HOMOGENEOUS DIFFERENCE SCHEMES ON NON-UNIFORM NETS*

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Non-uniform nets are widely used in the solution of various differential equations by difference methods. However, little has been done to study the convergence of the difference schemes.

The simplest examples show that the most frequently used criteria for judging the quality of difference schemes, a uniform estimate or a mean estimate of the approximation error of the scheme, are unsound for non-uniform nets and can give an untrue idea of the order of accuracy of the scheme.

For the differential equation

$$Lu = u'' + f(x) = 0 \quad (0 < x < 1), \quad u(0) = u_1, \quad u(1) = u_2 \tag{1}$$

let us consider two difference schemes on an arbitrary non-uniform net $\omega_N = \{x_i, i = 0, 1, \ldots, N, x_0 = 0, x_N = 1\}$:

$$1. \quad \Lambda_1 y_i = \frac{1}{\hbar_i}\left(\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right) + f_i = 0, \quad y_0 = u_1, \quad y_N = u_2,$$

where $y_i = y(x_i)$, $h_i = x_i - x_{i-1}$, $\hbar_i = 0.5\,(h_i + h_{i+1})$; the approximation error of this scheme is:

$$\psi_1 = \Lambda_1 u_i - Lu_i = \frac{h_{i+1} - h_i}{3} u_i''' + O(h_i^2) + O(h_{i+1}^2),$$

i.e. the scheme has first order approximation. It will be shown in § 2, however, that this scheme gives second order accuracy on an arbitrary

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non-uniform net

$$\|y - u\|_0 = \max_{1 \leqslant i \leqslant N-1} |y_i - u(x_i)| \leqslant M\bar{h}^2, \tag{2}$$

where $\bar{h}$ is the mean square mesh of the net:

$$\bar{h} = \left(\sum_{i=1}^{N} h_i^2 h_i\right)^{1/2}.$$

2. The following scheme is sometimes used:

$$\Lambda_2 y = \frac{1}{\hbar_{i+1}}\left(\frac{\bar{y}_{i+1} - \bar{y}_i}{\hbar_{i+1}} - \frac{\bar{y}_i - \bar{y}_{i-1}}{\hbar_i}\right) + f_i = 0, \qquad y(0) = u_1, \quad y(1) = u_2,$$

where

$$h_i = x_i - x_{i-1}, \quad \hbar_i = 0.5(h_i + h_{i+1}), \quad \bar{y}_i = y(\bar{x}_i), \quad \bar{x}_i = 0.5\,(x_i + x_{i+1}).$$

Calculation gives

$$\psi_i = \Lambda_2 \bar{u}_i - (Lu)\big|_{x=\bar{x}_i} = \frac{h_{i+2} - 2h_{i+1} + h_i}{4h_{i+1}} u_i'' + O(h_i) + O(h_{i+1}) + O(h_{i+2}),$$

i.e. generally speaking the scheme does not approximate to equation (1) if the net is arbitrary. However, it is shown in point 5 of § 2 that

$$\|\bar{y} - u(\bar{x})\|_0 \leqslant M\|h\|_0^2 \quad \|h\|_0 = \max h_i \text{ (Theorem 4)}, \tag{3}$$

i.e. scheme Λ_2 has second order accuracy on an arbitrary net.

It is not difficult to see that both schemes are the same for a uniform net $h_i = h = 1/N$.

It is shown in [1] that for homogeneous schemes for the differential equation

$$\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) - q(x)\,u + f(x) = 0$$

with discontinuous coefficients, the accuracy of the difference scheme is determined in the end by the integral approximation error, characterised by the norm

$$\|\psi\|_3 = \sum_{i=1}^{N} h\left|\sum_{k=1}^{i} \psi_k h\right|. \tag{4}$$

and not by the local error.

This type of norm, as we shall explain, is also suitable for estimating the error of homogeneous difference schemes on non-uniform nets.

We shall examine homogeneous difference schemes on non-uniform nets using the boundary problem

$$\left.\begin{aligned} L^{(k,q,f)}u = (k(x)u')' - q(x)u + f(x) = 0, \quad 0<x<1,\\ u(0) = u_1, \quad u(1) = u_2. \end{aligned}\right\} \tag{5}$$

as an example.

We shall consider the family of homogeneous conservative three-point difference schemes of standard type defined by the generating functional:

$$\Lambda y = L_h^{(k,q,f)} y = (ay_{\bar{x}})_{\hat{x}} - dy + \varphi = 0,$$

$$(ay_{\bar{x}})_{\hat{x}} = \frac{1}{\hbar}\left[\frac{a^{(+1)}(y^{(+1)} - y)}{h_{+1}} - \frac{a(y - y^{(-1)})}{h}\right], \qquad \hbar = 0.5(h + h_{+1}).$$

We determine the coefficients a, d, φ of the scheme with the help of the same pattern functionals $A[\mu(s)]$ $(-1 \leqslant s \leqslant 0)$, $D[\mu(s)]$, $F[\mu(s)]$ $(-0.5 \leqslant s \leqslant 0.5)$ as are used in the case of uniform nets in [1]:

$$a = A[k(x + sh)], \qquad d = D[q(x + (s + \Delta)\hbar)],$$

$$\varphi = F[f(x + (s + \Delta)\hbar)], \qquad \Delta = (h_{+1} - h)/4\hbar.$$

In § 1 we consider a family of homogeneous difference schemes on non-uniform nets, study the properties of Green's function and derive *a priori* estimates that will be needed later.

In § 2 we examine the accuracy of homogeneous difference schemes on an arbitrary sequence of non-uniform nets. In the class of smooth coefficients, a rational characteristic for a non-uniform net is its mean square mesh $\bar{h} = \|h\|_2$; in this case our schemes have second order accuracy with respect to $\bar{h}$, i.e. $\|y - u\|_0 \leqslant M\bar{h}^2$.

In point 4 it is shown that in the class of discontinuous coefficients our schemes have the same order of accuracy on non-uniform nets as on uniform nets. More precisely, if $k(x)$, $q(x)$ and $f(x)$ have discontinuities of the first kind in some neighbourhood of the point $\xi = x_n + \theta h_{n+1}$ $(0 \leqslant \theta \leqslant 1)$, then

$$\|y - u\|_0 \leqslant M\bar{h}^2 + M'(h_n^\varkappa + h_{n+1}^\varkappa + h_{n+2}^\varkappa),$$

where $\varkappa = 1$ for an arbitrary scheme of the given family, $\varkappa = 2$ for the scheme whose pattern functionals A, D, F have the form

$$A[\mu(s)] = \left[\int_{-1}^{0} \frac{ds}{\mu(s)}\right]^{-1}, \qquad D[\mu(s)] = F[\mu(s)] = \int_{-0.5}^{0.5} \mu(s)\, ds.$$

This estimate can be used to choose the net near the fixed points of discontinuity of the functions $\bar{k}$, q, f.

The results of § 2 enable us to drop the requirement $0 < M_1 \leqslant h_{i+1}/h_i \leqslant M_2$ used in [2].

Similar theorems concerning the accuracy of homogeneous difference schemes on non-uniform nets are obtained for the multidimensional equation of elliptic type

$$\sum_{\alpha=1}^{p} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x_1, \ldots, x_p)\frac{\partial u}{\partial x_\alpha}\right) - q(x_1, \ldots, x_p)\, u + f(x_1, \ldots, x_p) = 0.$$

In this case we use the method of integral (energy) inequalities to construct the corresponding *a priori* estimates.

It should be noted that special forms of difference schemes on non-uniform nets have been considered in a number of works (see, for example, [3]). However, to evaluate the accuracy of difference schemes on non-uniform nets the norm $\|\psi\|_0 = \max\limits_i |\psi_i|$, was used, and this does not enable the actual order of accuracy to be found.

1. Homogeneous difference schemes on a non-uniform net

1. The initial family of homogeneous schemes

Let us consider the first boundary problem for the differential equation

$$\left.\begin{aligned} L^{(k,q,f)} u = \frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)\,u + f(x) = 0, \qquad 0 < x < 1,\\ u(0) = u_1, \qquad u(1) = u_2, \end{aligned}\right\} \qquad (5)$$

where $k(x) \geqslant c_1 > 0$, $q(x) \geqslant 0$, and c_1 is a constant.

The class of boundary problems (5) is defined if we indicate the families of functions to which the functions $k(x)$, $q(x)$, $f(x)$ belong. Following [1] we shall denote by $C^{(m)}[a, b]$ the class of functions having a continuous m-th derivative on the segment $a \leqslant x \leqslant b$; by $Q^{(m)}[a, b]$ the class of functions which, together with its derivatives up to the m-th order inclusive are piece-wise continuous on $[a, b]$; by $C^{(m,1)}[a, b]$ the class of functions whose m-th derivative satisfies the Lipschitz condition on $[a, b]$; and by $Q^{(m,1)}[a, b]$ the class of functions of $Q^{(m)}$, the m-th derivative of which satisfies the Lipschitz condition on its continuous intervals. If at some point $\xi \in (0, 1)$ the function $k(x) \in Q^{(0)}[0, 1]$ has a discontinuity of the first kind $(k_l = k(\xi - 0) \neq k_r = k(\xi + 0))$

then at this point the usual junction conditions are satisfied:

$$[u] = u_r - u_l = 0, \quad \left[k\frac{du}{dx}\right] = 0 \text{ for } x = \xi. \tag{6}$$

Dividing the segment [0, 1] into N parts by the points $x_0 = 0$, $x_1, \ldots, x_i, \ldots, x_N = 1$, we obtain the difference net $\omega_N = \{x_i\}$. Generally speaking, the step $h_i = x_i - x_{i-1}$ is an arbitrary net function satisfying only the normalisation condition

$$\sum_{i=1}^{N} h_i = 1. \tag{7}$$

If all $h_i = h = 1/N$ ($i = 1, 2, \ldots, N$) then the net $\omega_N = \omega_h$ is uniform. Let $y_i = y(x_i)$ be some net function. We shall omit the suffix i as a rule, and write

$$y = y(x) = y_i, \quad y^{(+1)} = y_{i+1}, \quad y^{(-1)} = y_{i-1} \quad (x \in \omega_N).$$

Let us introduce notation for the "difference derivatives"

$$y_{\bar{x}} = \frac{y_i - y_{i-1}}{h_i} = \frac{y - y^{(-1)}}{h}, \quad y_x = \frac{y^{(+1)} - y}{h_{+1}}, \quad y_{\hat{x}} = \frac{y^{(+1)} - y}{\hbar} = \frac{h_{+1}}{\hbar} y_x,$$

where $\hbar = 0.5\,(h + h_{+1})$, $h = h_i$, $h_{+1} = h_{i+1}$. Then the difference operator

$$\Lambda y_i = \frac{1}{\hbar_i}\left[\frac{a_{i+1}(y_{i+1} - y_i)}{h_{i+1}} - \frac{a_i(y_i - y_{i-1})}{h_i}\right]$$

can be written in the convenient form

$$\Lambda y = (a y_{\bar{x}})_{\hat{x}}.$$

On a uniform net $\Lambda y = (a y_{\bar{x}})_x$.

