AN ECONOMIC CONTINUOUS CALCULATION SCHEME FOR THE STEFAN MULTIDIMENSIONAL PROBLEM*

A.A. SAMARSKII and B.D. MOISEYENKO

Moscow

(*Received* 26 *August* 1964)

IN this paper an economic continuous difference calculation scheme is put forward for the numerical solution of the Stefan problem in the case of several space variables and any number of phases.**

The continuous calculation scheme is characterized by the fact that the boundary of the phase division is not explicitly selected and uniform difference schemes are used. The principle of the "smearing" of the thermal capacity with temperature which in the same way does not depend on the number of dimensions plays an important part here. An explicit scheme for the one-dimensional problem has been considered in [1]. R.P. Fedorenko used another explicit scheme for the one-dimensional problem. Non-uniform implicit schemes have been used in [2]. For the one-dimensional Stefan problem a smearing algorithm using implicit schemes was tested by the authors with the cooperation of L.A. Vladimirov.

To solve the multidimensional quasilinear equation of heat conduction with smeared coefficients of thermal capacity and thermal conductivity a locally one-dimensional method is used which was put forward and proved in [3] and [4]. It consists of a stage by stage solution with different space variables of one-dimensional equations of heat conduction by means of unconditionally stable implicit schemes. The method is suitable for

---

* *Zh. vȳchisl. Mat. mat. Fiz.* 5, 5, 816 - 827, 1965.

** The basic content of the present paper was given at the Second All Union Congress on Theoretical and Applied Mechanics (February 1964 in Moscow). It was pointed out that similar work was being carried out by B.M. Budak, E.N. Sobol'eva and A.B. Uspenskii.

arbitrary regions in the case of boundary conditions of the first kind, and in the case of boundary conditions of the third kind for regions of special form [3].

The continuous calculation scheme for the Stefan problem was verified with selfmodelling solutions in the case of one and two space dimensions. We have not touched on those problems which are most favourable for the difference scheme. Thus, for example, we consider the problem with cylindrical symmetry (the boundary of the phase transition is circular) on a rectangular network. In all cases the method enables us to obtain a solution with sufficient accuracy.

1. The Stefan problem

1. In the study of thermal processes with phase transitions of a substance from one state to another we happen to encounter the following problem. In each of two or more phases we have the equation of heat conduction

$$c(u)\frac{\partial u}{\partial t}=\operatorname{div}(k(u)\operatorname{grad}u)+f, \tag{1}$$

where $u = u(\mathbf{r}, t)$ is the temperature at the instant t at the point with radius vector $\mathbf{r}(x_1, \ldots, x_p)$, $k = k(u)$ is the thermal conductivity, $c = c(u)$ the thermal capacity (per unit volume), and $f = f(\mathbf{r}, t)$ the density of the thermal sources. The boundary of the phase division is determined by the condition that the temperature along this boundary is equal to the temperature u^* of the phase transition, i.e. $u(\mathbf{r}, t) = u^*$. This relation is the equation for determining $\mathbf{R}(t)$ - the position of the boundary of the phase transition at the instant t. In the general form we can consider that the equation of the boundary of the phase transition takes the form $\Phi(u) = 0$, where $u(\mathbf{r}, t)$ must be substituted as the argument. We shall also write $\Phi(\mathbf{r}, t) = 0$. We now formulate the condition on the boundary of the phase transition. Let 1 be the phase index for which $u < u^*$, and 2 be the second phase index (for which $u > u^*$). Since grad Φ is directed along the normal to the surface Σ of the phase division, the normal component of the heat flux $\mathbf{Q} = -k$ grad u on Σ is given by

$$Q_{1,2}=-\left(k\operatorname{grad}u,\ \frac{\operatorname{grad}\Phi}{|\operatorname{grad}\Phi|}\right)_{1,2}$$

The difference between the heat fluxes $Q_2 - Q_1$ is equal to the product of the enthalpy of the phase transition λ and the normal component of the velocity $d\mathbf{R}/dt$ of motion of the boundary of the phase division

$$Q_2 - Q_1 = \lambda\left(\frac{d\mathbf{R}}{dt}, \quad \frac{\operatorname{grad}\Phi}{|\operatorname{grad}\Phi|}\right). \tag{2}$$

Using the fact that along Σ

$$\frac{d}{dt}\Phi(\mathbf{R}(t), t) = \frac{\partial\Phi}{\partial t} + \left(\frac{d\mathbf{R}}{dt}, \quad \operatorname{grad}\Phi\right) = 0,$$

we can write (2) in the form

$$\begin{gathered} u = u^*, \\ ((k \operatorname{grad} u)_1 - (k \operatorname{grad} u)_2, \operatorname{grad}\Phi) + \lambda\frac{\partial\Phi}{\partial t} = 0 \text{ where } \mathbf{r} = \mathbf{R}(t). \end{gathered} \tag{2'}$$

We shall assume that there are only 2 phases so that

$$c(u) = \begin{cases} c_1(u) & \text{if} \quad u < u^*, \\ c_2(u) & \text{if} \quad u > u^*; \end{cases} \qquad k(u) = \begin{cases} k_1(u), & u < u^*, \\ k_2(u), & u > u^*. \end{cases} \tag{3}$$

The functions $c_s(u)$ and $k_s(u)$, $s = 1, 2$, are differentiable a sufficient number of times and bounded below by the constants m_1 and m_2: $c_s(u) \geqslant m_1 > 0$, $k\ (u) \geqslant m_2 > 0$.

