# ON AN ECONOMICAL DIFFERENCE METHOD FOR THE SOLUTION OF A MULTIDIMENSIONAL PARABOLIC EQUATION IN AN ARBITRARY REGION*

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## Introduction

Let us consider the equation of parabolic type

$$\frac{\partial u}{\partial t} = Lu = \sum_{\alpha=1}^{p} L_\alpha u, \qquad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}, \quad x = (x_1, \ldots, x_p). \tag{1.B}$$

with several space variables $x_1, \ldots, x_\alpha, \ldots, x_p$. Let $\{x_i = (i_1 h_1, \ldots, i_\alpha h_\alpha, \ldots, i_p h_p),\ t_j = j \times \tau;\ i_\alpha = 0, \pm 1, \pm 2, \ldots;\ j = 0, 1, \ldots\}$ be a space-time difference net with steps $h_\alpha$ and $\tau$, $y^j = y(x_i, t_j)$ a net function, $\Lambda_\alpha$ the difference approximation of the operator $L_\alpha$ ($\Lambda_\alpha \sim L_\alpha$). We know that for the numerical solution of equation (1.B) the natural multidimensional schemes of the form

$$\frac{y^{j+1} - y^j}{\tau} = \sigma(\Lambda y)^{j+1} + (1-\sigma)(\Lambda y)^j, \qquad \Lambda = \sum_{\alpha=1}^{p} \Lambda_\alpha, \qquad 0 \leqslant \sigma \leqslant 1, \tag{2.B}$$

are not suitable: the explicit schemes ($\sigma = 0$) are conditionally stable only for sufficiently small values of the step $\tau$ in time, and although the implicit schemes ($\sigma \geqslant 0.5$) are absolutely stable, they necessitate the solution of a multidimensional system of algebraic equations which, even for two space variables ($p = 2$), takes many operations. In this connexion various "economical" schemes have been suggested in a number of works (see [1]-[11], [20]). The multidimensional equation is solved

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by stages by introducing intermediate (fractional, see [8]) steps in each of which there is no accurate approximation or even stability, generally speaking, these only occurring on going from the whole step $t_j$ to the step $t_{j+1}$. In the fractional steps the diagrams used lead to one-dimensional algebraic problems (in a spatial direction) for a three-point difference equation, the solution of which can be found using the well-known successive substitution formulae [12]. The simplest algorithm for the solution of the heat conductivity equation with constant coefficients

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} L_\alpha u, \qquad L_\alpha u = \frac{\partial^2 u}{\partial x_\alpha^2} \tag{3.B}$$

for the parallelepiped $G = (0 \leqslant x_\alpha \leqslant l_\alpha,\ \alpha = 1, 2, \ldots, p)$ and with the boundary condition of the first type $(u|_\Gamma = \mu)$ has been suggested by N.N. Yanenko (see [4]). This algorithm, which the author calls the splitting method, consists in approximating to the multidimensional scheme (2.B) by the scheme

$$\frac{y^{j+\alpha/p} - y^{j+(\alpha-1)/p}}{\tau} = \sigma\Lambda_\alpha y^{j+\alpha/p} + (1-\sigma)\Lambda_\alpha y^{j+(\alpha-1)/p} \qquad (\alpha = 1, \ldots, p). \tag{4.B}$$

It is clear that the values of $y^{j+\alpha/p}$ are determined from one-dimensional equations, to solve which it is sufficient to use the boundary conditions for $x_\alpha = 0$ and $x_\alpha = l_\alpha$

Despite the variety of algorithms and terminological differences in all the works listed above (method of variable directions, splitting method, fractional steps method, and so on) a study of the stability and convergence of their methods reduces to the study of the stability, approximation and convergence of a multidimensional scheme connecting the values of $y^j$ and $y^{j+1}$ at integral steps and obtained after eliminating the values of $y^{j+\alpha/p}$ $(\alpha = 1, 2, \ldots, p-1)$ at the intermediate steps.

If we do this, for example, for (4.B) we obtain

$$\prod_{\alpha=1}^{p} (E - \sigma\tau\Lambda_\alpha)\, y^{j+1} = \prod_{\alpha=1}^{p} (E + (1-\sigma)\,\tau\Lambda_\alpha)\, y^j \qquad (Ey = y). \tag{5.B}$$

This scheme is equivalent with respect to its order of approximation to the multidimensional scheme (2.B). In particular, for $p = 2$, $\sigma = 0.5$ scheme (5.B) has the form

$$\frac{y^{j+1} - y^j}{\tau} = 0.5\,(\Lambda y^{j+1} + \Lambda y^j) - \frac{\tau}{4}\,\Lambda_1\Lambda_2\,(y^{j+1} - y^j). \tag{6.B}$$

By the error of approximation of the scheme (4.B) we understand the error of approximation of the multidimensional scheme (5.B).

It has been shown in [9] that the scheme (4.B) can be considered as the result of factorising (splitting) equation (5.B), i.e. an approximate factorisation of equation (4.B). The difference scheme (5.B) has a troublesome feature: the space operator has order $2p$ and is not defined, in general, near the boundary of the given region. This leads to the problem of boundary conditions even for the simplest rectangular region (see [10] and [11]). It follows from our results, in particular, that for the splitting scheme (4.B) this boundary condition problem is removed, and that, when interpreted differently and with a corresponding definition of the concept of approximation this scheme is applicable to an arbitrary region and, when $\sigma = 1$, converges uniformly at the rate $O(h^2) + O(\tau)$.

We note also that the approximation requirement for the scheme (5.B) leads to an overstatement of the smoothness requirements for the solution $u = u(x, t)$ of the differential equation.

In this article we consider a local one-dimensional method for solving linear and quasilinear equations of parabolic type with any number $p$ of space variables, for an arbitrary region $G$. Let us give a brief description of the method using equation (1) as an example. In each layer

$$t_{j+(\alpha-1)/p} \leqslant t \leqslant t_{j+\alpha/p} = t_j + \tau\alpha/p \qquad (\alpha = 1, 2, \ldots, p)$$

the one-dimensional differential equation

$$\frac{1}{p}\frac{\partial u}{\partial t} - L_\alpha u = 0. \tag{7.B}$$

is solved. To do this we use the implicit homogeneous difference schemes

$$\Pi_\alpha y = \frac{y^{j+\alpha/p} - y^{j+(\alpha-1)/p}}{\tau} - \Lambda_\alpha y^{j+\alpha/p} = 0, \tag{8.B}$$

discussed in [13]-[18].

For the special case $k_\alpha = 1$, $r_\alpha = 0$ the scheme (8.B) is formally the same as (4.B) for $\sigma = 1$.

The difference scheme $\Pi y$ corresponding to equation (1) is the set (of blocks) $\Pi = \{\Pi_\alpha, \alpha = 1, \ldots, p\}$ of $p$ one-dimensional schemes $\Pi_\alpha$. An important characteristic of every scheme is its approximation error. In this case the ordinary approximation error of scheme $\Pi$, as we saw in the example of scheme (4.B), is not rational. Each of the schemes has an approximation error

$$\psi_\alpha = \Lambda_\alpha u^{j+\alpha/p} - \frac{u^{j+\alpha/p} - u^{j+(\alpha-1)/p}}{\tau}, \qquad \psi_\alpha = O(1),$$

where $u$ is a solution of equation (1.B). The approximation error of the scheme $\Pi = \{\Pi_\alpha\}$ is rationally defined as the sum

$$\Psi = \sum_{\alpha=1}^{p} \psi_\alpha. \tag{9.B}$$

If $\Lambda_\alpha$ is a scheme of the second order of approximation, i.e. $\Lambda_\alpha u - L_\alpha u = O(h_\alpha^2)$ then

$$\Psi = O(h^2) + O(\tau), \text{ where } h^2 = \frac{1}{p} \sum_{\alpha=1}^{p} h_\alpha^2 .$$

We shall show in § 2 that the order of accuracy of scheme $\Pi$ is the same as that of its approximation $\Psi$. This definition of the difference scheme $\Pi$ is independent of the shape of the region $G$ and the actual form of the operator $\Lambda_\alpha$.

At each moment of time $t_{j+\alpha/p}$ the equation (8.B) connects the values $y^{j+\alpha/p}$ at the nodes of the net lying on a straight line parallel to the axis $0x_\alpha$. Therefore at each moment of time at each point of the net we solve the one-dimensional heat conductivity equation for a segment with ends belonging to the boundary of the region. It follows at once from this that it is possible to use this method for an arbitrary region and for parabolic equations of a general form. It should be noted that in solving equation (8.B) along each direction $x_\alpha$ at the moment $t_{j+\alpha/p}$ we use the values $u|_\Gamma = \mu$ only at the points of intersection with $\Gamma$, the boundary of $G$, of straight lines parallel to $\Gamma$ and passing through the nodes of the net, and not over the whole of $\Gamma$.

To find $y^{j+\alpha/p}$ we can use the boundary data $\mu(x, t)$ and the values of the coefficients $k_\alpha(x, t)$, $r_\alpha(x, t)$ at any moment $t'_\alpha \in [t_j, t_{j+1}]$ (for example, $t'_\alpha = t_{j+1}$ for all $\alpha$). All the schemes obtained will have the same order of accuracy. For definiteness we take boundary data for $t'_\alpha = t_{j+\alpha/p}$ without loss of generality.

In § 1 we formulate the local one-dimensional method a) for a linear and b) for a quasilinear equation

$$c(x, t) \frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} L_\alpha u + f,$$

$$\text{a) } L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t) \frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x, t) \frac{\partial u}{\partial x_\alpha}, \qquad f = f(x, t) - q(x, t) u,$$

b) $L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x,t,u)\frac{\partial u}{\partial x_\alpha}, \qquad f = f(x,t,u)$

for an arbitrary region $G$ and for boundary conditions of the first kind. We consider some family of homogeneous difference schemes (see [13]-[18]), defined by a given class of pattern functionals with the help of which the coefficients of the difference scheme are calculated. In point 6 we consider the third boundary problem for a parallelepiped.

In point 1 of § 2 we prove the uniform stability of the local one-dimensional scheme with respect to the right-hand side, the boundary and initial data. The main result is Theorem 2 concerning the *uniform convergence* and accuracy of the method. It is shown that local one-dimensional schemes give the same accuracy $O(h^2) + O(\tau)$ as multi-dimensional implicit difference schemes [18].

We shall restrict ourselves here to the case of smooth coefficients. It must be emphasised that the maximum order of the derivatives required for the convergence of the method does not depend on the number of dimensions (see [11]). If the coefficients of the differential equation are discontinuous, then the order of accuracy of the scheme is reduced, by analogy with the one-dimensional case $p = 1$ (see [15], [17], [18]).

The results we obtain are applied to the case of arbitrary non-uniform nets.

In § 3 we give computing formulae for $p = 2$, and also the schemes for other equations (parabolic and hyperbolic).

## 1. The local one-dimensional method of variable directions

In this section we consider a homogeneous local one-dimensional difference scheme for the solution of a linear parabolic equation with any number of space variables. The reasoning is given for the first boundary problem and for an arbitrary region. In point 5 we consider a scheme for a quasilinear equation, and in point 6, the third boundary problem for a parallelepiped.

*1. The initial problem*

Let us consider the $p$-dimensional linear equation of parabolic type

$$c(x,t)\frac{\partial u}{\partial t} = Lu + f,$$
$$Lu = \sum_{\alpha=1}^{p}\left\{\frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right\} - q(x,t)\,u, \tag{1}$$

where $x = (x_1, \ldots, x_\alpha, \ldots, x_p)$ is a point of $p$-dimensional space with coordinates $x_1, x_2, \ldots, x_\alpha, \ldots, x_p$; and $c(x, t) = c(x_1, \ldots, x_p, t)$, $k = k_\alpha(x, t)$, $q = q(x, t)$, $f = f(x, t)$ are given functions. Let $G$ be an arbitrary closed $p$-dimensional region with boundary $\Gamma$,

$$\bar{G} = G + \Gamma, \qquad \bar{Q}_T = \bar{G} \times [0 \leqslant t \leqslant T], \qquad Q_T = G \times (0 < t \leqslant T).$$

It is required to find, in the cylinder $\bar{Q}_T$, a solution of the first boundary problem for equation (1):

$$c\,\frac{\partial u}{\partial t} = Lu + f \quad \text{in } Q_T, \tag{1}$$

$$u|_\Gamma = \mu(x, t) \quad \text{for } x \in \Gamma,\ 0 \leqslant t \leqslant T, \tag{2}$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in \bar{G}, \tag{3}$$

where $\mu(x, t)$ and $u_0(x)$ are given functions. The coefficients $k_\alpha$ and $c$ are bounded below:

$$k_\alpha(x, t) \geqslant c_1 > 0, \qquad c(x, t) \geqslant c_2 > 0 \text{ in } Q_T, \tag{4}$$

where $c_1$, $c_2$ are constants.

