1. In this section we consider first of all additive difference schemes (see [1] - [5]) for a system of second order hyperbolic equations which contain compound derivatives. There are schemes with variable directions which are absolutely stable and convergent at least with a speed

\[ O(|h|^2 + \tau), \]  

where \[ |h|^2 = \sum_{\alpha=1}^{p} h^2_\alpha, \]  

\( h_\alpha \) is the step in the variable \( a_\alpha \) and \( p \) is the number of dimensions. The numerical algorithm consists of the conversion of a three-point triangular operator, which reduces to the successive application of known formulae. The number of operations to determine a solution for a new time layer is proportional to the number of nodes of the space network and is a quantity of the same order as the number of operations for a purely explicit scheme. Thus the schemes put forward below are economic.

To construct economic schemes for an equation of the form

\[ \frac{\partial^2 u}{\partial t^2} + \sum_{\alpha, \beta=1}^{p} A_{\alpha\beta}(t) u = f \]

we use the common property of the operator

\[ A = \sum_{\alpha, \beta=1}^{p} A_{\alpha\beta}, \]

A hyperbolic system of equations with compound derivatives

i.e. its representation in the form of the sum of operators $A_{ab}$ of simpler structure.

Additive economic schemes for a general second order hyperbolic system are then used to describe economic schemes for a system of equations in the theory of elasticity in the case of two or three space variables $(p = 2, p = 3)$.

A resolving scheme is also constructed for equations of elasticity, which is absolutely stable and converges with a speed $O(|h|^2 + r^2)$. With regard to economy the resolving scheme is comparable with additive schemes, but for convergence it requires more smoothness for the solution of the differential equation.

In Section 7 an iteration scheme with alternating directions for the solution of a difference problem, corresponding to a stationary problem in the theory of elasticity, is considered.

The convergence of this scheme is proved for $p = 2, 3$, and it is shown that the number of iterations $\nu = O(h^{-2(p-1)/p} \ln (1/\varepsilon))$, where $\varepsilon$ is the required accuracy.

Economic schemes of another type are considered in the two-dimensional case $(p = 2)$ for a dynamic problem of the theory of elasticity in [6] and for a static problem in the theory of elasticity in [7].

2. Let $\bar{G} = G + \Gamma = \{0 \leq x_\alpha \leq l_\alpha, \alpha = 1, \ldots, p\}$ be a $p$-dimensional parallelepiped with boundary $\Gamma$. In the cylinder $\bar{Q}_T = \bar{G} \times [0 \leq t \leq T]$ there is a solution of the problem

$$\frac{\partial^2 u}{\partial t^2} = \sum_{\alpha, \beta=1}^p L_{ab} u + f(x, t), \quad L_{ab} u = \frac{\partial}{\partial x_{\alpha}} \left( k_{ab}(x, t) \frac{\partial u}{\partial x_{\beta}} \right), \quad (1)$$

$$u |_{\Gamma} = \nu(x, t), \quad 0 \leq t \leq T, \quad u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t} (x, 0) = \nu_1(x), \quad x \in \bar{G}. \quad (2)$$

Here $x = (x_1, \ldots, x_p); u = u(x, t) = (u^1, \ldots, u^s, \ldots, u^n), f, \nu, u_0, \nu_1$ are vectors of dimensionaly $n$, and $k = (k_{ab}) = (k_{ab}^{sm}), s, m = 1, \ldots, n$, is a partitioned $p \times p$ matrix with $n \times n$ submatrices, which satisfies the symmetry condition

$$k_{ab}^{mn}(x, t) = k_{ba}^{mn}(x, t) \quad (3)$$
and the condition of positive definiteness

\[ \sum_{s, m=1}^{n} \sum_{\alpha, \beta=1}^{p} k_{\alpha \beta}^{m}(x, t) \xi_{\beta}^{m} \xi_{\alpha}^{n} \geq c_{1} \sum_{s=1}^{n} \sum_{\alpha=1}^{p} (\xi_{\alpha}^{s})^{2}, \quad (x, t) \in \bar{Q}_{T}, \]  

where \( \xi_{\alpha} = (\xi_{\alpha}^{1}, \ldots, \xi_{\alpha}^{s}, \ldots, \xi_{\alpha}^{n}) \neq 0 \) is an arbitrary real vector and \( c_{1} \) is a positive constant.

We shall assume that the problem (1) - (2) has a unique solution \( u = u(x, t) \), which is continuous in \( \bar{Q}_{T} \) and differentiable as many times as necessary for this work. To infer the \( a \ priori \) evaluations it is assumed that \( k_{\alpha \beta}(x, t) \) satisfies a Lipschitz condition with respect to \( t \) and \( x_{\alpha'}, \alpha' = 1, \ldots, p \).

