

# ON THE CONVERGENCE OF THE FRACTIONAL STEP METHOD FOR HEAT CONDUCTIVITY EQUATIONS\*

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1. The splitting method (fractional step method) is suggested in [1] for the numerical solution of the multi-dimensional heat conductivity equation with constant coefficients. In [2] the difference scheme for the heat conductivity equation with constant coefficients is written down for the case of two space variables and a rectangular region, and a proof of the convergence (in the mean) of this scheme with speed  $O(\tau^2)$  with the additional condition  $\tau/h^2 = \text{const.}$  is given. In this note we shall show that the modified fractional step method can also be used in the case of an arbitrary region and a non-uniform difference net. We prove the convergence in the mean with speed  $O(h^2) + O(\tau)$  of the local one-dimensional schemes we examine; in contrast to [1], we shall not connect these schemes with the splitting of the multi-dimensional difference equation and this makes them suitable for an arbitrary region and quasi-linear parabolic equations. In contrast to [3], at each stage we use six-point homogeneous schemes which enables us, in particular, to take the scheme of [2] into account. As usual we give the argument for a family of homogeneous schemes characterised by the given pattern functionals of [3]-[5]. We use the methods developed in [3], [5] and [6]. Convergence is proved for arbitrary non-uniform nets.

2. Let  $G$  be an arbitrary two-dimensional region bounded by the contour  $\Gamma$ ,  $\bar{G} = G + \Gamma$ ,  $x = (x_1, x_2)$  a point with the coordinates  $x_1$  and  $x_2$ . In the cylinder  $\bar{Q}_T = \bar{G} \times [0 \leq t \leq T]$  we look for the solution of the problem

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^2 L_{\alpha} u, \quad L_{\alpha} u = \frac{\partial}{\partial x_{\alpha}} \left( k_{\alpha}(x, t) \frac{\partial u}{\partial x_{\alpha}} \right), \quad (x, t) \in Q_T = G \times (0 < t \leq T), \quad (1)$$

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$$u|_{\Gamma} = u_1(x, t), \quad t \in [0, T]; \quad u(x, 0) = u_0(x), \quad x \in \bar{G}, \quad (2)$$

$$k_{\alpha}(x, t) \geq c_1 > 0 \quad (c_1 = \text{const}), \quad \alpha = 1, 2. \quad (3)$$

We shall assume: (1) the problem (1)-(3) has a unique solution  $u = u(x, t)$ , continuous in  $\bar{Q}_T$ ; (2) the following conditions are satisfied.

*Conditions A.* In the region  $\bar{Q}_T$  the functions  $\partial^3 u / \partial x_{\alpha}^3$ ,  $\partial^2 k_{\alpha} / \partial x_{\alpha}^2$  satisfy the Lipschitz conditions for  $x_{\alpha}$ , and  $x_{\alpha}$ , a  $\partial u / \partial x_{\alpha}$ ,  $\partial^2 u / \partial x_{\alpha}^2$ ,  $\partial u / \partial t$ ,  $k_{\alpha}$ ,  $\partial k_{\alpha} / \partial x_{\alpha}$  - the Lipschitz conditions for  $t$  ( $\alpha = 1, 2$ ).

Below we use only one property of the region  $G$  explicitly: the intersection with the region  $\bar{G}$  of the straight line  $\mathcal{L}_{\alpha}$ , drawn through any point  $x \in G$  parallel to the coordinate axis  $Ox_{\alpha}$  consists of a finite number of intervals (see [3]). For simplicity we give our argument assuming that any straight line  $\mathcal{L}_{\alpha}$  intersects the contour  $\Gamma$  twice.

