

# EXAMPLES OF THE NUMERICAL CALCULATION OF TEMPERATURE WAVES\*

A.A. SAMARSKII and I.M. SOBOL'

(Moscow)

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## 1. Introduction

In this paper we are concerned with the numerical solution of the quasilinear heat conduction equation

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^p \frac{\partial}{\partial x_{\alpha}} \left( K_{\alpha}(u) \frac{\partial u}{\partial x_{\alpha}} \right) \quad (1.1)$$

for the cases  $p = 1, 2, 3$ . As a rule it is assumed everywhere that

$$K_{\alpha}(u) = \kappa_{\alpha} u^{\sigma_{\alpha}},$$

where  $\sigma_{\alpha} \geq 1$ ,  $\kappa_{\alpha} > 0$ . Although equation (1.1) arises in several fields of mathematical physics, for the sake of definiteness we shall call the function  $u = u(t, x_1, \dots, x_p)$  a temperature function.

It is shown in [1], [2] that when  $p = 1$  equation (1.1) has solutions whose derivatives are discontinuous at the points where  $u(t, x)$  is equal to zero, and the flow  $K(u) \partial u / \partial x$  is continuous, i.e. there exists a temperature front  $u = 0$  (Fig. 1) which is propagated with a finite velocity (see [3]). In this case there is no classical solution of the equation. The existence of a generalised solution of the Cauchy problem and of the boundary value problems is proved in [4]. In [5], [6] the convergence of one explicit difference scheme for an equation of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 F(u)}{\partial x^2} \quad (1.2)$$

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in the class of generalised solutions is proved (these results can probably be extended to the case of implicit schemes also). In [7] a generalised solution of an equation of the form (1.2) is calculated by A.A. Dorodnitsyn's integral relations method.

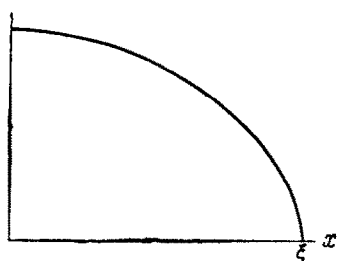


FIG. 1.

To calculate such generalised solutions (which we shall call *temperature waves*, or simply solutions, below) we use homogeneous difference schemes for 'through computation' which do not require that the points of weak discontinuity be picked out explicitly. The theory of schemes of this kind is elaborated in the papers [8], [9], [10], the last of which also gives a bibliography. However all the proofs of convergence assume that  $K_\alpha(u) \geq c > 0$  and, despite the fact that these theorems are very general [even admitting discontinuous functions  $K_\alpha = K_\alpha(t, x, u)$ ] they are inapplicable for the case when  $K_\alpha(u)$  is equal to zero.

Our purpose in this paper is to show that these schemes are also suitable for the calculation of temperature waves. They enable us to use a large time step in the computation, give a good idea of the velocity of propagation of the front and when the net is sufficiently fine give the profile of the front itself.

In the case of several space variables ( $p > 1$ ) we use the local one-dimensional method of variable directions described in [11], [12]. These papers also give a bibliography of work done on this problem (of which the work [13] should be noted). Let us give a brief description of the method of [11], [12] applied to equation (1.1).

The time step  $t^j \leq t \leq t^{j+1}$  is divided into  $p$  layers of identical width ("fractional steps")

$$t^{j+(\alpha-1)/p} \leq t \leq t^{j+\alpha/p}, \quad \alpha = 1, 2, \dots, p.$$

In the layer numbered  $\alpha$  the one-dimensional equation

$$\frac{1}{p} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x_\alpha} \left( K_\alpha(u) \frac{\partial u}{\partial x_\alpha} \right) \quad (1.3)$$

is solved. All coordinates  $x_\beta$  differing from  $x_\alpha$  are taken to be parameters. In this stage, as boundary conditions we use the values of the boundary functions at the points of intersection of straight lines parallel to the axis  $ox_\alpha$  with the boundary of the region of integration, and as initial values we take values obtained in the calculation of the preceding layer. In fact, the same one-dimensional program (OP) is used to solve all the equations (1.3), and this program replaces (1.3) by an implicit homogeneous difference scheme (Section 2, Para. 2).

In our opinion there is at the present time no more suitable method for the practical solution of many-dimensional quasilinear parabolic equations. This method can be applied to arbitrary regions (and not only to parallelepipeds) and the order of accuracy is preserved on non-uniform nets [12]. It is suitable for quasilinear parabolic equations of general form even when there are discontinuities (of the first kind) in the coefficients. With such a wide range of applicability the method of variable directions possesses in addition a whole number of practical advantages: simplicity of program; less rigorous requirements on the size of the working store (in comparison with most other schemes); stability of computation for very large time steps, enabling complex problems which do not require great accuracy to be solved quickly.

Instead of giving the exact profile of the wave, calculation using any difference scheme gives its own difference profile (the finer the net the more exact the profile). In order to study the structure of this profile for a very large net and to estimate the effective width of the front we construct the difference travelling wave in Section 5 for the case  $p = 1$ ; this is the analogue of the known solution of the form  $u = f(ct - x)$  called the travelling wave (the constant  $c$  is the velocity of the wave). The difference travelling wave was constructed in [14] for difference schemes of through computation in viscous gas dynamics.

It must be stressed that nowhere have we attempted to choose the most favourable computing conditions for our problem. On the contrary, in certain cases we have purposely chosen unfavourable conditions to make the divergence more noticeable. In some instances the space nets are large, and in others quite fine. The time step is always taken to be large.

The authors wish to express their deep gratitude to A.A. Dumova for writing the program and making the calculations.

## 2. One-dimensional problems

### 1. On the velocity of propagation of the temperature front

Let us consider the equation

$$\frac{\partial \varphi(u)}{\partial t} = \frac{\partial}{\partial x} \left[ K(u) \frac{\partial u}{\partial x} \right], \quad (2.1)$$

where  $K(0) = \varphi(0) = 0$ ;  $K(u) > 0$ ,  $\varphi'(u) > 0$  when  $u > 0$ ;

$$\lim_{u \rightarrow 0} [K(u)/\varphi'(u)] = 0.$$

The propagation of the hot front in this case takes place with finite velocity.

Let us denote the position of the front at time  $t$  by  $\xi(t)$  (Fig. 1).

