The fourth order difference scheme of approximation on the rectangular mesh \( (h_a 
eq h_b \text{ for } a 
eq b) \) will be considered for Poisson's equation. We shall prove the convergence in the mean of the schemes in question at the rate \( O(|h|^4) \), for the Dirichlet problem in the \( p \)-dimensional rectangular parallelepiped \( (p = 2, 3) \), where

\[
|h|^2 = \sum_{a=1}^{p} h_a^2,
\]

whatever the ratio \( h_a \) between the intervals. The conditions under which the maximum principle holds for the proposed schemes on a rectangular mesh will be discussed, and they will be shown to be uniformly convergent at a rate \( O(|h|^4) \) for \( p \leq 4 \).

An alternating directions iterational process will be considered, and the choice of sequence of iterational parameters \( \{\tau_n\} \) "reasonably high" speed of convergence of the process will be discussed. The choice of optimum ratios between the terms of the sequence \( \{\tau_n\} \), minimizing the number of iterations, will also be examined.

1. Given the \( p \)-dimensional parallelepiped

\[
\mathcal{D}_p = \{x = (x_1, \ldots, x_p) : 0 \leq x_a \leq l_a, \ a = 1, \ldots, p\}
\]

with boundary \( \Gamma \), we seek the solution in it of the problem

\[ * \text{ Zh. Vych. Mat. 4, No. 6, 1025-1036, 1964.} \]
Alternating direction iterational schemes

\[ Lu = \sum_{\alpha=1}^{p} \lambda_i u = -f(x), \quad L_\alpha u = \frac{\partial^2 u}{\partial x_\alpha^2}, \quad u |_\gamma = g(x). \] (1)

Let \( \omega_h = \{ x_i = (i_1h_1, \ldots, i_ph_p) \in D_p: \ i_\alpha = 0, \ldots, N_\alpha; \ h_\alpha = \frac{l_\alpha}{N_\alpha}, \) \( \alpha = 1, \ldots, p \) be a difference mesh, uniform with respect to each of directions \( x_\alpha, \) and \( \gamma = \{ x_i \in \Gamma \} \) the boundary of the mesh \( \omega_h. \)

Problem (1) - (2) was considered in [1] - [3] with \( p = 2, 3 \) on a difference mesh \( \omega_h \) with \( h_\alpha = h, \alpha = 1, \ldots, p, \) and difference schemes of the fourth order of approximation to the sufficiently smooth solution of equation (1) were proposed; provided the sides of the parallelepiped \( l_\alpha, \alpha = 1, \ldots, p \) were comparable in size, the schemes were shown to be uniformly convergent at the rate \( O(h^4) \).

An alternating directions iterational process (see (13) with \( \sigma = 1 \)) was proposed for these schemes in [4] with the number of iterations

\[ \nu = \nu_0 \log \frac{1}{h} \log \frac{1}{\varepsilon}, \] (3)

where \( \varepsilon > 0 \) is the required accuracy. The choice of optimum iterational parameters \( \{ \tau_n \} \) minimizing \( \nu_0 \) was likewise discussed. The expression for the rate of convergence was only proved in [4] for \( p = 2. \)

* In [4], note 1, there is an error in the evaluation of \( d \) and \( \varphi. \) The correct expressions are

\[ d = q + \frac{h^2}{12} q^2, \quad \varphi = f + \frac{h^2}{12} (qf + Lf) \) and \( q = \text{const.} \)
In the present paper we consider the more general family of itera
tional schemes with
\[ \sigma \geq \frac{1}{2}, \]  
(4)
where \( \sigma \) is a parameter characterizing the iterational process. A *one-
dimensional* procedure is proposed for choosing the iterational para-
ters \( \{\tau_n\} \), minimizing \( v_0 \) for difference schemes of the fourth order
of accuracy on the rectangular mesh \( h_{\alpha} \neq h_{\beta} \) for \( \alpha \neq \beta \). (Expression
(3) and the optimum value of \( v_0 \) are also obtained in passing for schemes
of the 2nd order of accuracy.)