2. The physical requirement, from which the boundary condition (2') follows, consists of the fact that with the temperature of the phase transition $u = u^*$ the energy w as a function of the temperature undergoes a jump of magnitude λ, which is called the heat (or enthalpy) of the phase transition. We can therefore write

$$w = \int_0^u c(u)\,du + \lambda\eta(u - u^*), \qquad \eta(\xi) = \begin{cases} 1, & \xi \geqslant 0; \\ 0, & \xi < 0. \end{cases} \tag{4}$$

Substituting expression (4) in the energy equation

$$\frac{\partial w}{\partial t} = \operatorname{div}(k \operatorname{grad} u) + f \tag{5}$$

and taking into consideration the fact that $d\eta(\xi)/d\xi = \delta(\xi)$ is the Dirac delta-function, we obtain

$$(c(u) + \lambda\delta(u - u^*))\frac{\partial u}{\partial t} = \operatorname{div}(k \operatorname{grad} u) + f. \tag{6}$$

This equation includes equation (1) and condition (2) on the phase surface. For the application of the continuous calculation method (without explicit choice of the boundary of the phase division) we find it convenient to use the equation in the form (6).

We can show that (2) follows from (6).

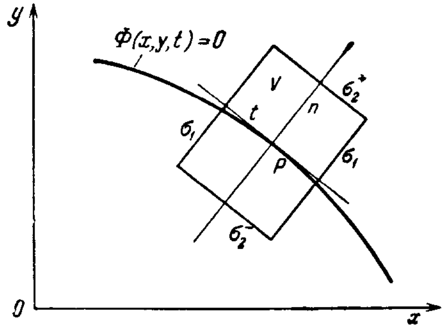

FIG. 1.

Let us consider the point P on the surface $\Phi = 0$ (see Fig. 1) and construct a cylinder which is symmetrical with respect to $\Phi = 0$, of sufficiently small volume with centre at the point P and with generator parallel to the normal $\mathbf{n}(P)$ to $\Phi = 0$ at the point P. Let σ_1 be the lateral surface, σ_2^- the lower and σ_2^+ the upper bases of the cylinder; their areas will be $|\sigma_1|$, $|\sigma_2| = |\sigma_2^-| = |\sigma_2^+|$, respectively. We integrate (6) over the volume V of the cylinder and let $|\sigma_1| \to 0$, and then $|\sigma_2| \to 0$ also. The integrals of $c\partial u/\partial t$ and f in the limit give zero. The volume integrals

$$J_1 = \int_V \lambda\delta(u - u^*)\frac{\partial u}{\partial t}dv = \int_V \lambda\frac{\partial\eta(u - u^*)}{\partial t}dv,$$

$$J_2 = \int_V \operatorname{div}(k \operatorname{grad} u)\,dv$$

are transformed into surface integrals over the cross-sections of the cylinder. We shall assume that $d\Phi/du > 0$; then $\eta(u - u^*) = \eta(\Phi)$ and $\partial\eta(\Phi)/\partial t = \delta(\Phi)(\partial\Phi/\partial t)$. The element of volume $dv = d\sigma dn$, where $d\sigma$ is an element of the area of the plane sections parallel to $\sigma_2^{\pm}$; integration along the normal can be replaced by integration with respect to Φ, since $d\Phi = (\operatorname{grad}\Phi, d\mathbf{n}) = |\operatorname{grad}\Phi|dn$. Then as $|\sigma_1| \to 0$

$$J_1 = \int_V \lambda\delta(\Phi)\frac{\partial\Phi}{\partial t}d\sigma\frac{d\Phi}{|\operatorname{grad}\Phi|} = \int_{\sigma_2(0)} \lambda\frac{\partial\Phi}{\partial t}\frac{d\sigma}{|\operatorname{grad}\Phi|} \to \lambda\frac{\partial\Phi}{\partial t}\frac{|\sigma_2|}{|\operatorname{grad}\Phi|},$$

where $\sigma_2^{(0)}$ is the average cross-section of the cylinder (on $\Phi = 0$).

Ostrogradskii's formula $|\sigma_1| \to 0$ gives

$$J_2 = \int_\sigma k \frac{\partial u}{\partial n} d\sigma \to \int_{\sigma_2(0)} \left((k \operatorname{grad} u)_2 - (k \operatorname{grad} u)_1, \frac{\operatorname{grad} \Phi}{|\operatorname{grad} \Phi|} \right) d\sigma =$$

$$= ((k \operatorname{grad} u)_2 - (k \operatorname{grad} u)_1, \operatorname{grad} \Phi) \frac{|\sigma_2|}{|\operatorname{grad} \Phi|}.$$

Considering that $\lim_{|\sigma_1| \to 0} (J_1 + J_2) = 0$, after dividing by $|\sigma_2| / |\operatorname{grad} \Phi|$ and letting $|\sigma_2| \to 0$ we obtain (2′).

