ON THE CONVERGENCE AND ACCURACY OF HOMOGENEOUS DIFFERENCE SCHEMES FOR ONE-DIMENSIONAL AND MULTIDIMENSIONAL PARABOLIC EQUATIONS*

A.A. SAMARSKII
(Moscow)

(Received 31 March 1962)

Homogeneous difference schemes, the general definition of which is given in [1], were considered from the point of view of their application to equations of the parabolic type with one space variable in [2]-[5]. Since the problem of the convergence of difference schemes can be reduced to the problem of the stability of the solution of a linear equation with respect to its right hand side, and to the boundary and initial data, *a priori* estimates were obtained in the first instance in [2] and [4] from which the stability follows. As in [1], special attention was paid to the choice of norms for the estimation of the right hand side of the difference equation with the help of which the convergence of homogeneous schemes could be proved in the class of discontinuous coefficients of the differential equation. In [3] the *a priori* estimates obtained in [2] were used in the proof of the uniform convergence and in the estimate of the order of accuracy of homogeneous difference schemes for the linear equation of heat conduction with discontinuous coefficients. In [5] homogeneous schemes were studied for a non-linear equation (1) of the parabolic type with boundary conditions of kind III.

In this paper we consider homogeneous schemes for quasilinear parabolic equations with one or more space variables. One-dimensional problems are studied in § 1 and multidimensional ones in § 2. The equation of heat conduction with the coefficient of heat conduction $k = k(x, t, u)$ is considered in § 1. The main *a priori* estimates of [2] and [4] for a four-point implicit scheme (forward scheme) and for a six-point symmetrical implicit scheme are improved in this section. This makes it

* *Zh. vych. mat.* 2, No. 4, 603-634, 1962.

possible, in particular, to obtain an estimate of the order of accuracy of homogeneous schemes for quasilinear parabolic equations with coefficients having mobile ("oblique") discontinuities on a finite number of curves $x = \eta_\nu(t)$. The results of [3] for the linear equation of heat conduction are also obtained.

It is shown (subsection 8, § 1) that there is a homogeneous difference scheme having in the class of discontinuous coefficients for the parabolic equation

$$\mathcal{P}_3 u = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) + r(x,t)\frac{\partial u}{\partial x} - f\left(x, t, u\frac{\partial u}{\partial t}\right) = 0$$

the same order of accuracy as in the class of smooth coefficients.

It should be remembered (see [1] [3] [5]) that we consider everywhere homogeneous difference schemes of "through computation" which do not change when discontinuous coefficients of the differential equation are used instead of smooth coefficients, and do not involve any changes in the scheme in the neighbourhood of the lines of discontinuity of the coefficient of heat conduction. All investigations are made for a wide class of schemes defined by specifying pattern functionals (see [1]) of a very general type, and the parameter α, $0 \leqslant \alpha \leqslant 1$ (weight of the row).

The methods used in [2]-[5] and here enable the convergence of homogeneous difference schemes to be proved for a system of parabolic equations with discontinuous coefficients. The *a priori* estimates obtained in [4] can be used for this purpose.

Implicit forward schemes, approximating a multidimensional equation of the parabolic type, are considered in § 2. The investigation, which is made on the analogy of the one-dimensional case, enables uniform convergence to be proved and an estimate to be found of the order of accuracy of homogeneous schemes in the class of smooth and discontinuous coefficients. Schemes of a special type for the case of partial coefficients were considered by a number of authors (cited in [6], see also [7]). In conclusion we would point out that for the solution of a multidimensional parabolic equation the use of the implicit schemes considered in § 2 involves a large amount of computation to be carried out for the solution of difference equations. In recent years a number of economic computational schemes have been proposed for the solution of multidimensional problems. With the methods used by us to study convergence it is possible to give a justification for (prove the uniform convergence of) the method of fractional steps (see [8]) in particular. A separate paper will be devoted to this question.

1. One-dimensional parabolic equations

1. Introduction

In [5] homogeneous schemes of through computation were considered for the non-linear equation

$$\mathcal{P}_1 u = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) + f\left(x, t, u\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0 \tag{1}$$

in the region $\overline{Д} = (0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T)$ and for boundary conditions of kind III

$$\left.\begin{aligned} k\frac{\partial u}{\partial x} - \sigma_1(t)\,u &= u_1(t) \qquad \text{for } x = 0, \\ k\frac{\partial u}{\partial x} + \sigma_2(t)\,u &= u_2(t) \qquad \text{for } x = 1. \end{aligned}\right\} \tag{2}$$

The results obtained in [5] are extended to the case of boundary conditions of a more general type:

$$\left.\begin{aligned} k\frac{\partial u}{\partial x} &= \varphi_1\left(t, u, \frac{\partial u}{\partial t}\right) \qquad \text{for } x = 0, \\ -k\frac{\partial u}{\partial x} &= \varphi_2\left(t, u, \frac{\partial u}{\partial t}\right) \qquad \text{for } x = 1, \end{aligned}\right\} \tag{3}$$

where $\varphi_s(t, u, q)$ $(s = 1, 2)$ are arbitrary functions satisfying the conditions

$$\frac{\partial \varphi_s}{\partial u} \geqslant 0, \qquad \frac{\partial \varphi_1}{\partial u} + \frac{\partial \varphi_2}{\partial u} \geqslant c_1 > 0, \qquad \frac{\partial \varphi_s}{\partial q} \geqslant c_2 > 0, \qquad s = 1,2, \tag{4}$$

where c_1 and c_2 are positive constants. The simplest example of conditions of this type are the conditions:

$$k\frac{\partial u}{\partial x} = C_1(t)\frac{\partial u}{\partial t} + \sigma_1(t)\,u + u_1(t) \qquad \text{for } x = 0, \tag{5}$$

$$-k\frac{\partial u}{\partial x} = C_2(t)\frac{\partial u}{\partial t} + \sigma_2(t)\,u + u_2(t) \qquad \text{for } x = 1. \tag{5'}$$

The physical significance of condition (5) is clear: at the boundary $x = 0$ there is a concentrated specific heat $C_1(t)$ and heat exchange occurs according to Newton's law with the external medium at the temperature $-u_1(t)/\sigma_1$.

Here we shall not be able to discuss the study of problems with conditions (3). We shall only note that the construction of difference boundary conditions with the second order of approximation is carried out according to the scheme proposed in [5]. The effect of the error of approximation of the boundary conditions on the accuracy of solution of

the difference boundary problem is estimated with the help of *a priori* estimates similar to the estimates obtained in [5] (theorem 1).

Since the method of calculating the error by means of approximation of the boundary conditions (2) (and therefore (3)) has been given in [5], we shall confine ourselves, for simplicity, to a detailed study of schemes for problems with boundary conditions of kind I.

In the rectangle $\bar{D}(0 \leqslant x \leqslant 1,\ 0 \leqslant t \leqslant T)$ we shall consider the following problem:

$$\mathcal{P}_2 u = \frac{\partial}{\partial x}\left(k(x,t,u)\frac{\partial u}{\partial x}\right) - c(x,t)\frac{\partial u}{\partial t} + f\left(x,t,u,\frac{\partial u}{\partial x}\right), \tag{6}$$

$$u(0,t) = u_1(t), \qquad u(1,t) = u_2(t), \tag{7}$$

$$u(x,0) = u_0(x), \tag{8}$$

$$k(x,t,u) \geqslant c_1 > 0, \qquad c(x,t) \geqslant c_2 > 0. \tag{9}$$

The function $k(x,\ t,\ u)$ and $f(x,\ t,\ u,\ p)$ have derivatives $\partial k/\partial u, \partial f/\partial u$, $\partial f/\partial p$ continuous with respect to the arguments u and p*.

The functions $k(x,\ t,\ u)$, $f(x,\ t,\ u,\ p)$ and $c(x,\ t)$ can have discontinuities of kind I with respect to the variables $(x,\ t)$ on a finite number of differentiable, mutually non-intersecting Γ_ν given by the equations $x = \eta_\nu(t)$, $\nu = 1,\ 2,\ \ldots,\ \nu_0$, $0 \leqslant t \leqslant T$ and $\eta_{\nu_1}(t) < \eta_{\nu_2}(t)$ for $\nu_1 < \nu_2$, $\eta_1(t) > \eta_0(t) = 1$, $\eta_{\nu_0}(t) < \eta_{\nu_0+1}(t) = 1$ on the segment $0 \leqslant t \leqslant T$. As usual (see [3], [5]), we denote by Δ_ν and D the regions:

$$\Delta_\nu = (\eta_\nu(t) < x < \eta_{\nu+1}(t),\quad 0 < t \leqslant T), \qquad \nu = 0,1,2,\ldots,\nu_0,$$

$$\bar{\Delta}_\nu = (\eta_\nu(t) \leqslant x \leqslant \eta_{\nu+1}(t),\quad 0 \leqslant t \leqslant T), \qquad D = \sum_{\nu=0}^{\nu_0} \Delta_\nu.$$

If the coefficient $k(x,\ t,\ u)$ is continuous on the curve $\Gamma_\nu(x = \eta_\nu(t))$, the following conjugation conditions are fulfilled

$$[u]_\nu = 0, \qquad \left[k\frac{\partial u}{\partial x}\right]_\nu = 0 \ \text{ for } x = \eta_\nu(t),\ 0 \leqslant t \leqslant T, \tag{10}$$

---

* The boundedness of the derivatives of $k(x,\ t,\ u)$ and $f(x,\ t,\ u,\ p)$ is made use of in the study of the convergence of the solution of difference problems to a given unique solution $u = u(x,\ t)$ of the problem (6)-(9). We also bear in mind the possibility of continuing suitably the functions k, f outside the region of variation of the arguments u, $p(u)$ corresponding to the given solution of u.

where $[u]_\nu = u(\eta_\nu(t) + 0, t) - u(\eta_\nu(t) - 0, t) = u_{\text{п},\nu} - u_{\text{л},\nu}$ etc.

If $\eta_\nu'(t) \equiv 0$ for all $0 \leqslant t \leqslant T$, i.e. $\eta_\nu(t) = \text{const.}$, we say that $k(x, t, u)$ has a fixed discontinuity. In the general case for $\eta_\nu'(t) \neq 0$ we say that k has a moving ("oblique") discontinuity.

2. *Homogeneous difference schemes*

Let $\bar\Omega = (x_i = ih, t_j = j\tau, i = 0, 1, \ldots, N, j = 0, 1, \ldots, K, h = 1/N, \tau = T/K)$ be a difference network, Ω the set of its internal points (x_i, t_j), $1 \leqslant i \leqslant N - 1$, $1 \leqslant j \leqslant K$; $\bar\omega_h = (x_i = ih, i = 0, 1, \ldots, N)$, $\omega_h = (x_i = ih, i = 1, 2, \ldots, N - 1)$ the network in x; $\bar\omega_\tau = (t_j = j \times \tau, j = 0, 1, \ldots, T)$, $\omega_\tau = (t_j = j\tau, j = 1, 2, \ldots, K)$ the network in t. The net function y_i^j given on $\bar\Omega$ or its parts will be denoted by $y(x, t)$ or simply y. The following notations will be used

$$y = y(x, t_{j+1}) = y^{j+1}, \qquad \check y = y(x, t_j) = y^j, \qquad y^{\pm 1} = y(x \pm h, t),$$
$$y_{\bar x} = (y - y^{(-1)})/h, \quad y_x = (y^{(+1)} - y)/h, \quad y_{\bar t} = (y - \check y)/\tau, \qquad y_{\overset{\circ}{x}} = 0.5(y_{\bar x} + y_x),$$

such that $(ay_{\bar x})_x = [a_{i+1}(y_{i+1} - y_i) - a_i(y_i - y_{i-1})]/h^2$,

$$(y, v) = \sum_{i=1}^{N-1} y_i v_i h, \qquad (y, v] = \sum_{i=1}^{N} y_i v_i h.$$

The following difference scheme will be associated with the differential equation (6)

$$\mathcal{P}_{h\tau}^{\alpha} y = (ay_{\bar x})_x^{(\alpha)} + \varphi(x, t^{(\alpha)}, y^{(\alpha)}, \lambda^{(\alpha)}(y)) - \rho^{(\alpha)} y_{\bar t}, \tag{11}$$

where $v^{(\alpha)} = \alpha v + (1 - \alpha)\check v$, α is an arbitrary parameter which can assume the following values in the segment $0 \leqslant \alpha \leqslant 1$

$$a = a(x, t, y^*), \qquad y^* = 0.5(y + y^{(-1)}), \qquad t^{(\alpha)} = \alpha t + (1 - \alpha)\check t, \qquad \check t = t - \tau,$$
$$\rho = \rho(x, t), \qquad \lambda(y) = y_{\overset{\circ}{x}}.$$

If $\alpha = 1$, then $\mathcal{P}_{h\tau}^{(1)} y$ is a four-point forward scheme; for $\alpha = 0.5$ we obtain a six-point implicit scheme $\mathcal{P}_{h\tau}^{(0.5)} y$. The scheme $\mathcal{P}_{h\tau}^{(\alpha)} y$ is defined by specifying the parameter α and the law for computing the coefficients a, φ and ρ in terms of the coefficients of the differential equation. We shall need the following properties of the coefficients a, φ and ρ:

1) $0 < c_1 \leqslant a \leqslant c_1'$, $0 \leqslant c_2 \leqslant \rho \leqslant c_2$, if $0 < c_1 < k < c_1'$, $0 < c_2 \leqslant c(x, t) \leqslant c_2'$;
2) $a(x, t, u) - k(x, t, u) = 0.5hk'(x, t, u) + O(h^2)$, $\qquad a_x(x, t, u) = k'(x, t, u) + O(h^2)$,

where the prime denotes differentiation with respect to x;

3) $$\varphi(x, t, u, \lambda(u)) - f(x, t, u, u') = O(h^2),$$
4) $$\rho(x, t) - c(x, t) = O(h^2),$$

if $k(x, t, u)$, $f(x, t, u, u')$, $c(x, t)$ and the solution $u(x, t)$ of equation (6) are differentiable a sufficient number of times, for example, k has three derivatives, f and c two derivatives each, and $u(x, t)$ four derivatives with respect to x. Conditions 1) - 4) are sufficient in order that scheme (11) may have the second order of approximation with respect to x. The concrete representation of a, φ and ρ in terms of k, f and c is not used in practice. Further, we require that the scheme $\mathcal{P}_{h\tau}^{(\alpha)} y$ should be a homogeneous scheme, i.e. its coefficients should be calculated according to the same law in terms of the coefficients of the differential equation at all points of the arbitrary network for the entire work of piecewise continuous functions. In [1] a method of calculating the coefficients of the scheme was given which makes use of the so-called pattern functionals

$$A^h[\mu(s)] \quad (-1 \leqslant s \leqslant 0), \qquad F^h[\mu(s)] \quad (-0.5 \leqslant s \leqslant 0.5)$$

according to the law

$$a(x, t, y^*) = A^h[k(x + sh, t, y^*)], \qquad \rho(x, t) = F^h[c(x + sh, t^*)],$$
$$\varphi(x, t, y, \lambda) = F^h[f(x + sh, t, y, \lambda)].$$

Without any loss of generality we can consider A^h and F^h to be canonical functionals independent of h, and denote them by A and F. Conditions 1) - 4) will be fulfilled if it is assumed that:

a) $A[\mu(s)]$ is a homogeneous non-decreasing functional of the first degree having differentials up to the third order of $A_m[\mu, f]$ (m = 1, 2, 3) and satisfying the conditions

$$A[1] = 1, \qquad A_1[s] = -0.5 \qquad (A_1[1, s] = A_1[s]);$$

b) $F[\mu\ s]$ is a linear non-negative functional and

$$F[1] = 1, \qquad F[s] = 0.$$

Thus the family of initial schemes $\mathcal{P}_{h\tau}^{(\alpha)}$ will be defined by specifying the parameter $\alpha \in [0, 1]$ and the class of pattern functionals A and F satisfying conditions a) and b). All the following investigations relate to the entire family of initial schemes for $0.5 \leqslant \alpha \leqslant 1$. In practice we are interested only in two values of the parameter α: $\alpha = 0.5$ and $\alpha = 1$. We do not consider explicit schemes ($\alpha = 0$).

The difference scheme with the pattern functionals (see [1])

$$A[\mu(s)] = \left[\int_{-1}^{0} \frac{ds}{\mu(s)}\right]^{-1}, \qquad F[\mu(s)] = \int_{-0.5}^{0.5} \mu(s)\, ds \tag{12}$$

ensures, as will be shown below, the highest order of accuracy in the class of discontinuous coefficients. The function f in (6) must be transformed to the form

$$f\left(x, t, u, 2k\frac{\partial u}{\partial x}\right),$$

which is always possible because $k \geqslant c_1 > 0$, and in formula (11) the approximation

$$\lambda(y) = a^{(+1)} y_x + a y_{\bar{x}}. \tag{13}$$

must be used instead of λ.

Below, when talking of scheme (12) we shall imply formula (13) also.

