LOCAL ONE-DIMENSIONAL DIFFERENCE SCHEMES FOR MULTI-DIMENSIONAL HYPERBOLIC EQUATIONS IN AN ARBITRARY REGION*

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In [1] and [2] an economical method for solving parabolic equations with several variables, called a local one-dimensional method, is described.

The purpose of this paper is to study local one-dimensional difference schemes for hyperbolic equations in an arbitrary region G. These schemes converge on arbitrary nonuniform nets ω_h.

If the region G is a parallelepiped, we can construct a number of other schemes which are splitting schemes [3] and [4]. Such schemes were first described in [3]. Splitting schemes of a higher order of accuracy are considered in [5].

1. The difference schemes

1. We shall consider the equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{\alpha=1}^{p} L_\alpha u + f(x, t), \quad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x, t)\frac{\partial u}{\partial x_\alpha} + q_\alpha(x, t)\, u, \tag{1}$$

where $x = (x_1, \ldots, x_p)$ is a point in p-dimensional space with coordinates $x_1, \ldots, x_\alpha, \ldots, x_p$. Let G be an arbitrary p-dimensional bounded region with boundary Γ, $\bar{Q}_T = (G + \Gamma) \times [0 \leqslant t \leqslant T]$, $Q_T = G \times (0 < t \leqslant T]$.

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In the cylinder $\overline{Q}_T$ we are looking for a solution of the problem

$$\frac{\partial^2 u}{\partial t^2} = \sum_{\alpha=1}^{p} L_\alpha u + f(x, t), \quad (x, t) \in Q_T; \quad u|_\Gamma = u_1(x, t);$$
$$u(x, 0) = u_0(x), \frac{\partial u}{\partial t}(x, 0) = \overline{u}_0(x). \tag{2}$$

As usual we assume that this problem has a unique solution which is continuous in the closed region $\overline{Q}_T$ and possesses all derivatives required in the course of the solution.

We make the same assumptions with respect to G as were made in [1] and [2].

2. We shall use the same nets $\omega_h^{(1)}, \omega_h^{(2)}$, as in [2]. We mainly consider the net $\omega_h = \omega_h^{(2)}$, the internal nodes of which are all the nodes $x_i \in G$ which lie inside G and all the boundary nodes $x_i \in \gamma_h$ which lie on Γ. If the region G is arbitrary, then the net $\omega_h^{(2)}$ is nonuniform near the boundary even when the basic lattice which covers G is uniform. The boundary conditions on this net are given without drift.

In contrast to [2], we take the net $\omega_\tau = \{t_j = j\tau \in [0 \leqslant t \leqslant T]\}$ to be uniform.

The notation is the same as in [2]. We introduce the intermediate steps $t_{j+\alpha/p}$ and the corresponding values $y^{j+\alpha/p} = y_\alpha$. We shall write

$$y = y^{j+1},\ \check{y} = y^j,\ \check{\check{y}} = y^{j-1},\quad \check{y}_\alpha = y^{(j-1)+\alpha/p},\ \ y_{\bar{t}} = (y - \check{y})/\tau,$$
$$y_{\bar{t}t} = (y - 2\check{y} + \check{\check{y}})/\tau^2, \quad y_{\bar{t}_\alpha} = (y_\alpha - y_{\alpha-1})/\tau.$$

In order to construct local one-dimensional schemes we proceed by analogy with [5]: we approximate to the operators

$$\mathcal{P}_\alpha u = \frac{1}{p}\frac{\partial^2 u}{\partial t^2} - (L_\alpha u + f_\alpha), \qquad \sum_{\alpha=1}^{p} f_\alpha = f, \qquad \alpha = 1, \ldots, p$$

separately. To approximate to $\partial^2 u/\partial t^2$ we use the expressions

$$u_{\bar{t}_\alpha \bar{t}_\alpha} = \frac{u_\alpha - 2u_{\alpha-1} + \check{u}_\alpha}{\tau^2} \sim \frac{1}{4}\frac{\partial^2 u}{\partial t^2}, \qquad \alpha = 1, 2 \quad (u_0 = \check{u},\ u_2 = u), \qquad p = 2, \tag{3}$$

$$u_{\bar{t}_\alpha \bar{t}_\alpha} = \frac{u_\alpha - u_{\alpha-1} - u_{\alpha-2} + \check{u}_\alpha}{\tau^2} \sim \frac{2}{9}\frac{\partial^2 u}{\partial t^2}, \quad \alpha = 1, 2, 3 \ (u_{-1} = \check{u}_2,\ u_{-2} = \check{u}_1), \ p = 3. \tag{4}$$

To approximate to $L_\alpha u + f_\alpha$ on ω_h we use the homogeneous difference scheme $\Lambda_\alpha y + \varphi_\alpha$ of second order of approximation, described in [2]. The coefficients Λ_α and φ_α will be taken at time $t_\alpha^* = 0.5\,(t_{j+\alpha/p} + t_{j-1+\alpha/p})$, so that $\Lambda_\alpha = \Lambda_\alpha(t_\alpha^*)$, $\varphi_\alpha = \varphi_\alpha(x, t_\alpha^*)$.

For the hyperbolic equations (1) the local one-dimensional schemes have the form

$$y_{\bar{t}_\alpha \bar{t}_\alpha} = \sigma\Lambda_\alpha\,(y_\alpha + \overset{\vee}{y}_\alpha) + 2\sigma\varphi_\alpha, \quad \alpha = 1, \ldots, p,\ p = 2, 3;$$
$$\sigma = \begin{cases} \frac{1}{4} & \text{when } p = 2, \\ \frac{1}{3} & \text{when } p = 3, \end{cases} \tag{5}$$

where $y_{\bar{t}_\alpha \bar{t}_\alpha}$ is given by one of the formulae (3) or (4).

