ON THE CONVERGENCE OF THE FRACTIONAL STEP METHOD FOR HEAT CONDUCTIVITY EQUATIONS*

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1. The splitting method (fractional step method) is suggested in [1] for the numerical solution of the multi-dimensional heat conductivity equation with constant coefficients. In [2] the difference scheme for the heat conductivity equation with constant coefficients is written down for the case of two space variables and a rectangular region, and a proof of the convergence (in the mean) of this scheme with speed $O(\tau^2)$ with the additional condition τ/h^2 = const. is given. In this note we shall show that the modified fractional step method can also be used in the case of an arbitrary region and a non-uniform difference net. We prove the convergence in the mean with speed $O(h^2) + O(\tau)$ of the local one-dimensional schemes we examine; in contrast to [1], we shall not connect these schemes with the splitting of the multi-dimensional difference equation and this makes them suitable for an arbitrary region and quasi-linear parabolic equations. In contrast to [3], at each stage we use six-point homogeneous schemes which enables us, in particular, to take the scheme of [2] into account. As usual we give the argument for a family of homogeneous schemes characterised by the given pattern functionals of [3]-[5]. We use the methods developed in [3], [5] and [6]. Convergence is proved for arbitrary non-uniform nets.

2. Let G be an arbitrary two-dimensional region bounded by the contour Γ, $\overline{G} = G + \Gamma$, $x = (x_1, x_2)$ a point with the coordinates x_1 and x_2. In the cylinder $\overline{Q}_T = \overline{G} \times [0 \leqslant t \leqslant T]$ we look for the solution of the problem

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{2} L_\alpha u, \quad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t)\frac{\partial u}{\partial x_\alpha}\right), \quad (x, t) \in Q_T = Gx\,(0 < t \leqslant T), \quad (1)$$

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$$u|_\Gamma = u_1(x, t), \quad t \in [0, T]; \qquad u(x, 0) = u_0(x), \quad x \in \overline{G}, \tag{2}$$

$$k_\alpha(x, t) \geqslant c_1 > 0 \quad (c_1 = \text{const}), \quad \alpha = 1, 2. \tag{3}$$

We shall assume: (1) the problem (1)-(3) has a unique solution $u = u(x, t)$, continuous in $\overline{Q}_T$; (2) the following conditions are satisfied.

Conditions A. In the region $\overline{Q}_T$ the functions $\partial^3 u/\partial x_\alpha^3$, $\partial^2 k_\alpha/\partial x_\alpha^2$ satisfy the Lipschitz conditions for x_α, and x_α, a $\partial u/\partial x_\alpha$, $\partial^2 u/\partial x_\alpha^2$, $\partial u/\partial t$, k_α, $\partial k_\alpha/\partial x_\alpha$ – the Lipschitz conditions for t ($\alpha = 1, 2$).

Below we use only one property of the region G explicitly: the intersection with the region $\overline{G}$ of the straight line $\mathscr{L}_\alpha$, drawn through any point $x \in G$ parallel to the coordinate axis Ox_α consists of a finite number of intervals (see [3]). For simplicity we give our argument assuming that any straight line $\mathscr{L}_\alpha$ intersects the contour Γ twice.