We now formulate a difference problem corresponding to the problem (6)(9):

$$\mathcal{P}_{h\tau}^{(\alpha)} y = 0 \text{ in } \Omega, \tag{14}$$

$$\left.\begin{aligned} y(0,t) = u_1(t), \qquad y(1,t) = u_2(t) \qquad &\text{for } t \in \omega_\tau, \\ y(x,0) = u_0(x) \qquad &\text{for } x \in \omega_h. \end{aligned}\right\} \tag{15}$$

From a), b) and (9) we obtain

$$0 < c_1 \leqslant a, \qquad 0 < c_2 \leqslant \rho. \tag{16}$$

3. The difference problem for the error

In solving problem (14)-(16) instead of problem (6)-(9) we introduce the error $z = y - u$. We shall find the conditions for the determination of z. Substituting $y = z + u$ in (14) and taking into account (6), (7), (8) and (15) we obtain the following conditions for z:

$$\overline{\mathcal{P}}_{h\tau}^{\alpha} z = (a(x,t,y^*)\, z_{\bar{x}})_x^{(\alpha)} - \rho^{(\alpha)} z_{\bar{t}} + Q(z) = -\Psi \text{ in } \Omega, \tag{17}$$

$$z(0,t) = 0, \qquad z(1,t) = 0 \quad \text{for } t \in \omega_\tau, \tag{18}$$

$$z(x,0) = 0 \qquad \text{for } x \in \omega_h, \tag{19}$$

$$Q(z) = (gz^*)_x^{(\alpha)} + b z_{\overset{\circ}{x}}^{(\alpha)} + d z^{(\alpha)}, \tag{20}$$

$$g = \overline{\frac{\partial a(x,t,u)}{\partial u}}, \qquad b = \frac{\partial \varphi}{\partial \lambda}, \qquad d = \overline{\frac{\partial \varphi}{\partial u}}, \qquad y^* = 0.5\,(y^{(-1)} + y).$$

The vinculum denotes that the derivatives are taken for certain mean values of the arguments u and λ (see [5]). The right hand side of Ψ equation (17) is, obviously, the error of approximation of scheme (14) in the solution of the differential equation (6).

It is determined from the formula

$$\begin{gathered}\Psi = \psi_a^{(\alpha)} + \overline{\psi}, \qquad \psi_a = (a(x,t,u^*)\,u_{\bar{x}})_x - \frac{\partial}{\partial x}\left(k(x,t,u)\frac{\partial u}{\partial x}\right), \\ \overline{\psi} = \varphi(x, t^{(\alpha)}, u^{(\alpha)}, u_{\overset{\circ}{x}}^{(\alpha)}) - \left(f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right)^{(\alpha)} - \left[\rho^{(\alpha)} u_{\bar{t}} - \left(c\frac{\partial u}{\partial t}\right)^{(\alpha)}\right].\end{gathered} \tag{21}$$

We shall assume that problem (6)-(10) has a unique solution $u = u(x,t)$ continuous in $\bar{D}$ and the following conditions are satisfied:

Conditions A_α: 1) the function $k(x, t, u)$ has a second derivative with respect to x, satisfying Lipshits's condition for x; 2) the function $f'(x, t, u, \lambda)$, $c'(x, t)$, $u'''(x, t)$ satisfy Lipshits's condition for x; 3) the derivatives $\partial^{m_\alpha - 1} c/\partial t^{m_\alpha - 1}$, $\partial^{m_\alpha} u/\partial t^{m_\alpha}$ satisfy Lipshits's condition for t, where $m_\alpha = 2$ for $\alpha = 0.5$ and $m_\alpha = 1$ for $\alpha \neq 0.5$.

If the conditions A_α are fulfilled in $\bar{D}$ (or in a fixed neighbourhood of the point (x, t), it is easy to show that

$$\Psi(x,t) = O(h^2) + O(\tau^{m_\alpha}), \quad m_\alpha = \begin{cases} 2 & \text{for } \alpha = 0.5, \\ 1 & \text{for } \alpha \neq 0.5. \end{cases} \tag{22}$$

4. The simplest a priori estimate

The question of the convergence and the order of accuracy of the difference problem can be reduced to the estimation of the solution of the problem (17)-(21) in terms of the error of approximation Ψ, i.e. to the proof of the stability of the solution with respect to the right hand side of Ψ. For this we use different *a priori* estimates depending on the type of the equation and the properties of the coefficients of the differential equation. The simplest to study is the case where $k = k(x, t)$ does not depend on $u(x, t)$ and the functions k, f and c are smooth. If $k = k(x, t)$, $g = 0$ in formula (20). We shall consider an implicit forward scheme, i.e. for $\alpha = 1$.

Let $z = z(x, t)$ be the solution of the following problem

$$\rho z_{\bar{t}} = (a z_{\bar{x}})_x + b_1 z_x + b_2 z_{\bar{x}} + d_1 z + d_2 \check{z} + \psi, \tag{23}$$

$$a^{(+1)}z_x = \mathscr{E}_1 z_{\bar{t}} + \sigma_1 z - \nu_1 \text{ for } x=0, \quad -az_{\bar{x}} = \mathscr{E}_2 z_{\bar{t}} + \sigma_2 z - \nu_2 \text{ for } x = x_N = 1, \tag{24}$$

$$z(x,0) = z_0(x), \tag{25}$$

$$\left.\begin{array}{llll} 0 < c_1 \leqslant a, & 0 < c_2 \leqslant \rho, & |d_s| \leqslant c_3, & |b_s| \leqslant c_4, \ s = 1,2, \\ \sigma_s \geqslant 0, & \mathscr{E}_s \geqslant 0, & \sigma_s + \mathscr{E}_s \geqslant c_5 > 0, & \sigma_1 + \sigma_2 \geqslant c_6 > 0. \end{array}\right\} \tag{26}$$

The boundary conditions (24) are obtained for a difference approximation of order $O(h^2)$ of boundary conditions III of kind (2) or conditions (5) (see [2] and [5]). For conditions (2) $\mathscr{E}_s = O(h)$. Boundary conditions I of the kind $z(0, t) = \nu_1$ and $z(1, t) = \nu_2$ follow from (24) on replacing in (24) ν_1 by $\sigma_1\nu_1$, ν_2 by $\sigma_2\nu_2$ and then passing on to the limiting transition $\sigma_1 \to \infty$ and $\sigma_2 \to \infty$.

Theorem 1. The solution of the problem (23)-(26) is stable with respect to the right hand side of ψ, the boundary data ν_1 and ν_2 and the initial values of $z(x, 0)$, so that for a sufficiently small $\tau < \tau_0$ and any h the following estimate is true

$$\|z(x,0)\|_0 \leqslant M\{\|z(x,0)\|_0 + \overline{|\nu_1(t)|} + \overline{|\nu_2(t)|}\} + M\left[\sum_{t'=\tau}^{t} \tau \|\psi(x,t')\|_0^2\right]^{1/2}, \tag{27}$$

where τ_0 and M are constants dependent on $c_1, \ldots, c_6$ and independent of the network;

$$\overline{|\nu_s(t)|} = \max_{\tau \leqslant t' \leqslant t} |\nu_s(t')|, \qquad s = 1, 2.$$

The theorem is proved by a method similar to that used in the theory of differential equations and based on the principle of the maximum for (23). We introduce a new function v^j, putting $z^j = v^j(1 + \bar{M}\tau)^j$ where $\bar{M}$ is an arbitrary positive constant which we shall select later. For the network function $v(x, t)$ we obtain the conditions

$$\left.\begin{array}{l} \bar{\rho} v_{\bar{t}} - (av_{\bar{x}})_x = b_1 v_x + b_2 v_{\bar{x}} - dv + \bar{d}_2 \check{v} + \bar{\psi}; \\ \bar{\mathscr{E}}_1 v_{\bar{t}} - a^{(+1)} v_x + \bar{\sigma}_1 v = \bar{\nu}_1 \text{ for } x = 0, \quad \bar{\mathscr{E}}_2 v_{\bar{t}} + av_{\bar{x}} + \bar{\sigma}_2 v = \bar{\nu}_2 \text{ for } x = 1; \end{array}\right\} \tag{28}$$

$$v(x,0) = z(x,0),$$

where

$$\bar{\rho} = \rho\gamma, \quad d = -d_1 + \rho\bar{M}\gamma, \quad \bar{d}_2 = d_2\gamma, \quad \bar{\psi}^j = \psi^j\gamma^j, \quad \bar{\nu}_s^j = \nu_s^j\gamma^j,$$
$$\bar{\sigma}_s = \sigma_s + \mathscr{E}_s\gamma\bar{M}, \qquad \bar{\mathscr{E}}_s = \gamma\mathscr{E}_s, \qquad \gamma = 1/(1 + \bar{M}\tau), \qquad s = 1, 2.$$

We formulate the conditions for v^2. Multiplying (28) by $2v$ and taking

into account the obvious identities

$$2v\,(av_{\bar{x}})_x = (a\,(v^2)_{\bar{x}})_x - av_{\bar{x}}^2 - a^{(+1)}v_x^2, \qquad 2v\cdot v_{\bar{t}} = (v^2)_{\bar{t}} + \tau v_{\bar{t}}^2,$$

$$2v\cdot v_x = (v^2)_x - hv_x^2, \qquad 2v\cdot v_{\bar{x}} = (v^2)_{\bar{x}} + hv_{\bar{x}}^2,$$

we obtain

$$\bar{\rho}\,(v^2)_{\bar{t}} - (a\,(v^2)_{\bar{x}})_x + Q\,(v) + 2dv^2 = 2v\,(b_1v_x + b_2v_{\bar{x}}) + 2\bar{d}_2\check{v}v + 2v\bar{\psi}, \tag{29}$$

$$\left.\begin{aligned} &\bar{\mathscr{E}}_1\,(v^2)_{\bar{t}} - a^{(+1)}\,(v^2)_x + 2\bar{\sigma}_1v^2 + R_1\,(v) = 2\bar{\nu}_1v \quad \text{for } x = 0,\\ &\bar{\mathscr{E}}_2\,(v^2)_{\bar{t}} + a\,(v^2)_{\bar{x}} + 2\bar{\sigma}_2v^2 + R_2\,(v) = 2\bar{\nu}_2v \quad \text{for } x = x_N = 1, \end{aligned}\right\} \tag{29'}$$

$$Q\,(v) = \bar{\rho}\tau v_{\bar{t}}^2 + av_{\bar{x}}^2 + a^{(+1)}v_x^2,$$

$$R_1\,(v) = (\tau\bar{\mathscr{E}}_1v_{\bar{t}}^2 + ha^{(+1)}v_{\bar{x}}^2)\,|_{x=0}, \qquad R_2\,(v) = (\tau\bar{\mathscr{E}}_2v_{\bar{t}}^2 + hav_{\bar{x}}^2)\,|_{x=1}.$$

Using the well-known inequality $|ab| \leqslant c_0a^2/2 + b^2/2c_0$, where c_0 is an arbitrary positive quantity we obtain

$$2\,|\,(b_1v_x + b_2v_{\bar{x}})\,v\,| \leqslant (2c_4^2/c_1)\,v^2 + av_{\bar{x}}^2 + a^{(+1)}v_x^2, \qquad 2\,|\,\bar{\psi}v\,| \leqslant c_0v^2 + \bar{\psi}^2/c_0,$$

$$2\,|\,\bar{d}_2\check{v}v\,| \leqslant 2\gamma c_3v^2 + 2\gamma c_3\,|\,v\,|\,\tau\,|\,v_{\bar{t}}\,| \leqslant 2\gamma\,(c_3 + 0.5c_3^2/c_*)\,v^2 + \gamma c_*\tau^2v_{\bar{t}}^2,$$

where $c_* = c_4^2/c_1 + c_3 + 0.5\,c_0$. Thus the right hand side of equation (29) is increased by the expression

$$2\,[\gamma\,(c_3 + 0.5c_3^2/c_*) + c_* - c_3]\,v^2 + \gamma c_*\tau^2v_{\bar{t}}^2 + \bar{\psi}^2/c_0$$

and instead of (29) the following inequality can be written

$$\bar{\rho}\,(v^2)_{\bar{t}} - (a\,(v^2)_{\bar{x}})_x + \gamma\,(c_2 - \tau c_*)\,v_{\bar{t}}^2 + 2d_*v^2 \leqslant \bar{\psi}^2/c_0, \tag{29''}$$

where $d_* = (c_2\bar{M} - c_3 - 0.5\,c_3^2/c_*)\gamma - c_*$. We now choose $\bar{M}$ such that $d_* > 0$. This condition will be satisfied if τ is sufficiently small ($\tau < \tau_0'$), and $\bar{M}$ is sufficiently large ($\bar{M} > M_*'$)

$$\tau < \tau_0' = \frac{c_2}{c_*}\left(1 - \frac{M_*'}{\bar{M}}\right), \qquad \bar{M} > M_*' = \frac{(c_3 + c_*)\,c_* + 0.5c_3^2}{c_2c_*} \tag{30}$$

(choosing, for example, $M = 2M_*'$ we obtain $\tau_0' \leqslant 0.5\,c_2/c_*$). Hence we obtain $c_2 - \tau c_* > 0$ and

$$\bar{\rho}\,(v^2)_{\bar{t}} - (a\,(v^2)_{\bar{x}})_x + 2d_*v^2 \leqslant \bar{\psi}^2/c_0. \tag{31}$$

Using the estimate $2|\nu_s z| \leqslant c_0' z^2 + \nu_s^2/c_0'$ $(s = 1, 2)$ where c_0' is an arbitrary positive quantity we obtain

$$\left.\begin{aligned} \bar{\mathscr{E}}_1 (v^2)_{\bar{t}} - a^{(+1)} (v^2)_x + 2\bar{\sigma}_{1*} v^2 &\leqslant \nu_1^2/c_0' \quad \text{for } x = 0, \\ \bar{\mathscr{E}}_2 (v^2)_{\bar{t}} + a (v^2)_{\bar{x}} + 2\bar{\sigma}_{2*} v^2 &\leqslant \nu_2^2/c_0' \quad \text{for } x = 1, \end{aligned}\right\} \tag{31'}$$

where $\bar{\sigma}_{1*} = \bar{\sigma}_1 - 0.5\, c_0'$, $\bar{\sigma}_{2*} = \bar{\sigma}_2 - 0.5\, c_0'$.

We shall now require that $\bar{\sigma}_{s*} \geqslant m_* > 0$ $(s = 1, 2)$, where m_* is a given constant. This condition will be satisfied if $\tau < \tau_0''$ (c_5, c_0', m_*), $\bar{M} > M_*'' > 0$. Taking (30) into account and denoting by τ_0 the least of τ_0' and τ_0'', and by M_* the greatest of M_*' and M_*'', we obtain

$$d_* > 0, \qquad \bar{\sigma}_{1*} \geqslant m_* > 0, \qquad \bar{\sigma}_{2*} \geqslant m_* > 0 \qquad \text{for } \tau < \tau_0,\ \bar{M} > M_* > 0. \tag{32}$$

We shall express $v = v^{(1)} + v^{(2)} + v^{(3)} + v^{(4)}$, where $v^{(1)}$ is the solution of equation (28) with homogeneous boundary and initial conditions, $v^{(2)}$ is the solution of the homogeneous equation $(\bar{\psi} = 0)$ with homogeneous boundary conditions $(\nu_1 = \nu_2 = 0)$ and the initial condition $v^{(2)}(x, 0) = z(x, 0)$, $v^{(3)}$ is the solution of problem (28) for $\bar{\nu}_2 = 0$, $\bar{\psi} = 0$, $v^{(3)}(x, 0 = 0$, and $v^{(4)}$ is the solution for $\bar{\nu}_1 = 0$, $\bar{\psi} = 0$, $v^{(4)}(x, 0) = 0$.

The function $w = (v^{(1)})^2 > 0$ can reach a maximum value only at the internal points of the network Ω. We assume, for instance, that it reaches a maximum at the point $(0, t)$. Then $w_{t,0} \geqslant 0$, $w_{x,0} \leqslant 0$ at this point and condition (31′) gives $2m_* w^2(0, t) \leqslant 0$, which is impossible.

We fix a $t \in \omega_\tau$. Let $w(x, t) = (v^{(1)})^2 > 0$ assume a maximum value for $x = \bar{x} \in \omega_h$. Then $(a\omega_{\bar{x}})_x \geqslant 0$ at the point $(\bar{x}, t)$ and the inequality (31) gives

$$\| v^{(1)} \|_0^2 \leqslant \| \check{v}^{(1)} \|_0^2 + \tau \| \bar{\psi} \|_0^2 / \bar{c}_0, \bar{c}_0 = c_0 c_2/(1 + \bar{M}\tau).$$

It follows hence that

$$\| v^{(1)}(x, t) \|_0 \leqslant \Big(\frac{1}{\bar{c}_0} \sum_{t'=\tau}^{t} \tau \| \bar{\psi}(x, t') \|_0^2 \Big)^{1/2}.$$

The function $v^{(2)}(x, t)$ obviously cannot have a maximum either inside Ω, or for $x = 0$ or $x = 1$, i.e.

$$\| v^{(2)}(x, t) \|_0 \leqslant \| z(x, 0) \|_0.$$

The function $v^{(3)}(x, t)$ can have a maximum only at some point $(0, t)$ of the boundary. At this point $w_{\bar{t},0} \geqslant 0$, $w_{x,0} \leqslant 0$, where $w = (v^{(3)})^2$, and the first of the inequalities (31′) gives

$$\|v^{(3)}(x,t)\|_0 \leqslant \frac{1}{\sqrt{c_0' m_*}} |\overline{\bar{\nu}_1(t)}|, \text{ where } |\overline{\bar{\nu}_1(t)}| = \max_{\tau \leqslant t' \leqslant t} |\bar{\nu}_1(t')|.$$

$\| v^{(4)} \|_0$ is estimated similarly. Collecting all the estimates and returning to the initial function $z^j = v^j(1 + \bar{M}\tau)^j$ we obtain inequality (27). It should be noted that c_0, c_0', m_* should be so chosen that the constant M in (27) is a minimum. The theorem is proved.