We shall assume everywhere that: 1) the problem (1)-(4) has a unique solution $u = u(x, t)$, continuous in the closed region $\bar{Q}_T$; and 2) the following conditions are satisfied:

*Conditions A*:

a) the functions

$$\frac{\partial^3 u}{\partial x_\alpha^2 \partial x_\beta}, \quad \frac{\partial^2 k_\alpha}{\partial x_\alpha \partial x_\beta}, \quad \frac{\partial r_\alpha}{\partial x_\beta}$$

where $\beta \leqslant \alpha$ for $\alpha \leqslant \left[\frac{p}{2}\right]$ and where $\beta > \alpha$ for $\alpha > \left[\frac{p}{2}\right]$ ($\alpha, \beta = 1, 2, \ldots, p$) and also the functions $\partial q/\partial x_\beta$, $\partial f/\partial x_\beta$, $\partial^2 u/\partial x_\beta \partial t$, $\partial c/\partial x_\beta$ satisfy the Lipschitz conditions in $x_\beta$ and $\bar{Q}_T$;

b) the functions $\partial u/\partial x_\alpha$, $\partial^2 u/\partial x_\alpha^2$, $\partial u/\partial t$, $c$, $q$, $f$, $r_\alpha$, $k_\alpha$, $\partial k_\alpha/\partial x_\alpha$ ($\alpha = 1, \ldots, p$) satisfy the Lipschitz conditions in $t$ in the closed region $\bar{Q}_T$.

These conditions are sufficient for the proof of the basic theorem, Theorem 2, concerning the accuracy of the difference schemes considered below and in a number of cases they can be replaced by weaker requirements. The smoothness requirements laid down for the surface $\Gamma$ are, as we know, connected with the properties of the solution $u = u(x, t)$ of the problem.

We make only one assumption: the intersection of the region $G$ by any straight line $\mathcal{L}_\alpha$ drawn through a point $x \in G$ parallel to the coordinate axis $Ox_\alpha$ consists of a finite number of intervals (we do not exclude the case when the intersection of this straight line with the boundary $\Gamma$ consists of whole segments, and not only of isolated points). To simplify the printing we shall always discuss the case when the intersection of $\mathcal{L}_\alpha$ and $G$ consists of one interval, or, more exactly, when the straight line $\mathcal{L}_\alpha$ intersects $\Gamma$ in two points. The general case is considered in a similar way.

## 2. *The difference nets*

We place the origin $O = (0, \ldots, 0)$ of a rectangular coordinate system $(x_1, \ldots, x_p)$ inside the region $G$ and draw $p$ families of hyperplanes

$$x_\alpha^{(i_\alpha)} = h_\alpha i_\alpha, \qquad i_\alpha = 0, \pm 1, \pm 2, \ldots, \qquad \alpha = 1, 2, \ldots, p.$$

The points of intersection $x_i = (x_1^{(i_1)}, \ldots, x_\alpha^{(i_\alpha)}, \ldots, x_p^{(i_p)})$ of these hyperplanes are the nodes. Two nodes are usually said to be adjacent if they are at a distance of $h_\alpha$ from one another in one of the directions $x_\alpha$ $(\alpha = 1, \ldots, p)$. The node $x_i \in G$ is said to be an internal node if all its $p$ neighbouring nodes belong to $\bar{G}$. A node is said to be a boundary node if at least one of its neighbouring nodes does not belong to $\bar{G}$. The set of all internal nodes forms the internal net region $\omega_h$, and the set of boundary nodes, the boundary $\gamma$ of the net $\omega_h$. It follows from the definition that $\bar{\omega}_h = \omega_h + \gamma \subset \bar{G}$. Through some point of the net $\omega_h$ let us draw the straight line $\mathcal{L}_\alpha$ parallel to the coordinate axis $Ox_\alpha$. The set of all nodes of the net $\bar{\omega}_h$ lying on this straight line is called the $x_\alpha$-chain and we denote it by $\Pi_\alpha$. The set of boundary nodes of all chains $\Pi_\alpha$ in a given direction $x_\alpha$ is denoted by $\gamma_\alpha$. It is obvious that $\gamma_\alpha \subset \gamma$, $\gamma_1 + \ldots + \gamma_p \subset \gamma$. Suppose $\bar{G} = \{0 \leqslant x_\alpha \leqslant l_\alpha, \ \alpha = 1, \ldots, p\}$ is a $p$-dimensional parallelepiped. Then all the chains $\Pi_\alpha$ for given $\alpha$ have the same number of nodes, and their boundary points are on the planes $x_\alpha = 0$, $x_\alpha = l_\alpha$. In this case $\gamma_\alpha$ is the set of nodes lying on the planes $x_\alpha = 0$, $x_\alpha = l_\alpha$, $0 < x_\beta < l_\beta$, $\alpha \neq \beta$.

Let us now introduce a net with respect to time $t$, by dividing the segment $0 \leqslant t \leqslant T$ into $K$ equal parts by points $t_j = j\tau$, $j = 0, 1, \ldots, K$. Each of the segments $[t_j, t_{j+1}]$ is divided into $p$ (the number of dimensions) equal parts by introducing the intermediate (fractional) moments of time $t_{j+\alpha/p} = t_j + \tau\alpha/p = (j + \alpha/p)\tau$, where $\alpha = 1, \ldots, p$; $j = 0, 1, \ldots, K-1$. We call the point $(x_i, t_{j+\alpha/p})$, where $x_i \in \bar{G}$, a node of the space-time net. The set of all nodes $(x_i, t_{j+\alpha/p})$, where $x_i \in \omega_h$, $j = 0, 1, \ldots, K-1$, $\alpha = 1, 2, \ldots, p$ is denoted by $\Omega$.

Let us now denote by $S_{(\alpha)}$ the set of points $(x_i, t_{j'+\alpha/p})$, where $x_i \subset \gamma_\alpha$, $t_{j'+\alpha/p} = (j' + \alpha/p)\tau$, $j' = 0, 1, \ldots, K-1$ and $\alpha$ is fixed $(1 \leqslant \alpha \leqslant p)$, $S_{(0)}$ is the set of points $(x_i, 0)$, where $x_i \in \gamma$, $S = S_{(0)} + S_{(1)} + \ldots + S_{(p)}$. The set of nodes $S \subset \bar{Q}_T$ is the boundary of the net region $\Omega$. We shall henceforth consider the net functions defined on the space-time net $\bar{\Omega} = \Omega + S$. Depending on the circumstances we shall use one of the notations:

$$y = y(x, t) = y(x_i, t_{j+\alpha/p}) = y^{j+\alpha/p}.$$

We shall also write

$$x^{(\pm m_\alpha)} = x_i^{(\pm m_\alpha)} = (x_1^{(i_1)}, \ldots, x_{\alpha-1}^{(i_{\alpha-1})}, x_\alpha^{(i_\alpha)} \pm mh_\alpha, \quad x_{\alpha+1}^{(i_{\alpha+1})}, \ldots, x_p^{(i_p)}), \quad m > 0,$$
$$y^{(\pm m_\alpha)} = y(x^{(\pm m_\alpha)}, t), \quad y_{\bar{x}_\alpha} = (y - y^{(-1_\alpha)}) / h_\alpha,$$
$$y_{x_\alpha} = (y^{(+1_\alpha)} - y) / h_\alpha, \qquad y_{\overset{\circ}{x}_\alpha} = 0.5\,(y_{\bar{x}_\alpha} + y_{x_\alpha}), y_{\bar{t}_\alpha} = (y^{j+\alpha/p} - y^{j+(\alpha-1)/p}) / \tau.$$

### 3. *Local one-dimensional homogeneous difference schemes*

Let us now formulate the difference algorithm for the solution of the problem (1)-(4) for an arbitrary region $G$.

Instead of writing down a multidimensional difference scheme which enables us to find the numerical solution of equation (1) for $t = t_{j+1}$ (at whole steps) we shall solve at each of the moments $t_{j+\alpha/p}$ the parabolic one-dimensional differential equation

$$\frac{1}{p} c(x, t) \frac{\partial u}{\partial t} = L_\alpha u + f_\alpha(x, t), \tag{5}$$

where

$$L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t) \frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x, t) \frac{\partial u}{\partial x_\alpha} - q_\alpha(x, t)\, u. \tag{6}$$

Here $q_\alpha$, $f_\alpha$ are arbitrary functions satisfying the same smoothness conditions as $q$, $f$ and connected with $q$, $f$ only by the conditions

$$\sum_{\alpha=1}^{p} q_\alpha(x, t) = q(x, t), \qquad \sum_{\alpha=1}^{p} f_\alpha(x, t) = f(x, t). \tag{7}$$

For example, $q_\alpha = q/p$, $f_\alpha = f/p$ or $q_\alpha = 0$, $f_\alpha = 0$ for $\alpha = 1, 2, \ldots, p-1$, $q_p = q$, $f_p = f$. For each $\alpha$ we look for a solution of equation (5) at the moment of time $t_{j+\alpha/p}$. In order to find the solution of equation (5) inside $G$ in the time interval $t_{j+(\alpha-1)/p} < t \leqslant t_{j+\alpha/p}$ it is sufficient to use the initial data for $t = t_{j+(\alpha-1)/p}$ and the boundary conditions (2) at only those points where the boundary $\Gamma$ intersects the straight lines parallel to the coordinate $Ox_\alpha$. Let us illustrate this point. Let $\bar{G} = \{0 \leqslant x_\alpha \leqslant l_\alpha,\ \alpha = 1,\ 2,\ p = 2\}$ be a rectangle in the plane $(x_1,\ x_2)$. Then to solve equation (5), for $\alpha = 1$ say, it is sufficient to know the boundary conditions on the sides $x_1 = 0$ and $x_1 = l_1$ only of this rectangle. The boundary conditions on the other part of the boundary, i.e. on the sides $x_2 = 0$ and $x_2 = l_2$ are used for $\alpha = 2$, i.e. to solve equation (5) in the direction $Ox_2$.

Thus at each moment $t_{j+\alpha/p}$ we must solve the first boundary problem for the one-dimensional equation (5). To solve it numerically we use the homogeneous difference schemes given in [13]-[18]. We approximate to the differential operator $L_\alpha u + f_\alpha$ by a three-point conservative difference scheme of the second order of approximation:

$$\Lambda_\alpha y + \varphi_\alpha (x, t), \text{where} \Lambda_\alpha y = (a_\alpha (x, t)\, y_{\bar{x}_\alpha})_{x_\alpha} + b_\alpha (x, t)\, y_{\overset{\circ}{x}_\alpha} - d_\alpha (x, t)\, y, \quad (8)$$

in which the coefficients $a_\alpha$, $b_\alpha$, $d_\alpha$, $\varphi_\alpha$ are functional of the corresponding coefficients $k_\alpha$, $r_\alpha$, $q_\alpha$, $f_\alpha$. We shall not describe in detail the properties of the pattern functionals which we use to express the coefficients of the scheme $\Lambda_\alpha$ in terms of the coefficients of the differential operator $L_\alpha$, but refer the reader to the articles [13] and [18]. We merely note that in the case of smooth coefficients we can use the simpler expressions

$$b_\alpha (x, t) = r_\alpha (x, t), \qquad d_\alpha = q_\alpha (x, t), \qquad \varphi_\alpha = f_\alpha (x, t),$$

$$a_\alpha (x, t) = k_\alpha (x^{(-0.5_\alpha)}, t) \ \text{ or } \ a_\alpha (x, t) = 0.5\, (k_\alpha^{(-1_\alpha)} + k_\alpha),$$

where

$$x^{(-m_\alpha)} = (x_1^{(i_1)}, \ldots, x_{\alpha-1}^{(i_{\alpha-1})}, x_\alpha^{(i_\alpha)} - mh_\alpha, x_{\alpha+1}^{(i_{\alpha+1})}, \ldots, x_p^{(i_p)}),$$

$m$ is any positive number (in the given case $m = 0.5$ and $m = 1$).