3. We now give a mathematical statement of the multidimensional Stefan problem in the case of boundary conditions of the first kind. Let G be a p-dimensional region of the space $x = (x_1, \ldots, x_p)$ with boundary Γ, $\bar{Q}_\Gamma = (G + \Gamma) \times [0 \leqslant t \leqslant T]$, $Q_\Gamma = G \times (0 < t \leqslant T]$, u_s^*, λ_s ($s = 1, 2, \ldots, s_0$) be constants. It is required to find a function $u(x, t)$ in $\bar{Q}_\Gamma$ and a vector function $\mathbf{R}_s(t)$ with $t \in [0, T]$ from the following conditions

$$\left[c(u) + \sum_{s=1}^{s_0} \lambda_s \delta(u - u_s^*) \right] \frac{\partial u}{\partial t} = \operatorname{div}(k \operatorname{grad} u) + f(x, t) \text{ in } Q_\Gamma; \tag{I}$$

$$u(x, 0) = u_0(x) \text{ if } x \in G + \Gamma, \quad u = \mu(x, t) \text{ if } x \in \Gamma, \quad t \in [0, T],$$

$$u = u_s^* \text{ if } \mathbf{r} = \mathbf{R}_s(t), \quad s = 1, 2, \ldots, s_0.$$

Here s_0 is the number of phases.

4. We shall limit ourselves to consideration of the two-phase problem. As is obvious from (6), $c(u)$ and $\lambda\delta(u - u^*)$ enter into the equation uniquely; $\lambda\delta(u - u^*)$ represents the concentrated thermal capacity (on the surface $u = u^*$). For transition to the difference scheme we replace the delta-function by the approximately deltaform, or smeared, delta-function $\delta(u - u^*, \Delta) \geqslant 0$, where Δ is the value of the semi-interval on which $\delta(u - u^*, \Delta)$ is different from zero. This smearing or smoothing is equivalent to the replacement in the interval $(u^* - \Delta, u^* + \Delta)$ of the discontinuous function $\eta(u - u^*)$ by the continuous function $\eta(u - u^*, \Delta)$, which is such that $\eta'(\xi, \Delta) = \delta(\xi, \Delta)$.

Thus we introduce a smoothed, or effective, thermal capacity $\tilde{c}(u) = c(u) + \lambda\delta(u - u^*, \Delta)$ from the conditions:

(1) $\tilde{c}(u) = c_1(u)$ if $u < u^* - \Delta$, $\tilde{c}(u) = c_2(u)$ if $u > u^* + \Delta$ (i.e. $\tilde{c}(u) = c(u)$ outside the interval $(u^* - \Delta,\ u^* + \Delta)$),

(2) the change of enthalpy in the interval $(u^* - \Delta,\ u^* + \Delta)$ is retained, i.e.

$$\int_{u^*-\Delta}^{u^*+\Delta} \tilde{c}(u)\,du = \lambda + \int_{u^*-\Delta}^{u^*} c_1(u)\,du + \int_{u^*}^{u^*+\Delta} c_2(u)\,du, \tag{7}$$

or $\tilde{w}(u^* + \Delta) - \tilde{w}(u^* - \Delta) = w(u^* + \Delta) - w(u^* - \Delta)$, where $w(u)$ is defined by formula (4). Here is a very simple example. Suppose that c_1 and c_2 do not depend on u. Then in the interval $(u^* - \Delta, u^* + \Delta)$ we can take $\tilde{c} = \lambda/2\Delta + (c_1 + c_2)/2$, which corresponds to a linear interpolation of w. We have also considered other interpolations of the thermal capacity (linear and quadratic with the condition of symmetry $\tilde{c}'(u^*) = 0$, etc.).

In the same interval $(u^* - \Delta,\ u^* + \Delta)$ the smoothing of the thermal capacity is also carried out (e.g. by means of a polynomial); an effective, or smoothed, coefficient $\tilde{k}(u)$ is introduced which is the same as $k_1(u)$ if $u < u^* - \Delta$ and as $k_2(u)$ if $u > u^* + \Delta$.

5. As a result, instead of (I) we obtain a problem for the equation of heat conduction with smoothed coefficients

$$\begin{aligned} &\tilde{c}(u)\,\frac{\partial u}{\partial t} = \operatorname{div}(\tilde{k}(u)\operatorname{grad} u) + f(t, x), \qquad (x, t) \in Q_\Gamma, \\ &u(x,0) = u_0(x), \qquad u|_\Gamma = \mu(x,t), \qquad t \in [0, T]. \end{aligned} \tag{$\tilde{\text{I}}$}$$

2. The difference scheme

1. For problem ($\tilde{\text{I}}$) we can now construct an algorithm for continuous calculation, since there are now no singular elements in the conditions of the problem; there is only one smearing parameter Δ. Such a formulation of the problem does not depend on the number of dimensions, or on the number of phases.

The question of the convergence of the solution of problem ($\tilde{\text{I}}$) to the solution of problem (I) as $\Delta \to 0$ has been considered in [5], and we shall not touch on it here. We shall assume that the problem has a unique solution $u = u(x,\ t)$, which is continuous in $\bar{Q}_\Gamma$ and possesses the derivatives necessary for further work.