Note. Theorem 1 and estimate (27) are obviously true for the first boundary problem

$$z(0,t) = \nu_1, \qquad z(1,t) = \nu_2,$$

This can be verified by making the limiting transition mentioned above or repeating the proof of the theorem, which becomes simpler in this case.

5. *Improved a priori estimates for a forward scheme*

In this subsection we shall follow the method described in [2] to obtain a number of new estimates for a four-point forward scheme. The order of accuracy of homogeneous schemes in the case of moving discontinuities (cf. [3]) can be improved by using these improved estimates.

Let us consider the following problem

$$\rho z_{\bar{t}} = (a z_{\bar{x}})_x + Q(z) + \psi, \tag{33}$$

$$Q(z) = (g_{11} z)_x + (g_{12} \check{z})_x + (g_{22} z)_{\bar{x}} + (g_{21} \check{z})_{\bar{x}} + b_1 z_x + b_2 z_{\bar{x}} + d_1 z + d_2 \check{z}, \tag{34}$$

$$a^{(+1)} z_x = \mathscr{E}_1 z_{\bar{t}} + \sigma_1 z - \nu_1 \text{ for } x = 0, \quad -a z_{\bar{x}} = \mathscr{E}_2 z_{\bar{t}} + \sigma_2 z - \nu_2 \text{ for } x = 1, \tag{35}$$

$$z(x, 0) = z_0(x), \tag{36}$$

where $\rho = \rho(x, t)$, $a = a(x, t)$, $\psi = \psi(x, t)$, $g_{sk} = g_{sk}(x, t)$, $b_s = b_s(t)$, $d_s = d_s(t)$, $\mathscr{E}_s = \mathscr{E}_s(t)$, $\sigma_s = \sigma_s(t)$, $\nu_s = \nu_s(t)$ $(s, k = 1, 2)$ are the given network functions. The coefficients of the problem satisfy the conditions:

$$\left.\begin{array}{l} 0 < c_1 \leqslant a, \quad 0 < c_2 \leqslant \rho, \quad |d_s| \leqslant c_3, \quad |b_s| \leqslant c_4, \quad |g_{sk}| \leqslant c_5 \quad (s, k = 1, 2), \\ \sigma_s \geqslant 0, \quad \sigma_1 + \sigma_2 \geqslant c_6 > 0, \quad 0 < c_7 h \leqslant \mathscr{E}_s, \end{array}\right\} \tag{37}$$

$$|\rho_{\bar{t}}| \leqslant c_8, \quad |(\mathscr{E}_s)_{\bar{t}}| \leqslant c_9 \mathscr{E}_s, \qquad s = 1, 2, \tag{38}$$

where $c_1 - c_9$ are positive constants independent of h and τ.

We introduce a new function v, writing

$$z^j = v^j (1 + \overline{M}\tau)^j,$$

where $\overline{M} > 0$ is an arbitrary constant. For v we obtain the conditions

$$\bar{\rho} v_{\bar{t}} - (a v_{\bar{x}})_x + dv = \Psi = \bar{\psi} + Q_1(v), \tag{39}$$

$$\left.\begin{aligned} a^{(+1)} v_x &= \overline{\mathscr{E}}_1 v_{\bar{t}} + \bar{\sigma}_1 v + \bar{\nu}_1 \quad \text{for } x = 0, \\ -a v_{\bar{x}} &= \overline{\mathscr{E}}_2 v_{\bar{t}} + \bar{\sigma}_2 v - \bar{\nu}_2 \quad \text{for } x = 1, \end{aligned}\right\} \tag{40}$$

$$v(x, 0) = z(x, 0) = z_0(x), \tag{41}$$

$$Q_1(v) = (g_{11} v)_x + (\bar{g}_{12} \check{v})_x + (g_{22} v)_{\bar{x}} + (\bar{g}_{21} \check{v})_{\bar{x}} + b_1 v_x + b_2 v_{\bar{x}} + \bar{d}_2 \check{v}, \tag{42}$$

where

$$\bar{\rho} = \rho\gamma, \quad d = \bar{\rho}\overline{M} - d_1, \quad \bar{d}_2 = d_2\gamma, \quad \bar{g}_{sk} = g_{sk} \cdot \gamma, \quad \overline{\mathscr{E}}_s = \mathscr{E}_s \cdot \gamma,$$

$$\bar{\sigma}_s = \sigma_s + \overline{M}\gamma\mathscr{E}_s, \quad \gamma = 1/(1 + \overline{M}\tau), \quad \bar{\psi}^j = \psi^j (1 + \overline{M}\tau)^{-j};$$

$$\bar{\nu}_s^j = \nu_s^j (1 + \overline{M}\tau)^{-j} \quad (s, k = 1, 2).$$

Henceforth we shall follow [2], improving on the estimates obtained there. Multiplying equation (39) and boundary conditions (40) by $v \times v^2 \ldots v^{2^{m-1}} = v^{\alpha_n}$, where $\alpha_n = 2^n - 1$ we obtain for $\overset{n}{v} = v^{2^n}$ the problem:

$$\bar{\rho} \overset{n}{v}_{\bar{t}} - \left(a \overset{n}{v}_{\bar{x}}\right)_x + \sum_{k=0}^{n-1} 2^{n-k-1} \left(a \left(\overset{k}{v}_{\bar{x}}\right)^2 + a^{(+1)} \left(\overset{k}{v}_x\right)^2 + \tau\bar{\rho} \left(\overset{k}{v}_{\bar{t}}\right)^2\right) v^{\alpha_n - \alpha_{k+1}} + \tag{43}$$
$$+ 2^n d \overset{n}{v} = 2^n v^{\alpha_n} \Psi \quad \text{(an equation of the } n\text{th order)},$$

$$a^{(+1)} \overset{n}{v}_x = \overline{\mathscr{E}}_1 \overset{n}{v}_{\bar{t}} + 2^n \bar{\sigma}_1 \overset{n}{v} + \sum_{k=0}^{n-1} 2^{n-k-1} \left(h a^{+(1)} \left(\overset{k}{v}_x\right)^2 + \tau \overline{\mathscr{E}}_1 \left(\overset{k}{v}_{\bar{t}}\right)^2\right) v^{\alpha_n - \alpha_{k+1}} + \tag{44}$$
$$+ 2^n \bar{\nu}_1 v^{\alpha_n} \quad \text{for } x = 0,$$

$$-a \overset{n}{v}_{\bar{x}} = \overline{\mathscr{E}}_2 \overset{n}{v}_{\bar{t}} + 2^n \bar{\sigma}_2 \overset{n}{v} + \sum_{k=0}^{n-1} 2_{n-k-1} \left(h a \left(\overset{k}{v}_{\bar{x}}\right)^2 + \tau \overline{\mathscr{E}}_2 \left(\overset{k}{v}_{\bar{t}}\right)^2\right) v^{\alpha_n - \alpha_{k+1}} - \tag{44'}$$
$$- 2^n \bar{\nu}_2 v^{\alpha_n} \quad \text{for } x = 1,$$

$$\overset{n}{v}(x,0) = \overset{n}{z}(x,0) \tag{45}$$

$$\left(\overset{k}{v}_{\bar t} = \left(\overset{k}{v} - \overset{\check k}{v}\right)/\tau, \quad \overset{k}{v}_{\bar x} = \left(\overset{k}{v} - \overset{k}{v}^{(-1)}\right)/h \qquad \text{etc.}\right).$$

Multiplying equation (43) by h, summing over the internal points $x = h,\ 2h,\ \ldots,\ (N-1)h$ of the network ω_h and taking into account (44) and (44′) we arrive at the integral identity of the n-th rank

$$[\bar\rho, \overset{n}{v}]_{\bar t} + 2I_n + P_n + 2^n[d, \overset{n}{v}] = 2^n[v^{\alpha_n}, \Psi] + [\bar\rho_{\bar t}, \overset{\check n}{v}], \tag{46}$$

where

$$[\bar\rho, \overset{n}{v}] = (\bar\rho, \overset{n}{v}) + \bar{\mathscr{E}}_1\overset{n}{v}_0 + \bar{\mathscr{E}}_2\overset{n}{v}_N, \quad [d, \overset{n}{v}] = (d, \overset{n}{v}) + \left(1 - \frac{1}{2^{n-1}}\right)(\bar\sigma_1\overset{n}{v}_0 + \bar\sigma_2\overset{n}{v}_N), \tag{47}$$

$$I_n = (a, \left(\overset{n-1}{v}_{\bar x}\right)^2] + \sum_{k=0}^{n-2} 2^{n-k-2}\left\{\left(a\left(\overset{k}{v}_{\bar x}\right)^2,\ v^{\alpha_n-\alpha_{k+1}}\right] + \left[a^{(+1)}\left(\overset{k}{v}_x\right)^2, v^{\alpha_n-\alpha_{k-1}}\right)\right\} + \\ + \bar\sigma_1\overset{n}{v}_0 + \bar\sigma_2\overset{n}{v}_N, \tag{48}$$

$$P_n = \sum_{k=0}^{n-1} 2^{n-k-1}\left[\bar\rho\left(\overset{k}{v}_{\bar t}\right)^2,\ v^{\alpha_n-\alpha_{k+1}}\right], \quad [\Psi, v^{\alpha_n}] = (\Psi, v^{\alpha_n}) + \bar\nu_1 v_0^{\alpha_n} + \bar\nu_2 v_N^{\alpha_n} \\ (v_0 = v(0,t), \quad v_N = v(1,t), \quad \alpha_n = 2^n - 1). \tag{49}$$

It will be noted that the term $\bar\sigma_1\overset{n}{v}_0 + \bar\sigma_2\overset{n}{v}_N$ has been left out in formula (17) from § 2 of [2]. This did not, however, affect the subsequent arguments and results.

The discussion below concerns the majorant estimate of the expressions on the right hand side of (46) by means of $(1, \overset{n}{v})$ and I_n. The arbitrariness in the choice of the constant $\bar M$ is made use of for this purpose. We use the estimate which follows from lemma 1* of [2],

$$\overset{n}{v} \leqslant 4\frac{\sigma_1\overset{n}{v}_0 + \sigma_2\overset{n}{v}_N}{c_6} + \frac{2}{c_1}\left\|\sqrt{a}\,\overset{n-1}{v}_{\bar x}\right\|_2^2 \leqslant M_* I_n, \tag{50}$$

where $M_* > 0$ is a constant dependent only on c_1 and c_6, Hölder's inequality

$$|[f, \psi]| \leqslant [1, |f|^p]^{1/p}[1, |\psi|^q]^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 0, \quad q > 0, \tag{51}$$

and the inequality

$$\prod_{k=1}^{m} c_k^{\mu_k} \leqslant \sum_{k=1}^{m} \mu_k c_k, \text{ where } \mu_k \geqslant 0, \qquad c_k \geqslant 0, \quad \sum_{k=1}^{m} \mu_k = 1. \tag{52}$$

We note, first of all, that because of conditions (37)-(38)

$$|[\bar{\rho}_{\bar{t}}, \overset{\vee}{v}^n]| \leqslant M [\overset{\vee}{\rho}, \overset{\vee}{v}^n], \text{ where } M = c_8/c_2. \tag{53}$$

We now consider the expression

$$2^n [\bar{\psi}, v^{\alpha_n}] = 2^n (\bar{\psi}, v^{\alpha_n}) + 2^n \bar{\nu}_1 v_0^{\alpha_n} + 2^n \bar{\nu}_2 v_N^{\alpha_n}.$$

Two types of estimate are possible for the sum $2^n(\bar{\psi}, v^{\alpha_n})$ depending on the choice of the norm for ψ

1) $2^n |(\bar{\psi}, v^{\alpha_n})| \leqslant 2^n (\bar{\psi}^{2^n}, 1)^{1/2^n} (1, \overset{n}{v})^{1-1/2^n} \leqslant 2^n (1, \overset{n}{v}) + (1, \bar{\psi}^{2^n})$; (54)

2) we choose for $\bar{\psi}$ the norm

$$\|\bar{\psi}\|_4 = \|\bar{\psi}\|_3 + |(\bar{\psi}, 1)|, \qquad \|\bar{\psi}\|_3 = \|\eta\|_2, \qquad \eta(x) = \sum_{x'=h}^{x} h\bar{\psi}(x').$$

We introduce the function η, writing

$$\eta_{\bar{x}} = \bar{\psi}, \qquad x = h, \ldots, (N-1)h = x_{N-1}, \qquad \eta_0 = 0,$$

and use the formula for summation by parts (see [4]):

$$(v^{\alpha_n}, \bar{\psi}) = (v^{\alpha_n}, \eta_{\bar{x}}) = -(\eta, (v^{\alpha_n})_x) + v_N^{\alpha_n} \eta_{N-1}, \qquad \eta_{N-1} = (1, \bar{\psi}).$$

The formula

$$(v^{\alpha_n})_x = \sum_{k=0}^{n-1} (v^{(+1)})^{\alpha_k} v^{\alpha_n - \alpha_k + 1} \overset{k}{v}_x. \tag{55}$$

was found for $(v^{\alpha_n})_x$ in [2].

Using inequalities (50)-(52) we obtain

$$2^n |(\eta, (v^{\alpha_n})_x)| \leqslant \frac{2^n}{\sqrt{c_1}} \sum_{k=0}^{n-1} (|v^{(+1)}|^{\alpha_k} |v|^{\alpha_{n-1} - \alpha_k} |\eta|, \ \sqrt{a^{(+1)}} |v|^{\alpha_{n-1} - \alpha_k} \left| \overset{k}{v}_k \right|) \leqslant$$

$$\leqslant M_*^{1/2-1/2^n} c_1^{-1/2} 2^n I_n^{1/2-1/2^n} \|\eta\|_2 \sum_{k=0}^{n-1} \frac{1}{\sqrt{2^{n-k-1}}} (2^{n-k-1} (a^{(+1)}, |v|^{\alpha_{n-1}-\alpha_k} (\overset{k}{v}_x)^2))^{1/2} \leqslant$$
$$\leqslant I_n^{1-1/2^n} \cdot 2^{n+1} M_*^{1/2-1/2^n} c_1^{-1/2} \|\eta\|_2.$$

Hence, because of (52), it follows that

$$2^n |(\eta, (v^{\alpha_n})_x)| \leqslant \tfrac{1}{8} I_n + (M 2^n \|\eta\|_2)^{2^n}, \tag{56}$$

where $M = M(c_1, c_6)$ is a positive constant dependent on c_1 and c_6.