If conditions A are satisfied, then at any internal point of the net

$$\Lambda_\alpha u - L_\alpha u = O\,(h_\alpha^2). \quad (9)$$

We put the differential equation (5) in correspondence with the four-point implicit homogeneous difference scheme (leading scheme or majorant scheme):

$$\rho y_{\bar{t}_\alpha} = \Lambda_\alpha y + \varphi_\alpha \qquad (\alpha = 1, 2, \ldots, p), \quad (10)$$

where

$$y_{\bar{t}_\alpha} = (y - \check{y}) / \tau, \qquad y = y^{j+\alpha/p}, \qquad \check{y} = y^{j+(\alpha-1)/p},$$
$$\Lambda_\alpha y = (a_\alpha (x, t^*) y_{\bar{x}_\alpha})_{x_\alpha} + b_\alpha (x, t^*) y_{\overset{\circ}{x}_\alpha} - d_\alpha (x, t^*) y, \tag{11}$$
$$\varphi_\alpha = \varphi_\alpha (x, t^*), \qquad \rho = \rho (x, t^{**}) \qquad (t_j \leqslant t^* \leqslant t_{j+1}, t_j \leqslant t^{**} \leqslant t_{j+1}).$$

The coefficient $\rho$ is, like $b_\alpha$, $d_\alpha$, and $\varphi_\alpha$ a linear functional of the coefficient $c(x, t)$, so that

$$\rho (x, t) - c (x, t) = O (h^2), \qquad h^2 = \frac{1}{p}\sum_{\alpha=1}^{p} h_\alpha^2 \tag{12}$$

(in the simplest case $\rho(x, t) = c(x, t)$). We note that $\rho$ does not depend on the suffix $\alpha$, i.e. the same value of $\rho$ is taken at some time $t^{**} \in [t_j, t_{j+1}]$ for all directions $x_\alpha$.

All the coefficients are taken at some arbitrary moments $t^*$; $t_j \leqslant t^* \leqslant t_{j+1}$ (for example, $t^* = t_j$, $t^* = t_{j+1}$, $t^* = t_{j+\alpha/p}$ etc.).* We can allow the choice of $t^*$ to be arbitrary since, as we shall show below, all the schemes obtained for different values of $t^*$ are equivalent with respect to the order of approximation (with an accuracy to $O(\tau)$) and, from Theorem 1, have the same order of accuracy. In every actual case $t^*$ must be selected from considerations of economy and convenience of calculation. There is also arbitrariness in the choice of $\varphi_\alpha$ and $d_\alpha$ since we require only that the conditions

$$\begin{aligned} \sum_{\alpha=1}^{p} \varphi_\alpha (x, t^*) &= f (x, t^*) + O (h^2) + O (\tau), \\ \sum_{\alpha=1}^{p} d_\alpha (x, t^*) &= q (x, t^*) + O (h^2) + O (\tau). \end{aligned} \tag{13}$$

shall be satisfied. In particular, we can put

$$\varphi_\alpha = 0 \text{ for } \alpha = 1, 2, \ldots, p-1, \qquad \varphi_p = \varphi (x, t), \qquad \varphi = f + O (h^2),$$
$$d_\alpha = 0 \text{ for } \alpha = 1, 2, \ldots, p-1, \qquad d_p = d (x, t), \qquad d = q + O (h^2).$$

We shall be concerned henceforth with the whole class of difference schemes described above.

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* Generally speaking, the values of $t^*$ are different for the different coefficients ($a_\alpha$, $b_\alpha$, $d_\alpha$, $\varphi_\alpha$).

By the difference scheme $\Pi = \{\Pi_\alpha,\ \alpha = 1, 2, \dots, p\}$ corresponding to equation (1) we mean the set (block) of homogeneous schemes $\Pi_\alpha y = \rho y_{\bar{t}_\alpha} - \Lambda_\alpha y - \varphi_\alpha$.

The difference scheme $\Pi = \{\Pi_\alpha\}$ is homogeneous not only with respect to space, but also with respect to time since the computational procedure is repeated on going from $j$ to $j + 1$.

In order to define the difference problem which corresponds to the problem (1)-(4) we must formulate the boundary conditions.

## 4. *Statement of the boundary problem*

Let us consider some chain $Ц_\alpha$. The difference equation (10) contains the values of $y$ at the points $x^{(-1_\alpha)}$, $x$, $x^{(+1_\alpha)}$ of this chain at time $t = t_{j+\alpha/p}$ and the value of $y$ at the point $x$ in the previous step $t = t_{j+(\alpha-1)/p}$. In order to define $y^{j+\alpha/p}$ at the nodes of $Ц_\alpha$, it is sufficient to give the values of $y$ at the ends of this chain only. It follows that at all points of the net $\omega_h$ the function $y^{j+\alpha/p}$ is determined merely by the boundary conditions on $\gamma_\alpha$, since each internal node of $\omega_h$ belongs to some chain of the given direction $0x_\alpha$. The boundary problem will be stated if we formulate the boundary conditions for the separate chain $Ц_\alpha$.

Let us draw a straight line $\mathscr{L}_\alpha$ parallel to the axis $0x_\alpha$ and passing through some node of the net. It will intersect the boundary $\Gamma$ in the points $x_1{}^*$ and $x_r{}^*$.

To simplify the argument we shall assume that straight lines parallel to the coordinate axes intersect the boundary $\Gamma$ in two points only. If the intersection of $\mathscr{L}_\alpha$ and $G$ consists of a finite number $m > 1$ of intervals, then $Ц_\alpha$ consists of $m$ parts $Ц_\alpha^{(k)}$ $(k = 1, 2, \dots, m)$ for each of which a boundary problem has to be solved.

Let us assume that the coordinate $x_\alpha$ increases from the point $x_r^*$ to the point $x_r$. We consider the node $x_1$ which is nearest the point $x_1^*$ and the node $x_1^{(+1_\alpha)}$ which is adjacent to the node $x_1$ on the chain $Ц_\alpha$. The node $x_\alpha$ is clearly a boundary node of the net $\omega_h (x_1 \in \gamma_\alpha)$. We can find the value of $y$ at the boundary node $x_1$ by using linear interpolation on the values of $y$ at the points $x_1^*$ and $x_1^{(+1_\alpha)}$, putting $y = \mu$ at the point $x_1^*$ :

$$y = \beta_1 y^{(+1_\alpha)} + (1 - \beta_1)\mu(x_1^*, t) \quad \text{for } x = x_1 \in \gamma_\alpha,\ t = t_{j+\alpha/p}, \tag{14}$$

where $\beta_1 = \kappa_1/(1 + \kappa_1)$, $\kappa_1 h_\alpha$ is the distance of the node $x_1$ from $x_1^*$ and $\mu(x, t)$ is the function given in (3). Similarly we can write down the condition on the right-hand end $x = x_r$ of the chain $Ц_\alpha$:

$$y = \beta_r y^{(-1_\alpha)} + (1 - \beta_r)\,\mu\,(x_1^*, t) \text{ for } x = x_r \in \gamma_\alpha, \quad t = t_{j+\alpha/p},$$

where $\beta_l = \kappa_r/(1 + \kappa_r)$, $\kappa_r h_\alpha$ is the distance of the node $x_r$ from $x_r^*$. It is clear from this that, for a uniform chain $Ц_\alpha$, we obtain boundary conditions of the third type, with

$$0 \leqslant \beta_l < 1, \qquad 0 \leqslant \beta_r < 1 \qquad (\varkappa_l \geqslant 0, \quad \varkappa_r \geqslant 0).$$

The value $\beta_l = 0$ ($\beta_r = 0$) corresponds to the case when the node $x_l$ ($x_r$) belongs to the boundary $\Gamma$ of the region $G$.

We know that this method of setting boundary conditions "with carry, using linear interpolation", has second order approximation. We shall not discuss here the simplest condition for first order approximation when the value of $y$ at the point $x_l$ is taken to be equal to the value of the function $u = \mu$ at the point $x_l^*$: $y = \mu(x_l^*, t)$ for $x = x_l$ and, similarly, $y = \mu(x_r^*, t)$ for $x = x_r$.

Let us pass now to the statement of the difference problem corresponding to the problem (1)-(4):

it is required to find a function $y$ defined on $\bar{\Omega}$ satisfying, inside $\Omega$, the difference equations

$$\rho\, y_{\bar{t}_\alpha} = \Lambda_\alpha y + \varphi_\alpha \text{ for } x \in \omega_h, \;, t = t_{j+\alpha/p} \text{ for } \alpha = 1, 2, \ldots, p,$$
$$j = 0, 1, \ldots, K-1 \tag{15}$$

the boundary conditions at time $t = t_{j+\alpha/p}$ for $x \in \gamma_\alpha$

$$\left.\begin{aligned} y &= \beta_l y^{(+1_\alpha)} + (1 - \beta_l)\,\mu\,(x_l^*, t), \qquad x = x_l \in \gamma_\alpha, \quad t = t_{j+\alpha/p}, \\ y &= \beta_r y^{(-1_\alpha)} + (1 - \beta_r)\,\mu\,(x_r^*, t), \qquad x = x_r \in \gamma_\alpha, \quad t = t_{j+\alpha/p}, \end{aligned}\right\} \tag{16}$$

and the initial condition

$$y\,(x, 0) = u_0\,(x), \qquad x \in \omega_h. \tag{17}$$

It is clear from this that for fixed $\alpha$ we solve, for each chain $Ц_\alpha$, the one-dimensional equation (15) with the boundary conditions of the third kind by using as initial data the values of $y(x, t_{j+(\alpha-1)/p})$ at internal nodes, these values being found by solving the same problem at time $t_{j+(\alpha-1)/p}$ for the chains $Ц_{\alpha-1}$ in the direction $Ox_{\alpha-1}$. Thus, in order to find the value of $y$ over the whole step $t_{j+1}$ from the data in step $t_j$, it is necessary stage by stage to solve $p$ one-dimensional problems in all the coordinate directions.

We could call this method the fractional step method, or the variable direction method. However, we consider that the name "local one-dimensional method" is a better reflection of the essential points: at any point of the net at any moment of time a one-dimensional equation is solved. Besides, the term "variable directions method" is used, for example, for the multidimensional scheme of [1] and the term "fractional step method" is used in [11] to characterise schemes which are one-dimensional for $\alpha > 1$ only and are multidimensional for $\alpha = 1$, and so on.

The local one-dimensional homogeneous difference scheme (15)-(17) is the most economical of the schemes suggested in [1]-[3], [5], [6], [11]. We note that for the special case of an equation with constant coefficients ($k_\alpha = 1$) and $r_\alpha = q = f = 0$ a similar scheme is considered in [4]. However, the method there is based on eliminating $y^{j+\alpha/p}$ for $\alpha = 1, 2, \ldots, p-1$ and studying the resulting equation which connects the values of $y$ at whole steps ($y^j$ and $y^{j+1}$). Since this equation contains a space operator of order $2p$, the transition to the case of an arbitrary region has no obvious solution, and the problem of boundary conditions arises even for the simplest region, a parallelepiped. In [4] the method was called a splitting method, the name originating, it follows from [10], in the splitting or (rough) factorisation of the multidimensional difference operator into one-dimensional operators.

## 5. *The approximation error*

Let $u = u(x, t)$ be the solution of problem (1)-(4), and $y = y(x, t)$ the solution of the difference problem (15)-(17). The difference

$$z = y - u.$$

is a characteristic of the accuracy of the local one-dimensional method. Writing $y = z + u$ in (15)-(17) we obtain the following conditions for the net function $z$ defined on $\bar{\Omega}$:

$$\rho z_{\bar{t}_\alpha} = \Lambda_\alpha z + \psi_\alpha \text{ on } \Omega, \tag{18}$$

$$\left.\begin{aligned} z &= \beta_1 z^{(+1_\alpha)} + \nu_1 \quad \text{for } x = x_1 \in \gamma_\alpha, \quad t = t_{j+\alpha/p}, \\ z &= \beta_r z^{(-1_\alpha)} + \nu_r \quad \text{for } x = x_r \in \gamma_\alpha, \quad t = t_{j+\alpha/p}, \end{aligned}\right\} \tag{19}$$

$$z(x, 0) = 0, \tag{20}$$

where

$$\psi_\alpha = \Lambda_\alpha u + \varphi_\alpha - \rho u_{\bar{t}_\alpha} \tag{21}$$

denotes the approximation error of the one-dimensional scheme $\Pi_\alpha$ and $\nu_1$ and $\nu_r$ are the approximation errors of the boundary conditions.

