AN ACCURATE HIGH-ORDER DIFFERENCE SYSTEM FOR A HEAT CONDUCTIVITY EQUATION WITH SEVERAL SPACE VARIABLES

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1. We shall consider that in $\overline{Q}_T = \overline{G} \times [0 \leqslant t \leqslant T]$, where $\overline{G} = \{0 \leqslant x_\alpha \leqslant l_\alpha,\ \alpha = 1, \ldots, p\}$ there is a p-dimensional parallelepiped with boundary Γ related to the problem (see [1]):

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} L_\alpha u + f(x, t), \qquad L_\alpha u = \frac{\partial^2 u}{\partial x_\alpha^2}, \tag{1}$$

$$u_\Gamma = \mu(x, t), \qquad u(x, 0) = u_0(x), \qquad x = (x_1, \ldots, x_p). \tag{2}$$

In [1] an economical double-layer difference system has been proposed for solving problem (1)-(2) and by the method of power inequalities it has been shown that this system is absolutely stable and converges in norm $\mathscr{L}_2(\omega_h)$ at a rate depending on $O(h^4 + \tau^2)$ when $p \leqslant 3$. The purpose of this paper is to show that this system retains these properties even when $p = 4$, i.e. is suitable for $p \leqslant 4$.

2. Let $\overline{\omega}_h$ be a spatial net, uniform in respect of each of the variables x_α with steps $h_\alpha = l_\alpha/N_\alpha$, $\alpha = 1, \ldots, p$ and $\overline{\omega}_\tau$ an arbitrary, non-uniform net in the segment $0 \leqslant t \leqslant T$, $\overline{\Omega} = \omega_h \times \overline{\omega}_\tau$. We shall also use the notation of [1]; the norm in $\mathscr{L}_2(\omega_h)$ is particularly suitable:

$$\| z \| = \left(\sum_{\omega_h} z^2 H \right)^{1/2}, \qquad H = \prod_{\alpha=1}^{p} h_\alpha. \tag{3}$$

The double-layer progressive scheme of [1] becomes

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$$Ay_{\bar{t}} = \Lambda \check{y} + \frac{1}{12} \sum_{\alpha=1}^{p} \sum_{\beta>\alpha} (h_\alpha^2 + h_\beta^2) \Lambda_\alpha \Lambda_\beta \check{y} + \varphi, \tag{4}$$

$$y|_\gamma = \mu, \qquad y(x, 0) = u_0(x), \tag{5}$$

where

$$y = y^{j+1}, \qquad \check{y} = y^j = y(x, t_j), \qquad y_{\bar{t}} = (y - \check{y})/\tau, \qquad \Lambda_\alpha y = y_{\bar{x}_\alpha x_\alpha},$$

$$\Lambda = \sum_{\alpha=1}^{p} \Lambda_\alpha, \qquad A = \prod_{\alpha=1}^{p} A_\alpha, \qquad A_\alpha = E - \sigma_\alpha \tau \Lambda_\alpha, \qquad \sigma_\alpha = \frac{1}{2}\left(1 - \frac{h_\alpha^2}{6\tau}\right),$$

E is the unit operator, and γ the boundary of the net $\bar{\omega}_h$.

The progressive scheme (4) for solving equation (1) has a maximum error of approximation

$$\Psi = O(|h|^4 + \tau^2), \qquad |h| = \left(\sum_{\alpha=1}^{p} h_\alpha^2\right)^{1/2}.$$

To solve problem (4)-(5) one of the one-dimensional computational algorithms of variable directions mentioned in [1] can be used. We emphasize that several computational algorithms correspond to the same progressive scheme.

We shall consider the problem

$$Az_{\bar{t}} = \Lambda \check{z} + \frac{1}{12} \sum_{\alpha=1}^{p} \sum_{\beta>\alpha} (h_\alpha^2 + h_\beta^2) \Lambda_\alpha \Lambda_\beta \check{z} + \Psi, \tag{6}$$

$$z|_\gamma = 0, \qquad z(x, 0) = z_0(x). \tag{7}$$

If y is the solution of problem (4)-(5), and $u = u(x, t)$ is the solution of the initial problem (1)-(2), their difference $z = y - u$ satisfies conditions (6)-(7) and $z_0(x) = 0$ and $\Psi = \Psi(x, t)$ is the error of approximation of scheme (4) in solving equation (1).