3. Consider the arbitrary non-uniform rectangular net $\overline{\omega}_h \{x_i \in \overline{G}\}$, where $x_i = (x_1^{(i_1)}, x_1^{(i_2)})$ is the node of the net with coordinates $x_1^{(i_1)}$ and $x_2^{(i_2)}$, $i_\alpha = 0, \pm 1, \pm 2, \ldots$, $\alpha = 1, 2$. The steps of the net $h_1^{(i_1)} = x_1^{(i_1)} - x_1^{(i_1-1)}$ and $h_2^{(i_2)} = x_2^{(i_2)} - x_2^{(i_2-1)}$ are functions of the coordinates $x_1^{(i_1)}$ and $x_2^{(i_2)}$ respectively. Let $\omega_h = \{x_i \in G\}$ be the internal net region, $\gamma = \{x_i \in \Gamma\}$ the boundary net region. Through the point $x_i \in \omega_h$ we draw the straight line $\mathscr{L}_\alpha$, parallel to the coordinate axis Ox_α; the set of all nodes of the net $\overline{\omega}_h$ lying on $\mathscr{L}_\alpha$, is called the chain $Ц_\alpha$. Let $x_л = x_л^{(\alpha)} \in \gamma$ and $x_п = x_п^{(\alpha)} \in \gamma$ be boundary points of the chain $Ц_\alpha$, where x_α increases on going from x_l to x_r, and $\gamma_{\alpha, l}$ is the set of the nodes x_l of all chains $Ц_\alpha$ in a given direction, $\gamma_{\alpha, r}$ is the set of nodes x_r. We put $\gamma_\alpha = \gamma_{\alpha, л} + \gamma_{\alpha, п}$, $\omega_h^{(+\alpha)} = \omega_h + \gamma_{\alpha, п}$, $\omega_h^{(-\alpha)} = \omega_h + \gamma_{\alpha, л}$. The segment $0 \leqslant t \leqslant T$ is divided into K equal parts of length τ by the points $t_0 = 0$, $\tau, \ldots, t_j = j\tau, \ldots, t_K = K\tau = T$, $\tau = T/K$ (uniform net with step τ) and we introduce the intermediate (fractional) steps $t_{j+1/2} = t_j + \tau/2 = (j + \frac{1}{2})\tau$, $j = 0, 1, \ldots, K - 1$. The net $\overline{\omega}_\tau$ with respect to time contains both integral and fractional steps, i.e. $\overline{\omega}_\tau = \{t_{j^*} \in [0 \leqslant t \leqslant T]\}$, where $t_{j^*} = j^*\tau$, $j^* = 0, \frac{1}{2}, 1, \ldots, j, j + \frac{1}{2}, \ldots, K - \frac{1}{2}, K$. The point $(x_i, t_{j^*}) \in \overline{Q}_T$ is a node of the space-time net $\overline{\Omega}$ and $\Omega = \{(x_i, t_{j^*}) \in Q_T\}$ is the set of internal nodes of the net $\overline{\Omega}$; it is clear that $\overline{\Omega} = \overline{\omega}_h \times \overline{\omega}_\tau$, $\Omega = \omega_h \times \omega_\tau$, where

$\omega_\tau = \{t_{j*} \in (0 < t \leqslant T]\}$.

We shall use the notation given in [3]. Let $y = y(x, t_{j*}) = y^{j*}$ be a net function given on $\bar{\Omega}$. We shall write $y^{(\pm m_\alpha)} = y(x^{(\pm m_\alpha)}, t)$, $x^{(\pm m_1)} = (x_1^{(i_1 \pm m_1)}, x_2^{(i_2)})$, $x_1^{(i_1 - 1/2)} = \frac{1}{2}(x_1^{(i_1)} + x_1^{(i_1 - 1)})$, $y_{\bar{x}_\alpha} = (y - y^{(-1_\alpha)})/h_\alpha$, $y_{x_\alpha} = (y^{(+1_\alpha)} - y)/h_{\alpha+}$, $y_{\hat{x}_\alpha} = (y^{(+1_\alpha)} - y)/\hbar_\alpha$, $h_\alpha = h_\alpha^{(i_\alpha)}$, $h_{\alpha+} = h_\alpha^{(i_\alpha+1)}$, $\hbar_\alpha = (h_\alpha + h_{\alpha+})/2$, $y_{\bar{t}_\alpha} = (y^{j+\alpha/2} - y^{j+(\alpha-1)/2})/\tau$.