We rewrite equation $(\tilde{\text{I}})$ in the form

$$\tilde{c}(u)\frac{\partial u}{\partial t}=\sum_{\alpha=1}^{p}L_\alpha u+f,\qquad L_\alpha u=\frac{\partial}{\partial x_\alpha}\left(\tilde{k}(u)\frac{\partial u}{\partial x_\alpha}\right). \tag{8}$$

Its solution on the difference network in the cylinder $\bar{Q}_\Gamma$, as is shown in [3, 4] can be reduced to the successive solution of the one-dimensional equations of heat conduction

$$\frac{1}{p}\tilde{c}(u)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_\alpha}\left(k(u)\frac{\partial u}{\partial x_\alpha}\right)+f_\alpha(x,t),\qquad \sum_{\alpha=1}^{p}f_\alpha=f. \tag{9}$$

Therefore it is natural to begin with difference schemes for the solution of the one-dimensional quasilinear equations (9).

2. Thus we shall consider in the region $0\leqslant x\leqslant 1$, $0\leqslant t\leqslant T$ the one-dimensional equation of the form (we omit the index α)

$$\tilde{c}(u)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\tilde{k}(u)\frac{\partial u}{\partial x}\right)+f(t,x),\qquad \tilde{k}(u)\geqslant c_1>0,\quad \tilde{c}(u)\geqslant c_2>0$$
$$(c_1,\ c_2=\text{const}) \tag{10}$$

with boundary conditions of the first kind and initial condition $u(x,\ 0)=u_0(x)$. The difference schemes for this case have been considered in [6] and [7]. Let us dwell on one method of solving (10). We introduce a network $\omega_h=\{x_i,\ i=0,\ 1,\ \dots,\ N\}$ in the segment $0\leqslant x\leqslant 1$ and $\omega_\tau=\{t_n=n\tau,\ n=0,\ 1,\ \dots\}$ a network with step τ in the segment $0\leqslant t\leqslant T$.

We now introduce a new function

$$v=\int_0^u\tilde{k}(u)\,du,$$

so that

$$\int_0^u\tilde{c}(u)\,du=\varphi(v),\qquad \varphi'(v)=c(u)/\tilde{k}(u)>0.$$

Then (10) assumes the form

$$\frac{\partial\varphi(v)}{\partial t}=\frac{\partial^2 v}{\partial x^2}+f(t,x). \tag{11}$$

For its solution we shall consider the scheme (see [6])

$$\frac{\varphi(y_i^{n+1})-\varphi(y_i^n)}{\tau}=\Lambda(\sigma y_i^{n+1}+(1-\sigma)y_i^n)+f(x_i,\ t_{n+1/2}),\qquad 0\leqslant\sigma\leqslant 1, \tag{12}$$

where Λ is the difference approximation of the operator L: on the uniform network $\omega_h = \{x_i = ih \in [0, 1]\}$

$$\Lambda y_i = (y_{i-1}-2y_i+y_{i+1})\,/\,h^2,$$

and on the non-uniform network ω_h

$$\Lambda y_i=\frac{1}{\hbar_i}\left(\frac{y_{i+1}-y_i}{h_{i+1}}-\frac{y_i-y_{i-1}}{h_i}\right),\text{where } h_i=x_i-x_{i-1},\ \hbar_i=\frac{1}{2}(h_i+h_{i+1}). \tag{13}$$

If $\sigma \geqslant 0.5$, scheme (12) is unconditionally stable and converges uniformly as $h \to 0$, $\tau \to 0$ on an arbitrary sequence of networks. We have shown that if $\sigma = 0.5$ scheme (12) can attain an accuracy of $O(h^2 + \tau^2)$. The three-point equation which is non-linear in y_i^{n+1} is solved by some iteration method such as Newton's; each of the iterations is found by the formulae of successive substitutions. We shall not dwell on the conditions which ensure the maximum order of accuracy and the convergence of the iterations.

In this work we make use of a purely implicit scheme, which is written for equation (10) as

$$\tilde c(y_i^{n+1})\,\frac{y_i^{n+1}-y_i^n}{\tau}=\Lambda' y_i^{n+1}+f(t_{n+1},x_i),$$

$$\Lambda' y_i=\frac{1}{\hbar_i}\left[\frac{1}{h_{i+1}}\tilde k(y_{i+1/2})(y_{i+1}-y_i)-\frac{1}{h_i}\tilde k(y_{i-1/2})(y_i-y_{i-1})\right], \tag{14}$$

$$y_{i-1/2}=\frac{y_i+y_{i-1}}{2}.$$

To determine $y_i = y_i^{n+1}$ by the iteration method the coefficients $\tilde c$ and $\tilde k$ can be taken from the previous iteration.

3. We now turn to the multidimensional problem ($\tilde{\text{I}}$) for an arbitrary region. Following [3] we use constructively only one assumption about G: the intersection with the region G of the line L_α, drawn through any inner point of G, parallel to the coordinate axis Ox_α, consists of a finite number of intervals. It is not necessary to give a detailed account of the locally one-dimensional scheme for equation ($\tilde{\text{I}}$) in the general case. It is sufficient to refer to [3] and [4], where two types of space network $\omega_h^{(1)}$ and $\omega_h^{(2)}$ are considered in the region G: on

$\omega_h^{(1)}$ the boundary conditions are given by means of linear interpolation in the direction in which the one-dimensional equation (9) is being solved at the given instant; on $\omega_h^{(2)}$ the boundary conditions are given without deflection.