Differentiating the identity  $\varphi(u(t, \xi(t))) \equiv 0$  and using the condition for continuity of flow at the front

$$[K(u) \partial u / \partial x]_{x=\xi(t)} \equiv 0,$$

we can derive the following expression for the velocity of the front:

$$\frac{d\xi}{dt} = - \lim_{x \rightarrow \xi} \left[ \frac{K(u)}{\varphi(u)} \frac{\partial u}{\partial x} \right]. \quad (2.2)$$

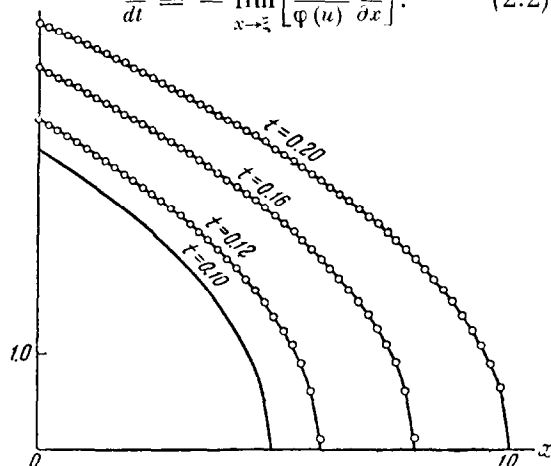


FIG. 2.

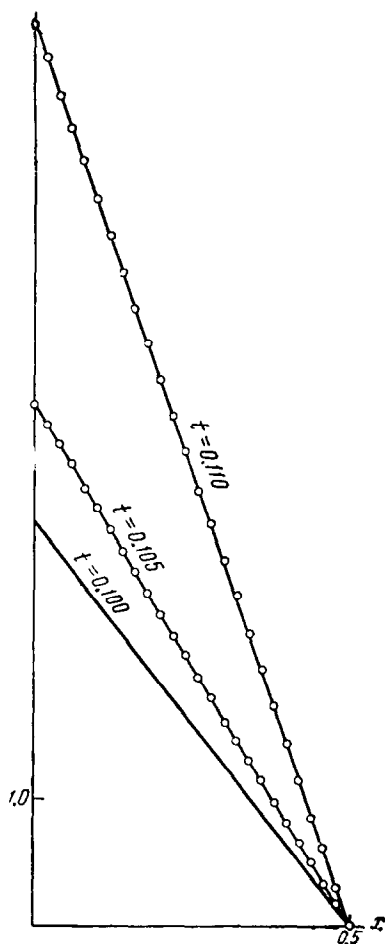


FIG. 3.

Below, in Example 1, the initial profile for  $t_0 = 0.1$  is

$$u = \begin{cases} \sqrt{10(1-2x)}, & 0 \leq x \leq 0.5, \\ 0, & 0.5 \leq x < 1. \end{cases}$$

Obviously  $\xi(t_0) = 0.5$  and, from formula (2.2),  $d\xi/dt = 5$ .

In Example 2 the initial profile for  $t_0 = 0.1$  is

$$u = \begin{cases} \sqrt{10(1-2x)}, & 0 \leq x \leq 0.5, \\ 0, & 0.5 \leq x < 1. \end{cases}$$

Here too  $\xi(t_0) = 0.5$  but from formula (2.2) we have  $d\xi/dt = 0$ . The corresponding solution in the first case (Fig. 2) is a wave propagated with constant velocity, but in the second case (Fig. 3), despite the rapid increase in temperature, the front is not propagated.

## 2. The computing scheme

The one-dimensional program OP was written to solve equation (2.1) as this is more general than equation (1.1) for  $p = 1$ , with boundary conditions of the first kind

$$u(t, 0) = \mu(t), \quad u(t, l) = \bar{\mu}(t).$$

It is assumed that  $\varphi(u) = u^r$ , but in all the examples given below  $r = 1$ .

We replace equation (2.1) by a homogeneous difference scheme (see [10]) with lead

$$\varphi(v_i) - \varphi(v_i) = A_{i+1}(v_{i+1} - v_i) - A_i(v_i - v_{i-1}), \quad (2.3)$$

where

$$A_i = \frac{\tau}{h^2} K \left( \frac{v_{i-1} + v_i}{2} \right); \quad (2.4)$$

the quantity with the invested circumflex is calculated in the  $j$ -th step, and the quantities without it in the  $(j + 1)$ -th step. The net is assumed to be uniform:  $x_i = ih$ ,  $0 \leq i \leq N$ ;  $t^j = j\tau$ . With this scheme there is stability for any step  $\tau$ .

We can solve the system of equations (2.3) for  $i = 1, 2, \dots, N - 1$

in every  $(j + 1)$ -th step by the following iterative method. Let  $s$  be the number of the iteration; writing

$$\varphi(v_i^{(s+1)}) = \varphi(v_i^{(s)}) + (v_i^{(s+1)} - v_i^{(s)}) \varphi'(v_i^{(s)}),$$

we can rewrite equation (2.3) in the form of a linear equation in  $v_i^{(s+1)}$ , omitting the indices  $(s)$  over all the other quantities for the sake of brevity:

$$A_{i+1}v_{i+1}^{(s+1)} - (A_{i+1} + A_i + B_i)v_i^{(s+1)} + A_iv_{i-1}^{(s+1)} + F_i = 0. \quad (2.5)$$

Here  $A_i$  are given by formula (2.4)

$$B_i = \varphi'(v_i), \\ F_i = \varphi(\check{v}_i) - \varphi(v_i) + v_i B_i.$$

Each iteration requires the solution of system (2.5) for  $i = 1, 2, \dots, N - 1$ . This is done by the method of successive substitution (see [15]) with the formulae:

$$\begin{aligned} \alpha_1 &= 0; & \alpha_{i+1} &= \frac{A_{i+1}}{A_{i+1} + A_i(1 - \alpha_i) + B_i}, & i &= 1, 2, \dots, N-1; \\ \beta_1 &= v_0; & \beta_{i+1} &= \frac{A_i\beta_i + F_i}{A_{i+1} + A_i(1 - \alpha_i) + B_i}, & i &= 1, 2, \dots, N-1; \\ v_N^{(s+1)} &= v_N; & v_i^{(s+1)} &= \alpha_{i+1}v_{i+1}^{(s+1)} + \beta_{i+1}, & i &= N-1, N-2, \dots, 1. \end{aligned}$$

The values of  $v_0$  and  $v_N$  are given by the boundary conditions:

$$v_0 = \mu(j\tau), \quad v_N = \bar{\mu}(j\tau).$$

For the zero iteration we take the values from the previous step:

$v_i^{(0)} = \check{v}_i$ . The condition for the iterations to end is

$$\max_{1 \leq i \leq N-1} |v_i^{(s+1)} - v_i^{(s)}| < \varepsilon.$$

In all our calculations we have put  $\varepsilon = 10^{-3}$  (see Section 3, Table 3).

For each example we calculate the actual number of iterations  $v^j$  and the "Courant ratio"

$$\chi = \max [K(u) \tau / h^2],$$

which characterises the size of the time step.

