ECONOMIC DIFFERENCE SCHEMES FOR A HYPERBOLIC SYSTEM OF EQUATIONS WITH COMPOUND DERIVATIVES AND THEIR APPLICATION TO EQUATIONS IN THE THEORY OF ELASTICITY*

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1. IN this section we consider first of all additive difference schemes (see [1] - [5]) for a system of second order hyperbolic equations which contain compound derivatives. There are schemes with variable directions which are absolutely stable and convergent at least with a speed

$O(|h|^2+\tau)$, where $|h|^2 = \sum_{\alpha=1}^{p} h_\alpha^2$, h_α is the step in the variable a_α

and p is the number of dimensions. The numerical algorithm consists of the conversion of a three-point triangular operator, which reduces to the successive application of known formulae. The number of operations to determine a solution for a new time layer is proportional to the number of nodes of the space network and is a quantity of the same order as the number of operations for a purely explicit scheme. Thus the schemes put forward below are economic.

To construct economic schemes for an equation of the form

$$\frac{\partial^2 u}{\partial t^2} + \sum_{\alpha,\beta=1}^{p} A_{\alpha\beta}(t)u = f$$

we use the common property of the operator

$$A = \sum_{\alpha,\beta=1}^{p} A_{\alpha\beta},$$

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i.e. its representation in the form of the sum of operators $A_{\alpha\beta}$ of simpler structure.

Additive economic schemes for a general second order hyperbolic system are then used to describe economic schemes for a system of equations in the theory of elasticity in the case of two or three space variables (p = 2, p = 3).

A resolving scheme is also constructed for equations of elasticity, which is absolutely stable and converges with a speed $O(|h|^2 + \tau^2)$. With regard to economy the resolving scheme is comparable with additive schemes, but for convergence it requires more smoothness for the solution of the differential equation.

In Section 7 an iteration scheme with alternating directions for the solution of a difference problem, corresponding to a stationary problem in the theory of elasticity, is considered.

The convergence of this scheme is proved for p = 2, 3, and it is shown that the number of iterations $\nu = O(h^{-2(p-1)/p}\ln(1/\varepsilon))$, where ε is the required accuracy.

Economic schemes of another type are considered in the two-dimensional case (p = 2) for a dynamic problem of the theory of elasticity in [6] and for a static problem in the theory of elasticity in [7].

2. Let $\bar{G} = G + \Gamma = \{0 \leqslant x_\alpha \leqslant l_\alpha,\ \alpha = 1, \ldots, p\}$ be a p-dimensional parallelepiped with boundary Γ. In the cylinder $\bar{Q}_T = \bar{G} \times [0 \leqslant t \leqslant T]$ there is a solution of the problem

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \sum_{\alpha,\beta=1}^{p} L_{\alpha\beta}\mathbf{u} + \mathbf{f}(x,t), \qquad L_{\alpha\beta}\mathbf{u} = \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}(x,t)\frac{\partial \mathbf{u}}{\partial x_\beta}\right), \tag{1}$$

$$\begin{gathered} \mathbf{u}|_\Gamma = \mathbf{v}(x,t), \quad 0 \leqslant t \leqslant T, \quad \mathbf{u}(x,0) = \mathbf{v}_0(x), \\ \frac{\partial \mathbf{u}}{\partial t}(x,0) = \mathbf{v}_1(x), \qquad x \in \bar{G}. \end{gathered} \tag{2}$$

Here $x = (x_1, \ldots, x_p)$; $\mathbf{u} = \mathbf{u}(x,t) = (u^1, \ldots, u^s, \ldots, u^n)$, $\mathbf{f}$, $\mathbf{v}$, $\mathbf{v}_0$, $\mathbf{v}_1$ are vectors of dimensionaly n, and $k = (k_{\alpha\beta}) = (k_{\alpha\beta}^{sm})$, $s, m = 1, \ldots, n$, is a partitioned $p \times p$ matrix with $n \times n$ submatrices, which satisfies the symmetry condition

$$k_{\alpha\beta}^{sm}(x,t) = k_{\beta\alpha}^{ms}(x,t) \tag{3}$$

and the condition of positive definiteness

$$\sum_{s,m=1}^{n} \sum_{\alpha,\beta=1}^{p} k_{\alpha\beta}^{sm}(x,t)\xi_\beta^m \xi_\alpha^n \geqslant c_1 \sum_{s=1}^{n}\sum_{\alpha=1}^{p} (\xi_\alpha^s)^2, \qquad (x,t)\in \bar{Q}_T, \qquad (4)$$

where $\xi_\alpha = (\xi_\alpha^1, \ldots, \xi_\alpha^s, \ldots, \xi_\alpha^n) \not\equiv 0$ is an arbitrary real vector and c_1 is a positive constant.

We shall assume that the problem (1) - (2) has a unique solution $u = u(x, t)$, which is continuous in $\bar{Q}_T$ and differentiable as many times as necessary for this work. To infer the *a priori* evaluations it is assumed that $k_{\alpha\beta}(x,t)$ satisfies a Lipschitz condition with respect to t and $x_{\alpha'}$, $\alpha' = 1, \ldots, p$.

