ECONOMICAL DIFFERENCE SCHEMES FOR PARABOLIC EQUATIONS WITH MIXED DERIVATIVES*

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1. Given the p-dimensional parallelepiped $\overline{G} = \{0 \leqslant x_\alpha \leqslant l_\alpha,\ \alpha = 1, \ldots, p\}$. In the cylinder $\overline{Q}_T = \overline{G} \times [0 \leqslant t \leqslant T]$ we consider the problem

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} \sum_{\beta=1}^{p} \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}(x, t) \frac{\partial u}{\partial x_\beta}\right) + \sum_{\alpha=1}^{p} r_\alpha (x, t) \frac{\partial u}{\partial x_\alpha} + f(x, t, u) = Lu + f; \tag{1}$$

$$u\,|_{x_\alpha=0} = u_{1\alpha}^{-}(x, t), \qquad u\,|_{x_\alpha=l_\alpha} = u_{1\alpha}^{+}(x, t), \qquad \alpha = 1, \ldots, p; \tag{2}$$

$$u(x, 0) = u_0(x), \qquad x = (x_1, \ldots, x_p). \tag{3}$$

The matrix $(k_{\alpha\beta})$ is positive definite

$$\sum_{\alpha,\beta=1}^{p} k_{\alpha\beta}(x, t)\, \xi_\alpha \xi_\beta \geqslant \gamma \sum_{\alpha=1}^{p} \xi_\alpha^2, \qquad (x, t) \in \overline{Q}_T, \tag{4}$$

where γ = const. > 0, $\xi = (\xi_1, \ldots, \xi_2, \ldots, \xi_p)$ is any real vector.

2. In [1] and [2] economical splitting schemes are described for the solution of equation (1) with constant coefficients $k_{\alpha\beta}$ = const., $r_\alpha = 0$, $f = f_{(x,t)}$ in the cases $p = 2$ and $p = 3$. When $p = 2$ one intermediate (fractional) step was introduced, and 4 fractional steps were introduced for $p = 3$.

Local one-dimensional schemes for $p = 2$ are given in [3] and a scheme of higher order accuracy $O(|h|^4 + \tau^2)$ is constructed in [4]. In [5] a splitting scheme for an equation with variable coefficients is studied: when $p > 2$ the matrix $(k_{\alpha\beta})$ satisfies a certain additional restriction besides (4) even in the case of constant coefficients.

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Below we describe economical difference schemes which converge only when the matrix $(k_{\alpha\beta})$ is positive definite.

3. We introduce a rectangular net, uniform with respect to each of the coordinates x_α, $\overline{\omega}_h = \{x_i = (x_1^{(i_1)}, \ldots, x_\alpha^{i_\alpha}, \ldots, x_p^{(i_p)}),\ x_\alpha^{i_\alpha} = i_\alpha h_\alpha,\ i_\alpha = 0, 1, \ldots, N_\alpha\}$, denoting the set of internal points by $\omega_h = \{x_i \in G,\ i_\alpha = 1, \ldots, N_\alpha - 1,\ \alpha = 1, \ldots, p\}$ and the set of boundary points of $\overline{\omega}_h$ by γ_h, so that $\overline{\omega}_h = \omega_h + \gamma_h$. Let $\omega_\tau = \{t_j,\ i = 0, 1, \ldots, k,\ t_k = T\}$ be an arbitrary non-uniform net on the segment $[0, T]$ with step $\tau = \tau_{j+1} = t_{j+1} - t_j$.