2. We consider the following difference scheme * for the approxima-
tion of problem (1) - (2):
\[ \Lambda' y + \varphi = 0, \quad y\big|_{\gamma} = g(x), \]  
(5)
where
\[ \Lambda' = \Lambda + \frac{\theta}{12} \sum_{\alpha=1}^{P} h_{\alpha}^2 \sum_{\beta \neq \alpha}^{1-p} \Lambda_{\alpha} \Lambda_{\beta}, \quad \Lambda = \sum_{\alpha=1}^{P} \Lambda_{\alpha}, \quad \Lambda_{\alpha} y = y_{\alpha} x_{\alpha}, \]  
(6)
\[ \varphi = f + \frac{\theta}{12} \sum_{\alpha=1}^{P} h_{\alpha}^2 \Lambda_{\alpha} f, \quad \theta = 0, 1. \]

Here (see [6])
\[ x = x_1 = (h_1 i_1, \ldots, h_P i_P), \]
\[ x^{(\pm i_\alpha)} = (h_1 i_1, \ldots, h_{\alpha-1} i_{\alpha-1}, h_\alpha (i_\alpha \pm 1), h_{\alpha+1} i_{\alpha+1}, \ldots, h_P i_P), \]
\[ y = y_1 = y(x), \quad y^{(\pm i_\alpha)} = y(x^{(\pm i_\alpha)}), \]
\[ y_{\alpha} = (y - y^{(i_\alpha)} / h_\alpha, \quad y_{\alpha} = (y^{(i_\alpha)} - y) / h_\alpha. \]

It is easily shown (see also [5]) that, for \( \theta = 1 \), scheme (5)-(6) (in
future scheme (5)-(6)) has the 4th order of approximation in \( |h| \) on the
class of sufficiently smooth solutions of (1), so that
\[ \psi = \Lambda' u + \varphi = O(|h|^4). \]
For \( \theta = 0 \) scheme (5)-(6) (scheme (5)-(6)) becomes the familiar scheme

* Scheme (5) - (6) for \( f = 0 \) and \( \rho = 2 \) was proposed without a proof of
convergence in [5].
of the second order of accuracy.

Let us show that (5)-(61) has the 4th order of accuracy in $|h|$. Let $u$ be a solution of problem (1)-(2) and $y$ a solution of problem (5)-(61).

We now obtain for the function $z = y - u$:

$$\Lambda'z + \psi = 0, \quad z|_\gamma = 0. \quad (7)$$

We require the scalar products (see [8])

$$(y, v) = \sum_{\omega_h} yvH, \quad (yv)_a = \sum_{c^+} yvH, \quad (y, v)_{a,\beta} = \sum_{c^+} yvH$$

and the corresponding norms

$$\|v\| = \sqrt{(v, v)}, \quad \|v_x^a\| = \sqrt{(v_x^a, v_x^a)}, \quad \|v_{x-x}^a\| = \sqrt{(v_{x-x}^a, v_{x-x}^a)}$$

where

$$H = \prod_{a=1}^p h_a, \quad \gamma^+_a = \{x_i \in \Gamma: i_a = N_a; i_\beta \neq 0, N_\beta \text{ for } \beta \neq a\},$$

$$\gamma^-_a = \{x_i \in \Gamma: i_a = 0; i_\beta \neq 0, N_\beta \text{ for } \beta \neq a\},$$

$$\gamma^+_{a\beta} = \{x_i \in \Gamma: i_a = N_a, i_\beta = N_\beta; i_\delta \neq 0, N_\delta \text{ for } \delta \neq a, \beta\},$$

$$\omega_h = \omega_h - \gamma, \quad \omega^+_a = \omega_h + \gamma^+_a, \quad \omega^+_{a+\beta} = \omega^+_a + \gamma^+_a + \gamma^+_{a\beta},$$

$$\gamma^+_a = \gamma^+_a + \gamma^-_a.$$
We consider \((\psi, z)\):

\[
(\psi, z) \leq \|\psi\| z \leq (M_0 I)^v \|\psi\| \leq c_0 I + \frac{M_0}{4c_0} \|\psi\|^2.
\]  

(10)

Substituting (9) and (10) in (8) and suitably fixing \(c_0\), we find that

\[
I \leq \frac{9M_0}{(4-p)} \|\psi\|^2,
\]

or, from (9),

\[
\|z\| \leq \frac{3M_0}{4-p} \|\psi\|.
\]

We have now proved:

**Theorem 1.** If the condition

\[
\|\psi\| \leq M|h|^4,
\]

is satisfied, the difference scheme (5)-(6) with \(p \leq 3\) is convergent in the mean at a rate \(O(1/h^4)\) so that

\[
\|y - u\| \leq M'|h|^4, \quad M' = M \frac{3M_0}{4-p},
\]

where \(M\) is a positive constant independent of \(|h|\).