We now give a locally one-dimensional scheme and computational formulae for problem ($\tilde{\text{I}}$) in the case where G is a rectangle (i.e. $0 \leqslant x \leqslant l_1$, $0 \leqslant y \leqslant l_2$, where we have assumed that $x_1 = x_2$, $x_2 = y$). We now introduce the arbitrary non-uniform network

$$\omega_h = \{(x_i, y_j),\ i = 1, 2, \ldots, N_1,\ j = 1, 2, \ldots, N_2,\ x_0 = 0,\ y_0 = 0,\ x_{N_1} = l_1,\ y_{N_2} = l_2\}$$

and denote by $v_{ij}{}^n$ the value of the required function at the point (x_i, y_i) at the instant $t_n = n\tau$. Let $\bar{v}_i = v_i{}^{n+1/2}$ be an intermediate value which is determined by the numerical solution of equation (9) with $x_\alpha = x$.

The locally one-dimensional scheme for the problem ($\tilde{\text{I}}$) is constructed from uniform schemes of the form (14) and has the form

$$\tilde{c}(\bar{v}_{ij})\frac{\bar{v}_{ij} - v_{ij}{}^n}{\tau} = \Lambda_1 \bar{v}_{ij} + f_1(x_i, y_j, t_{n+1/2}), \quad (x_i, y_j) \in G;$$

$$\bar{v}_{ij} = \mu|_{t=t_{n+1/2}} \quad \text{if} \quad i = 0, \quad i = N_1, \quad 0 < j < N_2;$$

$$\Lambda_1 \bar{v}_{ij} = \frac{1}{\hbar_{x,i}}\left[\tilde{k}(\bar{v}_{i+1/2,j})\frac{\bar{v}_{i+i,j} - \bar{v}_{i,j}}{h_{x,i+1}} - \tilde{k}(v_{i-1/2,j})\frac{\bar{v}_{i,j} - \bar{v}_{i-1,j}}{h_{x,i}}\right],$$

$$\tilde{c}(v_{ij}{}^{n+1})\frac{v_{ij}{}^{n+1} - v_{ij}}{\tau} = \Lambda_2 v_{ij}{}^{n+1} + f_2(x_i, y_j, t_{n+1/2}), \quad (x_i y_j) \in G;$$

$$v_{ij}{}^{n+1} = \mu|_{t=t_{n+1}} \quad \text{if} \quad j = 0, \quad j = N_2, \quad 0 < i < N_1;$$

$$\Lambda_2 v_{ij} = \frac{1}{\hbar_{y,j}}\left[\tilde{k}(v_{i,j+1/2})\frac{v_{i;j+1} - v_{i,j}}{h_{y,j+1}} - \tilde{k}(v_{i,j-1/2})\frac{v_{i,j} - v_{i,j-1}}{h_{y,j}}\right],$$

$$v^0_{i,j} = u_0(x_i, y_j) \quad \text{if} \quad t = 0, \quad 0 \leqslant i \leqslant N_1, \quad 0 \leqslant j \leqslant N_2.$$

Here $v_{i+1/2,j} = 0.5(v_{i,j} + v_{i+1,j})$, $v_{i,j+1/2} = 0.5(v_{i,j} + v_{i,j+1})$, $h_{x,i}$, $h_{y,j}$ — are network steps in x and y and $\hbar_{x,i} = 0.5(h_{x,i} + h_{x,i+1})$, $\hbar_{y,j} = 0.5(h_{y,j} + h_{y,j+1})$; the network is non-uniform and so h_x and h_y depend on the nodes of the network.

In the examples considered below c_1, c_2, k_1 and k_2 do not depend on the temperature but the smoothed coefficients $\tilde{c}$ and $\tilde{k}$ always depend on the temperature. Therefore, to determine v from the equations written above, iterations are necessary.

4. We have used the simplest iteration method: to calculate the $(s+1)$-th iteration of the required function $\overset{s+1}{\bar{v}}$ or $\overset{s+1}{v}{}^{n+1}$ the coefficients $\tilde{c}$ and $\tilde{k}$ are calculated from the values of $\overset{s}{\bar{v}}$ or $\overset{s}{\bar{v}}{}^{n+1}$ at the preceding iteration. For $\overset{s+1}{\bar{v}}$ and $\overset{s+1}{v}{}^{n+1}$ we obtain linear three-point equations which are solved by the same standard programme of successive substitutions.

We now give equations for $\overset{s+1}{\bar{v}}$ and $\overset{s+1}{v} = \overset{s+1}{v}{}^{n+1}$.