When $p = 2$ the scheme is a three-layer scheme, when $p = 3$, four-layer. This is where it differs from parabolic equations, for which the form of the local one-dimensional schemes did not depend on the number of dimensions.

We can write equations (5) in the form

$$(E - \sigma\tau^2\Lambda_\alpha)\,(y_\alpha + \overset{\vee}{y}_\alpha) = \begin{cases} 2y_{\alpha-1} + 2\sigma\tau^2\varphi_\alpha & \text{when } p = 2, \\ y_{\alpha-1} + y_{\alpha-2} + 2\sigma\tau^2\varphi_\alpha & \text{when } p = 3. \end{cases} \tag{6}$$

To find $y_\alpha + \overset{\vee}{y}_\alpha$ ($\overset{\vee}{y}_\alpha$ is known) we have to invert the three-point operator $E - \sigma\tau^2\Lambda_\alpha$, which can be done using the successive substitution formulae and the boundary condition

$$y_\alpha = u_1\,(x, t_{j+\alpha/p}) \quad \text{when } x \in \gamma_h^\alpha. \tag{7}$$

In the case of the net $\omega_h^{(1)}$ the boundary condition has the form

$$y_\alpha = \beta_\alpha^{\mp} y_\alpha^{(\pm 1_\alpha)} + (1 - \beta_\alpha^{\mp})\,u_{1\alpha}^{\mp} \quad \text{for } x \in \gamma_h^{\mp\alpha}$$

(see [1], [2]). If the operator $L_\alpha u$ contains the lowest terms $l_\alpha u = r_\alpha \frac{\partial u}{\partial x_\alpha} + q_\alpha u$, then when the successive substitution formulae are used it is generally necessary to make the steps of the net ω_h sufficiently small. In order to eliminate any restrictions on the steps of the net ω_h, we must, by analogy with [2], take the lowest terms on intermediate rows. Then y_α is found after inverting the operator $E - \sigma\tau^2\Lambda_\alpha^0$, where

$$\Lambda_\alpha^0 y = (a_\alpha y_{\bar{x}_\alpha})_{\hat{x}_\alpha} \sim L_\alpha^0 u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha}\right),$$

which is possible for any h_α.

It is clear from [1], [2], [6] and [7] that taking the lowest terms into account only complicates matters without changing the basic properties of the difference schemes. Therefore in future we shall take $\Lambda_\alpha = \Lambda_\alpha^0$ without loss of generality.

We take the initial conditions in the following form:

(a) if $p = 2$ then

$$y(x, 0) = u_0(x), \qquad (E - \sigma\tau^2\Lambda_1) y^{1/2} = F_1, \qquad y^{1/2} = y\left(x, \frac{\tau}{2}\right), \tag{8}$$
$$F_1 = u_0 + 0.5\bar{u}_0\tau + \tfrac{1}{4}\tau^2\Lambda_1 u_0 + \tau^2\left[\tfrac{1}{2} f_1 - \tfrac{1}{8}(\Lambda u + f)\right]_{t=0};$$

(b) if $p = 3$, then

$$y(x, 0) = u_0(x), \quad (E - \sigma\tau^2\Lambda_1) y^{1/3} = F_1, \quad (E - \sigma\tau^2\Lambda_2)(y^{2/3} + u_0) = 2y^{1/3} + F_2, \tag{9}$$
$$F_1 = u_0 + \tfrac{1}{3}\tau\bar{u}_0 + \tfrac{1}{3}\tau^2\Lambda_1 u_0 + \tau^2\left[\tfrac{2}{3} f_1 - \tfrac{1}{6}(\Lambda u + f)\right]_{t=0},$$
$$F_2 = \tau^2\left[\tfrac{2}{3} f_2 - \tfrac{1}{9}(\Lambda u + f)\right]_{t=0}.$$

Thus we associate problem (2) with the difference problem defined by conditions (5), (7), (8) or (9). We shall call it problem II.

3. We calculate the error of the difference scheme. Let u be the solution of problem (2) k and y the solution of problem II. The error $z_\alpha = y_\alpha - u^{j+\alpha/p}$ is given by the conditions

$$z_{\bar{t}_\alpha \bar{t}_\alpha} = \sigma\Lambda_\alpha(z_\alpha + \check{z}_\alpha) + \psi_\alpha \quad \text{for } t \geqslant \tau \ (j \geqslant 1),$$
$$z_\alpha = 0, \quad x \in \gamma_h^\alpha; \qquad z(x, 0) = 0, \quad x \in \bar{\omega}_h,$$
$$(E - \sigma\tau^2\Lambda_1) z^{1/p} = \tau^2\psi_1 \quad \text{for } t = \frac{\tau}{p}, \quad p = 2, 3,$$
$$(E - \sigma\tau^2\Lambda_2) z^{2/3} = 2z^{1/3} + \tau^2\psi_2 \quad \text{for } p = 3,$$

where $\psi_\alpha = \sigma\Lambda_\alpha(u_\alpha + \check{u}_\alpha) - u_{\bar{t}_\alpha \bar{t}_\alpha} + 2\sigma\varphi_\alpha$. The approximation error of the scheme is the sum

$$\Psi = \sum_{\alpha=1}^{p} \psi_\alpha .$$

Since (see [2])

$$\frac{1}{2}\Lambda_\alpha (u_\alpha + \check{u}_\alpha) = (L_\alpha u)_{t=t^*_\alpha} + (\mu_\alpha)_{\hat{x}_\alpha} + O(\hbar^2_\alpha),$$

where $\mu_\alpha = O(h^2_\alpha)$, we can write

$$\psi_\alpha = \mathring{\psi}_\alpha + \psi^*_\alpha, \qquad \mathring{\psi}_\alpha = 2\sigma\left[L_\alpha u - \frac{1}{p}\frac{\partial^2 u}{\partial t^2} + f_\alpha\right]^{(j-0.5)+\alpha/p}, \quad \alpha = 1, \ldots, p.$$

It is not difficult to see that

$$\sum_{\alpha=1}^{p} \mathring{\psi}_\alpha = O(\tau), \qquad \psi^*_\alpha = (\mu_\alpha)_{\hat{x}_\alpha} + O(\hbar^2_\alpha + \tau^2), \qquad \mu_\alpha = O(h^2_\alpha).$$

When $p = 2$ we have $\mathring{\psi}_2 + 2\mathring{\psi}_1 + \check{\mathring{\psi}}_1 = O(\tau^2)$.