The approximation error of the local one-dimensional scheme $\Pi = \{\Pi_\alpha, \alpha = 1, 2, \ldots, p\}$ is naturally defined as the sum of the errors

$$\Psi = \sum_{\alpha=1}^{p} \psi_\alpha. \tag{22}$$

In fact, using the one-dimensional schemes (15) an approximation to the multidimensional equation (1) can be found on the whole step $(t_j, t_{j+1})$ only.

*Lemma 1.* If the conditions A are satisfied, then

$$\Psi = O(h^2) + O(\tau) \quad \text{for all} \quad \Omega. \tag{23}$$

Let us consider first the local error $\psi_\alpha$ and put it in the form

$$\psi_\alpha = \psi_\alpha^0 + \psi_\alpha^*, \text{ where } \psi_\alpha^0 = \left(L_\alpha u + f_\alpha - \frac{1}{p} c \frac{\partial u}{\partial t}\right)^{j+\beta}, \tag{24}$$

$$\psi_\alpha^* = [\Lambda_\alpha u^{j+\alpha/p} - (L_\alpha u)^{j+\beta}] + \varphi_\alpha - f_\alpha^{j+\beta} - \left[\rho u_{\bar{t}_\alpha} - \frac{1}{p}\left(c \frac{\partial u}{\partial t}\right)^{j+\beta}\right], \tag{25}$$

where $\beta \in [0, 1]$ is an arbitrary number.

Summing the $\psi_\alpha^0$ over all $\alpha = 1, 2, \ldots, p$ and taking equation (1) into account for the function $u$, we obtain (for fixed $j$)

$$\sum_{\alpha=1}^{p} \psi_\alpha^0 = 0. \tag{26}$$

Let us now show that for any $t^* \in [t_j, t_{j+1}]$, $t^{**} \in [t_j, t_{j+1}]$

$$\psi_\alpha^* = O(h^2) + O(\tau). \tag{27}$$

In fact, if conditions A are satisfied, then

$$\left(L_\alpha u - \frac{1}{p} c \frac{\partial u}{\partial t}\right)^{j+\beta} = L_\alpha^* u^{j+\alpha/p} - \frac{1}{p} c^{**} \left(\frac{\partial u}{\partial t}\right)^{j+\alpha/p} + O(\tau), \tag{28}$$

where

$$L_\alpha^* u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha^* \frac{\partial u}{\partial x_\alpha}\right) + r_\alpha^* \frac{\partial u}{\partial x_\alpha}, \quad k_\alpha^* = k_\alpha(x, t^*),$$
$$r_\alpha^* = r_\alpha(x, t^*), \quad c^{**} = c(x, t^{**}).$$

Since the scheme $\Lambda_\alpha u$ has second order approximation, and

$$\rho - c^{**} = O(h^2), \qquad \rho u_{\bar{t}_\alpha} - \frac{1}{p} c^* \left(\frac{\partial u}{\partial t}\right)^{j+\alpha/p} =$$

$$= \frac{1}{p}(\rho - c^{**})\left(\frac{\partial u}{\partial t}\right)^{j+\alpha/p} + \rho\left(u_{\bar{t}_\alpha} - \frac{1}{p}\left(\frac{\partial u}{\partial t}\right)^{j+\alpha/p}\right) = O(h^2) + O(\tau),$$

(27) follows from (28) and (25) and the estimate (23) follows from (26) and (27).

Thus the local one-dimensional scheme (15) has first order approximation in time and second order in space. In § 2 we shall show that the order of accuracy of this scheme is the same as the order of approximation.

### 6. *Quasilinear equations*

The local one-dimensional method is also applicable to the solution of the quasilinear equations

$$\left.\begin{aligned} c(x, t)\frac{\partial u}{\partial t} &= \sum_{\alpha=1}^{p} L_\alpha u + f(x, t, u), \\ L_\alpha u &= \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t, u)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x, t, u)\frac{\partial u}{\partial x_\alpha} \end{aligned}\right\}. \tag{29}$$

In this article we shall consider the case where the "heat conductivity coefficient" $k_\alpha$ does not depend on $u$:

$$k_\alpha = k_\alpha(x, t),$$

since the basis of our method for such an equation can be laid down in the same way as for the linear equation (1). The proof of the convergence of our method for equation (29) requires a fairly cumbersome system of integral (energy) inequalities and will be given separately.

Thus, let us consider the following problem in the cylinder $\bar{Q}_T = \bar{G} \times [0 \leqslant t \leqslant T]$

$$c(x, t)\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} L_\alpha u + f(x, t, u) \quad (k_\alpha = k_\alpha(x, t)), \ (x, t) \in Q_T, \tag{30}$$

$$u|_\Gamma = \mu(x, t), \tag{31}$$

$$u(x, 0) = u_0(x). \tag{32}$$

The coefficients of the equation satisfy the conditions

$$1)\ k_\alpha(x, t) \geqslant c_1 > 0, \qquad c(x, t) \geqslant c_2 > 0 \text{ in } \bar{Q}_T, \tag{33}$$

$$2)\ \frac{\partial r_\alpha}{\partial u}, \quad \frac{\partial f}{\partial u} \ \text{are continuous functions of the argument } u. \tag{34}$$

We shall also assume that the solution $u = u(x, t)$ of the problem (30)-(34) and the coefficients $k_\alpha(x, t)$, $c(x, t)$, $r_\alpha(x, t, u)$, $f(x,t,u)$ as functions of $(x, t)$, satisfy the conditions A formulated in point 1.

We write the homogeneous local one-dimensional scheme in this case by analogy with (15):

It is required to find a net function $y$, defined on $\bar{\Omega}$, and satisfying on $\Omega$ the difference equations

$$\left.\begin{aligned} &\rho y_{\bar{t}_\alpha} = \Lambda_\alpha y + \varphi_\alpha, \\ &\Lambda_\alpha y = (a_\alpha(x, t^*) y_{\bar{x}_\alpha})_{x_\alpha} + b_\alpha(x, t^*, \check{y})\, y_{\overset{\circ}{x}_\alpha}, \qquad \varphi_\alpha = \varphi_\alpha(x, t^*, \check{y}), \end{aligned}\right\} \tag{35}$$

and the boundary conditions on $S$:

$$\left.\begin{aligned} &y = \beta_1 y^{(+1_\alpha)} + (1 - \beta_1)\,\mu(\overset{*}{x}_1, t), \qquad x = x_1 \in \gamma_\alpha, \quad t = t_{j+\alpha/p} \\ &y = \beta_r y^{(-1_\alpha)} + (1 - \beta_r)\,\mu(\overset{*}{x}_r, t), \qquad x = x_r \in \gamma_\alpha, \quad t = t_{j+\alpha/p}, \end{aligned}\right\} \tag{36}$$

$$y(x, 0) = u_0(x) \tag{37}$$

(see point 4, and compare (16) and (36)).

We recall that $y_{\bar{t}_\alpha} = (y - \check{y}) / \tau$, $y = y^{j+\alpha/p}$, $\check{y} = y^{j+(\alpha-1)/p}$. The difference equation (35) is linear with respect to the values of $y = y^{j+\alpha/p}$ at each step. Therefore, to determine $y^{j+\alpha/p}$ we can use the known formulae for the one-dimensional chain $\Pi_\alpha$ with boundary conditions of the third kind (36) at its ends.

This algorithm is convenient in that when finding the solution in the $(j+\alpha/p)$-th row it is necessary to remember only the values of the required function from the previous row.

The coefficients of the scheme are calculated according to the same formulae as the coefficients of scheme (15) for the linear equation (1).

Together with (35), we can consider a scheme with coefficients which depend on the values of the required function in the new row, so that

$$\Lambda_\alpha y = (a_\alpha(x, t^*)\, y_{\bar{x}_\alpha})_{x_\alpha} + b_\alpha(x, t^*, y)\, y_{\overset{\circ}{x}_\alpha}, \qquad \varphi_\alpha = \varphi_\alpha(x, t^*, y). \tag{35'}$$

In this case we need iterations to solve the resulting non-linear equations for $y$, and the computation must be done using the formulae. For certain problems the non-linear scheme (35′) is to be preferred, since in this case the step can be increased with respect to time $\tau$ (with a given accuracy).

In order to find an expression for the approximation error of the scheme, we write the solution of the problem (35)-(37) in the form of a sum $y = z + u$ where $u$ is the solution of the problem (30)-(34). Writing $y = z + u$ in (35) we obtain

$$\rho z_{\bar{t}_\alpha} = \bar{\Lambda}_\alpha z + \psi_\alpha, \tag{38}$$

$$\bar{\Lambda}_\alpha z = (a_\alpha(x, t^*)\, z_{\bar{x}_\alpha})_{x_\alpha} + b_\alpha(x, t^*, \check{y})\, z_{\overset{\circ}{x}_\alpha} + d_\alpha \check{z}, \tag{39}$$

$$\psi_\alpha = (a_\alpha(x, t^*)\, u_{\bar{x}_\alpha})_{x_\alpha} + b_\alpha(x, t^*, \check{u})\, u_{\overset{\circ}{x}_\alpha} + \varphi_\alpha(x, t^*, \check{u}) - \rho(x, t^*)\, u_{\bar{t}_\alpha}, \tag{40}$$

$$d_\alpha = \overline{\frac{\partial \varphi_\alpha}{\partial u}(x, t, u)} + \overline{\frac{\partial b_\alpha}{\partial u}(x, t^*, u)}$$

The bar means that the derivatives are taken at mean (between $\check{y}$ and $\check{u}$) values of the argument $u$.

In § 2 when we discuss the convergence of the solution of problem (35)-(37) to a fixed (unique) solution $u = u(x, t)$ ($|u| \leqslant M_0$) of problem (30)-(34) we shall assume that $d_\alpha$ and $b_\alpha(x, t, y)$ are bounded, since the functions $r_\alpha$ and $f$ can be continued in the region $|u| > M_0$ and therefore in this region $r_\alpha$, $\partial r_\alpha / \partial u$, $\partial f / \partial u$ are bounded.

For the non-linear scheme (35′) we obtain the equation (38) where

$$\bar{\Lambda}_\alpha z = (a_\alpha(x, t^*)\, z_{\bar{x}_\alpha})_{x_\alpha} + b_\alpha(x, t^*, y)\, z_{\overset{\circ}{x}_\alpha} + d_\alpha z. \tag{41}$$

Let us now formulate the conditions for the error $z = y - u$:

$$\rho z_{\bar{t}_\alpha} = \bar{\Lambda}_\alpha z + \psi_\alpha, \quad (x, t) \in \Omega, \tag{42}$$

$$\left.\begin{aligned} z &= \beta_1 z^{(+1_\alpha)} + \nu_{1,\alpha} \quad \text{for } x = x_1 \in \gamma_\alpha,\ t = t_{j+\alpha/p}, \\ z &= \beta_r z^{(-1_\alpha)} + \nu_{r,\alpha} \quad \text{for } x = x_r \in \gamma_\alpha,\ t = t_{j+\alpha/p}, \end{aligned}\right\} \tag{43}$$

$$z(x, 0) = 0, \tag{44}$$

where $\nu_{1,\alpha}$, $\nu_{r,\alpha}$ are the approximation errors of the boundary conditions.

The approximation error $\Psi$ of the homogeneous local one-dimensional scheme (35) is found as the sum of the local errors

$$\Psi = \sum_{\alpha=1}^{p} \psi_\alpha. \tag{45}$$

By analogy with point 4, we can establish the truth of Lemma 1:

$$\Psi = O(h^2) + O(\tau), \qquad (x, t) \in \Omega, \tag{46}$$

where

$$\psi_\alpha = \psi_\alpha^0 + \psi_\alpha^*, \quad \psi_\alpha^* = O(h^2) + O(\tau), \quad \sum_{\alpha=1}^{p} \psi_\alpha^0 = 0. \tag{47}$$

We recall that

$$\nu_{1,\alpha} = O(h_\alpha^2), \qquad \nu_{r,\alpha} = O(h_\alpha^2). \tag{48}$$

To estimate the accuracy of (15)-(17) and (30)-(34) it is necessary to prove the stability of the solution of problems (18)-(20) and (42)-(44) with respect to the right-hand sides $\psi_\alpha$ and $\nu_{1,\alpha}$, $\nu_{r,\alpha}$, using (47).

