) will be satisfied for a sufficiently small step $|h| \leq h_0$ and all the subsequent conclusions retain their validity.

4. Let us introduce the "triangular" operators L^{-} and L^{+} . For this we write the symmetrical matrix $k_{\alpha\alpha} = (k_{\alpha\alpha}^{sm})$ in the form of the sum of two triangular matrices $k_{\alpha\alpha} = k_{\alpha\alpha}^{-} + k_{\alpha\alpha}^{+}, k_{\alpha\alpha}^{-} = (k_{\alpha\alpha}^{-sm}), k_{\alpha\alpha}^{+} = (k_{\alpha\alpha}^{+sm}),$ assuming $k_{\alpha\alpha}^{-ss} = k_{\alpha\alpha}^{+ss} = \frac{1}{2}k_{\alpha\alpha}^{ss}, k_{\alpha\alpha}^{-sm} = k_{\alpha\alpha}^{sm}, k_{\alpha\alpha}^{+sm} = 0$ if $m < s, k_{\alpha\alpha}^{+sm} = k_{\alpha\alpha}^{sm}, k_{\alpha\alpha}^{-sm} = 0$ with m > s and any $\alpha = 1, \ldots, p$. The matrix $k_{\alpha\alpha}^{\pm}$ is a diagonal $p \ge p$ matrix with submatrices which are triangular $n \ge n$ matrices, conjugate to each other

$$k_{\alpha\alpha}^{-sm} = k_{\alpha\alpha}^{+ms} \qquad (a_{\alpha\alpha}^{-sm} = a_{\alpha\alpha}^{+ms}).$$
⁽¹³⁾

In accordance with the representation $k_{\alpha\alpha}=k^-_{\alpha\alpha}+k^+_{\alpha\alpha}$ we obtain

$$L_{\alpha\alpha} = L_{\alpha\alpha}^{-} + L_{\alpha\alpha}^{+}, \quad \Lambda_{\alpha\alpha} = \Lambda_{\alpha\alpha}^{-} + \Lambda_{\alpha\alpha}^{+},$$

where

$$L_{\alpha\alpha}^{\mp}\mathbf{u} = \frac{\partial}{\partial x_{\alpha}} \left(k_{\alpha\alpha}^{\mp} \frac{\partial \mathbf{u}}{\partial x_{\alpha}} \right)$$

etc. In view of condition (13) the operators $\Lambda_{\alpha\alpha}^{-}$ and $\Lambda_{\alpha\alpha}^{+}$ are conjugate to each other on the network ω_h in the sense of the scalar product

$$(\mathbf{y}, \mathbf{v}) = \sum_{\boldsymbol{\omega}_h} \mathbf{y}(x_i) \mathbf{v}(x_i) H, \quad H = h_1 \dots h_p$$

i.e.

$$(\Lambda_{\alpha\alpha}^{-}\mathbf{y}, \mathbf{v}) = (\Lambda_{\alpha\alpha}^{+}\mathbf{v}, \mathbf{y}), \quad a = 1, \ldots p,$$
 (14)

where **y** and **v** are arbitrary network functions, vanishing on the boundary γ_h of the network ω_h .

We shall put the operator

$$L = \sum_{\alpha, \beta=1}^{p} L_{\alpha\beta}$$

in the form of the sum of two triangular operators

$$L = L^{-} + L^{+}, \qquad L^{\mp} = \sum_{\alpha, \beta=1}^{p} L_{\alpha\beta}^{\mp} = \sum_{\alpha=1}^{p} L_{\alpha}^{\mp}, \qquad L_{\alpha}^{\mp} = \sum_{\beta=1}^{p} L_{\alpha\beta}^{\mp},$$

$$L_{\alpha\beta}^{\mp} = L_{\alpha\alpha}^{\mp} \text{ if } \beta = \alpha, \qquad L_{\alpha\beta} = L_{\alpha\beta}, \qquad (15)$$

$$L_{\alpha\beta}^{+} = 0 \text{ if } \beta \leq \alpha, \qquad L_{\alpha\beta}^{+} = L_{\alpha\beta},$$

$$L_{\alpha\beta} = 0$$
 if $\beta > \alpha$, $L_{\alpha}^- = L_{\alpha\alpha} + \sum_{\beta=i}^{i} L_{\alpha\beta}$, $L_{\alpha} = L_{\alpha\alpha} + \sum_{\beta=\alpha+i} L_{\alpha\beta}$.