We have considered problem (5) more than once using uniform nets (see [1] for example). It was shown in [1] that in the family of homogeneous schemes of standard type the only schemes that converge in the class of discontinuous coefficients are the conservative schemes

$$\Lambda y = L_h^{(k,\,q,\,f)} y = (a y_{\bar{x}})_x - dy + \varphi, \tag{8}$$

whose coefficients are defined with the help of the pattern functionals

$$A^h[\mu(s)] \quad (-1 \leqslant s \leqslant 0), \qquad D^h[\mu(s)] \quad (-0.5 \leqslant s \leqslant 0.5),$$
$$F_h[\mu(s)] \quad (-0.5 \leqslant s \leqslant 0.5)$$

by the formulae

$$a = a(x) = A^h[k(x + sh)], \quad d = D^h[q(x + sh)], \quad \varphi = F^h[f(x + sh)],$$

are in the class of discontinuous coefficients.*

On a non-uniform net too we shall only consider standard conservative schemes (each coefficient of which depends only on one coefficient of the differential equation):

$$\Lambda y = (ay_{\bar{x}})_{\hat{x}} - dy + \varphi. \tag{9}$$

As the scheme is conservative, the coefficient $a(x_i)$ depends only on the values of $k(x)$ on the segment $[x_i - h_i,\ x_i]$ of the net ω_N:

$$a_i = A^h\,[k\,(x_i + sh_i)] \quad \text{or} \quad a = A^h\,[k\,(x + sh)], \qquad -1 \leqslant s \leqslant 0.$$

The coefficient d (and φ) is determined by the values of the function $q(x)$ $(f(x))$ on the segment $[x - 0.5\,h,\ x + 0.5\,h_{+1}]$. Therefore, if we wish to use the same pattern function $D^h[\mu(s)]$ as for the uniform net, in deriving a formula for d we must place the centre $s = 0$ of the pattern $-0.5 \leqslant s \leqslant 0.5$ at the mean point $\bar{x} = x + 1/4(h_{+1} - h)$ of the segment $[x - 0.5\,h,\ x + 0.5\,h_{+1}]$, writing

$$d = D^h\,[q\,(\bar{x} + s\hbar)]. \tag{10}$$

The displacement transformation will have the form $x' = \bar{x} + s\hbar = x + (\Delta + s)\hbar$, so that, instead of (10), we can write

$$d = D^h\,[q\,(x + (s + \Delta)\,\hbar)], \qquad \Delta = (h_{+1} - h)/4\hbar \tag{10'}$$

and, similarly, $\varphi = F^h[f(x + (s + \Delta)\hbar)]$. The index h indicates that the pattern functionals depend on the net, i.e. in the case of a non-uniform net on the two parameters h, h_{+1}. If the pattern functionals of the scheme do not depend on the net, then we shall call them canonical functionals, by analogy with [1], and denote them by $A\,[\mu]$, $D\,[\mu]$, $F\,[\mu]$. We can call the corresponding scheme Λy a canonical scheme. All the discussion which follows will refer to the canonical schemes (9) for which

$$a = A\,[k\,(x + sh)], \quad d = D\,[q\,(x + (s + \Delta)\,\hbar], \quad \varphi = F\,[f(x + (s + \Delta)\,\hbar)].$$

Thus we shall consider the following family of conservative homogeneous difference schemes defined on non-uniform nets:

$$\Lambda y = (ay_{\bar{x}})_{\hat{x}} - dy + \varphi, \tag{11}$$

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* A homogeneous difference scheme written in a form without suffixes is, in essence, a generating functional (see [1]).

$$a = A\,[k\,(x + sh)] \qquad (-1 \leqslant s \leqslant 0), \qquad d = D\,[q\,(x + (s + \Delta)\,\hbar],$$
$$\varphi = F\,[f\,(x + (s + \Delta)\,\hbar)] \qquad (-0.5 \leqslant s \leqslant 0.5),$$

where $\Delta = (h_{+1} - h)/4\hbar$. The pattern functionals are defined on a class of piece-wise continuous functions $\mu \in Q^{(0)}$ and satisfy the conditions (see [1]):

1) $A[\mu(s)]$ is a non-decreasing ($A\,[\mu_2] \geqslant A\,[\mu_1]$ for $\mu_2 \geqslant \mu_1$), normalised ($A\,[1] = 1$) homogeneous functional of the first degree ($A\,[c\mu] = cA\,[\mu]$, $c = \text{const} > 0$) having a second differential;

2) $D\,[\mu(s)]$ and $F[\mu(s)]$ are linear normalised ($D[1] = 1$, $F[1] = 1$), non-negative functionals ($D[\mu] \geqslant 0$, $F[\mu] \geqslant 0$ for $\mu \geqslant 0$);

3) the necessary conditions of second-order approximation on uniform nets

$$A_1\,[s] = -\,0.5, \quad D\,[s] = F\,[s] = 0, \tag{12}$$

are satisfied, where $A_1[f] = A_1[1, f]$ is the first differential of the functional $A[\mu]$ at the point $\mu = 1$.

These conditions define the initial class of homogeneous difference schemes (11) on which this paper is based.

2. *Difference boundary problems*

Let us put the initial problem (5) in correspondence with the following difference problem

$$\Lambda y = (ay_{\bar{x}})_{\hat{x}} - dy + \varphi = 0 \text{ for } x = x_i, \quad 0 < i < N, \quad y\,(0) = u_1,\, y\,(1) = u_2. \tag{13}$$

It follows from the conditions $k \geqslant c_1 > 0$, $q \geqslant 0$ and the properties of the pattern functionals that

$$a \geqslant c_1 > 0, \qquad d \geqslant 0.$$

In order to determine the accuracy of the solution of problem (13) we must make an estimate of the net function $z = y - u$ given the limited division of the net, i.e. as $\|h\|_0 \to 0$. The function z is clearly a solution of the problem

$$\bar{\Lambda} z = (az_{\bar{x}})_{\hat{x}} - dz = -\,\psi, \qquad z\,(0) = 0, \qquad z\,(1) = 0, \tag{14}$$

where $\psi = \Lambda u - L^{(k,\,q,\,f)}u$ is the approximation error of our scheme Λu calculated for the solution $u = u(x)$ of the differential equation (5).

To answer the questions of the convergence and accuracy of scheme (13) we first calculate the approximation error ψ, and then find an estimate for the function z in terms of ψ (an *a priori* estimate).

3. *The approximation error on a non-uniform net*

Let us consider the approximation error

$$\psi = \Lambda u - L^{(k,q,f)} u = [(au_{\bar{x}})_{\hat{x}} - (ku')'] - (d-q)\,u + \varphi - f. \quad (15)$$

In our introduction we pointed out that, generally speaking, on a non-uniform net the order of approximation of difference schemes is reduced. For, consider the special case $k = 1$, $d = q$, $\varphi = f$ so that $\psi = u_{\bar{x}\hat{x}} - u''$. It follows from the formulae

$$u^{(+1)} = u + h_{+1}u' + 0.5\,h_{+1}^2 u'' + \tfrac{1}{6}\,h_{+1}^3 u''' + O(h_{+1}^4),$$
$$u^{(-1)} = u - hu' + 0.5\,h^2u'' - \tfrac{1}{6}h^3 u''' + O(h^4),$$

$$u_{\bar{x}} = u' - 0.5\,hu'' + \tfrac{1}{6}h^2u''' + O(h^3), \quad u_{\bar{x}\hat{x}} = u'' + \frac{h_{+1}^2 - h^2}{6\hbar}\,u''' + O(h^2) + O(h_{+1}^2)$$

that

$$\psi = \tfrac{1}{3}(h_{+1} - h)\,u''' + O(h^2) + O(h_{+1}^2),$$

i.e. $\psi = O(h) + O(h_{+1})$ for $h \neq h_{+1}$ and $\psi = O(h^2)$ on a uniform net $(h = h_{+1})$.

It was shown in [1] that in the case of a uniform net a scheme satisfying the necessary conditions for second order approximation (12) has only first order approximation if $k(x) \in C^{(1,1)}$ but that nevertheless this scheme has second order accuracy. A reduction in the order of ψ on a uniform net can also be associated with the reduction in the rank r_A (see [1]) of the functional $A[\mu(s)]$. If the rank $r_A = 3$ and $k(x) \in C^{(2,1)}[0,\ 1]$, q, $f \in C^{(1,1)}[0,\ 1]$ then $\psi = O(h^2)$ on a uniform net. For a non-uniform net, even with these conditions, $\psi = O(h) + O(h_{+1})$.