We need the following sums and norms:

$$(y, v)^* = \sum_{\omega_h} y_i v_i \hbar_1^{(i_1)} \hbar_2^{(i_2)}, \qquad (y, v) = \sum_{\omega_h} y_i v_i h_1^{(i_1)} h_2^{(i_2)},$$

$$(y, v]_\alpha = \sum_{\omega_h^{(+\alpha)}} y_i v_i h_\alpha^{(i_\alpha)} \hbar_{\alpha\pm1}^{(i_\alpha\pm1)} \quad (+ \text{ for } \alpha = 1, \; - \text{ for } \alpha = 2),$$

$$\|y\| = \sqrt{(y, y)^*}, \qquad \|y\|_2 = \sqrt{(y, y)}, \qquad \|y\|_{2_\alpha} = \sqrt{(1, y^2]_\alpha},$$

$$\|y\|_{3_\alpha} = \|\eta_\alpha\|_{2_\alpha}, \text{ where } (\eta_\alpha)_{\hat{x}_\alpha} = y,$$

$$\eta_\alpha = 0 \text{ for } x \in \gamma_{\alpha, \text{п}}.$$

4. The operator $L_\alpha u$ is replaced by the homogeneous uniform net scheme

$$\Lambda_\alpha y = (a_\alpha y_{\bar{x}_\alpha})_{\hat{x}},$$

having second order approximation on a uniform net (see [4] and [5]) so that $\Lambda_\alpha u - L_\alpha u = O(h_\alpha^2)$.

The class of pattern functionals a_α is described in [4]; in particular, for $a_\alpha = k_\alpha^{(-1/2)}$ we obtain the scheme used in [2]. We put the problem (1)-(3) in correspondence with the local one-dimensional scheme:

$$y_{\bar{t}_\alpha} = \Lambda_\alpha y^{(\sigma)} \text{ for } (x, t_{j+\alpha/2}) \in \Omega, \alpha = 1, 2; \; j = 0, 1, 2, \ldots, K-1; \tag{4}$$

$$y(x, t_{j+\alpha/2}) = u_1(x, t_{j+\alpha/2}) \text{ for } x \in \gamma_\alpha; \qquad y(x, 0) = u_0(x) \text{ for } x \in \bar{\omega}_h; \tag{5}$$

$$(a_\alpha(x, t) \geqslant c_1 > 0, \; (x, t) \in \omega_h^{(+\alpha)} \times [0 \leqslant t \leqslant T], \tag{6}$$

where $y^{(\sigma)} = \sigma y + (1-\sigma)\hat{y}$, $0 \leqslant \sigma \leqslant 1$, $y = y(x, t_{j+\alpha/2})$, $\hat{y} = y(x, t_{j+(\alpha-1)/2})$, $y_{\bar{t}_\alpha} = (y - \hat{y})/\tau$, $\Lambda_\alpha y = (a_\alpha(x, t^*) y_{\bar{x}_\alpha})_{\hat{x}_\alpha}$, $t^* \in [t_j, t_{j+1}]$ and for definiteness we shall take $t^* = t_{j+1/2}$. It is clear from (4)-(5) that at each moment $t_{j+\alpha/2}$ we solve the first boundary problem for all chains Π_α in the given direction x_α.

5. Let us now go on to study the convergence and accuracy of the scheme (4)-(6). Let u be the solution of problem (1)-(3), y the solution of problem (4)-(6). For $z = y - u$ we obtain the conditions

$$z_{\bar{t}_\alpha} = \Lambda_\alpha z^{(\sigma)} + \psi_\alpha, \qquad \psi_\alpha = \Lambda_\alpha u^{(\sigma)} - u_{\bar{t}_\alpha}, \tag{7}$$

$$z(x, t_{j+\alpha/2}) = 0, \quad x \in \gamma_\alpha; \qquad z(x, 0) = 0, \quad x \in \bar{\omega}_h, \tag{8}$$

$$a_\alpha(x, t_{j+\alpha/2}) \geqslant c_1 > 0. \tag{9}$$

Suppose conditions A are satisfied. By analogy with (7) we find

$$\Lambda_\alpha u - L_\alpha u = (\mu_\alpha)_{\hat{x}_\alpha} + \psi_{\alpha,a}, \qquad \|\mu_\alpha\|_{2_\alpha} = O(\|h_\alpha^2\|_{2_\alpha}), \qquad \|\psi_{\alpha,a}\| = O(\|h_\alpha^2\|_{2_\alpha}), \tag{10}$$

where $\|h_\alpha^2\|_{2_\alpha} = \sqrt{(1, h_\alpha^4]_\alpha}$ is the mean square value of h_α^2.