Here we are considering a class of boundary conditions more general than in subsection 4, since we allow one of the cases $\sigma_1 = 0$, $\mathscr{E}_1 = 0$ or $\mathscr{E}_1 = O(h)$; $\sigma_2 = 0$, $\mathscr{E}_2 = 0$ or $\mathscr{E}_1 = O(h)$

Taking into account the inequalities

$$2^n |\bar{\nu}_2 v_N^{\alpha_n}| \leqslant \tfrac{1}{16} I_n + (M \cdot 2^n |\bar{\nu}_2|)^{2^n}, \qquad 2^n |\eta_{N-1} v_N^{\alpha_n}| \leqslant \tfrac{1}{8} I_n + (M \cdot 2^n |\eta_{N-1}|)^{2^n}$$
$$\text{for } \sigma_2 = 0,$$
$$2^n |\bar{\nu}_2 v_N^{\alpha_n}| \leqslant 2^n \sigma_2 \bar{v}_N + (M |\bar{\nu}_2|)^{2^n} \qquad \text{for } \sigma_2 \geqslant c_6 > 0,$$

and also (54), (56) we obtain the estimates

$$2^n |[\bar{\psi}, v^{\alpha_n}]| \leqslant 2^n [1, \overset{n}{v}] + M^{2^n} \|\overset{n}{\bar{\psi}}\|_{1*}, \qquad \|\overset{n}{\bar{\psi}}\|_{1*} = \|\bar{\psi}^{2^n}\|_1 + 2^{\varepsilon_1 n} |\bar{\nu}_1| + 2^{\varepsilon_2 n} |\bar{\nu}_2|, \tag{57}$$

$$2^n |[\bar{\psi}, v^{\alpha_n}]| \leqslant \tfrac{1}{2} I_n + (M \cdot 2^n \|\bar{\psi}\|_5)^{2^n}, \tag{58}$$

where $\varepsilon_s = 1$ for $\sigma_s = 0$, $s = 1, 2$, $\varepsilon_s = 0$ for $\sigma_s \geqslant c_6 > 0$, $\varepsilon_1 + \varepsilon_2 = 1$

$$\|\bar{\psi}\|_5 = \|\bar{\psi}\|_4 + |\bar{\nu}_1| + |\bar{\nu}_2|, \qquad [1, \overset{n}{v}] = (1, \overset{n}{v}) + \sigma_1 \overset{n}{v}_0 + \sigma_2 \overset{n}{v}_N.$$

We now pass on to the estimation of the expression $2^n (Q_1(v), \overset{\alpha_n}{v})$. We consider first the term $2^n (b_1 v_x + b_2 v_{\bar{x}}, \overset{\alpha_n}{v})$. From the inequalities $[a^{(+1)}, v^{\alpha_{n-1}} v_x^2) \leqslant I_n / 2^{n-1}$, $(a, v^{\alpha_{n-1}} v_{\bar{x}}^2] \leqslant I_n / 2^{n-1}$,

$$2^n |(b_1 v_x, v^{\alpha_n})| \leqslant 2^n c_4 (\overset{n-1}{v}, |v|^{\alpha_{n-1}} |v_x|) \leqslant \tfrac{1}{8} I_n + M \cdot 2^n (1, \overset{n}{v})$$

it can be seen immediately that

$$2^n |(b_1 v_x + b_2 v_{\bar{x}}, v^{\alpha_n})| \leqslant \tfrac{1}{4} I_n + M \cdot 2^n (1, \overset{n}{v}), \qquad M = M(c_1, c_4, c_6) > 0. \tag{59}$$

The estimate

$$2^n|(\bar{d}_2, \check{v}v^{\alpha_n})| \leqslant 2^n c_3 (1, \overset{n}{v}) + c_3 (1, \overset{\check{n}}{v}). \tag{60}$$

does not require explanation. Estimation of the remaining terms in the expression $2^n(Q_1(v), \overset{\alpha_n}{v})$ presents the greatest difficulties. Without any loss of generality it may be considered that $g_{22} = 0$ when $x = 0$, $x = x_{N-1}$, and $g_{11} = 0$ for $x = h$, $x = x_N = 1$. If these conditions are not satisfied, then writing, for instance, $g = g^* + Ax + g(0, t)$, where $A = [g(x_{N-1}, t) - g(0, t)]/h$, we obtain $g^* = 0$ for $x = 0$ and $x = x_{N-1}$. The coefficients of $v_{\bar{x}}$ and v for $(g_{22}v)_{\bar{x}}$ (of v_x and v for $(g_{11}v)_x$)) vary for bounded values.

The formula of summation by parts (see [2]) gives

$$((g_{22}v)_{\bar{x}}, v^{\alpha_n}) = -(g_{22}v, (v^{\alpha_n})_x) \qquad (g_{22} = 0 \text{ for } x = 0,\ x = x_{N-1}).$$

We substitute here expression (55) for $(\overset{\alpha_n}{v})_x$.

$$2^n|((g_{22}v)_{\bar{x}}, v^{\alpha_n})| \leqslant \frac{2^n c_5}{\sqrt{c_1}} \sum_{k=0}^{n-1} (1, \overset{n}{v})^{1/2} \left(a^{(+1)} \binom{k}{v_x}^2, v^{\alpha_n - \alpha_{k+1}}\right)^{1/2} \leqslant$$

$$\leqslant \frac{1}{8} I_n + M \cdot 2^{2n} (1, \overset{n}{v}), \qquad M = M(c_1, c_5) > 0.$$

Similarly we find

$$2^n|((g_{11}v)_x, v^{\alpha_n})| \leqslant \tfrac{1}{8} I_n + M \cdot 2^{2n} (1, \overset{n}{v}), \qquad M = M(c_1, c_5) > 0.$$

If boundary conditions I of the kind $v_0 = 0$, $v_N = 0$ are given for $x = 0$ and $x = 1$, then

$$2^n|((\bar{g}_{21}\check{v})_{\bar{x}}, v^{\alpha_n})| = 2^n|(\bar{g}_{21}\check{v}, (v^{\alpha_n})_x)| \leqslant \tfrac{1}{8} I_n + M \cdot 2^n (1, \overset{n}{v}) + M' (\rho, \overset{\check{n}}{v}),$$

where $M = M(c_1, c_5) > 0$, $M' = M(c_2) > 0$. A similar estimate is obtained for $2^n|((\bar{g}_{12}\check{v})_x, \overset{\alpha_n}{v})|$.

Collecting all the estimates obtained above, we arrive at the following inequalities:

a) if all $g_{sk} = 0$, $s, k = 1, 2$ then

$$2^n|[\Psi, v^{\alpha_n}]| \leqslant I_n + M_1 \cdot 2^n [1, \overset{n}{v}] + M_2^{2^n} \left\| \overset{n}{\psi} \right\|_{1^*}, \tag{61}$$

$$2^n|[\Psi, v^{\alpha_n}]| \leqslant I_n + M_1 \cdot 2^n [1, \overset{n}{v}] + (M_2 2^n \|\bar{\psi}\|_5)^{2^n} \tag{62}$$
$$\left([1, \overset{n}{v}] = \left(1, \overset{n}{v}\right) + \bar{\sigma}_1 \overset{n}{v}_0 + \sigma_2 \overset{n}{v}_N\right);$$

b) if $g_{12} = g_{21} = 0$ then

$$2^n|[\Psi, v^{\alpha_n}]| \leqslant I_n + M_1 2^{2n} [1, \overset{n}{v}] + (M_2 2^n \|\bar{\psi}\|_5)^{2^n}; \tag{63}$$

c) if $v_0 = v_N = 0$ then

$$2^n|[\Psi, v^{\alpha_n}]| \leqslant I_n + M_1 \cdot 2^n [1, \overset{n}{v}] + (M_2 \cdot 2^n \|\bar{\psi}\|_5)^{2^n} + M_3 [\check{\rho}, \overset{n}{v}], \tag{64}$$

where M_1, M_2, M_3 are positive constants depending only on c_1, c_2, c_3, c_4, c_5, c_6.

We now choose $\bar{M}$ in such a way that for a)

$$\bar{M} - d_1(1 + \bar{M}\tau) - M_1(1 + \bar{M}\tau) \geqslant \bar{M} - (c_3 + M_1)(1 + \bar{M}\tau) \geqslant 0.$$

For this it is enough to require that the following conditions are fulfilled

$$\tau < \tau_0, \text{ where } \tau_0 < \frac{1}{c_3 + M_1} - \frac{1}{\bar{M}}, \quad \bar{M} > c_3 + M_1. \tag{65}$$

In case b) (and c)) we must write $\bar{M} = M^* \times 2^n$. From (65) it can be seen that for any $n \geqslant 1$ and $M^* > c_3 + M_1$ we shall have

$$\tau_0 < \frac{1}{2(c_3 + M_1)}.$$

It will be noted that τ_0 does not depend on n (compare with [2]).

Taking now the identity (46) of the n-th rank and remembering (53), (61)-(64) we obtain the following integral inequalities

$$\text{a}_1) \quad [\bar{\rho}, \overset{n}{v}] + \tau I_n \leqslant (1 + M\tau)[\check{\rho}, \check{\overset{n}{v}}] + M_2^{2^n} \left\|\overset{n}{\bar{\psi}}\right\|_{1^*}, \tag{66}$$

$$\text{a}_2) \quad [\bar{\rho}, \overset{n}{v}] + \tau I_n \leqslant (1 + M\tau)[\check{\rho}, \check{\overset{n}{v}}] + (M_2 \cdot 2^n \|\bar{\psi}\|_5)^{2^n}; \tag{67}$$

$$\text{b), c)} \quad [\bar{\rho}, \overset{n}{v}] + \tau I_n \leqslant (1 + M\tau)[\check{\rho}, \check{\overset{n}{v}}] + M_2 \cdot 2^n \|\bar{\psi}\|_5)^{2^n} \quad (\bar{M} = M^* \cdot 2^n). \tag{68}$$

Hence we find

$$[\bar{\rho}, \overset{n}{v}(x, t)] + \sum_{t'=\tau}^{t} \tau I_n(t') \leqslant M \left\{[\bar{\rho}(x, 0), \overset{n}{v}(x, 0)] + M_2^{2^n} \sum_{t'=\tau}^{t} \tau \left\|\overset{n}{\bar{\psi}}\right\|(x, t')\|_{1^*}\right\} \tag{69}$$

and correspondingly

$$[\bar{\rho}(x,t), \overset{n}{v}(x,t)] + \sum_{t'=\tau}^{t} \tau I_n(t') \leqslant M\Big\{[\bar{\rho}(x,0), \overset{n}{v}(x,0)] + \\ + (M_2\cdot 2^n)^{2^n} \sum_{t'=\tau}^{t} \tau \|\psi(x,t')\|_5^{2n}\Big\}. \tag{70}$$

We now return to the initial function $z^j = v^j(1+\bar{M}_T)^j$. Then from inequalities (69) and (70) we shall find

$$\text{a}_1)\quad H_n(t) \leqslant M_1[\rho(x,0), \overset{n}{z}(x,0)]^{1/2^n} + M_2\Big[\sum_{t'=\tau}^{t} \tau \|\overset{n}{\psi}(x,t')\|_{1^*}\Big]^{1/2^n}, \tag{71}$$

$$\text{a}_2)\quad H_n(t) \leqslant M_1[\rho(x,0), \overset{n}{z}(x,0)]^{1/2^n} + M_2\cdot 2^n\Big[\sum_{t'=\tau}^{t} \tau \|\psi(x,t')\|_5^{2^n}\Big]^{1/2^n}, \tag{72}$$

$$\text{b), c)}\quad H_n(t) \leqslant e^{M_1\cdot 2^n}[\rho(x,0), \overset{n}{z}(x,0)]^{1/2^n} + M_2\cdot 2^n e^{M_1\cdot 2^n}\Big[\sum_{t'=\tau}^{t} \tau \|\psi(x,t')\|_5^{2^n}\Big]^{1/2^n}, \tag{73}$$

where

$$H_n(t) = [\rho(x,t), \overset{n}{z}(x,t)]^{1/2^n} + \Big[\sum_{t'=\tau}^{t} \tau \|\sqrt{a(x,t')}\, \overset{n-1}{z}_{\bar{x}}(x,t')\|_2^2\Big]^{1/2^n},$$

$$\|\sqrt{a}\, \overset{n-1}{z}_{\bar{x}}\|_2^2 = \|\sqrt{a}\, \overset{n-1}{z}_{\bar{x}}\|_2 + \sigma_1 \overset{n}{z}_0 + \sigma_2 \overset{n}{z}_N, \qquad \|\overset{n}{\psi}\|_{1^*} = \|\psi^{2^n}\|_1 + (2^{\varepsilon_1 n}|\nu_1| + 2^{\varepsilon_2 n}|\nu_2|)^{2^n}.$$

All the constants M in (71)-(73) and below are positive and depend only on $c_1, \ldots, c_7$. We shall not write out their explicit expression in terms of c_s. As a rule we shall also omit the indices of the constant M.

Theorem 2. If the conditions

$$g_{sk} = 0, \qquad s, k = 1, 2, \tag{74}$$

are fulfilled, then for the solution $z = z(x, t)$ of the problem (33)-(38) the following uniform estimates are true

$$\|z(x,t)\|_0 \leqslant M\|z(x,0)\|_0 + M\|\overline{\psi(x,t)}\|_5 \ln^{\delta}\frac{1}{h} \quad \text{when } \tau < \tau_0,\ h < h_0. \tag{75}$$

$$\|z(x,t)\|_0 \leqslant M\|z(x,0)\|_0 + M\|\overline{\psi(x,t)}\|_5 \ln^{\delta}\frac{1}{\tau} \quad \text{when } \tau < \tau_0', \tag{76}$$

where $\delta = 1 + \varepsilon$, $\varepsilon > 0$ is any number $h_0 = h_0(\varepsilon)$, $\tau_0' = \tau_0'(\varepsilon)$,

$$\overline{\|\psi(x,t)\|}_5 = \max_{\tau \leqslant t' \leqslant t} \|\psi(x,t')\|_5,$$

h_0 and τ_0' are sufficiently small quantities, and τ_0 is determined from conditions (65).

1. First we shall prove (75). If $\psi = 0$, $\nu_1 = \nu_2 = 0$, (72) gives

$$\left[\rho, \overset{n}{z}\right]^{1/2^n} \leqslant M\,[\rho(x,0), \overset{n}{z}(x,0)]^{1/2^n}.$$

Hence

$$\|z(x,t)\|_0 \leqslant M\,\|z(x,0)\|_0. \tag{77}$$

If $z(x, 0) = 0$, it follows from (72) that

$$\|z(x,t)\|_0 \leqslant M \cdot 2^n h^{-1/2^n} \overline{\|\psi(x,t)\|}_5.$$

Now choosing $n = n(h)$ as a function of h such that

$$\frac{\log_2 \frac{1}{h}}{\varepsilon \log_2 \log_2 \frac{1}{h}} \leqslant 2^n \leqslant \log_2 \frac{1}{h} \quad \text{for } h < h_0(\varepsilon), \tag{78}$$

where $\varepsilon > 0$ is any number, and taking into account that $n + 1/2^n \log_2 1/h \leqslant \delta \log_2 \log_2 1/h$ $(\delta = 1 + \varepsilon)$ we find

$$\|z(x,t)\|_0 \leqslant M\,\overline{\|\psi(x,t)\|}_5 \ln^\delta \frac{1}{h} \qquad \text{for } h < h_0,\ \tau < \tau_0,. \tag{79}$$

Combining (77) and (79) we obtain inequality (75).

2. To prove the estimate (76) we use (72) and the inequalities

$$\left\|\overset{n}{z}\right\|_0 \leqslant M\left(\sigma_1 \overset{n}{z}_0 + \sigma_2 \overset{n}{z}_N + \left\|\sqrt{a}\,\overset{n-1}{z}_{\bar{x}}\right\|_2^2\right) = M\left\|\sqrt{a}\,\overset{n-1}{z}_{\bar{x}}\right\|_2^2, \qquad M = M(c_1, c_8),$$

$$\sum_{t'=\tau}^{t} \tau \left\|\overset{n}{z}(x,t')\right\|_0 \leqslant M \sum_{t'=\tau}^{t} \tau \left\|\sqrt{a}\,\overset{n-1}{z}_{\bar{x}}\right\|_2^2. \tag{80}$$

It is enough to confine oneself to the case $z(x, 0) = 0$. It follows from (72) and (80) that

$$\left(\sum_{t'=\tau}^{t} \tau \|z(x,t')\|_0^{2^n}\right)^{1/2^n} \leqslant M \cdot 2^n \overline{\|\psi(x,t)\|}_5, \quad \|z(x,t)\|_0 \leqslant M \cdot 2^n \tau^{-1/2^n} \overline{\|\psi(x,t)\|}_5.$$

Now choosing $n = n(\tau)$ as in (78) we obtain

$$\|z(x,t)\|_0 \leqslant M\,\overline{\|\psi(x,t)\|_5}\ln^\delta\frac{1}{\tau} \qquad \text{for } \tau<\tau_0'.$$

Note. For $n = 1$ from (72) follows the mean estimate

$$\|z(x,t)\|_2 \leqslant M\|z(x,0)\|_2 + M\Big[\sum_{t'=\tau}^{t}\tau\|\psi(x,t')\|_5^2\Big]^{1/2} \qquad \text{for } \tau<\tau_0. \tag{81}$$

For the first boundary condition $z_0 = z_N = 0$

$$\|z(x,t)\|_2 \leqslant M\|z(x,0)\|_2 + M\Big[\sum_{t'=\tau}^{t}\tau\|\psi(x,t')\|_3^2\Big]^{1/2} \qquad \text{for } \tau<\tau_0, \tag{81'}$$

if $g_{sk} \neq 0$, s, $k = 1, 2$.

Theorem 3. If the conditions (74) are fulfilled and $\delta_2 = 0$, then for the solution $z = z(x, t)$ of the problem (33)-(38) the following uniform estimates are fulfilled:

$$\|z(x,t)\|_0 \leqslant M\,(\|z(x,0)\|_0 + \overline{\|\psi(x_1t)\|_0} + \overline{|\nu_1(t)|}) + M\,\overline{|\nu_2(t)|}\ln^\delta\frac{1}{h} \tag{82}$$
$$\text{for } \tau<\tau_0,\ h<h_0,$$

$$\|z(x,t)\|_0 \leqslant M\,(\|z(x,0)\|_0 + \overline{\|\psi(x,t)\|_0} + \overline{|\nu_1(t)|}) + M\,\overline{|\nu_2(t)|}\ln^\delta\frac{1}{\tau} \quad \text{for } \tau<\tau_0', \tag{82'}$$

where

$$\overline{\|\psi(x,t)\|_0} = \max_{\tau\leqslant t'\leqslant t}\|\psi(x,t')\|_0, \qquad \overline{|\nu_s(t)|} = \max_{\tau\leqslant t'\leqslant t}|\nu_s(t')|, \qquad s=1,2,\ \delta>1.$$

1. Let $\nu_2(t) \equiv 0$. Then limiting transition can be carried out in (71) for $n \to \infty$, taking into account that τ_0 and the constants in (71) do not depend on n, and h and τ are fixed. Remembering that

$$\Big\|\overset{n}{\psi}\Big\|_{1^*}^{1/2^n} \leqslant \|\psi\|_0 + |\nu_1|,$$

we obtain (82) for $\nu_2 \equiv 0$.