Theorem 1 proves the convergence of scheme (5)-(6) in the mean on any sequence of rectangles uniform with respect to each mesh direction, provided only that \(|h| \to 0\). If we impose certain restrictions on the ratio between the intervals \(h_\alpha\) of the mesh \(\omega_h\), we can prove uniform convergence for scheme (5)-(6). We expand (5)-(6) in points

\[
\frac{7-p}{3} \sum_{\alpha=1}^{p} \frac{1}{h^2_\alpha} y = \frac{1}{6} \sum_{\alpha=1}^{p} \left( (7-p) \frac{1}{h^2_\alpha} - \sum_{\beta=1}^{1-p} \frac{1}{h^2_\beta} \right) (y^{(+1\alpha)} + y^{(-1\alpha)}) +
\]

\[
+ \frac{1}{12} \sum_{\alpha=1}^{p-1} \sum_{\beta=\alpha+1}^{p} \frac{1}{h^2_\alpha} + \frac{1}{h^2_\beta} \left( y^{(+1\alpha,+1\beta)} + y^{(+1\alpha,-1\beta)} + y^{(-1\alpha,+1\beta)} + y^{(-1\alpha,-1\beta)} \right) + \varphi.
\]

It is clear from (11) that the coefficient of \(y\) on the left-hand side is equal to the sum of all the coefficients on the right-hand side. Let \(h_\alpha\), \(\alpha = 1, \ldots, p\), be such that all the coefficients of (11) are non-negative, i.e.

\[
(7-p) \frac{1}{h^2_\alpha} - \sum_{\beta=\alpha+1}^{p} \frac{1}{h^2_\beta} \geq 0.
\]  

(12)
The maximum principle (see [2]) will then hold for equation (11), and we can prove, by the same arguments as in [2]*, that

**Theorem 2.**

If

$$
\| \psi \|_0 = \max_{\omega_h} |\psi| \leq M|h|^4
$$

and conditions (12) are satisfied, the difference scheme (5) - (6) with \( p \leq 4 \) is uniformly convergent at a rate \( O(h^4) \) so that

$$
\| y - u \|_0 \leq M'h|^4,
$$

where \( M' \) is a positive constant independent of \( |n| \).

For \( p = 2 \), conditions (12) become \( 1/\sqrt{5} \leq h_1/h_2 \leq \sqrt{5} \). For \( p = 3 \), the ratios \( h_1^2/h_2^2 \) and \( h_2^2/h_3^2 \), which satisfy conditions (12) are given by the coordinates of the part of the plane inside the triangle with vertices \( A(\frac{1}{3}, \frac{1}{3}), B(1, 3), C(3, 1) \). If \( p = 4 \), it follows from (12) that Theorem 2 only holds with \( h_\alpha = h \ (\alpha = 1, \ldots, 4) \), which is only possible if the sides \( l_\alpha, \alpha = 1, \ldots, 4 \) of the region \( D_p \) are commensurable.

**Note 1.** If \( D_p \) has commensurable sides \( l_\alpha \), we can introduce the difference scheme \( \check{\omega}_h \) with \( h_\alpha = h, \alpha = 1, \ldots, p \), into it. On this mesh, problem (1) - (2) can be associated with the difference scheme

$$
\Lambda^*y + \varphi^* = 0, \quad y|_\Gamma = g(z),
$$

where

$$
\Lambda^* = \Lambda + \frac{h^4}{6} \sum_{a=1}^{p-1} \sum_{\beta=\alpha+1}^{p} \Lambda_\alpha \Lambda_\beta + \frac{h^6}{30} \sum_{a=1}^{p-2} \sum_{\beta=\alpha+1}^{p-1} \sum_{\delta=\beta+1}^{p} \Lambda_\alpha \Lambda_\beta \Lambda_\delta,
$$

$$
\varphi^* = f + \frac{h^2}{12} Lf + \frac{h^4}{360} \left( L^4f + 2 \sum_{a=1}^{p} \sum_{\beta=\alpha+1}^{p} L_\alpha L_\beta f \right).
$$

Scheme (5*) - (6*) with \( p = 2, 3 \) was proposed in [3], [9], where its uniform convergence at a rate \( O(h^6) \) was proved.

3. We consider the alternating directions method for the approximate

* See [8] for the case \( p = 2 \).
solution of problem (5) - (66). Let \( u = v^{(n+1)} \) be the \((n + 1)\)-th iteration, \( \nu = v^{(n)} \), \( \tau = \tau_n \) the iterational parameter, which will be chosen later, and \( \nu_i = (\nu - \nu) / \tau \). For the derivative scheme (see [4]), connecting \( \nu \) and \( \tilde{\nu} \), we take

\[
Au_i = A\tilde{\nu} + \varphi, \quad \nu|_\gamma = g(x), \quad \nu^{(0)}(x) = v_0(x),
\]

(13)

where

\[
A = \prod_{\alpha=1}^{p} A_\alpha, \quad A_\alpha = E - \sigma \tau A_\alpha, \quad Ev = v, \quad \sigma > \frac{1}{2},
\]

and \( v_0(x) \) is the zero approximation.

