EFFECTIVE DIFFERENCE SCHEMES FOR PARABOLIC SYSTEMS OF EQUATIONS*

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1. To construct efficient difference schemes for parabolic and hyperbolic systems, we use the principle of additivity, according to which the solution of the operator equation

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{m} A_\alpha (t)\, u = f(t)$$

is reduced to the successive solution of equations

$$\frac{1}{m}\frac{\partial u}{\partial t} + A_\alpha u = f_\alpha, \qquad \alpha = 1, \ldots, m, \quad \sum_{\alpha=1}^{n} f_\alpha = f.$$

We used the principle of additivity when constructing local one-dimensional schemes [1], [2], [4]. We give here the additive schemes for parabolic systems of equations with several spatial variables. Relevant results were reported to the Soviet-American Symposium on Differential Equations in Novosibirsk (August, 1963) and partly discussed in papers [2]-[5].

Let G be the p-dimensional range of variation of $x = (x_1, \ldots, x_p)$ with limit Γ. The solution of the parabolic system is sought in the cylinder $\overline{Q}_T = (G + \Gamma)\times[0 \leqslant t \leqslant T]$ without mixed derivatives

$$\frac{\partial \mathbf{u}}{\partial t} = \sum_{\alpha=1}^{p} L_\alpha \mathbf{u} + \mathbf{f}(x, t) \text{and} Q_T = G\times(0 < t \leqslant T]; \qquad (1)$$
$$\mathbf{u}|_\Gamma = \boldsymbol{\mu}(x, t), \quad 0 \leqslant t \leqslant T; \quad \mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad x \in G + \Gamma,$$

where $\mathbf{u}_0$, $\mathbf{u} = (u^1, \ldots, u^n)$, $\mathbf{f} = (f^1, \ldots, f^n)$ and $\boldsymbol{\mu}$ - vectors.

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$$L_\alpha \mathrm{u} = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial \mathrm{u}}{\partial x_\alpha}\right),$$

$k_\alpha = (k_\alpha^{is})$ is the square matrix $n \times n$.

For editorial simplification the minor terms in L_α containing $\partial \mathrm{u}/\partial x_\alpha$ u, will be omitted. They are taken into account without any particular difficulties. We will assume, as usual, that problem (1) has a unique solution having all the derivatives necessary for presentation.

We present k_α as the sum of two triangular matrices $\bar{k}_\alpha$ and $\overset{+}{k}_\alpha$ and accordingly assume $L_\alpha = L_\alpha^- + L_\alpha^+$. We will assume that the matrices $\bar{k}_\alpha$ and $\overset{+}{k}_\alpha$ are positive, i.e.

$$\sum_{i,s=1}^{n} (k_\alpha^{\mp})^{is}\xi_i\xi_s \geqslant 0$$

for any vector $\xi = (\xi_1, \ldots, \xi_n)$, $\xi \neq 0$.

Let us introduce difference nets ω_h for G and $\omega_\tau = \{t_j = j\tau \in [0, T]\}$ in the segment $0 \leqslant t \leqslant T$; let $\gamma_h \in \Gamma$ be the limit ω_h, i.e. $\omega_h = \omega_h^{(2)}$ (see [2]). For other notations see paper [2]. We denote by $\mathrm{y} = (y^1, \ldots, y^i, \ldots, y^n)$ a network vector function given for $(\omega_h + \gamma_h)\times\omega_\tau$.

Let $\Lambda_\alpha^{\mp}\mathrm{y} = (a_\alpha^{\mp}\mathrm{y}_{\bar{x}_\alpha})_{\hat{x}_\alpha}$ be a homogeneous difference system of 2nd order of approximation for the operator $L_\alpha^{\mp}\mathrm{u}$; here $\bar{a}_\alpha$ and $\overset{+}{a}_\alpha$ are triangular matrices determined by $\bar{k}_\alpha$ and $\overset{+}{k}_\alpha$ by means of a linear positive model functional and therefore, satisfying the same conditions of positiveness as $\bar{k}_\alpha$ and $\overset{+}{k}_\alpha$:

$$\sum_{i,s=1}^{n} (a_\alpha^{\mp})^{is}\xi_i\xi_s \geqslant 0 \tag{2}$$

on any net of ω_h.