### 3. Example 1. A wave propagated with constant velocity

We use the analytic solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \kappa_0 u^\sigma \frac{\partial u}{\partial x} \right), \quad (2.6)$$

which is a travelling wave:

$$u = \begin{cases} [\sigma \kappa_0^{-1} (ct + x_1 - x)]^{1/\sigma}, & \text{for } x \leq x_1 + ct, \\ 0, & \text{for } x_1 + ct \leq x. \end{cases} \quad (2.7)$$

The parameters of the calculation are  $\sigma = 2$ ,  $\kappa_0 = 0.5$ ,  $c = 5$ ,  $x_1 = 0$ . From the solution (2.7) we selected initial values for  $t_0 = 0.100$  and the boundary conditions

$$u(t, 0) = 10 \sqrt{t}, \quad u(t, x_N) = 0.$$

The net is quite fine:  $h = 0.02$ ,  $N = 50$ . The calculation was carried as far as  $t = 0.200$  with the step  $\tau = 2 \times 10^{-4}$  ( $\chi = 5.0$ ). The analytic solution and the results of the computation are shown in Fig. 2. Except for certain nodes near the front the deviation between the calculated solution and the exact solution nowhere exceeds 0.002. The number of iterations  $v \leq 3$ .

### 4. Example 2. A non-propagated front

We use the analytic solution of equation (2.6) for  $-\infty < t < c$ :

$$u = \begin{cases} \left[ \frac{\sigma (x_1 - x)^2}{2\kappa_0 (\sigma + 2) (c - t)} \right]^{1/\sigma} & \text{for } x \leq x_1, \\ 0, & \text{for } x_1 \leq x. \end{cases} \quad (2.8)$$

The parameters of the calculation are  $\sigma = 2$ ,  $\kappa_0 = 0.5$ ,  $x_1 = 0.5$ ,  $c = 0.1125$ . The step of the net is  $h = 0.02$ , the number of nodes  $N = 50$ . From the solution (2.8) we took initial values for  $t_0 = 0.100$  and the boundary conditions

$$u(t, 0) = 1/\sqrt{0.9-8t}, \quad u(t, x_N) = 0.$$

The calculation was carried to  $t = 0.110$  with step  $\tau = 10^{-4}$  ( $\chi = 6.2$ ). The analytic solution and the results of the calculation are shown in Fig. 3.

The absolute error for  $t = 0.110$  does not exceed 0.03. The number of iterations  $v \leq 3$ .

It should be noted that the initial and boundary condition in Examples 1 and 2 are very similar, and the conditions of the calculation are identical.

### 5. Example 3. A wave propagated with variable velocity

We use a similarity solution of equation (2.6) put in the form of the series

$$u = \begin{cases} c(t + c_1)^n (1-s)^{1/\sigma} \frac{1 + a_1(1-s) + a_2(1-s)^2 + \dots}{1 + a_1 + a_2 + \dots} & \text{by } 0 \leq x \leq \xi, \\ 0, & \text{by } \xi \leq x, \end{cases} \quad (2.9)$$

where

$$s = x/\xi, \quad \xi = D(t + c_1)^m,$$

and the constants are

$$m = \frac{n\sigma + 1}{2}, \quad D^2 = \frac{\kappa_0 c^\sigma}{m\sigma(1 + a_1 + a_2 + \dots)^\sigma},$$

$$a_1 = \frac{n\sigma - m}{2m\sigma(\sigma + 1)}, \quad a_2 = -a_1 \times \frac{1 + 0.5a_1[(2\sigma + 1)(3\sigma + 1) - 4(\sigma + 1)]}{3(2\sigma + 1)}.$$

The parameters of the calculation are  $\sigma = 2$ ,  $n = 0$ ,  $c = 10$ ,  $c_1 = 0$ ,  $\kappa_0 = (1 + a_1 + a_2)^2$ . The step of the net  $h = 0.05$ , the number of nodes  $N = 80$ . The initial profile was calculated from the above terms of the solution (2.9) for  $t_0 = 0.010$ . The boundary conditions are

$$u(t, 0) = 10, \quad u(t, 4) = 0.$$

The calculation was done up to  $t = 0.090$ : (a) with step  $\tau = 2 \times 10^{-4}$



( $\chi = 6.7$ ) and (b) with step  $\tau = 10^{-3}$  ( $\chi = 33.6$ ). The difference between the results of the two calculations does not exceed 0.02 anywhere apart from a few nodes near the front.

TABLE 1.

$t$	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
a)	4	4	4	3	3	3	3	3
b)	8	7	6	6	6	6	5	5

Therefore in Fig. 4 we have put all the points obtained in variant (a) and only some of the crosses corresponding to the results of variant (b). The continuous curve was calculated from the terms in (2.9).

The number of iterations  $v$  is given in Table 1.

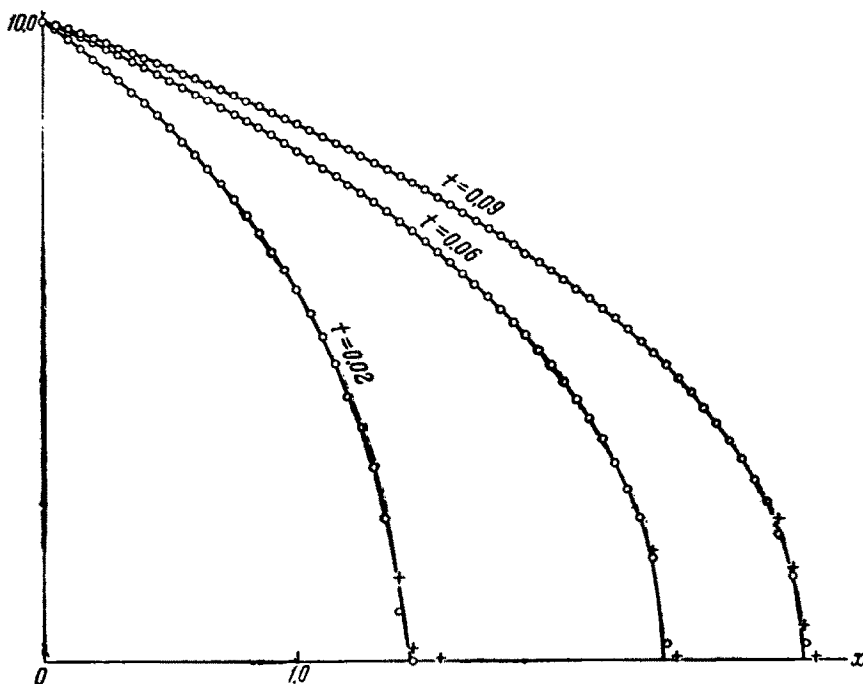


FIG. 4.