Let us find an expansion of the net function (15) in powers of h and h_{+1}. If q, $f \in C^{(1,1)}$ then, due to the linearity of the functionals D, F we can write

$$d(x) = D\,[q\,(x + (s + \Delta)\,\hbar)] = q(x) + \hbar q'(x)\,(D[s] + \Delta) + O(h^2) + O(h_{+1}^2) =$$
$$= q(x) + \tfrac{1}{4}(h_{+1} - h)\,q'(x) + O(h^2) + O(h_{+1}^2)\ \left(D[s] = 0,\ \Delta = \frac{1}{4\hbar}(h_{+1} - h)\right), \quad (16)$$
$$\varphi(x) = F\,[f\,(x + (s + \Delta)\,\hbar)] = f(x) + \tfrac{1}{4}(h_{+1} - h)\,f'(x) + O(h^2) + O(h_{+1}^2). \quad (17)$$

Lemma 1. The approximation error of any scheme Λu of the initial class for k, q, $f \in C^{(1,1)}$ and an arbitrary non-uniform net ω_N can be put in the form

$$\psi = \Lambda u - L^{(k,q,f)} u = \mu_{\hat{x}} + \psi^*, \tag{18}$$

where

$$\mu = au_{\bar{x}} - \overline{ku'} + \frac{h^2}{8} qu', \qquad \mu = O(h^2), \quad \psi^* = O(h^2) + O(h_{+1}^2), \tag{19}$$

$\overline{ku'} = ku'|_{x=\bar{x}}$, $\bar{x} = x - 0.5h$, and $au = u(x)$ is a solution of the differential $L^{(k,q,f)}u = 0$.

It follows from the equation $(ku')' = qu - f$ and conditions q, $f \in C^{(1,1)}$ that $(ku')'' = (qu - f)' \in C^{(0,1)}$. Therefore we can write

$$\overline{ku'} = ku' - 0.5h\,(ku')' + \frac{h^2}{8}(ku')'' + O(h^3),$$

$$(\overline{ku'})^{(+1)} = ku' + 0.5h_{+1}\,(ku')' + \tfrac{1}{8}\,h_{+1}^2\,(ku')'' + O(h_{+1}^3).$$

We have from this

$$\begin{aligned}(ku')' &= (\overline{ku'})_{\hat{x}} - \frac{1}{8\hbar}(h_{+1}^2 - h^2)\,(ku')'' + O(h^2) + O(h_{+1}^2) = \\ &= (\overline{ku'} - \tfrac{1}{8}h^2\,(ku')'')_{\hat{x}} + O(h^2) + O(h_{+1}^2),\end{aligned} \tag{20}$$

since $(h_{+1}^2 - h^2)\,(ku')''/8\hbar = \frac{1}{8}(h^2\,(ku')'')_{\hat{x}} + O(h_{+1}^2)$. Using (16) and (17), we transform $(d - q)u + \varphi - f$ similarly:

$$\begin{aligned}(d - q)\,u &= \tfrac{1}{4}(h_{+1} - h)\,q'u + O(h^2) + O(h_{+1}^2) = \frac{1}{8}\frac{h_{+1}^2 - h^2}{\hbar}q'u + O(h^2) + \\ &+ O(h_{+1}^2) = \tfrac{1}{8}(h^2 q'u)_{\hat{x}} + O(h^2) + O(h_{+1}^2),\end{aligned} \tag{21}$$

$$\varphi - f = \tfrac{1}{8}(h^2 f')_{\hat{x}} + O(h^2) + O(h_{+1}^2). \tag{22}$$

Inserting (20)-(22) in (15), we obtain (18) where

$$\mu = au_{\bar{x}} - \overline{ku'} + \tfrac{1}{8}h^2\,[(ku')'' - q'u + f']. \tag{23}$$

We find from the differential equation $L^{(k,q,f)}u = 0$, $(ku')'' - q'u + f' = qu'$. After substituting this expression in (23) the formula for μ takes the form

$$\mu = au_{\bar{x}} - \overline{ku'} + \tfrac{1}{8}h^2 qu'. \tag{19}$$

This transformation is valid for any a. Let us assume now that A has a second differential (is of second rank) and satisfies the necessary conditions for second order approximation $A\,[1] = 1$, $A_1\,[s] = -0.5$.

Expanding $au(x)$ in powers of h in the neighbourhood of the point $\bar{x} = x - 0.5\,h$ (see [1])

$$a\,(x) = A\,[k\,(\bar{x} + (s + 0.5)\,h)] = k\,(\bar{x}) + hA_1\,[s + 0.5]\,k'\,(\bar{x}) + \\ + O\,(h^2) = k\,(\bar{x}) + O\,(h^2)$$

and noting that $u_{\bar{x}} = u'\,(\bar{x}) + O\,(h^2)$, we find

$$au_{\bar{x}} - \overline{ku'} = (\bar{k} - O\,(h^2))\,(\bar{u}' + O\,(h^2)) - \overline{ku'} = O\,(h^2), \text{ i.e. } \mu = O(h^2).$$

If $A\,[\bar{k}(s)]$ is of third rank and $k(x) \in C^{(2.1)}$ where $m > 1$ then

$$a\,(x) = \bar{k} + h^2 \left\{\frac{\bar{k}''}{2}(A_1\,[s^2] - \tfrac{1}{4}) + \frac{(\bar{k}')^2}{\bar{k}} A_2\,[s]\right\} + O\,(h^3),$$

$$u_{\bar{x}} = \bar{u}' + \frac{h^2}{24}\bar{u}''' + O(h^3),$$

$$\mu = (\alpha_2 u' + \tfrac{1}{24}\overline{ku'''} + \tfrac{1}{8} qu')\,h^2 + O\,(h^3) = \mu_1\,(\bar{x})\,h^2 + O\,(h^3),$$

where

$$\alpha_2 = \frac{\bar{k}''}{2}(A_1\,[s^2] - \tfrac{1}{4}) + \frac{(\bar{k}')^2}{\bar{k}} A_2\,[s].$$

It follows that

$$\psi = (h_{+1} - h)\,\mu_1\,(x) + O\,(h^2) + O\,(h_{+1}^2),$$

i.e. the scheme always has first order approximation if the net ω_N is arbitrary. If the net is specially selected to satisfy the condition $h_{+1} - h = O(h^2)$ then $\psi = O(h^2)$.

4. *Green's difference function*

Let us now evaluate the solution of the problem

$$\bar{\Lambda} z = (az_{\bar{x}})_{\hat{x}} - dz = -\psi, \qquad z\,(0) = 0, \qquad z\,(1) = 0, \tag{14}$$
$$0 < c_1 \leqslant a \leqslant c_1', \quad 0 \leqslant d \leqslant c_2.$$

We shall be interested in the case where $\psi(x)$ has the form

$$\psi\,(x) = \mu_{\hat{x}} + \psi^*\,(x).$$

To construct *a priori* estimates for the solution of problem (14) we use Green's difference function, by analogy with [1].

We shall use the following notation for the sums and norms (see [4]):

$$(y, v) = \sum_{i=1}^{N-1} y_i v_i h_i, \qquad (y, v)^+ = \sum_{i=1}^{N-1} y_i v_i h_{i+1}, \qquad (y, v)^* = \sum_{i=1}^{N-1} y_i v_i \hbar_i,$$

$$(y, v] = \sum_{i=1}^{N} y_i v_i h_i, \qquad \|y\|_0 = \max |y_i|,$$

$$\|y\|_\sigma = (1, |y|^\sigma)^{1/\sigma} \text{ or } \|y\|_\sigma = (1, |y|^\sigma)^{+1/\sigma}, \|y_{\bar{x}}\|_\sigma = (1, |y_{\bar{x}}|^\sigma]^{1/\sigma}, \sigma = 1,2,$$

where $y = y(x)$, $v = v(x)$ are arbitrary net functions.

Let us give some of the simpler formulae (see [4]):

1) formula for summation by parts:

$$(y, v_{\hat{x}})^* = (y, v_x)^+ = -(v, y_{\bar{x}}) + y^{(-1)} v|_{x=1} - y v^{(+1)}|_{x=0} = \\ = -(v, y_{\bar{x}}] + yv|_{x=1} - y v^{(+1)}|_{x=0}; \tag{24}$$

2) Green's first difference formula:

$$(y, (av_{\bar{x}})_{\hat{x}})^* = (y, (av_{\bar{x}})_x)^+ = -(a, y_{\bar{x}} v_{\bar{x}}] + ayv_{\bar{x}}|_{x=1} - a^{(+1)} y v_x|_{x=0}; \tag{25}$$

3) Green's second difference formula:

$$(y, (av_{\bar{x}})_{\hat{x}})^* = (v, (ay_{\bar{x}})_{\hat{x}})^* + a(yv_{\bar{x}} - vy_{\bar{x}})|_{x=1} - a^{(+1)}(yv_x - vy_x)|_{x=0} \tag{26}$$

or

$$(y, \bar{\Lambda} v)^* = (v, \bar{\Lambda} y)^* + a(yv_{\bar{x}} - vy_{\bar{x}})|_{x=1} - a^{(+1)}(yv_x - vy_x)|_{x=0}. \tag{26'}$$

We introduce Green's difference function $G(x, \xi)$ of problem (14) with the help of the conditions

$$\left.\begin{aligned} \bar{\Lambda} G = (a(x) G_{\bar{x}}(x, \xi))_{\hat{x}} - d(x) G(x, \xi) = -\frac{\delta(x, \xi)}{\hbar}, \\ G(0, \xi) = 0, \qquad G(1, \xi) = 0, \end{aligned}\right\} \tag{27}$$

where $\delta(x, \xi) = 1$ for $x = \xi$, $\delta(x, \xi) = 0$ for $x \neq \xi$ (the dependence of $G(x, \xi)$ on the net is not indicated explicitly). The expression

$$G(x, \xi) = \begin{cases} \alpha(x)\beta(\xi)/\alpha(1) & \text{for } x \leqslant \xi, \\ \alpha(\xi)\beta(x)/\alpha(1) & \text{for } x \geqslant \xi \end{cases} \tag{28}$$

obtained in [1] is also valid for a non-uniform net, if we define $\alpha(x)$ and $\beta(x)$ as the solutions of the following problems with initial conditions:

$$\bar{\Lambda}\alpha = (a\alpha_{\bar{x}})_{\hat{x}} - d\alpha = 0, \qquad \alpha(0) = 0, \qquad a^{(+1)}\alpha_x\big|_{x=0} = 1, \tag{29}$$

$$\bar{\Lambda}\beta = (a\beta_{\bar{x}})_{\hat{x}} - d\beta = 0, \qquad \beta(1) = 0, \qquad a\beta_{\bar{x}}\big|_{x=1} = -1. \tag{30}$$