3. Let us introduce the intermediate values $\mathrm{y}_{(1)}, \ldots, \mathrm{y}_{(\alpha)}, \ldots, \mathrm{y}_{(p)}, \mathrm{y}_{(p+1)}, \ldots, \mathrm{y}_{(2p+1-\alpha)}, \ldots, \mathrm{y}_{(2p-1)}$, assuming that $\mathrm{y}_{(2p)} = \mathrm{y} = \mathrm{y}(x_i, t_{j+1})$, $\mathrm{y}_{(0)} = \check{\mathrm{y}} = \mathrm{y}(x_i, t_j)$. The additive scheme for problem (1) takes the form

$$\begin{gathered} y_{\bar{t}_\alpha} = \frac{\mathrm{y}_{(\alpha)} - \mathrm{y}_{(\alpha-1)}}{\tau} = \Lambda_\alpha^- \mathrm{y}_{(\alpha)} + \varphi_\alpha, \quad \alpha = 1, \ldots, p, \quad \mathrm{y}_{(\alpha)}|_{\gamma_h^\alpha} = \mu^* = \mu|_{\gamma_{h,t}^\alpha \in \omega_\tau}, \\ \mathrm{y}_{\bar{t}_{\alpha'}} = \Lambda_\alpha^+ \mathrm{y}_{(\alpha')} + \varphi_{\alpha'}, \end{gathered} \tag{3}$$

$$\alpha' = 2p + 1 - \alpha = p + 1, \ldots, 2p,$$
$$y_{(\alpha')} = \mu \text{ with } x \in \gamma_h^{\alpha}, \quad y(x, 0) = u_0(x). \tag{4}$$

Here $\varphi_\alpha = \varphi_\alpha(x, t^*)$ is the difference approximation f_α, $\sum_{\alpha=1}^{2p} f_\alpha = f$, and the coefficients $a_\alpha^{\mp} = a_\alpha^{\mp}(x, t^*)$ are taken at the moment $t^* \in [t_j, t_{j+1}]$, e.g. $t^* = t_{j+1}$.

To determine $y_{(\alpha)}$, $\alpha = 1, \ldots, p, p+1, \ldots, 2p$, the triangular three-point operators $E - \tau\Lambda_\alpha^-$ have to be changed if $\alpha \leqslant p$ and $E - \tau\Lambda_{2p+1-\alpha}^+$ if $\alpha > p$. This is achieved by formulae inverting three-diagonal matrices (one-dimensional rotation). Therefore scheme (3) is effective. To determine y on the new row $t = t_{j+1}$ it is necessary to have $O(n^2 N_p)$ operations, where N_p is the number of nodes of the net ω_h, the same way as for the explicit scheme.

Scheme (3) is absolutely stable and converges to the norm $\mathscr{L}_2(\omega_h)$ on an arbitrary non-uniform net $\omega_h \times \omega_\tau$. The rate of convergence was evaluated as $O(\|h^2\| + \sqrt{\tau})$, where $\|h\|$ is the mean square pitch of the net ω_h. The conditions which ensure the maximum order of approximation here and below, will not be given (see [1], [2]).

4. We will now consider problem (1) for a parabolic system of equations with mixed derivatives

$$\frac{\partial u}{\partial t} = \sum_{\alpha, \beta=1}^{p} L_{\alpha\beta} u + f(x, t), \qquad L_{\alpha\beta} u = \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}(x, t)\frac{\partial u}{\partial x_\beta}\right),$$
$$u|_\Gamma = \mu(x, t) \qquad u(x, 0) = u_0(x), \tag{5}$$

where $k_{\alpha\beta} = (k_{\alpha\beta}^{is}(x, t))$ are the square matrices $p \times p$ with squares $n \times n$, G, the region composed of parallelepipeds with boundaries parallel to the coordinate planes. We will assume that the net ω_h is uniform along each x_α, $\alpha = 1, \ldots, p$, and the following conditions are satisfied:

$$k_{\alpha\beta}^{ij} = k_{\beta\alpha}^{ji}; \tag{6}$$

$$\sum_{i, s=1}^{n} \sum_{\alpha, \beta=1}^{p} k_{\alpha\beta}^{is} \xi_\alpha^i \xi_\beta^j \geqslant c_1 \sum_{i=1}^{n} \sum_{\alpha=1}^{p} (\xi_\alpha^i)^2, \qquad c_1 = \text{const.} > 0. \tag{7}$$