We shall use the notation of [6]. To approximate to the operators

$$L_{\alpha\beta} u = \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}(x, t)\frac{\partial u}{\partial x_\beta}\right) \tag{5}$$

we use homogeneous difference schemes of second order approximation

$$\Lambda_{\alpha\beta} y = \frac{1}{2}\left[(a_{\alpha\beta} y_{\bar{x}_\beta})_{x_\alpha} + \left(a_{\alpha\beta}^{(+1_\beta)} y_{x_\beta}\right)_{\bar{x}_\alpha}\right],\ \beta \neq \alpha. \tag{6}$$

The coefficients $a_{\alpha\beta}$ are defined in terms of $k_{\alpha\beta}$ with the help of linear pattern functionals (see [6]). For example, if for our pattern functionals we select one-dimensional non-decreasing functionals $A_\beta[\mu(s_\beta)]$, $-1 \leqslant s_\beta \leqslant 0$, $A_\beta[1] = 1$, $A_\beta[s_\beta] = -0.5$, we define $a_{\alpha\beta}$ by the formula

$$a_{\alpha\beta}(x, t) = A_\beta[k_{\alpha\beta}(x_1, \ldots, x_{\beta-1}, x_\beta + s_\beta h_\beta, x_{\beta+1}, \ldots, x_p, t)]. \tag{7}$$

It follows from this and from (4) that the matrix $(a_{\alpha\beta})$ is positive definite

$$\sum_{\alpha,\beta=1}^{p} a_{\alpha\beta}\xi_\alpha\xi_\beta \geqslant \gamma_1 \sum_{\alpha=1}^{p} \xi_\alpha^2, \qquad \gamma_1 = \text{const.} > 0 \qquad (\gamma_1 < \gamma), \tag{8}$$

for sufficiently small $h_\alpha \leqslant h_0$. Besides this, we shall make the requirement that the conditions $h_\alpha \leqslant h_0$ ensure that

$$\sum_{\alpha,\beta=1}^{p} a_{\alpha\beta}^{(+1_\beta)}\xi_\alpha\xi_\beta \geqslant \gamma_1 \sum_{\alpha=1}^{p} \xi_\alpha^2. \tag{9}$$

is satisfied. The choice of h_0 obviously depends on

$$\max_{\overline{Q}_T,\ \alpha}\left|\frac{\partial k_{\alpha\beta}}{\partial x_\beta}\right|, \qquad \gamma \text{ and } \gamma_1.$$

We note that in the case $k_{\alpha\beta} = 0$ for $\alpha \neq \beta$ considered in [3], [6] the "parabolicity" condition (8) is satisfied on an arbitrary net $\overline{\omega}_h$. If

$a_{\alpha\beta} = k_{\alpha\beta}$ and $\Lambda_{\alpha\beta} y = (a_{\alpha\beta} y_{\bar{x}_\beta})_{x_\alpha}$, the condition $h_\alpha \leqslant h_0$ is missing, but $\Lambda_{\alpha\beta}$ is a scheme of first order approximation.

4. We consider first the case when $(k_{\alpha\beta})$ is a triangular matrix, so that $k_{\alpha\beta} = 0$ for $\beta > \alpha$. Then

$$\sum_{\beta=1}^{p} L_{\alpha\beta} u = L_\alpha u + \sum_{\beta=1}^{\alpha-1} L_{\alpha\beta} u = \hat{L}_\alpha u, \tag{10}$$

$$L_\alpha u = L_{\alpha\alpha} u = \frac{\partial}{\partial x_\alpha}\left(k_{\alpha\alpha}(x,t)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}, \tag{11}$$

and $L_{\alpha\beta}$ is defined by formula (5). Let Λ_α and $\Lambda_{\alpha\beta}$ be homogeneous difference schemes of second order approximation corresponding to L_α and $L_{\alpha\beta}$, and let $\Lambda_{\alpha\beta}$ have the form (6) and

$$\Lambda_\alpha y = (a_{\alpha\alpha}(x,t)\, y_{\bar{x}_\alpha})_{x_\alpha} + b_\alpha(x,t)\, y_{\overset{\circ}{x}_\alpha}, \qquad y_{\overset{\circ}{x}_\alpha} = 0.5\,(y_{\bar{x}_\alpha} + y_{x_\alpha}). \tag{12}$$