Clearly, by increasing the step five times we double the number of iterations. Thus computation with a large step is more profitable provided that it gives sufficient accuracy.

#### 6. Example 4. "Decay of a discontinuity" and calculation for steady flow

With the pre-set boundary conditions

$$u(t, 0) = u_0, \quad u(t, l) = 0$$

equation (2.6) has the stationary solution

$$u = u_0 (l - x)^{1/(\sigma+1)}, \quad 0 \leq x \leq l. \quad (2.10)$$

Let us choose the parameters for the calculation:  $\sigma = 2$ ,  $\kappa_0 = 3$ ,  $u_0 = 10$ ,  $l = 1$ . The step of the net  $h = 0.02$ , the number of nodes  $N = 50$ . The initial profile for  $t = 0$  is the same as in [7]:

$$u(0, x) = \begin{cases} 10, & 0 \leq x < 0.5, \\ 0 & 0.5 < x < 1. \end{cases}$$

The boundary conditions are

$$u(t, 0) = 10, \quad u(t, 1) = 0.$$

Using the comparison theorem of [3] we can estimate the time  $t^*$  when the front reaches the point  $x = 1$ . To do this we have to construct two analytic solutions of equation (2.6) for  $\sigma = 2$  and  $\kappa_0 = 3$  such that

$$u_1(0, x) \leq u(0, x) \leq u_2(0, x)$$

and

$$u_1(t, 0) \leq u(t, 0) \leq u_2(t, 0),$$

and to find the corresponding times  $t_1^*$  and  $t_2^*$ .

As  $u_1$  we can take a solution of the form (2.9) for  $n = 0$ , having found the arbitrary constants  $c$  and  $c_1$  from the conditions  $u_1(0, 0) = 10$ ,  $\xi_1(0) = 0.5$  (Fig. 5). We obtain  $t_1^* = 2 \times 10^{-3}$ . As  $u_2$  we choose a solution of the type (2.7), where the parameters  $c$  and  $x_1$  are found from the conditions  $u(0, \frac{1}{2}) = 10$  and  $t^* = \max$ . We obtain

$u_2 = 20 \sqrt{600t - x + 0.75}$  (Fig. 5) and  $t_2^* = 0.4 \times 10^{-3}$ . Thus we have the estimate

$$0.4 \times 10^{-3} < t^* < 2.1 \times 10^{-3}.$$

The calculation (a) was carried out with the step  $\tau = 10^{-5}$  ( $\chi = 7.5$ ) up to  $t = 5 \times 10^{-3}$ ; the time  $t^*$  turned out to be somewhat smaller than  $1.5 \times 10^{-3}$  (the results were printed out after 50 steps). At time  $t = 5 \times 10^{-3}$  the profile was still not steady. The number of iterations was:

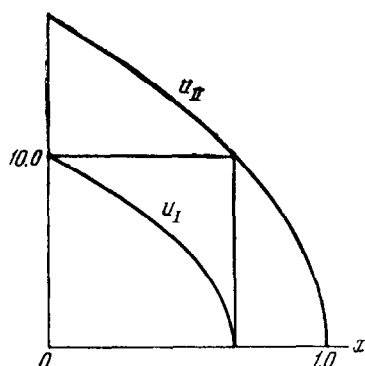


FIG. 5.

$10^4 t$	5	10	15	20	25	30	35	40	45	50
$\nu$	4	4	3	2	2	2	1	1	1	1

The results are shown in Fig. 6.

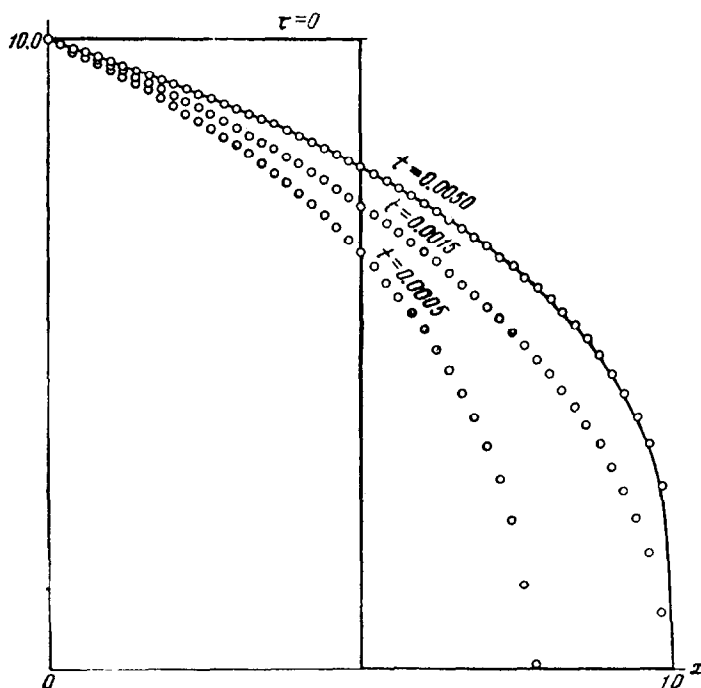


FIG. 6.

It is proved in Section 5, Para. 5 that in this example the steady profile does not depend on the time step. It is thus better to take a large step. The calculation (b) was carried out with the step  $\tau = 2 \times 10^{-4}$  ( $\chi = 150$ ) up to  $t = 2.5 \times 10^{-2}$ . The profile was already steady for  $t = 10^{-2}$ . \* For all  $t$ , starting from  $5 \times 10^{-3}$ , the number of iterations  $\nu = 1$ .

### 3. The two-dimensional problem

#### 1. Computing scheme

The program was written to solve equation (1.1) for  $p = 2$  with the boundary conditions

$$\begin{aligned} u(t, 0, y) &= \mu_1(t, y), & u(t, x, 0) &= \mu_2(t, x), \\ u(t, l_1, y) &= \bar{\mu}_1(t, y), & u(t, x, l_2) &= \bar{\mu}_2(t, x); \end{aligned}$$

for simplicity we shall use  $x, y$  instead of  $x_1, x_2$  as our independent variables.

The net is assumed to be uniform:  $x_i = ih_1, 0 \leq i \leq N_1; y_k = kh_2, 0 \leq k \leq N_2; t^j = j\tau$ .