The system of equations in the theory of elasticity

\[ \frac{\partial^{2}u}{\partial t^{2}} = \mu \Delta u + (\lambda + \mu) \text{grad} \text{div} u + f(x, t), \]  

where \( \Lambda u = \sum_{\alpha=1}^{p} \frac{\partial^{2}u}{\partial x_{\alpha}^{2}} \) is the Laplace operator, \( u = (u_{1}, \ldots, u_{p}) \), \( \lambda = \text{const} > 0 \) and \( \mu = \text{const} > 0 \) are Lamé's coefficients, is obviously a particular case of the system of equations (1) with \( n = p \) and

\[ k_{\alpha \beta}^{m} = \mu \delta_{\alpha \beta} \delta_{s m} + (\lambda + \mu) \delta_{s \alpha} \delta_{s \beta}, \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \]  

where \( \delta_{ij} \) is the Kronecker delta. Condition (3) is satisfied automatically. We shall show that condition (4) is also satisfied if \( c_{1} = \mu \). In fact

\[ \sum_{s, m=1}^{p} k_{\alpha \beta}^{m} \xi_{\alpha}^{s} \xi_{\beta}^{m} = \mu \sum_{s=1}^{p} (\xi_{s}^{s})^{2} + (\lambda + \mu) \sum_{s=1}^{p} \xi_{s}^{s} \xi_{s}^{s} = \]

\[ = \mu \sum_{s=1}^{p} (\xi_{s}^{s})^{2} + (\lambda + \mu) \left( \sum_{s=1}^{p} (\xi_{s}^{s})^{2} \right) = c_{1} \sum_{s=1}^{p} (\xi_{s}^{s})^{2}. \]

3. Let us introduce the difference networks \( \omega_{s} = \{ t_{j} = j \tau \in [0, T], \ j = 0, 1, \ldots \} \) and \( \bar{\omega}_{h} = \{ x_{i} = (i_{1} h_{1}, \ldots, i_{h_{n}} h_{n}, \ldots, i_{h_{p}} h_{p}) \in \bar{G} = G + \Gamma; \ i_{\alpha} = 0, 1, \ldots, N_{\alpha}, \ h_{\alpha} = l_{\alpha}/N_{\alpha}, \ \alpha = 1, 2, \ldots, p \} \) with steps \( \tau \) for the
variable $t$ and $h_\alpha$ for the variable $x_\alpha$, $\alpha = 1, \ldots, p$: let $\gamma_h = \{x_\alpha \in \Gamma\}$ be the boundary of the network $\omega_h$, $\omega_h \setminus \gamma_h = \{x_\alpha \in G\}$ be the set of inner nodes, $|h|^2 = \sum_{\alpha=1}^{p} h_\alpha^2$. Following [1], we shall introduce the notation

$$y - y(x_\alpha, t_{j+1}) = y^{j+1}, \quad \tilde{y} = y^j,$$

$$y^{(\pm 1)_\alpha} = (i_1 h_1, \ldots, i_{\alpha-1} h_{\alpha-1}, (i_\alpha \pm 1) h_\alpha, i_{\alpha+1} h_{\alpha+1}, \ldots, i_p h_p),$$

$$y^{(\pm 1)_\alpha} = y(x^{(\pm 1)_\alpha}, t_{j+1}), \quad y_{x_\alpha} = (y - y^{(-1)_\alpha})/h_\alpha, \quad y_{x_\alpha} = (y^{(+1)_\alpha} - y)/h_\alpha.$$

We shall replace the operator

$$L_{a_{\alpha\beta}} u = \frac{\partial}{\partial x_\alpha} \left( k_{a_{\alpha\beta}}(x, t) \frac{\partial u}{\partial x_\beta} \right)$$

in the difference network $\omega_h$ by the same scheme of the second order of approximation as in [2], assuming that

$$L_{a_{\alpha\beta}} y = \frac{1}{2} \left[ (a_{a_{\alpha\beta}} y_{\alpha} x_{\beta}) x_\alpha + (a_{a_{\alpha+1}} y_{\alpha} x_{\beta})^a \right] \text{ for } \beta \neq \alpha,$$

$$L_{a_{\alpha\alpha}} y = (a_{a_{\alpha\alpha}} y_{\alpha} x_{\alpha}),$$

where $(a_{a_{\alpha\beta}})$ is a matrix-functional of the matrix $(k_{a_{\alpha\beta}})$ with pattern ($-1 \leq s_\beta \leq 0, \beta = 1, \ldots, p$). The coefficients $a_{a_{\alpha\beta}} = (a_{a_{\alpha\beta}}^{sm})$ satisfy the conditions

$$a_{a_{\alpha\beta}}^{sm} = a_{a_{\alpha\beta}}^{ms},$$

for sufficiently small $|h| \leq h_0$, the condition

$$\sum_{s,m=1}^{n} \sum_{a,b=1}^{p} a_{a_{\alpha\beta}}^{sm} a_{a_{\alpha\beta}}^{ms} e_\alpha e_\beta \geq c_{1}' \sum_{s,m=1}^{n} (e_\alpha e_\beta)^2, \quad (x, t) \in \omega_h \times \omega_t,$$

where $c_{1}' \leq c_{1}$ is a positive constant which does not depend on the network, and the condition obtained from (10) after replacing $a_{a_{\alpha\beta}}^{sm}$ by the coefficient $(a_{a_{\alpha\beta}}^{sm})^{(1_{\beta})}$ at the point $x_{(\pm 1)^{\beta}}$.