We put ψ_α in the form

$$\psi_\alpha = \psi_\alpha^0 + \psi_\alpha^*, \qquad \psi_\alpha^0 = \left(L_\alpha u - \frac{1}{2}\frac{\partial u}{\partial t}\right)^{j+1}, \qquad \sum_{\alpha=1}^{2} \psi_\alpha^0 = 0, \tag{11}$$

$$\psi_\alpha^* = (\mu_\alpha)_{\hat{x}_\alpha} + \varphi_\alpha, \qquad \|\varphi_\alpha\|_2 = O(\|h_\alpha\|_2^2) + O(\tau). \tag{12}$$

6. With the estimate of the solution of the problem (7)-(9) we use the method of integral inequalities and a special method of summing the local errors ψ_α using the relation $\psi_1^0 + \psi_2^0 = 0$. Let us write down the basic integral identity with $\sigma = ½$. Let us multiply (7) by $(z + \check{z})\hbar_1\hbar_2$ and sum with respect to ω_h. Using Green's formula

$$(z + \check{z}, \Lambda_\alpha z^{(1/2)})^* = \frac{1}{2}\|\sqrt{a_\alpha}(z_{\bar{x}_\alpha} + \check{z}_{\bar{x}_\alpha})\|_{2_\alpha}^2 = I_\alpha, \tag{13}$$

we obtain the following identities:

$$(\|z\|^2)_{\bar{t}_\alpha} + I_\alpha = (\psi_\alpha, z + \check{z})^*, \tag{14}$$

$$(\|z\|^2)_{\bar{t}} + I_1 + I_2 = \sum_{\alpha=1}^{2} (\psi_\alpha, z + \check{z})^*, \tag{15}$$

where

$$(\|z\|^2)_{\bar{t}} = (\|z^{j+1}\|^2 - \|z^j\|^2)/\tau.$$

Putting $\psi_\alpha = 0$ in (15) we see that the scheme (4)-(6) is stable with respect to the initial data (for any h_α and τ):

$$\|z(x, t_{j^*})\| \leqslant \|z(x, 0)\| \qquad (t_{j^*} = j^*\tau, \ j^* = \frac{1}{2}, 1, \frac{3}{2}, \ldots, K - \frac{1}{2}, K). \tag{16}$$

7. *Lemma 1.* If ψ_α can be put in the form

$$\psi_\alpha = \psi_\alpha^0 + (\mu_\alpha)_{\hat{x}_\alpha} + \varphi_\alpha, \qquad \sum_{\alpha=1}^{2} \psi_\alpha^0 = 0, \tag{17}$$

then for the solution of problem (7)-(9) for $\sigma = ½$ and any h_α and τ we have the estimate

$$\| z(x, t_{j+1}) \| \leqslant \tau \| \psi^0(x, t_{j+1}) \| + \tau \sqrt{e} \overline{\| \psi^0(x, t_j) \|} + \sqrt{1+e} \sum_{j'=1}^{j+1} \tau Q^{j'}, \tag{18}$$

where

$$\psi^0 = \psi_2^0 = -\psi_1^0, \qquad \overline{\| \psi^0(x, t_j) \|} = \left(\frac{1}{t_j} \sum_{j'=1}^{j} \tau \| \psi^0(x, t_{j'}) \|^2 \right)^{1/2},$$

$$Q^{j'} = \tau^2 t_j \| \psi_{\bar{t}}^0(x, t_{j'}) \|^2 + \sum_{\alpha=1}^{2} (\| \mu_\alpha(x, t_{j'}) \|_{2_\alpha}^2 + \| \varphi_\alpha(x, t_{j'}) \|_{3_\alpha}^2). \tag{18'}$$