The system of equations in the theory of elasticity

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathbf{grad\,div\,u} + \mathbf{f}(x,t), \qquad (5)$$

where $\Delta\mathbf{u} = \sum_{\alpha=1}^{p} \partial^2\mathbf{u}/\partial x_\alpha^2$ is the Laplace operator, $\mathbf{u} = (u^1, \ldots, u^p)$, λ = const > 0 and μ = const > are Lamé's coefficients, is obviously a particular case of the system of equations (1) with $n = p$ and

$$k_{\alpha\beta}^{sm} = \mu\delta_{\alpha\beta}\delta_{sm} + (\lambda+\mu)\delta_{\alpha s}\delta_{\beta m}, \qquad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad (6)$$

where δ_{ij} is the Kronecker delta. Condition (3) is satisfied automatically. We shall show that condition (4) is also satisfied if $c_1 = \mu$. In fact

$$\sum_{s,m=1}^{p} \sum_{\alpha,\beta=1}^{p} k_{\alpha\beta}^{m}\xi_\alpha^s\xi_\beta^m = \mu \sum_{\alpha,s=1}^{p} (\xi_\alpha^s)^2 + (\lambda+\mu)\sum_{\alpha,s=1}^{p} \xi_\alpha^\alpha \xi_s^s =$$

$$= \mu \sum_{\alpha,s=1}^{p} (\xi_\alpha^s)^2 + (\lambda+\mu)\left(\sum_{\alpha=1}^{p} \xi_\alpha^\alpha\right)^2 \geqslant \mu \sum_{\alpha,s=1}^{p} (\xi_\alpha^s)^2.$$

3. Let us introduce the difference networks $\omega_\tau = \{t_j = j\tau \in [0, T]$, $j = 0, 1, \ldots\}$ and $\bar{\omega}_h = \{x_i = (i_1h_1, \ldots, i_\alpha h_\alpha, \ldots, i_p h_p) \in \bar{G} = G + \Gamma$; $i_\alpha = 0, 1 \ldots, N_\alpha$, $h_\alpha = l_\alpha / N_\alpha$, $\alpha = 1, 2, \ldots, p\}$ with steps τ for the

variable t and h_α for the variable x_α, $\alpha = 1, \ldots, p$: let $\gamma_h = \{x_i \in \Gamma\}$ be the boundary of the network $\bar{\omega}_h$, $\bar{\omega}_h \setminus \gamma_h = \{x_i \in G\}$ be the set of inner nodes, $|h|^2 = \sum_{\alpha=1}^{p} h_\alpha^2$. Following [1], we shall introduce the notation

$$\mathbf{y} = \mathbf{y}(x_i, t_{j+1}) = \mathbf{y}^{j+1}, \quad \check{\mathbf{y}} = \mathbf{y}^j,$$

$$x_i^{(\pm 1_\alpha)} = (i_1 h_1, \ldots, i_{\alpha-1} h_{\alpha-1}, (i_\alpha \pm 1) h_\alpha, i_{\alpha+1} h_{\alpha+1}, \ldots, i_p h_p),$$

$$\mathbf{y}^{(\pm 1_\alpha)} = \mathbf{y}(x_i^{(\pm 1_\alpha)}, t_{j+1}), \quad \mathbf{y}_{\bar{x}_\alpha} = (\mathbf{y} - \mathbf{y}^{(-1_\alpha)}) / h_\alpha, \quad \mathbf{y}_{x_\alpha} = (\mathbf{y}^{(+1_\alpha)} - \mathbf{y}) / h_\alpha.$$

We shall replace the operator

$$L_{\alpha\beta}\mathbf{u} = \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}(x, t)\frac{\partial \mathbf{u}}{\partial x_\beta}\right) \tag{7}$$

in the difference network ω_h by the same scheme of the second order of approximation as in [2], assuming that

$$\begin{aligned} \Lambda_{\alpha\beta}\mathbf{y} &= \tfrac{1}{2}\left[\left(a_{\alpha\beta}\mathbf{y}_{\bar{x}_\beta}\right)_{x_\alpha} + \left(a_{\alpha\beta}^{(+1_\beta)}\mathbf{y}_{x_\beta}\right)_{\bar{x}_\alpha}\right] \quad \text{for } \beta \neq \alpha, \\ \Lambda_{\alpha\alpha}\mathbf{y} &= \left(a_{\alpha\alpha}\mathbf{y}_{\bar{x}_\alpha}\right)_{x_\alpha}, \end{aligned} \tag{8}$$

where $(a_{\alpha\beta})$ is a matrix-functional of the matrix $(k_{\alpha\beta})$ with pattern $\{-1 \leqslant s_\beta \leqslant 0, \ \beta = 1, \ldots, p\}$. The coefficients $a_{\alpha\beta} = (a_{\alpha\beta}^{sm})$ satisfy the conditions

$$a_{\alpha\beta}^{sm} = a_{\beta\alpha}^{ms}, \tag{9}$$

for sufficiently small $|h| \leqslant h_0$, the condition

$$\sum_{s,m=1}^{n}\sum_{\alpha,\beta=1}^{p} a_{\alpha\beta}^{sm}\xi_\beta^{m}\xi_\alpha^{s} \geqslant c_1' \sum_{\alpha=1}^{p}\sum_{s=1}^{n}(\xi_\alpha^{s})^2, \qquad (x, t) \in \bar{\omega}_h \times \omega_\tau, \tag{10}$$

where $c_1' \leqslant c_1$ is a positive constant which does not depend on the network, and the condition obtained from (10) after replacing $a_{\alpha\beta}^{sm}$ by the coefficinet $(a^{sm}_{\beta})^{(1_\beta)}$ at the point $x_i^{(+1\,\beta)}$.