3. Consider the arbitrary non-uniform rectangular net  $\bar{\omega}_h (x_i \in \bar{G})$ , where  $x_i = (x_1^{(i)}, x_2^{(i)})$  is the node of the net with coordinates  $x_1^{(i)}$  and  $x_2^{(i)}$ ,  $i_{\alpha} = 0, \pm 1, \pm 2, \dots$ ,  $\alpha = 1, 2$ . The steps of the net  $h_1^{(i)} = x_1^{(i)} - x_1^{(i-1)}$  and  $h_2^{(i)} = x_2^{(i)} - x_2^{(i-1)}$  are functions of the coordinates  $x_1^{(i)}$  and  $x_2^{(i)}$  respectively. Let  $\omega_h = \{x_i \in G\}$  be the internal net region,  $\gamma = \{x_i \in \Gamma\}$  the boundary net region. Through the point  $x_i \in \omega_h$  we draw the straight line  $\mathcal{L}_{\alpha}$ , parallel to the coordinate axis  $Ox_{\alpha}$ ; the set of all nodes of the net  $\bar{\omega}_h$  lying on  $\mathcal{L}_{\alpha}$ , is called the chain  $I_{\alpha}$ . Let  $x_{\Pi} = x_{\Pi}^{(\alpha)} \in \gamma$  and  $x_{\Pi} = x_{\Pi}^{(\alpha)} \in \gamma$  be boundary points of the chain  $I_{\alpha}$ , where  $x_{\alpha}$  increases on going from  $x_l$  to  $x_r$ , and  $\gamma_{\alpha, l}$  is the set of the nodes  $x_l$  of all chains  $I_{\alpha}$  in a given direction,  $\gamma_{\alpha, r}$  is the set of nodes  $x_r$ . We put  $\gamma_{\alpha} = \gamma_{\alpha, \Pi} + \gamma_{\alpha, \Pi}$ ,  $\omega_h^{(+\alpha)} = \omega_h + \gamma_{\alpha, \Pi}$ ,  $\omega_h^{(-\alpha)} = \omega_h + \gamma_{\alpha, \Pi}$ . The segment  $0 \leq t \leq T$  is divided into  $K$  equal parts of length  $\tau$  by the points  $t_0 = 0$ ,  $\tau, \dots, t_j = j\tau, \dots, t_K = K\tau = T$ ,  $\tau = T/K$  (uniform net with step  $\tau$ ) and we introduce the intermediate (fractional) steps  $t_{j+\frac{1}{2}} = t_j + \tau/2 = (j + \frac{1}{2})\tau$ ,  $j = 0, 1, \dots, K-1$ . The net  $\bar{\omega}_T$  with respect to time contains both integral and fractional steps, i.e.  $\bar{\omega}_T = \{t_{j*} \in [0 \leq t \leq T]\}$ , where  $t_{j*} = j^*\tau$ ,  $j^* = 0, \frac{1}{2}, 1, \dots, j, j + \frac{1}{2}, \dots, K - \frac{1}{2}, K$ . The point  $(x_i, t_{j*}) \in \bar{Q}_T$  is a node of the space-time net  $\bar{\Omega}$  and  $\Omega = \{(x_i, t_{j*}) \in Q_T\}$  is the set of internal nodes of the net  $\bar{\Omega}$ ; it is clear that  $\bar{\Omega} = \bar{\omega}_h \times \bar{\omega}_T$ ,  $\Omega = \omega_h \times \omega_T$ , where

$$\omega_T = \{t_j \in (0 < t \leq T)\}.$$

We shall use the notation given in [3]. Let  $y = y(x, t_j) = y^{j*}$  be a net function given on  $\bar{\Omega}$ . We shall write  $y^{(\pm m_\alpha)} = y(x^{(\pm m_\alpha)}, t)$ ,  $x^{(\pm m_\alpha)} = (x_1^{(i_1 \pm m_1)}, x_2^{(i_2)})$ ,  $x_1^{(i_1 - 1/2)} = \frac{1}{2}(x_1^{(i_1)} + x_1^{(i_1 - 1)})$ ,  $y_{\bar{x}_\alpha} = (y - y^{(-1_\alpha)})/h_\alpha$ ,  $y_{x_\alpha} = (y^{(+1_\alpha)} - y)/h_{\alpha+}$ ,  $y_{\hat{x}_\alpha} = (y^{(+1_\alpha)} - y)/\hat{h}_\alpha$ ,  $h_\alpha = h_\alpha^{(i_\alpha)}$ ,  $h_{\alpha+} = h_\alpha^{(i_\alpha + 1)}$ ,  $\hat{h}_\alpha = (h_\alpha + h_{\alpha+})/2$ ,  $y_{\bar{t}_\alpha} = (y^{j+\alpha/2} - y^{j+(\alpha-1)/2})/\tau$ . We need the following sums and norms:

$$\begin{aligned} (y, v)^* &= \sum_{\omega_h} y_i v_i \hat{h}_1^{(i_1)} \hat{h}_2^{(i_2)}, & (y, v) &= \sum_{\omega_h} y_i v_i h_1^{(i_1)} h_2^{(i_2)}, \\ (y, v)_{1_\alpha} &= \sum_{\omega_h^{(+\alpha)}} y_i v_i h_\alpha^{(i_\alpha)} \hat{h}_{\alpha \pm 1}^{(i_\alpha \pm 1)} \quad (+ \text{ for } \alpha = 1, \quad - \text{ for } \alpha = 2), \\ \|y\| &= \sqrt{(y, y)^*}, & \|y\|_2 &= \sqrt{(y, y)}, & \|y\|_{2_\alpha} &= \sqrt{(1, y^2)_{1_\alpha}}, \\ \|y\|_{3_\alpha} &= \|\eta_\alpha\|_{2_\alpha}, & \text{where } (\eta_\alpha)_{\hat{x}_\alpha} &= y, \\ \eta_\alpha &= 0 \text{ for } x \in \Gamma_{\alpha, n}. \end{aligned}$$

4. The operator  $L_\alpha u$  is replaced by the homogeneous uniform net scheme

$$\Lambda_\alpha y = (a_\alpha y_{\bar{x}_\alpha})_{\hat{x}_\alpha},$$

having second order approximation on a uniform net (see [4] and [5]) so that  $\Lambda_\alpha u - L_\alpha u = O(h_\alpha^2)$ .

The class of pattern functionals  $a_\alpha$  is described in [4]; in particular, for  $a_\alpha = k_\alpha^{(-\frac{1}{2})}$  we obtain the scheme used in [2]. We put the problem (1)-(3) in correspondence with the local one-dimensional scheme:

$$y_{\bar{t}_\alpha} = \Lambda_\alpha y^{(0)} \text{ for } (x, t_{j+\alpha/2}) \in \Omega, \alpha = 1, 2; j = 0, 1, 2, \dots, K-1; \tag{4}$$

$$y(x, t_{j+\alpha/2}) = u_1(x, t_{j+\alpha/2}) \text{ for } x \in \Gamma_\alpha; \quad y(x, 0) = u_0(x) \text{ for } x \in \bar{\omega}_h; \tag{5}$$

$$(a_\alpha(x, t) \geq c_1 > 0, (x, t) \in \omega_h^{(+\alpha)} \times [0 \leq t \leq T]), \tag{6}$$

where  $y^{(0)} = \sigma y + (1 - \sigma)\hat{y}$ ,  $0 \leq \sigma \leq 1$ ,  $y = y(x, t_{j+\alpha/2})$ ,  $\hat{y} = y(x, t_{j+(\alpha-1)/2})$ ,

$y_{\bar{t}_\alpha} = (y - \hat{y})/\tau$ ,  $\Lambda_\alpha y = (a_\alpha(x, t^*) y_{\bar{x}_\alpha})_{\hat{x}_\alpha}$ ,  $t^* \in [t_j, t_{j+1}]$  and for definiteness we shall take  $t^* = t_{j+1/2}$ . It is clear from (4)-(5) that at each moment  $t_{j+\alpha/2}$  we solve the first boundary problem for all chains  $\Pi_\alpha$  in the given direction  $x_\alpha$ .

5. Let us now go on to study the convergence and accuracy of the scheme (4)-(6). Let  $u$  be the solution of problem (1)-(3),  $y$  the solution of problem (4)-(6). For  $z = y - u$  we obtain the conditions

$$z_{\bar{\Gamma}_\alpha} = \Lambda_\alpha z^{(0)} + \psi_\alpha, \quad \psi_\alpha = \Lambda_\alpha u^{(0)} - u_{\bar{\Gamma}_\alpha}, \tag{7}$$

$$z(x, t_{j+\alpha/2}) = 0, \quad x \in \Gamma_\alpha; \quad z(x, 0) = 0, \quad x \in \bar{\omega}_h, \tag{8}$$

$$a_\alpha(x, t_{j+\alpha/2}) \geq c_1 > 0. \tag{9}$$

Suppose conditions A are satisfied. By analogy with (7) we find

$$\Lambda_\alpha u - L_\alpha u = (\mu_\alpha)_{\hat{x}_\alpha} + \psi_{\alpha, \alpha}, \quad \|\mu_\alpha\|_{z_\alpha} = O(\|h_\alpha^2\|_{z_\alpha}), \quad \|\psi_{\alpha, \alpha}\| = O(\|h_\alpha^2\|_{z_\alpha}), \tag{10}$$

where  $\|h_\alpha^2\|_{z_\alpha} = \sqrt{(1, h_\alpha^2)_{z_\alpha}}$  is the mean square value of  $h_\alpha^2$ .