For \( \theta = 0 \), scheme (13) was proposed and investigated in [10] (see also [11] - [13]). One of the alternating directions algorithms (see [4], [7], [10] - [13]) may be used for determining \( \nu \) from (13); some of these only operate for \( \theta = 0 \). Let us prove, for instance, the algorithm proposed in [7]

\[
A_\alpha w_{(\alpha)} = A\tilde{\nu} + \varphi, \quad \Lambda_\alpha w_{(\alpha)} = w_{(\alpha-1)}, \quad \alpha = 2, \ldots, p,
\]

(14)

\[
w_{(\alpha)} = 0 \text{ for } x \in \gamma_\alpha, \quad \alpha = 1, \ldots, p, \quad \nu = \tilde{\nu} + \tau w_{(p)}.
\]

Notice that the algorithm proposed in [12] follows from (14) on carrying out the substitution \( w_{(\alpha)} = (v_{(\alpha)} - \nu) / \tau \) in (14); however, (14) is more economic, and in addition, the \( w_{(\alpha)}, \alpha = 1, \ldots, p, \) always satisfy the zero boundary conditions.

4. Let us consider the convergence of the iterational process (13).

We obtain for \( \omega = \nu - y \), where \( y \) is a solution of problem (5) - (66), and \( \nu \) a solution of problem (13)

\[
Aw_i = A\tilde{\nu}, \quad \omega|_\gamma = 0, \quad \omega^{(0)}(x) = v_0(x) - y(x).
\]

(15)

We apply Fourier's method for finding the solution of problem (15). Let \( \mu_\alpha = \mu_{k_\alpha}(x) \) and \( \lambda_\alpha = \lambda_{k_\alpha} \), \( k_\alpha = 1, \ldots, N_\alpha - 1, \alpha = 1, \ldots, p, \) be the eigenfunctions and eigenvalues of the one-dimensional Sturm-Liouville difference problem

\[
\Lambda_\alpha \mu_\alpha + \lambda_\alpha \mu_\alpha = 0, \quad \mu_\alpha(0) = \mu_\alpha(l_\alpha) = 0.
\]

(16)
The problem

\[ \Lambda \mu + \lambda \mu = 0, \quad \mu|_\gamma = 0 \]  

(17)

now has the solution

\[ \mu = \mu_k(x) = \prod_{a=1}^{p} \mu_{k_a}(x_a), \quad \lambda = \lambda_k = \sum_{a=1}^{p} \lambda_{k_a}, \quad k = (k_1, \ldots, k_p). \]

The eigenvalues of problem (15) are easily obtained:

\[ \lambda_{k_a} = \frac{4}{h_{a}^2} \sin^2 \frac{k_a \pi a}{2}, \quad k_a = 1, \ldots, N_a - 1, \]  

(18)

but we shall only require the maximum and minimum of them in what follows.

We shall seek the solution of problem (16) in the form

\[ w = w^{(n+1)} = \sum_k a_{k,n+1} \mu_k(x), \quad \bar{w} = \sum_k a_{k,n} \mu_k(x). \]  

(19)

Substituting (19) in (15) and recalling that the functions \( \mu_k(x) \) are orthogonal, we get

\[ a_{k,n+1} = \rho_{k,n+1} a_{k,n}, \]  

(20)

where

\[ \rho_{k,n+1} = 1 - \tau \left[ \frac{\lambda - \theta}{12} \sum_{a=1}^{p} h_{a} \sum_{\beta \neq a}^{1-p} \lambda_{a} \lambda_{\beta} \right] \prod_{a=1}^{p} (1 + \sigma \lambda_{a})^{-1}. \]  

(21a)

We obtain from (20)

\[ a_{k,n+1} = a_{k,0} \prod_{s=1}^{n+1} \rho_{k,s}, \]

and hence, from (19)

\[ w^{(n+1)} = \sum_k a_{k,0} \prod_{s=1}^{n+1} \rho_{k,s} \mu_k, \]

(22)

\[ \|w^{(n+1)}\| = \left( \sum_k \left[ a_{k,0} \prod_{s=1}^{n+1} \rho_{k,s} \mu_k \right]^2, 1 \right)^{1/2} \leq \|w^{(0)}\|, \]

where

\[ R_{n+1} = \max_k \prod_{s=1}^{n+1} \rho_{k,s}. \]  

(23)
Theorem 3. When conditions (4) are satisfied, the iterative process (13) with \( p = 2, 3 \) is convergent in the mean whatever the parameters \( \tau_h \) satisfying

\[
0 < c_1 \leq \tau_n \leq c_2,
\]

where \( c_1 \) and \( c_2 \) are constants independent of \( n \).