In the case of constant coefficients, $k_{\alpha\beta} =$ const., obviously $a_{\alpha\beta} = k_{\alpha\beta}$ and instead of (8) we obtain

$$\Lambda_{\alpha\beta}\mathbf{y} = \tfrac{1}{2} k_{\alpha\beta}(\mathbf{y}_{\bar{x}_\beta x_\alpha} + \mathbf{y}_{\bar{x}_\alpha x_\beta}), \qquad \Lambda_{\alpha\alpha}\mathbf{y} = k_{\alpha\alpha}\mathbf{y}_{\bar{x}_\alpha x_\alpha}; \tag{11}$$

condition (10) is satisfied on any network.

Note. For $\Lambda_{\alpha\beta}y$ instead of (8) we can consider other representations also, e.g.

$$\Lambda_{\alpha\beta}y = \frac{1}{2}\left[\left(a_{\alpha\beta}y_{\bar{x}\beta}\right)_{\bar{x}_\alpha} + \left(a_{\alpha\beta}^{(+1\beta)}y_{x_\beta}\right)_{x_\alpha}\right],$$
$$\Lambda_{\alpha\beta}y = \frac{1}{2}\left[\left(a_{\alpha\beta}y_{\bar{x}_\beta}\right)_{\overset{\circ}{x}_\alpha} + \left(a_{\alpha\beta}^{(+1\beta)}y_{x_\beta}\right)_{\overset{\circ}{x}_\alpha}\right], \tag{12}$$

where $v_{\overset{\circ}{x}_\alpha} = \frac{1}{2}(v_{\bar{x}_\alpha} + v_{x_\alpha})$ is the central difference derivative. In all cases the condition (10) will be satisfied for a sufficiently small step $|h| \leqslant h_0$ and all the subsequent conclusions retain their validity.

4. Let us introduce the "triangular" operators L^- and L^+. For this we write the symmetrical matrix $k_{\alpha\alpha} = (k_{\alpha\alpha}^{sm})$ in the form of the sum of two triangular matrices $k_{\alpha\alpha} = \overset{-}{k}_{\alpha\alpha} + \overset{+}{k}_{\alpha\alpha}$, $\overset{-}{k}_{\alpha\alpha} = (k_{\alpha\alpha}^{-sm})$, $\overset{+}{k}_{\alpha\alpha} = (k_{\alpha\alpha}^{+sm})$, assuming $k_{\alpha\alpha}^{-ss} = k_{\alpha\alpha}^{+ss} = \frac{1}{2}k_{\alpha\alpha}^{ss}$, $k_{\alpha\alpha}^{-sm} = k_{\alpha\alpha}^{sm}$, $k_{\alpha\alpha}^{+sm} = 0$ if $m < s$, $k_{\alpha\alpha}^{+sm} = k_{\alpha\alpha}^{sm}$, $k_{\alpha\alpha}^{-sm} = 0$ with $m > s$ and any $\alpha = 1, \ldots, p$. The matrix $k_{\alpha\alpha}^{\mp}$ is a diagonal $p \times p$ matrix with submatrices which are triangular $n \times n$ matrices, conjugate to each other

$$k_{\alpha\alpha}^{-sm} = k_{\alpha\alpha}^{+ms} \qquad (a_{\alpha\alpha}^{-sm} = a_{\alpha\alpha}^{+ms}). \tag{13}$$

In accordance with the representation $k_{\alpha\alpha} = \overset{-}{k}_{\alpha\alpha} + \overset{+}{k}_{\alpha\alpha}$ we obtain

$$L_{\alpha\alpha} = \overset{-}{L}_{\alpha\alpha} + \overset{+}{L}_{\alpha\alpha}, \qquad \Lambda_{\alpha\alpha} = \overset{-}{\Lambda}_{\alpha\alpha} + \overset{+}{\Lambda}_{\alpha\alpha},$$

where

$$\overset{\mp}{L}_{\alpha\alpha}u = \frac{\partial}{\partial x_\alpha}\left(\overset{\mp}{k}_{\alpha\alpha}\frac{\partial u}{\partial x_\alpha}\right)$$

etc. In view of condition (13) the operators $\Lambda_{\alpha\alpha}^-$ and $\Lambda_{\alpha\alpha}^+$ are conjugate to each other on the network ω_h in the sense of the scalar product

$$(y, v) = \sum_{\omega_h} y(x_i)v(x_i)H, \qquad H = h_1 \ldots h_p,$$

i.e.

$$(\overset{-}{\Lambda}_{\alpha\alpha}y, v) = (\overset{+}{\Lambda}_{\alpha\alpha}v, y), \qquad a = 1, \ldots p, \tag{14}$$

where $\mathbf{y}$ and $\mathbf{v}$ are arbitrary network functions, vanishing on the boundary γ_h of the network ω_h.