In the case of constant coefficients, $k_{a_{\alpha\beta}} = \text{const.}$, obviously $a_{a_{\alpha\beta}} = k_{a_{\alpha\beta}}$ and instead of (8) we obtain

$$L_{a_{\alpha\beta}} y = \frac{1}{2} k_{a_{\alpha\beta}} (y_{x_\beta x_\alpha} - y_{x_\alpha x_\beta}), \quad L_{a_{\alpha\alpha}} y = k_{a_{\alpha\alpha}} y_{x_\alpha x_\alpha}.$$
condition (10) is satisfied on any network.

**Note.** For \( \Lambda_{\alpha\beta}y \) instead of (8) we can consider other representations also, e.g.

\[
\Lambda_{\alpha\beta}y = \frac{1}{2} \left[ \left( a_{\alpha\beta} \frac{y}{\partial x_\beta} \right) \hat{z}_\alpha + \left( a^{(+\beta)}_{\alpha\beta} y x_\beta \right) \hat{x}_\alpha \right],
\]

where \( \hat{z}_\alpha = \frac{1}{2} \left( v_{x_\alpha} + v_{x_\alpha} \right) \) is the central difference derivative. In all cases the condition (10) will be satisfied for a sufficiently small step \( |h| \ll h_0 \) and all the subsequent conclusions retain their validity.

4. Let us introduce the "triangular" operators \( L^- \) and \( L^+ \). For this we write the symmetrical matrix \( k_{\alpha\alpha} = \left( k_{\alpha\alpha}^{sm} \right) \) in the form of the sum of two triangular matrices \( k_{\alpha\alpha} = k_{\alpha\alpha}^- + k_{\alpha\alpha}^+ \), \( k_{\alpha\alpha}^+ = \left( k_{\alpha\alpha}^{+sm} \right) \), assuming \( k_{\alpha\alpha} = k_{\alpha\alpha}^{-sm} = \frac{1}{2} k_{\alpha\alpha}, k_{\alpha\alpha}^{sm} = k_{\alpha\alpha}, k_{\alpha\alpha}^{+sm} = 0 \) if \( m < s \), \( k_{\alpha\alpha}^+ = k_{\alpha\alpha}, k_{\alpha\alpha}^{-sm} = 0 \) with \( m > s \) and any \( \alpha = 1, \ldots, p \). The matrix \( k_{\alpha\alpha}^{rs} \) is a diagonal \( p \times p \) matrix with submatrices which are triangular \( n \times n \) matrices, conjugate to each other

\[
k_{\alpha\alpha}^{sm} = k_{\alpha\alpha}^{+sm}, \quad (a_{\alpha\alpha}^{sm} = a_{\alpha\alpha}^{+sm}). \tag{13}
\]

In accordance with the representation \( k_{\alpha\alpha} = k_{\alpha\alpha}^- + k_{\alpha\alpha}^+ \) we obtain

\[
L_{\alpha\alpha} = L_{\alpha\alpha}^- + L_{\alpha\alpha}^+, \quad \Lambda_{\alpha\alpha} = \Lambda_{\alpha\alpha}^- + \Lambda_{\alpha\alpha}^+,
\]

where

\[
\frac{L_{\alpha\alpha}^+ u}{L_{\alpha\alpha} u} = \frac{\partial}{\partial x_\alpha} \left( k_{\alpha\alpha}^{+rs} \frac{\partial u}{\partial x_\alpha} \right)
\]

etc. In view of condition (13) the operators \( \Lambda_{\alpha\alpha}^- \) and \( \Lambda_{\alpha\alpha}^+ \) are conjugate to each other on the network \( \omega_h \) in the sense of the scalar product

\[
(y, v) = \sum_{\omega_h} y(x_i) v(x_i) H, \quad H = h_1 \ldots h_p,
\]

i.e.

\[
(\Lambda_{\alpha\alpha}^- y, v) = (\Lambda_{\alpha\alpha}^+ y, v), \quad a = 1, \ldots, p, \tag{14}
\]
A hyperbolic system of equations with compound derivatives

where \( y \) and \( v \) are arbitrary network functions, vanishing on the boundary \( \gamma_h \) of the network \( \omega_h \).