By virtue of the principle of additivity of $\begin{bmatrix} 1 & -5 \end{bmatrix}$ the solution of the system of equations

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \sum_{\alpha=1}^p [(L_{\alpha} + L_{\alpha}^+)\mathbf{u} + \mathbf{f}_{\alpha}], \qquad \sum_{\alpha=1}^p \mathbf{f}_{\alpha} = \mathbf{f}$$

reduces to the successive solution, on an $\omega_h \times \omega_{\tau}$ network with step τ/p , of the simpler equations

$$\frac{1}{p}\frac{\partial^2 \mathbf{u}}{\partial t^2} = L_{\alpha} \mathbf{u} + L_{\alpha}^+ \mathbf{u} + \mathbf{f}_{\alpha}.$$
 (16)

p

The case where $L_{\alpha\beta} = \delta_{\alpha\beta}L_{\alpha\alpha}$, i.e. compound derivatives are absent, is considered in [1].

We now introduce the values $y^{j+\alpha/p} = y_{(\alpha)}$, intermediate between $y^{j} = \check{y}$ and $\check{y}^{j+1} = y$ assuming that $y^{(j-1)+\alpha/p} = \check{y}_{(\alpha)}$, and use for the determination of $y_{(\alpha)}$ a difference scheme which approximates equation (16) with number α . By analogy with [1], for the approximation of $\partial^{2}u / \partial t^{2}$ we use the (p + 1)-th time layer

$$\frac{1}{p} \frac{\partial^2 \mathbf{u}}{\partial t^2} \sim \sigma_p \mathbf{u}_{\overline{t}_{\alpha} \overline{t}_{\alpha}}, \quad \alpha = 1, \dots, p; \quad (17)$$

$$\mathbf{u}_{\bar{t}_{\alpha}\bar{t}_{\alpha}} = \begin{cases} \left(\mathbf{u}_{(\alpha)} - 2\mathbf{u}_{(\alpha-1)} + \check{\mathbf{u}}_{(\alpha)}\right) / \tau^{2}, & \sigma_{p} = 2, & \mathbf{u}_{(0)} = \check{\mathbf{u}}_{(2)}, & p = 2\\ \left(\mathbf{u}_{(\alpha)} - \mathbf{u}_{(\alpha-1)} - \mathbf{u}_{(\alpha-2)} + \check{\mathbf{u}}_{(\alpha)}\right) / \tau^{2}, & \sigma_{p} = \frac{3}{2}, & \mathbf{u}_{(0)} = \check{\mathbf{u}}_{(3)}, \\ & \mathbf{u}_{(-1)} = \check{\mathbf{u}}_{(2)}, & p = 3. \end{cases}$$

The additive scheme of alternating directions for the problem (1) - (2) will have the form

$$\sigma_{p}\mathbf{y}_{\overline{t}_{\alpha}\overline{t}_{\alpha}} = \sum_{\beta=1}^{\alpha} \Lambda_{\alpha\beta}^{-} \mathbf{y}_{(\beta)} + \sum_{\beta=\alpha}^{p} \Lambda_{\alpha\beta}^{+} \check{\mathbf{y}}_{(\beta)} + \varphi_{\alpha}, \quad \alpha = 1, \ldots, p, \qquad (x, t) \in \omega_{h} \times \omega_{t},$$
(18)

$$y_{\alpha} = y(x, t_{j}^{*})$$
 with $x_{\alpha} = 0, l_{\alpha}, \quad \alpha = 1, ..., p; \quad y(x, 0) = y_{0}(x),$ (19)