2. Let $\nu_1 = 0$, $\psi = 0$, $z(x, 0) = 0$. Then it follows from (71)

$$\|z(x,t)\|_0 \leqslant M\cdot 2^n h^{-1/2^n}\,\overline{|\nu_2(t)|}.$$

Now choosing $n = n(h)$ in accordance with (78) we find

$$\|z(x,t)\|_0 \leqslant M\,\overline{|\nu_2(t)|}\ln^\delta\frac{1}{h} \text{ for } h<h_0, \quad \tau<\tau_0.$$

Hence and from 1 we get (82).

3. To prove (82′) we use (54) for $\nu_1 = \psi = z(x, 0) \equiv 0$ and (80). It follows from (71) and (80) that

$$\| z(x, t) \|_0 \leqslant M \cdot 2^n \tau^{-1/2^n} \overline{|\nu_2(t)|}.$$

Choosing $n = n(\tau)$ in accordance with (78) we obtain

$$\| z(x, t) \|_0 \leqslant M \overline{|\nu_2(t)|} \ln^\delta \frac{1}{\tau} \qquad \text{for } \tau < \tau_0'.$$

Note. Theorem 1 is proved for the conditions $\mathscr{E}_s + \sigma_s \geqslant c > 0$, $\sigma_1 + \sigma_2 \geqslant c_6 > 0$, $\sigma_s \geqslant 0$, $\mathscr{E}_s \geqslant 0$, $s = 1, 2$. Theorem 3 is true, for example, when $\mathscr{E}_2 + \sigma_2 = 0$ or $\mathscr{E}_2 + \sigma_2 = O(h)$ and hence theorem 1 cannot be used.

Theorem 4. Let the conditions

$$g_{12} = g_{21} = 0.$$

be fulfilled. Then for the solution of the problem (33)-(38) the following estimates are true

$$\| z(x, t) \|_0 \leqslant \{M_1 \| z(x, 0) \|_0 + M_2 \| \overline{\psi(x, t)} \|_5\} \exp\left(M_3 \sqrt{\ln \frac{1}{h}}\right) \text{ for } \tau < \tau_0,\ h < h_0, \tag{83}$$

$$\| z(x, t) \|_0 \leqslant \{M_1 \| z(x, 0) \|_0 + M_2 \| \overline{\psi(x, t)} \|_5\} \exp\left(M_3 \sqrt{\ln \frac{1}{\tau}}\right) \text{ for } \tau < \tau_0'. \tag{84}$$

1. From (73) we find for any n

$$\| z(x, t) \|_0 \leqslant e^{M \cdot 2^n} h^{-1/2^n} (\| z(x, 0) \|_0 + M \cdot 2^n \| \overline{\psi(x, t)} \|_5). \tag{85}$$

We now consider the expression

$$2^n \exp\left(M \cdot 2^n + \ln \frac{1}{h} \Big/ 2^n\right) = \exp\left(M \cdot 2^n + \ln \frac{1}{h} \Big/ 2^n - n \ln 2\right).$$

It can be seen from the condition $M \times 2^n \sim \ln(1/h)/2^n$ that $2^n \sim \sqrt{\ln(1/h)} / \sqrt{M}$.

Selecting correspondingly $n = n(h) \sim \ln \ln(1/h)$, we obtain $M \times 2^n + \ln(1/h)/2^n - n \ln 2 \leqslant M_3 \sqrt{\ln(1/h)}$ for a sufficiently small $h < h_0$. Hence and from (85) we get the inequality (83).

2. Estimate (84) is obtained similarly if (80) is taken into account. It is true for all values of h.

Note 1. Theorem 4 is true in the case of boundary conditions I of the kind ($z_0 = 0$, $z_N = 0$), even if the conditions $g_{12} = g_{21} = 0$ are not fulfilled.

Note 2. Writing $n = 1$ in (73) it is easy to see that for the solution of the problem (33)-(38) the mean estimate

$$\|z(x, t)\|_2 \leqslant M\|z(x, 0)\|_2 + M\Big[\sum_{t'=\tau}^{t} \|\psi(x, t')\|_5^2\Big]^{1/2}.$$

is true. It is valid even when $g_{sk} \neq 0$, $s, k = 1, 2$.

All *a priori* estimates in this subsection have been obtained on the assumption of the boundedness of the coefficients a, b_s, d_s, g_{sk} only. For this reason they can be used for the proof of the convergence in the case of moving (oblique) discontinuities of the coefficients of the differential equation (6).

6. *A priori estimates for a six-point scheme*

The following problem was considered in [2] and [4]

$$\overline{\mathcal{P}}_{h\tau}^{\alpha} z = \rho z_{\bar{t}} - (a z_{\bar{x}})_x^{(\alpha)} - Q(z) = \psi \qquad (0.5 \leqslant \alpha \leqslant 1), \tag{86}$$

$$l_1 z = (a^{(+1)} z_x - \sigma_1 z)^{(\alpha)} - \mathcal{E}_1 z_{\bar{t}} = \nu_1 \quad \text{for } x = 0;$$

$$l_2 z = (a z_{\bar{x}} + \sigma_2 z)^{(\alpha)} + \mathcal{E}_2 z_{\bar{t}} = -\nu_2 \quad \text{for } x = 1; \tag{87}$$

$$z(x, 0) = z_0(x), \tag{88}$$

$$Q(z) = b_{11} z_x + b_{12} \check{z}_x + b_{22} z_{\bar{x}} + b_{21} \check{z}_{\bar{x}} + d_1 z + d_2 \check{z}, \tag{89}$$

$$0 < c_1 \leqslant a \leqslant c_1', \qquad 0 < c_2 \leqslant \rho, \qquad |d_s| \leqslant c_3, \qquad |b_{sk}| \leqslant c_4,$$

$$\sigma_s \geqslant 0, \quad \sigma_1 + \sigma_2 \geqslant c_5 > 0, \qquad s, k = 1, 2, \tag{90}$$

$$|a_{\bar{t}}| \leqslant c_6, \quad |(\sigma_s)_{\bar{t}}| \leqslant c_7 \sigma_s, \quad 0 < c_8 h \leqslant \mathcal{E}_s \leqslant c_9. \tag{91}$$

The following *a priori* estimates were obtained

1) $$\|z(x, t)\|_0 \leqslant M\Big\{\|z_{\bar{x}}(x, 0)\|_2 + \Big(\sum_{t'=\tau}^{t} \tau \|\psi(x, t')\|_2^2\Big)^{1/2}\Big\}, \tag{92}$$

where

$\|\psi\|_{\bar{2}} = \|\psi\|_2 + |\nu_1| / \sqrt{\mathcal{E}_1} + |\nu_2| / \sqrt{\mathcal{E}_2}$;

2) if $\psi = \overline{\psi}^{(\alpha)}$, $\nu_s = \overline{\nu}_s^{(\alpha)}$, $s = 1, 2$, $z(x, 0) = z_0(x) = 0$, then

$$\|z(x,t)\|_0 \leqslant M\left\{\|\bar\psi(x,0)\|_5 + \|\bar\psi(x,t)\|_{5^*} + \left[\sum_{t'=\tau}^{t}\tau(\|\bar\psi(x,t')\|_{5^*} + \|\bar\psi_{\bar t}(x,t')\|_{5^*})^2\right]^{1/2}\right\} +$$
$$+ \bar M\left[\sum_{t'=\tau}^{t}\tau\|\bar\psi(x,t')\|_5^2\right]^{1/2}, \quad \bar M = 0 \text{ for } b_{sk} = 0. \tag{93}$$

Here $\|\bar\psi(x,t)\|_5 = \|\bar\psi(x,t)\|_4 + |\bar\nu_1| + |\bar\nu_2|$, $\|\bar\psi\|_{5^*} = \|\bar\psi\|_{4^*} + |\bar\nu_1| + |\bar\nu_2|$, $\|\psi\|_4 = \|\psi\|_3 + |(\psi,1)|$, $\|\psi\|_{4^*} = \|\psi\|_{3^*} + |(\psi,1)|$, $\|\psi\|_3 = \|\eta\|_2$, $\|\psi\|_{3^*} = \|\eta\|_1$, $\eta(x) = \sum_{x'=h}^{x} h\psi(x')$.

We shall show that estimate (93) can be improved if b_{sk} satisfy Lipshits's condition for t:

$$|(b_{sk})_{\bar t}| \leqslant c_{10}. \tag{94}$$

Theorem 5. Let $z(x,t)$ be the solution of the problem (86)-(91), where

$$\psi = \bar\psi^{(\alpha)}, \quad \nu_s = \bar\nu_s^{(\alpha)}, \quad Q(z) = (b_1 z_x + b_2 z_{\bar x})^{(\alpha)} + d_1 z + d_2 \check z, \quad |b_s| \leqslant c_4. \tag{95}$$

If the conditions

$$|(b_s)_{\bar t}| \leqslant c_{10}, \qquad 0.5 \leqslant \alpha \leqslant 1, \qquad \mathscr{E}_s + \sigma_s \geqslant c_* > 0, \quad s = 1, 2, \tag{96}$$

are fulfilled, for a sufficiently small $\tau < \tau_0$

$$\|z(x,t)\|_0 \leqslant M\left\{\|\bar\psi(x,0)\|_5 + \|\bar\psi(x,t)\|_{5^*} + \left[\sum_{t'=\tau}^{t}(\|\bar\psi(x,t')\|_{5^*} + \|\bar\psi_{\bar t}(x,t')\|_{5^*})^2\right]^{1/2}\right\}. \tag{97}$$

For an implicit scheme for $\alpha = 1$ the following estimate is true

$$\|z(x,t)\|_0 \leqslant M\left\{\|\bar\psi(x,0)\|_{5^*} + \|\bar\psi(x,t)\|_{5^*} + \left[\sum_{t'=\tau}^{t}\tau(\|\bar\psi(x,t')\|_{5^*} + \|\bar\psi_{\bar t}(x,t')\|_{5^*})^2\right]^{1/2}\right\}. \tag{98}$$

Let us express z in the form of a sum, $z = \mu + v + w$, where w is a solution of the stationary problem $(aw_{\bar x})_x = -\bar\psi$, $a_1 w_{\bar x,0} - \sigma_1 w_0 = \bar\nu_1$, $a_N w_{\bar x,N} + \sigma_2 w_N = -\bar\nu_2$, and μ and v are defined by the conditions

$$\bar{\mathscr{P}}^{\alpha}_{h\tau}\mu = \eta, \quad l_1\mu = \mathscr{E}_1 w_{\bar t}, \quad l_2\mu = -\mathscr{E}_2 w_{\bar t}, \qquad \mu(x,0) = -w(x,0), \tag{99}$$

$$\overline{\mathcal{P}}^{\alpha}_{h\tau}v = \overline{\eta}^{(\alpha)}, \qquad l_1 v = 0, \qquad l_2 v = 0, \qquad v(x, 0) = 0, \tag{100}$$
$$\eta = d_1 w + d_2 \check{w} - \rho w_{\bar{t}}, \qquad \overline{\eta} = b_1 w_x + b_2 w_{\bar{x}}.$$

In [4] it was shown that

$$\|w\|_0 \leqslant M\|\overline{\psi}\|_{5^*}, \quad \|w_{\bar{x}}\|_1 \leqslant M\|\overline{\psi}\|_{5^*}, \quad \|w_{\bar{x}\bar{t}}\|_1 \leqslant M\|\overline{\psi}_{\bar{t}}\|_{5^*}, \quad \|w_{\bar{t}}\|_0 \leqslant M\|\overline{\psi}_{\bar{t}}\|_{5^*},$$
$$\|w_{\bar{x}}\|_2 \leqslant M\|\overline{\psi}\|_5. \tag{101}$$

For estimating $\mu(x, t)$ we use (92), and for v the inequality (93), taking into account (101) and the obvious inequalities

$$\|\eta\|_2 \leqslant M(\|\overline{\psi}\|_{5^*} + \|\overline{\psi}_{\bar{t}}\|_{5^*}),$$
$$\|\overline{\eta}\|_5 = \|\overline{\eta}\|_4 \leqslant M\|w_{\bar{x}}\|_1 \leqslant M\|\overline{\psi}\|_{5^*}, \quad \|\dot{\eta}\|_{5^*} = \|\dot{\eta}\|_{4^*} \leqslant M\|\overline{\psi}\|_{5^*}, \quad \|\eta_{\bar{t}}\|_{5^*} \leqslant M\|\overline{\psi}_{\bar{t}}\|_{5^*}.$$

For μ we obtain (93) for $\overline{M} = 0$, and for v (97) for $\overline{\psi}(x, 0) = 0$. Collecting the estimates for w, μ and v we arrive at (97).

When $\alpha = 1$ we express μ in the form of a sum, $\mu = \overline{\mu} + \overline{\overline{\mu}}$, where $\overline{\mu}$ is the solution of the problem (99) with the initial condition $\overline{\mu}(x, 0) = 0$, and $\overline{\overline{\mu}}$ is found from the conditions $\overline{\mathcal{P}}^{\alpha}_{h\tau}\overline{\overline{\mu}} = 0$, $l_1\overline{\overline{\mu}} = l_2\overline{\overline{\mu}} = 0$, $\overline{\overline{\mu}}(x, 0) = -\overline{w}(x, 0)$. Now using theorem 1 we obtain

$$\|\overline{\overline{\mu}}(x, t)\|_0 \leqslant M\|w(x, 0)\|_0 \leqslant M\|\overline{\psi}(x, 0)\|_{5^*}.$$

Hence the inequality (98) is true for μ, as for v.

Note. If $Q(z)$ has the form (89) and conditions (94) are fulfilled, then an estimate of the same type as (97) is true. Also, instead of $\|\overline{\psi}\|_{5^*} + \|\overline{\psi}_{\bar{t}}\|_{5^*}$ under the summation sign we shall have the expression $\|\overline{\psi}\|_{5^*} + \|\overline{\psi}_{\bar{t}}\|_{5^*} + \tau\|\overline{\psi}\|_5$.

7. *Accuracy in the class of discontinuous coefficients*

To establish the order of accuracy of the scheme (14)-(16) it is enough to use one of the *a priori* estimates obtained in subsections 5 and 6. Together with (1) and (6) we shall consider the following equations of the parabolic type

$$\mathcal{P}_3 u = \frac{\partial}{\partial x}\left(k(x, t)\frac{\partial u}{\partial x}\right) + r(x, t)\frac{\partial u}{\partial x} + f\left(x, t, u, \frac{\partial u}{\partial t}\right) = 0,$$

$$\mathcal{P}_4 u = \frac{\partial}{\partial x}\left(k(x, t)\frac{\partial u}{\partial x}\right) + f\left(x, t, u, \frac{\partial u}{\partial x}\right) - c(x, t)\frac{\partial u}{\partial t} = 0.$$

Adding conditions (7)-(10) to the equation $\mathcal{P}_s u = 0$ ($s = 1, 2, 3, 4$) we shall obtain a problem which we shall denote by $(\mathrm{I}_s^{(\alpha)})$. The corresponding difference problem, which is formulated by analogy with (14)-(16) (for $(\mathrm{I}_1^{(\alpha)})$ see [5]), will be denoted by $(\mathrm{II}_s^{(\alpha)})$. Finally, the problem for $z = y - u$ will be denoted by $(\mathrm{III}_s^{(\alpha)})$.

To simplify the formulation, instead of using the words "the solution of the problem converges uniformly to the solution of the problem $(\mathrm{I}_s^{(\alpha)})$ and has the order of accuracy $O(h^m) + O(\tau)$", we shall say "the scheme converges uniformly at the rate $O(h^m) + O(\tau)$." Only unconditionally stable schemes, i.e. $0.5 \leqslant \alpha \leqslant 1$ are considered everywhere. Unless specified otherwise all statements relate to the entire family of schemes.

Theorem 6. The homogeneous difference schemes $(\mathrm{II}_s^{(\alpha)})$, $s = 1, 3, 4$ converge uniformly at the rate $O(h^2) + O(\tau^{m_\alpha})$ for $0.5 \leqslant \alpha \leqslant 1$ ($m_\alpha = 2$ for $\alpha = 0.5$ and $m_\alpha = 1$ for $\alpha \neq 0.5$), if the conditions A_α are fulfilled in the entire region $\bar{D}$, or more accurately, for a sufficiently small $\tau < \tau_0$ the following estimate is true

$$\|y - u\|_0 \leqslant M(h^2 + \tau^{m_\alpha}), \tag{102}$$

where M is a constant independent of h and τ. For the scheme $(\mathrm{II}_2^{(1)})$ (for $\alpha = 1$) the following estimates are fulfilled

$$\|y - u\|_2 \leqslant M(h^2 + \tau) \text{ for } \tau < \tau_0, \tag{103}$$

$$\|y - u\|_0 \leqslant M(h^{2-\rho(h)} + \tau^{1-\rho(\tau)}) \text{ for } \tau < \tau_0, h < h_0, \tag{104}$$

where $\rho(h) \sim 1/\sqrt{\ln(1/h)} \to 0$ for $h \to 0$, $\rho(\tau) \sim 1/\sqrt{\ln(1/\tau)} \to 0$ for $\tau \to 0$.