Writing $y = \alpha$, $v = \beta$ in the second Green formula we find

$$\alpha(1) = \beta(0), \qquad a(\beta\alpha_{\bar{x}} - \alpha\beta_{\bar{x}}) = \alpha(1) \quad \text{for} \quad 0 < x \leqslant 1.$$

It is clear from formula (28) that Green's function is symmetric:

$$G(x, \xi) = G(\xi, x).$$

We write $y(x) = G(x, \xi)$, $v(x) = z(x)$ in (26′), where $z(x)$ is a solution of problem (14). Both functions satisfy homogeneous boundary conditions, and so the substitutions for $x = 0$ and $x = 1$ are equal to zero. Using equations (29) and (30) and the symmetry of Green's function, after replacing x by ξ and ξ by x we obtain the formula

$$z(x) = (G(x, \xi), \psi(\xi))^{*} \quad \text{or} \quad z = (G, \psi)^{*},$$

which we shall use below to derive *a priori* estimates of the solution of problem (14). To do this we need estimates of Green's function $G(x, \xi)$ and of its difference derivatives $G_{\bar{x}}$, $G_{\bar{\xi}}$, $G_{\bar{x}\bar{\xi}}$.

Lemma 2. Green's difference function $G(x, \xi)$ of problem (14) and its difference derivatives $G_{\bar{x}}$, $G_{\bar{\xi}}$ satisfy the conditions

$$0 \leqslant G(x, \xi) \leqslant M_1, \qquad |G_{\bar{x}}(x, \xi)| \leqslant M_2, \qquad |G_{\bar{\xi}}(x, \xi)| \leqslant M_2, \tag{32}$$

$$|G_{\bar{x}\bar{\xi}}(x, \xi)| \leqslant M_3' \ \text{for}\ \xi \neq x, \quad |hG_{\bar{x}\bar{\xi}}(x, x)| \leqslant M_3'', (|G_{\bar{x}\bar{\xi}}(x, \xi)|, 1) \leqslant M_3, \tag{33}$$

where $M_1 = 1/c_1$, $M_2 = (c_1 + c_2)/c_1^2$, $M_3' = (c_1' + c_2)(1 + c_2 c_3')/c_1^2$, $c_3' =$
$= \sinh \sqrt{c_2/c_1} / \sqrt{c_1 c_2}$, $M_3'' = (1 + c_1' + c_2)/c_1$, $M_3 = M_3' + M_3''$.

Proof. Estimates (32) were obtained in [1] for a uniform net. Let G^0 be Green's function of problem (14) for $d = 0$. In this case

$$\alpha_i = \alpha_i^0 = \sum_{k=1}^{i} (h_k/a_k) \leqslant \frac{1}{c_1}, \quad \beta_i = \beta_i^0 = \sum_{k=i+1}^{N} (h_k/a_k) \leqslant \frac{1}{c_1}$$

and it follows from (28) that $G^0 \leqslant 1/c_1$, $|G^0_{\bar{\xi}}| \leqslant 1/c_1$, $|G^0_{\bar{x}}| \leqslant 1/c_1$, since

$\alpha^0(x) \leqslant \alpha^0(1) \leqslant 1/c_1$, $\beta^0(x) \leqslant \beta^0(0) \leqslant 1/c_1$. Then, using the inequality $G \leqslant G^0$ together with the formulae $G(x, \xi) = G^0(x, \xi) - (G^0(x, s), d(s)\,G(s, \xi))^*$, $G_{\bar{x}} = G^0_{\bar{x}} - (G^0_{\bar{x}}, dG)^*$, we obtain (32). For $G_{\bar{x}\bar{\xi}}$ we use formula (28) with the inequalities (see [1]): $0 < \alpha(x) \leqslant \alpha(1)$ for $0 < x \leqslant 1$, $0 < \beta(x) \leqslant \beta(0) = \alpha(1)$ for $0 \leqslant x < 1$, $1/c_1' \leqslant \alpha(1) \leqslant c_3' = \sinh \sqrt{c_2/c_1} / \sqrt{c_1 c_2}$, $\alpha_{\bar{x}}(x)/\alpha(1) \leqslant (c_1' + c_2)/c_1$. The calculations give

$$G_{\bar{x}\bar{\xi}}(x, \xi) = \begin{cases} \alpha_{\bar{x}}(x)\,\beta_{\bar{\xi}}(\xi)/\alpha(1) & \text{for } x < \xi, \\ \alpha_{\bar{\xi}}(\xi)\,\beta_{\bar{x}}(x)/\alpha(1) & \text{for } x > \xi, \end{cases} \quad G_{\bar{x}\bar{\xi}}(x, x) = \frac{1}{a(x)h(x)} + \frac{\alpha(x)\beta_{\bar{x}}(x)}{\alpha(1)},$$

$$|G_{\bar{x}\bar{\xi}}(x, \xi)| \leqslant (c_1' + c_2)(1 + c_2 c_3')/c_1^2 = M_3', \quad x \neq \xi,$$

$$h\,|G_{\bar{x}\bar{\xi}}(x, x)| \leqslant (1 + h(c_1' + c_2))/c_1 \leqslant (1 + c_1' + c_2)/c_1 = M_3'',$$

$$(|G_{\bar{x}\bar{\xi}}(x, \xi)|, 1) \leqslant M_3' + M_3'' = M_3.$$

This proves the lemma.

5. *The a priori estimates*

Lemma 3. If $\psi(x)$ has the form

$$\psi = \mu_{\hat{x}} + \psi^*, \tag{18}$$

then the solution of problem (14) satisfies the inequalities

$$\|z\|_0 \leqslant M_2\,(\|\mu\|_1 + \|\psi^*\|_3), \tag{34}$$

$$\|z\|_0 \leqslant M_1 \|\psi^*\|_1 + M_2 \|\mu\|_1, \tag{34'}$$

where

$$\|\psi^*\|_3 = \sum_{i=1}^{N-1} h_i \sum_{k=i}^{N-1} \psi_k^* \hbar_k, \qquad \|\mu\|_1 = (1, |\mu|] = \sum_{i=1}^{N} |\mu_i| h_i.$$

Using formula (31) we find

$$z = (G, \mu_{\hat{\xi}})^* + (G, \psi^*)^*. \tag{35}$$

Let us take each term separately. From (24) and (27) we find $(G, \mu_{\hat{\xi}})^* = (G, \mu_{\xi})^+ = -(G_{\bar{\xi}}, \mu]$, since the substitutions for $\xi = 0$ and $\xi = 1$ become zero.

Then using (32) we find

$$|(G, \mu_{\hat{\xi}})^*| \leqslant M_2\,(1, |\mu|] = M_2\,\|\mu\|_1. \tag{36}$$

Introducing the function $\eta(x)$ with the help of the conditions

$$\eta_{\hat{x}} = \psi^*, \qquad \eta_N = \eta(1) = 0,$$

we find $(G, \psi^*)^* = (G, \eta_\xi)^+ = -(G_{\bar{\xi}}(x, \xi), \eta(\xi))$ and therefore

$$|(G, \psi^*)^*| \leqslant M_2 (1, |\eta|) \leqslant M_2 \|\eta\|_1 = M_2 \|\psi^*\|_3. \tag{37}$$

where $\eta(x) = \eta_i = -\sum_{k=i}^{N-1} \psi_k^* \hbar_k$. (34) follows at one from (35), (37).

Lemma 4. If $\psi = \mu_{\hat{x}} + \psi^*$, then the difference derivative $z_{\bar{x}}$ of the solution of problem (14) satisfies the conditions

$$\|z_{\bar{x}}\|_0 \leqslant M_3 \|\mu\|_0 + M_2 \|\psi^*\|_1, \tag{38}$$

$$\|z_{\bar{x}}\|_1 \leqslant M_3 (\|\mu\|_1 + \|\psi^*\|_3), \qquad \|z_{\bar{x}}\|_2 \leqslant M_3 (\|\mu\|_2 + \|\eta\|_2). \tag{39}$$

Using the expression $z = -(G_{\bar{\xi}}(x, \xi), \mu(\xi)] + {}^*(G, \psi^*)^*$, obtained above, we find

$$z_{\bar{x}} = -(G_{\bar{x}\bar{\xi}}, \mu] + (G_{\bar{x}}, \psi^*)^* = -(G_{\bar{x}\bar{\xi}}, \tilde{\mu}] = -(G_{\bar{x}\bar{\xi}}, \tilde{\mu}]' - hG_{\bar{x}\bar{\xi}}(x, x) \tilde{\mu}(x),$$

where $\tilde{\mu} = \mu + \eta$, and the bar denotes that the summation is taken over all $\xi \neq x$. Now using the inequalities of (33) we obtain

$$\begin{aligned} &|z_{\bar{x}}(x)| \leqslant M_3' \|\mu\|_1 + M_3'' |\mu(x)| + M_2 \|\psi^*\|_1, \\ &|z_{\bar{x}}(x)| \leqslant M_3' (\|\mu\|_1 + \|\eta\|_1) + M_3'' (|\mu(x)| + |\eta(x)|), \\ &\|z_{\bar{x}}\|_0 \leqslant M_3' \|\mu\|_1 + M_3'' \|\mu\|_0 + M_2 \|\psi^*\|_1 \leqslant M_3 \|\mu\|_0 + M_2 \|\psi^*\|_1, \end{aligned} \tag{40}$$

since $\|\mu\|_1 \leqslant \|\mu\|_0$. The second inequality of (39) follows immediately from (40), since $(1, |\mu|] = \|\mu\|_1$, $(1, |\eta|] = (1, |\eta|) = \|\psi^*\|_3$.