We shall put the operator

$$L = \sum_{\alpha,\,\beta=1}^{p} L_{\alpha\beta}$$

in the form of the sum of two triangular operators

$$\begin{gathered}
L = L^- + L^+, \qquad L^{\mp} = \sum_{\alpha,\,\beta=1}^{p} \overset{\mp}{L}_{\alpha\beta} = \sum_{\alpha=1}^{p} \overset{\mp}{L}_{\alpha}, \qquad L_\alpha{}^{\mp} = \sum_{\beta=1}^{p} \overset{\mp}{L}_{\alpha\beta}, \\
\overset{\mp}{L}_{\alpha\beta} = \overset{\mp}{L}_{\alpha\alpha} \text{ if } \beta = \alpha, \qquad L_{\alpha\beta} = L_{\alpha\beta}, \\
\overset{+}{L}_{\alpha\beta} = 0 \text{ if } \beta \leqslant \alpha, \qquad \overset{+}{L}_{\alpha\beta} = L_{\alpha\beta}, \\
L_{\alpha\beta} = 0 \text{ if } \beta > \alpha, \qquad L_\alpha{}^- = L_{\alpha\alpha} + \sum_{\beta=1} L_{\alpha\beta}, \qquad L_\alpha = L_{\alpha\alpha} + \sum_{\beta=\alpha+1}^{p} L_{\alpha\beta}.
\end{gathered} \tag{15}$$

By virtue of the principle of additivity of [1 - 5] the solution of the system of equations

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \sum_{\alpha=1}^{p} [(\overset{-}{L}_\alpha + \overset{+}{L}_\alpha)\mathbf{u} + \mathbf{f}_\alpha], \qquad \sum^{p} \mathbf{f}_\alpha = \mathbf{f}$$

reduces to the successive solution, on an $\omega_h \times \omega_\tau$ network with step τ/p, of the simpler equations

$$\frac{1}{p}\frac{\partial^2 \mathbf{u}}{\partial t^2} = \overset{-}{L}_\alpha \mathbf{u} + \overset{+}{L}_\alpha \mathbf{u} + \mathbf{f}_\alpha. \tag{16}$$

The case where $L_{\alpha\beta} = \delta_{\alpha\beta} L_{\alpha\alpha}$, i.e. compound derivatives are absent, is considered in [1].

We now introduce the values $\mathbf{y}^{j+\alpha/p} = \mathbf{y}_{(\alpha)}$, intermediate between $\mathbf{y}^j = \check{\mathbf{y}}$ and $\mathbf{y}^{j+1} = \mathbf{y}$ assuming that $\mathbf{y}^{(j-1)+\alpha/p} = \check{\mathbf{y}}_{(\alpha)}$, and use for the determination of $\mathbf{y}_{(\alpha)}$ a difference scheme which approximates equation (16) with number α. By analogy with [1], for the approximation of $\partial^2 \mathbf{u} / \partial t^2$ we use the $(p + 1)$-th time layer

$$\frac{1}{p}\frac{\partial^2 \mathbf{u}}{\partial t^2} \sim \sigma_p \mathbf{u}_{\bar{t}_\alpha \bar{t}_\alpha}, \qquad \alpha = 1, \ldots, p; \tag{17}$$

$$u_{\bar{t}_\alpha \bar{t}_\alpha} = \begin{cases} (u_{(\alpha)} - 2u_{(\alpha-1)} + \check{u}_{(\alpha)})/\tau^2, & \sigma_p = 2, \quad u_{(0)} = \check{u}_{(2)}, \quad p = 2 \\ (u_{(\alpha)} - u_{(\alpha-1)} - u_{(\alpha-2)} + \check{u}_{(\alpha)})/\tau^2, & \sigma_p = \frac{3}{2}, \quad u_{(0)} = \check{u}_{(3)}, \\ & u_{(-1)} = \check{u}_{(2)}, \quad p = 3. \end{cases}$$

The additive scheme of alternating directions for the problem (1) - (2) will have the form

$$\sigma_p y_{\bar{t}_\alpha \bar{t}_\alpha} = \sum_{\beta=1}^{\alpha} \Lambda^-_{\alpha\beta} y_{(\beta)} + \sum_{\beta=\alpha}^{p} \Lambda^+_{\alpha\beta} \check{y}_{(\beta)} + \varphi_\alpha, \quad \alpha = 1, \ldots, p, \qquad (x, t) \in \omega_h \times \omega_t, \tag{18}$$

$$y_\alpha = \nu(x, t_j^*) \text{ with } x_\alpha = 0, l_\alpha, \qquad \alpha = 1, \ldots, p; \qquad y(x, 0) = \nu_0(x), \tag{19}$$

where $\varphi_\alpha = \varphi_\alpha(x, t_j^*)$ is a second order approximation on the network ω_h of the function $f_\alpha(x, t)$, $t_j^* \in [t_j, t_{j+1}]$, e.g. $t_j^* = t_{j+1/2} = t_j + 0.5\tau$. The coefficients $a_{\alpha\beta} = a_{\alpha\beta}(x, t_{(\alpha)}^*)$ are taken at the middle moment $t_{(\alpha)}^* = t_j + \frac{\alpha}{p}\tau - 0.5\tau$ (cf. [1]).