At the initial time  $t_0 = 0$  the matrix of initial values  $(v_{ik}^0)$  is given. Transition from the matrix  $(v_{ik}^j)$  in the step  $t^j$  to the matrix  $(v_{ik}^{j+1})$  is carried out in two stages:

1) the matrix  $(v_{ik}^{j+1/2})$  is calculated. The same one-dimensional program OP is used for this (Section 2, Para. 2) with  $N = N_1$

$$A_i = \tau h_1^{-2} K_1 [0.5 (v_{i-1} + v_i)],$$

$$v_0 = \mu_1((j + \frac{1}{2})\tau, kh_2),$$

$$v_{N_1} = \bar{\mu}_1((j + \frac{1}{2})\tau, kh_2).$$

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\* Subsequent changes in the values of  $v_i$  do not exceed one unit in the sixth significant figure.

This program calculates the row  $v_{ik}^{j+1/2}$  in turn for all  $k = 1, 2, \dots, N_2 - 1$ ; for each row  $v_{ik}^j$ .

2) we insert  $N = N_2$  in the OP program:

$$A_k = \tau h_2^{-2} K_2 [0.5 (v_{k-1} + v_k)],$$

$$v_0 = \mu_2 ((j + 1) \tau, ih_1),$$

$$v_{N_2} = \bar{\mu}_2 ((j + 1) \tau, ih_1),$$

and using this program for each column  $v_{ik}^{j+1/2}$  we calculate the column  $v_{ik}^{j+1}$  in turn for all  $i = 1, 2, \dots, N_1 - 1$ .

Thus the OP is used as a sub-routine which transforms the rows  $(v_{ik}^j)$  into the rows  $(v_{ik}^{j+1/2})$ , and then the columns  $(v_{ik}^{j+1/2})$  into the columns  $(v_{ik}^{j+1})$ . Altogether in each step the OP operates  $(N_2 - 1) + (N_1 - 1)$  times.\*

To give an idea of the step-length we calculate the quantities

$$\chi_1 = \max [K_1(u) \tau / h_1^2] \text{ and } \chi_2 = \max [K_2(u) \tau / h_2^2].$$

In order to observe the course of the computation easily at each step the quantity

$$\delta^j = \left\{ \sum_{i=1}^{N_1-1} \sum_{k=1}^{N_2-1} [v_{ik}^j - u(j\tau, ih_1, kh_2)]^2 \times h_1 h_2 \right\}^{1/2},$$

is calculated; in some measure this characterises the deviation from the exact solution.

## 2. Choice of the analytic solution

We can look for the solution of equation (1.1) in the form of a plane travelling wave  $u = f(\omega)$ , where

$$\omega = t - \sum_{\alpha=1}^p \lambda_\alpha x_\alpha,$$

and  $\lambda_1, \dots, \lambda_p$  are constants. Substituting this solution into equation

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\* We could take the boundary conditions in the first state at any other time  $t'$  lying between  $t^j$  and  $t^{j+1}$ .

(1.1) we obtain an ordinary differential equation whose integral has the form

$$\omega = C_1 + \sum_{\alpha=1}^p \kappa_\alpha \lambda_\alpha^2 \int \frac{u^{\sigma_\alpha} du}{u - C};$$

where  $C$  and  $C_1$  are constants of integration.

The family of solutions corresponding to  $C = 0$  has a very simple form:

$$\omega = C_1 + \sum_{\alpha=1}^p (\kappa_\alpha / \sigma_\alpha) \lambda_\alpha^2 u^{\sigma_\alpha}. \quad (3.1)$$

### 3. Example 5. An anisotropic plane wave

Let us consider the two-dimensional equation (1.1) with parameters

$$\sigma_1 = 4, \kappa_1 = 4; \quad \sigma_2 = 2, \kappa_2 = 0.25,$$

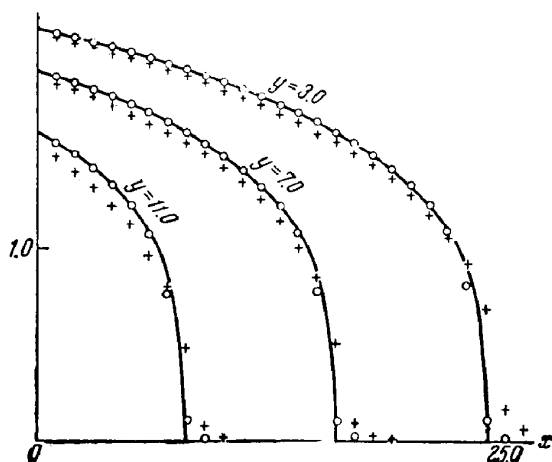


FIG. 7.

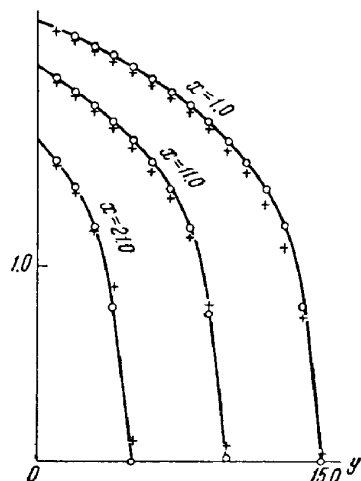


FIG. 8.

and for the calculation let us use the solution (3.1) with given constants  $C_1 = 0$ ,  $\lambda_1 = 1$ ,  $\lambda_2 = 2$ . The corresponding solution is

$$u = \begin{cases} 0.5 \sqrt{-\gamma + \sqrt{1 + 16(t - x - 2y)}} & \text{for } t \geq x + 2y, \\ 0 & \text{for } t \leq x + 2y. \end{cases} \quad (3.2)$$

the net is crude:  $h_1 = h_2 = 1$ ; the number of nodes  $N_1, N_2 = 30 \times 20 = 600$ . We take initial values from the solution (3.2), i.e.  $u(0, x, y) = 0$ ,

and boundary conditions on the straight lines  $x = 0$ ,  $x = 30$ ,  $y = 0$  and  $y = 20$ . The calculations were carried out up to  $t = 50$ ; (a) with step  $\tau = 0.2$  ( $\chi_1 = 37.2$ ,  $\chi_2 = 0.34$ ); (b) with step  $\tau = 1.0$  ( $\chi_1 = 186$ ,  $\chi_2 = 1.7$ ); (c) with step  $\tau = 2.0$  ( $\chi_1 = 372$ ,  $\chi_2 = 3.4$ ). Some of the results for  $t = 30$  are shown in Figs. 7 and 8, where the crosses denote

TABLE 2.

$t$		10	20	30	50
$\delta^j$	a)	0.290	0.438	0.543	0.457
	b)	0.538	0.935	1.270	1.019
	c)	0.825	1.55	2.13	1.85

TABLE 3.

$\epsilon \backslash t$	5	10	15	20	25
10	0.356	0.963	1.477	1.89	2.27
$10^{-3}$	0.217	0.290	0.392	0.438	0.509
$10^{-6}$	0.217	0.290	0.392	0.438	0.509

the results of variant (c), the dots those of variant (a); the continuous curves show the analytic solution.