Lemma 5. Let $z = z(x)$ be a solution of problem (14), where $\psi(x)$ has the form

$$\psi(x) = \mu_{\hat{x}} + \psi^*(x)$$

at all points of the net ω_N apart from the points $x = x_n$ and $x = x_{n+1}$. Then $z = z(x)$ satisfies the inequality

$$\|z\|_0 \leqslant M_2 \|\mu\|_1^* + M_1 (|\mu_n| + |\mu_{n+2}|) + M_1 (\|\psi^*\|_1^* + |\hbar_n \psi_n + \hbar_{n+1} \psi_{n+1}|) + \\ + M_2 h_{n+1} \hbar_n |\psi_n|, \tag{41}$$

where

$$M_1 = 1/c_1, \qquad M_2 = (c_1 + c_2)/c_1^2,$$

$$\|\mu\|_1' = \sum_{i=1}^{n} |\mu_i| h_i + \sum_{i=n+2}^{N} |\mu_i| h_i,$$

$$\|\psi^*\|_1'' = \sum_{i=1}^{n-1} |\psi_i^*| \hbar_i + \sum_{i=n+2}^{N-1} |\psi_i^*| \hbar_i.$$

Using formula (31), we put $z(x)$ in the form

$$z(x) = (G(x, \xi), \psi(\xi))^* = (G(x, \xi), \mu_{\hat{\xi}}(\xi) + \psi^*(\xi))^{*''} + \bar{z}(x),$$

where $\bar{z}(x) = G(x, x_n)\psi_n \hbar_n + G(x, x_{n+1})\psi_{n+1}\hbar_{n+1}$, and $(G, \mu_{\hat{\xi}} + \psi^*)^{*''}$ denotes the sum with respect to $\xi = x_1, \ldots, x_{n-1}, x_{n+2}, \ldots, x_{N-1}$ ($\xi \neq x_n, x_{n+1}$). Let us first transform $\bar{z}(x)$:

$$\bar{z}(x) = G(x, x_{n+1})(\hbar_n\psi_n + \hbar_{n+1}\psi_{n+1}) - G_{\bar{\xi}}(x, x_{n+1}) h_{n+1}\hbar_n\psi_n,$$

and apply Lemma 2. We obtain the estimate

$$\|\bar{z}\|_0 \leqslant M_1 |\hbar_n\psi_n + \hbar_{n+1}\psi_{n+1}| + M_2 h_{n+1}\hbar_n|\psi_n|. \tag{42}$$

The formula for summation by parts (24) gives

$$(G, \mu_{\hat{\xi}})^{*''} = -\sum_{k=1}^{n} G_{\bar{\xi}}(x, x_k)\mu_k h_k - \sum_{k=n+2}^{N} G_{\bar{\xi}}(x, x_k)\mu_k h_k + G(x, x_n)\mu_n - G(x, x_{n+1})\mu_{n+2},$$

so that

$$|(G, \mu_{\hat{\xi}})^{*''}| \leqslant M_2\|\mu\|_1' + M_1(|\mu_n| + |\mu_{n+2}|). \tag{43}$$

Using (42), (43) and the inequality $|(G, \psi^*)^{*''}| \leqslant M_1\|\psi^*\|_1''$, we arrive at the inequality (41).

Lemma 5 is used in estimating the order of accuracy in the class of discontinuous coefficients (see § 2, pt. 3).

Similar *a priori* estimates are obtained for the third boundary problem

$$\left.\begin{array}{c}\bar{\Lambda} z = (a z_{\bar{x}})_{\hat{x}} - dz = -\psi, \\ l_1 z = a^{(+1)} z_x - \sigma_1 z = -\nu_1 \text{ for } x=0, \qquad l_2 z = a z_{\bar{x}} + \sigma_2 z = \nu_2 \text{ for } x=1, \\ 0 < c_1 \leqslant a \leqslant c_1', \ 0 \leqslant d \leqslant c_2, \ \sigma_1 \geqslant 0, \ \sigma_2 \geqslant 0, \ \sigma_1 + \sigma_2 \geqslant c_3 > 0.\end{array}\right\} \quad (44)$$

In this case Green's function of problem (44) defined as the solution of equation (14) satisfying the boundary conditions $l_1 G = 0$, $l_2 G = 0$ has the form

$$G(x, \xi) = \begin{cases}\alpha(x)\beta(\xi)/\Delta & \text{for } x \leqslant \xi, \\ \alpha(\xi)\beta(x)/\Delta & \text{for } x \geqslant \xi.\end{cases} \qquad (45)$$

The functions $\alpha(x)$ and $\beta(x)$ are determined by the conditions

$$\bar{\Lambda}\alpha = 0, \quad a^{(+1)}\alpha_x = 1, \ l_1\alpha = 0 \text{ for } x = 0,$$
$$\bar{\Lambda}\beta = 0, \quad a\beta_{\bar{x}} = -1, \quad l_2\beta = 0 \text{ for } x = 1.$$

We can find an expression for $\Delta = a(\beta\alpha_{\bar{x}} - \alpha\beta_{\bar{x}})$ using Green's formula (26') (see [4]):

$$\Delta = \alpha(1) + \frac{1}{\sigma_2}(1 + (d, \beta)^*) = \beta(0) + \frac{1}{\sigma_1}(1 + (d, \alpha)^*),$$

with $\alpha(x) > 0$, $\beta(x) > 0$. If, for example, $\sigma_1 = 0$ then $\alpha(x)$ can be determined from the conditions $\alpha(0) = 1$, $\alpha_x(0) = 0$. In this case

$$\Delta = 1 + (\alpha, \beta)^* = \alpha(1) + \frac{1}{\sigma_2}(1 + (d, \alpha)^*).$$

The solution of problem (44) can be put in the form

$$z(x) = (G(x, \xi), \psi(\xi))^* + G(x, 0)\nu_1 + G(x, 1)\nu_2.$$

For Green's function G and its difference derivatives we have

$$0 < G \leqslant M_1, \qquad |G_{\bar{x}}| \leqslant M_2, \qquad |G_{\bar{\xi}}| \leqslant M_2,$$
$$|G_{\bar{x}\bar{\xi}}(x, \xi)| \leqslant M_3' \text{ for } x \neq \xi, \ |hG_{\bar{x}\bar{\xi}}(x, x)| \leqslant M_3'', \qquad (|G_{\bar{x}\bar{\xi}}(x, \xi)|, 1] \leqslant M_3 = $$
$$= M_3' + M_3'',$$

where M_1, M_2, M_3', M_3'' are positive constants depending on c_1, c_2, c_3 and c_1'. Lemma 3 can be generalised to the case of the third boundary problem (44).

Lemma 6. If $\psi = \mu_{\hat{x}} + \psi^*$, then the solution of problem (44) satisfies the *a priori* estimates

$$\|z\|_0 \leqslant M_2(\|\mu\|_1 + \|\psi^*\|_4) + M_1(|\nu_1| + |\nu_2| + |\mu(x_1)| + |\mu(1)|), \quad (46)$$

where

$$\|\psi^*\|_4 = \|\psi^*|_3 + |(\psi^*, 1)^*|, \qquad \|\psi^*\|_3 = \|\eta\|_1, \qquad \eta_i = \sum_{k=i}^{N-1} \psi_k^* \hbar_k.$$

The proof of this lemma is analogous to that of Lemma 3. Lemmas 4 and 5 can similarly be generalised to the case of the third boundary problem.

2. Concerning the accuracy of homogeneous difference schemes on non-uniform nets

1. *Concerning accuracy in the class of smooth coefficients*

Using representation (18) for $\psi(x)$ (Lemma 1), and also Lemma 2, it is not difficult to see that Theorem 1 is true.

Theorem 1. Let $\Lambda y = (ay_{\bar{x}})_{\hat{x}} - dy + \varphi$ be any homogeneous scheme of the initial family. If k, q, $f \in C^{(1,1)}[0, 1]$ then scheme Λy has second order accuracy on any sequence of non-uniform nets, or more exactly

$$\|y - u\|_0 \leqslant M\bar{h}^2, \quad (47)$$

where y is a solution of the difference problem (13), $u = u(x)$ is a solution of problem (5) and M is a constant which does not depend on the net, and

$$\bar{h} = \|h\|_2 = (1, h^2]^{1/2} \quad (48)$$

is the mean square mesh of the net.

To prove the theorem we need an estimate for the difference $z = y - u$ which is determined from conditions (14):

$$(az_{\bar{x}})_{\hat{x}} - dz = -\psi, \quad z(0) = 0, \quad z(1) = 0, \qquad 0 < c_1 \leqslant a, \qquad 0 \leqslant d \leqslant c_2.$$

From Lemma 1

$$\psi = \mu_{\hat{x}} + \psi^*, \text{ where } \mu = O(h^2), \qquad \psi^* = O(h^2) + O(h_{+1}^2). \quad (18)$$

Using estimate (34) of Lemma 3:

$$\|z_0\| \leqslant M_2(\|\mu\|_1 + \|\psi^*\|_3).$$

Then since $\|\mu\|_1 = O(h^2)$, $\|\psi^*\|_3 \leqslant \|\psi^*\|_1 = O(\bar{h}^2)$, we have $\|z\|_0 \leqslant M\bar{h}^2$.