We shall seek the solution of problem (6)-(7) by a method of separation of variables assuming

$$z^{j+1} = \sum_k T_k^{j+1} V_k(x), \quad k = (k_1, \ldots, k_p), \quad k_\alpha = 1, 2, \ldots, N_\alpha - 1, N_\alpha = l_\alpha / h_\alpha, \tag{8}$$

where

$$V_k(x) = \prod_{\alpha=1}^{p} v_{k_\alpha}(x_\alpha), \qquad v_{k_\alpha}(x_\alpha) = \sin \frac{\pi k_\alpha x_\alpha}{l_\alpha}, \tag{9}$$

and $v_{k_\alpha}(x_\alpha)$ is the eigenfunction of the one-dimensional problem

$$\Lambda_\alpha v_{k_\alpha} + \frac{4}{h_\alpha^2}\xi_\alpha v_{k_\alpha} = 0, \qquad v_{k_\alpha}(0) = v_{k_\alpha}(l_\alpha) = 0, \qquad \alpha = 1, \ldots, p, \tag{10}$$

$$\xi_\alpha = \sin^2\frac{\pi k_\alpha h_\alpha}{2l_\alpha}. \tag{11}$$

Substituting (8) in (6) and taking into account (9)-(11), we obtain

$$(\rho - 1)\prod_{\alpha=1}^{p}(1 + 4\gamma_\alpha\sigma_\alpha\xi_\alpha) = -4\sum_{\alpha=1}^{p}\gamma_\alpha\xi_\alpha + \frac{4}{3}\sum_{\alpha=1}^{p}\sum_{\beta>\alpha}(\gamma_\alpha + \gamma_\beta)\xi_\alpha\xi_\beta + \tau A/T^j, \tag{12}$$

where $\rho = \rho_k = T_k^{j+1}/T_k^{\,j}$, $\gamma_\alpha = \tau/h_\alpha^2$, and A is the Fourier coefficient of function Ψ in relation to $\{V_k\}$:

$$A = A_k^{j+1} = (\Psi^{j+1}, V_k) = \sum_{\omega_h}\Psi^{j+1}V_k H. \tag{13}$$

4. From (8) and (12) we find

$$z^{j+1} = \sum_k q_k T_k^{\,j} V_k(x) + \tau\sum_k B_k^{j+1}V_k(x), \tag{14}$$

where

$$B_k = A_k/\delta, \qquad \delta = \prod_{\alpha=1}^{p}(1 + 4\gamma_\alpha\sigma_\alpha\xi_\alpha), \tag{15}$$

$$q_k = 1 - \left(4\sum_{\alpha=1}^{p}\gamma_\alpha\xi_\alpha - \frac{4}{3}\sum_{\alpha=1}^{p}\sum_{\beta\neq\alpha}\gamma_\alpha\xi_\alpha\xi_\beta\right)\Big/\delta. \tag{16}$$

Further, it will be shown that

$$-1 < q_k \leqslant 1, \text{ i.e. } |q_k| \leqslant 1 \quad \text{if } p \leqslant 4. \tag{17}$$

We assume that condition (17) is satisfied. Noting that

$$1 + 4\gamma_\alpha\sigma_\alpha\xi_\alpha = 1 + \left(2\gamma_\alpha - \tfrac{1}{3}\right)\xi_\alpha \geqslant \tfrac{2}{3} + 2\gamma_\alpha \geqslant \tfrac{2}{3},$$

$$\delta \geqslant \left(\tfrac{2}{3}\right)^p, \tag{18}$$

from (14)-(15) we obtain

$$\| z^{j+1} \| \leqslant \| z^j \| + \tau_{j+1}\left(\tfrac{3}{2}\right)^p \| \Psi^{j+1} \|.$$

Hence follows the *a priori* estimate

$$\| z^{j+1} \| \leqslant \| z_0(x) \| + \left(\tfrac{3}{2}\right)^p \| \Psi^{j+1} \|, \tag{19}$$

$$\| \Psi^{j+1} \| = \sum_{j'=1}^{j+1}\tau_{j'}.$$

This proves the absolute stability (for all values of $\gamma_\alpha = \tau/h_\alpha^2$) of scheme (6)-(7) with respect to the initial data and the right-hand side.