From (22), to prove the theorem we have to show that \( R_{n+1} \to 0 \) as \( n \to \infty \). But to do this, it is sufficient to show that

\[
|\rho_{k,s}| < \rho < 1,
\]

where \( \rho \) is a constant independent of \( n \), since we then have from (23)

\[ R_{n+1} \leq p^{n+4}. \]

By (18),

\[
h_a^2 \lambda_a < \lambda_a \quad \text{and} \quad \frac{1}{12} \sum_{a=1}^{p} h_a^2 \sum_{\beta=1}^{1-p} \lambda_a \lambda_\beta < \frac{p-1}{3} \sum_{a=1}^{p} \lambda_a = \frac{p-1}{3} \lambda.
\]

It follows from this, and (21a), that

\[
\rho_{k,s} < 1 - \left(1 + \theta \frac{1-p}{3}\right) \tau \lambda \prod_{a=1}^{p} \left(1 + \sigma \lambda_a \right)^{-1}, \quad (26a)
\]

\[
\rho_{k,s} > 1 - \tau \lambda \prod_{a=1}^{p} \left(1 + \sigma \lambda_a \right)^{-1} > 1 - \frac{\tau \lambda}{1 + \sigma \lambda}. \quad (27)
\]

On now using condition (24), we find that

\[
|\rho_{k,s}| < \rho,
\]

where

\[
\rho = \max \left\{ \left|1 - \left(1 + \theta \frac{1-p}{3}\right) c_1 \lambda \prod_{a=1}^{p} \left(1 + \sigma \lambda_a \right)^{-1}\right|, \left|1 - \frac{c_2 \lambda}{1 + \sigma \lambda}\right| \right\},
\]

i.e., \( \rho \) is independent of \( n \). Recalling (4), we find that in fact \( \rho < 1 \).

(Theorem 3 was proved for \( \theta = 0 \) in [10].)

Note 2. The iterative scheme for problem (5) - (6) is

\[
A v_T = \Lambda v + q^*, \quad v \mid_{\gamma} = g(x), \quad v^{(0)} (x) = v_0 (x), \quad (13^a)
\]

while the corresponding function is

\[
\rho_{k,n+1} = 1 - \left[ \lambda - \frac{a}{6} \sum_{a=1}^{p-1} \sum_{\beta=1}^{p-1} \lambda_a \lambda_\beta + \frac{h_s}{30} \sum_{a=1}^{p-2} \sum_{\beta=1}^{p-1} \sum_{\gamma=1}^{p-1} \lambda_a \lambda_\beta \lambda_\gamma \right] \prod_{a=1}^{p} \left(1 + \sigma \lambda_a \right)^{-1} \quad (21^a)
\]
Recalling (18), it can easily be seen from (21*) that, with (4) and (24),
the upper bound for the function \( \rho_{h,n+1}^* \) is of the same form as for the
function \( \rho_{h,n+1} \) with \( \theta = 1 \). Theorem 1 therefore holds for scheme (13*)
also.

5. To estimate the rate of convergence (number of iterations) of the
iterational process (13), we require a more exact upper bound for
\( |\rho_{h,n+1}| \).

**Lemma 1.** For the function \( \rho_{h,n+1} \), defined by (21g) with

\[
\sigma > \sigma_p, \quad \sigma_{1,1} = \frac{5}{6}, \quad \sigma_{2,1} = \frac{1}{2}, \quad \sigma_{p,0} = \frac{1}{2} \left[ 1 + \left( \frac{p-1}{p} \right)^{p-1} \right]
\]

we have

\[
|\rho_{h,n+1}| < \bar{\rho}(a),
\]

where

\[
0 < \bar{\rho}(a) = 1 - \frac{1}{\sigma} \left( 1 + \frac{1-p}{3} \right) \frac{pa}{(1+a)^p}, \quad a = \frac{\sigma \lambda}{p} > 0.
\]

For, by the theorem on the arithmetic mean and geometric mean (see
[14], p. 29), we have

\[
\prod_{a=1}^{p} (1 + \sigma \lambda a) \leq \left( \frac{1}{1 + \frac{\sigma \lambda}{p}} \right)^p.
\]

We find from this and (26g) that

\[
\rho_{h,n+1} < 1 - \left( 1 + \frac{1-p}{3} \right) \frac{\tau \lambda}{\left( 1 + \frac{\sigma \lambda}{p} \right)^p} = \bar{\rho}(a)
\]

whatever the positive \( \sigma_p \).