We shall prove the corresponding stability theorem in § 2.

*7. The third boundary problem for a parallelepiped*

Let $\bar{G} = \{0 \leqslant x_\alpha \leqslant l_\alpha,\ \alpha = 1, 2, \ldots, p\}$ be a $p$-dimensional parallelepiped on the boundary of which the boundary conditions of the third kind apply:

$$\left.\begin{aligned} k_\alpha \frac{\partial u}{\partial x_\alpha} &= \sigma_{1\alpha} u + \mu_{1\alpha}(x, t) \quad \text{for } x_\alpha = 0, \quad \alpha = 1, 2, \ldots, p, \\ -k_\alpha \frac{\partial u}{\partial x_\alpha} &= \sigma_{2\alpha} u + \mu_{2\alpha}(x, t) \quad \text{for } x_\alpha = l_\alpha. \end{aligned}\right\} \tag{49}$$

The coefficients $\sigma_{1\alpha}(x, t)$, $\sigma_{2\alpha}(x, t)$ satisfy the conditions

$$\sigma_{1\alpha}(x, t) \geqslant c_6 > 0, \qquad \sigma_{2\alpha}(x, t) \geqslant c_6 > 0, \tag{50}$$

where $c_6$ is some constant.

The simplest difference analogue of these conditions are the boundary conditions

$$\left.\begin{aligned} a_\alpha^{(+1_\alpha)} y_{x_\alpha} &= \sigma_{1\alpha} y + \mu_{1\alpha}(x, t) \quad \text{for } x_\alpha = 0, \quad t = t_{j+\alpha/p}, \\ -a_\alpha y_{\bar{x}_\alpha} &= \sigma_{2\alpha} y + \mu_{2\alpha}(x, t) \quad \text{for } x_\alpha = l_\alpha, \quad t = t_{j+\alpha/p}. \end{aligned}\right\} \tag{51}$$

These boundary conditions have first order approximation. Let $u = u(x, t)$ be the solution of equation (1) (or (29)) with boundary conditions (49) and the initial condition

$$u(x, 0) = u_0(x), \tag{3}$$

and let $y = y(x, t)$ $((x, t) \in \bar{\Omega})$ be the solution of (15) (or (35)) with boundary conditions (51) and initial condition

$$y(x, 0) = u_0(x), \qquad x \in \omega_h. \tag{52}$$

Then we obtain equation (18) for the difference $z = y - u$ with the boundary conditions

$$\left.\begin{aligned} a_\alpha^{(+1_\alpha)} z_{x_\alpha} &= \sigma_{1\alpha} z - \overset{*}{\nu}_{1\alpha} \quad \text{for } x_\alpha = 0, \quad t = t_{j+\alpha/p}, \\ - a_\alpha z_{\bar{x}_\alpha} &= \sigma_{2\alpha} z + \overset{*}{\nu}_{2\alpha} \quad \text{for } x_\alpha = l_\alpha, \quad t = t_{j+\alpha/p}, \end{aligned}\right\} \tag{53}$$

where

$$\overset{*}{\nu}_{1\alpha} = \left(a_\alpha^{(+1_\alpha)} u_{x_\alpha} - k_\alpha \frac{\partial u}{\partial x_\alpha}\right)_{x_\alpha=0} = O(h_\alpha),$$

$$\overset{*}{\nu}_{2\alpha} = \left(a_\alpha u_{\bar{x}_\alpha} - k_\alpha \frac{\partial u}{\partial x_\alpha}\right)_{x_\alpha=l_\alpha} = O(h_\alpha).$$

Let us rewrite conditions (53) in the form

$$\begin{aligned} z &= \beta_l z^{(+1_\alpha)} + \nu_{1\alpha} \quad \text{for } x_\alpha = 0, \quad 0 < x_\beta < l_\beta, \ \beta \neq \alpha, \ t = t_{j+\alpha/p}, \\ z &= \beta_r z^{(-1_\alpha)} + \nu_{2\alpha} \quad \text{for } x_\alpha = l_\alpha, \ 0 < x_\beta < l_\beta, \ \beta \neq \alpha, \ t = t_{j+\alpha/p}, \end{aligned} \tag{53'}$$

where

$$\beta_l = \frac{a_{\alpha,0}^{(+1_\alpha)}}{\Delta_{1\alpha}}, \quad \beta_r = \frac{a_{\alpha,N_\alpha}}{\Delta_{2\alpha}}, \quad \nu_{1\alpha} = \frac{h_\alpha \overset{*}{\nu}_{1\alpha}}{\Delta_{1\alpha}}, \quad \nu_{2\alpha} = -\frac{h_\alpha \overset{*}{\nu}_{2\alpha}}{\Delta_{2\alpha}},$$

$$a_{\alpha,0} = a_\alpha |_{x_\alpha=0}, \quad a_{\alpha,N_\alpha} = a_\alpha |_{x_\alpha = N_\alpha h_\alpha = l_\alpha}, \quad \Delta_{1\alpha} = a_{\alpha,0}^{(+1_\alpha)} + \sigma_{1\alpha} h_\alpha,$$

$$\Delta_{2\alpha} = a_{\alpha,N_\alpha} + \sigma_{2\alpha} h_\alpha.$$

Thus the error of the solution of the difference boundary problem (15), (51), (52) also satisfies equation (18), the zero initial condition $z(x, 0) = 0$ and the boundary conditions (53) analogous in form to the conditions (19). We can therefore examine at the same time the convergence of our difference method for the first boundary condition in the case of an arbitrary region and for the third boundary problem for a parallelepiped.

## 2. On the uniform convergence and accuracy of the local one-dimensional scheme

In this section we shall prove the uniform convergence of the local one-dimensional method for the linear equation (1) and the quasilinear equation (29) for an arbitrary region, and we shall give a uniform estimate for the order of accuracy.

### *1. Uniform stability*

Let us show that the difference problem (15)-(18) has been correctly described, i.e. its solution depends continuously on the right-hand side of (15), the initial and the boundary values. Let us consider the following problem: to find a function $z$, defined on the net $\bar{\Omega}$ and satisfying $\Omega$ the equations

$$\left.\begin{aligned} \rho z_{\bar{t}_\alpha} &= \Lambda_\alpha z + \psi(x, t) \quad \text{for } x \in \omega_h, \quad t = t_{j+\alpha/p}, \\ \Lambda_\alpha z &= (a_\alpha(x, t)\, z_{\bar{x}_\alpha})_{x_\alpha} + b_{1\alpha} z_{x_\alpha} + b_{2\alpha} z_{\bar{x}_\alpha} - d_{1\alpha} z - d_{2\alpha} \check{z}, \end{aligned}\right\} \tag{54}$$

the boundary conditions on the boundary $S$ of the net region

$$\left.\begin{aligned} z &= \beta_l z^{(+1_\alpha)} + \nu_l \quad \text{for } x = x_l \in \gamma_\alpha, \quad t = t_{j+\alpha/p}, \\ z &= \beta_r z^{(-1_\alpha)} + \nu_r \quad \text{for } x = x_r \in \gamma_\alpha, \quad t = t_{j+\alpha/p} \end{aligned}\right\}. \tag{55}$$

and the initial condition

$$z(x, 0) = z_0(x). \tag{56}$$

The coefficients $a_\alpha$, $b_{1\alpha}$, $b_{2\alpha}$, $d_{1\alpha}$, $d_{2\alpha}$, $\beta_l$, $\beta_r$ satisfy the conditions

$$a_\alpha \geqslant c_1 > 0, \qquad \rho \geqslant c_2 > 0, \qquad |d_{s\alpha}| \leqslant c_3, \qquad |b_{s\alpha}| \leqslant c_4, \qquad s = 1, 2, \tag{57}$$

$$0 \leqslant \beta_l \leqslant \beta^* < 1, \qquad 0 \leqslant \beta_r \leqslant \beta^* < 1, \tag{58}$$

where $c_1$, $c_2$, $c_3$, $c_4$, $\beta^*$ are positive constants which do not depend on the steps $h_\alpha$ ($\alpha = 1, 2, \ldots, p$) and $\tau$ of the difference net. Two cases arise:

1) a boundary problem of the first kind for an arbitrary region: $\beta = \kappa/(1 + \kappa)$ (we omit the suffixes $l$, $r$, $\alpha$ for the time being) where $0 \leqslant \kappa \leqslant c_5$, $c_5 > 0$ is a constant which does not depend on the net; then

$$\begin{gathered} 1 - \beta = 1/(1 + \kappa) \geqslant 1/(1 + c_5) = \beta_* > 0, \\ 1 - \beta_l \geqslant \beta_*, \qquad 1 - \beta_r \geqslant \beta_*; \end{gathered} \tag{59}$$

2) a boundary problem of the third kind for a parallelepiped:

$$\beta = \frac{a}{a + \sigma h}$$

(omitting all suffixes).

For a parallelepiped in the first boundary problem ($z = \nu_l$, $z = \nu_r$)

$$\beta_l = 0, \quad \beta_r = 0, \quad \beta_* = 1.$$

All the schemes $\Lambda_\alpha z$ considered in § 1 are special cases of the scheme (54). Therefore the results obtained below will be valid for the problems (18)-(20), (38)-(40) and (38), (53), (40).

*Theorem 1*. Let $z = z(x, t)$ be the solution of the problem (54)-(58).

Then for sufficiently small $\tau < \tau_0$ and any $h_\alpha$

$$\|z(x, t)\|_0 \leqslant M_1 (\|\dot{z}(x, 0)\|_0 + \|\nu(x, t)\|_{0,s}) + M_2 \left( \sum_{t'=\tau/p}^{t} \tau \|\psi(x, t')\|_0^2 \right)^{1/2}, \qquad (60)$$

where

$$\|z(x, t)\|_0 = \max_{x \in \bar{\omega}_h} |z(x, t)|, \qquad \|\nu(x, t)\|_{0,s} = \max_{(x,t) \in S} |\nu(x, t)|,$$

$$\nu(x, t) = \nu_{1,\alpha}; \quad x = x_1 \in \gamma_\alpha; \qquad \nu = \nu_{r,\alpha}, \qquad x = x_r \in \gamma_\alpha, \quad t = t_{j+\alpha/p},$$

$M_1$, $M_2$ are positive constants which depend only on $c_1$, $c_2$, $c_3$, $c_4$, $\beta_*$, $T$.