Inequality (102) follows from (92), and (103) and (104) from (83)-(85) (theorem 4).

We now pass on to the problem of accuracy in the class of discontinuous coefficients. We shall distinguish two cases: a) all discontinuities fixed (FD), i.e. $\eta_\nu'(t) \equiv 0$ for $\nu = 1, 2, \ldots, \nu_0$; b) at least one discontinuity moving (MD), i.e. $\eta_\nu'(t) \neq 0$ for at least one $\nu = 1, 2, \ldots, \nu_0$. In [5] an estimate is given of the accuracy of the scheme $(\mathrm{II}_1^{(\alpha)})$ in the case FD. It was assumed in this case that conditions A_α are satisfied in each of the regions $\bar{\Delta}_\nu$, and besides that the limiting values of the functions k, k', k'', u', u'', u''', $\partial^2 u/\partial x \partial t$ satisfy Lipshits's conditions for t along each of the straight lines Γ_ν ($\nu = 1, 2, \ldots, \nu_0$). It has been shown (theorem 4 in [5]) that

$$\|y-u\|_0 \leqslant M(h^{\varkappa_1}+\tau^{m_\alpha}) \text{ for } \tau<\tau_0, \tag{105}$$

where $\kappa_1 = 0.5$ for the entire family of schemes, and $\kappa_1 = 1.5$ for the scheme (12)-(13). From theorem 5 and the estimates for the error of approximation near the line of discontinuity given in [5] we have the following:

Theorem 7. The difference scheme $(\mathrm{II}_3^{(1)})$ in the class of coefficients of the equation $\mathcal{P}u = 0$ having fixed discontinuities converges uniformly at the rate $O(h^{\kappa_2}) + O(\tau)$, i.e.

$$\|y-u\|_0 \leqslant M(h^{\varkappa_2}+\tau) \text{ for } \tau<\tau_0,$$

where $\kappa_2 = 1$ for the entire family of schemes $(\mathrm{II}^{(1)})$, $\kappa_2 = 2$ for the scheme (12)-(13), if the coefficient $r(x, t)$ satisfies Lipshits's condition for t in $\bar{\Delta}_\nu$, $\nu = 1, 2, \ldots, \nu_0$.

It can be seen from (106) that the scheme (12) has the same accuracy in the case of discontinuous coefficients as for smooth coefficients.

Using theorem 4 it can be easily shown that scheme $(\mathrm{II}_2^{(1)})$ in the case FD converges in the mean at the rate $O(h^{\kappa_1}) + O(\tau)$ and converges uniformly $O(h^{\varkappa_1-\rho(h)}) + O(\tau^{1-\rho(\tau)})$, i.e.

$$\|y-u\|_0 \leqslant M(h^{\varkappa_1-\rho(h)}+\tau^{1-\rho(\tau)}) \text{ for } h<h_0, \quad \tau<\tau_0,$$

where $\rho(h) \sim 1/\sqrt{\ln(1/h)}$, $\rho(\tau) \sim 1/\sqrt{\ln(1/\tau)}$.

For the scheme $(\mathrm{II}_2^{(\alpha)})$ for $\alpha \neq 1$ it is not possible to prove uniform convergence even in the class of uniform coefficients.

Now let us consider the problem of the convergence and the order of accuracy in the class of coefficients having moving discontinuities (MD). It is assumed everywhere that in each of the regions $\bar{\Delta}_\nu$ the conditions A_α are fulfilled. Convergence is proved only for the schemes $\mathrm{II}_2^{(1)}$ and $\mathrm{II}_4^{(1)}$ $(\alpha = 1)$. We shall assume that $c(x, t)$ can have only moving discontinuities. In [3] the convergence was proved for a special case of the scheme $(\mathrm{II}_4^{(1)})$ for $f = f(x, t) - q(x, t)u$, $q \geqslant 0$. Theorems 2 and 4 can be used, first of all, to improve the estimate of the order of accuracy obtained in [3], and secondly to prove the following theorem:

Theorem 8. The difference schemes $(\mathrm{II}_2^{(1)})$ and $(\mathrm{II}_4^{(1)})$ converge uniformly in the class of coefficients having moving discontinuities, so that the following estimates are true:

1) for the scheme ($\mathrm{II}_4^{(1)}$)

$$\|y-u\|_0 \leqslant M\left(h^{\varkappa_3}\ln^\delta\frac{1}{h}+\tau\ln^\delta\frac{1}{\tau}\right) \text{ for } h<h_0,\ \ \tau<\tau_0;$$

2) for the scheme ($\mathrm{II}_2^{(1)}$)

$$\|y-u\|_0 \leqslant M\left(h^{\varkappa_3-\rho(h)}+\tau^{1-\rho(\tau)}\right) \text{ for } h<h_0,\ \ \tau<\tau_0,$$

where $\kappa_3 = 0.5$ for the entire family of schemes, $\kappa_3 = 1$ for the scheme (12), (13), $\rho(h)\sim 1/\sqrt{\ln(1/h)}$, $\rho(\tau)\sim 1/\sqrt{\ln(1/\tau)}$, $\delta>1$.

To prove theorem 8 all arguments given in [3] are repeated, lemmas 2, 3 and 4 of [3] being improved by means of theorems 2 and 4.

Note. All results of this subsection, in accordance with [5], remain true in the case of boundary conditions of a more general type (for example, for (2) and (5)). It is not necessary to deal with the proof of the corresponding theorems. We would only mention that *a priori* estimates (theorems 1-5) have been obtained for boundary conditions of a very general type, and the method of describing difference boundary conditions with order of approximation $O(h^2) + O(\tau^{m_\alpha})$ is described in [5].

Here we shall not deal with the question of the methods of solving non-linear difference equations for the schemes ($\mathrm{II}_1^{(\alpha)}$), ($\mathrm{II}_2^{(\alpha)}$), ($\mathrm{II}_3^{(1)}$) (see [5]).

2. Multidimensional parabolic equations

1. Formulation of the problem

Let $x = (x_1, \ldots, x_p)$ be a point in a p-dimensional space with the coordinates $x_1, \ldots, x_p$, $\bar{G} = \{0 \leqslant x_\alpha \leqslant l_\alpha,\ \alpha = 1, 2, \ldots, p\}$ a p-dimensional parallelepiped with the boundary Γ, $G = \bar{G} - \Gamma$, $\bar{Q}_T = \bar{G} \times [0 \leqslant t \leqslant T]$, $Q_T = G \times (0 < t \leqslant T)$. The solution of the following problem is sought in the cylinder $\bar{Q}_T$

$$c(x,t)\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} L_\alpha u + f\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_p}\right), \quad (x,t)\in Q_T, \tag{1}$$

$$u|_\Gamma = \chi(x,t) \qquad (x\in\Gamma,\ 0\leqslant t\leqslant T), \tag{2}$$

$$u(x,0) = u_0(x), \qquad x\in\bar{G}, \tag{3}$$

where

$$L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t,u)\frac{\partial u}{\partial x_\alpha}\right), \tag{4}$$

$c(x,t) = c(x_1, x_2, \ldots, x_p, t)$, $k_\alpha = k_\alpha(x,t,u)$, $f = f(x, t, u, \lambda_1, \ldots, \lambda_p)$,

$\chi = \chi(x, t)$, $u_0(x)$ are given functions. We shall assume that

$$k_\alpha(x, t, u) \geqslant c_1 > 0, \qquad c(x, t) \geqslant c_2 > 0; \tag{5}$$

$k_\alpha(x, t, u)$ and $f(x, t, u, \lambda_1, \ldots, \lambda_p)$ have continuous derivatives with respect to $u, \lambda_1, \ldots, \lambda_p$. If the coefficient $k_\alpha = k_\alpha(x, t, u)$ has a discontinuity of kind I on the plane $x_\alpha = \xi_\alpha = \text{const.}$, the following conjugation conditions (continuity of u and $k_\alpha(\partial u/\partial x_\alpha)$ are fulfilled:

$$\begin{gathered}[u]_\alpha = u(x_1, \ldots, x_{\alpha-1}, x_\alpha + 0, x_{\alpha+1}, \ldots, x_p, t) - \\ - u(x_1, \ldots, x_{\alpha-1}, x_\alpha - 0, x_{\alpha+1}, \ldots, x_p, t) = 0, \\ \left[k_\alpha \frac{\partial u}{\partial x_\alpha}\right] = 0, \qquad x_\alpha = \xi_\alpha.\end{gathered} \tag{6}$$

As usual we assume that the problem (1)-(5) has a unique solution continuous in $\bar{Q}_T$ and having the number of derivatives required in the course of description. The following conditions will be used.

Conditions A: 1) the solution of the problem (1)-(5) has derivatives $\partial^4 u/\partial x_\alpha^4$, $\partial^2 u/\partial t^2$, $\alpha = 1, 2, \ldots, p$ uniformly bounded inside Q_T; 2) the functions $c(x, t)$, $k_\alpha(x, t, u)$, $f(x, t, u, \lambda_1, \ldots, \lambda_p)$ have derivatives $\partial^2 c/\partial x_\alpha^2$, $\partial^3 k_\alpha/\partial x_\alpha^3$, $\partial^2 f/\partial x_\alpha^2$, $\alpha = 1, 2, \ldots, p$ uniformly bounded in Q_T.

Conditions B: if c, k and f have a finite number of discontinuities of kind I on the hyperplanes $x_\alpha = \xi_\alpha^{(s)} = \text{const.}$, $s = 1, 2, \ldots, p$, $\alpha = 1, 2, \ldots, p$, then $\partial^4 u/\partial x_\alpha^4$, $\partial^2 u/\partial t^2$, $\partial^2 c/\partial x_\alpha^2$, $\partial^2 f/\partial x_\alpha^2$, $\partial^3 k_\alpha/\partial x_\alpha^3$ are uniformly bounded inside each of the regions into which $\bar{Q}_T$ is partitioned by these hyperplanes.

These conditions are sufficient for proving the theorem on the accuracy of the difference schemes considered below. In a number of cases they can be replaced by weaker requirements. But here we do not intend to carry out our investigations on the solution and the coefficients of the problem (1)-(5) under the minimum conditions. This would require more complex methods of investigation and would be beyond the scope of the paper.

2. *Difference networks and network functions*

The planes $x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha$, $i_\alpha = 0, 1, \ldots, N_\alpha$, $\alpha = 1, 2, \ldots, p$, $h_\alpha = l_\alpha/N_\alpha$ partition the region $\bar{G}$ into parallelepipeds with vertices at the points $x_i = (x_1^{(i_1)}, \ldots, x_\alpha^{(i_\alpha)}, \ldots, x_p^{(i_p)})$. We introduce the notations $\omega_h = \{x_i \in \bar{G}\}$, $\omega_h = \{x_i \in G\}$, $\gamma = \{x_i \in \Gamma\}$, $\bar{\omega}_h = \omega_h + \gamma$.

We break up the segment $0 \leqslant t \leqslant T$ by the points $t_j = j \times \tau$,

$j = 0, 1, \ldots, K$ into K intervals of length $\tau = T/K$. Let $\bar{\omega}_\tau = \{t_j = j \times \tau \in [0 \leqslant t \leqslant T]\}$, $\omega_\tau = \{t_j = j \times \tau,\ j = 1, 2, \ldots, K\}$, $\bar{\Omega} = \bar{\omega}_h \times \bar{\omega}_\tau = \{(x_i, t_j) \in \bar{Q}_T\}$, $\Omega = \omega_h \times \omega_\tau = \{(x_i, t_j) \in Q_T\}$ be a space-time network. Below we shall as a rule use a system of notation without indices, writing

$$x = x_i, \qquad t = t_{j+1}, \qquad \check{t} = t_j, \qquad y = y(x, t) = y(x_i, t_{j+1}) = y_i^{j+1},$$

$$x^{(\pm 1_\alpha)} = x_i^{(\pm 1_\alpha)} = (x_1^{(i_1)}, \ldots, x_{\alpha-1}^{(i_{\alpha-1})}, x_\alpha^{(i_\alpha)} \pm h_\alpha, x_{\alpha+1}^{(i_{\alpha+1})}, \ldots, x_p^{(i_p)}),$$

$$y^{(\pm 1_\alpha)} = y(x^{(\pm 1_\alpha)}, t),$$

$$y_{\bar{x}_\alpha} = \frac{y - y^{(-1_\alpha)}}{h_\alpha}, \quad y_{x_\alpha} = \frac{y^{(+1_\alpha)} - y}{h_\alpha} = y_{\underset{x_\alpha}{(+1_\alpha)}}, \quad \check{y} = y(x, t - \tau), \quad y_{\bar{t}} = \frac{y - \check{y}}{\tau}$$

$$(y, v) = \sum_{\omega_h} y \cdot v h_1 \ldots h_p = \sum_{i_1=1}^{N_1-1} h_1 \ldots \sum_{i_\alpha=1}^{N_\alpha-1} h_\alpha \ldots \sum_{i_p=1}^{N_p-1} h_p y_i v_i,$$

$$(y, v]_\alpha = \sum_{i_1=1}^{N_1-1} h_1 \ldots \sum_{i_{\alpha-1}=1}^{N_{\alpha-1}-1} h_{\alpha-1} \sum_{i_\alpha=1}^{N_\alpha} h_\alpha \sum_{i_{\alpha+1}=1}^{N_{\alpha+1}-1} h_{\alpha+1} \ldots \sum_{i_p=1}^{N_p-1} h_p y_i v_i,$$

where y and v are any functions given on the network $\bar{\Omega}$. We use the following norms (compare § 1) for the estimation of network functions

$$\|y\|_0 = \max_{\bar{\omega}_h} |y|, \qquad \|y\|_\sigma = (|y|^\sigma, 1)^{1/\sigma} \text{ or } \|y\|_{\sigma_\alpha} = (|y|^\sigma, 1]_\alpha^{1/\sigma}, \qquad \sigma = 1, 2,$$

$$\|y\|_{3_\alpha} = \|\eta_\alpha\|_2, \qquad \eta_\alpha(x, t) = \sum_{x'_\alpha = h_\alpha}^{x_\alpha} y(x_1, \ldots, x_{\alpha-1}, x'_\alpha, x_{\alpha+1}, \ldots, x_p, t) h_\alpha,$$

$$\|y\|_3 = \sum_{\alpha=1}^{p} \|y\|_{3_\alpha}.$$

3. *Homogeneous difference schemes*

We shall consider implicit forward homogeneous schemes, which we shall construct by analogy with the one-dimensional case $p = 1$. We introduce the pattern functionals $A^{(\alpha)}[\mu(s)]$, $\alpha = 1, 2, \ldots, p$, $s = (s_1, s_2, \ldots, s_p)$ defined in the class of piecewise continuous functions $\mu(s)$ given in a p-dimensional parallelepiped $-0.5 \leqslant s_\beta \leqslant 0.5$, $\beta \neq \alpha$, $-1 \leqslant s_\alpha \leqslant 0$. For simplicity we consider that $A^{(\alpha)}$ is independent of $h_1, \ldots, h_p$, i.e. we consider canonical functionals (and canonical schemes). We shall assume that $A^{(\alpha)}[\mu]$ is a non-decreasing ($A^{(\alpha)}[\mu_2] \geqslant A^{(\alpha)}[\mu_1]$ for $\mu_2 \geqslant \mu_1$),

normalized ($A^{(\alpha)}[1] = 1$), homogeneous functional of the first degree ($A^{(\alpha)}[c\mu] = cA^{(\alpha)}[\mu]$, $c = \text{const} > 0$) having a differential of the third order and satisfying the conditions $A_1^{(\alpha)}[s_\beta] = 0$ for $\beta \neq \alpha$, $A_1^{(\alpha)}[s_\alpha] = -0.5$, where $A_1^{(\alpha)}[\eta] = A_1^{(\alpha)}[1, \eta]$ is the first differential of the functional $A^{(\alpha)}[\mu]$ at the point $\mu = 1$. We associate the family of homogeneous three-point difference schemes

$$\Lambda_\alpha y = (a_\alpha(x, t, y^*) y_{\bar{x}_\alpha})_{x_\alpha}, \qquad y^* = 0.5(y^{(-1_\alpha)} + y), \tag{7}$$

with the difference operator $L_\alpha u$. The coefficients of the schemes are determined with the help of the functional $A^{(\alpha)}$ from the law

$$a_\alpha = a_\alpha(x, t, y^*) = A^{(\alpha)}[k_\alpha(x_1 + s_1 h_1, \ldots, x_\alpha + s_\alpha h_\alpha, \ldots, x_p + s_p h_p, t, y^*)]. \tag{8}$$