The local one-dimensional alternating direction scheme has the form

$$y_{\bar{t}_\alpha} = \Lambda_\alpha y_\alpha + \sum_{\beta=1}^{\alpha-1} \Lambda_{\alpha\beta} y_\beta + \varphi_\alpha(x, t, y_{\alpha-1}), \qquad \alpha = 1, \ldots, p, \tag{13}$$

$$y_\alpha = u_{1\alpha}^-(x,t) \text{ when } x_\alpha = 0, \qquad y_\alpha = u_{1\alpha}^-(x,t) \text{ when } x_\alpha = l_\alpha, \tag{14}$$

$$y(x, 0) = u_0(x). \tag{15}$$

Here, as usual, $y_1, \ldots, y_\alpha, \ldots, y_{p-1}$ are intermediate values (on the "fractional" steps $t_j + \tau_{j+1}\alpha/p$), $y_p = y^{j+1}$, $y_{\bar{t}_\alpha} = (y_\alpha - y_{\alpha-1})/\tau$, $\tau = \tau_{j+1} = t_{j+1} - t_j$.

It should be borne in mind that the difference derivative of the net function z_α is always taken with respect to its direction x_α, so that $z_{x_\alpha} = (z_\alpha^{(+)} - z_\alpha)/h_\alpha$ and so on. Therefore the expression $(a_{\alpha\beta} z_{\bar{x}_\beta})_{\bar{x}_\alpha}$ means

$$(a_{\alpha\beta} z_{\bar{x}_\beta})_{\bar{x}_\alpha} = a_{\alpha\beta}\left(\frac{z_\beta - z_\beta^{(-1\beta)}}{h_\beta}\right)_{\bar{x}_\alpha}.$$

To find y_α from (13) and (14) we have to invert the operator $E - \tau\Lambda_\alpha$, and this is done by one-dimensional successive substitution formulae. Therefore the algorithm is economical.

5. In the case of an arbitrary matrix $(k_{\alpha\beta})$ equation (1) can be transformed to "triangular" form

$$\frac{\partial u}{\partial t} = Lu + f(x, t, u), \qquad Lu = \sum_{\alpha=1}^{p} \hat{L}_\alpha u, \qquad \hat{L}_\alpha u = L_\alpha u + 2\sum_{\beta=1}^{\alpha-1} \hat{L}_{\alpha\beta} u \tag{16}$$

with the triangular matrix $(\hat{k}_{\alpha\beta})$. In fact,

$$\sum_{\alpha=1}^{p}\sum_{\beta=1}^{p}\frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}\frac{\partial u}{\partial x_\beta}\right)=\sum_{\alpha=1}^{p}\sum_{\beta=1}^{\alpha-1}\frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}\frac{\partial u}{\partial x_\beta}\right)+\sum_{\alpha=1}^{p}\sum_{\beta=\alpha}^{p}\frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}\frac{\partial u}{\partial x_\beta}\right).$$

Changing the order of summation in the second term on the right-hand side, replacing α by β and β by α and noting that

$$\frac{\partial}{\partial x_\beta}\left(k_{\beta\alpha}\frac{\partial u}{\partial x_\alpha}\right)=\frac{\partial}{\partial x_\alpha}\left(k_{\beta\alpha}\frac{\partial u}{\partial x_\beta}\right)+\frac{\partial k_{\beta\alpha}}{\partial x_\beta}\frac{\partial u}{\partial x_\alpha}-\frac{\partial k_{\beta\alpha}}{\partial x_\alpha}\frac{\partial u}{\partial x_\beta},$$

we obtain

$$\sum_{\alpha=1}^{p}\sum_{\beta=\alpha}^{p}\frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}\frac{\partial u}{\partial x_\beta}\right)=\sum_{\alpha=1}^{p}\sum_{\beta=1}^{\alpha}\frac{\partial}{\partial x_\alpha}\left(k_{\beta\alpha}\frac{\partial u}{\partial x_\beta}\right)+\sum_{\alpha=1}^{p}\sum_{\beta=1}^{\alpha-1}\left(\frac{\partial k_{\beta\alpha}}{\partial x_\beta}\frac{\partial u}{\partial x_\alpha}-\frac{\partial k_{\beta\alpha}}{\partial x_\alpha}\frac{\partial u}{\partial x_\beta}\right).$$