5. We shall now prove the inequality (17) used when deriving the *a priori* estimate (19). Index k is omitted. At first we show that

$$q \leqslant 1 \quad \text{if} \quad p \leqslant 4. \tag{20}$$

Indeed, considering that $\xi_\beta < 1$, we find

$$\frac{4}{3} \sum_{\alpha=1}^{p} \sum_{\beta \neq \alpha} \gamma_\alpha \xi_\alpha \xi_\beta < \frac{4(p-1)}{3} \sum_{\alpha=1}^{p} \gamma_\alpha \xi_\alpha \leqslant 4 \sum_{\alpha=1}^{p} \gamma_\alpha \xi_\alpha, \qquad p \leqslant 4 \tag{21}$$

(20) follows from (21) and (16).

The main difficulty arises in proving the result

$$q > -1. \tag{22}$$

We derive expressions for $p = 4$. For $p < 4$ all the conclusions remain valid and the arguments are simplified. We write in detail the expression for δ:

$$\delta = \prod_{\alpha=1}^{p} (1 + 4\gamma_\alpha \sigma_\alpha \xi_\alpha) = 1 + 2 \sum_{\alpha=1}^{p} \nu_\alpha \xi_\alpha + 4 \sum_{\alpha=1}^{p} \sum_{\beta>\alpha} \nu_\alpha \nu_\beta \xi_\alpha \xi_\beta + 8 \sum_{\alpha_1<\alpha_2<\alpha_3}^{1-p} \nu_{\alpha_1} \nu_{\alpha_2} \nu_{\alpha_3} \xi_{\alpha_1} \xi_{\alpha_2} \xi_{\alpha_3} + 16 \prod_{\alpha=1}^{p} \nu_\alpha \xi_\alpha, \qquad p = 4, \tag{23}$$

where

$$\nu_\alpha = \gamma_\alpha - \frac{1}{6}. \tag{24}$$

From (22) and (15), (16) it can be seen that the condition $q + 1 > 0$ will be satisfied if

$$2F = 2\delta - 4 \sum_{\alpha=1}^{4} \gamma_\alpha \xi_\alpha + \frac{4}{3} \sum_{\alpha=1}^{4} \sum_{\beta>\alpha} (\gamma_\alpha + \gamma_\beta) \xi_\alpha \xi_\beta > 0. \tag{25}$$

Substituting expression (23) in (25) for δ, we obtain

$$F = 1 - \frac{1}{3} \sum_{\alpha=1}^{4} \xi_\alpha + \sum_{\alpha=1}^{4} \sum_{\beta>\alpha} \left(4\gamma_\alpha \gamma_\beta + \frac{1}{9}\right) \xi_\alpha \xi_\beta + 8 \sum_{\alpha_1<\alpha_2<\alpha_3}^{1-4} \nu_{\alpha_1} \nu_{\alpha_2} \nu_{\alpha_3} \xi_{\alpha_1} \xi_{\alpha_2} \xi_{\alpha_3} + 16 \prod_{\alpha=1}^{4} \nu_\alpha \xi_\alpha. \tag{26}$$

6. As well as F we shall consider the expression

$$F_1 = \prod_{\alpha=1}^{4}\left(1-\frac{1}{3}\xi_\alpha\right) = 1-\frac{1}{3}\sum_{\alpha=1}^{4}\xi_\alpha + \frac{1}{9}\sum_{\alpha=1}^{4}\sum_{\beta>\alpha}\xi_\alpha\xi_\beta - \frac{1}{27}\sum_{\alpha_1<\alpha_2<\alpha_3}\xi_{\alpha_1}\xi_{\alpha_2}\xi_{\alpha_3} + \\ + \frac{1}{81}\prod_{\alpha=1}^{4}\xi_\alpha.$$

It is easy to see that $F_1 > (2/3)^4$ since $\xi_\alpha < 1$, $1 - 1/3\ \xi_\alpha > 2/3$.

We derive the difference

$$F - F_1 = 4\sum_{\alpha=1}^{4}\sum_{\beta>\alpha}\gamma_\alpha\gamma_\beta\xi_\alpha\xi_\beta + \sum_{\alpha_1<\alpha_2<\alpha_3}^{1-4}\left(8\nu_{\alpha_1}\nu_{\alpha_2}\nu_{\alpha_3} + \frac{1}{27}\right)\xi_{\alpha_1}\xi_{\alpha_2}\xi_{\alpha_3} + \\ + 16\prod_{\alpha=1}^{4}\nu_\alpha\xi_\alpha - \frac{1}{81}\prod_{\alpha=1}^{4}\xi_\alpha$$

and show that it is non-negative if

$$\gamma_\alpha \geqslant \frac{1}{6}, \quad \alpha = 1, \ldots, p.$$

Indeed, observing that $\nu_\alpha \geqslant 0$, $\prod_{\alpha=1}^{4}\xi_\alpha < \frac{1}{4}\sum_{\alpha_1<\alpha_2<\alpha_3}\xi_{\alpha_1}\xi_{\alpha_2}\xi_{\alpha_3}$, we at once obtain $F - F_1 > 0$, i.e. $F \geqslant F_1 > (2/3)^4$.