It is easy to see that the scheme $\Lambda_\alpha y$ has, because of the assumptions made above regarding $A^{(\alpha)}$, the second order of approximation

$$\Lambda_\alpha u - L_\alpha u = O(h^2), \text{ where } h^2 = \frac{1}{p}\sum_{\alpha=1}^{p} h_\alpha^2. \tag{9}$$

The simplest family of schemes $\Lambda_\alpha y$ is formed by schemes, the coefficients of which are calculated by using the functional $A[\mu(s_\alpha)]$, where $\mu(s_\alpha)$ is a function of one variable s_α, $-1 \leqslant s_\alpha \leqslant 0$, so that

$$a_\alpha = A[k_\alpha(x_1, \ldots, x_{\alpha-1}, x_\alpha + s_\alpha h_\alpha, x_{\alpha+1}, \ldots, x_p, t, y^*)], \tag{8'}$$

Here $A[\mu(s_\alpha)]$ is the pattern functional used in § 1, subsection 2, for the construction of one-dimensional difference schemes of the second order of approximation. Evidently

$$A[\mu(s_\alpha)] = A^{(\alpha)}[\mu(0, 0, \ldots, 0, s_\alpha, 0, \ldots, 0)].$$

For the approximation of the function $f(x, t, u, \lambda_1, \ldots, \lambda_p)$ on the network we use by analogy with the case $p = 1$ (§ 1, subsection 2) the linear functionals

$$F[\mu(s)] = F[\mu(s_1, \ldots, s_p)], \qquad -0.5 \leqslant s_\alpha \leqslant 0.5, \quad \alpha = 1, 2, \ldots, p,$$

assuming that $F[\mu(s)] \geqslant 0$ for $\mu(s) \geqslant 0$, $F[1] = 1$, $F[s_\alpha] = 0$, $\alpha = 1, 2, \ldots, p$. The function $f(x, t, u, \lambda_1, \ldots, \lambda_p)$ is associated with the network function

$$\varphi = \varphi(x, t, u, \lambda_1, \ldots, \lambda_p) = F[f(x_1 + s_1 h_1, \ldots, x_p + s_p h_p, t, u, \lambda_1, \ldots, \lambda_p)], \tag{10}$$

$$\varphi(x, t, u, \lambda_1, \ldots, \lambda_p) - f(x, t, u, \lambda_1, \ldots, \lambda_p) = O(h^2), \quad h^2 = \frac{1}{p}\sum_{\alpha=1}^{p} h_\alpha^2.$$

An analogue of the one-dimensional scheme (12), § 1 (see [1]) for $p > 1$ is, in particular, a scheme with a pattern functional of the type

$$\left.\begin{aligned} A^{(\alpha)}[\mu(s_1, \ldots, s_p)] &= B_{p-1}^{(\alpha)}[A_\alpha^*[\mu(s_1, \ldots, s_p)]], \\ A_\alpha^*[\mu(s_1, \ldots, s_p)] &= \left[\int_{-1}^{0} \frac{ds_\alpha}{\mu(s_1, \ldots, s_\alpha, \ldots, s_p)}\right]^{-1}, \end{aligned}\right\} \tag{11}$$

where $B_{p-1}^{(\alpha)}[\eta(s_1, \ldots, s_{\alpha-1}, s_{\alpha+1}, \ldots, s_p)]$ is a linear normalized ($B_{p-1}^{(\alpha)}[1] = 1$), non-negative ($B_{p-1}^{(\alpha)}[\eta] \geqslant 0$ for $\eta \geqslant 0$) functional defined in a class of piecewise continuous functions of the $(p-1)$-th variable given inside a $(p-1)$-dimensional cube $-0.5 \leqslant s_\beta \leqslant 0.5$, $\beta \neq \alpha$, $\beta = 1, 2, \ldots, \alpha - 1, \alpha + 1, \ldots, p$. Simultaneously $B_{p-1}^{(\alpha)}$ satisfies the condition

$$B_{p-1}^{(\alpha)}[s_\beta] = 0 \text{ for } \beta \neq \alpha, \quad \beta = 1, 2, \ldots, \alpha - 1, \alpha + 1, \ldots, p.$$

For example, $B_{p-1}^{(\alpha)}[\eta] \equiv 1$, or

$$B_{p-1}^{(\alpha)}[\eta] = \int_{-0.5}^{0.5} ds_1 \ldots \int_{-0.5}^{0.5} ds_{\alpha-1} \int_{-0.5}^{0.5} ds_{\alpha+1} \ldots \int_{-0.5}^{0.5} ds_p \eta(s_1, \ldots, s_{\alpha-1}, s_{\alpha+1}, \ldots, s_p).$$

The analogue of the one-dimensional functional $F[\mu(s)] = \int_{-0.5}^{0.5} \mu(s)\,ds$ is the functional

$$F[\mu(s)] = \int_{-0.5}^{0.5} ds_1 \ldots \int_{-0.5}^{0.5} ds_p \mu(s_1, \ldots, s_p). \tag{12}$$

In [5] it was explained that for the construction of difference schemes with maximum accuracy in a class of discontinuous coefficients it is convenient to transform the function f to the form

$$f = f\left(x, t, u, 2k_1 \frac{\partial u}{\partial x_1}, \ldots, 2k_p \frac{\partial u}{\partial x_p}\right), \tag{13}$$

This is always possible because $k_\alpha \geqslant c_1 > 0$. The flux $2k_\alpha(\partial u/\partial x_\alpha)$ is approximated by the expression

$$\lambda_\alpha(y) = a_\alpha^{(+1_\alpha)} y_{x_\alpha} + a_\alpha y_{\bar{x}_\alpha} \sim \lambda_\alpha^0(u) = 2k_\alpha \frac{\partial u}{\partial x_\alpha}. \tag{14}$$

We shall also consider schemes for which

$$f = f\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_\alpha}, \ldots, \frac{\partial u}{\partial x_p}\right)$$
$$\lambda_\alpha(y) = y_{\overset{\circ}{x}_\alpha} = 0.5\,(y_{x_\alpha} + y_{\bar{x}_\alpha}) \sim \lambda_\alpha^0(u) = \frac{\partial u}{\partial x_\alpha}. \tag{15}$$

4. The difference boundary problem

We now pass on to the formulation of the difference problem corresponding to the initial problem (1)-(5). We have to find the network function $y(x, t)$ defined in Ω and satisfying the equation

$$\rho y_{\bar{t}} = \Lambda y + \varphi(x, t, y, \lambda_1(y), \ldots, \lambda_p(y)) \text{ in } \Omega, \tag{16}$$

the boundary condition

$$y = \chi(x, t) \text{ for } x \in \gamma, \quad t \in \bar{\omega}_\tau \tag{17}$$

and the initial condition

$$y(x, 0) = u_0(x) \text{ for } t = 0, \quad x \in \bar{\omega}_h. \tag{18}$$

Here

$$\Lambda y = \sum_{\alpha=1}^{p} \Lambda_\alpha y, \qquad \Lambda_\alpha y = (a_\alpha(x, t, y^*)\, y_{\bar{x}_\alpha})_{x_\alpha}, \tag{19}$$

$\lambda_\alpha(y)$ is defined by one of the expressions (14) or (15), $\rho = \rho(x, t) = F[c(x_1 + s_1 h_1, \ldots, x_p + s_p h_p, t)]$.

Let u be the solution of the problem (1)-(5), y the solution of the problem (16)-(19). We shall find the equation for the error $z = y - u$. Substituting $y = z + u$ in (16) and reasoning as in § 1, subsection 4, we obtain

$$\rho z_{\bar{t}} = \bar{\Lambda} z - dz + Q(z) + \psi, \tag{20}$$

$$\left.\begin{aligned} z &= 0 \text{ for } x \in \gamma, \quad t \in \bar{\omega}_\tau, \\ z &= 0 \text{ for } t = 0, \quad x \in \omega_h, \end{aligned}\right\} \tag{21}$$

$$\bar{\Lambda} z = \sum_{\alpha=1}^{p} \bar{\Lambda}_\alpha z, \qquad \bar{\Lambda}_\alpha z = (a_\alpha(x, t, y^*)\, z_{\bar{x}_\alpha})_{x_\alpha}, \tag{22}$$

$$Q(z) = \sum_{\alpha=1}^{p} Q_\alpha(z), \qquad Q_\alpha(z) = b_\alpha \lambda_\alpha(z) + (g_\alpha z^*)_{x_\alpha}, \tag{23}$$

$$d = -\overline{\frac{\partial \varphi}{\partial u}}, \qquad b_\alpha = \overline{\frac{\partial \varphi}{\partial \lambda_\alpha}}, \qquad g_\alpha = \overline{\frac{\partial a_\alpha}{\partial u}} u_{\bar{x}_\alpha}.$$

The vinculum indicates that the derivatives are taken for some mean arguments u, λ_α. The error of approximation has the form:

$$\psi = \sum_{\alpha=1}^{p} \left[(a_\alpha(x, t, u^*) u_{\bar{x}})_{x_\alpha} - \frac{\partial}{\partial x_\alpha} \left(k_\alpha(x, t, u) \frac{\partial u}{\partial x_\alpha} \right) \right] - \left[\rho(x, t) u_{\bar{t}} - c(x, t) \frac{\partial u}{\partial t} \right] +$$
$$+ [\varphi(x, t, u, \lambda_1(u), \ldots, \lambda_p(u)) - f(x, t, u, \lambda_1^0(u), \ldots, \lambda_p^0(u))].$$

Because of the conditions for f, c, k_α, u the coefficients a_α, ρ, d, b_α, g_α are bounded:

$$a_\alpha \geqslant c_1 > 0, \quad \rho \geqslant c_2 \geqslant 0, \quad |d| \leqslant c_3, \quad |b_\alpha| \leqslant c_4, \quad |g_\alpha| \leqslant c_5, \tag{24}$$

where c_1, c_2, c_3, c_4, c_5 are positive constants independent of h_α and τ. ($u = u(x, t)$ is a given solution of the problem (1)-(5). Outside the region of variation of u, $\lambda_\alpha(u)$ it is possible to define $k(x, t, u)$, $f(x, t, u, \lambda_1, \ldots, \lambda_p)$ in such a way that $\partial x/\partial u$, $\partial f/\partial u$, $\partial f/\partial \lambda_\alpha$ ($\alpha = 1, \ldots, p$) are bounded.)

If the conditions A (§ 2, subsection 1) are fulfilled,

$$\psi = O(h^2) + O(\tau) \quad \text{or} \quad \|\psi\|_0 \leqslant M(h^2 + \tau). \tag{25}$$

5. A priori estimates

To determine the order of accuracy of the scheme (16) the solution of the problem (20)-(24) must be determined in terms of the error of approximation. We shall consider a problem of a more general type:

$$\rho z_{\bar{t}} = \Lambda z + Q(z) + \psi \text{ in } \Omega, \tag{26}$$

$$z = 0 \text{ for } x \in \gamma, \quad t \in \omega_\tau, \qquad z = 0 \text{ for } t = 0, \quad x \in \omega_h, \tag{27}$$

$$\Lambda z = \sum_{\alpha=1}^{p} (a_\alpha z_{\bar{x}_\alpha})_{x_\alpha} - d_1 z, \tag{28}$$

$$Q(z) = \sum_{\alpha=1}^{p} [\bar{b}_\alpha z_{x_\alpha} + \bar{\bar{b}}_\alpha z_{\bar{x}_\alpha} + (g_{11}^{(\alpha)} z)_{x_\alpha} + (g_{22}^{(\alpha)} z)_{\bar{x}_\alpha} + (g_{12}^{(\alpha)} \check{z})_{x_\alpha} + (g_{21}^{(\alpha)} \check{z})_{\bar{x}_\alpha}] + d_2 \check{z}, \tag{29}$$

$$\rho \geqslant c_1 > 0, \quad a_\alpha \geqslant c_1 > 0, \quad |d_s| \leqslant c_2, \quad |\bar{b}_\alpha| \leqslant c_3, \quad |\bar{\bar{b}}_\alpha| \leqslant c_3, \quad |g_{sk}^{(\alpha)}| \leqslant c_4,$$
$$s,k = 1, 2, \quad \alpha = 1, \ldots, p. \tag{30}$$

We shall assume that

$$|\rho_{\bar{t}}| \leqslant c_5. \tag{31}$$

Theorem 9. If

$$g_{sk}^{(\alpha)} = 0, \qquad s, k = 1, 2, \quad \alpha = 1, \ldots, p, \tag{32}$$

then the following estimate is true for the solution of the problem (26)-(30)

$$\|z(x, t)\|_0 \leqslant M \overline{\|\psi(x, t)\|}_0 \quad \text{for } \tau < \tau_0, \tag{33}$$

where

$$M = M(c_1, c_2, c_3), \qquad \overline{\|\psi(x, t)\|}_0 = \max_{(\tau \leqslant t' \leqslant t)} \|\psi(x, t')\|_0.$$

If, moreover, condition (31) is satisfied, then for sufficiently small $\tau < \tau_0$ and $h < h_0$ the inequality

$$\|z(x, t)\|_0 \leqslant M \overline{\|\psi(x, t)\|}_{3_\alpha} \ln^\delta (1/H), \qquad \alpha = 1, 2, \ldots, p, \tag{34}$$

is true. Here $H = h_1 \ldots h_p$, $a\|\psi(x, t)\|_{3_\alpha}$ is given in subsection 1, $\delta > 1$.

Theorem 10. Let $z(x, t)$ be the solution of the problem (26)-(31). Then for a sufficiently small $\tau < \tau_0$ the following estimate in the mean is true

$$\|z(x, t)\|_2 \leqslant M \left[\sum_{t'=\tau}^{t} \tau \|\psi(x, t')\|_{3_\alpha}^2 \right]^{1/2}, \tag{35}$$

where α is any of the numbers $\alpha = 1, 2, \ldots, p$. If, furthermore, $h < h_0$ is sufficiently small and $g_{12}^{(\alpha)} = g_{21}^{(\alpha)} = 0$,

$$\|z(x, t)\|_0 \leqslant M \overline{\|\psi(x, t)\|}_{3_\alpha} \exp(M \sqrt{\ln(1/H)}), \qquad H = h_1 \ldots h_p. \tag{36}$$

These theorems are a simple generalization of theorems 2 and 4 proved in § 1. The method of proving them is the same as for $p = 1$, only the description is somewhat complicated. First the transformation

$z^j = v^j(1 + \bar{M}\tau)^j$ is carried out. For v we obtain the same problem (26)-(30), and the coefficient $d_1 \geqslant M^* > 0$, if $\tau < \tau_0$, where M^* is any preassigned number. $\bar{M}$ is expressed in terms of M^* and the constants $c_1 - c_5$. To simplify the account we shall assume that this transformation has been carried out already, and shall retain the former notation z for the required function. By analogy with the one-dimensional case we shall write the integral identity of the n-th rank for the function $\overset{n}{z} = z^{2^n}$:

$$(\rho, \overset{n}{z})_{\bar{t}} + 2\sum_{\alpha=1}^{p} I_n^{(\alpha)} + \sum_{\alpha=1}^{p} P_n^{(\alpha)} + 2^n(d_1, \overset{n}{z}) = 2^n(\Psi, z^{\alpha_n}) + (\rho_{\bar{t}}, \overset{n}{z}), \quad (37)$$

$$\Psi = Q(z) + \psi, \qquad P_n^{(\alpha)} = \tau\sum_{k=0}^{n-1} 2^{n-k-1}(\rho(\overset{k}{z}_{\bar{t}})^2, z^{\alpha_n - \alpha_k + 1}), \quad (38)$$

$$I_n^{(\alpha)} = (a, (\overset{n-1}{z}_{\bar{x}_\alpha})^2]_\alpha + \sum_{k=0}^{n-2} 2^{n-k-2}\{(a_\alpha(\overset{k}{z}_{\bar{x}_\alpha})^2, z^{\alpha_n - \alpha_k + 1}) + \quad (39)$$
$$+ (a_\alpha^{(+1_\alpha)}(\overset{k}{z}_{x_\alpha})^2, z^{\alpha_n - \alpha_k + 1})\}.$$

The integral inequality of the n-th rank and all estimates of the right hand side are derived as in subsection 5 § 1 for $p = 1$. Since we are considering the first boundary problem here, the arguments are considerably simplified. To avoid unnecessary repetition (of § 1) we shall not give the proofs of theorems 9 and 10.

It should be noted that the specific form of the region G (parallelepiped) is not used in deriving the identity of the n-th rank. This identity is true on the rectangular network ω_h which approximates the arbitrary region G.

We now pass on to the question of the stability of the solution of equation (26) with respect to the boundary and initial data.

The equation of the first rank has the form

$$\rho w_{\bar{t}} - \Lambda w + R(z) + 2d_1 w = 2zQ(z) + 2z\psi \qquad (w = z^2),$$
$$R(z) = \tau\rho z_{\bar{t}}^2 + \sum_{\alpha=1}^{p}(a_\alpha z_{\bar{x}_\alpha}^2 + a_\alpha^{(+1_\alpha)} z_{x_\alpha}^2). \quad (40)$$

The principle of the maximum is true for this equation when $g_{sk}^{(\alpha)} = 0$ and therefore the following is true.