It follows from this and (1) that (16) holds, where $\hat{L}_{\alpha\beta}u=\frac{\partial}{\partial x_\alpha}\left(\hat{k}_{\alpha\beta}\frac{\partial u}{\partial x_\beta}\right)$, L_α is given by formula (11) with a new expression $\hat{r}_\alpha$ for the coefficient of $\partial u/\partial x_\alpha$, which we do not give, and

$$\hat{k}_{\alpha\beta}=0.5\,(k_{\alpha\beta}+k_{\beta\alpha}). \tag{17}$$

We have assumed here the existence of the derivatives $\partial k_{\beta\alpha}/\partial x_\beta$ and $\partial k_{\beta\alpha}/\partial x_\alpha$ for $\alpha\neq\beta$. We note that the symmetry of the matrix $(k_{\alpha\beta})$ is not assumed.

6. If the matrix $(k_{\alpha\beta})$ is symmetric, $k_{\alpha\beta}=k_{\beta\alpha}$, and there is another scheme which is suitable. We introduce the intermediate values $y\ ,\dots,y_p$, $y_{p+1},\ \dots,\ y_{2p-1}$ and represent $L_{\alpha\beta}u$ in the form of the sum

$$L_{\alpha\beta}u=L^-_{\alpha\beta}u+L^+_{\alpha\beta}u,\qquad L^\mp_{\alpha\beta}u=\frac{\partial}{\partial x_\alpha}\left(k^\mp_{\alpha\beta}\frac{\partial u}{\partial x_\beta}\right), \tag{18}$$

where $(k^-_{\alpha\beta})$ and $(k^+_{\alpha\beta})$ are, respectively, left and right triangular matrices ($k^-_{\alpha\beta}=0$ when $\beta>\alpha$, $k^+_{\alpha\beta}=0$ when $\beta<\alpha$). Let $\Lambda^-_{\alpha\beta}$ and $\Lambda^+_{\alpha\beta}$ be the corresponding homogeneous difference schemes. We consider the following local one-dimensional economical scheme:

$$y_{\bar{t}_\alpha}=\tfrac{1}{2}\Lambda_\alpha y_\alpha+\sum_{\beta=1}^{\alpha-1}\Lambda^-_{\alpha\beta}y_\beta+\varphi_\alpha\,(x,t,y_{\alpha-1}),\qquad \alpha=1,\dots,p, \tag{19}$$

$$y_{\bar{t}_{\alpha'}}=\tfrac{1}{2}\Lambda_\alpha y_\alpha+\sum_{\beta=\alpha+1}^{p}\Lambda^+_{\alpha\beta}y_{\beta'}+\varphi_{\alpha'}\,(x,t,y_{\alpha'-1}), \tag{19'}$$

$$\alpha'=2p+1-\alpha,\quad \beta'=2p+1-\beta,\qquad \alpha'=p+1,\dots,2p,\quad y_{2p}=y,$$

$$y_\alpha = y_{\alpha'} = u_{1\alpha}^-(x, t) \text{ when } x_\alpha = 0; \qquad y_\alpha = y_{\alpha'} = u_{1\alpha}^+(x, t) \text{ when } x_\alpha = l_\alpha,$$
$$\alpha = 1, \ldots, p, \tag{20}$$
$$y(x, 0) = u_0(x) \qquad (y_{2p} = y^{j+1}), \tag{21}$$

Here the boundary conditions are taken in the simplest form $y_\alpha = y_{\alpha'}$ for $x \in \gamma_h^\alpha$, $t = t_{j+1}$. It would be simplest of all to put $\varphi_2 = 0$ for $\alpha = 2, \ldots, 2p$, $\varphi_1 = \varphi(x, t, y^j)$. It is clear from (19)-(20) that as before it is sufficient to invert the operator $E - 0.5\tau\Lambda_\alpha$ to find y_α, $\alpha = 1, \ldots, 2p$.