It follows from (27) that, to complete the proof of Lemma 1, we have
to show that

\[
-1 + \left( 1 + \frac{1-p}{3} \right) \frac{\tau \lambda}{\left( 1 + \frac{\sigma \lambda}{p} \right)^p} < 1 - \frac{\tau \lambda}{1 + \sigma \lambda}.
\]

To this end, we consider the function

\[
F_{p,0} = \Sigma - \frac{pa}{1+pa} \left( 1 + \frac{1-p}{3} \right) \frac{pa}{(1+a)^p},
\]
the fact that this is positive being equivalent to (31). We transform
\( F_{p,1} \) to the form
\[
F_{3,1} = 2\sigma - 1 + \frac{a^3 + 2a + 1}{(a + 1)^3(3a + 1)}, \quad F_{3,1} = 2\sigma - 1 + \frac{a^3 + 2a + 1}{(a + 1)^3(3a + 1)}.
\]
Given (28), the fact that \( F_{3,1} \) is positive is now obvious. Given (28),
the fact that \( F_{2,1} \) is positive is equivalent to the numerator being
positive, and this can easily be proved by considering its minimum. When
investigating \( F_{p,0} \), we shall be satisfied with a crude estimate. In fact,
we shall estimate separately \( pa/(1+a)p \) and \( pa/(1+pa) \). Now,
\[
F_{p,0} > 2\sigma - 1 - \left( \frac{p-1}{p} \right)^{p-1},
\]
and the lemma follows from this and (28).

The expression involving \( \rho_k, n+1 \), established by Lemma 1 holds under
stronger restrictions on \( \sigma \) (except for the case \( p = 3, \theta = 1 \)) then does
Theorem 3. For \( p = 2 \) it is possible to obtain an estimate rather differ-
ent from (29) for \( \rho_k, n+1 \), which holds for \( \sigma \geq \frac{1}{2} \).

**Lemma 2.** Given the function
\[
\rho(a_1, a_2) = 1 - x \frac{a_1 + a_2 - a_1 a_2}{(1 + a_1)(1 + a_2)}, \quad x \geq 0, \quad a \geq 0, \quad a_a > 0.
\]
If the condition
\[
x \leq 2,
\]
is satisfied, we have
\[
\rho(a_a, a_a) > 0, \quad \rho^2(a_1, a_2) \leq \rho(a_1, a_1) \rho(a_2, a_2).
\]
The first inequality may be proved immediately
\[
\rho(a_a, a_a) > \frac{1 - 2(x - 1) a_a + a_a^2}{(1 + a_a)^3} \geq 0 \quad \text{for } x \leq 2.
\]
We consider the difference
\[
\rho(a_1, a_1) \rho(a_2, a_2) - \rho^2(a_1, a_2) = \frac{J}{(1 + a_1)^3(1 + a_2)^3},
\]
where
\[
J = [(1 + a_1)^2 + x a_1 a_2 - 2x a_1] [(1 + a_2)^2 + x a_2 a_1 - 2x a_2] -
\]
Alternating direction iteration schemes

\[- \left[ (1 + a_1)(1 + a_2) + \kappa a_1 a_2 - \kappa (a_1 + a_2) \right]^2.\]

Removing the brackets and collecting like terms, we get

\[ J = \kappa (2 - \kappa + \alpha)(a_1 - a_2)^2. \]

This leads us to (33), provided (32) is satisfied.

A fairly simple corollary of Lemma 2 is

**Lemma 3.** If \( p = 2 \), we have for the function \( \rho_{k,n+1} \) defined by (219), provided condition (4) is satisfied

\[ (\rho_{k,n+1})^2 \leq \prod_{a=1}^{2} \bar{\rho}(a_a), \tag{34} \]

where

\[ \bar{\rho}(a_a) = 1 - \frac{1}{\sigma} \left( 1 - \frac{\theta}{3} \right) \frac{2a_a}{(1 + a_a)^3}, \quad a_a = \sigma \lambda_a. \tag{35} \]

For, it follows from Lemma 2 that

\[ (\rho_{k,n+1})^2 \leq \prod_{a=1}^{2} \rho_{k_a, n+1}, \quad \rho_{k_a, n+1} = 1 - \frac{2\tau \lambda_a - \theta \frac{h^2}{6} \tau \lambda_a^2}{(1 + \sigma \lambda_a)^3}, \]

since \( \rho_{k_a, n+1} > 0 \) for \( \sigma > \frac{1}{2} \). But we have, by (18),

\[ \frac{h^2}{6} \tau \lambda_a^2 \leq \frac{2}{3} \tau \lambda_a \text{ and } \rho_{k_a, n+1} \leq \bar{\rho}(a_a). \]

(This lemma was proved in [12] for \( \theta = 0 \) and \( \sigma = 1 \).)