We shall put the operator

\[
L = \sum_{\alpha, \beta=1}^{p} L_{\alpha \beta}
\]

in the form of the sum of two triangular operators

\[
L = L^- + L^+, \quad L^- = \sum_{\alpha, \beta=1}^{p} L_{\alpha \beta}^- = \sum_{\alpha=1}^{p} L_{\alpha}^-, \quad L^+ = \sum_{\beta=1}^{p} L_{\alpha \beta}^+, \quad L_{\alpha \beta}^+ = I_{\alpha \alpha} \text{ if } \beta = \alpha, \quad L_{\alpha \beta}^- = L_{\alpha \beta},
\]

(15)

\[
L_{\alpha \beta}^- = 0 \text{ if } \beta \leq \alpha, \quad L_{\alpha \beta}^+ = L_{\alpha \beta}, \quad L_{\alpha}^+ = L_{\alpha} = L_{\alpha \alpha} + \sum_{\beta=\alpha+1}^{p} L_{\alpha \beta}.
\]

By virtue of the principle of additivity of [1 - 5] the solution of the system of equations

\[
\frac{\partial^2 u}{\partial t^2} = \sum_{\alpha=1}^{p} [(L^- + L^+) u + f_{\alpha}], \quad \sum_{\alpha=1}^{p} f_{\alpha} = f
\]

reduces to the successive solution, on an \( \omega_h \times \omega_r \) network with step \( \tau/p \), of the simpler equations

\[
\frac{1}{p} \frac{\partial^2 u}{\partial t^2} = L^- u + L^+ u + f_a.
\]

(16)

The case where \( L_{\alpha \beta} = \delta_{\alpha \beta} L_{\alpha \alpha} \), i.e. compound derivatives are absent, is considered in [1].

We now introduce the values \( y^{j+a/p} = y(\alpha) \), intermediate between \( y^j = y \) and \( y^{j+1} = y \) assuming that \( y^{(j-1)+a/p} = y(\alpha) \), and use for the determination of \( y(\alpha) \) a difference scheme which approximates equation (16) with number \( \alpha \). By analogy with [1], for the approximation of \( \partial^2 u / \partial t^2 \) we use the \((p+1)\)-th time layer

\[
\frac{1}{p} \frac{\partial^2 u}{\partial t^2} \approx \sum_{\alpha=1}^{p} f_{\alpha} \tau_{\alpha}, \quad \alpha = 1, \ldots, p;
\]

(17)
The additive scheme of alternating directions for the problem (1) - (2) will have the form

\[ u_{\alpha \beta} = \sum_{\beta=0}^{\alpha-1} \Lambda_{\alpha \beta} \tilde{y}(\beta) + \sum_{\beta=\alpha}^{\alpha} \Lambda_{\alpha \beta} \tilde{y}(\beta) + \varphi_{\alpha}, \quad \alpha = 1, \ldots, p, \quad (x, t) \subseteq \omega_h \times \omega_t, \]

\[ y_a = y(x, t_j) \text{ with } x_a = 0, 1, \ldots, a = 1, \ldots, p; \quad y(x, 0) = y_0(x), \]

where \( \varphi_{\alpha} = \varphi_{\alpha}(x, t_j) \) is a second order approximation on the network \( \omega_h \) of the function \( f_a(x, t) \), \( t_j \in [t_j, t_{j+1}] \), e.g. \( t_j = t_{j+1} \). The coefficients \( a_{\alpha \beta} = a_{\alpha \beta}(x, t_{\alpha \beta}) \) are taken at the middle moment \( t_{(\alpha \beta)} = t_j + \frac{\alpha \beta}{p} - 0.5 \tau \) (cf. [1]).

The second initial condition can be approximated by analogy with [1], or more simply by assuming

\[ y_{(\alpha \beta \gamma)} = \varphi_{(\alpha \beta \gamma)}(x) + \frac{\alpha \beta \gamma}{p} \varphi_{(\alpha \beta \gamma)}(x), \quad \alpha = 1, \ldots, p - 1, \quad p = 2, 3. \]

Such a condition is sufficient for an accuracy \( O(\tau + |h|^2) \). Let us rewrite (18) in the form

\[ (E - \frac{\tau^2}{\sigma_p} \Lambda_{\alpha \alpha}) y_{(\alpha \beta \gamma)} = R_{\alpha}(y) + \frac{\tau^2}{\sigma_p} \sum_{\beta=1}^{\alpha-1} \Lambda_{\alpha \beta} \tilde{y}(\beta) + F_{\alpha} = \Phi_{\alpha}, \]

where

\[ F_{\alpha} = \frac{\tau^2}{\sigma_p} \left[ \sum_{\beta=\alpha}^{p} \Lambda_{\alpha \beta} \tilde{y}(\beta) + \varphi_{\alpha} \right], \]

\[ R_{\alpha}(y) = 2y_{(\alpha-1)} - \tilde{y}_{(\alpha)} \text{ if } p = 2, \quad R_{\alpha}(y) = y_{(\alpha-1)} + y_{(\alpha-2)} - \tilde{y}_{(\alpha)} \text{ if } p = 3 \]

and \( E \) is the operator of identity.