Let us now discuss the question of the stability and convergence of the scheme II.

2. Convergence of the difference schemes

1. We consider the cases $p = 2$, $p = 3$ separately.

We shall use the scalar products and norms introduced in [2]

$$(y, z) = \sum_{\omega_h} yzH, \qquad \|y\|^2 = (y, y) \text{ etc.}, \tag{10}$$

where

$$H = \prod_{\alpha=1}^{p} \hbar_\alpha, \qquad \hbar_\alpha = 0.5\,(h_\alpha + h_{\alpha+}).$$

The only assumption we make about Λ_α is that

$$\begin{gathered}(-\Lambda_\alpha(z_\alpha + \check{z}_\alpha),\ z_\alpha - \check{z}_\alpha) \geqslant I_\alpha - (1 + c_*\tau)\,\check{I}_\alpha,\\ I_\alpha \geqslant c_1\|z_\alpha\|^2,\end{gathered} \tag{11}$$

where c_* and c_1 are positive constants which do not depend on the net. It is not difficult to see that the scheme $\overset{0}{\Lambda}_\alpha z = (a_\alpha z_{\bar{x}_\alpha})_{\hat{x}_\alpha}$ satisfies these requirements if $z|_{\gamma_h^\alpha} = 0$ and $a_\alpha = a_\alpha(x, t)$ satisfies the Lipschitz condition with respect to t. In future we shall assume that conditions (11) are satisfied without specifying the form of Λ_α.

2. Let $p = 2$. We consider the problem

$$z_{\bar{t}_\alpha \bar{t}_\alpha} = \sigma\Lambda_\alpha (z_\alpha + \check{z}_\alpha) + \psi_\alpha, \tag{12}$$

$$z(x, 0) = 0, \qquad \frac{1}{\tau} z_{\bar{t}_1} = \sigma\Lambda_1 z_1 + \psi_1 \quad \text{when } t = \frac{\tau}{2}, \tag{13}$$

$$z_\alpha = 0 \qquad \text{for } x \in \gamma_h^\alpha,\ t \in \omega_{\tau},\quad \alpha = 1, 2. \tag{14}$$

We make a scalar multiplication of (12) by $\tau(z_{\bar{t}_\alpha} + z_{\bar{t}_{\alpha-1}}) = z_\alpha - \check{z}_\alpha = \tau(z_\alpha)_{\bar{t}}$ and use condition (11)

$$\begin{gathered} \| z_{\bar{t}_\alpha} \|^2 + \sigma I_\alpha \leqslant \| z_{\bar{t}_{\alpha-1}} \|^2 + \sigma(1 + c_* \tau)\check{I}_\alpha + \tau(\psi_\alpha, (z_\alpha)_{\bar{t}}), \\ \| z_{\bar{t}_1} \|^2 + \sigma I_1 \leqslant \tau(\psi_1, z_{\bar{t}_1}). \end{gathered} \tag{15}$$

Here ψ_α have the form indicated in Section 1. Below it will be more convenient to take

$$\begin{gathered} \psi_\alpha = \mathring{\psi}_\alpha + \overset{*}{\psi}_\alpha, \qquad \sum_{\alpha=1}^{p} \mathring{\psi}_\alpha = 0; \\ \overset{*}{\psi}_\alpha = (\mu_\alpha)_{\hat{x}_\alpha} + O(\hbar_\alpha^2) + O(\tau), \qquad \mu_\alpha = O(h_\alpha^2). \end{gathered} \tag{16}$$

We write the solution of the problem (12)-(14) in the form of the sum $z = \xi + v$ where ξ is the solution of the problem (12)-(14) with the right-hand side $\psi_\alpha = \overset{\circ}{\psi}_\alpha$ and v the solution of the same problem with the right-hand side $\psi_\alpha = \overset{*}{\psi}_\alpha$. We write the energy inequality for ξ

$$\| \xi_{\bar{t}_\alpha} \|^2 + \sigma I_\alpha[\xi] \leqslant \| \xi_{\bar{t}_{\alpha-1}} \|^2 + \sigma(1 + c_*\tau)\check{I}_\alpha[\xi] + \tau(\mathring{\psi}_\alpha, (\xi_\alpha)_{\bar{t}}).$$

We sum over $\alpha = 1, 2$ and use the fact that

$$\begin{aligned} \tau \sum_{\alpha=1}^{2} (\mathring{\psi}_\alpha, (\xi_\alpha)_{\bar{t}}) &= \tau^2 (\mathring{\psi}_2, \xi_{\bar{t}_2 \bar{t}}) = \tau^2 (\mathring{\psi}_2, \xi_{\bar{t}_2})_{\bar{t}} - \tau^2 (\mathring{\psi}_{2\bar{t}}, \check{\xi}_{\bar{t}_2}) \leqslant \\ &\leqslant \tau^2 (\mathring{\psi}_2, \xi_{\bar{t}_2})_{\bar{t}} + c_*\tau \| \check{\xi}_{\bar{t}_2} \|^2 + \frac{\tau^3}{4c_*} \| \mathring{\psi}_{2\bar{t}} \|^2. \end{aligned}$$

We then obtain

$$\| \xi \|_*^2 = \| \xi_{\bar{t}_2} \|^2 + \sigma(I_2 + I_1) \leqslant (1 + c_*\tau) \| \check{\xi} \|_*^2 + \tau^2 (\mathring{\psi}_2, \xi_{\bar{t}_2})_{\bar{t}} + \frac{\tau^3}{4c_*} \| \mathring{\psi}_{2\bar{t}} \|^2.$$