Theorem 11. If $g_{sk}^{(\alpha)} = 0$, $s, k = 1, 2$, $\alpha = 1, 2, \ldots, p$, the solution of equation (26) defined in $\bar{\Omega}$ is stable with respect to the right hand

side, the initial and boundary values, so that for sufficiently small $\tau < \tau_0$ the inequality

$$\|z(x,t)\|_0 \leqslant M_1(\|z(x,0)\|_0 + \|\overline{z(x,t)}\|_{0,\gamma}) + M_2\left[\sum_{t'=\tau}^{t} \tau \|\psi(x,t')\|_0^2\right]^{1/2}, \quad (41)$$

is true. Here τ_0 and M_1, M_2 are positive constants depending only on c_i and τ, $i = 1, \ldots, 4$ $\|z(x,t)\|_{0,\gamma} = \max\limits_{(x\in\gamma)} |z(x,t)|$.

This theorem is proved by analogy with the proof of theorem 1. There is no need, therefore, to dwell on it.

It will be noted that the first statement (33) of theorem 3 follows from (41).

We now consider the general equation (26), when $g_{sk}^{(\alpha)} \neq 0$. We express its solution in the form of the sum $z = v + w$, where v and w are determined by the conditions

$$v_{\bar{t}} - \Lambda v + d_1 v = Q_1(v) + \psi, \quad (42)$$

$$v = z \text{ for } x \in \gamma,\ t \in \bar{\omega}_\tau; \qquad v(x,0) = z(x,0) \text{ for } x \in \bar{\omega}_h,$$

$$Q_1(v) = \sum_{\alpha=1}^{n} (\bar{b}_\alpha v_{x_\alpha} + \bar{\bar{b}}_\alpha v_{\bar{x}_\alpha}) = Q(v) - Q_2(v),$$

$$w_{\bar{t}} - \Lambda w + d_1 w = Q(w) + Q_2(w),$$

$$w = 0 \text{ for } x \in \gamma,\ t \in \omega_\tau, \qquad w(x,0) = 0 \text{ for } x \in \omega_h.$$

The estimate (41) is evidently true for $v(x, t)$. To estimate w we use the integral estimate (37) of the n-th rank. We transform terms of the type $2^n((gv)_{x_\alpha}, w^{\alpha_n})$ on the right hand side as follows:

$$2^n((gv)_{x_\alpha}, w^{\alpha_n}) = -2^n(gv, (w^{\alpha_n})_{\bar{x}_\alpha}), \qquad \alpha_n = 2^n - 1.$$

Hence, using the formula for $(w^{\alpha_n})_{\bar{x}_\alpha}$ we find

$$2^n|((gv)_{x_\alpha}, w^{\alpha_n})| \leqslant 2^n M(1, \overset{n}{w}) + \tfrac{1}{4} I_n^{(x)} + (M \cdot 2^n \|v\|_0)^{2^n}.$$

As a result we have the following estimates:

$$\left.\begin{aligned} &\|w(x,t)\|_2 \leqslant M\overline{\|v(x,t)\|_0}, \\ &\|w(x,t)\|_0 \leqslant M\overline{\|v(x,t)\|_0} \exp(M\sqrt{\ln(1/H)}), \qquad H = h_1 \ldots h_p \end{aligned}\right\} \quad (43)$$

Combining (41) and (43) we find that the following is true.

Theorem 12. Let $z(x, t)$ be the solution of equation (26) defined on the network $\bar{\Omega}$. If conditions (30) and (31) are fulfilled, the following inequalities are true:

1) for a sufficiently small $\tau < \tau_0$

$$\|z(x, t)\|_2 \leqslant M\left\{\|z(x, 0)\|_0 + \overline{\|z(x, t)\|}_{0,\gamma} + \left[\sum_{t'=\tau}^{t} \tau \|\psi(x, t')\|_{3\alpha}^2\right]^{1/2}\right\}; \quad (44)$$

2) for sufficiently small $\tau < \tau_0$ and $h < h_0$

$$\|z(x, t)\|_0 \leqslant M\{\|z(x, 0)\|_0 + \overline{\|z(x, t)\|}_{0,\gamma} + \overline{\|\psi(x, t)\|}_{3\alpha}\}\exp(M\sqrt{\ln(1/H)}). \quad (45)$$

The second estimate contains a factor dependent on $H = h_1, \ldots, h_p$. If will be used in dealing with the problem of the uniform convergence of difference schemes corresponding to quasilinear equations of the parabolic type with the coefficient of heat conduction $k = k(x, t, u)$ dependent on the required function $u = u(x, t)$.

6. *The convergence and order of accuracy of homogeneous schemes for a quasilinear equation*

We now return to the problem (20)-(24). It is a special case of the problem (26)-(30) considered in subsection 5. We can therefore use the *a priori* estimates used in that subsection and enunciate a number of theorems on the convergence and the order of accuracy of the solution of the problem (16)-(19). Particular attention is paid to the proof of uniform convergence.

Theorem 13. If conditions A are satisfied in Q_T and $k_\alpha = k_\alpha(x, t)$ does not depend on u, the solution $y = y(x, t)$ of the problem (16)-(19) converges uniformly to the solution $u = u(x, t)$ of the problem (1)-(5) as h_α ($\alpha = 1, \ldots, p$) and τ independently tend to zero in such a way that for sufficiently small $\tau < \tau_0$

$$\|y - u\|_0 \leqslant M(h^2 + \tau), \qquad h^2 = \frac{1}{p}\sum_{\alpha=1}^{p} h_\alpha^2. \quad (46)$$

Theorem 14. If the conditions A are fulfilled in Q_T and $k_\alpha = k_\alpha(x, t, u)$ depends on u, the solution of the problem (16)-(19) converges to the solution of the problem (1)-(5) in such a way that

1) for a sufficiently small $\tau < \tau_0$

$$\|y - u\|_2 \leqslant M(h^2 + \tau) \text{ (convergence in the mean)}; \quad (47)$$

2) for sufficiently small $\tau < \tau_0$ and $h < h_0$

$$\|y-u\|_0 \leqslant M_1(h^2+\tau)\exp(M_2\sqrt{\ln(1/H)})\,\text{(uniform convergence)}\,. \quad (48)$$

Both theorems are true for any $p \geqslant 1$. Estimate (48) cannot be improved even for $p = 1$.

To prove theorem (13) it is enough to use condition (25) and estimate (33) of theorem 9. Theorem 14 follows from theorem 12 and (25).

The *a priori* estimates (34), (35) and (44), (45) (theorems 9, 10, 12) can be used to prove the convergence of the difference scheme (16) in the case of discontinuous coefficients k_α and f, because these estimates were obtained on the assumption of the boundedness of the coefficients of equation (1) only. Let us assume that the coefficient $k_\alpha(x, t, u)$ has a finite number of discontinuities of kind I for $x_\alpha = \xi_\alpha = \text{const}$. Then the conjugation conditions (5) are fulfilled along the hyperplane $x_\alpha = \xi_\alpha$, and the scheme $\Lambda_\alpha u$, as we have seen in [1], [3], [5] does not approximate the operator $L_\alpha u$ in the neighbourhood $x_\alpha = \xi_\alpha$. It is sufficient to consider the case when, for example, the coefficient $k_1(x, t, u)$ has a discontinuity for $x_1 = \xi_1 = x_1^{(n_1)} + \theta_1 h_1$, $0 \leqslant \theta_1 \leqslant 1$, $x_1^{(n_1)} = n_1 h_1$. We shall assume that in each of the regions into which the hyperplane $x_1 = \xi_1$ divides the cylinder $\bar{Q}_T$ the conditions A are fulfilled, i.e. the conditions B are valid. In calculating the error of approximation at the points $x_{n_1} = (x_1^{(n_1)}, x_2, \ldots, x_p)$ and $x_{n_1+1} = (x_1^{(n_1+1)}, x_2, \ldots, x_p)$ it is sufficient to confine oneself to a study of the term

$$\psi_1 = \Lambda_1 u - L_1 u$$

(for simplicity we assume that $c(x, t)$ and f are continuous for $x_1 = \xi_1$), since $\psi = \psi_1 = Q(h^2) + Q(\tau)$ at the points $x = x_{n_1}$ and $x = x_{n_1+1}$.

Moreover, we have $\psi = Q(h^2) + Q(\tau)$ everywhere in Ω, in addition to the points (x_{n_1}, t) and (x_{n_1+1}, t), where $t \in \omega_\tau$. Noting that ψ_1 is the error of approximation of the one-dimensional scheme $\Lambda_1 u$, and using the results of [1] and [5] we obtain

$$1)\ h_1\psi_1(x_{n_1}, t) = O(1), \qquad \psi_1(x_{n_1}, t) + \psi_1(x_{n_1+1}, t) = O(1) \qquad (49)$$

for an arbitrary scheme from the initial family of difference schemes defined in subsection 2,

$$2)\ \psi_1(x_{n_1}, t) = O(1), \qquad \psi_1(x_{n_1}, t) + \psi_1(x_{n_1+1}, t) = O(h_1) \qquad (50)$$

for the best schemes defined by pattern functionals of the type (11) or (12).

It follows from (25), (49)-(50) that:

for any scheme (16)

$$h_1 \psi(x_{n_1}, t) = O(1), \quad \psi(x_{n_1}, t) + \psi(x_{n_1+1}, t) = O(1) \tag{51}$$

for the scheme (11)-(12)

$$h_1 \psi(x_{n_1}, t) = O(h_1), \quad \psi(x_{n_1}, t) + \psi(x_{n_1+1}, t) = O(h_1). \tag{52}$$

If the functions $f(x, t, u, \lambda_1, \ldots, \lambda_p)$ and $c(x, t)$ also have discontinuities of kind I for $x_1 = \xi_1 = x_1^{(n_1)} + \theta_1 h_1$, $0 \leqslant \theta_1 \leqslant h_1$, $x_1^{(n_1)} = n_1 h_1$, conditions (51) are satisfied for any scheme and any of the representations (14) or (15). For a scheme defined by the functionals (11) and (12), and also the expression (14) for $\lambda_\alpha(y)$, conditions (52) are satisfied.

At the remaining points of the network

$$\psi = O(h^2) + O(\tau) \text{ for } x \neq x_{n_1}, \quad x \neq x_{n_1+1}$$

for any of the schemes of the family considered.

By analogy with § 3 of [5] we find

$$\|\psi\|_{3_1} \leqslant M\{h^2 + \tau + h_1^{3/2} |\psi(x_{n_1}, t)| + h_1 |\psi(x_{n_1}, t) + \psi(x_{n_1+1}, t)|\}, \tag{53}$$

where M is a constant independent of h and τ (and dependent, in particular, on the volume of the region G).

Now taking theorems 9 and 10 in turn we see that the following statements are true:

a) if none of the coefficients $k_\alpha = k_\alpha(x, t)$ depends on u, and each of the functions k_α, c, f has a finite number of discontinuities of kind I for $x_\alpha = \xi_\alpha^{(s)} = \text{const.}$, $s = 1, 2, \ldots, m_\alpha$, $\alpha = 1, 2, \ldots, p$, the solution of the problem (16)-(19) converges uniformly to the solution of the problem (1)-(5) as $h_\alpha \to 0$ and $\tau \to 0$ such that for sufficiently small $h < h_0$ and $\tau < \tau_0$ the estimate

$$\|y - u\|_0 \leqslant M(h_1^\varkappa \ln^\delta(1/H) + h^2 + \tau), \tag{54}$$

is true. Here $\varkappa = 0.5$ for any initial scheme, $\varkappa = 1.5$ for schemes (11), (12), $\delta > 1$.

b) if even one function $k_\alpha = k_\alpha(x, t, u)$ depends on u, for sufficiently small $\tau < \tau_0$ we have

$$\|y-u\|_2 \leqslant M(h_1^{\varkappa}+h^2+\tau) \tag{55}$$

and

$$\|y-u\|_0 \leqslant M(h_1^{\varkappa}+h^2+\tau)\exp(M\sqrt{\ln(1/H)}) \text{ for } \tau<\tau_0,\ \ h<h_0. \tag{56}$$

In a number of particular cases the estimates given of the order of accuracy of the schemes considered can be improved.

With the method of studying the convergence described here similar estimates can be easily obtained for an arbitrary region with a sufficiently smooth boundary. The main difficulty here will arise in the case where the conditions on the boundary Γ of the region G are transferred to the boundary γ of the network ω_h by means of linear interpolation (see [9]). This method of specifying boundary conditions is easier to use for the so-called method of fractional steps (see [8]). This method will be formulated and its convergence and accuracy studied in a subsequent paper.

We now take the third boundary problem. Instead of (2) let the following conditions be given

$$k_\alpha \frac{\partial u}{\partial x_\alpha} - \sigma_{1\alpha}u = f_{1\alpha} \text{ for } x_\alpha = 0, \quad k_\alpha \frac{\partial u}{\partial x_\alpha} + \sigma_{2\alpha}u = f_{2\alpha} \text{ for } x_\alpha = l_\alpha, \alpha = 1, 2, \ldots, p.$$

The difference boundary conditions $a_\alpha^{(+1_\alpha)} y_{x_\alpha} - \sigma_{1\alpha}y = f_{1\alpha}$ for $x_\alpha = 0$, $a_\alpha y_{\bar{x}_\alpha} + \sigma_{2\alpha}y = f_{2\alpha}$ for $x_\alpha = l_\alpha$ have a first order of approximation. The following estimates are true for the corresponding difference problem

$$\|y-u\|_0 \leqslant M(h+\tau) \text{ instead of (46)},$$
$$\|y-u\|_0 \leqslant M(h_1^{\varkappa}\ln^{\delta}(1/H)+h+\tau) \text{ instead of (54)},$$
$$\|y-u\|_0 \leqslant M(h_1^{\varkappa}+h+\tau)\exp(M\sqrt{\ln(1/H)}) \text{ instead of (56)}.$$

It is interesting to study boundary conditions of kind III with the second order of approximation (compare [5]).

7. *Some remarks*

1. If $k_\alpha = k_\alpha(x, t)$ has a derivative $\partial k_\alpha/\partial t$ bounded in Q_T, and hence $|(a_\alpha)_{\bar{t}}| \leqslant M$, theorem 5 can be generalized for the multidimensional case. In particular it follows from this that for the linear equation (1) for

$$k_\alpha = k_\alpha(x, t), \qquad f = f(x, t) + \sum_{\alpha=1}^{p} r_\alpha(x, t)\frac{\partial u}{\partial x_\alpha} - q(x, t)\, u$$

the estimates (54) and (55) can be improved. In particular instead of (54) we obtain

$$\|y - u\|_0 \leqslant M\left(h_1^{\varkappa_2} \ln \frac{1}{H} + h^2 + \tau\right),$$

where $\kappa_2 = 1$ for the entire family of initial schemes and $\kappa_2 = 2$ for the scheme (11)-(12). Without going into the proof of this we shall only note that for $\beta \neq \alpha$, $\partial k_\alpha/\partial x_\beta$ is bounded along the hyperplane $x_\alpha = \xi_\alpha =$ const. on which k_α has a discontinuity of kind I. It is also assumed that f, r_α and $q(x, t)$ have finite derivatives with respect to t.

2. For the linear parabolic equation

$$\sum_{\alpha=1}^{p} L_\alpha u + f\left(x, t, u, \lambda_1^0(u), \ldots, \lambda_p^0(u), \frac{\partial u}{\partial t}\right) = 0,$$

$$L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t)\frac{\partial u}{\partial x_\alpha}\right),$$

where

$$\lambda_\alpha^0(u) = \frac{\partial u}{\partial x_\alpha} \quad \text{or} \quad \lambda_\alpha^0(u) = 2k_\alpha \frac{\partial u}{\partial x_\alpha},$$

the investigation is carried out as for the one-dimensional case (see [5]).

Translated by R. Feinstein

REFERENCES

1. Tikhonov, A.N. and Samarskii, A.A., *Zh. vych. mat.* 1, No. 1, 4-63, 1961.
2. Samarskii, A.A., *Zh. vych. mat.* 1, No. 3, 441-460, 1961.
3. Samarskii, A.A. and Fryazinov, I.V., *Zh. vych. mat.* 1, No. 5, 806-824, 1961.
4. Samarskii, A.A., *Zh. vych. mat.* 1, No. 6, 971-1000, 1961.
5. Samarskii, A.A., *Zh. vych. mat.* 2, No. 1, 25-56, 1962.

6. Saul'ev, V.K., Integrirovaniye uravnenii parabolicheskogo tipa metodom setok (Integration of equations of the parabolic type by the network method). Fizmatgiz, Moscow-Leningrad, 1960.

7. Ladyzhenskaya, O.A., *Uspeke matem. nauk*, 13, 123-148, 1957.

8. Yanenko, N.N., *Dokl. Akad. Nauk SSSR*, 125, No. 6, 1207-1210, 1959.

9. Berezin, I.S. and Zhidkov, N.P., Metody vychislenii (Computational methods), Vol. 2. Fizmatgiz, Moscow, 1960.