Here it is obvious that to determined \( y_{(\alpha)} \) (all \( y_{(\beta)} \) for \( \beta < \alpha \) and all \( \tilde{y}_{(\beta)} \) for \( \beta = 1, \ldots, p \) are already known) we obtain a system of three-
A hyperbolic system of equations with compound derivatives

Point equations with a triangular matrix for their coefficients. Its solution reduces to an inversion of the operator \( E - (\alpha^2 / \sigma_p) \Lambda_{(a)} \), which is attained by an \( n \)-fold use of the ordinary formulae for each chain \( y(a) \) (see [1]) for fixed \( a = 1, \ldots, p \). To realize the algorithm (21) we must bear in mind the values of \( y(a) \) on \( p \) layers.

Let us write the equation for the \( s \)-th components \( y(a,s) \) of the vector \( Y(a) \)

\[
y(a,s) - \frac{\alpha^2}{\sigma_p} (a^{-s} y(a,s))_{x_a} = \Phi(a,s) + \frac{\alpha^2}{\sigma_p} \sum_{m=1}^{s-1} (a^{-s} y(a,s))_{x_a}.
\]

(22)

Here \( \Phi(a,s) \) is known, since the calculation takes place in the direction from \( a \) to \( a + 1 \), \( a = 1, \ldots, p \); the second term is also known if we determine successively \( y(a,1), \ldots, y(a,s), \ldots, y(a,p) \), i.e. carry out the calculation from \( s \) to \( s + 1 \). Hence it is obvious that we can find \( y(a,s) \) by solving the first boundary value problems for the three-point equations on segments parallel to the axis \( \partial x_a \).

If we interchange the roles of \( L_a^- \) and \( L_a^+ \), respectively,

\[
\Lambda_a^- = \sum_{\beta=1}^{a-1} \Lambda_{a\beta} = \sum_{\beta=a}^{a} \Lambda_{a\beta} \quad \text{and} \quad \Lambda_a^+ = \sum_{\beta=a+1}^{p} \Lambda_{a\beta} = \sum_{\beta=a}^{p} \Lambda_{a\beta},
\]

we obtain a second additive scheme

\[
\sigma_p Y_{i_a} \tau_a = \sum_{\beta=a}^{p} \Lambda_{a\beta} y_{(\beta)} + \sum_{\beta=a}^{a} \Lambda_{a\beta} \dot{y}_{(\beta)} + \Phi_a
\]

(23)

with the same initial and boundary conditions as in the first scheme. Here to determine \( y_{(a)} \) we must invert the triangular three-point operator \( E - (\alpha^2 / \sigma_p) \Lambda_{a2}^+ \). Here the calculation proceeds from \( a + 1 \) to \( a \) and from \( s + 1 \) to \( s \).

The alternation of these two schemes gives a third scheme. Introducing the intermediate value \( y^{1+\alpha/2} \), \( \alpha = 1, \ldots, 2p - 1 \), we obtain (cf. [2, 5])

\[
\sigma_p Y_{i_a} \tau_a = \sum_{\beta=1}^{a} \Lambda_{a\beta} y_{(\beta)} + \sum_{\beta=a}^{p} \Lambda_{a\beta} \dot{y}_{(\beta)} + \Phi_a, \quad \alpha = 1, \ldots, p.
\]

(24)
\[
\sigma_p Y_{\alpha'} Y_{\alpha'} = \sum_{\beta=\alpha}^{p} \Lambda_{\alpha\beta} Y_{(\beta')} + \sum_{\beta=1}^{p} \Lambda_{\alpha\beta} Y_{(\beta')} + \varphi_{\alpha'},
\]

where \( \alpha' = p + 1, \ldots, 2p \), \( \varphi_{\alpha'} = \varphi_{\alpha}, \; \alpha = p, p - 1, \ldots, 2, 1. \)

5. Schemes (18) and (23) are stable for sufficiently small \( |h| \leq h_0 \), ensuring that the requirement (10) of the positive definiteness of the matrices \( (a_{\alpha\beta}) \) and \( (a_{\alpha\beta}^{(\alpha+\beta)}) \) for any \( \tau \) is satisfied, and converge at least with a speed \( O(\|h\|^2 + \tau) \). The proof of these statements is carried out by analogy with [1, 4] by the method of energy inequalities.