We put  $\psi_{\alpha, \alpha}$  in the form

$$\psi_\alpha = \psi_\alpha^0 + \psi_\alpha^*, \quad \psi_\alpha^0 = \left( L_\alpha u - \frac{1}{2} \frac{\partial u}{\partial t} \right)^{j+1}, \quad \sum_{\alpha=1}^2 \psi_\alpha^0 = 0, \tag{11}$$

$$\psi_\alpha^* = (\mu_\alpha)_{\hat{x}_\alpha} + \varphi_\alpha, \quad \|\varphi_\alpha\|_2 = O(\|h_\alpha\|_2^2) + O(\tau). \tag{12}$$

6. With the estimate of the solution of the problem (7)-(9) we use the method of integral inequalities and a special method of summing the local errors  $\psi_{\alpha, \alpha}$  using the relation  $\psi_1^0 + \psi_2^0 = 0$ . Let us write down the basic integral identity with  $\sigma = \frac{1}{2}$ . Let us multiply (7) by  $(z + \bar{z})h_1h_2$  and sum with respect to  $\omega_h$ . Using Green's formula

$$(z + \bar{z}, \Lambda_\alpha z^{(j/2)})^* = \frac{1}{2} \| \sqrt{a_\alpha} (z_{x_\alpha} + \bar{z}_{x_\alpha}) \|_{z_\alpha}^2 = I_\alpha, \tag{13}$$

we obtain the following identities:

$$(\|z\|^2)_{\bar{\Gamma}_\alpha} + I_\alpha = (\psi_\alpha, z + \bar{z})^*, \tag{14}$$

$$(\|z\|^2)_{\bar{\Gamma}} + I_1 + I_2 = \sum_{\alpha=1}^2 (\psi_\alpha, z + \bar{z})^*, \tag{15}$$

where

$$(\|z\|^2)_{\bar{\Gamma}} = (\|z^{j+1}\|^2 - \|z^j\|^2) / \tau.$$

Putting  $\psi_{\alpha, \alpha} = 0$  in (15) we see that the scheme (4)-(6) is stable with respect to the initial data (for any  $h_{\alpha, \alpha}$  and  $\tau$ ):

$$\|z(x, t_{j^*})\| \leq \|z(x, 0)\| \quad (t_{j^*} = j^* \tau, \quad j^* = \frac{1}{2}, 1, \frac{3}{2}, \dots, K - \frac{1}{2}, K). \tag{16}$$

7. Lemma 1. If  $\psi_\alpha$  can be put in the form

$$\psi_\alpha = \psi_\alpha^0 + (\mu_\alpha)_{\hat{x}_\alpha} + \varphi_\alpha, \quad \sum_{\alpha=1}^2 \psi_\alpha^0 = 0, \tag{17}$$

then for the solution of problem (7)-(9) for  $\sigma = \frac{1}{2}$  and any  $h_\alpha$  and  $\tau$  we have the estimate

$$\|z(x, t_{j+1})\| \leq \tau \|\psi^0(x, t_{j+1})\| + \tau \sqrt{e} \|\overline{\psi^0(x, t_j)}\| + \sqrt{1+e} \sum_{j'=1}^{j+1} \tau Q^{j'}, \tag{18}$$

where

$$\begin{aligned} \psi^0 &= \psi_2^0 = -\psi_1^0, \quad \|\overline{\psi^0(x, t_j)}\| = \left( \frac{1}{t_j} \sum_{j'=1}^j \tau \|\psi^0(x, t_{j'})\|^2 \right)^{1/2}, \\ Q^{j'} &= \tau^2 t_j \|\psi_t^0(x, t_{j'})\|^2 + \sum_{\alpha=1}^2 (\|\mu_\alpha(x, t_{j'})\|_{2_\alpha}^2 + \|\varphi_\alpha(x, t_{j'})\|_{3_\alpha}^2). \end{aligned} \tag{18'}$$