The theorem is proved by the method described in § 1 of [18], based on the use of the principle of the maximum for the parabolic difference equations. We introduce a new function $v$ by writing

$$z^{j+\alpha/p} = (1 + \bar{M}\tau)^{pj+\alpha} v^{j+\alpha/p}, \qquad (61)$$

where $\bar{M}$ is an arbitrary positive constant which we select later. For the net function $v$ we obtain the equation

$$\bar{\rho} v_{\bar{t}_\alpha} = \Lambda^*_\alpha v - d_\alpha v + \bar{\psi}, \ \Lambda^*_\alpha v = (a_x v_{\bar{x}_\alpha})_{x_\alpha} + b_{1\alpha} v_{x_\alpha} + b_{2\alpha} v_{\bar{x}_\alpha} - \bar{d}_{2\alpha} \check{v}, \qquad (62)$$

the boundary conditions

$$\left.\begin{aligned} v &= \beta_1 v^{(+1_\alpha)} + \bar{\nu}_1 \qquad \text{for} \quad x = x_1 \in \gamma_\alpha, \quad t = t_{j+\alpha/p}, \\ v &= \beta_r v^{(-1_\alpha)} + \bar{\nu}_r \qquad \text{for} \quad x = x_r \in \gamma_\alpha, \quad t = t_{j+\alpha/p}, \end{aligned}\right\} \qquad (63)$$

and the initial condition

$$v(x, 0) = z(x, 0). \qquad (64)$$

where we have used the notation

$$\left.\begin{aligned} &\bar{\rho} = \rho\varepsilon, \qquad \bar{d}_{2\alpha} = d_{2\alpha}\varepsilon, \qquad d_\alpha = d_{1\alpha} + \bar{M}\bar{\rho}, \qquad \varepsilon = (1 + \bar{M}\tau)^{-1}, \\ &\bar{\psi} = \psi\varepsilon^{jp+\alpha} \qquad (\psi = \psi^{j+\alpha/p} = \psi(x, t_{j+\alpha/p}), \qquad \bar{\nu} = \nu\varepsilon^{jp+\alpha}. \end{aligned}\right\} \qquad (65)$$

Multiplying (62) and (63) by $2v$ and using the same arguments as in [18] we obtain the first rank equation

$$\bar{\rho}(v^2)_{\bar{t}_\alpha} - (a_\alpha (v^2)_{\bar{x}_\alpha})_{x_\alpha} + R_\alpha(v) + 2d_\alpha v^2 = 2v(b_{1\alpha} v_{x_\alpha} + b_{2\alpha} v_{\bar{x}_\alpha}) -$$

$$-(2\bar{d}_{2\alpha} \check{v} v + 2v\bar{\psi},$$

$$\left.\begin{aligned} 2\,(1-\beta_l)\,v^2+\beta_l h_\alpha v_{x_\alpha}^2 &= \beta_l\,(v^2)_{x_\alpha}+2v\bar{v}_l,\\ 2\,(1-\beta_r)\,v^2+\beta_r h_\alpha v_{\bar{x}_\alpha}^2 &= \beta_r\,(v^2)_{\bar{x}_\alpha}+2v\bar{v}_r, \end{aligned}\right\} \tag{66}$$

$$R_\alpha(v) = \tau\bar{\rho} v_{\bar{t}_\alpha}^2 + a_\alpha v_{\bar{x}_\alpha}^2 + a_\alpha^{(+1_\alpha)} v_{x_\alpha}^2 \quad (v_{\bar{t}_\alpha}^2 = (v_{\bar{t}_\alpha})^2,\ v_{x_\alpha}^2 = (v_{x_\alpha})^2 \quad \text{etc.}).$$

By analogy with [18] we arrive at the inequality

$$\rho\,(v^2)_{\bar{t}_\alpha} - (a_\alpha\,(v^2)_{\bar{x}_\alpha})_{x_\alpha} + 2d_* v^2 \leqslant \frac{\bar{\psi}^2}{c_0},\ \text{where } d_*>0, \tag{67}$$

or

$$\bar{\rho} v^2 - \tau\,(a_\alpha\,(v^2)_{\bar{x}_\alpha})_{x_\alpha} + 2\,d_*\tau v^2 \leqslant \bar{\rho}\check{v}^2 + \frac{\bar{\psi}^2}{c_0}\,\tau, \tag{68}$$

which is satisfied for sufficiently small values of $\tau$ and sufficiently large $\bar{M}$:

$$\tau<\tau_0, \qquad \tau_0 \leqslant \frac{c_2}{c_*}\left(1-\frac{M_*}{\bar{M}}\right), \quad \bar{M}>M_* = \frac{(c_3+c_*)\,c_*+0.5\,c_3^2}{c_2 c_*},$$

$$c_* = \frac{c_4^2}{c_1}+c_3+0.5\,c_0.$$

Here $c_0$ is an arbitrary positive constant which is selected to make the constant $M_2$ in (60) a minimum.

Using the fact that $2\,|v\bar{v}| \leqslant \beta_* v^2 + \bar{v}^2/\beta_*$, we obtain from (14)

$$\left.\begin{aligned} v^2 - \beta_l\beta_*^{-1}(v^2)_{x_\alpha} &\leqslant \bar{v}_l^2/\beta_*^2,\\ v^2 + \beta_r\beta_*^{-1}(v^2)_{\bar{x}_\alpha} &\leqslant \bar{v}_r^2/\beta_*^2, \end{aligned}\right\} \tag{69}$$

where $\beta_* = 1-\beta^* > 0$.

We put $v$ in the form of the sum $v = v^{(1)} + v^{(2)}$, where $v^{(1)}$ is the solution of the problem (62)-(64) for $\bar{\psi} = 0$ and $v^{(2)}$ is the solution of the same problem with homogeneous boundary and initial conditions ($\nu_l = 0$, $\nu_r = 0$, $v^{(2)}(x,0) = 0$). The function $w = (v^{(1)})^2$ takes its maximum value on the boundary of the region $\Omega$. Thus, suppose $w$ has a maximum at some point $P$ inside $\Omega$. Then at this point $w_{\bar{t}_\alpha} \geqslant 0$, $(a_\alpha w_{\bar{x}_\alpha})_{x_\alpha} \leqslant 0$, which contradicts the inequality $\bar{\rho} w_{\bar{t}_\alpha} - (a_\alpha w_{\bar{x}_\alpha})_{x_\alpha} + 2d_* w \leqslant 0$. If $P = P(x,0)$ then $\|w\|_0 \leqslant \|z\,(x,0)\|_0^2$. If $P = P(x_l, t_{j+\alpha/p})$ then at this point $w_{x_\alpha} \leqslant 0$ and the first of the inequalities (69) gives $w \leqslant \|\bar{v}_l\|_0^2 / \beta_*^2$. Writing then $P = P(x_r, t_{j+\alpha/p})$ and using the fact that $w_{\bar{x}_\alpha} \geqslant 0$, we obtain from the second inequality of (69) $w \leqslant \|\bar{v}_r\|_0^2/\beta_*^2$.

It follows that the inequality

$$\| v^{(1)}(x, t)\|_0 \leqslant \| z(x, 0)\|_0 + \frac{1}{\beta_*} \|\bar{\nu}(x, t)\|_{0, *}. \tag{70}$$

is always satisfied.

We have inequality (68) for the function $\bar{w} = (v^{(2)})^2$. Due to the homogeneity of the boundary conditions the function $\bar{w}$ must take its maximum value at some internal point of the region $\omega_h$. Since at this point $(a_\alpha w_{\bar{x}_\alpha})_{x_\alpha} \leqslant 0$, it must follow from inequality (68) that

$$\|\bar{w}\|_0 \leqslant \|\check{\bar{w}}\|_0 + \bar{c}\tau \|\bar{\psi}\|^2, \qquad \bar{c} = (c_0 c_2)^{-1}(1 + \overline{M}\tau),$$

where

$$\bar{w} = \bar{w}(x, t_{j+\alpha/p}), \quad \check{\bar{w}} = \bar{w}(x, t_{j+(\alpha-1)/p}), \quad \begin{matrix} j = 0, 1, \ldots, \\ \alpha = 1, 2, \ldots, p. \end{matrix}$$

Summing these inequalities for all values of $t' \leqslant t_{j+\alpha/p} = t$ and using the initial condition $\bar{w}(x, 0) = 0$ we find

$$\| v^{(2)}(x, t)\|_0 \leqslant \sqrt{\bar{c}} \left[ \sum_{t'=\tau/p}^{t} \tau \|\bar{\psi}(x, t')\|_0^2 \right]^{1/2}. \tag{71}$$

It follows from (18) and (20) that

$$\| v(x, t)\|_0 \leqslant \| v(x, 0)\|_0 + \frac{1}{\beta} \|\bar{\nu}(x, t)\|_{0, *} + \sqrt{\bar{c}} \left[ \sum_{t'=\tau/p}^{t} \tau \|\bar{\psi}(x, t')\|_0^2 \right]^{1/2}. \tag{72}$$

Returning to the function $z(x, t)$, in accordance with (61) and (65) we obtain, from (72), the estimate (60).

This proves the theorem.

*Notes*. 1. Theorem 1 is true for the first boundary problem ($z = \nu_1$, $z = \nu_2$). In this case

$$\beta_1 = \beta_r = 0, \quad \beta_* = 1.$$

2. Theorem 1 is true for the third boundary problem in the parallelepiped $\bar{G}(0 \leqslant x_\alpha \leqslant l_\alpha,\ \alpha = 1, 2, \ldots, p)$:

$$\left.\begin{aligned} a_\alpha^{(+1_\alpha)} z_{x_\alpha} &= \sigma_{1\alpha} z - \nu_{1\alpha} \quad \text{for } x_\alpha = 0, \quad t = t_{j+\alpha/p}, \\ -a_\alpha z_{\bar{x}_\alpha} &= \sigma_{2\alpha} z + \nu_{2\alpha} \quad \text{for } x_\alpha = l_\alpha, \quad t = t_{j+\alpha/p}, \end{aligned}\right\} \tag{73}$$

if $\sigma_{1\alpha} \geqslant c_6 > 0$, $\sigma_{2\alpha} \geqslant c_6 > 0$. In this case we must formally put $\beta_* = c_6 h_\alpha$ in (70), replacing $\bar{\nu}$ by $\bar{\nu} h_\alpha$. Therefore, those $\nu$ which enter in conditions (73) will figure in (60).

3. If $b_{1\alpha} = b_{2\alpha} = d_{2\alpha} = 0$, $d_{1\alpha} \geqslant c_7 > 0$ where $c_7$ is some positive constant which does not depend on $h_\alpha$ and $\tau$, then the inequality (60) of Theorem 1 is valid for any $\tau$ and $h_\alpha$ ($\alpha = 1, 2, \ldots, p$). In this case there is no need to pass to the function $v$ using (61), and we can, for example, put $c_0$ equal to $0.5\, c_7$.

*2. The accuracy of the local one-dimensional scheme*

Let us turn now to the problem (18)-(20) for the error $z = y - u$, where $u$ is the solution of the initial problem (1)-(4) and $y$ the solution of the difference problem (15)-(18). In order to estimate the function $z$, write

$$z = v + \eta,$$

where $\eta$ is a net function defined by the conditions

$$\left.\begin{array}{ll} \rho\eta_{\bar{t}_\alpha} = \psi_\alpha^0 & \text{inside } \Omega, \\ \eta(x, 0) = 0, & x \in \omega_h. \end{array}\right\} \tag{74}$$

The function $\eta$ is introduced so that the property

$$\sum_{\alpha=1}^{p} \psi_\alpha^0 = 0. \tag{75}$$

can be used. Equation (74) enables us to find $\eta = \eta^{j+\alpha/p}$ only inside the region $\Omega$. We find $\eta$ on the boundary $S$ of the net $\bar{\Omega}$ by writing

$$(a_\alpha \eta_{\bar{x}_\alpha})_{x_\alpha} = 0 \tag{76}$$

at the nodes $x_l^{(+1_\alpha)}$, $x_r^{(-1_\alpha)}$ of the chain $Ц_\alpha$, these being adjacent to the boundary nodes $x_l$ and $x_r$. Using (76) we can find $\eta$ at the boundary nodes $x_l$ and $x_r$:

$$\begin{array}{ll} \eta = \eta^{(+1_\alpha)} - a_\alpha^{(+2_\alpha)} \left(\eta^{(+2_\alpha)} - \eta^{(+1_\alpha)}\right)/a_\alpha^{(+1_\alpha)} & \text{for } x = x_l \in \gamma_\alpha, \\ \eta = \eta^{(-1_\alpha)} + a_\alpha^{(-1_\alpha)} \left(\eta^{(-1_\alpha)} - \eta^{(-2_\alpha)}\right)/a_\alpha & \text{for } x = x_r \in \gamma_\alpha. \end{array} \tag{77}$$

*Lemma 2.* If condition (75) is satisfied, then the function $\eta$ defined on $\Omega$ by conditions (74) and (76) has the form

$$\eta(x, t_{j+\alpha/p}) = \frac{\tau}{\rho} \sum_{\sigma=1}^{\alpha} \psi_\sigma^0 = -\frac{\tau}{\rho} \sum_{\sigma=\alpha+1}^{p} \psi_\sigma^0, \qquad \eta(x, t_{j+1}) = 0 \tag{78}$$

for any $j$, i.e. $\eta = 0$ on whole steps and $\eta = O(\tau)$ on fractional steps.

We note first of all that for fixed $j$ the coefficient $\rho$ does not change when $\alpha$ changes, i.e. on all the steps $j + \alpha/p$, $\alpha = 1, 2, \ldots, p$ ($j$ fixed) $\rho$ has the same value. It follows from equation (74) that

$$\eta^{j+\alpha/p} = \eta^{j+(\alpha-1)/p} + \psi_\alpha^0/\rho.$$

Writing $j = 0$, summing with respect to $\alpha$ and using conditions (75) and $\eta(x, 0) = 0$, we find $\eta^1 = 0$. Continuing the argument we arrive at (78).