The second initial condition can be approximated by analogy with [1], or more simply by assuming

$$y^{\alpha/p} = \nu_1(x) + \frac{\alpha\tau}{p} \tilde{\nu}_1(x), \qquad \alpha = 1, \ldots, p-1, \qquad p = 2, 3. \tag{20}$$

Such a condition is sufficient for an accuracy $O(\tau + |h|^2)$. Let us re-write (18) in the form

$$\left(E - \frac{\tau^2}{\sigma_p} \Lambda^-_{\alpha\alpha}\right) y_{(\alpha)} = R_\alpha(y) + \frac{\tau^2}{\sigma_p} \sum_{\beta=1}^{\alpha-1} \Lambda_{\alpha\beta} y_{(\beta)} + F_\alpha = \Phi_\alpha, \tag{21}$$

where

$$F_\alpha = \frac{\tau^2}{\sigma_p} \left[\sum_{\beta=\alpha}^{p} \Lambda^+_{\alpha\beta} \check{y}_{(\beta)} + \varphi_\alpha \right],$$

$R_\alpha(y) = 2y_{(\alpha-1)} - \check{y}_{(\alpha)}$ if $p = 2$, $R_\alpha(y) = y_{(\alpha-1)} + y_{(\alpha-2)} - \check{y}_{(\alpha)}$ if $p = 3$ and E is the operator of identity.

Here it is obvious that to determined $y_{(\alpha)}$ (all $y_{(\beta)}$ for $\beta < \alpha$ and all $\check{y}_{(\beta)}$ for $\beta = 1, \ldots, p$ are already known) we obtain a system of three-point equations with a triangular matrix for their coefficients. Its solution reduces to an inversion of the operator $E - (\tau^2/\sigma_p)\Lambda_{\alpha\alpha}^-$, which is attained by an n-fold use of the ordinary formulae for each chain $y_{(\alpha)}$ (see [1]) for fixed $\alpha = 1, \ldots, p$. To realize the algorithm (21) we must bear in mind the values of $\mathbf{y}_{(\alpha)}$ on p layers.

Let us write the equation for the s-th components $y_{(\alpha)}^s$ of the vector $\mathbf{y}_{(\alpha)}$

$$y_{(\alpha)}^s - \frac{\tau^2}{\sigma_p}(a_{\alpha\alpha}^{-ss} y_{\bar{x}_\alpha}^s)_{x_\alpha} = \Phi_{(\alpha)}^s + \frac{\tau^2}{\sigma_p}\sum_{m=1}^{s-1}(a_{\alpha\alpha}^{-sm} y_{\bar{x}_\alpha}^m)_{x_\alpha}. \tag{22}$$

Here Φ_α^s is known, since the calculation takes place in the direction from α to $\alpha + 1$, $\alpha = 1, \ldots, p$; the second term is also known if we determine successively $y_{(\alpha)}^1, \ldots, y_{(\alpha)}^s, \ldots, y_{(\alpha)}^p$, i.e. carry out the calculation from s to $s + 1$. Hence it is obvious that we can find $y_{(\alpha)}^s$ by solving the first boundary value problems for the three-point equations on segments parallel to the axis Ox_α.

If we interchange the roles of L_α^- and L_α^+, respectively,

$$\Lambda_\alpha^- = \Lambda_{\alpha\alpha}^- + \sum_{\beta=1}^{\alpha-1}\Lambda_{\alpha\beta} = \sum_{\beta=1}^{\alpha}\Lambda_{\alpha\beta}^- \text{ and } \Lambda_\alpha^+ = \Lambda_{\alpha\alpha}^+ + \sum_{\beta=\alpha+1}^{p}\Lambda_{\alpha\beta} = \sum_{\beta=\alpha}^{p}\Lambda_{\alpha\beta}^+,$$

we obtain a second additive scheme

$$\sigma_p \mathbf{y}_{\bar{t}_\alpha \bar{t}_\alpha} = \sum_{\beta=\alpha}^{p}\Lambda_{\alpha\beta}^+ \mathbf{y}_{(\beta)} + \sum_{\beta=1}^{\alpha}\Lambda_{\alpha\beta}^+ \check{\mathbf{y}}_{(\beta)} + \varphi_\alpha \tag{23}$$

with the same initial and boundary conditions as in the first scheme. Here to determine y_α we must invert the triangular three-point operator $E - (\tau^2/\sigma_p)\Lambda_{\alpha\alpha}^+$. Here the calculation proceeds from $\alpha + 1$ to α and from $s + 1$ to s.

The alternation of these two schemes gives a third scheme. Introducing the intermediate value $y^{j+\alpha/2p}$, $\alpha = 1, \ldots, 2p - 1$, we obtain (cf. [2, 5])

$$\sigma_p \mathbf{y}_{\bar{t}_\alpha \bar{t}_\alpha} = \sum_{\beta=1}^{\alpha}\Lambda_{\alpha\beta}^- \mathbf{y}_{(\beta)} + \sum_{\beta=\alpha}^{p}\Lambda_{\alpha\beta}^+ \check{\mathbf{y}}_{(\beta)} + \varphi_\alpha, \qquad \alpha = 1, \ldots, p; \tag{24}$$

$$\sigma_p y_{\bar{t}_{\alpha'}\bar{t}_{\alpha'}} = \sum_{\beta=\alpha}^{p} \Lambda^{+}_{\alpha\beta} y_{(\beta')} + \sum_{\beta=1}^{\alpha} \Lambda^{-}_{\alpha\beta'} \check{y}_{(\beta')} + \varphi_{\alpha'},$$
$$\alpha' = 2p + 1 - \alpha, \quad \beta' = 2p + 1 - \beta;$$

where $\alpha' = p + 1, \ldots, 2p,\ \ \varphi_{\alpha'} = \varphi_{\alpha},\ \ \alpha = p, p-1, \ldots, 2, 1.$

5. Schemes (18) and (23) are stable for sufficiently small $|h| \leqslant h_0$, ensuring that the requirement (10) of the positive definiteness of the matrices $(a_{\alpha\beta})$ and $(a_{\alpha\beta}^{(+1\beta)})$ for any τ is satisfied, and converge at least with a speed $O(|h|^2 + \tau)$. The proof of these statements is carried out by analogy with [1, 4] by the method of energy inequalities.