Using Lemma 4 from [6] and the inequality

$$\tau(\mathring{\psi}_2, \xi_{\bar{t}_2}) \leqslant c_0 \|\xi_{\bar{t}_2}\|^2 + \frac{\tau^2}{4c_0}\|\mathring{\psi}_2\|^2 \leqslant c_0 \|\xi\|_*^2 + \frac{\tau^2}{4c_0}\|\mathring{\psi}_2\|^2,$$

we have

$$\|\xi^{j+1}\|_*^2 \leqslant M_1 \|\xi^1\|_*^2 + M_2 \max_{\omega_\tau} (\|\mathring{\psi}_2\|^2 + \|\mathring{\psi}_{2\bar{t}}\|^2)\,\tau^2 \quad (c_0,\ M_1,\ M_2 = \text{const.} > 0).$$

Combining this with the inequalities

$$\|\xi^1\|_*^2 \leqslant \tau(\mathring{\psi}_2, (\xi_2)_{\bar{t}}) + \tau(\mathring{\psi}_1, \xi_{\bar{t}_1}) = \tau(\mathring{\psi}_2, \xi_{\bar{t}_2}) \leqslant$$
$$\leqslant 0.5\,\tau^2 \|\mathring{\psi}_2^1\|^2 + 0.5\|\xi_{\bar{t}_2}\|^2 \leqslant 0.5\,\tau^2\|\mathring{\psi}_2^1\|^2 + 0.5\|\xi^1\|_*^2 \quad \text{when } t = \tau,$$
$$\|\xi^1\|_*^2 \leqslant \tau^2 \|\mathring{\psi}_2^1\|^2,$$

we obtain

$$\|\xi^{j+1}\|_*^2 \leqslant M_1 \tau^2 \max_{\omega_\tau} \|\mathring{\Psi}\|^2, \qquad \|\mathring{\Psi}\|^2 = \|\mathring{\psi}_2\|^2 + \|\mathring{\psi}_{2\bar{t}}\|^2.$$

It follows from this and from (11) that

$$\|\xi^{j+1}\| \leqslant M\tau \max_{\omega_\tau} \|\mathring{\Psi}\|. \tag{17}$$

Let us now turn to the estimate of v. We write inequality (15) for v and make the estimate

$$\tau(\psi_\alpha^*, (v_\alpha)_{\bar{t}}) = \tau(\psi_\alpha^*, v_\alpha)_{\bar{t}} - \tau(\psi_{\alpha\bar{t}}^*, \check{v}_\alpha) \leqslant \tau(\psi_\alpha^*, v_\alpha)_{\bar{t}} + \frac{c_0\tau}{2c_1}\check{I}_\alpha[v] + \frac{\tau}{2c_0}\|\psi_{\alpha\bar{t}}^*\|^2,$$
$$(\psi_\alpha^*, v_\alpha) \leqslant \frac{c_0}{2}\|v_\alpha\|^2 + \frac{1}{2c_0}\|\psi_\alpha^*\|^2 \leqslant \frac{c_0}{2c_1} I_\alpha + \frac{1}{2c_0}\|\psi_\alpha^*\|^2.$$

We insert these estimates in (15)

$$\|v\|_*^2 \leqslant (1 + M_1\tau)\|\check{v}\|_*^2 + M_2\tau \sum_{\alpha=1}^{2} \|\psi_{\alpha\bar{t}}^*\|^2 + \tau \sum_{\alpha=1}^{2} (\psi_\alpha^*, v_\alpha)_{\bar{t}}. \tag{18}$$

We take the equations for $t = 0.5\,\tau$ and $t = \tau$. After the usual reasoning we obtain an *a priori* estimate of the form

$$\|v_{\bar{t}_2}^{j+1}\|^2 + \|v^{j+1}\|^2 \leqslant M \max_{\omega_\tau} \|\Psi^*\|^2, \tag{19}$$

where

$$\|\Psi^*\|^2 = \sum_{\alpha=1}^{2} (\|\psi_\alpha^*\|^2 + \|\psi_{\alpha\bar{t}}^*\|^2). \tag{20}$$

The estimate of ψ_α^* in the norm $\|\psi_\alpha^*\|^2$ is too rough, because of the term $(\mu_\alpha)_{\hat{x}_\alpha}$ which appears on a nonuniform net. It is therefore necessary to introduce the norm $\|\psi_\alpha\|_{3_\alpha}$ (see [2]).

To do this we replace conditions (11) by the conditions

$$(-\Lambda_\alpha(z_\alpha+\check{z}_\alpha),\ z_\alpha-\check{z}_\alpha)\geqslant I_\alpha-(1+c_*\tau)\check{I}_\alpha,\ I_\alpha\geqslant\frac{1}{c_2^2}\|z_{\bar{x}_\alpha}\|^2, \tag{21}$$

and this gives (see [2])

$$(z_\alpha,\psi_\alpha^*)\leqslant c_2 I_\alpha^{1/2}\|\psi_\alpha^*\|_{3_\alpha}. \tag{22}$$

As a result instead of (19) we have the estimate

$$\|v_{\bar{t}_2}^{j+1}\|^2+\|v^{j+1}\|^2\leqslant M\max_{\omega_\tau}\|\Psi^*\|_3^2, \tag{23}$$

where

$$\|\Psi^*\|_3^2=\sum_{\alpha=1}^{2}[\|\psi_\alpha^*\|_{3_\alpha}^2+\|(\psi_\alpha^*)_{\bar{t}}\|_{3_\alpha}^2].$$

3. We have thus proved the following theorem:

Theorem 1. If conditions (11) and (16) are satisfied, the solution of the problem (12)-(14) satisfies the estimate

$$\|z^{j+1}\|+\|z_{\bar{t}}^{j+1}\|\leqslant M_1\tau\max_{\omega_\tau}\|\mathring{\Psi}\|+M_2\max_{\omega_\tau}\|\Psi^*\|. \tag{24}$$

If conditions (21) and (16) are satisfied, then the *a priori* estimate

$$\|z^{j+1}\|+\|z_{\bar{t}}^{j+1}\|\leqslant M_1\tau\max_{\omega_\tau}\|\mathring{\Psi}\|+M_2\max_{\omega_\tau}\|\Psi^*\|_3, \tag{24'}$$

holds, where M_1, M_2 are positive constants which are independent of the net.