For practical purposes the accuracy with which the flow ku', the difference expression of which has the form $ay_{\bar{x}}$, is determined, is often as important as the accuracy of the solution of problem (5). Let us find an estimate for the order of accuracy with which the flow is determined on a non-uniform net using the given homogeneous difference schemes. We are interested in the error

$$v = ay_{\bar{x}} - \overline{ku'} = az_{\bar{x}} + \mu^*,$$

where $\mu^* = au_{\bar{x}} - \overline{ku'}$ is the approximation error of the flow. We see that

$$\| v \|_0 \leqslant \| a \|_0 \| z_{\bar{x}} \|_0 + \| \mu^* \|_0, \qquad \| v \|_1 \leqslant \| a \|_0 \| z_{\bar{x}} \|_1 + \| \mu^* \|_1,$$

i.e. the estimate of v converges to the estimate of $\| z_{\bar{x}} \|$, since, from Lemma 1, $\mu^* = O(h^2)$ and therefore

$$\| \mu^* \|_0 \leqslant M \| h \|_0^2, \qquad \| \mu^* \|_1 \leqslant M\bar{h}^2,$$

where $\bar{h} = \| h \|_2$. From Lemma 4

$$\| z_{\bar{x}} \|_0 \leqslant M_3 (\| \mu \|_0 + \| \psi^* \|_1) \leqslant M \| h \|_0^2,$$

$$\| z_{\bar{x}} \|_1 \leqslant M_3 (\| \mu \|_1 + \| \psi^* \|_3) \leqslant M \| h \|_2^2, \qquad \| z_{\bar{x}} \|_3 \leqslant M \| h \|_2^2. \tag{49}$$

As a result we obtain the following inequalities for the flow:

$$\| ay_{\bar{x}} - \overline{ku'} \|_0 \leqslant M \| h \|_0^2, \tag{50}$$

$$\| ay_{\bar{x}} - \overline{ku'} \|_1 \leqslant M \| h \|_2^2. \tag{51}$$

We can see from inequality (50) that the scheme (13) gives second order accuracy for the flow also, but a uniform estimate for the error in determining the flow contains the maximum value of the mesh, i.e. $\|h\|_0$ and not $\|h\|_2 = \bar{h}$ as for the function itself. Therefore good accuracy cannot be obtained for the flow on just any sequence of non-uniform nets. To find the solution we can use any nets for which the mean square mesh $\bar{h} = \| h \|_2$ is sufficiently small. It sometimes happens that high accuracy for the flow is only required at individual fixed points (for example, at the boundaries of regions with different physical parameters). In this case the estimates for the flow can be made more precise. Without giving the reasoning, which is based on the use of Lemmas 1, 2 and 4 as well as inequality (33), we give an estimate for $|z_{\bar{x}}|$ at some fixed point $x^* = x_{i_0}$ of the net:

$$|(ay_{\bar{x}} - \overline{ku'})_{i_0}| \leqslant Mh_{i_0}^2 + M\bar{h}^2. \tag{52}$$

By selecting the net so that the point x^* is a nodal point of the net ($x^* = x_{i_0}$) and $h_{i_0} = O(\bar{h})$ we obtain $|v(x^*)| \leqslant \bar{M}h^2$, where $v = ay_{\bar{x}} - \overline{ku'}$.

2. *The third boundary problem*

Let us consider now the boundary problem

$$\left.\begin{aligned} &L^{(k,q,f)}u = (ku')' - qu + f = 0, \qquad 0 < x < 1, \\ &k(0)\,u'(0) - \sigma_1 u(0) = u_1, \qquad k(1)\,u'(1) + \sigma_2 u(1) = u_2, \\ &k(x) \geqslant c_1 > 0, \quad q(x) \geqslant 0, \quad \sigma_1 \geqslant 0, \quad \sigma_2 \geqslant 0, \quad \sigma_1 + \sigma_2 \geqslant c_3 > 0. \end{aligned}\right\} \tag{53}$$

The corresponding difference problem will become:

$$\left.\begin{aligned} &\Lambda y = (ay_{\bar{y}})_{\hat{x}} - dy + \varphi = 0, \qquad 0 < x_i < 1, \\ &a^{(+1)} y_x - \bar{\sigma}_1 y = \bar{u}_1 \text{ for } x = 0, \qquad ay_{\bar{x}} + \bar{\sigma}_2 y = \bar{u}_2 \text{ for } x = 1, \\ &\bar{\sigma}_1 = \sigma_1 + 0.5h_1 q(0), \qquad \bar{\sigma}_2 = \sigma_2 + 0.5h_N q(1), \\ &\bar{u}_1 = u_1 + 0.5h_1 f(0), \qquad \bar{u}_2 = u_2 + 0.5h_N f(1). \end{aligned}\right\} \tag{54}$$

The difference boundary conditions (54) have second order approximation

$$\begin{aligned} \nu_1 &= a^{(+1)}u_x - k(0)\,u'(0) + 0.5h_1 (q(0)\,u(0) - f(0)) = O(h_1^2), \\ \nu_2 &= au_{\bar{x}} - k(1)\,u'(1) - 0.5h_N\,(q(1)\,u(1) - f(1)) = O(h_N^2) \end{aligned}$$

for the solution $u = u(x)$ of the differential equation.

Let $u(x)$ be a solution of problem (53), and y a solution of (54). Then their difference $z = y - u$ satisfies the condition

$$\left.\begin{aligned} &(az_{\bar{x}})_{\hat{x}} - dz = -\psi, \\ &a^{(+1)}z_y - \bar{\sigma}_1 z = -\nu_1 \text{ for } x = 0, \qquad az_{\bar{x}} + \bar{\sigma}_2 z = \nu_2 \text{ for } x = 1. \end{aligned}\right\} \tag{55}$$

where $\psi = \Lambda u - L^{(k,q,f)} u$ is the approximation error of the scheme Λu calculated in pt. 4 of § 1, and ν_1 and ν_2 are the approximation errors of the boundary conditions. To estimate z we must use Lemmas 1 and 6. Then, repeating the argument given in pt. 1 in the proof of Theorem 1, we arrive at

Theorem 2. If the conditions of Theorem 1 are satisfied, then the difference problem (54) has second order accuracy on any sequence of nets, or more precisely

$$\|y-u\|_0 \leqslant M\|h\|_{\bar{2}}^2, \tag{56}$$

where

$$\|h\|_{\bar{2}} = \|h\|_2 + h_1 + h_N.$$

In this case the estimate for the error $z = y - u$ involves the steps h_1 and h_N near the boundary as well as the mean square mesh $\bar{h} = \|h\|_2$ of the net. It follows that the solution of the third boundary problem (54) satisfies the inequality

$$\|y-u\|_0 \leqslant M\|h\|_{\bar{2}}^2 = M\bar{h}^2, \tag{57}$$

if $h_1 = O(\bar{h})$, $h_N = O(\bar{h})$, i.e. the order of smallness of h_1 and h_N is not lower than that of $\bar{h}$ (thus (57) is true for the first boundary problem even if $h_1 = O(\bar{h}^{2/3})$, $h_N = O(\bar{h}^{2/3})$).

3. *The approximation error in the neighbourhood of a point of discontinuity of the coefficients*

Let us calculate the approximation error

$$\psi = \Lambda u - L^{(k,q,f)}u$$

on a non-uniform net in the neighbourhood of a point of discontinuity of the coefficients k, q, f of the differential equation. We shall assume that $k, q, f \in Q^{(1,1)}$. Let $\xi = x_n + \theta h_{n+1}$ $(0 \leqslant \theta \leqslant 1,\ h_{n+1} = x_{n+1} - x_n)$ be a point of the interval $0 < x < 1$ at which the coefficient $k(x)$ (and also $q(x)$ and $f(x)$) has a discontinuity of the first kind. Let $k_l = k(\xi - 0)$, $k_r = k(\xi + 0)$ denote the limiting values of $k(x)$ from the left and from the right at the point $x = \xi$. At this point the solution $u = u(x)$ of the differential equation $L^{(k,q,f)}u = 0$ satisfies the junction conditions

$$[u] = 0, \qquad [ku'] = (ku')_r - (ku')_l = 0.$$

To simplify printing we shall assume that there is only one point $x = \xi$ at which k, q, f are discontinuous. Since the scheme is a three-point one, at all points of the net apart from $x = x_n$ and $x = x_{n+1}$ it is possible, according to Lemma 2, to put the function $\psi(x)$ in the form

$$\psi = \mu_{\hat{x}} + \psi^*, \qquad \mu = au_{\bar{x}} - \overline{ku'} + \tfrac{1}{8}h^2 qu' \qquad \text{for } x \neq x_n,\ x \neq x_{n+1},$$
$$\mu = O(h^2), \qquad \psi^* = O(h^2) + O(h_{+1}^2). \tag{58}$$

Let us calculate $\psi(x)$ for $x = x_n$ and $x = x_{n+1}$. We write ψ in the form of the sum $\psi = \psi_a + \psi_d$, where $\psi_a = (au_{\bar{x}})_{\hat{x}} - (ku')'$, $\psi_d = -(d-q)u + \varphi - f$. Expanding $k(x)$ and $u(x)$ in a neighbourhood of the point $x = \xi$ we find

$$a_n = a(x_n) = k_1 - (0.5h_n + \theta h_{n+1})k_1' + O(h_n^2) + O(h_{n+1}^2),$$
$$a_{n+2} = k_r + (0.5h_{n+2} + (1-\theta)h_{n+1})k_r' + O(h_{n+1}^2) + O(h_{n+2}^2),$$
$$u_{\bar{x},n} = u_1' - (0.5h_n + \theta h_{n+1})u_1'' + 0.5(0.5h_n + \theta h_{n+1})^2 u_1''' + O(h_n^3) + O(h_{n+1}^3),$$
$$u_{x,n} = u_r' + (0.5h_{n+2} + (1-\theta)h_{n+1})u_r'' + 0.5(0.5h_{n+2} + (1-\theta)h_{n+1})^2 u_r''' + O(h_{n+1}^3) + O(h_{n+2}^3),$$

$$u_{\bar{x},n+1} = u_{x,n} = \theta u_1' + (1-\theta)u_r' + O(h_n) + O(h_{n+1}) = w\left(\frac{\theta}{k_1} + \frac{1-\theta}{k_r}\right) + O(h_n) + O(h_{n+1}), \qquad w = k_1 u_1' = k_r u_r'.$$