We put (17) in (15) and make the following transformations:

$$\begin{aligned} \sum_{\alpha=1}^2 (\psi_\alpha^0, z + \bar{z})^* + (\psi^0, z^{j+1} - z^j) &= \tau (\psi^0(x, t_{j+1}), z(x, t_{j+1}))_t^* - \\ &- (\psi_t^*(x, t_{j+1}), z(x, t_j))^* \tau, \end{aligned}$$

$$\begin{aligned} |(\mu_\alpha)_{\hat{x}_\alpha}, z + \bar{z})^*| &= |(\mu_\alpha, z_{x_\alpha} + \bar{z}_{x_\alpha})_\alpha| \leq \frac{1}{2} J_\alpha + \frac{1}{c_1} \|\mu_\alpha\|_{2_\alpha}^2, \quad |(\varphi_\alpha, z + \bar{z})^*| \leq \\ &\leq \frac{1}{2} J_\alpha + \frac{1}{c_1} \|\varphi_\alpha\|_{3_\alpha}^2, \quad \tau |(\psi_t^0, z^j)^*| \leq \frac{\tau^2}{2c_0} \|\psi_t^0\|^2 + \frac{c_0}{2} \|z^j\|^2 \end{aligned}$$

where  $c_0$  is an arbitrary positive constant which we shall put equal to  $1/t_j$  later. As a result we obtain the inequality

$$\|z(x, t_{j+1})\|^2 \leq (1 + 0.5c_0\tau) \|z(x, t_j)\|^2 + 0.5\tau Q^{j+1} + \tau^2 (\psi^0(x, t_{j+1}), z(x, t_{j+1}))_t^*.$$

Using the inequality  $\tau |(\psi^0, z)^*| \leq 0.5 \|z\|^2 + 0.5\tau^2 + 0.5\tau^2 \|\psi^0\|^2$ , and also the initial condition  $z(x, 0) = 0$  we find

$$\|z(x, t_{j+1})\|^2 \leq c_0\tau \sum_{j'=1}^j \|z(x, t_{j'})\|^2 + \tau^2 \|\psi^0(x, t_{j+1})\|^2 + \sum_{j'=1}^j \tau Q^{j'}. \tag{19}$$

Let us now use the following simple lemma, the proof of which we omit:

Lemma 2. If  $\rho(t_j)$  and  $\sigma(t_j)$  are non-negative functions ( $t_j = j\tau$ ,  $j = 1, 2, \dots$ ) then the inequality

$$\rho(t_{j+1}) \leq c_0 \tau \sum_{j'=1}^j \rho(t_{j'}) + \sigma(t_{j+1}), \quad j = 1, 2, \dots, \quad (c_0 > 0)$$

gives

$$\rho(t_{j+1}) \leq \sigma(t_{j+1}) + c_0 e^{c_0 t_j} \sum_{j'=1}^j \tau \rho(t_{j'}). \tag{20}$$

It follows from the inequality (19), from Lemma 2, that

$$\|z(x, t_{j+1})\|^2 \leq \tau^2 \|\psi^0(x, t_{j+1})\|^2 + \tau^2 e \overline{\|\psi^0(x, t_j)\|^2} + (1 + e) \sum_{j'=1}^{j+1} \tau Q^{j'},$$

$$\overline{\|\psi^0(x, t_j)\|^2} = \frac{1}{t_j} \sum_{j'=1}^j \tau \|\varphi^0(x, t_{j'})\|^2.$$

This proves Lemma 1.

8. *Theorem 1.* If conditions A are satisfied, then the difference scheme (5)-(7) converges in the mean with speed  $O(\|h^2\|_2) + O(\tau)$  for  $\sigma = 0.5$  and as  $h_\alpha, \tau$  tend independently to zero on any non-uniform space net:

$$\|y(x, t_{j+\alpha/2}) - u(x, t_{j+\alpha/2})\| \leq M(\|h^2\|_2 + \tau), \quad \alpha = 1, 2, \quad j = 0, 1, \dots, K-1, \tag{21}$$

$$\|h^2\|_2 = \|h_1^2\|_{2_1} + \|h_2^2\|_{2_2}, \quad \|h_\alpha^2\|_{2_\alpha} = \sqrt{(1, h_\alpha^1)_\alpha}, \quad \alpha = 1, 2,$$

where  $M$  is a positive constant which does not depend on the choice of net.