This, together with (16), gives the following theorem:

Theorem 2. When $p = 2$ scheme II converges at the rate $O(\|h^2\|+\tau)$ on the arbitrary sequence of nets $\omega_h\times\omega_\tau$:

$$\|y^{j+\alpha/2}-u^{j+\alpha/2}\|=O(\|h^2\|+\tau),\qquad\|h^2\|=\sum_{\alpha=1}^{2}\|h_\alpha^2\|,\qquad\alpha=1,2, \tag{25}$$

if conditions (21) and conditions ensuring maximum order of approximation

$\|\overset{\circ}{\Psi}\| = O(1)$, $\|\Psi^*\|_3 = O(\|h^2\| + \tau)$ are satisfied.

Note. If the operator L_α (and, therefore Λ_α) contains lower terms, then conditions (11) and (21) are satisfied only for sufficiently small $\tau \leqslant \tau_0$. Therefore the estimates of Theorem 1 will also be true for $\tau \leqslant \tau_0$.

4. The estimate of the order of accuracy given by Theorem 2 is, generally speaking, too low. If the region G is a rectangle, our scheme has second order accuracy on the arbitrary nonuniform net ω_h

$$\| y^{j+1} - u^{j+1} \| = O(\|h^2\| + \tau^2) \quad \text{for } \tau \leqslant \tau_0. \tag{26}$$

We rewrite equation (6) in the form

$$A_\alpha (y_\alpha + \check{y}_\alpha) = 2y_{\alpha-1} + 0.5\tau^2 \varphi_\alpha, \qquad A_\alpha = E - \sigma\tau^2\Lambda_\alpha, \quad \alpha = 1, 2. \tag{27}$$

We eliminate y_1 and $\check{y}_1$ from the equations $A_1 (y_1 + \check{y}_1) = 2\check{y} + 0.5\,\tau^2\varphi_1$, $A_2 (y + \check{y}) = 2y_1 + 0.5\tau^2\varphi_2$, $\check{A}_2 (\check{y} + \check{\check{y}}) = 2\check{y}_1 + 0.5\tau^2\check{\varphi}_2$.

$$A_1A_2 (y + \check{y}) + A_1\check{A}_2 (\check{y} + \check{\check{y}}) = 4\check{y} + \tau^2\Phi; \qquad \Phi = \varphi_1 + 0.5A_1(\varphi_2 + \check{\varphi}_2). \tag{28}$$

Equations (27) are written along the boundaries and can be used when $t = t_{j+1/2}$ to find the boundary conditions in terms of the boundary conditions when $t = t_{j+1}$ and $t = t_j$. This is the difference between this and the previous formulation (7) of the boundary conditions.

When $t = \tau$ we obtain the equation

$$A_1A_2 (y + \check{y}) = 2\check{y} + \tau^2\Phi_1 + \tau u_0, \quad \Phi_1 = 2\varphi_1 + A_1\varphi_2. \tag{29}$$

We rewrite equations (28) and (29) in the form

$$y_{\bar{t}t} = \sigma(\Lambda_1 + \Lambda_2)(y + \check{y}) + \sigma (\Lambda_1 + \check{\Lambda}_2) (\check{y} + \check{\check{y}}) - \sigma^2\tau^2\Lambda_1\Lambda_2 (y + \check{y}) - \tag{30}$$
$$- \sigma^2\tau^2\Lambda_1\check{\Lambda}_2 (\check{y} + \check{\check{y}}) + \Phi \quad \text{when } t > \tau,$$

$$\frac{1}{\tau} (y_{\bar{t}} - 0.5\,\bar{u}_0) = \sigma (\Lambda_1 + \Lambda_2) (y + \check{y}) - \sigma^2\tau^2\Lambda_1\Lambda_2 (y + \check{y}) + \Phi_1, \quad t = \tau, \tag{31}$$

$$y|_{\gamma_h} = u_1, \qquad y (x, 0) = u_0 (x). \tag{32}$$

It is clear from this that we must interpret the scheme (27) as a splitting scheme in the case of the simplest region. For $z = y - u$ we obtain the conditions

$$z_{\bar{t}t} = \sigma (\Lambda_1 + \Lambda_2) (z + \check{z}) + \sigma (\Lambda_1 + \check{\Lambda}_2) (\check{z} + \check{\check{z}}) - \sigma^2\tau^2\Lambda_1\Lambda_2 (z + \check{z}) - \tag{33}$$
$$- \sigma^2\tau^2\Lambda_1\check{\Lambda}_2 (\check{z} + \check{\check{z}}) + \Psi \quad \text{for } t > \tau,$$

$$\frac{1}{\tau} z_{\bar{t}} = \sigma(\Lambda_1 + \Lambda_2)(z + \check{z}) - \sigma^2\tau^2\Lambda_1\Lambda_2(z + \check{z}) + \Psi, \quad t = \tau, \tag{34}$$

$$z(x, 0) = 0, \qquad z(x, t)|_{\gamma_h} = 0. \tag{35}$$

The approximation error $\Psi = O(|h|^2 + \tau^2)$ for $t > \tau$ and $\Psi = O(|h|^2 + \tau)$ for $t = \tau$.