We note that by analogy with [6] the lower terms containing $\partial u/\partial x_\alpha$, can conveniently be referred to the previous layer.

7. We pass now to the question of the stability and convergence of the scheme (13)-(15). Let u be the solution of the initial problem (1)-(3), y the solution of the difference problem (13)-(15). Putting $y_\alpha = u^{j+1} + z_\alpha$ for $\alpha = 1, \ldots, p$, $y_0 = y^j = u^j + z^j$, we obtain for the net function z_α the problem

$$z_{\bar{t}_\alpha} = \Lambda_\alpha^* z + \psi_\alpha, \qquad \Lambda_\alpha^* z = \Lambda_\alpha z + \sum_{\beta=1}^{\alpha-1} \Lambda_{\alpha\beta} z_\beta + d_\alpha z_{\alpha-1}, \qquad \alpha = 1, \ldots, p, \tag{22}$$
$$z_\alpha = 0 \text{ when } x_\alpha = 0, \qquad x_\alpha = l_\alpha, \qquad z(x, 0) = 0, \ |d_\alpha| \leqslant c^*, \quad c^* = \text{const.} > 0,$$

where ψ_α is the local error of approximation, equal to

$$\psi_\alpha = \Lambda_\alpha^* u^{j+1} - \delta_{\alpha,1} u_{\bar{t}}, \qquad \delta_{\alpha,1} = \begin{cases} 1, & \alpha = 1, \\ 0, & \alpha \neq 1; \end{cases} \qquad u_{\bar{t}} = (u^{j+1} - u^j) / \tau_{j+1}.$$

It is not difficult to put ψ_α in the form of the sum

$$\psi_\alpha = \mathring{\psi}_\alpha + \psi_\alpha^*, \qquad \mathring{\psi}_\alpha = \left(L_\alpha u + \sum_{\beta=1}^{\alpha-1} L_{\alpha\beta} u + f_\alpha(x, t, u) - \frac{1}{p}\frac{\partial u}{\partial t}\right)^{j+1} \tag{23}$$
$$\sum_{\alpha=1}^{p} \mathring{\psi}_\alpha = 0, \qquad \psi_\alpha^* = O(|h|^2 + \tau), \qquad |h|^2 = \sum_{\alpha=1}^{p} h_\alpha^2, \tag{24}$$

since $\Lambda_{\alpha\beta}$ and Λ_α have second order approximation.

The question of the stability and convergence of the scheme (13)-(15) as usual reduces to the *a priori* estimate of the solution of problem (22) with the additional conditions (23)-(24).

8. *Lemma 1.* Let z_α be a net function which satisfies the boundary conditions (22). If conditions (8) and (9) are satisfied, then

$$-\sum_{\alpha=1}^{p}(\Lambda_\alpha^* z,\ z_\alpha)\geqslant\frac{\gamma_1}{2}\sum_{\alpha=1}^{p}\|z_{\bar{x}_\alpha}\|^2-M\sum_{\alpha=1}^{p}\|z_\alpha\|^2, \tag{25}$$

where M = const. > 0 does not depend on the net.

We write the energy identity for problem (22). To do this we perform a scalar multiplication of equation (22) by $2z_\alpha$, apply Green's formula and sum over $\alpha = 1, \ldots, p$ (see [6])

$$(\|z\|^2)_{\bar{t}}+\tau\sum_{\alpha=1}^{p}\|z_{\bar{t}_\alpha}\|^2+J=2\sum_{\alpha=1}^{p}(\psi_\alpha+b_\alpha z_{\overset{\circ}{x}_\alpha}+d_\alpha z_{\alpha-1},\ z_\alpha), \tag{26}$$

where

$$J=2\sum_{\alpha=1}^{p}(a_{\alpha\alpha},\ z^2_{\bar{x}_\alpha}]_\alpha-2\sum_{\alpha=1}^{p}\sum_{\beta=1}^{\alpha-1}(\Lambda_{\alpha\beta}z_\beta,\ z_\alpha). \tag{27}$$