For the estimate (21) for  $\alpha = 2$  (on a whole step) follows from (18) since, from (10) and (12) we have  $Q^{j'} = O(\|h^2\|_2^2) + O(\tau^2)$ . To estimate  $\|z(x, t_{j+1/2})\|$  on a fractional step we use (18) with  $\alpha = 2$  and the identity (14) for  $\alpha = 1$ . Thus on fractional steps the scheme (5)-(7) has the same order of accuracy as on integral steps.

The convergence of the scheme (5)-(7) can be proved with considerably weaker requirements than conditions A.

*Theorem 2.* If the following conditions are satisfied then the scheme (4)-(6) converges in the mean for  $\sigma = 0.5$  and for any  $h_\alpha$  and  $\tau$  so that

$$\|y(x, t_{j+\alpha/2}) - u(x, t_{j+\alpha/2})\| \leq M(\|h^\gamma\|_2 + \tau^\gamma). \tag{22}$$

The conditions are (1)  $\partial k_\alpha / \partial x_\alpha, \partial^2 u / \partial x_\alpha^2$  satisfy in  $\bar{Q}_\tau$  the Hölder conditions of order  $\gamma, 0 < \gamma \leq 1$ , (2)  $\partial u / \partial x_\alpha, \partial^2 u / \partial x_\alpha^2, \partial u / \partial t, k_\alpha, \partial k_\alpha / \partial x_\alpha$  satisfy in  $\bar{Q}_\tau$

the Hölder conditions of order  $\gamma > 0$  with respect to  $t$ .

To prove Theorem 2 we have to use the fact that  $\|\Phi_\alpha\|_{3_\alpha} = O(\|h_\alpha^\gamma\|_{2_\alpha}) + O(\tau^\gamma)$ ,  $\|\mu_\alpha\|_{2_\alpha} = O(\|h_\alpha^{1+\gamma}\|_{2_\alpha})$  and make use of Lemma 1.

9. We have given all the preceding reasoning for the simplest equation of heat conductivity, (1). However, Theorems 1 and 2 remain true for equations of the general form

$$c(x, t) \frac{\partial u}{\partial t} = \sum_{\alpha=1}^2 L_\alpha u + f(x, t) - g(x, t)u; \quad L_\alpha u = \frac{\partial}{\partial x_\alpha} \left( k_\alpha(x, t) \frac{\partial u}{\partial x_\alpha} \right) + r_\alpha(x, t) \frac{\partial u}{\partial x_\alpha},$$

$$c(x, t) \frac{\partial u}{\partial t} = \sum_{\alpha=1}^2 L_\alpha u + f(x, t, u); \quad L_\alpha u = \frac{\partial}{\partial x_\alpha} \left( k_\alpha(x, t, u) \frac{\partial u}{\partial x_\alpha} \right) + r_\alpha(x, t, u) \frac{\partial u}{\partial x_\alpha}.$$

The corresponding expressions for  $\Lambda_\alpha y$  are given in [3] and we shall not give them here. It is proved there that the local one-dimensional method is uniformly convergent for  $\sigma = 1$ . If  $r_\alpha \neq 0$ ,  $q_\alpha \neq 0$ , then the estimate is true for  $\tau < \tau_0$  where  $\tau_0$  depends on  $c_1$  and  $\max |r_\alpha|$ ,  $\max |q_\alpha|$ .

10. The method of summation of local errors  $\psi_\alpha$  used here, which leads to the *a priori* estimate (18), cannot be extended to the case  $p > 2$ . In this case instead of (21) we have only been able to obtain the estimate

$$\|y - u\| \leq M(\sqrt{\tau} + h^2).$$

If we suppose that  $\tau/h_\alpha^2 \leq [2(1-\sigma)\max a_\alpha]^{-1}$ , then using the method of [3] we can prove the uniform convergence of the scheme (5)-(7) for  $0 \leq \sigma \leq 1$  and for any  $p > 1$  with speed  $O(\bar{h}^2) + O(\tau)$ , where  $h^2 = \bar{h}_1^2 + \bar{h}_2^2$ ,  $\bar{h}_\alpha$  is the maximum value of the step  $h_\alpha^{(i_\alpha)}$  on the net  $\omega_h^{(+1_\alpha)}$ . For the case  $\sigma = 1$  this estimate holds for any  $h_\alpha$  and  $\tau$  (Theorem 2 of [3]).

It must be stressed that we are using here boundary conditions without approximation (in contrast to [3]). This formulation of the difference boundary conditions always leads to different schemes on non-uniform nets. We have proved that in this case also second order accuracy is attained with respect to space.

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