The deviation from the exact solution is given in Table 2.

The reduction in the deviation for  $t = 50$  is explained by the fact that the wave already embraces almost the whole rectangle, and the process of equalizing temperature stabilization is calculated better than the process of propagation.

TABLE 4.

Score	a)	b)	c)
По $x$	5	10	15
По $y$	2	3	4

To exclude the possibility of "underiteration" we calculated part of variant (a) with three different values of  $\epsilon$  (see Section 2, Para. 2).

The corresponding values of  $\delta^j$  are given in

Table 3.\* The maximum number of iterations  $v_{max}$  are given in Table 4.\*\*

The comment made at the end of Para. 5, Section 2, remains valid.

\* When  $\epsilon = 10$  only one iteration is computed in each step.

\*\* For all  $t$  for which there are columns the values of  $v$  are equal to 1 for those rows (columns) which the wave has not yet reached, and to  $v_{max}$  for most of the other rows (columns).

#### 4. Comments

Because, in the computation for  $p = 2$ , we used boundary values taken for  $t = t^{j+1/2}$ , the profiles  $v(t^{j+1}, x, y_k)$  agree with the boundary conditions for  $x = 0$  less well than the profiles  $v(t^{j+1}, x_i, y)$  agree with the boundary conditions for  $y = 0$ .

At first sight it appears that all the values of  $v(t^{j+1}, x, y)$  on the line  $x + 2y = \text{const.}$  must be included between the corresponding values of  $u(t^{j+1/2}, x, y)$  and  $u(t^j, x, y)$ . However, in the second step of the computing process w.r.t.  $y$ , certain values, mainly in the neighbourhood of the front, can become smaller than the values of  $u(t^{j+1/2}, x, y)$ . In principle, this phenomenon can lead to the appearance of non-monotonic behaviour in the computation. In our example, even when the step is large, the non-monotonic effects are very small, and in the variant (a) they do not exceed two to three thousandths.

Finally we note that in the anisotropic problem changing the order in which the directions are computed – first w.r.t.  $y$  and then w.r.t.  $x$  in general affects the accuracy of the result.

### 4. The three-dimensional problem

#### 1. The computing scheme

The program was written for the solution of equation (1.1) with  $p = 3$  in the cube  $0 \leq x \leq l_1$ ,  $0 \leq y \leq l_2$ ,  $0 \leq z \leq l_3$  (we shall write  $x, y, z$  instead of  $x_1, x_2, x_3$ ) on the surface of which the values of the unknown function are given.

The net is uniform:  $x_i = ih_1$ ,  $0 \leq i \leq N_1$ ;  $y_k = kh_2$ ,  $0 \leq k \leq N_2$ ;  $z_m = mh_3$ ,  $0 \leq m \leq N_3$ ;  $t^j = j\tau$ .

At the initial time  $t_0 = 0$  we have the initial table of  $(v_{ikm}^0)$ . The transition from  $(v_{ikm}^j)$  to  $(v_{ikm}^{j+1})$  is done in three stages:

1) first the sub-routine OP transforms the "rows"  $0 \leq i \leq N_1$  of the table  $(v_{ikm}^j)$  into the "rows"  $(v_{ikm}^{j+1/2})$ , using the boundary values  $u(t^{j+1/2}, 0, y_k, z_m)$  and  $u(t^{j+1/2}, l_1, y_k, z_m)$ ; in this stage  $N = N_1$  and

$$A_i = \tau h_1^{-2} K_1 [0.5 (v_{i-1} + v_i)];$$



2) the "columns"  $0 \leq k \leq N_2$  of the table  $(v_{ikm}^{j+1/2})$  are transformed by the OP sub-routine into the "columns"  $(v_{ikm}^{j+2/2})$  with the help of the boundary values  $u(t^{j+1/2}, x_i, 0, z_m)$  and  $u(t^{j+1/2}, x_i, l_2, z_m)$ ; in this stage  $N = N_2$  and

$$A_k = \tau h_2^{-2} K_2 [0.5 (v_{k-1} + v_k)];$$

3) the "heights"  $0 \leq m \leq N_3$  of the table  $(v_{ikm}^{j+2/2})$  are transformed into the "heights"  $(v_{ikm}^{j+1})$  using the values of  $u(t^{j+1}, x_i, y_k, 0)$  and  $u(t^{j+1}, x_i, y_k, l_3)$  on the assumption that the number of nodes in the OP is  $N = N_3$  and

$$A_m = \tau h_3^{-2} K_3 [0.5 (v_{m-1} + v_m)].$$

In all, in each step, the sub-routine OP operates  $(N_2 - 1)(N_3 - 1) + (N_1 - 1)(N_3 - 1) + (N_1 - 1)(N_2 - 1)$  times.

## 2. Example 6. An isotropic plane wave in space

Let us consider the three-dimensional equation (1.1) with parameters

$$\sigma_1 = \sigma_2 = \sigma_3 = 2, \quad \kappa_1 = \kappa_2 = \kappa_3 = 0.06$$

and for the calculation we use the solution (3.1) with given constants:  $C_1 = 0$ ,  $\lambda_1 = \lambda_2 = \lambda_3 = 1$ . The corresponding solution is

$$u = \begin{cases} (10/3) \sqrt{t - x - y - z}, & \text{for } t \geq x + y + z, \\ 0 & \text{for } t \leq x + y + z. \end{cases} \quad (4.1)$$

The net is coarse:  $h_1 = h_2 = h_3 = 1$ ; the number of nodes  $N_1 \times N_2 \times N_3 = 10^3$ . From the solution (4.1) we take our initial values, i.e.  $u(0, x, y, z) \equiv 0$ , and the boundary conditions on the planes  $x = 0$ ,  $x = 10$ ,  $y = 0$ ,  $y = 10$ ,  $z = 0$  and  $z = 10$ .

The calculation was carried out from  $t_0 = 0$  to  $t = 9.0$ :\* (a) with step  $\tau = 0.2$  ( $\chi = 1.2$ ) and (b) with step  $\tau = 1.0$  ( $\chi = 6.0$ ).

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\* By this time the wave has not reached the plane  $x + y + z = 10$ , and so we can use zero boundary conditions on the boundaries  $x = 10$ ,  $y = 10$  and  $z = 10$ .

We judge the results to be good: at a distance of two computing intervals from the front the computed values are already near the exact ones. As an example, in Table 5 we give the values of  $v_{i21}$  (i.e. the values of the function at nodes lying on the straight line  $y = 2, z = 1$ ) at time  $t = 9$ .

TABLE 5.