Using Green's difference formula and the fact that $z_{x_\beta}=0$ for $x_\alpha=0$, $x_\alpha=l_\alpha$ when $\alpha\neq\beta$ we find

$$-(2\Lambda_{\alpha\beta}z_\beta,\ z_\alpha)=2(a_{\alpha\beta}z_{\bar{x}_\beta},\ z_{\bar{x}_\alpha})+2(a_{\alpha\beta}^{(+1_\beta)}z_{x_\beta},\ z_{x_\alpha}).$$

Therefore the expression for J takes the form

$$J=\left(\sum_{\alpha=1}^{p}\sum_{\beta=1}^{\alpha}a_{\alpha\beta}z_{\bar{x}_\beta}z_{\bar{x}_\alpha},\ 1\right)+\left(\sum_{\alpha=1}^{p}\sum_{\beta=1}^{\alpha}a_{\alpha\beta}^{(+1_\beta)},\ z_{x_\beta}z_{x_\alpha},\ 1\right)+$$

$$+\sum_{\alpha=1}^{p}\{a_{\alpha\alpha}z^2_{\bar{x}_\alpha}h_\alpha|_{x_\alpha=l_\alpha}+a_{\alpha\alpha}^{(+1_\alpha)}z^2_{x_\alpha}h_\alpha|_{x_\alpha=0}.$$

It follows from this and from (8) and (9) that

$$J\geqslant 2\gamma_1\sum_{\alpha=1}^{p}\|z_{\bar{x}_\alpha}\|^2, \tag{28}$$

Therefore instead of (26) we can write

$$(\|z\|^2)_{\bar{t}}+\tau\sum_{\alpha=1}^{p}\|z_{\bar{t}_\alpha}\|^2+2\gamma_1\sum_{\alpha=1}^{p}\|z_{\bar{x}_\alpha}\|^2\leqslant 2\sum_{\alpha=1}^{p}(\psi_\alpha+b_\alpha z_{\overset{\circ}{x}_\alpha}+d_\alpha z_{\alpha-1},\ z_\alpha). \tag{29}$$

We have the estimate

$$2\sum_{\alpha=1}^{p}(b_\alpha z_{\overset{\circ}{x}_\alpha}+d_\alpha z_{\alpha-1},\ z_\alpha)\leqslant\gamma_1\sum_{\alpha=1}^{p}\|z_{\bar{x}_\alpha}\|^2+M_1\sum_{\alpha=1}^{p}\|z_\alpha\|^2, \tag{30}$$

where M_1 is a positive constant which depends on $\max\limits_{\alpha,\overline{Q}_T}|d_\alpha|$ and $\max\limits_{\alpha,\overline{Q}_T}|b_\alpha|$. Lemma (25) follows from this and from (28).

9. *Lemma* 2. Let $\overset{\circ}{\psi}_\alpha$ satisfy the condition

$$\sum_{\alpha=1}^{p} \mathring{\psi}_\alpha = 0. \tag{31}$$

Then

$$2\sum_{\alpha=1}^{p} (\mathring{\psi}_\alpha, z_\alpha) = 2\sum_{\alpha=1}^{p}\left(\mathring{\psi}_\alpha, \tau\sum_{\sigma=1}^{\alpha} z_{\bar{t}_\sigma}\right) \leqslant \frac{1}{2}\tau\sum_{\alpha=1}^{p}\| z_{\bar{t}_\alpha}\|^2 + \tau p^2 \sum_{\alpha=1}^{p}\|\mathring{\psi}_\alpha\|^2. \tag{32}$$

The lemma is proved in [6].