Using the notation

$$\overset{s}{A}_{ij} = \frac{1}{h_{x,i}} \widetilde{k}\left(\overset{s}{\bar{v}}_{i-1/2,j}\right), \qquad \overset{s}{B}_{ij} = \frac{1}{\hbar_{x,i}} \widetilde{k}\,(\overset{s}{v}_{i,j-1/2}),$$

we obtain

$$\overset{s}{A}_{ij} \overset{s+1}{\bar{v}}_{i-1,j} - \left[\overset{s}{A}_{ij} + \overset{s}{A}_{i+1,j} + \frac{\hbar_{x,i}}{\tau}\tilde{c}\left(\overset{s}{\bar{v}}_{ij}\right)\right]\overset{s+1}{\bar{v}}_{ij} + \overset{s}{A}_{i+1,j}\overset{s+1}{\bar{v}}_{i+1,j} =$$
$$= -\frac{\hbar_{x,i}}{\tau}\tilde{c}\,(\bar{v}_{ij})\,v_{ij}^{n} + h_{x,i} f_{1,ij}^{n+1/2}$$

with boundary conditions for $i = 0$ and $i = N_1$, and

$$\overset{s}{B}_{ij}\overset{s+1}{v}_{i,j-1} - \left[\overset{s}{B}_{ij} + \overset{s}{B}_{i,j+1} + \frac{\hbar_{y,j}}{\tau}\tilde{c}\left(\overset{s}{v}_{ij}\right)\right]\overset{s+1}{v}_{ij} + \overset{s}{B}_{i,j+1}\overset{s+1}{v}_{i,j+1} =$$
$$= -\frac{\hbar_{y,j}}{\tau}\tilde{c}\left(\overset{s}{v}_{ij}\right)\bar{v}_{ij} + \hbar_{y,j} f_{2,ij}^{n+1/2}$$

with boundary conditions for $i = 0$ and $i = N_2$.

3. The solution of the problem with a plane boundary for the phase division

Let us consider the known (see [8]) one-dimensional Stefan problem of freezing. In the region $(0 \leqslant x < \infty,\ t \geqslant 0)$ a solution of the equations

$$c_s \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k_s \frac{\partial u}{\partial x}\right), \qquad s = 1,2, \tag{15}$$

is sought, with constant coefficients (c_s and k_s do not depend on u) and constant initial and boundary data; phase 1 occurs in the region $0 < x < \xi(t)$, phase 2 in the region $x > \xi(t)$. If $x = \xi(t)$ the given conditions are

$$u(\xi + 0, t) = u(\xi - 0, t) = u^*,$$

$$k_1 \frac{\partial u}{\partial x}\bigg|_{x=\xi+0} - k_2 \frac{\partial u}{\partial x}\bigg|_{x=\xi-0} = \lambda \frac{d\xi}{dt}. \tag{16}$$

The problem has a selfmodelling solution

$$u(x, t) = A_s\Phi\left(\frac{x}{2a_s\sqrt{t}}\right) + B_s, \qquad s = 1, 2, \qquad a_s = \sqrt{\frac{k_s}{c_s}}, \tag{17}$$

where a is a constant determined from some transcendental equation (see [8]), and A_s and B_s are constants whose expressions are given in [8, p. 264].

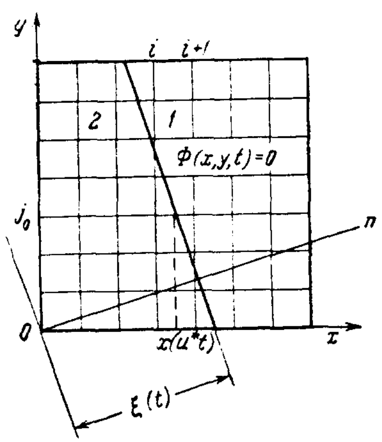

FIG. 2.

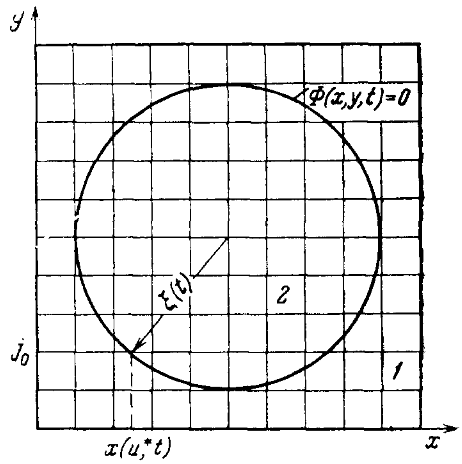

FIG. 3.

We shall consider this problem in the plane (x, y) choosing the coordinate axes such that the boundary of the division of the regions is not parallel to either of the axes (see Fig. 2). To solve it numerically we shall use the method put forward in Section 2. As the region G we shall choose the rectangle $(0 \leqslant x \leqslant l_1,\ 0 \leqslant y \leqslant l_2)$. The initial (with $t = t_{in}$) and boundary data for the difference problem will be taken to coincide with the selfmodelling solution (17).

In the variant considered $l_1 = l_2 = 2$, $h_x = h_y = 0.1$; $\tau = 1$, the number of points 21 x 21, $t_{in} = 4$, $u^* = 0$, $\lambda = 1$, $k^{(1)} = 1.25$, $k^{(2)} = 2$,

$c^{(1)} = 0.75$, $c^{(2)} = 0.5$, $u_0 = 1$, $a = 0.2$, $\Delta = 0.08$.