$i$		1	2	3	4	5	6	7
$u$		7.4536	6.6667	5.7735	4.7140	3.3333	0	0
$v_{i21}$	a)	7.4541	6.6673	5.7740	4.7154	3.3587	0.735	0.001
	b)	7.4547	6.6710	5.7832	4.7352	3.3901	1.119	0.002

To show how well the velocity of propagation of the wave is given, in Table 6 we show the values of the function  $v$  at all the internal nodes lying in the plane  $x + y + z = 5$  for  $t = 8$  and in the plane  $x + y + z = 6$  for  $t = 9$  (variant (a)).

TABLE 6.

$t = 8$							$t = 9$			

The maximum number of iterations  $v_{\max}$  is equal to 3 or 4 in variant (a) and to 4 to 6 in variant (b)).

### 3. Comparison with the calculation using an explicit scheme

We calculated the same problem using "Euler's scheme":

$$\begin{aligned} v_{ikm}^{j+1} = & v_{ikm} + A_{i+1km}^{(1)} (v_{i+1km} - v_{ikm}) - A_{ikm}^{(1)} (v_{ikm} - v_{i-1km}) + \\ & + A_{ik+1m}^{(2)} (v_{ik+1m} - v_{ikm}) - A_{ikm}^{(2)} (v_{ikm} - v_{ik-1m}) + \\ & + A_{ikm+1}^{(3)} (v_{ikm+1} - v_{ikm}) - A_{ikm}^{(3)} (v_{ikm} - v_{ikm-1}), \end{aligned}$$

where

$$\begin{aligned} A_{ikm}^{(1)} &= \tau h_1^{-2} K_1 [0.5 (v_{i-1km} + v_{ikm})], \\ A_{ikm}^{(2)} &= \tau h_2^{-2} K_2 [0.5 (v_{ik-1m} + v_{ikm})], \\ A_{ikm}^{(3)} &= \tau h_3^{-2} K_3 [0.5 (v_{ikm-1} + v_{ikm})]. \end{aligned}$$

All the quantities on the right hand side are calculated in the  $j$ -th step. The net and the boundary conditions are the same as in Para. 2. To obtain roughly the same accuracy as in the method of variable directions we had to reduce the step length 20 times ( $\tau = 0.01$  and  $\tau = 0.05$ ). The computing time with the explicit scheme was almost four times longer.\*

## 5. The difference travelling wave

### 1. Definition

Let us construct the solution of equation (2)3) satisfying the condition

$$v_i^j = v_{i-\beta}^{j-1}, \quad (5.1)$$

where  $\beta \geq 1$  is an integer. Obviously, such a solution preserves its

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\* We know that as  $\chi$  increases the explicit scheme loses its stability. The need to carry out the calculation with a small step can be avoided, but at much greater cost (more than four times), when the heat conduction equation is part of a system of differential equations.

profile from step to step (w.r.t.  $j$ ), shifting to the right by  $\beta$  computing intervals. Thus the velocity of motion of all the points of the profile is constant and equal to  $c = \beta h / \tau$ .

Inserting (5.1) in (2.3) we obtain the equation

$$\varphi(v_i) - \varphi(v_{i+\beta}) = A_{i+1}(v_{i+1} - v_i) - A_i(v_i - v_{i-1}),$$

which contains all the quantities in one step. This equation has a "first integral":

$$A_{i+1}(v_{i+1} - v_i) + \sum_{k=1}^{\beta} \varphi(v_{i+k}) = C_1. \quad (5.2)$$

All solutions of equation (5.2) are called *difference travelling waves*.

These solutions can be very useful in the study of various interpolation methods (i.e. various methods of calculating  $A_i$ ).

We shall put  $\varphi(v) \equiv v$  below.

## 2. A difference travelling wave propagated along a null background

The solution of equation (5.2) which is a wave moving over a null background (Fig. 9) can be found as follows. Let  $\xi_0 = x_{i_0+1}$  be the position of the difference front. We shall take

$$v_{i_0+1} = v_{i_0+2} = \dots = v_{i_0+\beta} = 0,$$

and  $v_{i_0} \neq 0$ . Let us find this value with the help of formula (2.2) on the assumption that the equation

$$-\frac{K(u)}{u} \frac{\partial u}{\partial x} = \frac{\beta h}{\tau}$$

is valid at the fictitious point  $x_{i_0} + \frac{1}{2}$ :

$$-\frac{(h/\tau) A_{i_0+1}(v_{i_0+1} - v_{i_0})}{0.5(v_{i_0+1} + v_{i_0})} = \frac{\beta h}{\tau}.$$

This gives us the condition for finding  $v_{i_0}$ :

$$A_{i_0+1} = 0.5 \beta \quad (5.3)$$

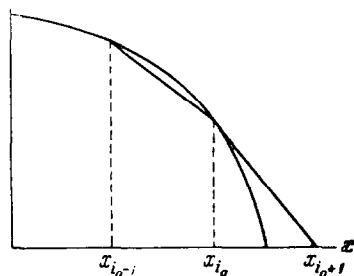


FIG. 9.

We shall find all the values of  $v_i$  for  $i < i_0$  from the equation

$$A_{i+1}(v_{i+1} - v_i) + \sum_{k=1}^{\beta} v_{i+k} = C_1, \quad (5.4)$$

where, clearly,  $C_1 = -0.5 \beta v_{i_0}$ .

When  $K(u) = \kappa_0 u^\sigma$  it follows from (5.3) that

$$v_{i_0} = 2 (hc/2\kappa_0)^{1/\sigma},$$

where  $c = \beta h/\tau$  is the velocity. It is not difficult to verify that this value differs from the corresponding value on the analytic travelling wave (see (2.7)) by the factor  $0.5 (2\sigma)^{1/\sigma}$ . In particular, when  $\sigma = 1$  (Boussinesque's equation) and also when  $\sigma = 2$ , the profile of the wave we have constructed, (5.3)-(5.4), is very near that of the wave (2.7).

### 3. The difference front in the case of a through computation scheme

In the through computation of a wave propagated over a null background we have to deal with a solution which, strictly speaking, satisfies the condition at infinity

$$v_i \rightarrow 0, \quad \text{when } i \rightarrow \infty.$$

instead of the solution (5.3)-(5.4). The appearance of the front is associated with the appearance of the "machine zero" or, in other words, is determined by the allowable accuracy of the computations. This means that in order to study the through computation scheme (Para. 2, Section 2) we must consider the solutions of the equation

$$A_{i+1}(v_{i+1} - v_i) + \sum_{k=1}^{\beta} v_{i+k} = 0. \quad (5.5)$$

It is easy to prove that all the positive solutions of equation (5.5) are monotonic, tend to zero as  $i \rightarrow \infty$  and tend to  $\infty$  as  $i \rightarrow -\infty$ .