Let all values of $\gamma_\alpha < 1/6$, i.e. $1/6 < \nu_\alpha < 0$, $\alpha = 1, \ldots, p$. Then the following inequality is valid:

$$F > F_1 + 4\sum_{\alpha=1}^{4}\sum_{\beta>\alpha}\gamma_\alpha\gamma_\beta\xi_\alpha\xi_\beta - \frac{4}{27} - \frac{2}{81} > \left(\frac{2}{3}\right)^4 - \frac{14}{81} = \frac{2}{81} > 0,$$

i.e. $F > 0$.

Various combinations of signs are possible for ν_α and ν_β, $\alpha \neq \beta$. For example, let $\nu_1 < 0$ and $\nu_\alpha \geqslant 0$, $\alpha = 2, 3, 4$, i.e. $\gamma_1 < 1/6$, $|\nu_1| < 1/6$, and $\gamma_\alpha \geqslant 1/6$, $\alpha > 1$. Writing $\mu_\alpha = \nu_\alpha\xi_\alpha$ and considering that $|\mu_1| < 1/6$, we obtain

$$\sum_{\alpha_1<\alpha_2<\alpha_3}\left(8\mu_{\alpha_1}\mu_{\alpha_2}\mu_{\alpha_3} + \frac{1}{27}\xi_{\alpha_1}\xi_{\alpha_2}\xi_{\alpha_3}\right) + 16\prod_{\alpha=1}^{4}\mu_\alpha - \frac{1}{81}\prod_{\alpha=1}^{4}\xi_\alpha > \\ > \left(8-\frac{8}{3}\right)\mu_2\mu_3\mu_4 + \left(\frac{1}{27}-\frac{1}{81}\right)\xi_2\xi_3\xi_4 - \frac{4}{3}\xi_1(\mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4) + \\ + \frac{1}{27}\xi_1(\xi_2\xi_3 + \xi_2\xi_4 + \xi_3\xi_4)$$

and consequently $F - F_1 \geqslant 8/3(\gamma_2\gamma_3\xi_2\xi_3 + \gamma_2\gamma_4\xi_2\xi_4 + \gamma_3\gamma_4\xi_3\xi_4) > 0$.

It is easy to be convinced that with any other combination of signs of ν_α, $\alpha = 1, \ldots, 4$, $F > 0$. Thus it is proved that inequality (16) is satisfied with any γ_α value, $\alpha = 1, \ldots, 4$, and consequently the *a priori* estimate (19) is correct for $p \leqslant 4$.

Theorem 1. The progressive scheme (4)-(5) is absolutely stable in the norm $\mathscr{L}_2(\omega_h)$ with respect to the initial data and the right-hand side, so that for any values of τ and h_α, $\alpha = 1, 2, \ldots, p$, for the solution of problem (6)-(7) the *a priori* evaluation is correct

$$\| z^{j+1} \| \leqslant \| z(x, 0) \| + \left(\tfrac{3}{2}\right)^p \| \Psi^{j+1} \|, \quad \text{if} \quad p \leqslant 4. \tag{27}$$

Theorem 2. Let the solution of problem (1)-(2) satisfy conditions such that system (4)-(5) has a maximum order of approximation, or more precisely

$$\| \Psi \| = O(|h|^4) + O(\tau^2). \tag{28}$$

Then the progressive scheme (4)-(5) with $p \leqslant 4$ for an arbitrary sequence of nets $\bar{\Omega}$ has fourth order accuracy in respect of $|h|$ and second order accuracy in respect of τ:

$$\| y^{j+1} - u^{j+1} \| = O(|h|^4) + O(\| \tau^2 \|_{j+1}), \tag{29}$$

where

$$\| \tau^2 \|_{j+1} = \sum_{j'=1}^{j+1} \tau_{j'}^2 \tau_{j'}.$$

Theorem 1 has been proved above. Theorem 2 follows from Theorem 1 and condition (28) for the error of approximation Ψ of scheme (41-(5). It is easy to see that condition (28) will be satisfied if $u(x, t) \in C^{(6)}$, $f(x, t) \in C^{(4)}$.