Finally, we require

**Lemma 4.** Given

\[ 0 < m < M. \tag{36} \]

The maximum of the function \( \bar{\rho}(a) \) defined by (30) and (35) in the interval \([m, M]\) is now equal to

\[ \rho_p = \max_{m \leq a \leq M} \bar{\rho}(a) = \max \left[ 1 - \frac{1}{\sigma} \left( 1 + \theta \frac{1-p}{3} \right) \frac{pm}{(1+m)^p}, 1 - \frac{1}{\sigma} \left( 1 + \theta \frac{1-p}{3} \right) \frac{pM}{(1+M)^p} \right]. \tag{37} \]
For, it follows from

$$\tilde{\rho}(a) = \frac{1}{a} \left( 1 + \frac{1-p}{3} \right) \rho \left( \frac{p-1}{p-1} \right) \frac{a-1}{(1+a)^{p+1}} = \begin{cases} \leq 0 & \text{for } a \leq \frac{1}{p-1} \\ \geq 0 & \text{for } a > \frac{1}{p-1} \end{cases}$$

that $\tilde{\rho}(a)$ takes its maximum value at either the left- or the right-hand end of the interval $[m, M]$.

6. We shall now estimate the rate of convergence of the iterational process (13). To be more precise, we shall find a sequence of iterational parameters $\{\tau_n\}$ such that a "reasonably high" rate of convergence is obtained. It follows from Theorem 3 that the parameter $\tau_n$ may vary within fairly wide limits. We shall therefore try to find a sequence $\{\tau_n\}$ such that, given any value of $\lambda$, there is at least one value of $\tau$ such that $|\rho_{k,n+1}| < \rho < 1$, where $\rho$ is independent of both $n$ and $|h|$. If we then perform the cycle of iterations (13) with the given system of parameters, we shall obtain, in view of (22) - (23), a $p^{-1}$ times reduction in the norm of the error. It is desirable for the total number of parameters in the sequence $\{\tau_n\}$ to be "not very great" (obviously, in the worst case we can avoid a number of parameters equal to the number of distinct eigenvalues $\lambda$), i.e. for one parameter $\tau$ to be "stipulated" by a whole series of eigenvalues and not just one. In fact, let the sequence of intervals $(\xi_{(n-1)}, \xi_{(m)})$, $n = 1, \ldots, n_0$, cover the interval $[\lambda_1, \lambda_{N-1}]$, where

$$\xi_{(0)} = \lambda_1, \quad \xi_{(n-1)} < \lambda_{N-1}, \quad \xi_{(m)} > \lambda_{N-1},$$

(38)

the coordinates $\xi_{(n)}$ and the number $n_0$ being subject to definition. Let $\tau_n$ "stipulate" the $\lambda_k$ which satisfy

$$\xi_{(n-1)} < \lambda_k < \xi_{(n)},$$

(39)

i.e. for the $k$ given by (39), the functions $\rho_{k,n+1}$ satisfy (25) with a $p$ independent of either $n$ or $|h|$. This means in our case, by Lemmas 1 and 4, that

$$pm \leq \tau_n \sigma \xi_{(n-1)} \leq \tau_n \sigma \lambda_k \leq \tau_n \sigma \xi_{(n)} \leq pM,$$

where $m < M$ are positive constants independent of either $n$ or $|h|$. If $m$ and $M$ are chosen, let

$$pm = \sigma \tau_n \xi_{(n-1)}, \quad pM = \sigma \tau_n \xi_{(n)}.$$

(40)

It now follows from this and (38) that
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\[ \xi(n) = q^{-n} \lambda_1, \quad \tau_n = \frac{pm}{c} \lambda_1^{1/q-1}, \quad q = \frac{m}{M}, \quad (41) \]

\[ \lg \frac{\lambda_1}{\lambda_{N-1}} \lg q \leq n_0 \leq \frac{\lambda_1}{\lambda_{N-1}} \lg q + 1. \quad (42) \]

Using Lemma 4, we arrive from (41), (42) and (22) - (23) at

**Lemma 5.** If a cycle of \( n_0 \) iterations is carried out in accordance with method (13) with a system of parameters \( \{\tau_n\} \) given by (41), then, if conditions (28) are satisfied,

\[ \|z^{(n_0)}\| \leq \rho \|z^{(0)}\|, \quad (43) \]

where \( \rho \) is given by (37).