We take $(k_{\alpha\beta}) = (k_{\alpha\beta-}) + (k_{\alpha\beta+})$, where $k_{\alpha\beta\mp}$ are triangular along α and

β of the matrix, $k_{\alpha\alpha-}=k_{\alpha\alpha+}=\frac{1}{2}k_{\alpha\alpha}$. We make the operator $L_{\alpha\beta\mp}$ correspond to the difference scheme of the 2nd order of approximation, e.g. for the type (see [3], [1])

$$\Lambda_{\alpha\beta}y_{(\beta)}=\frac{1}{2}[(a_{\alpha\beta}y_{\bar{x}_\beta})_{x_\alpha}+(a_{\alpha\beta}^{(+1\beta)}y_{x_\beta})_{\bar{x}_\alpha}],\quad a_{\alpha\beta}=a_{\alpha\beta}(x,t^*),\quad t^*\in[t_j,t_{j+1}]. \tag{8}$$

To solve problem (5) we use a scheme of the same type as in (3)

$$y_{\bar{t}_\alpha}=\sum_{\beta=1}^{\alpha}\Lambda_{\alpha\beta-}y_{(\beta)}+\varphi_\alpha,\quad y_{\bar{t}_{\alpha_1}}=\sum_{\beta=\alpha}^{p}\Lambda_{\alpha\beta+}y_{(\beta_1)}+\varphi_{\alpha_1}, \tag{9}$$

where $\alpha_1=2p+1-\alpha$, $\beta_1=2p+1-\beta$, $\alpha=1,2,\ldots,p$;

$$y_{(\alpha)}=y_{(\alpha_1)}=\mu \text{ with } x\in\gamma_h^\alpha,\quad y(x,0)=u_0(x). \tag{10}$$

To determine y, $2p$ of the one-dimensional parabolic system has to be solved by a matrix rotation method. $O(n^3N_p)$ operations are required. It was shown that scheme (9) is stable and converges to the norm $\mathscr{L}_2(\omega_h)$ at a rate of $O(|h|^2+\sqrt{\tau})$ under one additional condition: h_α steps of the net ω_h are so small, $|h|\leqslant h_0, |h|=\left(\sum_{\alpha=1}^{p}h_\alpha^2\right)^{1/2}$ that for $a_{\alpha\beta}^{is}$ and $a_{\alpha\beta}^{is}$ inequality (7) is satisfied in which $k_{\alpha\beta}^{is}$ is replaced by $a_{\alpha\beta\pm}^{is}$ and c_1 by a constant $c_1'<0.5c_1$. If $k_{\alpha\beta}=k_{\alpha\beta}(t)$, condition $|h|\leqslant h_0$ is not required.

A more efficient scheme is given below with $O(n^2N_p)$ operations which uses only one-dimensional rotation.

5. Matrices $\varkappa_\mp=(k_{\alpha\beta\mp}^{is})$ are the elements of matrices $(k_{\alpha\beta-})$ and $(k_{\alpha\beta+})$. We present them in the form of the sum of triangular matrices $\varkappa_\mp=\overset{\pm}{_+}\varkappa+\overset{\pm}{_-}\varkappa$ and $\varkappa_\mp^{is-}=k_{\alpha\beta\mp}^{is-}=0$ if $s>i$, $\varkappa_\mp^{is+}=k_{\alpha\beta\mp}^{is+}=0$ if $s<i$, $k_{\alpha\beta\pm}^{ii+}=\frac{1}{2}k_{\alpha\beta\pm}^{ii}$. The operator $L_{\alpha\beta}$ with matrix $(k_{\alpha\beta\pm}^{is\pm})$ will be denoted by $L_{\alpha\beta\pm}^{is\pm}$. The difference scheme $\Lambda_{\alpha\beta\pm}^{is\pm}$ corresponds to this. Let

$$(\Lambda_{\alpha\beta\pm}^{-}y)_i=\sum_{s=1}^{i}\Lambda_{\alpha\beta\pm}^{is-}y^s,\quad (\Lambda_{\alpha\beta\pm}^{+}y)_i=\sum_{s=i}^{n}\Lambda_{\alpha\beta\pm}^{is+}y^s$$

(index i will be omitted from the left, writing down the equations for the vectors).