Noting that

$$2\sum_{\alpha=1}^{p} (\psi^*_\alpha, z_\alpha) \leqslant \frac{\gamma_1}{2}\sum_{\alpha=1}^{p}\| z_{\bar{x}_\alpha}\|^2 + M_2\sum_{\alpha=1}^{p}\|\psi^*_\alpha\|^2_{3\alpha},$$

and using Lemmas 1 and 2 we can transform inequality (29) for sufficiently small $\tau \leqslant \tau_0$ to the form

$$\| z^{j+1}\|^2 + 0.5\gamma_1\tau\sum_{\alpha=1}^{p}\| z_{\bar{x}_\alpha}\|^2 \leqslant (1+M\tau)\| z^j\|^2 + M\tau\|\psi^{j+1}\|^2, \tag{33}$$

$$\|\psi\|^2 = \tau p^2\sum_{\alpha=1}^{p}\|\mathring{\psi}_\alpha\|^2 + \frac{1}{2\gamma_1}\sum_{\alpha=1}^{p}\|\psi^*_\alpha\|^2_{3\alpha}, \tag{34}$$

where M are positive constants which do not depend on the net. We now apply Lemma 4a from [7]

$$\| z^{j+1}\| \leqslant M\| z(x, 0)\| + M'\,\overline{\|\psi^{j+1}\|}, \tag{35}$$

where

$$\overline{\|\psi^{j+1}\|} = \left(\sum_{j'=1}^{j+1}\tau_{j'}\|\psi^{j'}\|^2\right)^{1/2}. \tag{36}$$

We have thus proved the following theorem.

Theorem 1. The difference scheme (13)-(15) for sufficiently small $|h| < h_0$ and $\tau < \tau_0$ is stable in the mean with respect to the right-hand side and the initial data. If the conditions securing maximum order of approximation (23) and (24) are satisfied, the scheme (13)-(15) converges at the rate $O(|h|^2 + \sqrt{\tau^*})$, so that

$$\| y^{j+1} - u^{j+1}\| \leqslant M(|h|^2 + \sqrt{\tau^*}) \quad \text{for} \quad \tau^* \leqslant \tau_0, \quad |h| \leqslant h_0, \tag{37}$$

where $\tau^* = \max_{\omega_\tau} \tau_j$.

10. Theorem 1 is also valid for the scheme (19)-(21). The condition

$|h| < h_0$ is missing if the coefficients $k_{\alpha\beta}$ are constant or if the scheme $\Lambda_{\alpha\beta}y = (a_{\alpha\beta}y_{\bar{x}_\beta})_{x_\alpha}$, $a_{\alpha\beta} = k_{\alpha\beta}$, with first order approximation is used to approximate to $L_{\alpha\beta}$. We note that a different approximation was used in [1], [5] for $L_{\alpha\beta}$.

The condition $\tau \leqslant \tau_0$ is missing if $r_\alpha \equiv 0$ or the division of L into a sum is done so that L_α contains the derivative $\partial u/\partial x_{\alpha-1}$; then Λ_α contains the term $y_{\overset{\circ}{x}_{\alpha-1}}$, which is taken on the preceding "fractional" step, i.e., is calculated from the values of $y_{\alpha-1}$ (see [6]).

The estimate $O(|h|^2 + \sqrt{\tau^*})$ for $z = y - u$ is probably too crude and is associated with the method of proof used; even for a diagonal matrix $(k_{\alpha\beta})$, when $k_{\alpha\beta} = 0$ for $\alpha \neq \beta$ we have not succeeded in getting rid of $O(\sqrt{\tau^*})$ Meanwhile the maximum principle in this case gives $\|z\|_0 = O(|h|^2 + \|\tau\|_0)$, $\|\tau\|_0 = \max\limits_{\omega_\tau} \tau_j = \tau^*$.