Using the methods developed in [2] and [5] it is not difficult to obtain an *a priori* estimate by the method of power inequalities of the form

$$\| z^{j+1} \| \leqslant M \max_{\omega_\tau} (\| \Psi \|_3 + \| \Psi_{\bar{t}} \|_3), \quad i = 1, 2, \ldots, \qquad \| z^1 \| \leqslant M_\tau \| \Psi^1 \|_3, \tag{36}$$

which is true for sufficiently small $\tau \leqslant \tau_0$. This also gives (26). To obtain the energy identity we have to make a scalar multiplication of (33) by $\tau(z + \check{z})_{\bar{t}}$ (for $t > \tau$) and of (34) by $\tau z_{\bar{t}}$ (for $t = \tau$) and use an explicit expression for $\Lambda_\alpha z = (a_\alpha z_{\bar{x}_\alpha})_{\hat{x}_\alpha} + b_\alpha^+ z_{\hat{x}_\alpha} + b_\alpha^- z_{\check{x}_\alpha} + d_\alpha z$ (see [2]) assuming that $|(a_\alpha)_{\bar{t}}|$, $|(a_\alpha)_{\bar{x}_\beta}|$ $(\alpha, \ \beta = 1, 2)$ are bounded. There is no need to reproduce the proof of the estimate (36), but we shall discuss the conditions under which the scheme (30)-(32) has the maximum approximation error.

5. If we consider the net $\omega_h^{(1)}$ introduced in [1] and [2], which is uniform with respect to each of the variables x_1 and x_2, we can show that the solution of problem II when $p = 2$ in the arbitrary region G satisfies the relation

$$\| y - u \|^{j+1} = O(|h|^2) + O(\tau^2 h_1^{-1/2}), \qquad |h|^2 = h_1^2 + h_2^2. \tag{37}$$

Let us give the main points of the derivation of this estimate. We first represent ψ_α in the form

$$\psi_\alpha = \mathring{\psi}_\alpha + \psi_\alpha^*, \text{ where } \mathring{\psi}_1 = -0.5\,(\mathring{\psi}_2 + \check{\mathring{\psi}}_2), \mathring{\psi}_2 = O(1), \tag{38}$$

so that

$$\psi_\alpha^* = O(\tau^2) + O(h_\alpha^2) \text{ for } t > \tau, \qquad \psi_1^* = O(\tau) + O(h_1^2) \text{ when } t = \tau/2. \tag{39}$$

By analogy with [1] we look for the solution of problem (12)-(14) in the form of the sum

$$z = \eta + v, \tag{40}$$

where η is found from the conditions

$$\eta_{\bar{t}_\alpha \bar{t}_\alpha} = \mathring{\psi}_\alpha, \qquad \eta(x, 0) = 0, \qquad \eta(x, \tau) = 0, \tag{41}$$

so that $\eta_2 - 2\eta_1 + \check{\eta}_2 = \tau^2 \mathring{\psi}_2, \qquad \eta_1 - 2\check{\eta} + \check{\eta}_1 = -0.5\,\tau^2 (\mathring{\psi}_2 + \check{\mathring{\psi}}_2),$ $\check{\eta} - 2\check{\eta}_1 + \check{\check{\eta}}_2 = \tau^2 \check{\mathring{\psi}}_2$. We eliminate η_1 and $\check{\eta}_1$: $\eta - 2\check{\eta} + \check{\check{\eta}} = 0$ and, from (41), $\eta^j = 0$ for all $j = 0, 1, \ldots$. When $\alpha = 2$ equation (41) at once gives

$$\eta^{j+1/2} = -0.5\,\tau^2 \mathring{\psi}_2^{j+1} = O(\tau^2).$$

We define η at the points γ_h^1 by analogy with [1] so that $\Lambda_1 \eta = O(\eta)$ near the boundary. For the net function v we obtain the conditions

$$\left.\begin{array}{l} v_{\bar{t}_\alpha \bar{t}_\alpha} = \sigma \Lambda_\alpha (v_\alpha + \check{v}_\alpha) + \Psi_\alpha, \qquad \Psi_\alpha = \overset{*}{\psi}_\alpha + \sigma \Lambda_\alpha (\eta_\alpha + \check{\eta}_\alpha)\,\delta_{\alpha,1}, \quad \alpha = 1, 2, \\ v = \beta_\alpha^{\pm} v^{(\mp 1_\alpha)} + \nu_\alpha^{\pm}, \qquad \nu_\alpha^{\pm} = O(h_\alpha^2) + O(\eta)\,\delta_{\alpha,1} \quad \text{for } x \in \gamma_h^{\pm\alpha} \text{ (see. [2])}, \\ v(x, 0) = 0, \qquad (E - \sigma\tau^2 \Lambda_1)\, v\left(x, \frac{\tau}{2}\right) = \tau^2 \Psi_1, \qquad \Psi_1 = \overset{*}{\psi}_1 + \sigma \Lambda_1 \eta^1, \\ \qquad \text{or} \quad \frac{1}{\tau} v_{\bar{t}_1} = \sigma \Lambda_1 v_1 + \Psi_1, \qquad \overset{*}{\psi}_1 = O(\tau + h_1^2). \end{array}\right\} \tag{42}$$

Since v satisfies nonhomogeneous boundary conditions, we cannot use (11) and (21). We need a concrete expression for Λ_1. It is sufficient to carry out the reasoning for a segment (for a single chain c_1) omitting summation with respect to x_2. The factor $h_1^{-1/2}$ appears because of the nonhomogeneity of the boundary conditions and the need to use inequalities of the type

$$\nu_{\bar{t}} v_{\bar{x}_1} \leqslant \frac{1}{2c_0} \left(\frac{\nu_{\bar{t}}}{\sqrt{h_1}} \right)^2 + \frac{c_0}{2} v_{\bar{x}_1}^2 h_1, \qquad c_0 = \text{const.} > 0.$$

In [1] we estimated v with the help of the maximum principle. If we also use the energy method there, instead of the estimate $\| v \|_0 = O(\tau)$ we obtain $\| v \| = O(\tau / \sqrt{h_1})$.