Here the basic part is played by an identity of the form

$$\sum_{\alpha=1}^{p} \{(a^{-}_{\alpha\beta}\xi_{\beta},\ \xi_{\alpha} - \check{\xi}_{\alpha}) + (a^{+}_{\alpha\beta}\check{\xi}_{\beta},\ \xi_{\alpha} - \check{\xi}_{\alpha}) = J - \check{J}\ \ (1 + O(\tau)) + R,$$

where

$$J = \sum_{\alpha=1}^{p} \sum_{\beta=1}^{\alpha} (a^{-}_{\alpha\beta}\xi_{\beta},\ \xi_{\alpha}) = \sum_{\alpha=1}^{p} \sum_{\beta=\alpha}^{p} (a^{+}_{\alpha\beta}\xi_{\beta},\ \xi_{\alpha}),$$
$$R = \sum_{\alpha=1}^{p} \left\{ \sum_{\beta=1}^{\alpha} (a^{+}_{\alpha\beta}\check{\xi}_{\beta},\ \xi_{\alpha}) - (a^{-}_{\alpha\beta}\check{\xi}_{\beta},\ \xi_{\alpha}) \right\}.$$

Using (3) it is not difficult to see that $R = 0$.

Note. In the case of constant coefficients. $k_{\alpha\beta} =$ const., the given values of the accuracy of the schemes considered, (18) and (23), are valid for any h_{α} and τ.

6. We now turn to the equations of elasticity (5). In this case as we have seen in Section 2, $n = p$, and $k^{sm}_{\alpha\beta} =$ const.,

$$k^{sm}_{\alpha\beta} = \mu\delta_{\alpha\beta}\delta_{sm} + (\lambda + \mu)\,\delta_{\alpha s}\delta_{\beta m}, \qquad \delta_{sm} = \begin{cases} 1, & s = m, \\ 0, & s \neq m. \end{cases} \tag{25}$$

For the equations of elasticity (5) we can use any of the schemes considered in Section 4, bearing in mind that $a_{\alpha\beta} = k_{\alpha\beta}$, where $k_{\alpha\beta}$ is determined by formula (25).

We shall write in more detail the difference equations (18) in the case where $p = 2$; here $\mathbf{y} = (y^{(1)}, y^{(2)})$ and

$$2y^{(1)}_{\bar{t}_1 t_1} = \frac{1}{2}(\lambda + 2\mu)(y^{(1)}_{(1)\bar{x}_1 x_1} + \check{y}^{(1)}_{(1)\bar{x}_1 x_1}) + \frac{1}{2}(\lambda + \mu)(\check{y}^{(2)}_{(2)\bar{x}_2 x_1} + \check{y}^{(2)}_{(2)x_2\bar{x}_1}) + \varphi^{(1)}_{(1)}$$

$$2y^{(2)}_{\bar{t}_1 t_1} = \frac{1}{2}\mu(y^{(2)}_{(1)\bar{x}_1 x_1} + \check{y}^{(2)}_{(1)\bar{x}_1 x_1}) + \varphi^{(2)}_{(1)},$$

$$2y^{(1)}_{\bar{t}_2 t_2} = \frac{1}{2}\mu(y^{(1)}_{(2)\bar{x}_2 x_2} + \check{y}^{(1)}_{(2)\bar{x}_2 x_2}) + \varphi^{(1)}_{(2)},$$

$$2y^{(2)}_{\bar{t}_2 t_2} = \frac{1}{2}(\lambda + 2\mu)(y^{(2)}_{(2)\bar{x}_2 x_2} + \check{y}^{(2)}_{(2)\bar{x}_2 x_2}) + \frac{1}{2}(\lambda + \mu)(y^{(1)}_{(1)\bar{x}_2 x_1} + y^{(1)}_{(1)\bar{x}_1 x_2}) + \varphi^{(2)}_{(2)}.$$

We must remember that here the upper index means the number of the component and the lower one the number of the vector $(y^{(1)}_{(1)} = (y^{(1)})^{j+1/2}$, $y^{(1)}_{(2)} = (y^{(1)})^{j+1} = y^{(1)}$, $\check{y}^{(2)}_{(2)} = (y^{(2)})^{j+1} = \check{y}^{(2)}$ etc.). Using (18) and (25) it is not difficult to write down the scheme for $p = 3$.

7. For the equations of elasticity we can also construct a series of resolving schemes which are absolutely stable, economic and convergent with speed $O(\tau + |h|^2)$ or $O(\tau^2 + |h|^2)$.

Let us first consider the two multidimensional schemes

$$\mathbf{y}_{\bar{t}t} = \Lambda^-\mathbf{y} + \Lambda^+\check{\check{\mathbf{y}}} + \check{\boldsymbol{\varphi}} \qquad (\mathbf{y}_{\bar{t}t} = (\mathbf{y}^{j+1} - 2\mathbf{y}^j + \mathbf{y}^{j-1})/\tau^2); \tag{I$^-$}$$

$$\mathbf{y}_{\bar{t}t} = \Lambda^+\mathbf{y} + \Lambda^-\check{\check{\mathbf{y}}} + \check{\boldsymbol{\varphi}}, \tag{I$^+$}$$

where Λ^- and Λ^+ are triangular operators, which approximate the triangular differential operator L^- and L^+, $y = y^{j+1}$, $\check{\check{y}} = y^{j-1}$, $\check{y} = y^j$.