Using these expressions we obtain

$$\begin{aligned} \hbar_n\psi_{a,n} = \hbar_n\psi_a(x_n) = w\left[a_{n+1}\left(\frac{\theta}{k_1} + \frac{1-\theta}{k_r}\right) - 1\right] + a_{n+1}\frac{h_{n+1}}{2} \times \\ \times [(1-\theta)^2 u_r'' - \theta^2 u_1''] - (0.5-\theta)h_{n+1}(ku')_1' + O(h_n^2) + O(h_{n+1}^2), \\ \hbar_n\psi_{a,n} + \hbar_{n+1}\psi_{a,n+1} = (0.5-\theta)[(ku')_r' - (ku')_1'] + O(h_n^2) + O(h_{n+1}^2) + O(h_{n+2}^2). \end{aligned} \tag{59}$$

Consider now the term

$$\psi_d = -(d-q)u + \varphi - f.$$

If $D[\bar{q}(s)]$, $F[\bar{f}(s)]$ are arbitrary functionals of the family defined in § 1 pt. 1, we have

$$\psi_{d,n} = O(1), \qquad \psi_{d,n+1} = O(1).$$

Thus for an arbitrary scheme of the initial family the conditions

$$\left.\begin{aligned} \hbar_n\psi_n = O(1), \qquad \hbar_{n+1}\psi_{n+1} = O(1), \\ \hbar_n\psi_n + \hbar_{n+1}\psi_{n+1} = O(h_n) + O(h_{n+1}) + O(h_{n+2}), \end{aligned}\right\} \tag{60}$$

must be satisfied, and these are analogous to the conditions which were obtained in [1] for a uniform net.

The scheme with pattern functionals

$$A[\bar{k}(s)] = \left[\int_{-1}^{0} \frac{ds}{\bar{k}(s)}\right]^{-1}, \qquad D[\bar{f}(s)] = F[\bar{f}(s)] = \int_{-0.5}^{0.5} \bar{f}(s)\,ds, \tag{61}$$

for which

$$a_i = \left[\frac{1}{h_i}\int\limits_{x_{i-1}}^{x_i} \frac{dx}{k(x)}\right]^{-1}, \qquad d_i = \frac{1}{\hbar_i}\int\limits_{x_i-0.5h_i}^{x_i+0.5h_{i-1}} q(x)\,dx, \qquad \varphi_i = \frac{1}{\hbar_i}\int\limits_{x_i-0.5h_i}^{x_i+0.5h_{i+1}} f(x)\,dx.$$

will play a special role in our work. In this case calculation gives

$$a_{n+1} = \left(\frac{\theta}{k_l} + \frac{1-\theta}{k_r}\right)^{-1} + O(h_{n+1}),$$

$$\hbar_n(d_n - q_n)\,u_n = (0.5-\theta)(q_r - q_l)\,u(\xi) + O(h_n) + O(h_{n+1}),$$

$$\hbar_{n+1}(d_{n+1} - q_{n+1})\,u_{n+1} = O(h_{n+1}^3) + O(h_{n+2}^3) \quad \text{for } \theta < 0.5,$$

$$\hbar_n(d_n - q_n)\,u_n = O(h_n^3) + O(h_{n+1}^3), \qquad \hbar_{n+1}(d_{n+1} - q_{n+1})\,u_{n+1} =$$
$$= (0.5-\theta)(q_r - q_l)\,u(\xi) + O(h_{n+1}) + O(h_{n+2}) \quad \text{for } \theta > 0.5.$$

Similar expressions are obtained for $\hbar_n(\varphi_n - f_n)$, $\hbar_{n+1}(\varphi_{n+1} - f_{n+1})$. It follows from this and from (59) that

$$\hbar_n\psi_n = O(h_n) + O(h_{n+1}), \qquad \hbar_{n+1}\psi_{n+1} = O(h_{n+1}) + O(h_{n+2}). \tag{62}$$

Instead of (60) we obtain

$$\hbar_n\psi_n + \hbar_{n+1}\psi_{n+1} = (0.5-\theta)\,h_{n+1}\,[(L^{(k,q,f)}u)_r - (L^{(k,q,f)}u)_l] + O(h_n^2) +$$
$$+ O(h_{n+1}^2) + O(h_{n+2}^2),$$

i.e.

$$\hbar_n\psi_n + \hbar_{n+1}\psi_{n+1} = O(h_n^2) + O(h_{n+1}^2) + O(h_{n+2}^2). \tag{62'}$$

4. *Concerning accuracy in the class of discontinuous coefficients*

Let us now find the order of accuracy of our homogeneous difference schemes in the class of discontinuous coefficients on an arbitrary sequence of non-uniform nets ω_N. We shall assume that $k(x)$, $q(x)$, $f(x) \in Q^{(1,1)}[0, 1]$. It is sufficient to consider the case of one discontinuity at the point $\xi = x_n + \theta h_{n+1}$, $0 \leqslant \theta \leqslant 1$. From pt. 3 of § 2 and Lemma 1 we have

$$\psi = \mu_{\hat{x}} + \psi^*, \qquad \psi^* = O(h^2) + O(h_{+1}^2) \qquad \text{for } x \neq x_n,\ x \neq x_{n+1},$$

$$\mu = au_{\bar{x}} - \overline{ku'} + \frac{h^2}{8}qu' = O(h^2) \qquad \text{for } x \neq x_{n+1}.$$

Lemma 5 gives

$$\|z\|_0 \leqslant M(\bar{h}^2 + h_n^2 + h_{n+2}^2) + M_1\,|\hbar_n\psi_n + \hbar_{n+1}\psi_{n+1}| + M_2 h_{n+1}\hbar_n\,|\psi_n|, \tag{63}$$

where $z = y - u$ is a solution of problem (14). It follows from this and from (62) that for any scheme of the initial family

$$\|z\|_0 \leqslant M\,(\bar{h}^2 + h_n + h_{n+1} + h_{n+2}), \tag{64}$$

where M is a positive constant which does not depend on the net.

For scheme (61), according to (62)-(62′),

$$|\hbar_n\psi_n + \hbar_{n+1}\psi_{n+1}| + h_{n+1}\hbar_n|\psi_n| \leqslant M\,(h_n^2 + h_{n+1}^2 + h_{n+2}^2), \tag{65}$$
$$\|z\|_0 \leqslant M\bar{h}^2 + M\,(h_n^2 + h_{n+1}^2 + h_{n+2}^2).$$

If $k,\ q,\ f$ have discontinuities of the first kind at the points $\xi_j = x_{n_j} + h_{n_j+1}\,\theta_j,\ 0 \leqslant \theta_j \leqslant 1,\ j = 1,\ 2,\ \ldots,\ j_0$ then instead of (63) and (64) we obtain

$$\|z\|_0 \leqslant M\bar{h}^2 + M'\sum_{j=1}^{j_0}(h_{n_j}^{\varkappa} + h_{n_j+1}^{\varkappa} + h_{n_j+2}^{\varkappa}),$$

where $\varkappa = 2$ for the scheme (61), and $\varkappa = 1$ for the whole family of schemes. By the same token we can prove the following theorem.

Theorem 3. The homogeneous difference scheme (13) is uniformly convergent in the class of discontinuous coefficients on any sequence of non-uniform nets ω_N. If $k,\ q,\ f \in Q^{(1,1)}\,[0,\ 1]$ then

$$\|y - u\|_0 \leqslant M\bar{h}^2 + M'\sum_{j=1}^{j_0}(h_{n_j}^{\varkappa} + h_{n_j+1}^{\varkappa} + h_{n_j+2}^{\varkappa}), \tag{66}$$

where y is a solution of problem (13), u is a solution of the initial problem (5), M and M' are positive constants which do not depend on the nets, j_0 is the number of points $\xi_j = x_{n_j} + \theta_j h_{n_j+1}$ of the interval $0 < x < 1$ at which at least one of the coefficients $k(x),\ q(x),\ f(x)$ has a discontinuity, $\varkappa = 2$ for scheme (61) and $\varkappa = 1$ for the whole initial family of homogeneous difference schemes.

Note 1. Unlike the case of smooth coefficients, in this case the error $z = y - u$ depends not only on the mean square mesh $\bar{h} = \|h\|_2$ which is an integral characteristic of the net, but also on the meshlengths of the net in the neighbourhood of the point of discontinuity $\xi_j = x_{n_j} + \theta_j h_{n_j+1}$ of the coefficients of the differential equation. It follows in particular that if the position of the points of discontinuity ξ_j of fixed $k(x)$, $q(x)$ and $f(x)$ is known then, by choosing a finer net in the neighbourhood of the points ξ_j, the order of accuracy of the scheme can be increased. Thus, for example, if we choose $h_{n_j} = O(\bar{h}^2)$,

$h_{n_j+1}=O(\bar{h}^2)$, $h_{n_j+2}=O(\bar{h}^2)$, we obtain

$$\| y-u \|_0 \leqslant M\bar{h}^2, \qquad \bar{h}^2=\| h \|_2^2,$$

i.e. on these nets any initial scheme will have second order accuracy.

Note 2. If the point of discontinuity $\xi_j = x_{n_j}$ is a nodal point of the net ($\theta_j = 0$) then it is not difficult to show that

$$\hbar_{n_j}\psi_{n_j}=O(h_{n_j}^2)+O(h_{n_j+1}^2), \qquad \hbar_{n_j+1}\psi_{n_j+1}=O(h_{n_j+1}^2)+O(h_{n_j+2}^2).$$

Clearly in this case we have, instead of (66),

$$\| y-u \|_0 \leqslant M\bar{h}^2+M'\sum_{j=1}^{j_0}(h_{n_j}^2+h_{n_j+1}^2), \tag{67}$$

i.e. any initial scheme has second order accuracy on a sequence of nets for which the points of discontinuity of the fixed coefficients k, q, f are nodal points. If k, q, f are known, then clearly we can always change the net near the points of discontinuity ξ_j and arrange for all the ξ_j to be nodal points of the net. The nets obtained in this way will no longer be arbitrary, but will depend on the actual functions k, q, f. It is convenient to use these nets in cases where the positions of the discontinuities of the coefficients of the differential equation are known beforehand. It must be stressed that we have been discussing above the whole class of discontinuous coefficients and any sequence of nets ω_N obtained by an arbitrary division of the segment $0\leqslant x\leqslant 1$ into N parts by the points $x_0 = 0$, x_1, ..., x_i, ..., $x_N = 1$.