It is therefore reasonable to expect that the appearance of the factor $h_1^{-1/2}$ in (37) is a result of the method of proof.

6. Let us now turn to the three-dimensional problem ($p = 3$)

$$z_{\bar{t}_\alpha \bar{t}_\alpha} = \frac{1}{\tau}(w_\alpha - w_{\alpha-2}) = \sigma\Lambda_\alpha (z_\alpha + \check{z}_\alpha) + \psi_\alpha, \qquad t \geqslant \tau, \tag{43}$$

$$\frac{1}{\tau} w_1 = \sigma\Lambda_1 z_1 + \psi_1 \quad \text{when } t = \tau/3, \tag{44}$$

$$\frac{1}{\tau}(w_2 - w_1) = \sigma\Lambda_2 z_2 + \psi_2 \quad \text{when } t = \frac{2\tau}{3}, \tag{45}$$

$$z(x, 0) = 0, \quad \text{for } x \in \omega_h, \qquad z_\alpha = 0 \quad \text{for } x \in \gamma_h^\alpha, \tag{46}$$

where

$$w_\alpha = z_{\bar{t}_\alpha} = (z_\alpha - z_{\alpha-1})/\tau. \tag{47}$$

To obtain the energy inequality we multiply equation (43) by $z_\alpha - \check{z}_\alpha$ and use (11). We have the formula

$$(z_{\bar{t}_\alpha \bar{t}_\alpha}, z_\alpha - \check{z}_\alpha) = \| z_{\bar{t}_\alpha} \|_*^2 - \| z_{\bar{t}_{\alpha-1}} \|_*^2, \tag{48}$$

where

$$\| z_{\bar{t}_\alpha} \|_*^2 = \| w_\alpha \|^2 + (w_\alpha, w_{\alpha-1}) + \| w_{\alpha-1} \|^2. \tag{49}$$

In fact, $z_\alpha - \check{z}_\alpha = \tau(w_\alpha + w_{\alpha-1} + w_{\alpha-2})$ and $(z_{\bar{t}_\alpha \bar{t}_\alpha}, \ z_\alpha - \check{z}_\alpha) = (w_\alpha - w_{\alpha-2}, w_\alpha + w_{\alpha-1} + w_{\alpha-2}) = [\| w_\alpha \|^2 + (w_\alpha, w_{\alpha-1})] - [\| w_{\alpha-2} \|^2 + (w_{\alpha-1}, w_{\alpha-2})] = \| z_{\bar{t}_\alpha} \|_*^2 - \| z_{\bar{t}_{\alpha-1}} \|_*^2$, adding and subtracting $\| w_{\alpha-1} \|^2$. We have the estimate

$$(\psi_\alpha, z_\alpha - \check{z}_\alpha) = \tau(\psi_\alpha, z_\alpha)_{\bar{t}} - \tau(\psi_{\alpha\bar{t}}, \check{z}_\alpha) \leqslant \tau(\psi_\alpha, z_\alpha)_{\bar{t}} + 0.5 c_2 \tau \check{I}_\alpha + \\ + 0.5(\tau/c_2) \| \psi_\alpha \|_{3_\alpha}, \tag{50}$$

if condition (21) is satisfied.

We write the power inequality $\| z_{\bar{t}_\alpha} \|_*^2 + \sigma I_\alpha \leqslant \| z_{\bar{t}_{\alpha-1}} \|_*^2 + (1 + c_*\tau) \ \sigma \check{I}_\alpha \ + \ \tau(\psi_\alpha, (z_\alpha)_{\bar{t}})$. Summation for α = 1, 2, 3 gives

$$\| z_{\bar{t}_3} \|_*^2 + \sigma I \leqslant \| \check{z}_{\bar{t}_3} \|_*^2 + (1 + c_*\tau) \sigma \check{I} + \tau \sum_{\alpha=1}^{3} (\psi_\alpha, (z_\alpha)_{\bar{t}}), \tag{51}$$

where

$$I = \sum_{\alpha=1}^{3} I_\alpha.$$

We note that $\| z_{\bar{t}_\alpha} \|_*^2 \geqslant \frac{1}{2}(\| z_{\bar{t}_\alpha} \|^2 + \| z_{\bar{t}_{\alpha-1}} \|^2)$.

By analogy with the case p = 2, we represent z in the form of the sum

$$z = \xi + v,$$

where ξ is the solution of the problem with the right-hand side $\psi_\alpha = \mathring{\psi}_\alpha$ and

$$\sum_{\alpha=1}^{3} \mathring{\psi}_\alpha = 0.$$

To estimate ξ we use the inequality (51), replacing z by ξ and ψ_α by $\mathring{\psi}_\alpha$. The condition $\mathring{\psi}_1 + \mathring{\psi}_2 + \mathring{\psi}_3 = 0$ gives

$$\begin{aligned}\sum_{\alpha=1}^{3} \tau\,(\mathring{\psi}_\alpha, (z_\alpha)_{\bar t}) = \tau^2\,[(\mathring{\psi}_3, z_{\bar{t}_3\bar t}) - (\mathring{\psi}_1, z_{\bar{t}_3\bar t})] = \tau^2\,[(\mathring{\psi}_3, z_{\bar{t}_3}) - \\ - (\mathring{\psi}_1, z_{\bar{t}_3})]_{\bar t} - \tau^2\,[(\mathring{\psi}_{3\bar t}, \check{z}_{\bar{t}_3}) - (\mathring{\psi}_{1\bar t}, \check{z}_{\bar{t}_3})] \leqslant \tau^2\,[(\mathring{\psi}_3, z_{\bar{t}_3}) - \\ - (\mathring{\psi}_1, z_{\bar{t}_3})]_{\bar t} + \tau c_*\|\check{z}_{\bar{t}_\alpha}\|_*^2 + 0.5\frac{\tau^3}{c_*}[\|\mathring{\psi}_{1\bar t}\|^2 + \|\mathring{\psi}_{3\bar t}\|^2].\end{aligned}$$