Let us formulate now the conditions for the function $v = z - \eta$. To do this we write $z = v + \eta$ in (18) and (19) and use conditions (74). Then we have

$$\rho v_{\bar{t}_\alpha} = \Lambda_\alpha v + \tilde{\psi}_\alpha \qquad \text{on } \Omega, \tag{79}$$

$$\left.\begin{aligned} v &= \beta_1 v^{(+1_\alpha)} + \tilde{\nu}_1 \qquad \text{for } x = x_1 \in \gamma_\alpha,\ t = t_{j+\alpha/p}, \\ v &= \beta_r v^{(-1_\alpha)} + \tilde{\nu}_r \qquad \text{for } x = x_r \in \gamma_\alpha,\ t = t_{j+\alpha/p}, \end{aligned}\right\} \tag{80}$$

$$v(x, 0) = 0, \tag{81}$$

where

$$\tilde{\psi}_\alpha = \psi_\alpha^* + \Lambda_\alpha \eta,$$
$$\tilde{\nu}_1 = \nu_1 + (\beta_1 \eta^{(+1_\alpha)} - \eta), \qquad \tilde{\nu}_r = \nu_r + (\beta_r \eta^{(-1_\alpha)} - \eta).$$

The values of $\eta$ for $x = x_1$ and $x = x_r$ are found from formulae (77). At the node $x_1^{(+1_\alpha)}$ adjacent to $x_1$ on the chain $\Pi_\alpha$ we have

$$\eta_{\mathring{x}_\alpha} = (\eta^{(+2_\alpha)} - \eta)\frac{1}{2h_\alpha} = \frac{1}{2}\left(1 + \frac{a_\alpha^{(+1_\alpha)}}{a_\alpha}\right)\eta_{x_\alpha}$$

and hence

$$\Lambda_\alpha \eta = \frac{1}{2} b_\alpha \left(1 + \frac{a_\alpha^{(+1_\alpha)}}{a_\alpha}\right)\eta_{x_\alpha} - d_\alpha \eta. \tag{82}$$

We obtain a similar expression for $\Lambda_\alpha \eta$ at the node $x_r^{(-1_\alpha)}$. It follows from this and from (78) that $\Lambda_\alpha \eta = O(\tau)$ if conditions A are satisfied.

From Lemmas 1 and 2 we have

$$\tilde{\psi}_\alpha = O(h^2) + O(\tau), \qquad \tilde{\nu}_1 = O(h^2) + O(\tau), \qquad \tilde{\nu}_r = O(h^2) + O(\tau). \tag{83}$$

To estimate the function $v$ we use Theorem 1, taking into account the fact that it follows from conditions A and the properties of the pattern functionals of the scheme that the coefficients

$$a_\alpha \geqslant c_1 > 0, \qquad \rho \geqslant c_2 > 0, \qquad |d_\alpha| \leqslant c_3, \qquad |b_\alpha| \leqslant c_4,$$
$$0 \leqslant \beta_l \leqslant \beta^* < 1, \qquad 0 \leqslant \beta_r \leqslant \beta^* < 1$$

are bounded and so Theorem 1 is applicable. Using estimates (60), (83) and Lemma 2, we conclude that the following theorem is true.

*Theorem* 2. The local one-dimensional difference scheme (15)-(17) converges uniformly in $\bar{\Omega}$ at a rate $O(h^2) + O(\tau)$ as $h$ and $\tau$ tend independently to zero; more exactly, if conditions A (point 1, § 1) are satisfied, then for sufficiently small $\tau < \tau_0$

$$\|y(x, t) - u(x, t)\|_0 \leqslant M(h^2 + \tau), \qquad h^2 = \frac{1}{p}\sum_{\alpha=1}^{p} h_\alpha^2 \tag{84}$$

for all $t = t_{j+\alpha/p}$, $j = 0, 1, \ldots, k-1$, $\alpha = 1, 2, \ldots, p$ where $u$ is the solution of the problem (1)-(4), $y$ the solution of the difference problem (15)-(18), and $M$ is a positive constant which does not depend on the choice of the difference net.

Thus from Lemma 2

$$\eta = O(\tau) \quad \text{or} \quad \|\eta\|_0 < M\tau.$$

It follows from Theorem 1 and (83) that

$$\|v(x, t)\|_0 \leqslant M(h^2 + \tau).$$

We conclude that

$$\|z(x, t)\|_0 = \|y(x, t) - u(x, t)\|_0 \leqslant \|\eta(x, t)\|_0 + \|v(x, t)\|_0 \leqslant M(h^2 + \tau).$$

The next theorem can be proved similarly.

*Theorem 3.* If conditions A are satisfied, then the local one-dimensional difference scheme (35)-(37) uniformly converges at the rate $O(h^2) + O(\tau)$ so that for sufficiently small $\tau < \tau_0$ estimate (84) holds, where $u$ is the solution of the problem (30)-(34) and $y$ the solution of the corresponding problem (35)-(37).

*Notes.* 1. The solution of the difference boundary problem for a parallelepiped with boundary conditions of the third kind (51) converges uniformly at the rate $O(h) + O(\tau)$ both in the case of equation (1) and in the case of equation (29).

The order of accuracy with respect to $h$ can in this case be increased by selecting a finer net near the boundary. If, for example, the first step near the boundary is of order $O(h^2)$, then the magnitude of the error arising from the approximation to the boundary conditions will be $O(h^2)$. As a result, we obtain a non-uniform net on which the order of approximation for the scheme is, generally speaking, lowered. In this connexion a new examination of the integral order of accuracy of the difference scheme is called for. To do this special *a priori* estimates are needed, and these will be obtained in the second part of this article. The question of the convergence of homogeneous difference schemes for arbitrary non-uniform nets will be considered separately.

2. If $p = 2$, the conditions A can be relaxed. In this case we do not make use of the existence of the mixed derivatives of the functions $u$, $k_\alpha$.

## 3. Appendix

1. To illustrate the use of the local one-dimensional method, let us write out the computing formulae for the two-dimensional heat conductivity equation and for a rectangle ($0 < x_\alpha < l_\alpha$, $\alpha = 1, 2$, $p = 2$):

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(k_1(x_1, x_2, t)\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(k(x_1, x_2, t)\frac{\partial u}{\partial x_2}\right) + f(x_1, x_2, t),\\ &u(0, x_2, t) = \mu_{11}(x_2, t), \qquad u(l_1, x_2, t) = \mu_{12}(x_2, t),\\ &u(x_1, 0, t) = \mu_{21}(x_1, t), \qquad u(x_1, l_2, t) = \mu_{22}(x_1, t),\\ &u(x_1, x_2, 0) = u_0(x_1, x_2). \end{aligned}\right\} \quad (85)$$

Consider the net $x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha$, $\alpha = 1, 2$, $t_j = j\tau$ where $i_\alpha = 0, 1, \ldots, N_\alpha$, $h_\alpha = l_\alpha/N_\alpha$, $j = 0, 1, \ldots, K$, $\tau = T/K$. In this case one fractional step $t_{j+1/2} = t_j + 0.5\,\tau$ is introduced. Let $y_{i_1 i_2}^{j+\alpha/2} = y^{j+\alpha/2} = y(i_1h_1, i_2h_2, (j + 0.5\,\alpha)\tau)$, $\alpha = 1, 2$ denote the net function.

We write the local one-dimensional scheme

$$\left.\begin{aligned} &\frac{y^{j+1/2} - y^j}{\tau} = \Lambda_1 y^{j+1/2} + \varphi_1,\\ &\frac{y^{j+1} - y^{j+1/2}}{\tau} = \Lambda_2 y^{j+1} + \varphi_2, \end{aligned}\right\} \quad (86)$$

$$\left.\begin{aligned} &y_{0i_2}^{j+1/2} = \mu_{11}(i_2h_2, t_{j+1/2}), \qquad y_{N_1,i_2}^{j+1/2} = \mu_{12}(i_2h_2, t_{j+1/2}),\\ &y_{i_1,0}^{j+1} = \mu_{21}(i_1h_1, t_{j+1}), \qquad y_{i_1,N_2}^{j+1} = \mu_{22}(i_1h_1, t_{j+1}), \end{aligned}\right\} \quad (87)$$

$$y_{i_1 i_2}^0 = u_0(i_1h_1, i_2h_2), \quad (88)$$

$$\Lambda_\alpha y = (a_\alpha y_{\bar{x}_\alpha})_{x_\alpha} = \frac{a_\alpha^{(+1_\alpha)}(y^{(+1_\alpha)} - y) - a_\alpha\,(y - y^{(-1_\alpha)})}{h_\alpha^2},$$

$$\varphi_\alpha = \frac{1}{2} f(x_1, x_2, t_{j+0.5\alpha}) \text{ or } \varphi_1 = 0,\ \varphi_2 = f(x_1, x_2, t_{j+1}) \text{ etc.},$$

$$a_1(x_1, x_2, t) - k_1(x_1 - 0.5\,h_1, x_2, t) = O(h^2),\ a_2 - k_2^{(-0.5\alpha)} = O(h^2).$$

Let us give the simplest expressions for $a_\alpha$:

$$a_\alpha = k_\alpha^{(-0.5_\alpha)},\ \text{i. e. } a_1 = k_1(x_1 - 0.5\,h_1, x_2, t),\ a_2 = k_2(x_1, x_2 - 0.5\,h_2, t)\quad (x_\alpha = i_\alpha h_\alpha),$$

$$a_\alpha = 2k_\alpha^{(-1_\alpha)} k_\alpha / (k_\alpha^{(-1_\alpha)} + k_\alpha),\qquad a_1 = \frac{1}{2}\,[k_1(x_1 - 0.5\,h_1, x_2 - 0.5\,h_2, t) + k_1(x_1 - 0.5\,h_1, x_2 + 0.5\,h_2, t)] \text{ etc.}$$

We shall give the computing formulae for problem (86)-(88). We can re-write the difference equations (86) in the form

$$\left.\begin{aligned}
&(a_1)_{i_1 i_2} y_{i_1-1, i_2}^{j+1/2} - \left((a_1)_{i_1 i_2} + (a_1)_{i_1+1, i_2} + \frac{h_1^2}{\tau}\right) y_{i_1 i_2}^{j+1/2} + (a_1)_{i_1+1, i_2} y_{i_1+1, i_2}^{j+1/2} = -F_{i_1 i_2}^{j+1/2}\\
&\qquad (0 < i_1 < N_1),\\
&y_{i_1 i_2}^{j+1/2} = \mu_{11}(i_2 h_2, t_{j+1/2}),\qquad y_{N_1, i_2}^{j+1/2} = \mu_{12}(i_2 h_2, t_{j+1/2}),\\
&F_{i_1 i_2}^{j+1/2} = \frac{h_1^2}{\tau}\, y_{i_1 i_2}^{j} + h_1^2 \varphi_{1, i_1 i_2},
\end{aligned}\right\}\tag{89}$$

$$\left.\begin{aligned}
&(a_2)_{i_1 i_2} y_{i_1, i_2-1}^{j+1} - \left((a_2)_{i_1 i_2} + (a_2)_{i_1 i_2+1} + \frac{h_2^2}{\tau}\right) y_{i_1 i_2}^{j+1} + (a_2)_{i_1, i_2+1} y_{i_1, i_2+1}^{j+1} = -F_{i_1 i_2}^{j+1}\\
&\qquad (0 < i_2 < N_2),\\
&y_{i_1, 0}^{j+1} = \mu_{21}(i_1 h_1, t_{j+1}),\qquad y_{i_1, N_2}^{j+1} = \mu_{22}(i_1 h_1, t_{j+1}),\\
&F_{i_1 i_2}^{j+1} = \frac{h_2^2}{\tau}\, y_{i_1 i_2}^{j+1/2} + h_2^2 (\varphi_2)_{i_1 i_2}.
\end{aligned}\right\}\tag{90}$$