Thus, the double-layer scheme (4)-(5) is applicable for the same number of dimensions ($p \leqslant 4$) as the triple layer scheme (see [2]). It is obvious that scheme (4)-(5) is more economical (requires fewer arithmetical operations for finding $y = y^{j+1}$) compared with the scheme of [2], and that it converges for any values of γ_α, $\alpha = 1, \ldots, p$ whereas the scheme of [2] restricts $\gamma_\alpha = \gamma$ in the form $\gamma \geqslant \text{const.} > 0$. In [2] the case of a quadratic net $h_\alpha = h_\beta = h$ was studied. With a non-quadratic net ($h_\alpha \neq h_\beta$) the scheme of paper [2] becomes considerably more complicated. Scheme (4)-(5) in [1] was generalized for the case of an equation with variable coefficients.

8. As in [1] it is easy to write down accurate high-order difference schemes $O(|h|^4 + \tau^2)$ for a differential equation of hyperbolic type. We shall consider in $\bar{Q}_T$ the following problem:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{\alpha=1}^{p} L_\alpha u + f(x, t), \qquad (x, t) \in Q_T, \quad L_\alpha u = \frac{\partial^2 u}{\partial x_\alpha^2}, \tag{30}$$

$$u|_{\Gamma} = \mu(x, t), \qquad u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = \overline{u}_0(x). \tag{31}$$

The initial scheme $O(|h|^4 + \tau^2)$ takes the form

$$y_{\overline{t}t} = 0.5\,\Lambda(y + \check{\check{y}}) - \frac{1}{12}\sum_{\alpha=1}^{p} h_\alpha^2 \Lambda_\alpha y_{\overline{t}t} + \frac{1}{12}\sum_{\alpha=1}^{p}\sum_{\beta>\alpha}(h_\alpha^2 + h_\beta^2)\Lambda_\alpha\Lambda_\beta \check{y} + \varphi, \tag{32}$$

where $y = y^{j+1}$, $\check{y} = y^j$, $\check{\check{y}} = y^{j-1}$, $y_{\overline{t}t} = (y - 2\check{y} + \check{\check{y}})/\tau^2$, $\bar{\omega}_\tau$ is the uniform net ($\tau_{j+1} = \tau = \text{const.}$).

This can be made to correspond to several progressive schemes of the same order of accuracy. For example we give a progressive scheme

$$Ay_{\overline{t}} = \check{y}_{\overline{t}} + \sum_{\alpha=1}^{p}(0.5 - \sigma_\alpha)\,\tau\Lambda_\alpha \check{y}_{\overline{t}} + 0.5\tau\Lambda(\check{y} + \check{\check{y}}) + \tag{33}$$

$$+ \sum_{\alpha=1}^{p}\sum_{\beta>\alpha}(1 - \sigma_\alpha - \sigma_\beta)\,\tau^3\Lambda_\alpha\Lambda_\beta\check{y} + \tau\varphi = \Phi[\check{y}],$$

$$y|_\gamma = \mu, \qquad y(x, 0) = u_0(x), \qquad y_{\overline{t}}(x, \tau) = \overline{u}_0(x) - 0.5\tau\left[\sum_{\alpha=1}^{p} L_\alpha u_0 + f(x, 0)\right], \tag{34}$$

where

$$A = \prod_{\alpha=1}^{p} A_\alpha, \qquad A_\alpha = E - \tau^2\sigma_\alpha\Lambda_\alpha, \qquad \sigma_\alpha = \frac{1}{2} - \frac{h_\alpha^2}{12\tau^2}.$$

To define $y = y^{j+1}$ from here the following variable direction algorithm is suitable:

$$A_1 w_{(1)} = \Phi, \qquad A_\alpha w_{(\alpha)} = w_{(\alpha-1)}, \quad \alpha = 2, \ldots, p, \qquad y = \check{y} + \tau w_{(p)}. \tag{35}$$

At the limit (if $x \in \gamma$) $w_{(\alpha)}$ are defined by conditions

$$w_{(\alpha)} = A_{\alpha+1}\ldots A_p \mu_{\overline{t}}^{j+1} \quad \text{for} \quad x_\alpha = 0, \qquad x_\alpha = l_\alpha \tag{36}$$

(cp. with [1]).

If $\sigma_\alpha = 0.5$, $\alpha = 1, \ldots, p$, scheme (33) is converted into a known scheme $O(|h|^2 + \tau^2)$.

For scheme (33)-(34) theorems similar to Theorems 1 and 2, are valid.

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