The power inequality will take the form

$$\begin{aligned}\|z_{\bar{t}_3}\|_*^2 + \sigma I \leqslant (1 + c_*\tau)(\|\check{z}_{\bar{t}_3}\|_*^2 + \sigma\check{I}) + \tau^2\,[(\mathring{\psi}_3, z_{\bar{t}_3}) - (\mathring{\psi}_1, z_{\bar{t}_3})]_{\bar t} + \\ + 0.5\,\frac{\tau^3}{c_*}\,[\|\mathring{\psi}_{1\bar t}\|^2 + \|\mathring{\psi}_{3\bar t}\|^2].\end{aligned} \tag{52}$$

Using the standard argument, taking the inequality for $t = \tau/3$, $t = 2\tau/3$ and the estimate

$$\tau\,[(\mathring{\psi}_3, z_{\bar{t}_3}) - (\mathring{\psi}_1, z_{\bar{t}_3})] \leqslant 0.5\|z_{\bar{t}_3}\|_*^2 + 0.5\tau^2(\|\mathring{\psi}_1\|^2 + \|\mathring{\psi}_3\|^2),$$

we can see that the estimate

$$\|\xi^{j+1}\| \leqslant M\tau\max_{\omega_\tau}\|\mathring{\Psi}\|, \qquad \|\mathring{\Psi}\|^2 = \|\mathring{\psi}_1\|^2 + \|\mathring{\psi}_3\|^2 + \|\mathring{\psi}_{1\bar t}\|^2 + \|\mathring{\psi}_{3\bar t}\|^2, \tag{53}$$

holds, where M is a positive constant which depends on c_1, c_* and c_2. In deriving the *a priori* estimate for v we use inequality (51).

As a result we see that Theorem 1 and Theorem 2 hold for $p = 3$.

7. If the region G is a parallelepiped, we can construct a large number of economical schemes with accuracy $O(|h|^2 + \tau^2)$. In [5] we even construct a scheme $O(|h|^4 + \tau^2)$. It is not possible to dwell in detail on splitting schemes here. We note only the different variants of the generating schemes. We first write the initial scheme

$$y_{\bar{t}t} = 0.5\Lambda\,(y + \check{\check{y}}) + \varphi, \quad \Lambda = \sum_{\alpha=1}^{p} \Lambda_\alpha.$$

Replacing the operator $E - 0.5\tau^2\Lambda$ by the product $\prod_{\alpha=1}^{p}(E - 0.5\tau^2\Lambda_\alpha)$, we can obtain the following generating schemes:

$$Ay_{\bar{t}} = \check{y}_{\bar{t}} + 0.5\tau\Lambda\,(\check{y} + \check{\check{y}}) + \tau\varphi, \qquad A = \prod_{\alpha=1}^{p} A_\alpha,\ A_\alpha = E - 0.5\tau^2\Lambda_\alpha,$$

$$Ay_{\mathring{t}} = \check{y}_{\bar{t}} + 0.5\tau\Lambda\check{y} + 0.5\tau\varphi, \qquad y_{\mathring{t}} = 0.5\,(y_{\bar{t}} + \check{y}_{\bar{t}}), \quad Ay = 2\check{y} - \check{\check{y}} + \tau^2\varphi.$$

For each of these we can propose a number of alternatin direction computing algorithms. Such a large number of splitting algorithms requires a comparison of the various algorithms from the point of view of economy, simplicity, etc.

In contrast to local one-dimensional schemes, the splitting schemes are, generally speaking, stable only for sufficiently small values of τ, and in practice this may lead to a reduction in the step τ, not required by the accuracy, and to an increase in the number of calculations involved.

8. For the system of hyperbolic equations

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \sum_{\alpha=1}^{p} L_\alpha \mathbf{u}, \qquad L_\alpha \mathbf{u} = \frac{\partial}{\partial x_\alpha}\left(k_\alpha\,(x,\,t)\,\frac{\partial \mathbf{u}}{\partial x_\alpha}\right),$$

where $k_\alpha = (k_\alpha^{is})$ is a symmetric positive definite n x n matrix, $\mathbf{u} = (u_1, \ldots, u_n)$, we can write down economic local one-dimensional schemes by analogy with [1]. Consider, for example, the case $p = 2$. We represent the matrix k_α in the form of a sum of triangular matrices $k_\alpha = k_\alpha^- + k_\alpha^+$, with $(k_\alpha^-)^{ii} = (k_\alpha^+)^{ii} = 0.5k_\alpha^{ii}$, so that k_α^- and k_α^+ are conjugate matrices. Then

$$L_\alpha \mathbf{u} = L_\alpha^- \mathbf{u} + L_\alpha^+ \mathbf{u}$$

and, accordingly, $\Lambda_\alpha = \Lambda_\alpha^- + \Lambda_\alpha^+$.

The simplest local one-dimensional scheme has the form

$$y_{\bar{t}_\alpha \bar{t}_\alpha} = 2\sigma\,(\Lambda_\alpha^- y_\alpha + \Lambda_\alpha^+ \check{y}_\alpha).$$

Determination of the vector $\mathbf{y}_\alpha$ reduces to the application of successive substitution n times, due to the triangularity of the operator Λ_α^-. For this scheme estimate (26) holds.

The local one-dimensional schemes considered above converge in the class of discontinuous coefficients having discontinuities of the same type as in [2].

All these results can be applied to the case of the equation

$$c(x,t)\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} L_\alpha u + f, \qquad c(x,t) \geqslant c_3 > 0,$$

where $L_\alpha u$ is given by formula (1), $f = f(x, t, u)$, $b = b(x, t, u)$, $r_\alpha = r_\alpha(x, t, u)$.

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