Here the basic part is played by an identity of the form

\[
\sum_{\alpha=1}^{p} \left((a_{\alpha\beta} \xi_{\alpha} - \bar{\xi}_{\alpha}) + (a_{\alpha\beta}^+ \bar{\xi}_{\alpha}, \xi_{\alpha} - \bar{\xi}_{\alpha}) = J - \bar{J} \right) (1 + O(\tau)) + R,
\]

where

\[
J = \sum_{\alpha=1}^{p} \sum_{\beta=1}^{p} (a_{\alpha\beta} \xi_{\alpha}, \bar{\xi}_{\alpha}) = \sum_{\alpha=1}^{p} \sum_{\beta=1}^{p} (a_{\alpha\beta}^+ \xi_{\alpha}, \bar{\xi}_{\alpha}),
\]

\[
R = \sum_{\alpha=1}^{p} \left( \sum_{\beta=1}^{p} (a_{\alpha\beta}^+ \bar{\xi}_{\alpha}, \xi_{\alpha}) - (a_{\alpha\beta} \xi_{\alpha}, \bar{\xi}_{\alpha}) \right).
\]

Using (3) it is not difficult to see that \( R = 0. \)

Note. In the case of constant coefficients, \( k_{\alpha\beta} = \text{const.} \), the given values of the accuracy of the schemes considered, (18) and (23), are valid for any \( h, \alpha \), and \( \tau. \)

6. We now turn to the equations of elasticity (5). In this case as we have seen in Section 2, \( n = p \), and \( k_{\alpha\beta}^{en} = \text{const.} \),

\[
k_{\alpha\beta}^{en} = \mu \delta_{\alpha\beta} \delta_{sm} + (\lambda + \mu) \delta_{\alpha s} \delta_{\beta m}, \quad \delta_{sm} = \begin{cases} 1, & s = m, \\ 0, & s \neq m. \end{cases}
\]

For the equations of elasticity (5) we can use any of the schemes considered in Section 4, bearing in mind that \( a_{\alpha\beta} = k_{\alpha\beta} \), where \( k_{\alpha\beta} \) is determined by formula (25).

We shall write in more detail the difference equations (18) in the
case where $p = 2$: here $y = (y^{(1)}, y^{(2)})$ and

$$2y^{(1)}_{i,t} = \frac{1}{2} (\lambda + 2\mu) (y^{(1)}_{(1)^{+}x_{x_{1}}} + y^{(1)}_{(1)^{-}x_{x_{1}}} + \frac{1}{2} (\lambda + \mu) (y^{(2)}_{(2)^{+}x_{x_{1}}} + y^{(2)}_{(2)^{-}x_{x_{1}}} + \varphi^{(1)}_{(1)}

$$

$$2y^{(2)}_{i,t} = \frac{1}{2} \mu (y^{(2)}_{(1)^{+}x_{x_{2}}} + y^{(2)}_{(1)^{-}x_{x_{2}}} + \varphi^{(2)}_{(2)}),

$$

$$2y^{(1)}_{i,t} = \frac{1}{2} \mu (y^{(1)}_{(2)^{+}x_{x_{2}}} + y^{(1)}_{(2)^{-}x_{x_{2}}} + \varphi^{(1)}_{(1)}),

$$

$$2y^{(2)}_{i,t} = \frac{1}{2} (\lambda + 2\mu) (y^{(2)}_{(2)^{+}x_{x_{2}}} + y^{(2)}_{(2)^{-}x_{x_{2}}} + \frac{1}{2} (\lambda + \mu) (y^{(1)}_{(1)^{+}x_{x_{1}}} + y^{(1)}_{(1)^{-}x_{x_{1}}} + \varphi^{(2)}_{(2)}).

We must remember that here the upper index means the number of the component and the lower one the number of the vector $y^{(1)}_{(1)} = (y^{(1)})^{j+s}$, $y^{(2)}_{(2)} = (y^{(2)})^{j+s} = y^{(1)}$ etc.). Using (18) and (25) it is not difficult to write down the scheme for $p = 3$.

7. For the equations of elasticity we can also construct a series of resolving schemes which are absolutely stable, economic and convergent with speed $O(\tau + |h|^{2})$ or $O(\tau^{2} + |h|^{2})$.

Let us first consider the two multidimensional schemes

$$y_{i,t}^{(1)} = \Lambda^{-}y + \Lambda^{+}y_{t} - \varphi^{(1)} (y_{i,t}^{(1)} = (y^{(1)} - 2y^{(1)} + y^{(1)})/\tau^{2});

$$

$$y_{i,t}^{(1)} = \Lambda^{+}y + \Lambda^{-}y_{t} + \varphi^{(1)},

$$

where $\Lambda^{-}$ and $\Lambda^{+}$ are triangular operators, which approximate the triangular differential operator $L^{-}$ and $L^{+}$, $y = y^{(1)}$, $\dot{y} = y^{(1)}$, $\ddot{y} = y^{(1)}$.