where $\varphi_{\alpha} = \varphi_{\alpha}(x, t_{j}^{*})$ is a second order approximation on the network ω_{h} of the function $f_{\alpha}(x, t), t_{j}^{*} \in [t_{j}, t_{j+1}], \text{ e.g. } t_{j}^{*} = t_{j+1/2} = t_{j} + 0.5 \tau.$ The coefficients $a_{\alpha\beta} = a_{\alpha\beta}(x, t_{(\alpha)}^{*})$ are taken at the middle moment $t_{(\alpha)}^{*} = t_{j} + \frac{\alpha}{p}\tau - 0.5\tau$ (cf. [1]).

The second initial condition can be approximated by analogy with [1], or more simply by assuming

$$\mathbf{y}^{\alpha/p} = \mathbf{v}_1(x) + \frac{\alpha \tau}{\rho} \widetilde{\mathbf{v}}_1(x), \qquad \alpha = 1, \dots, p-1, \qquad p = 2, 3. \tag{20}$$

Such a condition is sufficient for an accuracy $O(\tau + |h|^2)$. Let us rewrite (18) in the form

$$\left(E - \frac{\tau^2}{\sigma_p} \Lambda_{\alpha\alpha}^{-}\right) \mathbf{y}_{(\alpha)} = \mathbf{R}_{\alpha} (\mathbf{y}) + \frac{\tau^2}{\sigma_p} \sum_{\beta=1}^{\alpha-1} \Lambda_{\alpha\beta} \mathbf{y}_{(\beta)} + \mathbf{F}_{\alpha} = \mathbf{\Phi}_{\alpha}, \qquad (21)$$

where

$$\mathbf{F}_{\alpha} = \frac{\tau^2}{\sigma_p} \left[\sum_{\beta=\alpha}^p \Lambda^+_{\alpha\beta} \check{\mathbf{y}}_{(\beta)} + \varphi_{\alpha} \right],$$

 $R_{\alpha}(\mathbf{y}) = 2\mathbf{y}_{(\alpha-1)} - \mathbf{y}_{(\alpha)}$ if p = 2, $R_{\alpha}(\mathbf{y}) = \mathbf{y}_{(\alpha-1)} + \mathbf{y}_{(\alpha-2)} - \mathbf{y}_{(\alpha)}$ if p = 3and E is the operator of identity.

Here it is obvious that to determined $y_{(\alpha)}$ (all $y_{(\beta)}$ for $\beta \leq \alpha$ and all $\tilde{y}_{(\beta)}$ for $\beta = 1, \ldots, p$ are already known) we obtain a system of three-

point equations with a triangular matrix for their coefficients. Its solution reduces to an inversion of the operator $E = (\tau^2 / \sigma_p) \Lambda_{\alpha \alpha}$, which is attained by an *n*-fold use of the ordinary formulae for each chain $y_{(\alpha)}$ (see [1]) for fixed $\alpha = 1, \ldots, p$. To realize the algorithm (21) we must bear in mind the values of $y_{(\alpha)}$ on *p* layers.

Let us write the equation for the s-th components $y_{(\alpha)}{}^s$ of the vector $\mathbf{y}_{(\alpha)}$

$$y^{\mathfrak{s}}_{(\alpha)} - \frac{\mathfrak{r}^{\mathfrak{s}}}{\sigma_{p}} (a^{-\mathfrak{s}\mathfrak{s}}_{\alpha\alpha} y^{\mathfrak{s}}_{\overline{x}_{\alpha}})_{x_{\alpha}} = \Phi^{\mathfrak{s}}_{(\alpha)} + \frac{\mathfrak{r}^{\mathfrak{s}}}{\sigma_{p}} \sum_{m=1}^{\mathfrak{s}-1} (a^{-\mathfrak{s}m}_{\alpha\alpha} y^{m}_{\overline{x}_{\alpha}})_{x_{\alpha}}. \tag{22}$$

Here Φ_{α}^{s} is known, since the calculation takes place in the direction from α to $\alpha + 1$, $\alpha = 1, \ldots, p$; the second term is also known if we determine successively $y_{(\alpha)}^{1}, \ldots, y_{(\alpha)}^{s}, \ldots, y_{(\alpha)}^{p}$, i.e. carry out the calculation from s to s + 1. Hence it is obvious that we can find $y_{(\alpha)}^{s}$ by solving the first boundary value problems for the three-point equations on segments parallel to the axis Ox_{α} .