The efficient additive scheme takes the form

$$\mathbf{y}_{\bar{t}_\alpha} = \sum_{\beta=1}^{\alpha} \Lambda^-_{\alpha\beta-}\mathbf{y}_{(\beta)} + \varphi_\alpha, \quad \mathbf{y}_{\bar{t}_{\alpha_1}} = \sum_{\beta=1}^{\alpha} \Lambda^+_{\alpha\beta-}\mathbf{y}_{(\beta_1)} + \varphi_{\alpha_1},$$

$$\mathbf{y}_{\bar{t}_{\alpha_2}} = \sum_{\beta=\alpha}^{p} \Lambda^+_{\alpha\beta+}\mathbf{y}_{(\beta_2)} + \varphi_{\alpha_2}, \quad \mathbf{y}_{\bar{t}_{\alpha_3}} = \sum_{\beta=\alpha}^{p} \Lambda^-_{\alpha\beta+}\mathbf{y}_{(\beta_3)} + \varphi_{\alpha_3}, \quad (x, t) \in \omega_h \times \omega_\tau, \tag{11}$$

$$\mathbf{y}_{(\alpha)} = \mathbf{y}_{(\alpha_k)} = \mu \text{ with } x \in \gamma_h^\alpha, \quad t \in \omega_\tau, \; k = 1, 2, 3; \; \mathbf{y}(x, 0) = \mathbf{u}_0(x),$$

where $\alpha = 1, 2, \ldots, p$, $\alpha_k = kp + 1, \ldots, (k+1)p$, $k = 1, 2, 3$; $\alpha_k = (k+1)p + 1 - \alpha$, $\beta_k = (k+1)p + 1 - \beta$, φ_{α_k} approximate functions $\mathbf{f}_{\alpha_k}$, $\alpha_0 = \alpha$, $k = 0, 1, 2, 3$,

$$\sum_{k=0}^{3} \sum_{\alpha=1}^{p} \mathbf{f}_{\alpha_k} = \mathbf{f}; \quad \mathbf{y}_{(0)} = \check{\mathbf{y}} = \mathbf{y}(x, t_j), \quad \mathbf{y}_{(4p)} = \mathbf{y} = \mathbf{y}(x\, t_{j+1}).$$

Only $4p - 1$, the intermediate value of $\mathbf{y}_{(\alpha_k)}$ is introduced. To determine the vector $\mathbf{y}$ on the layer $t = t_{j+1}$, four triangular-three-point operators $E - \tau \sum_\beta \Lambda^\pm_{\alpha\beta\pm}$, have to be changed successively by ω_h which requires $O(n^2 N_p)$ operations.

Let the following conditions be satisfied

$$\sum_{i,s=1}^{n} \sum_{\alpha,\beta=1}^{p} k^{is\pm}_{\alpha\beta-} \xi^i_\alpha \xi^j_\beta \geqslant c_1 \sum_{i=1}^{n} \sum_{\alpha=1}^{p} (\xi^i_\alpha)^2, \quad c_1 = \text{const.} > 0. \tag{12}$$

Then, by virtue of (6) it will be satisfied also for matrices $(k^{is\mp}_{\alpha\beta+})$. From the definition of $(a^{is\pm}_{\alpha\beta\pm})$ and from (12) it follows (see [3]) that

$$\sum_{i,s=1}^{n} \sum_{\alpha,\beta=1}^{p} a^{is\pm}_{\alpha\beta-} \xi^i_\alpha \xi^i_\beta \geqslant c_1' \sum_{i=1}^{n} \sum_{\alpha=1}^{p} (\xi^i_\alpha)^2, \quad c_1' = \text{const.}, \; c_1' < c_1 \tag{13}$$

with a sufficiently small $|h| \leqslant h_0(c_1')$.

By methods described in paper [2] it can be shown that scheme (11) under condition (13) is absolutely stable to the norm $\mathscr{L}_2(\omega_h)$ and converges at a rate of $O(|h|^2 + \sqrt{\tau})$. If the coefficients $k_{\alpha\beta}$ do not depend on x, the condition $|h| \leqslant h_0$ becomes superfluous.

Remark 1. The methods of separation make it possible to reduce the many-dimensional problem (see, e.g. [6]) of (1) or (5) to p one-dimensional systems, for the solution of which, e.g. matrix rotation has to be used. For the separation scheme the requirements as regards the coefficients of the equation are much more rigid than for additive schemes (9).

Remark 2. The principle of additivity makes it possible to write down efficient schemes also for second order hyperbolic systems (see [4]).

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