5. *Homogeneous difference schemes of the second type*

The homogeneous difference schemes we have been considering above can be obtained by the integral interpolation method (see [1]) by writing down the balance equation for the interval ($\bar{x}_{i-1}\leqslant x\leqslant \bar{x}_i = 0.5\,(x_i + x_{i+1})$). If we write down the balance equation for ($x_i \leqslant x \leqslant x_{i+1}$) and refer the values of the function y to the point $\bar{x}_i$ by putting $\bar{y}_i = y(\bar{x}_i)$ then we can obtain homogeneous difference schemes of the second type

$$\Lambda\bar{y}=(a\bar{y}_{\check{x}})_x-d\bar{y}+\varphi=0, \qquad y(0)=u_1, \qquad y(1)=u_2, \tag{68}$$

where

$$\bar{y}_{\check{x}}=(\bar{y}-\bar{y}^{(-1)})\,\hbar, \qquad \hbar=0.5\,(h+h_{+1}), \qquad h=h_i=x_i-x_{i-1},$$
$$(a\bar{y}_{\check{x}})_{x,i}=\frac{1}{h_{i+1}}\left[a_{i+1}\frac{\bar{y}_{i+1}-\bar{y}_i}{\hbar_{i+1}}-a_i\frac{\bar{y}_i-\bar{y}_{i-1}}{\hbar_i}\right].$$

The values of the function $\bar{y}$ refer to the points $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{N-1}$, i.e. there are N internal points and not $N-1$.

The coefficients a, d, φ are defined with the help of the same pattern functionals A, D and F as in pt. 1 of § 1, by the formulae

$$a = A[k(\bar{x} + s\hbar)], \; d = D[q(\bar{x} + sh_{+1})], \; \varphi = F[f(\bar{x} + sh_{+1})] \qquad (\bar{x} = x + 0.5h_{+1})$$

We can obtain equations (68) at the points $\bar{x}_0 = 0.5h_1$, $\bar{x}_{N-1} = x_N - 0.5h_N$ by putting formally $h_0 = h_{N+1} = 0$, $\bar{y}_N = y(1)$, $\bar{y}_{-1} = y(0)$ so that $\hbar_0 = 0.5\, h_1$, $\hbar_N = 0.5\, h_N$.

Writing $\bar{y} = z + u(\bar{x})$ we obtain the conditions

$$\overline{\Lambda} z = (az_{\check{x}})_x - dz = -\psi, \qquad z(0) = z(1) = 0 \tag{69}$$
$$(0 < c_1 \leqslant a \leqslant c_1', \quad 0 \leqslant d \leqslant c_2),$$

for z, where $\psi = \Lambda\bar{u} - (L^{(k,q,f)}u)_{x=\bar{x}}$ is the approximation error of the scheme (68).

Let us assume that $k, q, f \in C^{(1,1)}$ and therefore $u(x) \in C^{(3,1)}$; arguing by analogy with pt. 3 of § 1 and using the relations

$$(\overline{ku'})' = (ku')_x + O(h_{+1}^2), \qquad d = \bar{q} + O(h_{+1}^2), \qquad \varphi = \bar{f} + O(h_{+1}^2),$$

we find

$$\psi = \mu_x + \psi^*, \qquad \mu = au_{\check{x}} - ku',$$
$$\psi^* = O(h^2) + O(\hbar_{+1}^2) + O(h_{+2}^2). \tag{70}$$

It follows from the expansions

$$u_{\check{x}} = u' + \tfrac{1}{4}(h_{+1} - h)\, u'' + O(h^2) + O(h_{+1}^2),$$
$$a = k + \tfrac{1}{4}(h_{+1} - h)\, k' + O(h^2) + O(h_{+1}^2)$$

that

$$\mu = \eta_{\check{x}} + \mu^*, \qquad \eta = \tfrac{1}{8} h_{+1}^2 (ku')'^{(+1)} = \tfrac{1}{8} h_{+1}^2 (qu - f)^{(+1)},$$
$$\mu^* = O(h^2) + O(h_{+1}^2), \tag{71}$$

Thus formula (70) takes the form

$$\psi = \mu_x^* + \psi^* + \eta_{\check{x}x}, \tag{72}$$

where μ^*, η and ψ^* are of second order with respect to $\|h\|_0 = \max\limits_{(i)} h_i$.

The solution of problem (69) is given by the formula

$$z(x) = (G(x, \xi), \psi(\xi))^+ = \bar{z} + v, \qquad \bar{z} = (G, \mu_{\xi}^* + \psi^*)^+, \qquad v = (G, \eta_{\check{\xi}\xi})^+ \tag{73}$$

$$\left((y, v)^+ = \sum_{i=0}^{N-1} \bar{y}_i \bar{v}_i h_{i+1}, \qquad (y, v) = \sum_{i=0}^{N-1} \bar{y}_i \bar{v}_i h_i \qquad \text{etc.}\right),$$

where $G(x, \xi)$ is Green's difference function, defined by the conditions

$$(aG_{\check{x}}(x, \xi))_x - d(x)\, G(x, \xi) = -\delta(x, \xi) / h_{+1}, \quad G(0, \xi) = G(1, \xi) = 0. \tag{74}$$

The inequalities of Lemma 2 apply to $G(x, \xi)$. Therefore Lemma 3 is applicable and so

$$\|\bar{z}\|_0 \leqslant M_2 (\|\mu^*\|_1 + \|\psi^*\|_3). \tag{75}$$

Green's second formula (26) gives

$$v(x) = (G(x, \xi), \eta_{\check{\xi}\xi}(\xi))^+ = (G_{\check{\xi}\xi}(x, \xi), \eta(\xi))^+ + G_{\check{\xi}}\eta\,|_{\xi=x_N} - \eta G_{\hat{\xi}}\,|_{\xi=0}.$$

Let us insert here the expression for $G_{\check{\xi}\xi}$ found from equation (74):

$$v(x) = -\frac{\eta(x)}{a(x)} - \left(\frac{a_{\xi}(\xi)}{a(\xi)} G_{\hat{\xi}}(x, \xi) - \frac{d(\xi)}{a(\xi)} G(x, \xi), \eta(\xi)\right)^+ + G_{\check{\xi}}\eta\,|_{\xi=x_N} - \eta G_{\hat{\xi}}\,|_{\xi=0}.$$

If $k(x) \in C^{(0,1)}$ then $|a_x| \leqslant c_4$ (c_4 is a constant which does not depend on the net) and

$$\|v\|_0 \leqslant M \|\eta\|_0 \qquad (M = M(c_1, c_2, c_4)). \tag{76}$$

Let $k(x) \in Q^{(0,1)}$ $(0 < c_1 \leqslant a \leqslant c_1')$ and $\xi = \bar{x}_n + \theta \hbar_n$ be a point of discontinuity of the function $k(x)$. Then $|a_x| \leqslant c_4$ for $x \neq \bar{x}_n$ and $x \neq \bar{x}_{n+1}$. It is clear from the formula

$$\left(\frac{a_{\xi}}{a} G_{\hat{\xi}}, \eta\right)^+ = \frac{1}{a_n}(a_{n+1} - a_n) G_{\hat{\xi}}(x, x_n) + \frac{1}{a_{n+1}}(a_{n+2} - a_{n+1}) G_{\hat{\xi}}(x, x_{n+1}) +$$
$$+ \sum_{i=0}^{n-1} \frac{1}{a_i} a_{x,i} G_{\hat{\xi}}(x, \bar{x}_i) h_{i+1} + \sum_{i=n+2}^{N-1} \frac{1}{a_i} a_x(\bar{x}_i) G_{\hat{\xi}}(x, \bar{x}_i) h_{i+1}$$

that the estimate (76) is valid. In this case $M = M(c_1, c_1', c_2, c_4)$.

By the same token we prove Lemma 7.

Lemma 7. If $0 < c_1 \leqslant a \leqslant c_1'$, $0 \leqslant d \leqslant c_2$ and $|a_x| \leqslant c_4$ everywhere apart from a finite number of points of the net (for any N), and $\psi(x)$ has the form (72), then the solution of problem (69) satisfies the relation

$$\|z\|_0 \leqslant M_2 (\|\mu^*\|_1 + \|\psi^*\|_3) + M_4 \|\eta\|_0,$$

where $M_4 > 0$ is a constant depending only on c_1, c_1, c_2, c_4, $M_2 = (c_1 + c_2)/c_1^2$.

Theorem 4. If k, q, $f \in C^{(1,1)}$ $[0, 1]$ then any homogeneous difference scheme (68) has second order accuracy on an arbitrary sequence of nets ω_N:

$$\|\bar{y} - u(\bar{x})\|_0 \leqslant M \|h\|_0^2, \tag{77}$$

where $\bar{y}$ is a solution of problem (68), $u(x)$ is a solution of the initial problem (5) and $M > 0$ is a constant which does not depend on the net. If k, q, $f \in Q^{(1,1)}$ $[0, 1]$ then

$$\|\bar{y} - u(\bar{x})\|_0 \leqslant M \|h\|_0^{\varkappa}, \tag{78}$$

where $\kappa = 1$ for the whole family of schemes (68), $\kappa = 2$ for scheme (61).

The relation (77) follows from Lemma 7 and the estimates for μ^*, η, ψ^*. When $q = f = 0$ we have instead of (77) $\|\bar{y} - u(\bar{x})\|_0 \leqslant M \|h\|_2^2$. The estimate (78) can be derived by using the analogue of Lemma 6.

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