If we interchange the roles of L_{α}^{-} and L_{α}^{+} , respectively,

$$\Lambda_{\alpha}^{-} = \Lambda_{\alpha\alpha}^{-} + \sum_{\beta=1}^{\alpha-1} \Lambda_{\alpha\beta} = \sum_{\beta=1}^{\alpha} \Lambda_{\alpha\beta}^{-} \text{ and } \Lambda_{\alpha}^{+} = \Lambda_{\alpha\alpha}^{+} + \sum_{\beta=\alpha+1}^{p} \Lambda_{\alpha\beta} = \sum_{\beta=\alpha}^{p} \Lambda_{\alpha\beta}^{+},$$

we obtain a second additive scheme

$$\sigma_{p}\mathbf{y}_{\tilde{t}_{\alpha}\tilde{t}_{\alpha}} = \sum_{\beta=\alpha}^{p} \Lambda_{\alpha\beta}^{+}\mathbf{y}_{(\beta)} + \sum_{\beta=1}^{\alpha} \Lambda_{\alpha\beta}^{+}\tilde{\mathbf{y}}_{(\beta)} + \varphi_{\alpha}$$
(23)

with the same initial and boundary conditions as in the first scheme. Here to determine y_{α} we must invert the triangular three-point operator $E = (\tau^2 / \sigma_p) \Lambda_{\alpha\alpha}^+$. Here the calculation proceeds from $\alpha + 1$ to α and from s + 1 to s.

The alternation of these two schemes gives a third scheme. Introducing the intermediate value $y^{j+\alpha/2p}$, $\alpha = 1, \ldots, 2p-1$, we obtain (cf. [2, 5])

$$\sigma_{p}\mathbf{y}_{\tilde{t}_{\alpha}\tilde{t}_{\alpha}} = \sum_{\beta=1}^{\alpha} \Lambda_{\alpha\beta}^{-} \mathbf{y}_{(\beta)} + \sum_{\beta=\alpha}^{p} \Lambda_{\alpha\beta}^{+} \check{\mathbf{y}}_{(\beta)} + \boldsymbol{\varphi}_{\alpha}, \qquad \alpha = 1, \ldots, p;$$
⁽²⁴⁾

$$\begin{split} \sigma_{p}\mathbf{y}_{\bar{t}_{\alpha'}\bar{t}_{\alpha'}} &= \sum_{\beta=\alpha}^{p} \Lambda_{\alpha\beta}^{+}\mathbf{y}_{(\beta')} + \sum_{\beta=1}^{\alpha} \Lambda_{\alpha\beta'}^{-} \mathbf{y}_{(\beta')} + \boldsymbol{\varphi}_{\alpha'}, \\ \alpha' &= 2p + 1 - \alpha, \quad \beta' = 2p + 1 - \beta; \end{split}$$

where $\alpha' = p + 1, ..., 2p$, $\varphi_{\alpha'} = \varphi_{\alpha}, \ \alpha = p, \ p - 1, ..., 2, 1$.

5. Schemes (18) and (23) are stable for sufficiently small $|h| \leq h_0$, ensuring that the requirement (10) of the positive definiteness of the matrices $(a_{\alpha\beta})$ and $(a_{\alpha\beta}^{(+1\beta)})$ for any τ is satisfied, and converge at least with a speed $O(|h|^2 + \tau)$. The proof of these statements is carried out by analogy with [1, 4] by the method of energy inequalities.