We put (17) in (15) and make the following transformations:

$$\sum_{\alpha=1}^{2} (\psi_\alpha^0, z + \check{z})^* + (\psi^0, z^{j+1} - z^j) = \tau (\psi^0(x, t_{j+1}), z(x, t_{j+1}))_{\bar{t}}^* - (\psi_{\bar{t}}^*(x, t_{j+1}), z(x, t_j))^* \tau,$$

$$|((\mu_\alpha)_{\hat{x}_\alpha}, z + \check{z})^*| = |(\mu_\alpha, z_{\bar{x}_\alpha} + \check{z}_{\bar{x}_\alpha}]_\alpha| \leqslant \frac{1}{2} I_\alpha + \frac{1}{c_1} \| \mu_\alpha \|_{2_\alpha}^2, \quad |(\varphi_\alpha, z + \check{z})^*| \leqslant \frac{1}{2} I_\alpha + \frac{1}{c_1} \| \varphi_\alpha \|_{3_\alpha}^2, \quad \tau |(\psi_{\bar{t}}^0, z^j)^*| \leqslant \frac{\tau^2}{2c_0} \| \psi_{\bar{t}}^0 \|^2 + \frac{c_0}{2} \| z^j \|^2$$

where c_0 is an arbitrary positive constant which we shall put equal to $1/t_j$ later. As a result we obtain the inequality

$$\| z(x, t_{j+1}) \|^2 \leqslant (1 + 0.5 c_0 \tau) \| z(x, t_j) \|^2 + 0.5 \tau Q^{j+1} + \tau^2 (\psi^0(x, t_{j+1}), z(x, t_{j+1}))_{\bar{t}}^*.$$

Using the inequality $\tau |(\psi^0, z)^*| \leqslant 0.5 \| z \|^2 + 0.5 \tau^2 + 0.5 \tau^2 \| \psi^0 \|^2$, and also the initial condition $z(x, 0) = 0$ we find

$$\| z(x, t_{j+1}) \|^2 \leqslant c_0 \tau \sum_{j'=1}^{j} \| z(x, t_{j'}) \|^2 + \tau^2 \| \psi^0(x, t_{j+1}) \|^2 + \sum_{j'=1}^{j} \tau Q^{j'}. \tag{19}$$

Let us now use the following simple lemma, the proof of which we omit:

Lemma 2. If $\rho(t_j)$ and $\sigma(t_j)$ are non-negative functions ($t_j = j\tau$, $j = 1, 2, \ldots$) then the inequality

$$\rho(t_{j+1}) \leqslant c_0\tau \sum_{j'=1}^{j} \rho(t_{j'}) + \sigma(t_{j+1}), \qquad j = 1, 2, \ldots, \quad (c_0 > 0)$$

gives

$$\rho(t_{j+1}) \leqslant \sigma(t_{j+1}) + c_0 e^{c_0 t_j} \sum_{j'=1}^{j} \tau\sigma(t_{j'}). \tag{20}$$

It follows from the inequality (19), from Lemma 2, that

$$\|z(x, t_{j+1})\|^2 \leqslant \tau^2 \|\psi^0(x, t_{j+1})\|^2 + \tau^2 e \overline{\|\psi^0(x, t_j)\|}^2 + (1 + e) \sum_{j'=1}^{j+1} \tau Q^{j'},$$

$$\overline{\|\varphi^0(x, t_j)\|}^2 = \frac{1}{t_j} \sum_{j'=1}^{j} \tau \|\varphi^0(x, t_{j'})\|^2.$$

This proves Lemma 1.

8. *Theorem 1.* If conditions A are satisfied, then the difference scheme (5)-(7) converges in the mean with speed $O(\|h^2\|_2) + O(\tau)$ for $\sigma = 0.5$ and as h_α, τ tend independently to zero on any non-uniform space net:

$$\|y(x, t_{j+\alpha/2}) - u(x, t_{j+\alpha/2})\| \leqslant M(\|h^2\|_2 + \tau), \qquad \alpha = 1, 2, \quad j = 0, 1, \ldots, K-1, \tag{21}$$

$$\|h^2\|_2 = \|h_1^2\|_{2_1} + \|h_2^2\|_{2_2}, \quad \|h_\alpha^2\|_{2_\alpha} = \sqrt{(1, h_\alpha^4]_\alpha}, \qquad \alpha = 1, 2,$$

where M is a positive constant which does not depend on the choice of net.