A simple consequence of Lemma 5 is

**Theorem 4.** In order to reduce the norm \( L_2 \) of the error \( \|z^{(0)}\| \) by a factor \( 1/\varepsilon \) with the aid of method (13), it is sufficient, if conditions (28) are satisfied, to perform a cycle of \( n_0 \) iterations with the system of parameters \( \{\tau_n\} \) given by (41) \( k_0 \) times, where \( n_0 \) is given by (42), and \( k_0 \) by

\[ k_0 \geq \lg \varepsilon \lg -1 \rho. \quad (44) \]

The following asymptotic formula holds here for the total number of iterations \( v = n_0 k_0 \):

\[ v \asymp v_0 \frac{\lambda_1}{\lambda_{N-1}} \lg \varepsilon, \quad v_0 = \frac{4}{\lg q \lg \rho}. \quad (45) \]

**Note 3.** We have by (18):

\[ \frac{\lambda_1}{\lambda_{N-1}} = \sum_{\alpha=1}^{p} \frac{1}{h_{\alpha}^2} \sin^2 \frac{\pi h_{\alpha}}{2l_{\alpha}}. \]

If \( \tilde{D} \) is the \( p \)-dimensional cube with side \( l \) and the mesh \( \omega_h \) is square, i.e. \( h_{\alpha} = h, \alpha = 1, \ldots, p \), then

\[ \frac{\lambda_1}{\lambda_{N-1}} = \lg \frac{\pi h}{2l} = O(h^2) \text{ and } \lg \frac{\lambda_1}{\lambda_{N-1}} = O(\lg h). \]
The constructions used in the proofs of Lemma 5 and Theorem 4 are based on Lemma 1 and therefore hold only if conditions (23) are satisfied.

We can prove with \( p = 2 \), from Lemma 3, and by analogy with [11]:

**Theorem 5.** In the case \( p = 2 \), in order to reduce the norm \( L_2 \) of the error \( \|x^{(0)}\| \) by a factor \( 1/\varepsilon \) with the aid of method (13), it is sufficient, given any \( \sigma \geq 0.5 \), to carry out a cycle of \( n_0 \) iterations with the system of parameters

\[
\tau_n = \frac{m}{c} c_q^n q^{-1} \quad (46)
\]

\( k_0 \) times, where \( k_0 \) is given by (44), while

\[
\lg \frac{c^*}{c_0} - \lg -1 \frac{1}{q} \leq n_0 < \lg \frac{c^*}{c_0} - \lg -1 \frac{1}{q} + 1 \quad (47)
\]

and

\[
c_0 = \min_{k_0} \lambda_{k_0}, \quad c^* = \max_{k_0} \lambda_{k_0}.
\]

The following asymptotic formula holds here for the total number of iterations \( \nu = n_0 k_0 \):

\[
\nu = \nu_0 \lg \frac{c^*}{c_0} \lg \frac{1}{\varepsilon}, \quad \nu_0 = \frac{1}{\lg q \lg p}.
\]
Notice that, in a square region and on a square mesh, (46) is the same as (41), (47) as (42) and (48) as (45).

**Note 4.** Using Note 1, it is easily shown that Theorem 4 also holds for the iterational scheme (13').

We now consider the minimization of the coefficient $v_0$. Using (37) and (41), it is clear from (45) and (48) that, with $\theta$ fixed, $v_0$ is a function of the three variables $m$, $M$ and $\sigma$. Since $q$ and $\rho_p$ are always less than unity, $v_0$ will decrease with $q$ and $\rho_p$. Hence, if $\rho_p$ is fixed, $v$ will be a minimum if $q$ is a minimum. But it follows from (41) that $q$ is a minimum if the first and second terms on the right-hand side of (37) are the same, i.e. $m / (1 + m)^p = M / (1 + M)^p$. Hence

$$M_2 = \frac{1}{m}, \quad M_3 = \frac{V(3+m)^4 + 4/m - (3+m)^2}{2}.$$

It is clear from (37) that $\rho_p$ is an increasing function with respect to $\sigma$. Hence, for $v_0$ to be as small as possible, $\sigma$ must also be a minimum. After $M$ and $\sigma$ have been fixed, $v_0$ remains a function of $m$ only and its minimum can be found numerically to any degree of accuracy. Table 1 gives the numerical values of the parameters occurring in $v_0$, optimum with respect to $m$ for minimum $\sigma$. Table 2 gives for comparison the same parameters for $\sigma = 1$. (The values of the parameters were obtained in [12] for $p = 2$ and $\theta = 0$, and in [4] for $p = 2$, $3$ and $\theta = 1$. The numbers quoted there correspond to natural logarithms in (45) and (47), whereas we use logarithms to base 10.)

Translated by D.E. Brown

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