Here the basic part is played by an identity of the form

$$\sum_{\alpha=1}^{p} \{ (a_{\alpha\beta}\xi_{\beta}, \xi_{\alpha} - \check{\xi}_{\alpha}) + (a_{\alpha\beta}^{+}\check{\xi}_{\beta}, \xi_{\alpha} - \check{\xi}_{\alpha}) = J - \check{J} (1 + O(\tau)) + R,$$

where

$$J = \sum_{\alpha=1}^{p} \sum_{\beta=1}^{\alpha} (a_{\alpha\beta}^{-} \xi_{\beta}, \xi_{\alpha}) = \sum_{\alpha=1}^{p} \sum_{\beta=\alpha}^{p} (a_{\alpha\beta}^{+} \xi_{\beta}, \xi_{\alpha}),$$
$$R = \sum_{\alpha=1}^{p} \left\{ \sum_{\beta=1}^{\alpha} (a_{\alpha\beta}^{+} \check{\xi}_{\beta}, \xi_{\alpha}) - (a_{\alpha\beta}^{-} \check{\xi}_{\beta}, \xi_{\alpha}) \right\}.$$

Using (3) it is not difficult to see that R = 0.

Note. In the case of constant coefficients. $k_{\alpha\beta} = \text{const.}$, the given values of the accuracy of the schemes considered, (18) and (23), are valid for any h_{α} and τ .

6. We now turn to the equations of elasticity (5). In this case as we have seen in Section 2, n = p, and $k_{aB}^{sm} = \text{const.}$,

$$k_{\alpha\beta}^{sm} = \mu \delta_{\alpha\beta} \delta_{sm} + (\lambda + \mu) \delta_{\alpha s} \delta_{\beta m}, \qquad \delta_{sm} = \begin{cases} 1, & s = m, \\ 0, & s \neq m. \end{cases}$$
(25)

For the equations of elasticity (5) we can use any of the schemes considered in Section 4, bearing in mind that $a_{\alpha\beta} = k_{\alpha\beta}$, where $k_{\alpha\beta}$ is determined by formula (25).

We shall write in more detail the difference equations (18) in the

case where
$$p = 2$$
; here $\mathbf{y} = (y^{(1)}, y^{(2)})$ and
 $2y_{t_1t_1}^{(1)} = \frac{1}{2} (\lambda + 2\mu) (y_{(1)}^{(1)} \frac{1}{x_1x_1} + \check{y}_{(1)}^{(1)} \frac{1}{x_1x_1}) + \frac{1}{2} (\lambda + \mu) (\check{y}_{(2)}^{(2)} \frac{1}{x_2x_1} + \check{y}_{(2)}^{(3)} \frac{1}{x_2x_1}) + \varphi_{(1)}^{(1)}$
 $2y_{t_1t_1}^{(2)} = \frac{1}{2} \mu (y_{(1)}^{(2)} \frac{1}{x_1x_1} + \check{y}_{(1)}^{(2)} \frac{1}{x_1x_1}) + \varphi_{(1)}^{(2)}$,
 $2y_{t_1t_2}^{(1)} = \frac{1}{2} \mu (y_{(2)}^{(1)} \frac{1}{x_2x_2} + \check{y}_{(2)}^{(1)} \frac{1}{x_2x_2}) + \varphi_{(2)}^{(1)}$,
 $2y_{t_1t_2}^{(2)} = \frac{1}{2} \mu (y_{(2)}^{(1)} \frac{1}{x_2x_2} + \check{y}_{(2)}^{(1)} \frac{1}{x_2x_2}) + \varphi_{(2)}^{(1)}$,

We must remember that here the upper index means the number of the component and the lower one the number of the vector $(y_{(1)}^{(1)} = (y^{(1)})^{j+1/s}$, $y_{(2)}^{(1)} = (y^{(1)})^{j+1} = y^{(1)}$, $\check{y}_{(2)}^{(2)} = (y^{(2)})^{j+1} = \check{y}^{(2)}$ etc.). Using (18) and (25) it is not difficult to write down the scheme for p = 3.