For each row $j = j_0$, where j_0 is the number of some row, we can find $x(u^*, t)$ by means of linear interpolation, and then by the formula for transformation of coordinates by rotations calculate $\xi(t)$.

In Table 1 exact and computed values of the function $\xi(t)$ are given for $j_0 = 11$.

TABLE 1.

t	Exact values	Computed values	t	Exact values	Computed values	t	Exact values	Computed values
10	0.633	0.640	50	1.414	1.422	90	1.897	1.911
20	0.894	0.903	60	1.549	1.563	100	2.000	2.013
30	1.095	1.110	70	1.673	1.684	120	2.191	2.202
40	1.265	1.277	80	1.789	1.798			

TABLE 2.

Number j of row	$t = 70$ $\xi = 1.673$	$t = 125$ $\xi = 2.236$	Number j of row	$t = 70$ $\xi = 1.673$	$t = 125$ $\xi = 2.236$	Number j of row	$t = 70$ $\xi = 1.673$	$t = 125$ $\xi = 2.236$
1	1.681	—	8	1.686	—	15	1.684	2.247
2	1.687	—	9	1.683	—	16	1.688	2.250
3	1.684	—	10	1.686	—	17	1.685	2.248
4	1.687	—	11	1.683	2.236	18	1.689	2.250
5	1.684	—	12	1.687	2.247	19	1.686	2.249
6	1.686	—	13	1.684	2.246	20	1.690	2.250
7	1.683	—	14	1.687	2.248	21	1.675	2.237

TABLE 3.

Exact values	Computed values	Exact values	Computed values	Exact values	Computed values	Exact values	Computed values
$t=70$				$t=125$			
0.7294	0.7294	0.1944	0.1975	0.7974	0.7974	0.3949	0.3957
0.6756	0.6758	0.1417	0.1450	0.7570	0.7571	0.3550	0.3558
0.6215	0.6223	0.0892	0.0927	0.7166	0.7168	0.3151	0.3159
0.5677	0.5688	0.0369	0.0406	0.6762	0.6765	0.2754	0.2761
0.5140	0.5154	−0.0097	−0.0088	0.6358	0.6363	0.2357	0.2363
0.4604	0.4621	−0.0430	−0.0426	0.5955	0.5960	0.1962	0.1967
0.4069	0.4089	−0.0762	−0.0761	0.5553	0.5558	0.1567	0.1571
0.3536	0.3559	−0.1094	−0.1093	0.5151	0.5158	0.1173	0.1175
0.3004	0.3029	−0.1425	−0.1424	0.4750	0.4757	0.0781	0.0780
0.2473	0.2501	−0.1755	−0.1754	0.4349	0.4357	0.0390	0.0388
		−0.2084	−0.2084			0.0000	0.0000

In Table 2 values of ξ are given which were calculated for $j_0 = 1, 2, \ldots, 21$, for two instants $t = 70$ and 125. They characterize the value of the deviation of the boundary of the section from the plane.

In Table 3, for the same instants t, profiles of the function u, obtained from the selfmodelling solution and in the process of computation of $j_0 = 11$, are given.

In conclusion we note that in the region $2(u > u^*)$

$$\frac{k_2\tau}{c_2h^2} = 400.$$

4. The solution of the problem with a cylindrical boundary for the phase division

We shall consider the case where the boundary of the division of the two phases is a circle of radius $\xi(t)$. The temperature distribution is given in the form

$$u_s = B_s - A_s\frac{r^2}{t_0 - t} \qquad (s = 1, 2); \qquad t < t_0, \qquad r^2 = (x - x_0)^2 + (y - y_0)^2, \tag{18}$$

and so the function $\xi(t) = a\sqrt{t_0 - t}$.

We relate the values $s = 1$ to the points $r > \xi(t)$. Suppose that for this phase $u < u^*$; then $s = 2$ corresponds to the inner points of the circle for which $u > u^*$.

From the conditions of the equality of temperature and fluxes, which for our case are of the form

$$\begin{gathered} u|_{\xi+0} = u|_{\xi-0} = u^*, \\ k_1\frac{\partial u}{\partial r}\Big|_{\xi+0} - k_2\frac{\partial u}{\partial r}\Big|_{\xi-0} = \lambda\frac{d\xi}{dt}, \end{gathered} \tag{19}$$

we can obtain an expression for A_s and B_s in terms of k_1, k_2, a, λ and u_0 (the temperature when $r = 0$)

$$A_1 = \frac{k_2}{k_1}\frac{u_0 - u^*}{a^2} + \frac{a\lambda}{4k_1}, \quad A_2 = \frac{u_0 - u^*}{a^2}, \quad B_1 = u^* + a^2A_1, \quad B_2 = u_0$$

The temperature distribution in the form (18) satisfies the following differential equations in polar coordinates

$$c_s \frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(k_s r \frac{\partial u}{\partial r}\right) + A_s\left(\frac{4k_s}{t_0 - t} - \frac{c_s r^2}{(t_0 - t)^2}\right), \tag{20}$$

$$0 < r < r_{\max}, \qquad 0 < t \leqslant t_{\max} < t_0,$$

$$s = 1 \quad \text{if} \quad u < u^*, \qquad s = 2 \quad \text{if} \quad u > u^*.$$

We shall consider this problem in the variables (x, y) and choose as the region G a square with side l (see Fig. 3) such that the circle is wholly situated inside it and the origin of the coordinates is at one of its vertices. We shall rewrite equation (20) in a rectangular system of coordinates and use the difference approximation of the equation obtained together with boundary conditions (19). The values of the required network function on the boundary of the square and at the initial instant are calculated by formula (18).