Since in the general case the maximum principle for the scheme (13) does not apply, the method of [3] enables us to obtain only the estimate $O(|h|^2 + \tau^*/\sqrt{h_*})$, where $h_* = \min\limits_{1\leqslant\alpha\leqslant p} h_\alpha$ (see [8]). The scheme (13)-(15) was experimentally verified for $p = 2$ on various nets $\omega_h \times \omega_\tau$; it was shown that it can have first order accuracy on arbitrary sequences of non-uniform nets.

When $p = 2$ we can write a number of difference schemes [3] (three-parameter family of schemes). We cannot consider these schemes in detail on this note. The results are still valid if in (1) we replace $\partial u/\partial t$ by $c(x, t)\, \partial u/\partial t$, where $c(x, t) \geqslant c_1 > 0$.

11. We have been describing the case when the region of change of the space variables $(x_1, \ldots, x_p)$ is a parallelepiped. All our results can be applied without change to the case of regions G formed by parallelepipeds with boundaries parallel to the coordinate planes. The construction and study of local one-dimensional algorithms suitable for arbitrary regions G is of interest.

We have also assumed that the space net ω_h is uniform in each of the variables. The study of similar schemes on non-uniform nets ω_h would be of interest. We note that in this case there would probably appear certain restrictions on $h_\alpha/h_{\alpha+}$, because of the condition for the matrix $(a_{\alpha\beta})$ to be positive definite.

We have restricted ourselves to the study of the first boundary problem.

An algorithm for the third boundary problem can be constructed similarly.

12. The term "local one-dimensional scheme" applied to this case means that in order to find y_α we obtain one-dimensional algebraic problems. In fact the schemes (13) and (19), like our other economical schemes [3], [6], [8], are concrete realizations of the general principle of constructing economical schemes for the equations

$$\frac{\partial u}{\partial t} = Lu + f, \qquad \frac{\partial^2 u}{\partial t^2} = Lu + f,$$

which uses only the additive property of the operator L

$$Lu = \sum_\alpha L_\alpha u,$$

where L_α are linear non-bounded operators.

Schemes constructed on the basis of this principle are called additive schemes. The operators L_α in [3], [6], [8] are one-dimensional elliptic operators; in this work L_α has a more complex form. For additive schemes to be applicable it is sufficient that the difference approximations Λ_α of the operator L_α should be positive definite operators on the net ω_h. A description can be given for operator equations. In this direction, in particular, it is not difficult to obtain economical schemes for systems of parabolic equations of general form assuming that in (1) u is a vector $u = (u^{(1)}, \ldots, u^{(i)}, \ldots, u^{(n)})$, and $(k_{\alpha\beta}) = (k_{\alpha\beta}^{ij})$ is a cell matrix whose elements are the matrices $k_{\alpha\beta}^{ij}$ with respect to i, j. In this case $2p - 1$ (or $4p - 1$) intermediate values y_α, $\alpha = 1, 2, \ldots, 2p - 1$, $y_{(2p)} = y^{j+1}$ are introduced. The basic requirement which is then made in the algorithm for determining the vector y_α is that the operator Λ_α should be a three-point operator with respect to space and that the matrix of coefficients should be triangular. For all local additive schemes estimate (37) holds. The region G here is arbitrary if the operator L does not contain mixed derivatives, and the region G has the special form indicated in Para. 11, if L does contain mixed derivatives.

13. For hyperbolic equations containing mixed derivatives,

$$\frac{\partial^2 u}{\partial t^2} = Lu + f, \tag{38}$$

where Lu is (1), in the cases $p = 2$ and $p = 3$ it is not difficult to write similar schemes which are a generalization of the schemes of [8] for an equation with a diagonal coefficient matrix.

In this case the local one-dimensional schemes converge at the rate $O(|h|^2 + \tau)$ or $O(|h|^2 + \tau^2/\sqrt{h_k})$. The splitting method enables us to construct three-layer schemes $O(|h|^4 + \tau^2)$ (see [4]) with more strict requirements on the smoothness of the coefficients of the differential equation (38), on the assumption that G is a parallelepiped.

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