For the estimate (21) for $\alpha = 2$ (on a whole step) follows from (18) since, from (10) and (12) we have $Q^{j'} = O(\|h^2\|_2^2) + O(\tau^2)$. To estimate $\|z(x, t_{j+1/2})\|$ on a fractional step we use (18) with $\alpha = 2$ and the identity (14) for $\alpha = 1$. Thus on fractional steps the scheme (5)-(7) has the same order of accuracy as on integral steps.

The convergence of the scheme (5)-(7) can be proved with considerably weaker requirements than conditions A.

Theorem 2. If the following conditions are satisfied then the scheme (4)-(6) converges in the mean for $\sigma = 0.5$ and for any h_α and τ so that

$$\|y(x, t_{j+\alpha/2}) - u(x, t_{j+\alpha/2})\| \leqslant M(\|h^\gamma\|_2 + \tau^\gamma). \tag{22}$$

The conditions are (1) $\partial k_\alpha/\partial x_\alpha$, $\partial^2 u/\partial x_\alpha^2$ satisfy in $\overline{Q}_T$ the Hölder conditions of order γ, $0 < \gamma \leqslant 1$, (2) $\partial u/\partial x_\alpha$, $\partial^2 u/\partial x_\alpha^2$, $\partial u/\partial t$, k_α, $\partial k_\alpha/\partial x_\alpha$ satisfy in $\overline{Q}_T$

the Hölder conditions of order $\gamma > 0$ with respect to t.

To prove Theorem 2 we have to use the fact that $\|\varphi_\alpha\|_{3_\alpha} = O(\|h_\alpha^\gamma\|_{2_\alpha}) + O(\tau^\gamma)$, $\|\mu_\alpha\|_{2_\alpha} = O(\|h_\alpha^{1+\gamma}\|_{2_\alpha})$ and make use of Lemma 1.

9. We have given all the preceding reasoning for the simplest equation of heat conductivity, (1). However, Theorems 1 and 2 remain true for equations of the general form

$$c(x, t)\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{2} L_\alpha u + f(x, t) - q(x, t)u; \qquad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x, t)\frac{\partial u}{\partial x_\alpha},$$

$$c(x, t)\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{2} L_\alpha u + f(x, t, u); \qquad L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x, t, u)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x, t, u)\frac{\partial u}{\partial x_\alpha}.$$

The corresponding expressions for $\Lambda_\alpha y$ are given in [3] and we shall not give them here. It is proved there that the local one-dimensional method is uniformly convergent for $\sigma = 1$. If $r_\alpha \not\equiv 0$, $q_\alpha \not\equiv 0$, then the estimate is true for $\tau < \tau_0$ where τ_0 depends on c_1 and $\max|r_\alpha|$, $\max|q_\alpha|$.

10. The method of summation of local errors ψ_α used here, which leads to the *a priori* estimate (18), cannot be extended to the case $p > 2$. In this case instead of (21) we have only been able to obtain the estimate

$$\|y - u\| \leqslant M(\sqrt{\tau} + h^2).$$

If we suppose that $\tau/h_\alpha^2 \leqslant [2(1-\sigma)\max a_\alpha]^{-1}$, then using the method of [3] we can prove the uniform convergence of the scheme (5)-(7) for $0 \leqslant \sigma \leqslant 1$ and for any $p > 1$ with speed $O(\bar{h}^2) + O(\tau)$, where $h^2 = \bar{h}_1^2 + \bar{h}_2^2$, $\bar{h}_\alpha$ is the maximum value of the step $h_\alpha^{(i_\alpha)}$ on the net $\omega_h^{(+1_\alpha)}$ For the case $\sigma = 1$ this estimate holds for any h_α and τ (Theorem 2 of [3]).

It must be stressed that we are using here boundary conditions without approximation (in contrast to [3]). This formulation of the difference boundary conditions always leads to different schemes on non-uniform nets. We have proved that in this case also second order accuracy is attained with respect to space.

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