Let $\mathring{\Lambda}_\alpha \mathbf{y} = \mathbf{y}_{\bar{x}_\alpha x_\alpha}$, and $\mathring{\Lambda}_{sk}$ denote the difference approximation of the compound derivative $\partial^2\mathbf{u}/\partial x_s \partial x_k$, e.g. $\mathring{\Lambda}_{sk}\mathbf{y} = \frac{1}{2}(\mathbf{y}_{\bar{x}_s x_k} + \mathbf{y}_{x_s \bar{x}_k})$ or $\mathring{\Lambda}_{sk}\mathbf{y} = \frac{1}{2}(\mathbf{y}_{\bar{x}_s \bar{x}_k} + \mathbf{y}_{x_s x_k})$. Then the expressions for the triangular operators Λ^- and Λ^+ can be written in the form (the upper index s or k is the number of the component)

$$(\Lambda^-\mathbf{y})^s = \frac{1}{2}\sum_{\alpha=1}^{p} \varkappa_{s\alpha}\mathring{\Lambda}_\alpha y^s + (\lambda + \mu)\sum_{k=1}^{s-1}\mathring{\Lambda}_{sk} y^k, \qquad \varkappa_{s\alpha} = \mu + (\lambda + \mu) + \delta_{s\alpha}, \tag{26}$$

$$(\Lambda^+\mathbf{y})^s = \frac{1}{2}\sum_{\alpha=1}^{p} \varkappa_{\rho\alpha}\mathring{\Lambda}_\alpha y^s + (\lambda + \mu)\sum_{k=s+1}^{p}\mathring{\Lambda}_{sk} y^k. \tag{27}$$

The boundary conditions on γ_h are exactly satisfied

$$\mathbf{y}|_{\gamma_h} = \boldsymbol{\nu}(x, t),$$

and the initial conditions have the form

$$\mathbf{y}(x, 0) = \boldsymbol{\nu}_0(x), \qquad \mathbf{y}_{\bar{t}}(x, \tau) = \boldsymbol{\nu}_1(x) + \tau\tilde{\boldsymbol{\nu}}_1(x),$$

where $\tilde{\boldsymbol{\nu}}_1(x)$ is chosen so that the initial condition $\partial\mathbf{u}/\partial t = \boldsymbol{\nu}_1$ is approximated with accuracy $O(\tau^2)$; for this, for instance, it is sufficient to assume that $\tilde{\boldsymbol{\nu}}_1 = -(L\mathbf{u} + \mathbf{f})|_{t=0}$.

Each of the schemes I^- and I^+ is absolutely stable and has an accuracy $O(\tau + |h^2|)$. Applying these schemes alternately (e.g. scheme (26) on odd and scheme (27) on even layers), we obtain an accuracy $O(\tau^2 + |h|^2)$.

Following the principle given in (8), we shall write the generating scheme II^- for the scheme I^-

$$\left.\begin{aligned} A^s y^s_{\bar{t}} &= \check{\Phi}^s + F^s, \qquad A^s = \prod_{\alpha=1}^{p} A^s_\alpha, \qquad A^s_\alpha = E - 0.5\tau^2 \varkappa_{s\alpha}\hat{\Lambda}_\alpha, \\ \check{\Phi}^s &= \check{y}^s_{\bar{t}} + \tau\left[(\Lambda^+\check{y})^s + 0.5\sum_{\alpha=1}^{p}\varkappa_{s\alpha}\mathring{\Lambda}_\alpha\check{y}^s + \check{\varphi}^s\right], \\ F^s &= \tau(\lambda + \mu)\sum_{k=1}^{s-1}\mathring{\Lambda}_{sk}y^k. \end{aligned}\right\} \tag{28}$$

Similarly the generating scheme II^+ is written for the scheme I^+. In this case only the formulae for $\check{\Phi}^s$ and F^s are changed

$$\check{\Phi}^s = \check{y}^s_{\bar{t}} + \tau\left[(\Lambda^-\check{y})^s + 0.5\sum_{\alpha=1}^{p}\varkappa_{s\alpha}\mathring{\Lambda}_\alpha\check{y}^s + \check{\varphi}^s\right], \qquad F^s = \tau(\lambda + \mu)\sum_{k=s+1}^{p}\mathring{\Lambda}_{sk}y^k. \tag{29}$$

To determine y^s on a new layer we can write some numerical algorithms for alternating directions. We give for scheme II^- only the algorithm which we put forward in [8] for the equation of heat conduction: this algorithm has the form

$$A_1^s v^s_{(1)} = \check{\Phi}^s + F^s, \qquad A_\alpha v^s_{(\alpha)} = v^s_{(\alpha-1)}, \quad \alpha = 1, 2, \ldots, p, \ s = 1, \ldots, p; \qquad y^s = \check{y}^s + \tau v^s_{(p)}. \tag{30}$$

We shall take the boundary conditions with $x_\alpha = 0$, $x_\alpha = l_\alpha$ for $v_{(\alpha)}{}^s$ in the form

$$v^s_{(\alpha)} = A^s_{\alpha+1} \ldots A_p{}^s \mathbf{v}_{\bar{t}}{}^s \text{ for } x_\alpha = 0, \quad x_\alpha = l_\alpha, \quad \alpha = 1, \ldots, p-1, \quad v^s_{(p)} = \mathbf{v}_{\bar{t}}{}^s. \tag{31}$$

The order of calculation is as follows: the components $y^{(1)}, \ldots, y^{(p)} = y^{j+1}$ are determined in turn.