Let $\tilde{\Lambda}_{a}y = y_{x_{a}x_{a}}$, and $\tilde{\Lambda}_{ak}$ denote the difference approximation of the compound derivative $\partial^{2}u / \partial x_{a} \partial x_{k}$, e.g. $\tilde{\Lambda}_{ak}y = \frac{1}{2} (y_{x_{a}x_{k}} + y_{x_{k}x_{a}})$ or $\tilde{\Lambda}_{ak}y = \frac{1}{2} (y_{x_{a}x_{k}} + y_{x_{k}x_{a}})$. Then the expressions for the triangular operators $\Lambda^{-}$ and $\Lambda^{+}$ can be written in the form (the upper index $s$ or $k$ is the number of the component)

$$(\Lambda^{-}y)^{s} = \frac{1}{2} \sum_{a=1}^{p} \kappa_{sa} \tilde{\Lambda}_{a}y^{s} + (\lambda + \mu) \sum_{k=1}^{s-1} \tilde{\Lambda}_{ak}y^{k}, \quad \kappa_{sa} = \mu + (\lambda + \mu) + \delta_{sa},

$$

$$(\Lambda^{+}y)^{s} = \frac{1}{2} \sum_{a=1}^{p} \kappa_{sa} \tilde{\Lambda}_{a}y^{s} + (\lambda + \mu) \sum_{k=s+1}^{p} \tilde{\Lambda}_{ak}y^{k}.

$$

The boundary conditions on $y_{h}$ are exactly satisfied
y|_{\nu h} = \nu(x, t),

and the initial conditions have the form

\[ y(x, 0) = \nu_0(x), \quad y_{\tilde{v}_1}(x, \tau) = \nu_1(x) + \tilde{v}_1(x), \]

where \( \tilde{v}_1(x) \) is chosen so that the initial condition \( \partial u / \partial t = v_1 \) is approximated with accuracy \( O(\tau^2) \); for this, for instance, it is sufficient to assume that \( \tilde{v}_1 = -(Lu + f)|_{t=0} \).

Each of the schemes \( I^- \) and \( I^+ \) is absolutely stable and has an accuracy \( O(\tau + |h|^2) \). Applying these schemes alternately (e.g. scheme (26) on odd and scheme (27) on even layers), we obtain an accuracy \( O(\tau^2 + |h|^2) \).

Following the principle given in (8), we shall write the generating scheme \( II^- \) for the scheme \( I^- \)

\[
A^s y^s_i = \Phi^s + F^s, \quad A^s = \prod_{a=1}^P A^s_a, \quad A^s_a = E - 0.5\tau^2 \kappa_a \Lambda_a, \\
\Phi^s = \gamma^s_i + \tau \left[ (A^s \gamma^s)^s + 0.5 \sum_{a=1}^P \kappa_a \Lambda_a \gamma^s + \phi^s \right], \\
F^s = \tau (\lambda + \mu) \sum_{k=1}^{s-1} \Lambda_{sk} y^k.
\] (28)

Similarly the generating scheme \( II^+ \) is written for the scheme \( I^+ \). In this case only the formulae for \( \Phi^s \) and \( F^s \) are changed

\[
\Phi^s = \gamma^s_i + \tau \left[ (A^s \gamma^s)^s + 0.5 \sum_{a=1}^P \kappa_a \Lambda_a \gamma^s + \phi^s \right], \\
F^s = \tau (\lambda + \mu) \sum_{k=s+1}^P \Lambda_{sk} y^k.
\] (29)

To determine \( y^s \) on a new layer we can write some numerical algorithms for alternating directions. We give for scheme \( II^- \) only the algorithm which we put forward in [8] for the equation of heat conduction; this algorithm has the form

\[
A^s v^s_i (\alpha) = \Phi^s + F^s, \quad A^s v^s_\alpha (\alpha) = v^s_{(\alpha-1)}, \quad \alpha = 1, 2, \ldots, p, \quad s = 1, \ldots, p; \\
y^s = \gamma^s + \nu^s_{(p)}. \]

We shall take the boundary conditions with \( x_\alpha = 0, \quad x_\alpha = l_\alpha \) for \( v^s_{(\alpha)} \) in the form
\[ \psi_a = A_{a,1}^* \ldots A_p^* \psi_i^* \quad \text{for} \quad x_a = 0, \ x_a = l_a, \ \alpha = 1, \ldots, p-1, \ \psi_{(p)} = \psi_i^*. \]

The order of calculation is as follows: the components \( y^{(1)}, \ldots, y^{(p)} = y^{j+1} \) are determined in turn.

For the algorithm which corresponds to scheme II\(+\) the formulae (30) remain unchanged, but the order of calculation is reversed: the components \( y^{(p)}, \ldots, y^{(1)} \) are determined successively.

Alternating the schemes II\(-\) and II\(+) we find the solution of the problem with accuracy to within \( O(\|h\|^2 + \tau^2) \). This evaluation is obtained by the method of \([1, 2, 8]\).

From the formulae for \( \psi_{(a)}^* \) it is obvious that the solution \( y^* \) of the difference problem is determined by means of successive inversion of the triangular matrices (according to the formulae given in \([9]\)). Therefore the resolving schemes are economic: to calculate the vector \( y^{j+1} \) operations of the order \( O(p^2/h^2) \) are required.