7. For the equations of elasticity we can also construct a series of resolving schemes which are absolutely stable, economic and convergent with speed $O(\tau + |h|^2)$ or $O(\tau^2 + |h|^2)$.

Let us first consider the two multidimensional schemes

$$\begin{aligned} \mathbf{y}_{\bar{t}\bar{t}} &= \Lambda^{-}\mathbf{y} + \Lambda^{+}\check{\mathbf{y}} + \check{\mathbf{\varphi}} \quad (\mathbf{y}_{\bar{t}\bar{t}} = (\mathbf{y}^{j+1} - 2\mathbf{y}^{j} + \mathbf{y}^{j-1})/\tau^{2}); \quad (I^{-}) \\ \mathbf{y}_{\bar{t}\bar{t}} &= \Lambda^{+}\mathbf{y} + \Lambda^{-}\check{\mathbf{y}} + \check{\mathbf{\varphi}}, \quad (I^{+}) \end{aligned}$$

where Λ^- and Λ^+ are triangular operators, which approximate the triangular differential operator L^- and L^+ , $y = y^{j+1}$, $\check{y} = y^{j-1}$, $\check{y} = y^j$.

Let $\Lambda_{\alpha} \mathbf{y} = \mathbf{y}_{\overline{x}_{\alpha} x_{\alpha}}$, and Λ_{sk} denote the difference approximation of the compound derivative $\partial^2 \mathbf{u} / \partial x_s \partial x_k$, e.g. $\Lambda_{sk} \mathbf{y} = \frac{1}{2} (\mathbf{y}_{\overline{x}_s x_k} + \mathbf{y}_{x_s \overline{x}_k})$ or $\Lambda_{sk} \mathbf{y} = \frac{1}{2} (\mathbf{y}_{\overline{x}_s x_k} + \mathbf{y}_{x_s \overline{x}_k})$. Then the expressions for the triangular operators Λ^- and Λ^+ can be written in the form (the upper index s or k is the number of the component)

$$(\Lambda^{-}\mathbf{y})^{s} = \frac{1}{2} \sum_{\alpha=1}^{p} \varkappa_{s\alpha} \mathring{\Lambda}_{\alpha} y^{s} + (\lambda + \mu) \sum_{k=1}^{s-1} \mathring{\Lambda}_{sk} y^{k}, \quad \varkappa_{s\alpha} = \mu + (\lambda + \mu) + \mathring{\delta}_{s\alpha}, \quad (26)$$

$$(\Lambda^{+}\mathbf{y})^{s} = \frac{1}{2} \sum_{\alpha=1}^{p} \varkappa_{\rho\alpha} \mathring{\Lambda}_{\alpha} y^{s} + (\lambda + \mu) \sum_{k=s+1}^{p} \mathring{\Lambda}_{sk} y^{k}.$$
(27)

The boundary conditions on γ_h are exactly satisfied

$$\mathbf{y}|_{\mathbf{v}_h} = \mathbf{v}(x, t),$$

and the initial conditions have the form

$$\mathbf{y}(x, 0) = \mathbf{v}_0(x), \qquad \mathbf{y}_{\overline{t}}(x, \tau) = \mathbf{v}_1(x) + \tau \mathbf{v}_1(x),$$

where $\tilde{\mathbf{v}}_{1}(x)$ is chosen so that the initial condition $\partial \mathbf{u} / \partial t = \mathbf{v}_{1}$ is approximated with accuracy $O(\tau^{2})$; for this, for instance, it is sufficient to assume that $\tilde{\mathbf{v}}_{1} = -(L\mathbf{u} + \mathbf{f})|_{t=0}$.

Each of the schemes I⁻ and I⁺ is absolutely stable and has an accuracy $O(\tau + |h^2|)$. Applying these schemes alternately (e.g. scheme (26) on odd and scheme (27) on even layers), we obtain an accuracy $O(\tau^2 + |h|^2)$.