Let us introduce the new variables

$$w_i = \chi_0^{1/\sigma} v_i,$$

where  $\chi_0 = \kappa_0 \tau h^{-2}$ . In the new variables we obtain instead of (5.5) the

equation

$$[0.5 (w_{i+1} + w_i)]^\sigma (w_{i+1} - w_i) + \sum_{k=1}^{\beta} w_{i+k} = 0. \quad (5.6)$$

Therefore, the profiles of the required solutions depend mainly on the non-linearity ( $\sigma$ ) and on the velocity ( $\beta$ ) and not on the net or  $\kappa_0$ .

Let  $\eta$  denote the smallest admissible number in the computation (in other words, any number less than  $\eta$  is considered to be zero). If  $w_{i+1} = \eta$  then it follows from (5.6) that

$$w_i = (2^\sigma \eta)^{1/(\sigma+1)} - \frac{\sigma-1}{\sigma+1} \eta + o(\eta).$$

Therefore, if we denote the number of the last non-zero number  $w_i$  by  $i_0$  we can say that the only values which are possible in the computation are\*

$$\eta < w_{i_0} < (2^\sigma \eta)^{1/(\sigma+1)} - \frac{\sigma-1}{\sigma+1} \eta.$$

In fact, if  $w_{i_0}$  is less than  $\eta$  the last number of a non-zero  $w_i$  will be less than  $i_0$  and if  $w_{i_0}$  is greater than the right-hand side we have  $w_{i_0+1} \geq \eta$  and the last number is greater than  $i_0$ .

In Table 7 we give the values of  $w_i$  corresponding to the case  $\sigma = 2$ ,  $\eta = 2.5 \times 10^{-10}$  for  $\beta = 1$  and  $\beta = 2$ ;  $i_0 = 10$ .

TABLE 7.

$i$	0	1	2	3	4	5	6	7	8	9	10
$\beta=1$	3.359	3.062	2.736	2.372	1.955	1.454	0.820	0.158	$1.0 \cdot 10^{-3}$	$2.5 \cdot 10^{-10}$	0
$\beta=2$	4.253	3.814	3.333	2.801	2.205	1.530	0.822	0.158	$1.0 \cdot 10^{-3}$	$2.5 \cdot 10^{-10}$	0

#### 4. Numerical example

The problem considered in Example 1 (Section 2) was calculated with

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\* For simplicity we give the argument for  $w_i$  and not for  $v_i$ .

a very coarse net:  $N = 50$ ,  $h = 0.25$ , from  $t_0 = 0.10$  to  $t = 2.00$  with step  $\tau = 0.05$  ( $\chi = 80$ ) and with step  $\tau = 0.10$  ( $\chi = 160$ ). These parameters satisfy the condition  $c\tau/h = \beta$  for  $\beta = 1$  and  $\beta = 2$  so that, with a special choice of the boundary conditions, the solution could turn out to be a difference travelling wave.

In our example, after a few steps the computed profile becomes very close to the difference travelling wave. Subsequently this profile is displaced with a velocity near  $c$ , being somewhat rearranged so that

$|v_i^j - v_{i-\beta}^{j-1}| \rightarrow 0$  as  $j$  increases. In Fig. 10 we give, on a large scale, the analytic solution (2.7) in the neighbourhood of the front at time  $t = 2.00$  (the continuous curve) and the results obtained in the computation (the dots for  $\tau = 0.05$ , and the crosses for  $\tau = 0.10$ ); the broken curves are the profiles of the difference travelling waves calculated using the table at the end of Para. 3, from the formula  $v_i = w_i/\sqrt{\chi_0}$ .

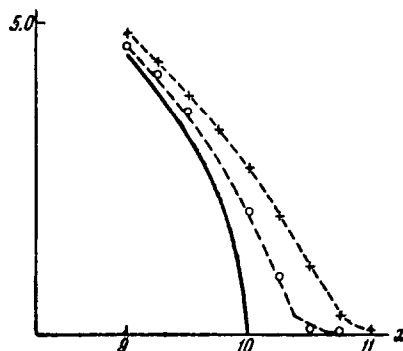


FIG. 10.

Since the front is "blurred" for a very coarse net, the error in the neighbourhood of the front is quite considerable. As the distance from the front increases the error rapidly decreases.

For the first of three variants the number of iterations at each step  $\nu = 9$ , and for the second  $\nu = 12$  (cf. the end of Para. 5, Section 2).

## 5. A difference stationary solution

The stationary solution of equation (2.3) given by the condition

$$v_i^j = v_i^{j-1}, \quad (5.7)$$

can be considered as a special case of a difference travelling wave for  $\beta = 0$ , when the velocity of the profile is equal to zero. The difference stationary solutions satisfy the equation

$$A_{i+1}(v_{i+1} - v_i) = C_1, \quad (5.8)$$

which is completely analogous to (5.2), where  $C_1$  is an arbitrary constant.

In Example 4 we found a stationary solution satisfying the boundary conditions  $v_0$  given,  $v_N = 0$ . It is not difficult to verify that this solution does not depend on the time step  $\tau$  (this is a property of homogeneous difference schemes).

In fact, since, in this example,  $\sigma = 2$  we can rewrite equation (5.8) (using (2.4)) in the form

$$(v_{i+1} + v_i)^2 (v_{i+1} - v_i) = -C^3,$$

where  $C = -\sqrt[3]{4C_1 h^2 / \kappa_0 \tau}$  is again an arbitrary constant. Let us make the substitution  $v_i = Cw_i$ . To find  $w_i$  we have the equation

$$(w_{i+1} + w_i)^2 (w_i - w_{i+1}) = 1.$$

Knowing  $w_N = 0$  we can find all the  $w_i$ ,  $i = N - 1, N - 2, \dots, 0$ . We have then to choose the value of the arbitrary constant  $\bar{C} = v_0/w_0$  and find the required values  $v_i = \bar{C}w_i$ . It is clear that this solution depends only on  $N$ , i.e. on the size of the space net.

In the calculation of example 4(b) we obtained values which coincide with the stationary solution calculated in this way to an accuracy of 0.0000001.

TABLE 8.

$x_i$		0.2	0.4	0.6	0.8	0.9	0.98
$u(x_i)$		9.283	8.434	7.368	5.848	4.642	2.714
$v_i$	$N=50$	9.289	8.447	7.393	5.900	4.733	2.981
	$N=10$	9.308	8.495	7.486	6.091	5.052	—
	$N=5$	9.332	8.551	7.592	6.298	—	—



In Table 8 we give the values of the analytic and difference stationary solutions for this problem for  $N = 50$ ,  $N = 10$ ,  $N = 5$ .

Translated by R. Feinstein

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