For the algorithm which corresponds to scheme II$^+$ the formulae (30) remain unchanged, but the order of calculation is reversed: the components $y^{(p)}, \ldots, y^{(1)}$ are determined successively.

Alternating the schemes II$^-$ and II$^+$ we find the solution of the problem with accuracy to within $O(|h|^2 + \tau^2)$. This evaluation is obtained by the method of [1, 2, 8].

From the formulae for $v_{(\alpha)}{}^s$ it is obvious that the solution y^s of the difference problem is determined by means of successive inversion of the triangular matrices (according to the formulae given in [9]). Therefore the resolving schemes are economic: to calculate the vector $\mathbf{y}^{j+1}$ operations of the order $O(p^2 / h^p)$ are required.

8. We turn now to a stationary problem of the theory of elasticity

$$L\mathbf{u} = \mu \Delta \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} = -\mathbf{f}(x), \quad x \in G, \quad \mathbf{u}|_\Gamma = \mathbf{v}(x). \tag{32}$$

Its solution reduces to the solution of the difference problem for establishing the parabolic system of equations

$$\frac{\partial \mathbf{u}}{\partial t} = L\mathbf{u} + f(x), \quad \mathbf{u}|_\Gamma = \mathbf{v}(x)$$

with arbitrary initial data

$$\mathbf{u}(x, 0) = \mathbf{v}_0(x).$$

For this we make use of the resolving scheme. Let the difference scheme for (32) take the form

$$\Lambda^- \mathbf{v} + \Lambda^+ \mathbf{v} = \varphi, \quad \mathbf{v}|_{\gamma_h} = \mathbf{v}_0(x).$$

We write the generating scheme for the determination of $\mathbf{y} = \mathbf{y}(x_i, (j+1)\tau)$, where $j+1$ is the number of the iteration and τ the iteration parameter, as

$$A^s y_{\bar{t}}{}^s = \check{\Phi}^s + F^s, \qquad A^s = \prod_{\alpha=1}^{p} A_\alpha{}^s, \qquad A^s_\alpha = E - 0.5\, \tau \varkappa_{s\alpha} \mathring{\Lambda}_\alpha,$$

$$\check{\Phi}^s = \sum_{\alpha=1}^{p} \varkappa_{s\alpha} \mathring{\Lambda}_\alpha \check{y}^s + (\lambda + \mu) \sum_{k=s+1}^{p} \mathring{\Lambda}_{sk} \check{y}^k + \varphi^s,$$

$$F^s = (\lambda + \mu) \sum_{k=1}^{s-1} \mathring{\Lambda}_{sk} y^k.$$

In this case the numerical algorithm for alternating directions of Section 6 takes the form

$$A_1{}^s w_{(1)}{}^s = \check{\Phi}^s + F^s, \quad A_\alpha{}^s w_{(\alpha)}{}^s = w^{s}_{(\alpha-1)}, \quad \alpha > 1,$$
$$y^s = \check{y}^s + \tau w_{(p)}{}^s, \quad w_{(\alpha)}{}^s|_{\gamma_h} = 0,$$

i.e. for $w_{(\alpha)}{}^s$ the boundary conditions are always zero.

For comparison we quote one more numerical algorithm (two-layered)

$$A_1{}^s y_{(1)}{}^s = \tau(\check{\Phi}^s + F^s) + A^s \check{y}^s, \quad A_\alpha{}^s y^{s}_{(\alpha-1)} = y^{s}_{(\alpha-1)}, \quad \alpha > 1,$$

$$y_{(\alpha)}{}^s = A^{s}_{(\alpha+1)} \ldots A_p{}^s \nu^s \quad \text{for} \quad x_\alpha = 0, \quad x_\alpha = l_\alpha, \quad \alpha = 1, 2, \ldots, p-1:$$
$$y_{(p)}{}^s|_{\gamma_h} = \nu^s.$$

The order of the computation is the same as before: initially the first component $y^{(1)}$ is determined, then the second $y^2 (s = 2)$ etc.

To find y^s, at all nodes of the network ω_h operations of the order $O(1/h_1 h_2 \ldots h_p)$ are required. The iteration process converges for $\tau = O(h_*)$, $h_* = \min h_\alpha$. The rate of convergence is determined by the number of iterations $\nu \approx O((1/h_*) \times \ln(1/\varepsilon))$ for $p = 2$ and $O((1/h_*{}^{4/3}) \ln(1/\varepsilon))$ for $p = 3$, where ε is the required accuracy.

The evaluation for the number of iterations is obtained by the method of energy inequalities by analogy with [10, 7, 4].

Switching the roles of the operators Λ^- and Λ^+, we obtain the second iteration scheme II$^+$. The same value is obtained for the rate of convergence of the iterations with it as for the scheme II$^-$ described above. It is hoped that this is the basis which the interchange of these two iteration algorithms II$^-$ and II$^+$ can bring to speeding up to convergence.

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