Following the principle given in (8), we shall write the generating scheme II⁻ for the scheme I⁻

$$A^{s}y_{\tilde{t}}^{s} = \check{\Phi}^{s} + F^{s}, \qquad A^{s} = \prod_{\alpha=1}^{p} A_{\alpha}^{s}, \qquad A_{\alpha}^{s} = E - 0.5\tau^{2}\varkappa_{s\alpha}\dot{\Lambda}_{\alpha},$$
$$\check{\Phi}^{s} = \check{y}_{\tilde{t}}^{s} + \tau \left[(\Lambda^{+}\check{y})^{s} + 0.5\sum_{\alpha=1}^{p}\varkappa_{s\alpha}\dot{\Lambda}_{\alpha}\check{y}^{s} + \check{\phi}^{s} \right],$$
$$F^{s} = \tau (\lambda + \mu) \sum_{k=1}^{s-1} \dot{\Lambda}_{sk}y^{k}.$$
$$(28)$$

Similarly the generating scheme II⁺ is written for the scheme I⁺. In this case only the formulae for $\check{\Phi^s}$ and F^s are changed

$$\check{\Phi}^{s} = \check{y}^{s}_{i} + \tau \left[(\Lambda^{-\check{y}})^{s} + 0.5 \sum_{\alpha=1}^{p} \varkappa_{s\alpha} \mathring{\Lambda}_{\alpha} \check{y}^{s} + \check{\phi}^{s} \right], \qquad F^{s} = \tau (\lambda + \mu) \sum_{k=s+1}^{p} \mathring{\Lambda}_{sk} y^{k}.$$

(29)

To determine y^s on a new layer we can write some numerical algorithms for alternating directions. We give for scheme II only the algorithm which we put forward in [8] for the equation of heat conduction: this algorithm has the form

$$A_{1}^{s}v_{(1)}^{s} = \check{\Phi}^{s} + F^{s}, \qquad A_{\alpha}v_{(\alpha)}^{s} = v_{(\alpha-1)}^{s}, \quad \alpha = 1, 2, \dots, p, \ s = 1, \dots, p; y^{s} = \check{y}^{s} + \tau v_{(p)}^{s}.$$
(30)

We shall take the boundary conditions with $x_{\alpha} = 0$, $x_{\alpha} = l_{\alpha}$ for $v_{(\alpha)}$ ^s in the form

$$v_{(\alpha)}^{s} = A_{\alpha+1}^{s} \dots A_{p}^{s} v_{\bar{t}}^{s} \text{ for } x_{\alpha} = 0, \quad x_{\alpha} = l_{\alpha}, \quad \alpha = 1, \dots, p-1, \quad v_{(p)}^{s} = v_{\bar{t}}^{s}.$$
(31)

The order of calculation is as follows: the components $y^{(1)}, \ldots, y^{(p)} = y^{j+1}$ are determined in turn.

For the algorithm which corresponds to scheme II^+ the formulae (30) remain unchanged, but the order of calculation is reversed: the components $y^{(p)}, \ldots, y^{(1)}$ are determined successively.

Alternating the schemes II⁻ and II⁺ we find the solution of the problem with accuracy to within $O(|h|^2 + \tau^2)$. This evaluation is obtained by the method of [1, 2, 8].

From the formulae for $v_{(\alpha)}^s$ it is obvious that the solution y^s of the difference problem is determined by means of successive inversion of the triangular matrices (according to the formulae given in [9]). Therefore the resolving schemes are economic: to calculate the vector $|\mathbf{y}^{j+1}|$ operations of the order $O(p^2/h^p)$ are required.