8. We turn now to a stationary problem of the theory of elasticity

\[ Lu = \mu \Delta u + (\lambda + \mu) \text{grad div } u = -f(x), \quad x \in G, \quad u|_\Gamma = v(x). \quad (32) \]

Its solution reduces to the solution of the difference problem for establishing the parabolic system of equations

\[ \frac{\partial u}{\partial t} = Lu + f(x), \quad u|_\Gamma = v(x) \]

with arbitrary initial data

\[ u(x, 0) = v_0(x). \]

For this we make use of the resolving scheme. Let the difference scheme for (32) take the form

\[ \Lambda^+ v + \Lambda^+ v = \varphi, \quad v|_{\gamma} = v_0(x). \]

We write the generating scheme for the determination of \( y = y(x, (j + 1)\tau) \), where \( j + 1 \) is the number of the iteration and \( \tau \) the iteration parameter, as

\[ A_s^* \psi_i^* = \Phi^s + F^s, \quad A_s = \prod_{a=1}^P A_a, \quad A_s^* = E - 0.5 \nu_{sa}^\gamma \Delta_a. \]
In this case the numerical algorithm for alternating directions of Section 6 takes the form

\[ A_{1}^{s}w_{(1)}^{s} = \Phi^{s} + F^{s}, \quad A_{\alpha}^{s}w_{(\alpha-1)}^{s} = w_{(\alpha-1)}, \quad \alpha > 1, \]

\[ y^{s} = \dot{y}^{s} + \tau w_{(p)}^{s}, \quad w_{(\alpha)}^{s}|_{\gamma_{k}} = 0, \]

i.e. for \( w_{(\alpha)}^{s} \) the boundary conditions are always zero.

For comparison we quote one more numerical algorithm (two-layered)

\[ A_{1}^{t}y_{(1)}^{t} = \tau(\ddot{y}^{t} + F^{t}) + A_{\alpha}^{t}y^{t}, \quad A_{\alpha}^{t}y_{(\alpha-1)}^{t} = y_{(\alpha-1)}, \quad \alpha > 1, \]

\[ y_{(\alpha)}^{t} = A_{(\alpha+1)}^{t} \ldots A_{\alpha}^{t}y_{(\alpha)}^{t} \quad \text{for} \quad x_{\alpha} = 0, \quad x_{\alpha} = l_{\alpha}, \quad \alpha = 1, 2, \ldots, p - 1: \]

\[ y_{(p)}^{t}|_{\gamma_{k}} = v^{t}. \]

The order of the computation is the same as before: initially the first component \( y^{(1)} \) is determined, then the second \( y^{2}(s = 2) \) etc.

To find \( y^{s} \), at all nodes of the network \( \omega_{h} \) operations of the order \( O(1/h_{1}h_{2} \ldots h_{p}) \) are required. The iteration process converges for \( \tau = O(h_{\alpha}), \quad h_{\alpha} = \min h_{\alpha} \). The rate of convergence is determined by the number of iterations \( \nu \approx O((1/h_{\alpha})\times \ln (1/\varepsilon)) \) for \( p = 2 \) and \( O((1/h_{\alpha})^{2}\ln (1/\varepsilon)) \) for \( p = 3 \), where \( \varepsilon \) is the required accuracy.

The evaluation for the number of iterations is obtained by the method of energy inequalities by analogy with [10, 7, 4].

Switching the roles of the operators \( \Lambda^{-} \) and \( \Lambda^{+} \), we obtain the second iteration scheme \( II^{+} \). The same value is obtained for the rate of convergence of the iterations with it as for the scheme \( II^{-} \) described above. It is hoped that this is the basis which the interchange of these two iteration algorithms \( II^{-} \) and \( II^{+} \) can bring to speeding up to convergence.

Translated by H.F. Cleaves
REFERENCES

1. SAMARSKII, A.A., Local one-dimensional difference schemes for multi-
dimensional hyperbolic equations in an arbitrary region. Zh. vychisl.

2. SAMARSKII, A.A., Economic difference schemes for parabolic equations
with compound derivatives. Zh. vychisl. Mat. mat. Fiz. 4, 4, 753 -

3. SAMARSKII, A.A., Economic difference method for the solution of a
multi-dimensional parabolic equation in an arbitrary region. Zh.

4. SAMARSKII, A.A., Local one-dimensional schemes on non-uniform net-

5. SAMARSKII, A.A., Economic difference schemes for systems of parabolic

6. KONOVALOV, A.N., Application of resolving methods to the numerical
solution of dynamic problems in the theory of elasticity. Zh.

7. KONOVALOV, A.N., Iteration scheme for the solution of static prob-
lems in the theory of elasticity. Zh. vychisl. Mat. mat. Fiz., 4,
5, 942 - 945, 1964.

8. SAMARSKII, A.A., Schemes for increasing the order of accuracy for a

9. BEREZIN, I.S. and ZHIDKOV, N.P., Numerical Methods (Metody vychis-

10. LEES, W., A note on the convergence of alternating direction methods.