It is clear from this that similar problems are presented by $y_{i_1 i_2}^{j+1/2}$ and $y_{i_1 i_2}^{j+1}$: problem (89) is solved for the segments $0 \leqslant x_1 \leqslant l_1$ for different $i_2 = 1, 2, \ldots, N_2 - 1$, and problem (90) for segments $0 \leqslant x_2 \leqslant l_2$ for different $i_1 = 1, 2, \ldots, N_1 - 1$. Using the notation $y^{(\alpha)} = y_{i_1 i_2}^{j+\alpha/2}$ $(\alpha = 1, 2)$, $a_\alpha = (a_\alpha)_{i_1 i_2}$, $a_1^{(\pm 1_1)} = (a_1)_{i_1 \pm 1, i_2}$, $a_2^{(\pm 1_2)} = (a_2)_{i_1, i_2 \pm 1}$ we can write the formulae (89)-(90) in the form

$$\left.\begin{aligned}
&a_\alpha y^{(-1_\alpha)} - \left(a_\alpha + a_\alpha^{(+1_\alpha)} + \frac{h_\alpha^2}{\tau}\right) y + a_\alpha^{(+1_\alpha)} y^{(+1_\alpha)} = -\overset{(\alpha)}{F} \qquad (y = \overset{(\alpha)}{y}),\\
&\overset{(\alpha)}{y}\big|_{x_\alpha=0} = \mu_{\alpha 1}^{j+\alpha/2}, \qquad \overset{(\alpha)}{y}\big|_{x_\alpha=l_\alpha} = \mu_{\alpha 2}^{j+\alpha/2}, \qquad \alpha = 1, 2,\\
&\overset{(\alpha)}{F} = \frac{h_\alpha^2}{\tau}\overset{(\alpha-1)}{y} + h_\alpha^2 \varphi_\alpha, \qquad \overset{(0)}{y} = y^j.
\end{aligned}\right\} \quad (91)$$

The boundary problems (91) can be solved using the recurrence formulae

$$\xi_\alpha^{(+1_\alpha)} = \frac{a_\alpha^{(+1_\alpha)}}{a_\alpha^{(+1_\alpha)} + (1 - \xi_\alpha) a_\alpha + h_\alpha^2/\tau}, \qquad \xi_\alpha = 0 \text{ for } x_\alpha = h_\alpha, \quad (92)$$

$$\eta_\alpha^{(+1_\alpha)} = \frac{\xi_\alpha^{(+1_\alpha)}}{\overset{(+1_\alpha)}{a}} (a_\alpha \eta_\alpha + \overset{(\alpha)}{F}), \qquad \eta_\alpha = \overset{(\alpha)}{\mu}_{\alpha 1} \text{ for } x_\alpha = h_\alpha \quad (93)$$

$$y = \xi_\alpha^{(+1_\alpha)} y^{(+1_\alpha)} + \eta_\alpha^{(+1_\alpha)}, \; y = \overset{(\alpha)}{y} = \overset{(\alpha)}{\mu}_{\alpha 2} \text{ for } x_\alpha = l_\alpha = N_\alpha h_\alpha. \quad (94)$$

The formulae as given are suitable for programming. At time $t = t_{j+0.5}$ the computations follow formulae (92)-(94) with $\alpha = 1$, at time $t = t_{j+1}$ we put $\alpha = 2$ in formulae (92)-(94).

In the special case of an equation with constant coefficients $k_\alpha = 1$ we have $a_\alpha = 1$ and formulae (92)-(94) are simplified.

2. For the sake of comparison, let us give other economical schemes, discussed in [1], [2], [4], for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}.$$

with constant coefficients.

In [1]: $\dfrac{y^{j+1/2} - y^j}{\tau} = 0.5\,(\Lambda_1 y^{j+1/2} + \Lambda_2 y^j), \quad \dfrac{y^{j+1} - y^{j+1/2}}{\tau} = 0.5\,(\Lambda_1 y^{j+1/2} + \Lambda_2 y^{j+1}).$

In [2]: $\dfrac{y^{j+1/2} - y^j}{\tau} = \Lambda_1 y^{j+1/2} + \Lambda_2 y^j, \quad \dfrac{y^{j+1} - y^{j+1/2}}{\tau} = \Lambda_2 y^{j+1} - \Lambda_2 y^j.$

In [4]: $\dfrac{y^{j+1/2} - y^j}{\tau} = \Lambda_1 (\sigma y^{j+1/2} + (1 - \sigma) y^j),$

$$\frac{y^{j+1} - y^{j+1/2}}{\tau} = \Lambda_2 (\sigma y^{j+1} + (1 - \sigma) y^{j+1/2}), \qquad \text{where } 0 \leqslant \sigma \leqslant 1. \quad (95)$$

Since each equation contains the variables in two layers $j + \alpha/2$ and $j + (\alpha - 1)/2$ $(\alpha = 1, 2)$, we need the boundary conditions (for $\sigma \neq 1$) at the points $x_1 = 0$, $x_1 = l_1$ for $t = t_j$ and $t = t_{j+1/2}$ to determine $y^{j+1/2}$ and the values at $x_2 = 0$, $x_2 = l_2$ for $t = t_{j+1/2}$ and $t = t_{j+1}$ to determine $y^{j+1}$.

In [11]: $$\frac{y^{j+1/2}-y^j}{\tau} = 0.5\,(\Lambda_1 y^{j+1/2} + \Lambda_1 y^j) + 0.5\,\Lambda_2 y^j + \frac{\tau}{4}\,\Lambda_1\Lambda_2 y^j,$$
$$\frac{y^{j+1}-y^{j+1/2}}{\tau} = 0.5\ \Lambda_2 y^{j+1/2}.$$

In this case, for $t = t_j$ and $t = t_{j+1}$ we use the boundary conditions over the whole boundary, and for $t = t_{j+1/2}$ the boundary conditions for $x_1 = 0$ and $x_1 = l_1$ are determined in terms of $\mu_{21}^{j+1}$ and $\mu_{22}^{j+1}$ from the formula

$$y^{j+1/2} = y^{j+1} - 0.5\,\tau\Lambda_2 y^{j+1}.$$

These values are used to find $y^{j+1/2}$ from the first equation. The first equation for $y^{j+1/2}$ is complicated because we have attempted to increase the order of accuracy with respect to $\tau$.

All these schemes are absolutely stable and convergent in the mean.

3. For the parabolic equation of general form

$$c\,(x, t)\,\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p}\sum_{\beta=1}^{p}\frac{\partial}{\partial x_\alpha}\left(k_{\alpha\beta}\,(x, t)\,\frac{\partial u}{\partial x_\beta}\right) + \sum_{\alpha=1}^{p} r_\alpha\,(x, t)\,\frac{\partial u}{\partial x_\alpha},$$

where

$$\sum_{\alpha,\beta=1}^{p} k_{\alpha\beta}\xi_\alpha\xi_\beta \geqslant c_0 \sum_{\alpha=1}^{p}\xi_\alpha^2, \qquad c_0 = \text{const} > 0,$$

we can also write down a number of absolutely stable and (mean) convergent economic schemes.

We give a few of the schemes for the case $p = 2$, $k_{\alpha\beta} = \text{const}$, $k_{12} = k_{21}$:

1) $y_{\bar{t}_\alpha} = \Lambda_{11}\bar{y} + \sigma\overset{*}{\Lambda}_{12}y, \qquad y_{\bar{t}_2} = \Lambda_{22}y + (1-\sigma)\,\overset{**}{\Lambda}_{12}\bar{y}, \qquad 0 \leqslant \sigma \leqslant 1,$

where $y = y^{j+1}, \quad \bar{y} = y^{j+1/2}, \quad \check{y} = y^j, \quad \Lambda_{\alpha\alpha}y = k_{\alpha\alpha}y_{\bar{x}_\alpha x_\alpha} + r_\alpha y_{\overset{\circ}{x}_\alpha}, \quad y_{\overset{\circ}{x}_\alpha} = (y^{(+1_\alpha)} - y^{(-1_\alpha)})\,/\,2h_\alpha, \quad \overset{*}{\Lambda}_{12}y = 2\,k_{12}y_{x_1\bar{x}_2}, \quad \overset{**}{\Lambda}_{12}y = 2k_{12}y_{\bar{x}_1 x_2}, \quad y_{\bar{t}_\alpha} = (y^{j+\alpha/p} - y^{j+(\alpha-1)/p})\,/\,\tau$
$(\alpha = 1, 2;\ p = 2)$;

2) $y_{\bar{t}_1} = \Lambda_{11}\bar{y} + \sigma\Lambda_{12}\check{y}, \ \ y_{\bar{t}_2} = \Lambda_{22}y + (1-\sigma)\,\Lambda_{12}\bar{y}, \qquad 0 \leqslant \sigma \leqslant 1, \quad \Lambda_{12} = \overset{*}{\Lambda}_{12} + \overset{**}{\Lambda}_{12};$

3) $y_{\bar{t}_1} = 0.5\,\Lambda_{11}\bar{y}, \qquad y_{\bar{t}} = (y - \check{y})\,/\,\tau = 0.5\,\Lambda_2\,(y - \check{y}) + \Lambda\bar{y}, \quad \Lambda\bar{y} = \Lambda_{11}y + \Lambda_{22}y + \Lambda_{12}y.$

The scheme

$$y_{\bar{t}_1} = \Lambda_{11}\bar{y} + \Lambda_{12}\check{y}, \quad y_{\bar{t}_2} = \Lambda_{22}y + \Lambda_{12}\bar{y}, \qquad \Lambda_{12}y = k_{12}y_{\overset{\circ}{x}_2\overset{\circ}{x}_1}.$$

(for $r_\alpha = 0$) is suggested in [7]. If $p = 3$ and $k_{\alpha\beta} = \text{const}$, then we can use the following absolutely stable and mean convergent scheme:

$$y_{\bar{t}_1} = \Lambda_{11}\bar{y}, \qquad y_{\bar{t}_2} = \Lambda_{22}\bar{\bar{y}} + \Lambda_{12}\bar{y}, \qquad y_{\bar{t}_3} = \Lambda_{33}y + \Lambda_{23}\bar{\bar{y}} + \Lambda_{13}\bar{y},$$

where

$$\Lambda_{\alpha\beta}y = k_{\alpha\beta}(y_{\bar{x}_\alpha x_\beta} + y_{x_\alpha \bar{x}_\beta}), \qquad k_{\alpha\beta} = k_{\beta\alpha} \text{ for } \alpha \neq \beta, \quad \bar{y} = y^{j+1/3}, \quad \bar{\bar{y}} = y^{j+2/3}.$$

If $p = 3$ and $k_{\alpha\beta} = \text{const}$, $r_\alpha = 0$ the scheme suggested in [9] uses five fractional steps (where we use two). The schemes for an equation with variable coefficients are written similarly.

4. For the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = \sum_{\alpha=1}^{2} L_\alpha u, \qquad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x,t)\frac{\partial u}{\partial x_\alpha} \qquad (p = 2)$$

we can use the local one-dimensional scheme

$$\frac{y^{j+\alpha/2} - 2y^{j+(\alpha-1)/2} + y^{j-1+\alpha/2}}{\tau^2} = 0.5\,[(\Lambda_\alpha y)^{j+\alpha/2} + (\Lambda_\alpha y)^{j-1+\alpha/2}], \qquad \alpha = 1, 2,$$

where $\Lambda_\alpha y = (a_\alpha y_{\bar{x}_\alpha})_{x_\alpha} + b_\alpha y_{\overset{\circ}{x}_\alpha}$, with boundary conditions (2) and an arbitrary region $G$. This scheme is absolutely stable and (for a rectangular region $G$) has second order accuracy in $\tau$ and $h^2$. Other types of schemes for an equation with constant coefficients are discussed in [19].

5. Finally let us give an absolutely stable and mean convergent scheme for the quasilinear equation

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} L_\alpha u + f\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_p}\right), \qquad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right).$$

This scheme has the form

$$y_{\bar{t}_\alpha} = \Lambda_\alpha y^{j+\alpha/p}, \qquad \alpha = 1, 2, \ldots, p-1,$$

$$y_{\bar{t}_p} = \Lambda_p y^{j+1} + f(x, t_{j+1}, y^{j+(p-1)/p}, y_{\overset{\circ}{x}_1}^{j+1/p}, \ldots, y_{\overset{\circ}{x}_{p-1}}^{j+(p-1)/p}, y_{\overset{\circ}{x}_p}^{j}), \qquad \alpha = p.$$

We shall give the proof of convergence and stability for the schemes in points 3, 4 and 5 elsewhere.

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