8. We turn now to a stationary problem of the theory of elasticity

$$L\mathbf{u} = \mu\Delta\mathbf{u} + (\lambda + \mu) \text{ grad div } \mathbf{u} = -\mathbf{f}(x), \quad x \in G, \quad \mathbf{u}|_{\Gamma} = \mathbf{v}(x). \quad (32)$$

Its solution reduces to the solution of the difference problem for establishing the parabolic system of equations

$$\frac{\partial \mathbf{u}}{\partial t} = L\mathbf{u} + f(x), \quad \mathbf{u} \mid \mathbf{r} = \mathbf{v}(x)$$

with arbitrary initial data

$$\mathbf{u}(x,0) = \mathbf{v}_0(x).$$

For this we make use of the resolving scheme. Let the difference scheme for (32) take the form

$$\Lambda^{-\mathbf{v}} + \Lambda^{+\mathbf{v}} = \boldsymbol{\varphi}, \quad \mathbf{v}|_{\boldsymbol{\gamma}_h} = \boldsymbol{v}_0(\boldsymbol{x}).$$

We write the generating scheme for the determination of $y = y(x_i, (j + 1)\tau)$, where j + 1 is the number of the iteration and τ the iteration parameter, as

$$A^{s}y_{\overline{t}}{}^{s} = \check{\Phi}^{s} + F^{s}, \qquad A^{s} = \prod_{\alpha=1}^{p} A_{\alpha}{}^{s}, \qquad A^{s}_{\alpha} = E - 0.5 \operatorname{traga}_{s\alpha} \mathring{\Lambda}_{\alpha},$$

$$\check{\Phi}^{s} = \sum_{\alpha=1}^{p} \varkappa_{s\alpha} \mathring{\Lambda}_{\alpha} \check{y}^{s} + (\lambda + \mu) \sum_{k=s+1}^{p} \mathring{\Lambda}_{sk} \check{y}^{k} + \varphi^{s},$$
$$F^{s} = (\lambda + \mu) \sum_{k=1}^{s-1} \mathring{\Lambda}_{sk} y^{k}.$$

In this case the numerical algorithm for alternating directions of Section 6 takes the form

$$A_{1}^{s}w_{(1)}^{s} = \Phi^{s} + F^{s}, \quad A_{\alpha}^{s}w_{(\alpha)}^{s} = w^{s}_{(\alpha-1)}, \quad \alpha > 1,$$
$$y^{s} = y^{s} + \tau w_{(p)}^{s}, \quad w_{(\alpha)}^{s}|_{\gamma_{h}} = 0,$$

i.e. for $w_{(\alpha)}^{s}$ the boundary conditions are always zero.

For comparison we quote one more numerical algorithm (two-layered)

$$A_{1}^{s}y_{(i)}^{s} = \tau(\tilde{\Phi^{s}} + F^{s}) + A^{s}\tilde{y^{s}}, \quad A_{\alpha}^{s}y_{(\alpha-1)}^{s} = y_{(\alpha-1)}^{s}, \quad \alpha > 1,$$

$$y_{(\alpha)}^{s} = A_{(\alpha+1)}^{s} \dots A_{p}^{s}v^{s} \quad \text{for} \quad x_{\alpha} = 0, \quad x_{\alpha} = l_{\alpha}, \quad \alpha = 1, 2, \dots, p-1:$$

$$y_{(p)}^{s}|_{\gamma_{h}} = v^{s}.$$

The order of the computation is the same as before: initially the first component $y^{(1)}$ is determined, then the second $y^2(s=2)$ etc.

To find y^s , at all nodes of the network ω_h operations of the order $O(1/h_1h_2 \ldots h_p)$ are required. The iteration process converges for $\tau = O(h_*)$, $h_* = \min h_{\alpha}$. The rate of convergence is determined by the number of iterations $v \approx O((1/h_*) \times \ln(1/\epsilon))$ for p = 2 and $O((1/h_*)^{4/2}) \ln(1/\epsilon))$ for p = 3, where ϵ is the required accuracy.

The evaluation for the number of iterations is obtained by the method of energy inequalities by analogy with [10, 7, 4].

Switching the roles of the operators Λ^- and Λ^+ , we obtain the second iteration scheme II⁺. The same value is obtained for the rate of convergence of the iterations with it as for the scheme II⁻ described above. It is hoped that this is the basis which the interchange of these two iteration algorithms II⁻ and II⁺ can bring to speeding up to convergence.

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