For $j = j_0$ (j_0 is a number of the row y, y_{j0} = const.) the value of $x(u^*, t)$ and from it $\xi(t)$ is determined by linear interpolation with respect to the temperature. The calculation of a series of variants of this problem was carried out with various parameters. Common elements for the variants were $t_0 = 64$, $u^* = 0$, $u_0 = 1$, $a = 0.2$, $l = 4$, $h_x = h_y = h$, $c_1 = 2$, $c_2 = 1.25$, $k_1 = 0.5$, $k_2 = 0.75$.

In Table 4 exact and calculated values (three iterations) of ξ are given for $j_0 = 16$ for a variant with $\lambda = 1$, $\Delta = 0.15$, $h = 4/30$, $\tau = 0.5$.

TABLE 4.

t	Exact values	Computed values	t	Exact values	Computed values
10	1.4697	1.4690	40	0.980	0.968
20	1.327	1.324	50	0.748	0.724
30	1.166	1.158			

In Table 5 the value of ξ is given for all rows $j = 1, 2, \ldots, 31$ at time $t = 30$ for the same variant.

TABLE 5.

№ строки	8	9	10	11	12	13	14	15	16
ξ	1.161	1.160	1.157	1.161	1.158	1.159	1.151	1.155	1.158

The picture is symmetrical with respect to $j = 16$, $i = 16$, i.e. with respect to the lines parallel to the axes and passing through the centre of the circle.

5. Conclusions

From the given tables it is obvious that the method is entirely suitable for practical purposes. Distinctive features of this method are its logical simplicity and the possibility of using a large time step.

From the analysis of the results, which are not quoted in the present paper, we can make more remarks about this method. This first of all touches on the choice of some δ-form function (in the form of a "step", "peak" or parabola etc.). The results depended slightly on the method of

TABLE 6.

$t=30$	Exact	Variant		
		Without interation	Interation	Interation
ξ	1.166	1.149	1.151	1.151
u	1.000	0.985	0.992	0.992

choosing the functions; it is best to choose the δ-form functions, having no sharp maximum as, for instance, in the case of the "peak". From our point of view it is simpler and more natural to choose the "step". The region of definition of the δ-form function is chosen so that it embraces 2 - 3 calculational points. We must supplement this after evaluating the characteristic temperature gradients. Thus in our variants of the problems with cylindrical boundaries to the phase division for preference $\Delta = 0.15$, since for too small Δ non-monotonicity in the smoothing may occur and with too large Δ strong divergence from the original problem. The process of choosing Δ can be made automatic. In our examples the coefficients in the equation of heat conduction were constant. We do not necessarily require this, however, and no complications of the method arise here.

The iterations which are used to solve the problems converge rapidly. In Table 6 values of ξ and U are given for one row and one instant of time in a variant without iterations, and with three and seven iterations.

The decrease of the space step by a factor of 1.5 gave better results than the decrease of the time step by a factor of two. In general analysis shows that, as is obvious, the scheme has an order of accuracy of $O(\tau) + O(h^2)$.

In conclusion we wish to thank I.V. Fryazinov for his valuable help and advice.

Translated by H.F. Cleaves

REFERENCES

1. KAMENOMOSTSKAYA, S.L. On the Stefan problem. *Mat. Sb.* 53(95), 4, 489 - 514, 1961.

2. EHRLICH, L.W. A numerical method of solving a heat flow problem with moving boundary. *J. Ass. Comput. Mach.* 5, 2, 161 - 176, 1958.

3. SAMARSKII, A.A. An economic difference method of solving a multi-dimensional parabolic equation in an arbitrary region. *Zh. vȳchisl. Mat. mat. Fiz.* 2, 5, 787 - 811, 1962.

4. SAMARSKII, A.A. Locally one-dimensional difference schemes on non-uniform networks. *Zh. vȳchisl. Mat. mat. Fiz.* 3, 3, 431 - 466, 1963.

5. OLEINIK, O.A. A method of solving the general Stefan problem. *Dokl. Akad. Nauk SSSR*, 135, 5, 1054 - 1057, 1960.

6. SAMARSKII, A.A. Equations of parabolic type with discontinuous co-efficients and difference methods for their solution. *Tr. Bses. soveshchaniya po differ yr-niyam* (Erevan, Nov. 1958). Erevan, Izd-vo AN ArmSSR, 148 - 160, 1960.

7. SAMARSKII, A.A. Uniform difference schemes for non-linear equations of parabolic type. *Zh. vȳchisl. Mat. mat. Fiz.* 2, 1, 25 - 56, 1962.

8. TIKHONOV, A.N. and SAMARSKII, A.A. *Equations of mathematical physics* (Uravneniya matematicheskoi fiziki). 2nd Ed. Gostekhizdat, Moscow, 1953.